Department of Atmospheric Sciences
National Central University
Advanced Applied Mathematics
Homework IV
2021/10/27-2021/11/10

(1) 10 pts. 一質點被限制在 $\phi(x,y,z)=0$ 所代表的曲面上運動。假設在 0 到 T 的時間內，該質點由(x_1,y_1,z_1)移動到(x_2,y_2,z_2)，且其動能$\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)$隨時間的累積量為最小。試證明此質點運動的座標滿足:

$$\frac{\ddot{x}}{\phi_x}=\frac{\ddot{y}}{\phi_y}=\frac{\ddot{z}}{\phi_z}$$

A particle is confined to move only along the surface represented by $\phi(x,y,z)=0$. Suppose from time *0* to *T*, this particle moves from (x_1,y_1,z_1) to (x_2,y_2,z_2). During this period of time, its kinematic energy $\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)$, after being accumulated over time, is a minimum. Please prove that the coordinate of this particle's movement will satisfy:

$$\frac{\ddot{x}}{\phi_x}=\frac{\ddot{y}}{\phi_y}=\frac{\ddot{z}}{\phi_z}$$

(2) 30 pts. $J=\int_{x_1}^{x_2}\left[p(x)y'^2-q(x)y^2\right]dx$

(a) 約束條件$\int_{x_1}^{x_2} r(x)y^2dx=1$，邊界條件在 $x = x_1, x_2$ 處已給定。

證明使$\delta J=0$的 Euler eq.為 $\frac{d}{dx}\left(p\frac{dy}{dx}\right)+(q+\lambda r)y=0$，其中$\lambda$為一常數。

If the constraint is $\int_{x_1}^{x_2} r(x)y^2dx=1$, with *x* being specified at the boundaries $x = x_1, x_2$, prove that in order to let $\delta J=0$, the Euler equation is $\frac{d}{dx}\left(p\frac{dy}{dx}\right)+(q+\lambda r)y=0$ where λ is a constant.

(b) 利用(a)的結果，如欲使 $\delta\int_0^{\pi} y'^2dx=0$，且受到 $\int_0^{\pi} y^2dx=1$ 的約束，已知邊界條件 $y(0) = 0$, $y(\pi) = 0$

證明 $y=\sqrt{\frac{2}{\pi}}\sin nx$，為本題的 Euler eq. ，其中 n 是一不為零的整數。

Using the result from (a), suppose we want to let $\delta\int_0^{\pi} y'^2 dx = 0$, and subject to the constraint $\int_0^{\pi} y^2 dx = 1$, with known boundaries conditions y(0) = 0, y(π) = 0, prove that $y=\sqrt{\frac{2}{\pi}}\sin nx$ is the Euler equation for this problem, where *n* is a non-zero integer.

(c) 如果沒有 $\int_0^{\pi} y^2 dx = 1$ 這個約束條件，證明(b)中的解變成 $y \equiv 0$.(a trivial solution)

If the constraint $\int_0^{\pi} y^2 dx = 1$ does not exist, prove that the solution in (b) becomes a trivial solution $y \equiv 0$.

(3) 10 pts. 一物體質量 m，在空間中受重力 g 牽引鉛直下降。

令 $L = T - V = m\left(\frac{1}{2}\dot{x}^2 + gx\right)$ ，x 為下降距離。

請證明會使得 L 由時間 t_1 到 t_2 的累積量為最小的 Euler-Lagrange Equation，就是控制此物體移動的運動方程式。

A particle with mass *m* falls down due to the gravity.

Let $L = T - V = m\left(\frac{1}{2}\dot{x}^2 + gx\right)$ where *x* is the falling distance. Prove that the Euler-Lagrange Equation which can make *L* accumulating from time t_1 to t_2 reach a minimum is also the governing equation for the movement of this particle.