NOTES AND CORRESPONDENCE

Inconsistent Finite Differencing Errors in the Variational Adjustment of Horizontal Wind Components

By Y. K. Sasaki, P. S. Ray, J. S. Goerss and P. Soliz

University of Oklahoma, Norman, Oklahoma 73019

(Manuscript received 28 July 1978, in revised form 30 October 1978)

Abstract

Inconsistency in writing a set of partial differential equations in finite difference form results in error in the numerical solution. An example is shown in which horizontal wind components are variationally adjusted to remove the surface pressure tendency. Three Euler-Lagrange equations are solved simultaneously using two different finite difference forms, one of which is consistent and the other inconsistent. Derived quantities (divergence and ω-velocity) are examined using both consistently and inconsistently adjusted horizontal wind components.

1. Introduction

Significant errors can arise in numerical computations when a set of model differential equations is inconsistently written. For example, the inconsistency appears when the first derivative $\partial\lambda/\partial x$ is approximated by the centered difference $(\lambda_{i+1}-\lambda_{i-1})/2d$ and the second derivative $\partial^2\lambda/\partial x^2$ is approximated by $(\lambda_{i+1}-2\lambda_i+\lambda_{i-1})/d^2$ where λ is a variable, d is the grid size, and i represents the i-th grid point on the x-coordinate. The finite difference analog of the second derivative operator consistent with the first derivative finite differencing is $(\lambda_{i+2}-2\lambda_i+\lambda_{i-2})/4d^2$. The truncation error is smaller for the inconsistent scheme than for the consistent one. However, the inconsistent scheme produces errors which warrant careful consideration in designing numerical schemes.

Although the problem of inconsistency has been described quite clearly by Miyakoda and Moyer (1968), the authors feel that further illumination of the problem would be useful. In the next section a variational analysis problem is posed in which the grid-point horizontal wind velocities on a pressure coordinate system are variationally adjusted subject to a constraint which removes the surface pressure tendency. In the final section divergences and ω-velocities computed from upper winds that have been adjusted consistently are compared with those produced when the adjustment was inconsistent. It is found that the constraint is not satisfied when the finite difference scheme is inconsistent.

2. Variational adjustment of horizontal wind components

The problem to be examined is the variational adjustment of gridded wind velocities on a pressure coordinate system so that the surface pressure tendency will be zero everywhere. Mass continuity is expressed by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0 \qquad (1)$$

where u and v are the horizontal wind components on the x and y coordinates, respectively. When (1) is integrated from the bottom to the top of the atmosphere and ω is assumed to be zero at both places, the following integral constraint results,

$$\int_0^{p_s}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dp=0 \qquad (2)$$

This constraint will enforce the condition that $dp_s/dt=0$. At any pressure level the value of ω can be computed by

$$\omega=\int_0^p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dp=-\int_{ps}^p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dp \tag{3}$$

The divergences and ω-velocities computed from the u- and v-components after adjustment to satisfy constraint (2) are identical to those one would obtain using the techniques of Kurihara (1961) and O'Brien (1970). The advantage of the approach presented here is that rather than adjusting only the derived quantities of divergence and ω-velocity, the horizontal wind components are adjusted and are thus consistent with those derived quantities.

A number of analyses have been conducted in past years using the variational method to accommodate necessary constraints (*e.g.*, McGinley, 1975; McFarland, 1975). Consistent finite difference formulation may be accomplished when the functionals in the variational methods are written in a finite difference form (Sasaki, 1970). We will follow this approach in order to study the errors arising from inconsistency in finite differencing. If the adjusted and unadjusted grid values are denoted by u, v, and $\tilde{u}$, $\tilde{v}$, respectively, then a functional may be written (for an $im\times jm$ grid with km levels)

$$J=\sum_{i=1}^{im}\sum_{j=1}^{jm}\sum_{k=1}^{km}\tilde{\alpha}(p)[(u-\tilde{u})^2+(v-\tilde{v})^2] +\sum_{i=1}^{im}\sum_{j=1}^{jm}\lambda m^2\sum_{k=1}^{km}[\nabla_x(u/m)+\nabla_y(v/m)] \tag{4}$$

where λ is the Lagrange multiplier, $\tilde{\alpha}(p)$ is a weight which may vary with height, and m is the map factor defined by $1+\sin 60°/1+\sin\theta$ where θ is the latitude. The finite difference operators ∇_x and ∇_y are defined as

$$\nabla_x(\ )=\{(\ )_{i+1}-(\ )_{i-1}\}/2d$$

and

$$\nabla_y(\ )=\{(\ )_{j+1}-(\ )_{j-1}\}/2d \tag{5}$$

where d is the grid size at latitude 60°N and i and j are the grid-point numbers in the x and y coordinates, respectively.

Kurihara (1961) presents two schemes for adjusting ω-velocities computed from upper wind observations. One is a simple scheme where the correction of the horizontal divergence is uniform with height while the other is more complicated and involves non-uniform divergence corrections based on the accuracy of upper wind data. If $\alpha(p)$ in (4) is set to some constant, then the ω-velocities computed from the variationally adjusted components will be identical to those obtained using Kurihara's simple correction scheme. On the other hand if $\alpha(p)$ is permitted to increase with pressure based on the accuracy of the upper wind data, then the results would parallel the second correction scheme of Kurihara.

In order to simplify the discussion of inconsistency error in finite differencing analogs, $\alpha(p)$ will be given the constant value 1. The extremum of the functional (4) is given by the condition that J is stationary and its first variation vanishes, *i.e.*,

$$\delta J=\sum_{i=1}^{im}\sum_{j=1}^{jm}\sum_{k=1}^{km}[2(u-\tilde{u})\delta u+2(v-\tilde{v})\delta v] +\sum_{i=1}^{im}\sum_{j=1}^{jm}\lambda_m{}^2\sum_{k=1}^{km}[\nabla_x(\delta u/m)+\nabla_y(\delta y/m)] +\sum_{i=1}^{im}\sum_{j=1}^{jm}\delta\lambda m^2\sum_{k=1}^{km}[\nabla_x(u/m)+\nabla_y(v/m)]=0$$

Using the finite difference equivalent of integration by parts (Sasaki, 1970) the above equation becomes

$$\delta J=\sum_{i=2}^{im-1}\sum_{j=2}^{jm-1}\sum_{k=1}^{km}\{[2(u-\tilde{u})\delta u+2(v-\tilde{v})\delta v] -[\nabla_x(\lambda m^2)(\delta u/m)+\nabla_y(\lambda m^2)(\delta v/m)]\} +\sum_{i=1}^{im}\sum_{j=1}^{jm}\delta\lambda m^2\sum_{k=1}^{km}[\nabla_x(u/m)+\nabla_y(v/m)] +B=0 \tag{6}$$

where B represents the terms which are uncommutable and appear at the boundary grid points and also the grid points next to the boundary.

The boundary terms vanish under various conditions. A set of such conditions may be expressed by

$$\lambda_{1,j}=\lambda_{2,j}=\lambda_{im-i,j}=\lambda_{im,j}=0 \text{ for } 1\leqq j\leqq jm$$

and

$$\lambda_{i,1}=\lambda_{i,2}=\lambda_{i,jm-1}=\lambda_{i,jm}=0 \text{ for } 1\leqq i\leqq im \tag{7}$$

Note that m and λ are not functions of pressure.

Using the boundary conditions (7) together with the condition that δu and δv are arbitrary functions, we obtain the so-called Euler-Lagrange equations:

$$2(u-\tilde{u})-\frac{1}{m}\nabla_x(m^2\lambda)=0 \tag{8}$$

$$2(v-\tilde{v})-\frac{1}{m}\nabla_y(m^2\lambda)=0 \tag{9}$$

$$\sum_{k=1}^{km}[\nabla_x(u/m)+\nabla_y(v/m)]=0 \qquad (10)$$

Substituting u from (8) and v from (9) into (10), we obtain an equation for λ:

$$\nabla_x\left(\frac{1}{m^2}\nabla_x m^2\lambda\right)+\nabla_y\left(\frac{1}{m^2}\nabla_y m^2\lambda\right)$$

$$=-\frac{2}{km}\sum_{k=1}^{km}[\nabla_x(\tilde{u}/m)+\nabla_y(\tilde{v}/m)] \qquad (11)$$

In order to facilitate the discussion of inconsistency errors, m is assumed to be 1 so that (11) becomes

$$\nabla^2_x\lambda+\nabla^2_y\lambda=-\frac{2}{km}\sum_{k=1}^{km}[\nabla_x\tilde{u}+\nabla_y\tilde{v}] \qquad (12)$$

From the definition of ∇_x and ∇_y given in (5), $\nabla_x{}^2$ and $\nabla_y{}^2$ are expressed as

$$\nabla^2_x\lambda=(\lambda_{i+2,j}-2\lambda_{i,j}+\lambda_{i-2,j})/4d^2$$

and

$$\nabla^2_y\lambda=(\lambda_{i,j+2}-2\lambda_{i,j}+\lambda_{i,j-2})/4d^2 \qquad (13)$$

One can see that the right-hand side of (12) involves the vertically summed divergence of the unadjusted winds. A solution for λ is found using (12) and substituted into (8) and (9) in order to obtain the adjusted horizontal wind components. From (10) it can be seen that the vertically summed divergence of the adjusted winds should be zero.

3. Results

An inconsistency error will be induced when the finite difference analogs of the second derivatives are written

$$\nabla^2_x\lambda=(\lambda_{i+i,j}-2\lambda_{i,j}+\lambda_{i-1,j})/d^2$$
$$\nabla^2_y\lambda=(\lambda_{i,j+1}-2\lambda_{i,j}+\lambda_{i,j-1})/d^2, \qquad (14)$$

and substituted for $\nabla^2_x\lambda$ and $\nabla^2_y\lambda$ in (12). In practice the use of this inconsistent finite difference operator in (12) is equivalent to incorrectly formulating the right-hand side of the equation.

In order to fully examine the effects of inconsistent finite differencing, the integral constraint described previously was imposed on an actual wind analysis using the finite difference schemes (13) and (14). For the *consistent* scheme (13) the boundary condition $\lambda=0$ must be imposed at the boundary and at one additional row of grid points immediately adjacent to the boundary. The *inconsistent* scheme (14) requires the boundary condition only at the actual boundary points.

The gridded values of $\tilde{u}$ and $\tilde{v}$ used in this test were produced by the U.S. Air Force Global Weather Central's (AFGWC) operational objective analysis program (Moreno, 1973), which uses a successive correction method (Cressman, 1959). Our study employed a 12×10 subset of the AFGWC grid located approximately over the eastern half of the United States. To simplify the vertical finite differencing the wind fields were interpolated from the standard pressure levels to 10 levels separated by 100 mb and beginning at 1,000 mb.

The variational objective analysis was performed upon three different data sets using both inconsistent and consistent finite difference schemes. In all cases both m and $\tilde{\alpha}(p)$ were set to 1. The three data sets were Case 1: 00GMT February 25, 1975, Case 2: 12GMT February 17, 1975, and Case 3: 12GMT February 28, 1975. Table 1 shows the average absolute error $\bar{\varepsilon}$ in the divergence calculation for the adjusted winds as defined by

$$\bar{\varepsilon}=\overline{\left|\frac{1}{km}\sum_{k=1}^{km}(\nabla_x u+\nabla_y v)\right|}$$

for each case and finite difference scheme. The overbar denotes an average over the entire grid.

Table 1. Average absolute error, $\bar{\varepsilon}$, for consistent and inconsistent finite difference schemes.

Case	Finite Difference Scheme	$\bar{\varepsilon}=\overline{\left\lvert\frac{1}{km}\sum_{k=1}^{km}(\nabla_x u+\nabla_y v)\right\rvert}$ [sec^{-1}]
1		1.06×10^{-10}
2	Consistent	$8.95+10^{-11}$
3		7.84×10^{-11}
1		1.98×10^{-5}
2	Inconsistent	1.62×10^{-5}
3		1.03×10^{-5}

From (10) it can be seen that u and v should be adjusted so that $\bar{\varepsilon}=0$. Thus, $\bar{\varepsilon}$ can be used as an index of the inconsistency error.

It can be seen in Table 1 that for each case the value of $\bar{\varepsilon}$ is essentially zero (since only 6 significant digits were carried by the computer used) when consistent differencing was used. With inconsistent differencing the values of ε were in every case about half as large as the average of the absolute value of the vertically summed divergences for the unadjusted winds! Clearly, it is essential that consistent finite difference schemes be employed in order to accurately enforce the constraint.

Since the results for the three cases examined were quite similar, only those for Case 1 are displayed in Figs. 1-3. Figs. 1 and 2 compare the 550 mb ω-velocity fields with the analyzed

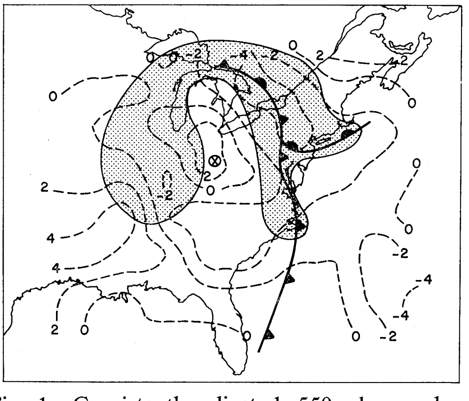

Fig. 1 Consistently-adjusted 550-mb ω-velocities (μb sec^{-1}) and analyzed surface frontal positions and areas of active precipitation at 00Z February 25, 1975.

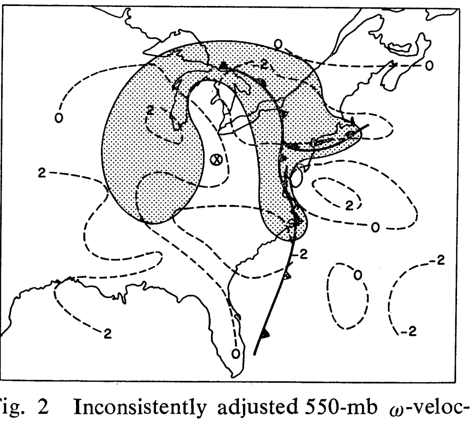

Fig. 2 Inconsistently adjusted 550-mb ω-velocities (μb sec^{-1}) and analyzed surface frontal positions and areas of active precipitation at 00Z February 25, 1975.

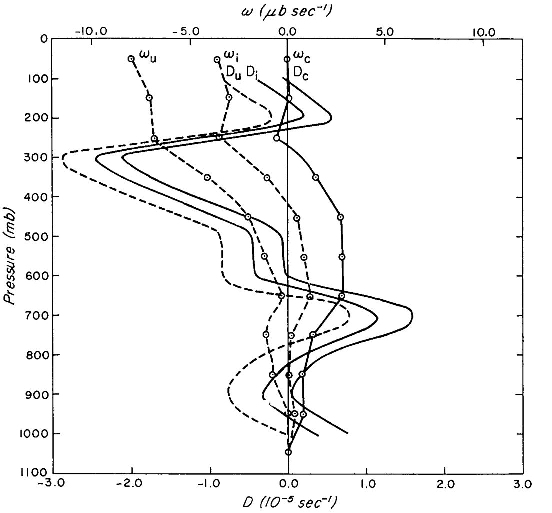

Fig. 3 Vertical profiles of unadjusted (D_u and ω_u), inconsistently adjusted (D_i and ω_i), and consistently adjusted (D_c and ω_c) divergence and ω-velocity at grid point X, 00Z February 25, 1975.

frontal positions and areas of precipitation at 00Z February 25, 1975 for the consistent and inconsistent schemes, respectively. Although any comparison of vertical velocities with areas of precipitation that one might make using these figures is quite subjective, it is readily apparent that the consistent and inconsistent ω-fields are quite different. The ω-velocities shown in Fig. 1 are consistent with horizontal winds which satisfy the mass constraint (2) while those in Fig. 2 are not. This point is illustrated by Fig. 3 which shows divergence and ω-velocity profiles computed before adjustment and after consistent and inconsistent adjustment at the grid point whose location is given by X in Figs. 1 and 2. The results shown in this figure are typical of those found at other grid points and point out the fact that the constraint (2) is not satisfied by the inconsistently adjusted winds. On the other hand it can be seen that for the consistently adjusted winds, the vertically summed divergence as well as the ω-velocity at the top of the model atmosphere are zero.

References

Cressman, G. P., 1959: An Operational Objective Analysis System. *Mon. Wea. Rev.*, **87**, 367–374.

Kurihara, Y., 1961: Accuracy of winds-aloft data and estimation of error in numerical analysis of atmospheric motions. *J. Meteor. Soc. Japan*, **39**, 331–345.

McGinley, J. A., 1973: Environmental energy fields associated with severe storms. Master's Thesis, Univ. of Oklahoma, Dept. of Meteorology.

McFarland, M. J., 1975: Variational optimization analysis of temperature and moisture in a severe storm environment. Ph.D. Dissertation, Univ. of Oklahoma, Dept. of Meteorology.

Miyakoda, K., and R. W. Moyer, 1968: A method of initialization for dynamic weather forecasting. *Tellus*, **20**, 115–128.

Moreno, Capt. Dennis, 1973: "AFGWC Macro-scale Upper Air Analysis Model (Revised)". AFGWCT M 73–1, October 1, 1973.

O'Brien, J., 1970: Alternative solution to the classical vertical velocity problem. *J. Appl. Meteor.*, **9**, 197–203.

Sasaki, Y. K., 1970: Some basic formalisms in numerical variational analysis. *Mon. Wea. Rev.*, **98**, 875–883.

水平風成分の変分法的調節における不調和差分誤差について

Y. K. Sasaki・P. S. Ray・J. S. Goerss・P. Soliz

オクラホマ大学，アメリカ合衆国

偏微分方程式を差分形で書きあらわすとき，もし差分形の不調和があると数値解に誤差が生じる。その例として，地表面気圧変化傾向を無くすように，水平風成分を変分的に調節する場合を示す。調和したものと不調和なものと二つの相異る差分形を用いて，3つのオイラー・ラグランジュ方程式を連立して解く。そして調和，不調和の二通りの場合について，調節した水平風成分を用いた物理的導出量（すなわち，発散と速度）を調べた。