Derivation of Eq. (2.30)

From (2.20), we linearize the integrand with respect to h to get equation (2.30)

$$S(Z)=\int_0^Z \frac{acdh}{(a+h)\left[(a+h)^2 n^2(h)-C^2\right]^{1/2}} \tag{2.20}$$

$$\frac{dn}{dh}=\beta_0., \quad n=\beta_0 h+c', \quad n(0)=1=c'$$

$$c=an(0)\cos\theta_e=a\cos\theta_e$$

$$n=1+\beta_0 h, \quad n^2=(1+\beta_0 h)^2\sim 1+2\beta_0 h \qquad (\because h\le a, \quad \beta_0 h<<1)$$

example: $\because \dfrac{dN}{dh}=-300km^{-1} \quad N=(n-1)\times 10^6, \quad \dfrac{dN}{dh}\times 10^{-6}=\dfrac{dn}{dh}$

$$\therefore \frac{dn}{dh}=\beta_0=-3\times 10^{-4}km^{-1}$$

substitute into (2.20)

$$\int_0^Z \frac{a^2\cos\theta_e dh}{a\left[a^2(1+2\frac{h}{a})(1+2\beta_0 h)-a^2\cos^2\theta_e\right]^{1/2}}$$

$$=\int_0^Z \frac{a\cos\theta_e dh}{\left[a^2(1+2\frac{h}{a}+2\beta_0 h+4\frac{\beta_0 h^2}{a})-a^2\cos^2\theta_e\right]^{1/2}}$$

$$\cong\int_0^Z \frac{a\cos\theta_e dh}{\left(a^2+2ah+2a^2\beta_0 h-a^2\cos^2\theta_e\right)^{1/2}}$$

$$=\int_0^Z \frac{a\cos\theta_e dh}{\left[\sin^2\theta_e a^2+2ah(1+a\beta_0)\right]^{1/2}}$$

let $y=\sin^2\theta_e a^2+2ah(1+a\beta_0)$ and $dy/dh=2a(1+a\beta_0)$, the above equation becomes:

$$\int_{\sin^2\theta_e a^2}^{\sin^2\theta_e a^2+2aZ(1+a\beta_0)} \frac{a\cos\theta_e dy}{2a(1+a\beta_0)y^{1/2}}$$

$$=\frac{\cos\theta_e}{(1+a\beta_0)}\, y^{1/2}\Big|_{\sin^2\theta_e a^2}^{\sin^2\theta_e a^2+2aZ(1+a\beta_0)}$$

$$=\frac{\cos\theta_e}{(1+a\beta_0)}\left\{\left[\sin^2\theta_e a^2+2aZ(1+a\beta_0)\right]^{1/2}-\sin\theta_e a\right\} \tag{2.30}$$