Faculty of Natural and Mathematical Sciences
Department of Information

King's College London
Strand Campus, London,
United Kingdom

7CCSMPRJ

Individual Project Submission 2023/08

Name:	Yingming Luo
Student Number:	K2257345
Degree Programme:	MSc Computational Finance
Project Title:	Pricing of Exotic Options Using Monte Carlo Simulation
Supervisor:	Riaz Ahmad
Word Count:	Word count goes here

RELEASE OF PROJECT

Following the submission of your project, the Department would like to make it publicly available via the library electronic resources. You will retain copyright of the project.

☑ I **agree** to the release of my project

☐ I **do not** agree to the release of my project

Signature: Signature **Date:** August 11, 2023

Department of Information
King's College London
United Kingdom

7CCSMPRJ Individual Project

Pricing of Exotic Options Using Monte Carlo Simulation

Name: **Yingming Luo**
Student Number: K2257345
Course: MSc Computational Finance

Supervisor: Riaz Ahmad

This dissertation is submitted for the degree of MSc in MSc Computational Finance.

The content of “Acknowledgements” is in “\contents\acknowledgements.tex”

Acknowledgements

It is a short paragraph to thank those whose have contributed to the project work.

Abstract

It is a precis of the report (normally in one page), which should include:

- A brief introduction to the project objectives
- A brief description of the main work of the project
- A brief description of the contributions, major findings, results achieved and principal conclusion of the project

Nomenclature

$S0$ Initial asset price
E Strike price (in equations)
K Strike price (in code)
r risk-free interest rate
σ volatility
T_0 current time
t current time
T maturity day
$T - t$ time to expiry
N number of time steps
n number of time steps
M number of simulations
$N(.)$ cumulative distribution function of standard normal distribution
dt single time-step
Sb barrier price
S_u upper barrier price
S_d lower barrier price
S_u^* adjusted upper barrier price
S_d^* adjusted lower barrier price

Contents

List of Figures

List of Tables

Abstract

It is a precis of the report (normally in one page), which should include:

- A brief introduction to the project objectives
- A brief description of the main work of the project
- A brief description of the contributions, major findings, results achieved and principal conclusion of the project

1 Introduction

The primary objective of this research endeavor is to delve into the utilization of Monte Carlo (MC) simulation techniques for the valuation of exotic options, focusing particularly on Asian options and Barrier options. This investigation will further encompass the integration of the renowned Black-Scholes model, a extensively employed pricing framework within the financial domain, to capture the dynamics of the underlying asset's price behavior.

The current challenge lies in valuing exotic options, which possess intricate payoffs influenced by non-linear characteristics and path dependencies. This complexity often poses considerable difficulties for financial institutions and investors alike. Established frameworks like the Black-Scholes model may falter in effectively pricing such options, primarily due to their reliance on numerous unrealistic assumptions. Furthermore, the absence of closed-form mathematical solutions further complicates matters, as these options exhibit multifaceted structures that are hard to encapsulate through traditional methods. Consequently, the adoption of Monte Carlo simulation, a robust numerical technique, has witnessed a surge in popularity within the financial sector as a means to accurately price exotic options.

This study aims to assess the efficacy of Monte Carlo simulation in pricing Barrier and Asian options. To enhance the efficiency and precision of the simulations, the Antithetic Variate method, a variance reduction technique, will be employed. The project's technical requirements involve utilizing Python for implementing the Monte Carlo simulation and conducting an in-depth analysis of the generated data.

2 Background Theories

Before getting into exotic options it is best to understand what are financial derivatives. As the term suggests, derivatives are financial contracts negotiated and agreed upon between two or more parties, where its values are derived from the price movements of one or more underlying assets or some benchmarks [1]. Derivatives are vital to the financial industry for several reasons with the most important one being risk management. Investors often trade derivatives, either through the exchange or over the counter, to hedge against risks in the market [1].

2.1 Vanila Options

One particular category of financial derivatives is the Options contract, which has two subclass: Call option and Put option. Ever since 1973, organised exchanges have facilitated the trading of modern call and put options using stocks as their underlying asset [2]. The purchase of a call option on a stock grants the buyer the right to buy the stock at a predetermined price, called the strike price (denoted as E), on or before a predetermined expiration date (denoted as T), but it does not impose an obligation to do so [3]. Conversely, the purchase of a put option grants the buyer the right to sell the stock at a predetermined strike price on or before an expiration date, but again, it does not impose an obligation to do so. When the buyers do decide to use their right to perform the action of buying or selling, it is said that the option is exercised [3].

The buyer, also known as the option holder, incurs the cost of purchasing the option, which is referred to as the premium. The seller on the other hand is known as the option writer, is obligated to sell or buy the underlying asset at the strike price to the holder if the option is exercised. Thus the study of option pricing hinges on this central aspect: without knowing the future changes in the underlying asset's price, how can one determine the premium of an option.

Plain vanilla options are often used to refer to basic call and put options that lack any special or complex features. There are two general classes of vanilla options, one is the European Option, where the buyer can only exercise their right to buy or sell on the expiration date; the other one is the American Option, where the buyers can exercise their right anytime before the expiration date [4]. In this chapter, more emphasis will be placed on European options because the exotic options investigated later on are also European style options.

2.2 European Option Payoffs

The payoff of an European option refers to the profit or loss gained by the option holder at the maturity date, based on the underlying asset price at expiry (denotes as S_T) and the terms of the contract.

European Call Option:
For European Call option, the payoff function is given by:

$$C = max(S_T - E, 0) \tag{2.1}$$

This means that, if:

- $S_T > E$, the option holder should exercise the option to buy the underlying asset at the lower strike price E and sell at the higher market price S_T to generate an positive payoff of $S_T - E > 0$.
- $S_T \leq E$, the option holder should not exercise the option, and payoff is 0.

Thus, at expiry, this call option would give the holder a value of $max(S_T - E, 0)$, which is never negative. Whereas the option seller has a payoff of $-max(S_T - E, 0)$, which is never positive. Thus the option writer must be paid a premium for the contract.

European Put Option:
For European Call option, the payoff function is given by:

$$P = max(E - S_T, 0) \tag{2.2}$$

This means that, if:

- $S_T < E$, the option holder should exercise the option to sell the underlying asset at the higher strike price E to generate an positive payoff of $E - S_T > 0$.
- $S_T \geq E$, the option holder should not exercise the option, and payoff is 0.

Based on the payoff function, a European option can be in one of the three states at expiry: in the money, at the money, or out of the money. Summarised in Table 1. It is clear that only when the option is in the money, it will be exercised by the holder [5].

	Call Option	Put Option
At the money	$S_T = E$	$S_T = E$
In the money	$S_T > E$	$S_T < E$
Our of the money	$S_T < E$	$S_T > E$

Table 1: Three states of European Option at expiry.

2.3 Put-Call parity

Given the payoff functions discussed above, under the condition of no arbitrage opportunities, one can construct a portfolio of a long Call and short Put, which is guaranteed to receive a payoff of holding a long asset S and short cash $Ee^{-r(T-t)}$ position [3]. This important relationship between the two types of options is called the Put-Call parity, given by:

$$C - P = S - Ee^{-r(T-t)} \tag{2.3}$$

This parity is valid at any time up to expiry of the options contract and is independent of both the future movements of the underlying asset price and the model used to simulate the underlying [3].

Using the Put-Call parity, it reveal that if the current price of the underlying, the strike price and maturity date is known, then given the price of a European Call option, one can easily obtain the price of European Put option that shares the same contract conditions and vice versa.

2.4 Further reviews of vanilla options

However, there are many limitations to these vanilla options due to its simplicity and fixed payoff structure, which limits their flexibility to meet the needs of different investors. As discussed above, when investors want to use options to hedge against certain risks, they must pay a premium upfront in order to acquire the option, which can be expensive. Furthermore, vanilla options can be limited in their effectiveness as a hedging tool in volatile markets. One reason is that they have fixed strike prices, which means that if the market moves significantly in one direction, the option may become out of the money and lose much of its value. Additionally, vanilla options are often priced using models that assume constant volatility, such as the original Black-Scholes model, which "often fail when compared to the real market data" [6]. Therefore, in order to address these limitations, a new category of options called Exotic options has been developed.

Unlike the fixed and straightforward framework of vanilla options, exotic options are designed to offer more customised or tailored risk management solutions to investors. They can provide more flexibility in terms of the payoff structure and can be tuned to meet the specific needs of an investor. Moreover exotic options can also be used to hedge against more complex risks, which cannot be hedged using standard vanilla options, and enhance the returns on an investment portfolio by taking on more sophisticated and complex trading strategies.

Aside from the nuanced details in payoff structure between the vanilla and exotic options, there is also a fundamental difference between the two that we must emphasise. That is, vanilla options are path-independent [7]. The payoff of the American and European options depends only on the price of the underlying asset at the time of exercise, not on the path that the asset price took to get there. Exotic options on the other hand are path-dependent [3], which means their value is not only determined by the underlying asset's price at expiration but also the path it takes to get there. As Wilmott [3] has pointed out, there are two types of path dependency, namely strong and weak, with the difference being strong path-dependence options work in higher dimensions that require us to introduce an additional variable to deal with path dependency, whereas the weak path-dependence ones do not. In order to understand the effectiveness of MC simulation methods in pricing both weakly and strongly path-dependent options, two exotic options will be considered in this thesis: Barrier Options (weakly path-dependent) and Asian Options (strongly path-dependent).

3 Black-Scholes Model

This chapter aims to present a succinct overview of the Wiener Process in subsection 3.1 and Ito's Lemma in subsection 3.2, as they constitute the fundamental building blocks necessary for deriving the Black-Scholes model. The subsequent sections present a detailed derivation of the Black-Scholes partial derivative equation, culminating in the final equations for pricing vanilla European options.

One of the very popular models used in industry to price options is the Black-Scholes model developed and made publicly available by Fischer Black and Myron Scholes in 1973 [8]. The model is a mathematical formula used to determine the fair price for a European Call or Put option, in other words the premium, by considering the following factors: underlying asset price (S), expiration date (T), volatility of the underlying (σ), drift rate (μ), risk-free interest rate (r) and strike price (E) [9]. Hence, for option pricing, the value V of an option can be a function denoted as:

$$V(S, t; \sigma, \mu; E, T; r) \tag{3.1}$$

where t is the current time, and $T - t$ represents time to expiry. To simplify notations, in the derivation of the Black-Scholes equation below, the option value will simply be written as $V(S, t)$ or V.

3.1 Brownian Motion / Wiener Process

According to the efficient market hypothesis, asset prices move randomly following two key principles. The first principle emphasises that the present price fully incorporates the entire past history. The second principle asserts that markets instantaneously respond to any new information concerning an asset [10]. This means that the asset prices should behaves the same way as a Markov process.

Suppose at time t, the price of an asset is S, with a small change in time interval dt, the asset price would change to $S + dS$. The associated return of the asset price change can be calculated as dS/S, which is the change in price divided by the original price. In order to simulate such price return, a simple model called Brownian motion, or Wiener process, is developed. This model consist of two parts:

- **deterministic part (μdt)**: a predictable return comparable to the interest gained from investing in a risk-free bank, where μ is the average growth rate of the asset price, also known as the drift rate.
- **random part (σdW_t)**: a random part that mimics the arrival of unanticipated news that causes the price to change. Here the dW_t term is a random variable drawn from a normal distribution, which represents the Wiener process with a mean of 0 and variance dt. σ is the volatility, a constant term that measures the standard deviation of the price returns, also known as the diffusion rate.

By adding the terms together we can form a **stochastic differential equation** capable of generating random price movements for assets, resembling random walks:

$$\frac{dS}{S} = \mu dt + \sigma dW_t \tag{3.2}$$

To be more precise, Equation 3.2 is also called a Geometric Brownian Motion (GBM) because asset's price changes proportionally to its current value. This implies that the percentage returns are normally distributed with a constant drift and volatility, and the asset's price trajectory resembles a geometric growth or decay over time[3].

3.2 Ito's Lemma

For the Brownian motion defined above, it exhibits the property of being continuous everywhere but differentiable no where. This is due to the Wiener process, being a random variable, create "jumps" equivalent to infinitesimal variations along its path. Therefore the standard calculus cannot be applied to stochastic differential equations, when we want to model the random walk in continuous time limit $dt \to 0$.

However, according to [10], "Ito's lemma relates small changes in a function of a random variable to the small change in the random variable itself." Thus allowing us to compute the derivative of a function that depends on a stochastic variable, providing a way of handling the Wiener process dW_t as $dt \to 0$.

In general form:
If a function $G(t)$ satisfies a GBM, such that it can be represented in the following form of a stochastic differential equation:

$$dG_t = A(t, G_t)dt + B(t, G_t)dW_t \tag{3.3}$$

where A and B are functions of G_t and t, corresponding to the drift and diffusion rates.

Now consider a function $F = F(t, G_t)$, according to Ito's lemma, the derivative dF can be represented as:

$$dF = \left(\frac{\delta F}{\delta t} + A \frac{\delta F}{\delta G_t} + \frac{1}{2} B^2 \frac{\delta^2 F}{\delta G_t^2} \right) dt + B \frac{\delta F}{\delta G_t} dW_t \tag{3.4}$$

where $\frac{\delta F}{\delta t} + A \frac{\delta F}{\delta G_t} + \frac{1}{2} B^2 \frac{\delta^2 F}{\delta G_t^2}$ is the drift rate and $B \frac{\delta F}{\delta G_t}$ is the diffusion rate.

Apply to option value V:
Suppose $V = V(t, S_t)$ where the underlying asset price S_t at time t satisfies GBM, such that:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{3.5}$$

Applying Ito's lemma, the change in option value dV can be calculate by:

$$dV = \left(\frac{\delta V}{\delta t} + \mu S \frac{\delta V}{\delta S} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} \right) dt + \sigma S \frac{\delta V}{\delta S} dW_t \tag{3.6}$$

3.3 Deriving Black-Scholes Equation

The foundation of the model is based on many assumptions, to name a few, the underlying asset price follows a log-normal random walk, short selling is allowed, interest rate and volatility of the market are known constants, no transaction costs, no dividend payout on the underlying and no risk-free arbitrage opportunities [9]. Under these assumptions, the model is able to construct a delta-hedged portfolio consisting of a long position in the stock and a short position in the option. The derivation of the Black-Scholes equation below is based on the book written by P. Wilmott (2007)[3].

First, construct a delta-hedged portfolio, with Π denoting the value of the portfolio, which consist of two components, a long option position $V(S,t)$, and a short position in some quantity delta (Δ) of the underlying asset S.

$$\Pi = V(S,t) - \Delta S \tag{3.7}$$

Then across one time step: $t \to t + dt$, $\Pi \to \Pi + d\Pi$, we hold Δ fixed across each time step so Δ is treated as a constant, meaning:

$$d(\Delta S) = \Delta dS \tag{3.8}$$

hence:

$$\begin{aligned} d\Pi &= dV - d(\Delta S) \\ d\Pi &= dV - \Delta dS \end{aligned} \tag{3.9}$$

Using the results from Ito's lemma for dV in Equation 3.6 and substitute into $d\Pi$ gives:

$$\begin{aligned} d\Pi &= \left(\frac{\delta V}{\delta t} + \mu S \frac{\delta V}{\delta S} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} \right) dt + \sigma S \frac{\delta V}{\delta S} dW_t - \Delta \left(\mu S dt + \sigma S dW_t \right) \\ &= \left(\frac{\delta V}{\delta t} + \mu S \frac{\delta V}{\delta S} - \Delta \mu S + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} \right) dt + \sigma S \left(\frac{\delta V}{\delta S} - \Delta \right) dW_t \end{aligned} \tag{3.10}$$

In order to eliminate risk, which is the dW_t term, we set $\Delta = \frac{\delta V}{\delta S}$.

Sub $\Delta = \frac{\delta V}{\delta S}$ back into $d\Pi$ in Equation 3.10 gives:

$$d\Pi = \left(\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} \right) dt \tag{3.11}$$

By following the no risk-free arbitrage opportunities assumption, the portfolio must earn at the risk-free interest rate r:

$$d\Pi = r\Pi dt \tag{3.12}$$

Finally equating the two $d\Pi$ equations from Equation 3.11 and Equation 3.12:

$$\begin{aligned} \left(\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2}\right) dt = r\Pi dt \\ \frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} = r\left(V - S\frac{\delta V}{\delta S}\right) \end{aligned} \tag{3.13}$$

By rearranging, we arrive at the famous 1973 Black-Scholes partial differential equation (PDE):

$$\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} + rS\frac{\delta V}{\delta S} - rV = 0 \tag{3.14}$$

3.4 Black-Scholes formula for European Options

In order to obtain the formulas for pricing European options, we must solve the Black-Scholes equation with final conditions depending on the option payoffs. A full derivation can be found in the textbook written by P. Wilmott (2007)[3]. In this thesis, we will simply make use of the resulting analytical formulas which can be solved to calculate the exact price of European options under the Black-Scholes framework.

Call option value:

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2) \tag{3.15}$$

Put option value:

$$P(S,t) = -SN(d_1) + Ee^{-r(T-t)}N(-d_2) \tag{3.16}$$

where $N(.)$ is the cumulative distribution function for the standard normal distribution $\mathcal{N}(0,1)$:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\phi^2} d\phi \tag{3.17}$$

and:

$$\begin{aligned} d_1 = \frac{log\left(\frac{S}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = \frac{log\left(\frac{S}{E}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \end{aligned} \tag{3.18}$$

3.5 Further observations

The Black-Scholes equation contains all the obvious variables and parameters such as the underlying (S), time (t), volatility (σ) and risk-free interest rate (r), but there is no mention of the drift rate (μ). This is because delta-hedging has cancelled out the μ

term at Equation 3.11, meaning the return of the underlying stock is no longer whatever the stock return for that particular share is, as there is no more risks involved since the dW_t term is eliminated. Thus the stock cannot grow at any rate other than the risk-free return, which is why the model only has the risk-free interest rate (r) to represent the growth rate.

In mathematical terms, originally the asset price is evolving according to Equation 3.2. Now under the Black-Scholes framework, the asset price is evolving according to:

$$\frac{dS}{S} = rdt + \sigma dW_t \tag{3.19}$$

By continuously hedging, the fluctuations in the value of a long stock position can be completely balanced out by the corresponding changes in the value of a short option position, leading to a risk-less portfolio and a certain return under the Black-Scholes assumptions.[9]. Although there have been many criticisms of the unrealistic assumptions made by the model, nonetheless, the easy to use closed form solution for pricing the European option proposed by the model, together with its framework for understanding the relationship between the price of an option and the underlying asset were invaluable to the development of quantitative finance. In this thesis, the analytical solutions to pricing options derived from the Black-Scholes model will be used as benchmarks to validate the results obtained from numerical methods, and help improve the accuracy of simulations.

4 Barrier Options

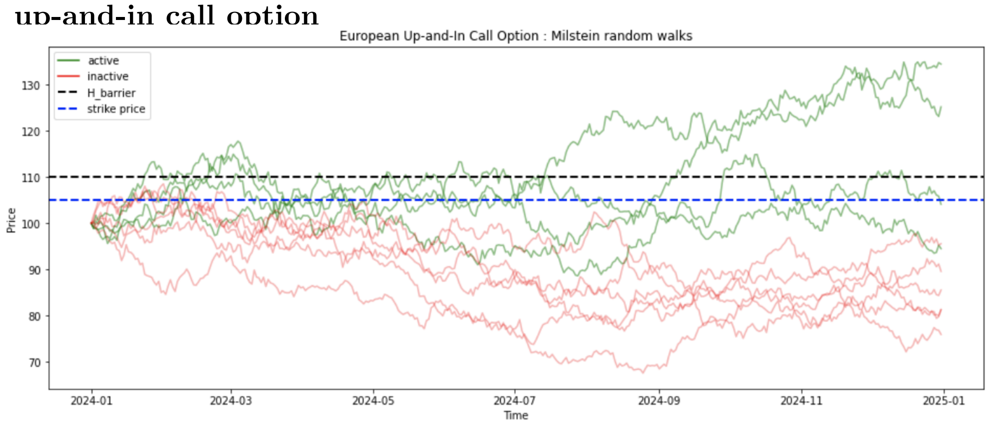
As previously mentioned, Barrier options are path-dependent options. The payoff of these contracts is linked to the level of the realised asset path, and specific conditions in the contract are activated if the asset price reaches a certain "barrier", whether it is too high or too low [3].

The barrier can be specified as either "in" or "out," and it can be set at different levels, such as "up" or "down." In general, an "in" barrier option only becomes active or "knocks in" if the price of the underlying asset reaches the barrier level during the life of the option. Conversely, an "out" barrier option becomes void or "knocks out" if the price of the underlying asset reaches the barrier level during the life of the option.

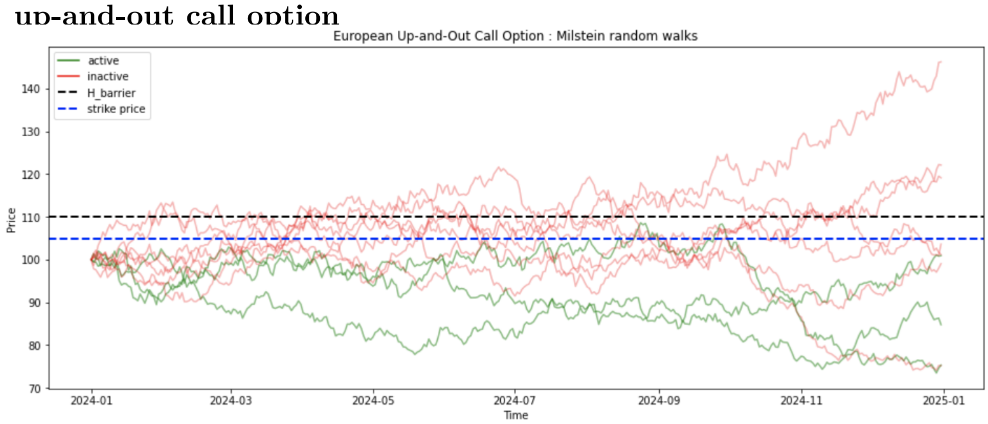
The classification of barrier options can be further extended according to the barrier's position S_b relative to the initial value of the underlying asset price S_0. Specifically, an "up" barrier option has a barrier level set above the current price of the underlying asset, $S_b > S_0$, while a "down" option satisfies $S_b < S_0$.

The mix and match of these features provides a set of Barrier options listed below:

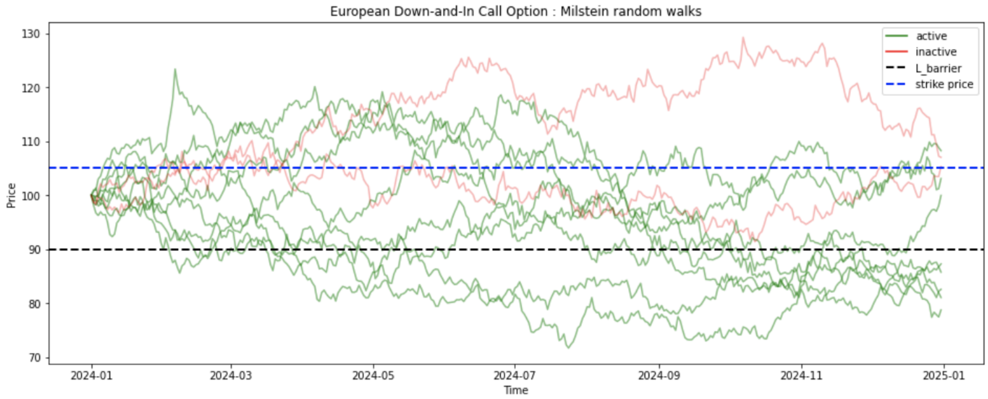
- **up-and-in call option**



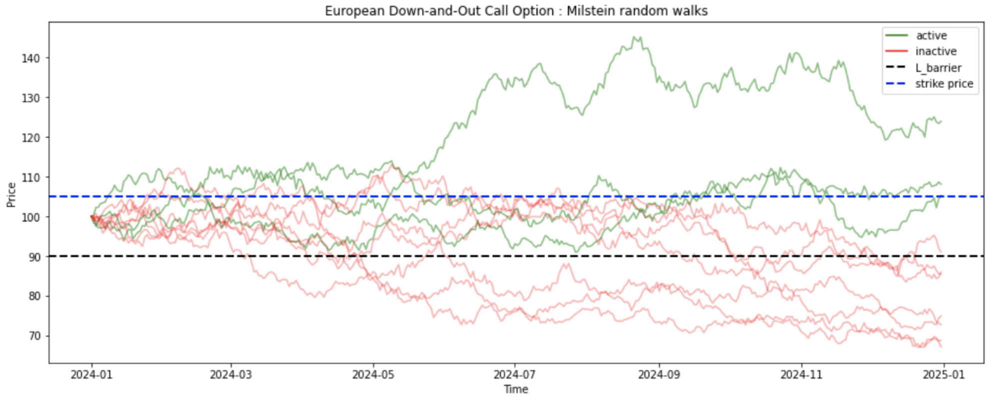
- **up-and-out call option**



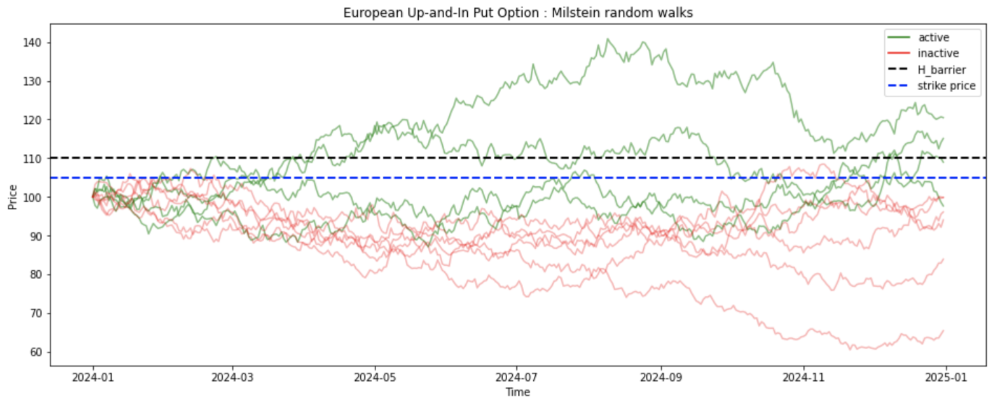
- **down-and-in call option**



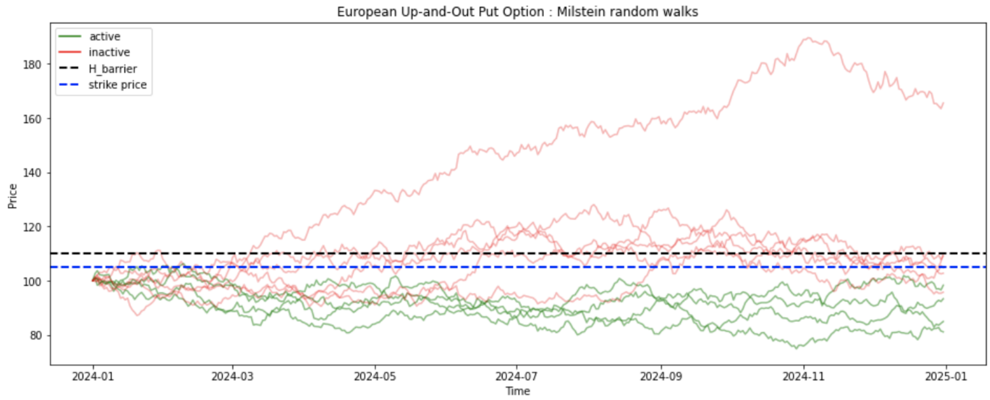
- **down-and-out call option**



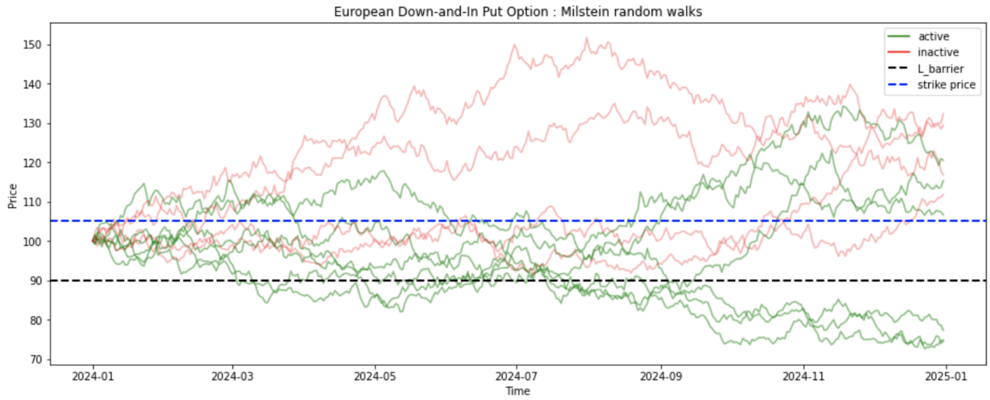
- **up-and-in put option**



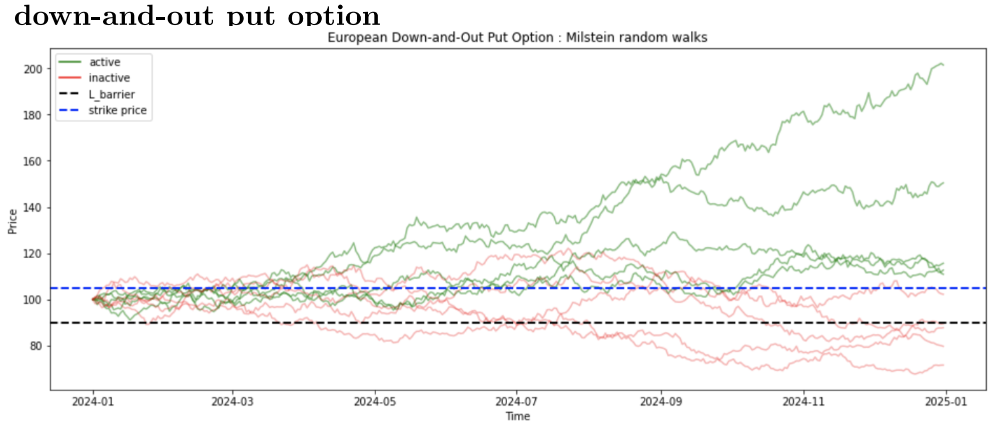
- **up-and-out put option**



- **down-and-in put option**



- **down-and-out put option**



Furthermore, Barrier option is also **weakly** path-dependent, because it works in low dimension. In other words, under the Black-Scholes framework, the value of the contract only depends on two independent variables: the current price level of the underlying asset S, and time to expiry t, which makes it a two dimensional option [3].

Take the up-and-out call option as an example. Before hitting the barrier level, the option works the same way as a plain vanilla call option. As it gives the holder the right (but not the obligation) to buy the underlying asset at the strike price. However, once the underlying asset reaches or surpasses the specified barrier level at any point during the option's lifetime, the option is knocked out and becomes worthless. While the up-and-out call option provides the potential for profit if the price of the underlying asset rises and stays below the barrier level, it also restricts the potential gains if the price surpasses the barrier. This restricting feature naturally makes the up-and-out call option to become cheaper than a regular vanilla call option.

4.1 In-Out parity of Barrier options

Similar to the put-call parity of European vanilla options defined in subsection 2.3, an in-out parity also exists between the value of Barrier options. Given a portfolio comprising one European in-option and one European out-option, both sharing the same barrier, strike price, and expiration date. The total value of this portfolio is equivalent to that of a corresponding European option with the same strike price and expiration date. This is because, as one option gets knocked-out at the barrier level, the other becomes knocked-in. Thus at expiry, only one of the two barrier options will remain active, and its payoff will be identical to that of the European option [11]. The exact in-out parities are listed below:

$$\begin{aligned} C_{vanilla}(S,t) &= C_{up-and-in}(S,t,S_b) + C_{up-and-out}(S,t,S_b) \\ &= C_{down-and-in}(S,t,S_b) + C_{down-and-out}(S,t,S_b) \end{aligned} \tag{4.1}$$

$$\begin{aligned} P_{vanilla}(S,t) &= P_{up-and-in}(S,t,S_b) + P_{up-and-out}(S,t,S_b) \\ &= P_{down-and-in}(S,t,S_b) + P_{down-and-out}(S,t,S_b) \end{aligned} \tag{4.2}$$

Consequently, once the value of the corresponding in-option is known, the value of an out-option can be readily determined, and vice versa.

4.2 PDE for Barrier Options

In practice, the vast majority of barrier options traded in markets are monitored discretely. This means that they establish specific and fixed intervals for checking the barrier status, often on a daily closing basis [12]. Under this monitoring mode, it is difficult to derive any analytical solutions for pricing the Barrier options, as the model need to take into account the discontinuities in the option's payoff function. However, if the asset price is monitored continuously, meaning if the asset price does hit the barrier level, the option immediately becomes "knocked-in" or "knocked-out", then it is possible to derive analytical formulas by extending the Black-Scholes model with new boundary conditions incorporating the barrier level[4].

According to Hu et al., (2006), as the Barrier options are only weakly path-dependent, the Black-Scholes partial differential equation governing the value of the option is the same as the one defined for European vanilla options in Equation 3.14.

Subject to final conditions:

$$V\left(S,t\right)=\begin{cases}\left(S-E\right)^{+}, & \text{knock-out call option;}\\ 0, & \text{knock-in call option;}\\ \left(E-S\right)^{+}, & \text{knock-out put option;}\\ 0, & \text{knock-in put option;}\end{cases}$$

and boundary conditions:

$$V\left(S_b,t\right)=\begin{cases}0, & \text{knock-out option;}\\ V_{vanilla}\left(S_b,t\right), & \text{knock-in option;}\end{cases}$$

within the different domains D:

$$D=\begin{cases}\{\left(S,t\right)|0\leq S\leq S_b,0\leq t\leq T\}, & \text{up option;}\\ \{\left(S,t\right)|S_b\leq S<\infty,0\leq t\leq T\}, & \text{down option;}\end{cases}$$

4.3 Analytical solutions for pricing Barrier options

A full derivation of the formulas is presented in the paper presented by Reiner & Rubinstein (1991) [13]. Presented below are the set of analytical solutions for solving the different types of Barrier options under the Black-Scholes framework, summarised by P. Wilmott (2013) [14]:

Notation:

- $N(.)$**:** the cumulative distribution function for the standard normal distribution $\mathcal{N}(0,1)$ defined in Equation 3.17.
- q**:** the dividend yield on stocks or the foreign interest rate for FX. **Note:** within this thesis, we assume that there is no dividend payout on the underlying, thus this variable is set to $q=0$ in all computations. It is only included here for completeness.
- S_b**:** the barrier position.
- a **and** b**:**

$$a=\left(\frac{S_b}{S}\right)^{-1+\frac{2(r-q)}{\sigma^2}},$$

$$b=\left(\frac{S_b}{S}\right)^{1+\frac{2(r-q)}{\sigma^2}}$$

- d_* **terms:**

$$\begin{aligned}
d_1 =& \frac{log\left(\frac{S}{E}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 =& \frac{log\left(\frac{S}{E}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_3 =& \frac{log\left(\frac{S}{S_b}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_4 =& \frac{log\left(\frac{S}{S_b}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_5 =& \frac{log\left(\frac{S}{S_b}\right) - \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_6 =& \frac{log\left(\frac{S}{S_b}\right) - \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_7 =& \frac{log\left(\frac{SE}{S_b^2}\right) - \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_8 =& \frac{log\left(\frac{SE}{S_b^2}\right) - \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}$$

Up-and-out call:

$$\begin{aligned}
C_{up-and-out} =& Se^{-q(T-t)}\left(N(d_1) - N(d_3) - b\left(N(d_6) - N(d_8)\right)\right) \\
&- Ee^{-r(T-t)}\left(N(d_2) - N(d_4) - a\left(N(d_5) - N(d_7)\right)\right)
\end{aligned} \tag{4.3}$$

Up-and-in call:

$$\begin{aligned}
C_{up-and-in} =& Se^{-q(T-t)}\left(N(d_3) + b\left(N(d_6) - N(d_8)\right)\right) \\
&- Ee^{-r(T-t)}\left(N(d_4) + a\left(N(d_5) - N(d_7)\right)\right)
\end{aligned} \tag{4.4}$$

Down-and-out call:

1. $E > S_b$:

$$\begin{aligned}
C_{down-and-out} =& Se^{-q(T-t)}\left(N(d_1) - b\left(1 - N(d_8)\right)\right) \\
&- Ee^{-r(T-t)}\left(N(d_2) - a\left(1 - N(d_7)\right)\right)
\end{aligned} \tag{4.5}$$

2. $E < S_b$:

$$\begin{aligned}
C_{down-and-out} =& Se^{-q(T-t)}\left(N(d_3) - b\left(1 - N(d_6)\right)\right) \\
&- Ee^{-r(T-t)}\left(N(d_4) - a\left(1 - N(d_5)\right)\right)
\end{aligned} \tag{4.6}$$

Down-and-in call:

1. $E > S_b$:

$$\begin{aligned} C_{down-and-in} =& Se^{-q(T-t)}b\left(1 - N\left(d_8\right)\right) \\ & - Ee^{-r(T-t)}a\left(1 - N\left(d_7\right)\right) \end{aligned} \tag{4.7}$$

2. $E < S_b$:

$$\begin{aligned} C_{down-and-in} =& Se^{-q(T-t)}\left(N\left(d_1\right) - N\left(d_3\right) + b\left(1 - N\left(d_6\right)\right)\right) \\ & - Ee^{-r(T-t)}\left(N\left(d_2\right) - N\left(d_4\right) + a\left(1 - N\left(d_5\right)\right)\right) \end{aligned} \tag{4.8}$$

Down-and-out put:

$$\begin{aligned} P_{down-and-out} =& - Se^{-q(T-t)}\left(N\left(d_3\right) - N\left(d_1\right) - b\left(N\left(d_8\right) - N\left(d_6\right)\right)\right) \\ & + Ee^{-r(T-t)}\left(N\left(d_4\right) - N\left(d_2\right) - a\left(N\left(d_7\right) - N\left(d_5\right)\right)\right) \end{aligned} \tag{4.9}$$

Down-and-in put:

$$\begin{aligned} P_{down-and-in} =& - Se^{-q(T-t)}\left(1 - N\left(d_3\right) + b\left(N\left(d_8\right) - N\left(d_6\right)\right)\right) \\ & + Ee^{-r(T-t)}\left(1 - N\left(d_4\right) + a\left(N\left(d_7\right) - N\left(d_5\right)\right)\right) \end{aligned} \tag{4.10}$$

Up-and-out put:

1. $E > S_b$:

$$\begin{aligned} P_{up-and-out} =& - Se^{-q(T-t)}\left(1 - N\left(d_3\right) - bN\left(d_6\right)\right) \\ & + Ee^{-r(T-t)}\left(1 - N\left(d_4\right) - aN\left(d_5\right)\right) \end{aligned} \tag{4.11}$$

2. $E < S_b$:

$$\begin{aligned} P_{up-and-out} =& - Se^{-q(T-t)}\left(1 - N\left(d_3\right) - bN\left(d_8\right)\right) \\ & + Ee^{-r(T-t)}\left(1 - N\left(d_2\right) - aN\left(d_7\right)\right) \end{aligned} \tag{4.12}$$

Up-and-in put:

1. $E > S_b$:

$$\begin{aligned} P_{up-and-in} =& - Se^{-q(T-t)}\left(N\left(d_3\right) - N\left(d_1\right) + bN\left(d_6\right)\right) \\ & + Ee^{-r(T-t)}\left(N\left(d_4\right) - N\left(d_2\right) + aN\left(d_5\right)\right) \end{aligned} \tag{4.13}$$

2. $E < S_b$:

$$\begin{aligned} P_{up-and-in} =& - Se^{-q(T-t)}bN\left(d_8\right) \\ & + Ee^{-r(T-t)}aN\left(d_7\right) \end{aligned} \tag{4.14}$$

4.4 Correction formula for Discretely monitored Barrier options

Barrier option pricing models in the previous section work well when the barrier is continuously monitored, meaning knock-in or knock-out events are assumed to happen immediately upon barrier breach. Under this assumption, Merton (1973) [15] was first able to develop a partial differential equation for pricing knock-out call barrier options. Subsequent work followed by Reiner and Rubinstein (1991) [13], Kunitomo and Ikeda (1992) [16] has contributed to the invention of other analytical solutions defined in the previous section. However, continuous monitoring is not always possible in the real financial market and instead practitioners need to specify how often the barrier is going to be monitored with discrete monitoring. According to Kat and Verdonk (1995) [17], there could be considerable price differences between options with discrete and continuous barrier monitoring, even with daily monitoring. Their research suggested that knock-in options might be cheaper, and knock-out options more expensive, when using discrete monitoring compared to continuous monitoring. This poses challenging issues because there are virtually no analytical solutions available for discretely monitored barrier options due to its complexity and uncertainty from delays in data. Thus practitioners had to continue to use analytical solutions derived from continuous monitoring to price real world discretely monitored barrier options with significant errors.

In order to address this issue, Broadie, Glasserman, and Kou (1997) [18] developed a simple and effective formula for approximating the price of discretely monitored barrier options. This is done by introducing a basic continuity correction to the continuous barrier option formulas, shifting the barrier across each time step to account for discrete monitoring. The adjustment is solely based on the monitoring frequency, asset volatility, and a constant β, approximately equal to 0.5826 [18]. In the original paper it states that

THEOREM 1.1. Let $V_n(S_b)$ be the price of a discretely monitored knock-in or knock-out down call or up put with barrier S_b where n is the monitoring frequency over the period of $[0,T]$. Let $V(S_b)$ be the price of the continuously monitored barrier option. Then

$$V_n(S_b) = V\left(S_b e^{\pm\beta\sigma\sqrt{dt}}\right) + O\left(\frac{1}{\sqrt{n}}\right) \tag{4.15}$$

where $+$ applies if $S_b > S_0$, $-$ applies if $S_b < S_0$, $\beta = -\xi\left(\frac{1}{2}\right)/\sqrt{2\pi} \approx 0.5826$, and ξ is the Riemann zeta function.

Therefore when using the analytical solutions defined in the previous section, the barrier values will take the following new form to improves accuracy of pricing:

$$S_b^* = S_b e^{\pm\beta\sigma\sqrt{dt}} \tag{4.16}$$

5 Asian Options

For Asian options the payoff is based on the average price of the underlying asset over a period of time before expiry, rather than the price at a specific point in time. The average can be calculated over different time intervals, such as the entire life of the option or a specific subset of the option's life [2]. The Asian option is said to be **strongly** path-dependent, because it is a high dimensional option. In order to price such option using the Black-Scholes framework, on top of the asset price (S) and time to expiry (t), it needs to keep track of a third independent variable, such as the average price of the underlying asset over a certain period [3].

5.1 PDE for Asian options

5.1.1 state variables

In the general case, we first need to introduce a path-dependent quantity $I(T)$, called the **state variable**, which varies based on the asset's path. This can be expressed as the integral of a certain function $f\left(S,\tau\right)$ of the asset over the time period from zero to t:

$$I\left(t\right)=\int_0^t f\left(S,\tau\right)d\tau \tag{5.1}$$

The function $f(S,t)$ can be changed accordingly depending on which aspect of the path-dependency is required. Hence the derivative of the state variable is simply:

$$\begin{aligned} dI =& I\left(t+dt\right)-I(t)\\ =&\int_0^{t+dt} f(S,\tau)d\tau-\int_0^t f\left(S,\tau\right)d\tau\\ =&\left(\int_0^t f\left(S,\tau\right)d\tau+\int_t^{t+dt} f(S,\tau)d\tau\right)-\int_0^t f\left(S,\tau\right)d\tau\\ =&\int_t^{t+dt} f(S,\tau)d\tau\\ =&f(S_{t+dt},t+dt)-f(S_t,t)\\ =&f\left(S,t\right)dt \end{aligned} \tag{5.2}$$

5.1.2 Derivation

Assuming that the underlying asset evolves according to the same lognormal random walk defined in Equation 3.5. In order to keep track of the state variable in our model, the value of the option need to be redefined as $V(S,I(t),t)$, a function with three variables.

Now construct a delta-hedged portfolio with Π denoting the value of the portfolio, holding one of the path-dependent option and short some quantity Δ of the underlying asset:

$$\Pi = V(S, I, t) - \Delta S \tag{5.3}$$

Across each time step we hold Δ fixed, and by applying Ito's lemma together with results from Equation 5.2, the change in portfolio value is given by:

$$\begin{aligned} d\Pi &= \left(\frac{\delta V}{\delta t} + \mu S\frac{\delta V}{\delta S} + \frac{1}{2}\sigma^2 S^2\frac{\delta^2 V}{\delta S^2}\right)dt + \frac{\delta V}{\delta I}dI + \sigma S\frac{\delta V}{\delta S}dW_t - \Delta\left(\mu S dt + \sigma S dW_t\right) \\ &= \left(\frac{\delta V}{\delta t} + \mu S\frac{\delta V}{\delta S} - \Delta\mu S + \frac{1}{2}\sigma^2 S^2\frac{\delta^2 V}{\delta S^2}\right)dt + f\left(S,t\right)\frac{\delta V}{\delta I}dt + \sigma S\left(\frac{\delta V}{\delta S} - \Delta\right)dW_t \\ &= \left(\frac{\delta V}{\delta t} + \mu S\frac{\delta V}{\delta S} - \Delta\mu S + \frac{1}{2}\sigma^2 S^2\frac{\delta^2 V}{\delta S^2} + f\left(S,t\right)\frac{\delta V}{\delta I}\right)dt + \sigma S\left(\frac{\delta V}{\delta S} - \Delta\right)dW_t \end{aligned} \tag{5.4}$$

Set $\Delta = \frac{\delta V}{\delta S}$ and sub back into Equation 5.4 to eliminate risk:

$$d\Pi = \left(\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2\frac{\delta^2 V}{\delta S^2} + f\left(S,t\right)\frac{\delta V}{\delta I}\right)dt \tag{5.5}$$

By following the no risk-free arbitrage opportunities assumption, the portfolio must earn at the risk-free interest rate r:

$$d\Pi = r\Pi dt \tag{5.6}$$

Equating the two equations from Equation 5.5 and Equation 5.6:

$$\begin{gathered} \left(\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2\frac{\delta^2 V}{\delta S^2} + f\left(S,t\right)\frac{\delta V}{\delta I}\right)dt = r\Pi dt \\ \frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2\frac{\delta^2 V}{\delta S^2} + f\left(S,t\right)\frac{\delta V}{\delta I} = r\left(V - S\frac{\delta V}{\delta S}\right) \end{gathered} \tag{5.7}$$

Finally by rearranging, the general PDE for an Asian option becomes:

$$\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2\frac{\delta^2 V}{\delta S^2} + f\left(S,t\right)\frac{\delta V}{\delta I} + rS\frac{\delta V}{\delta S} - rV = 0 \tag{5.8}$$

5.2 Asian option types

How the average should be calculated in an Asian option can be flexible depending on the buyer's need. This means focusing on defining the function $f(S,t)$ mentioned in Equation 5.1 of the state variable. Below are some commonly used approaches and their corresponding PDEs.

Arithmetic Asian:
set $f(S,t) = S(t)$, the state variable becomes:

$$I = \int_0^t S(\tau) d\tau \tag{5.9}$$

Substitute into Equation 5.8 the resulting PDE is:

$$\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} + S\frac{\delta V}{\delta I} + rS\frac{\delta V}{\delta S} - rV = 0 \tag{5.10}$$

Geometric Asian:
set $f(S,t) = log(S(t))$, the state variable becomes:

$$I = \int_0^t log(S(\tau)) d\tau \tag{5.11}$$

Substitute into Equation 5.8 the resulting PDE is:

$$\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} + log(S)\frac{\delta V}{\delta I} + rS\frac{\delta V}{\delta S} - rV = 0 \tag{5.12}$$

Note the above state variables consist of integrals of the asset price over the averaging period, which makes them **continuously sampled averages**. Alternatively, we could take only data points at specific time intervals, such as summing the closing prices of each day and divide by the total number of days, which are thought to be more reliable, known as **discretely sampled averages**. In real world practice, the latter form of sampling is adopted because data can be unreliable and the exact time of a trade may not be known accurately. Therefore from a legal point of view, it is preferred to use only key prices such as the closing price which can be guaranteed to be a genuine traded price, for calculating the averages of Asian options. However, there are no closed-form solutions for pricing Asian options using discretely sampled averages, generally they can only be solved with numerical methods.

After defining the average function for calculating the respective average price A, Asian options can be further categorised depending on its payoff function. For example, when compared with the payoff function of a vanilla European Call option [14]:

$$C = max(S - E, 0) \tag{5.13}$$

average strike (floating strike) call:
Replace the strike price E with the average price A, payoff is

$$C = max(S - A, 0) \tag{5.14}$$

average strike (floating strike) put:
Again, replace the strike price E with the average price A, payoff is

$$P = max(A - S, 0) \tag{5.15}$$

average rate (fixed strike) call:
Replace the expiry price S with the average price A, payoff is

$$C = max(A - E, 0) \tag{5.16}$$

average rate (fixed strike) put:
Again, replace the expiry price S with the average price A, payoff is

$$P = max(E - A, 0) \tag{5.17}$$

5.3 Analytical solution for pricing Asian options

There are only a handful of closed-form analytical solutions to the pricing of Asian options due to the option's value is sensitive not only to the final price at expiration but also to the price path taken by the underlying asset during the averaging period. In fact, there are no real closed-form analytical solutions for pricing Asian options, only closed-form approximation solutions. This meant that it can only be priced using numerical solutions such as Monte Carlo [19].

However, there does exist some closed-form solutions for pricing continuously sampled geometric average rate Asian options under certain conditions. This is because the geometric average of lognormal random variables is still lognormally distributed [3]. This is a crucial feature as it allows mathematicians to apply standard mathematical techniques of lognormal distributions to forming the closed-form solution.

Kemna and Vorst (1990)
One of the earliest research effort in this field is work done by Kemna and Vorst (1990) [20], where they developed Black-Scholes style expressions for estimating the value of geometric averaging Asian call and put options, subject to two conditions:

1. The volatility must be replaced with $\sigma/\sqrt{3}$.
2. The dividend yield must be replaced with $D + \sigma^2/6$.

The estimating solutions are given by:

$$C \approx Se^{-r(b-r)(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{5.18}$$

and

$$P \approx -Se^{(b-r)(T-t)}N(-d_1) + Ke^{(T-t)}N(-d_2) \tag{5.19}$$

where,

$$
\begin{aligned}
d_1 =& \frac{ln\left(\frac{S}{K}\right) + \left(b + \frac{\sigma_A^2}{2}\right)T}{\sigma_A\sqrt{T}} \\
d_2 =& \frac{ln\left(\frac{S}{K}\right) + \left(b - \frac{\sigma_A^2}{2}\right)T}{\sigma_A\sqrt{T}} \\
=& d_1 - \sigma_A\sqrt{T}
\end{aligned}
$$

and

$\sigma_A = \dfrac{\sigma}{\sqrt{3}}$ is the adjusted volatility, with σ being the observed volatility

$b = \dfrac{1}{2}\left(r - D - \dfrac{\sigma^2}{6}\right)$ as the adjusted dividend payout, which is assumed to be 0 in this thesis.

Values obtained from Kemna and Vorst's method will be used as a benchmark in this thesis to compare with the results obtained from Monte Carlo method.

Continuously Monitored Geometric Averaging Options
There are also other closed form formulas for continuously monitored geometric averaging options where the full derivation of these formulas can be found in the book written by Kwok (2001) [4]. Below are two examples formulas summarised by Wilmott (2013) [14].

Geometric average rate call:

$$
C = e^{-r(T-t)}\left(Gexp\left(\frac{\left(r - D - \frac{\sigma^2}{2}\right)(T-t)^2}{2T} + \frac{\sigma^2 (T-t)^3}{6T^2}\right)N(d_1) - EN(d_2)\right) \tag{5.20}
$$

Geometric average rate put:

$$
P = e^{-r(T-t)}\left(EN(-d_2) - Gexp\left(\frac{\left(r - D - \frac{\sigma^2}{2}\right)(T-t)^2}{2T} + \frac{\sigma^2 (T-t)^3}{6T^2}\right)N(d_1)\right) \tag{5.21}
$$

where,

$$
\begin{aligned}
I =& \int_0^t log\left(S\left(\tau\right)\right) d\tau, \\
G =& e^{\frac{I}{T}} S^{\frac{(T-t)}{T}}, \\
d_1 =& \frac{Tlog\left(\frac{G}{E}\right) + \left(r - D - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2} + \sigma^2\frac{(T-t)^3}{3T}}{\sigma\sqrt{\frac{(T-t)^3}{3}}}, \\
d_2 =& \frac{Tlog\left(\frac{G}{E}\right) + \left(r - D - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2}}{\sigma\sqrt{\frac{(T-t)^3}{3}}}
\end{aligned}
\tag{5.22}
$$

and D is the dividend payout of the underlying

Aside from the technical details, Asian options are popular in the market due to their ability to lower the likelihood of market manipulation close to the expiry date, and provide more hedging opportunities for businesses with a series of exposures [21]. For example businesses involving thinly traded commodities such as rare earth metals and agricultural products.

6 Monte Carlo Simulations

Complex mathematical problems often lack closed-form solutions, particularly when dealing with exotic options that possess unique features, as discussed in previous chapters. In such cases, numerical methods offer an approach to estimate the solution through approximations and iterative processes. [22]. This chapter will be based on understanding the concepts of the Monte Carlo method, which is a type of numerical method and explore its application to exotic option pricing.

The Monte Carlo Method was created by John von Neumann and Stanislaw Ulam during World War II as a means of enhancing decision making in situations where outcomes were uncertain. They have named this method after a well-known casino destination, Monaco, as probability plays a central role in the simulation process, much like in games of roulette [23].

6.1 Basic concepts

MC simulation is based on the idea of using random numbers to solve complex mathematical problems. The method involves simulating a large number of possible outcomes, each with a different set of randomly generated inputs. By repeating this process many times, the simulation produces a distribution of possible outcomes, allowing for the calculation of probabilities and expected values. Based on the law of large numbers, the accuracy of MC methods can be improved by simply increasing the number of simulations because the estimated results should converge to the real value [24].

Suppose there is some arbitrary function $g(X)$ where X is a random variable, with an unknown mean and variance, which we denote as $\mathbb{E}[g(X)] = \mu$ and $var\,[g(X)] = \sigma^2$. In order to estimate the value of the mean a, we can draw M independent random sample values $X_1, X_2, X_3, ..., X_M$ from the same distribution as X using a probability density function $f(x)$, and calculate an estimated value called sample mean μ_* by:

$$\mu_* = \frac{1}{M}\sum_{i=1}^{M} g(X_i) \tag{6.1}$$

Due to the law of large numbers, as we increase our sample points in the estimation, it follows that:

$$\frac{1}{M}\sum_{i=1}^{M} g(X_i) \to \mathbb{E}[g(X)], \text{ or } \mu_* \to \mu, \text{ with probability 1, as } M \to \infty$$

Thus our sample mean μ_* gradually converges to the real mean μ given sufficiently large number of samples. For the variance, since $var[g(X)] = \mathbb{E}\Big[(g(X) - \mathbb{E}[g(X)])^2\Big]$, we can estimate an unbiased sample variance value σ_*^2 by:

$$\sigma_*^2 = \frac{1}{M-1}\sum_{i=1}^{M}(g(X_i) - \mu_*)^2$$

The approximations made by repeatedly drawing random samples from the same distribution as the target random variable X is indeed a typical use of the Monte Carlo method. Subsequently, by the Central Limit Theorem, we could say that $\mu_* - \mu$ behaves similar to a random variable with distribution $N\left(0, \frac{\sigma^2}{M}\right)$. The **standard error** resulted from this approximation is its standard deviation, more commonly referred to as $\varepsilon = \frac{\sigma}{\sqrt{M}}$. This will be an important indicator to measure the accuracy of the simulated result. Thus as n increases, the variance becomes smaller and the error of approximation reduces [25].

The implementation of the MC method is very straightforward to code, and we can make adjustments to how the payoffs should be calculated for different exotic options. For example when pricing Barrier Options, we can examine whether each realisation has hit the barrier and decide whether the contract should be nullified or not. For Asian options on the other hand, we can keep track of the full path of each realisation and calculate the required averages to be used in the final payoffs. However the disadvantages of the MC method are also very apparent. In order to get something close to the accurate answer, tens of thousands of realisations need to be run, which is time consuming [3]. Furthermore, the Greeks in the Black-Scholes equation are calculated by taking the partial derivatives of the option price with respect to some parameters, such as asset price, time, volatility and interest rate. In MC simulations, the partial derivatives are estimated by making small changes to the simulated inputs and taking the average, which can be computationally expensive and inaccurate [26]. This makes it harder to get accurate estimates of the Greeks.

6.2 Monte Carlo recipe

The MC method is simple and yet very powerful for pricing European style options. The concept readily carries over to exotic and path-dependent contracts. The key steps required are simply:

1. Simulate the risk-neutral random walk starting at today's value over the required time frame, which gives one realisation of the price path of the underlying asset.

2. Calculate the payoff of the option for this realisation.

3. Repeat many more realisations of the asset's price movement over the same time frame.

4. Estimate the average payoff of all the realisations.

5. Calculate the present value discounting the average payoff and that will be the option's value today.

In mathematical terms, when valuing the price of options using Monte Carlo method, we simulate the underlying under the risk-neutral measure and discount the expected payoff depending on the option type. The general option's value can be written in the form:

$$V(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[Payoff(S')] \tag{6.2}$$

where $\mathbb{Q}$ represents the risk-neutral density, and

$$\mathbb{E}^{\mathbb{Q}}[Payoff(S')] = \int_0^{\infty} \tilde{p}\left(S,t;S',T\right) P\left(S'\right) dS'$$

with $\tilde{p}\left(S,t;S',T\right)$ representing the probability density function.

6.3 Generating sample paths

Following the recipe above, the first step is to simulate sample paths of the underlying price movement. For vanilla European options, this step is not necessary, because under the Black-Scholes framework, the closed form solution only requires the price at maturity to calculate an exact value of the option by discounting the value of the final payoff. How the price has evolved across the time period is irrelevant. However, for pricing Barrier and Asian options, their payoffs are path-dependent, hence the whole sample path between current and maturity date is required.

In order to simulate the price movement, we assume that the underlying asset follows a GBM defined in Equation 3.2, but with the risk-free interest rate as its growth rate, written as:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{6.3}$$

There are many different methods for approximating the stochastic process described by the GBM above, this thesis will focus on three approaches: Euler Scheme, Forward Euler-Maruyama method and the Milstein method. The full derivation of each method can be found in the paper by Rouah (2011) [27].

6.3.1 Euler Scheme

For this method, we first assume a very special case, the value of the option contract $V(S)$ changes by the function $V(S) = log(S)$. Also notice, $V(S)$ is one dimensional and the function value only depends on the asset price S, and independent of time. Hence we have the following:

$$\frac{\delta V}{\delta S} = \frac{1}{S}, \quad \frac{\delta^2 V}{\delta S^2} = -\frac{1}{S^2}, \quad \frac{\delta V}{\delta t} = 0$$

Apply Ito's lemma defined in Equation 3.6 to work out dV:

$$\begin{aligned} dV = d\left(logS\right) &= \left(\mu S\left(\frac{1}{S}\right) + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\right)\right)dt + \left(\sigma S\left(\frac{1}{S}\right)\right)dW_t \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t \end{aligned}$$

Let $W_t \sim \phi\sqrt{t}$, where $\phi \sim N(0,1)$ is a standard normal random variable. Integrate over $[t, t+\delta t]$ and by using Euler Scheme, also known as the Euler discretisation method, find the solution in discrete time stepping form:

$$\begin{aligned} \int_t^{t+\delta t} d\left(logS\right) &= \int_t^{t+\delta t}\left(\mu - \frac{1}{2}\sigma^2\right)d\tau + \int_t^{t+\delta t}\sigma dW_\tau \\ log\left(S_{t+\delta t}\right) - log\left(S_t\right) &= \left(\mu - \frac{1}{2}\sigma^2\right)\delta t + \sigma\left(W_{t+\delta t} - W_t\right) \\ log\left(S_{t+\delta t}\right) &= log\left(S_t\right) + \left(\mu - \frac{1}{2}\sigma^2\right)\delta t + \sigma\phi\sqrt{\delta t} \end{aligned}$$

exponentiating both sides gives the following expression:

$$S_{t+\delta t} = S_t e^{\left(\mu - \frac{1}{2}\sigma^2\right)\delta t + \sigma\phi\sqrt{\delta t}} \tag{6.4}$$

which is an exact solution that can be used to simulate asset path.

6.3.2 Forward Euler-Maruyama method

Consider a stochastic process of the form:

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t \tag{6.5}$$

then simulate the asset price S_t over the time interval $[0, T]$ by discretising the time interval from $0 = t_1 < t_2 < ... < t_m = T$, such that $t_i - t_{i-1} = dt$. Thus when we integrate dS_t over the interval $[t, t+\delta t]$, the equation becomes:

$$S_{t+\delta t} = S_t + \int_t^{t+\delta t}\mu(S_\tau, \tau)d\tau + \int_t^{t+\delta t}\sigma(S_\tau, \tau)dW_\tau \tag{6.6}$$

Applying Euler scheme gives:

$$S_{t+\delta t} = S_t + \mu(S_t, t)dt + \sigma(S_t, t)\phi\sqrt{dt} \tag{6.7}$$

Thus apply Euler-Maruyama to our GBM in Equation 6.3, the resulting forward Euler-Maruyama method is:

$$\begin{aligned} S_{t+\delta t} &= S_t + rS_t\delta t + \sigma S_t\phi\sqrt{\delta t} \\ &= S_t\left(1 + r\delta t + \sigma\phi\sqrt{\delta t}\right) \end{aligned} \tag{6.8}$$

6.3.3 Milstein Method

Take the exact solution obtained in Equation 6.4, apply Taylor expansion to the exponential term:

$$e^{\left(\mu-\frac{1}{2}\sigma^2\right)\delta t+\sigma\phi\sqrt{\delta t}} \sim 1+\left(r-\frac{1}{2}\sigma^2\right)\delta t+\sigma\phi\sqrt{\delta t}+\frac{1}{2}\sigma^2\phi^2\delta t \tag{6.9}$$

gives:

$$S_{t+\delta t} \sim S_t\left(1+r\delta t+\sigma\phi\sqrt{\delta t}+\frac{1}{2}\sigma^2\left(\phi^2-1\right)\delta t+...\right) \tag{6.10}$$

which we can use to simulate the asset path. Furthermore, if we compare with the Euler-Maruyama method in Equation 6.8, we see that there is an extra term $\frac{1}{2}\sigma^2\left(\phi^2-1\right)\delta t$. The term

$$\frac{1}{2}\left(\phi^2-1\right)\delta t \tag{6.11}$$

is called the Milstein correction term.

6.4 Confidence interval

As well as measuring the standard error ε resulted from the simulations, another way of determining the reliability of the result is to look at its confidence interval. Since MC method is based on probability, where the underlying is generated from a pool of normally distributed samples, the end results in theory may have large variances, and we cannot bound the error exactly. Instead we could impose a confidence interval, which bounds the error within a 95% range.

Given that for a random variable Y, such that $(Y-\mu)/\sigma \sim N(0,1)$, its confidence interval is defined as [25]:

$$\mathbb{P}(\mu-1.96\sigma \leq Y \leq \mu+1.96\sigma)=0.95 \tag{6.12}$$

When using MC, the option value V is expected to have a distribution of $N\left(0,\frac{\sigma^2}{M}\right)$. Thus substituting into Equation 6.12:

$$\mathbb{P}\left(-1.96\sqrt{\frac{\sigma^2}{M}} \leq V \leq 1.96\sqrt{\frac{\sigma^2}{M}}\right)=0.95 \tag{6.13}$$

From the equation, we observe that the width of the confidence interval is inversely proportional to $\sqrt{M}$, hence as we increase the number of simulations M, the confidence interval should becomes smaller. In fact in order to shrink the interval by a factor of 10, the MC method requires 100 times as many samples, which can be computationally expensive. Another approach to shrink the interval is to reduce variance, covered in the next section.

6.5 Variance Reduction

The Monte Carlo method is inefficient because as observed, the standard error

$$\varepsilon = \frac{\sigma}{\sqrt{M}} \tag{6.14}$$

is proportional to $\sqrt{var(X_i)}$, and requires large number of simulations to reduce its variance, which can be computationally expensive. Therefore it order to minimise the error and help the simulations to converge faster, variance reduction techniques have been invented. This thesis will make use of the antithetic variate method. and control variate

6.5.1 antithetic variate

In this technique, we compute two approximations for the option value utilising a single set of random numbers. This is achieved by drawing a set of random samples from the normal distribution to generate a realization of the asset price path, along with the associated option payoff and its present value. Subsequently, the same set of random numbers is utilised, but with their signs reversed, effectively replacing the variable ϕ with its negative counterpart, $-\phi$. Another realization is then simulated, and the option payoff and its present value are calculated accordingly. The final estimate for the option value is obtained as the average of these two computed values. By repetitively performing this operation, a precise and reliable estimate for the option value is derived [3].

The effectiveness of this technique stems from the symmetry inherent in the Normal distribution, if $\phi^{(n)} \sim N(0,1)$, then $-\phi^{(n)}$ also has a standard Normal distribution. This is a property which the antithetic variables were based on [25]. Hence, the utilization of antithetic variables enables the generation of pairs of sample paths that exhibit mirror-like behavior, resulting in a reduction of variance and leading to more accurate estimations of the option value.

In the case of pricing path-dependent options, if we update each price increment with a $N(0,1)$ random variable ϕ_j coming from the $i.i.d.$ sequence $\{\phi_0, \phi_1, \phi_2, ...\phi_{n-1}\}$ represented by Φ_1. Also the negative variate $\{-\phi_0, -\phi_1, -\phi_2, ... - \phi_{n-1}\}$ represented by Φ_2. Denote the payoff function as $U(.)$. The estimated payoff for the antithetic variate method becomes:

$$Pay\tilde{o}ff = \frac{1}{n}\sum_{i=1}^{n}\frac{U(\phi_1)+U(\phi_2)}{2} \tag{6.15}$$

The variance of the antithetic method can be calculated by:

$$var\left[\frac{U(\Phi_1)+U(\Phi_2)}{2}\right] = \frac{1}{4}\left(var(U(\Phi_1)) + var(U(\Phi_2)) + 2cov(U(\Phi_1), U(\Phi_2))\right) \tag{6.16}$$

which will be smaller than the variance of standard method $var\left[U(\Phi_1)\right]$, because since $\Phi_2 = -\Phi_1$, their covariance $cov(U(\Phi_1), U(\Phi_2))$ will be negative.

The distribution of the pairs $(\Phi_1, \Phi 2)$ with size of n samples each, exhibits more regularity compared to a collection of $2n$ independent samples, where the sample mean over the antithetic pairs consistently matches the population mean of 0. As a result, the data set demonstrates lower variance.

7 Results, Analysis and Evaluation

This chapter presents a comprehensive series of computational results employing the Monte Carlo method for pricing Barrier and Asian options. A thorough analysis of the efficacy and precision of these outcomes will be provided.

To help differentiate the results, for asset paths which did not make use of variance reduction techniques, they are referred to as plain realisations or crude Monte Carlo. "Euler Scheme", "Euler Maruyama" and "Milstein" columns each holds the option value obtained with the respective random walk models. "Error" columns are referred to the standard error of Monte Carlo method. "Time" columns refers to the runtime of the corresponding simulations. To simplify calculation, assuming all option contracts starts on the first day of year 2024, and ends on the last day, with each day being a trading day, thus number of monitoring frequency or time steps $N = 365$, with M denoting the number of simulations.

7.1 Barrier Options

Example: up-and-in call option
Consider pricing an up-and-in call option with the following parameters: $S_0 = 100, \sigma = 0.2, r = 0.03, S_u = 110, S_u^* = 110.6772, E = 105,\ t_0 = 01/01/2024,\ T = 31/12/2024,\ N = 365$.

Using a closed formula defined in Equation 4.3, we are able to obtain the exact price of the Barrier option in Table 2, under continuous monitoring, with the corrected up Barrier value of $S_u^* = 110.6772$.

Exact value
7.1055

Table 2: Exact value of up-and-in Call Barrier option

In Table 3 is a set of results obtained from Monte Carlo method. In this case, models of the three different random walks with plain realisations were used, showing their respective standard errors and runtimes. To ensure a fair comparison between the models, across each time step, the underlying movement has been updated by the same random variable ϕ_i. It can be observed that as number of simulations M increases, the estimated option price decreases from approximately 7.3 down to 7.09, getting closer and closer to the exact value.

Table 4 displays the estimated values obtained using antithetic variates method. The random walk models have also used the same set of random variables as with the plain realisations in Table 3 to ensure a fair comparison.

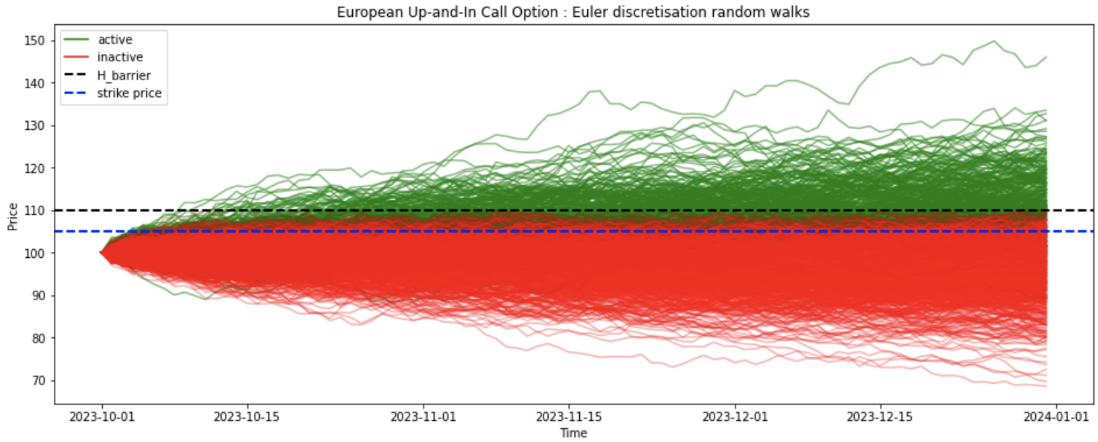


Figure 1: Monte Carlo simulation of up-and-in Call option

M	Euler scheme	Error	Time	Euler Maruyama	Error	Time	Milstein	Error	Time
100	7.3361	1.17796	0.0068	7.3124	1.17361	0.0016	7.3353	1.17784	0.0024
250	8.0450	0.88356	0.0035	8.0445	0.88335	0.0012	8.0442	0.88347	0.0020
500	6.9126	0.56032	0.0081	6.9152	0.56018	0.0065	6.9119	0.56026	0.0062
750	7.5408	0.45783	0.0142	7.5512	0.45807	0.0040	7.5400	0.45778	0.0051
1000	7.7562	0.41264	0.0162	7.7626	0.41227	0.0057	7.7554	0.41260	0.0066
2500	6.8556	0.24776	0.0389	6.8512	0.24775	0.0146	6.8549	0.24774	0.0221
5000	7.2991	0.18025	0.1466	7.2972	0.18010	0.0317	7.2984	0.18023	0.0554
7500	7.0519	0.14458	0.4341	7.0512	0.14449	0.0503	7.0512	0.14457	0.0655
10000	7.2341	0.12742	0.1918	7.2336	0.12737	0.0825	7.2334	0.12740	0.0874
25000	7.0314	0.07930	0.5213	7.0310	0.07927	0.2310	7.0307	0.07929	0.2801
50000	7.1689	0.05640	0.9893	7.1685	0.05638	0.4349	7.1682	0.05640	0.6582
75000	7.1440	0.04570	1.2655	7.1441	0.04568	0.6715	7.1433	0.04569	0.9131
100000	7.0941	0.03938	1.6225	7.0940	0.03936	0.9272	7.0934	0.03938	1.3184

Table 3: Pricing of up-and-in Call option with crude Monte Carlo

By comparing the figures from the two tables, it appears that for the same random walk model, using antithetic variates method has helps the estimated price to be more stable and converges to the true value more quickly. For example, consider the Milstein model. When using plain realisations, it first obtained a close estimate at $M = 7500$, with a value of 7.0512. Whereas when using the antithetic variates reduction technique, a better approximate value of 7.1011 was achieved at $M = 5000$. Similar conclusions can be drawn for the other models, which proves that variance reduction technique does indeed allow Monte Carlo method to converge faster.

Another observation from the results is that there are only minute differences between the estimated values by the three random walk models when the same set of random variables were used. Thus when using Monte Carlo method to option pricing, the choice of the random walk model may have little effect on the final outcome, and practitioners should focus more on applying variance reduction techniques.
In Figure 2, Figure 2 and Figure 4, a plot of the confidence level for each random walk model is displayed. The grey shaded areas indicate the 95% confidence interval of the crude Monte Carlo method with plain realisations, with the black dots representing the

M	Euler scheme	Error	Time	Euler Maruyama	Error	Time	Milstein	Error	Time
100	6.3359	0.64786	0.0026	6.3272	0.64573	0.0010	6.3353	0.64781	0.0013
250	7.6119	0.50358	0.0036	7.6092	0.50341	0.0023	7.6111	0.50353	0.0036
500	7.1962	0.33642	0.0104	7.1957	0.33611	0.0063	7.1955	0.33639	0.0104
750	7.1380	0.26230	0.0113	7.1430	0.26218	0.0078	7.1373	0.26227	0.0104
1000	7.2509	0.23101	0.0143	7.2554	0.23080	0.0100	7.2502	0.23099	0.0150
2500	6.9221	0.14252	0.0395	6.9193	0.14251	0.0264	6.9214	0.14250	0.0444
5000	7.1018	0.10341	0.0851	7.1003	0.10335	0.0554	7.1011	0.10340	0.0850
7500	7.1884	0.08518	0.1297	7.1876	0.08512	0.0956	7.1877	0.08517	0.1355
10000	7.1517	0.07375	0.1572	7.1512	0.07369	0.1109	7.1510	0.07375	0.1671
25000	7.0373	0.04645	0.3866	7.0369	0.04643	0.2722	7.0365	0.04645	0.5347
50000	7.1493	0.03268	1.3931	7.1482	0.03266	1.0032	7.1486	0.03268	2.1447
75000	7.1079	0.02666	2.3588	7.1079	0.02664	1.2882	7.1072	0.02666	4.0066
100000	7.0940	0.02297	2.9149	7.0935	0.02295	3.1759	7.0933	0.02296	4.6564

Table 4: Pricing of up-and-in Call option with antithetic variates

estimated values. The orange shaded area represent 95% confidence interval of Monte Carlo with antithetic variate technique, and the red dots are the respective estimated option price. From the plots, it is clear that antithetic variate is a very effective method of reducing variance, as the bounds of its confidence interval is much smaller than of crude Monte Carlo method. For all three random walk models, the exact value from closed form solution successfully lies within the bounds of their confidence interval. This visualisation further proves that the three random walk models are almost equally as good for simulating the underlying movement.

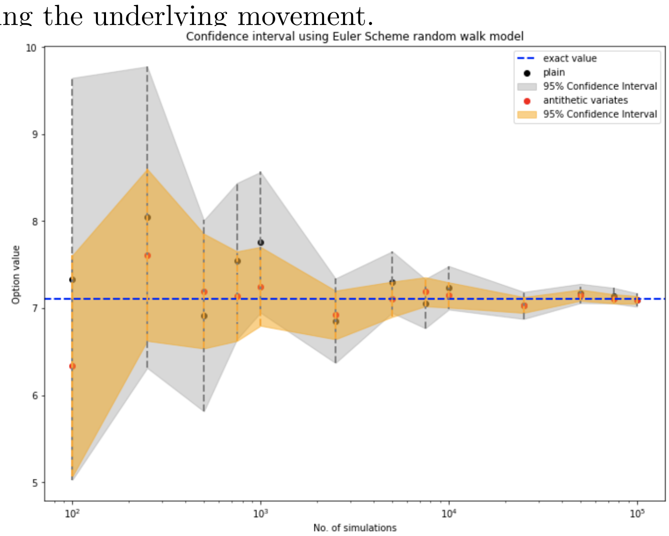


Figure 2: Confidence Interval of Euler Scheme model for up-and-in Barrier option

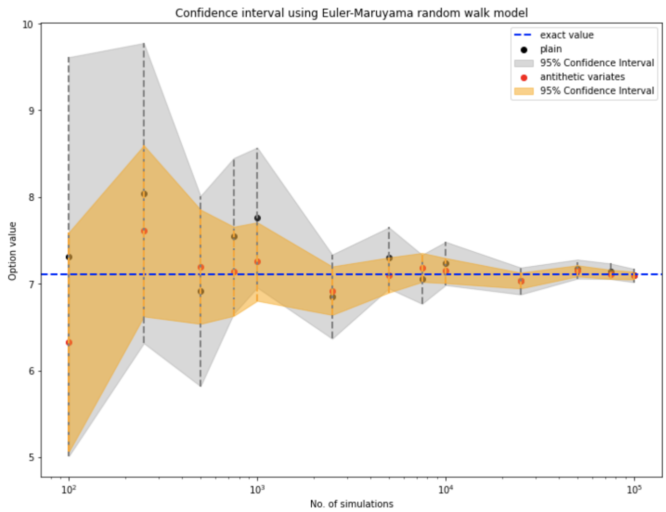


Figure 3: Confidence Interval of Euler Maruyama model for up-and-in Barrier option

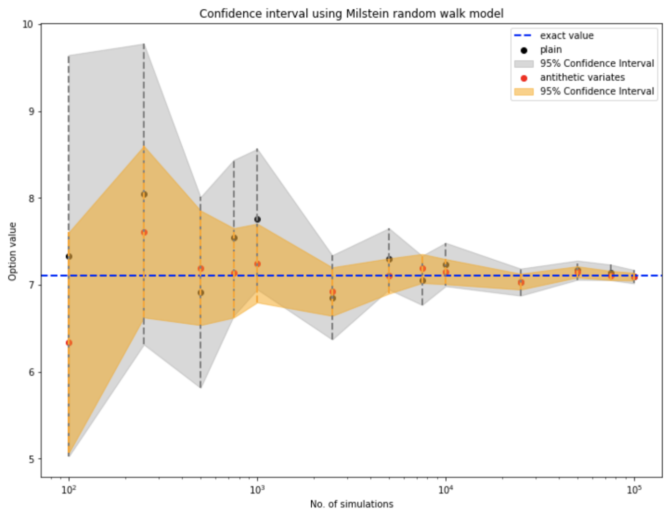


Figure 4: Confidence Interval of Milstein model for up-and-in Barrier option

7.2 Asian Options

Example: geometric average rate Call option
This example shows the pricing of an Geometric average rate Call option with the following parameters: $S_0 = 100, \sigma = 0.2, r = 0.03, E = 105$, $T_0 = 01/01/2024$, $T_1 = 01/01/2024$, $T_2 = 31/12/2024$ and $N = 365$, where T_0 is the contract start date, T_1 is the start date of the period from which to calculate the average price, T_2 is the expiration date, where the averaging period also ends. Thus in this example, average taken is the average price across the whole year.

This example is chosen so that we can make use of the exact closed form solution defined in Equation 5.18, and the exact value is shown in Table 5.

Exact value
2.9849

Table 5: Exact value of geometric average rate call option

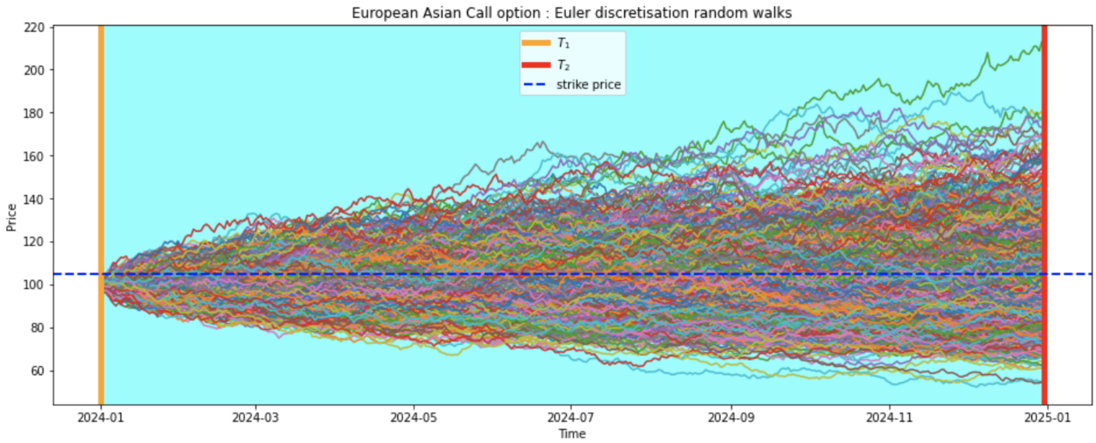


Figure 5: Monte Carlo simulation of geometric average rate Call option

Figure 5 is a plot of the Monte Carlo simulations using Euler Scheme random walk model. The orange band labelled as T_1 denotes the start date of the averaging period. The red band labelled as T_2 denotes the end date of the averaging period. The highlighted area in cyan colour is to help with visualising the effective time interval for calculating the geometric average price.

Table 6 shows the option value estimated by crude Monte Carlo method. It appears that with sufficient number of simulations, the estimated value does also converge to the exact value with closed form solution. Once again, values obtained from different random walk models show very slight differences, which suggests that for the pricing of geometric average rate call option, the choice of random walk has little impact on the final outcome.

M	Euler scheme	Error	Time	Euler Maruyama	Error	Time	Milstein	Error	Time
100	2.1525	0.47836	0.0035	2.1531	0.47860	0.0014	2.1523	0.47831	0.0016
250	2.7379	0.34885	0.0053	2.7322	0.34837	0.0017	2.7376	0.34882	0.0043
500	3.1371	0.27773	0.0119	3.1342	0.27763	0.0064	3.1367	0.27770	0.0051
750	2.8139	0.20103	0.0135	2.8122	0.20080	0.0071	2.8135	0.20100	0.0083
1000	2.8906	0.17842	0.0166	2.8903	0.17843	0.0075	2.8903	0.17840	0.0085
2500	3.2076	0.12140	0.0438	3.2073	0.12134	0.0218	3.2072	0.12139	0.0275
5000	3.0177	0.08419	0.1018	3.0171	0.08415	0.0503	3.0173	0.08419	0.0559
7500	2.9533	0.06519	0.1910	2.9532	0.06516	0.0675	2.9530	0.06518	0.0765
10000	2.9803	0.05811	0.1825	2.9796	0.05807	0.0930	2.9799	0.05810	0.1459
25000	2.9928	0.03701	0.4715	2.9921	0.03699	0.2367	2.9924	0.03701	0.2948
50000	3.0103	0.02625	1.0332	3.0100	0.02624	0.6747	3.0099	0.02625	0.9024
75000	2.9819	0.02139	1.5308	2.9816	0.02137	1.0966	2.9815	0.02138	1.3826
100000	2.9827	0.01852	2.0763	2.9820	0.01851	1.3152	2.9824	0.01852	1.8602

Table 6: Pricing of geometric average rate call option with crude Monte Carlo

Table 7 on the other hand displays the estimated value with antithetic variate technique applied. It seems that with $M = 750$, the simulation has already converged to some relatively accurate value of 2.9037, 2.9017 and 2.9034 from the three random walk models. Whereas for crude Monte Carlo, similar accuracy was achieved when $M = 1000$. Meanwhile the runtime for crude Monte Carlo and antithetic variate methods are relatively similar. This observation agrees with the fact that antithetic variate does indeed improve the efficiency of Monte Carlo method.

M	Euler scheme	Error	Time	Euler Maruyama	Error	Time	Milstein	Error	Time
100	2.7454	0.30910	0.0029	2.7471	0.30954	0.0021	2.7451	0.30907	0.0038
250	3.5352	0.25368	0.0065	3.5310	0.25318	0.0048	3.5347	0.25366	0.0045
500	3.1747	0.16898	0.0102	3.1749	0.16888	0.0077	3.1743	0.16896	0.0094
750	2.9037	0.12693	0.0166	2.9017	0.12679	0.0112	2.9034	0.12692	0.0135
1000	2.9173	0.11129	0.0182	2.9174	0.11120	0.0138	2.9170	0.11128	0.0178
2500	3.0791	0.07167	0.0562	3.0788	0.07164	0.0428	3.0788	0.07167	0.0526
5000	3.0068	0.05115	0.1729	3.0065	0.05113	0.2296	3.0064	0.05114	0.1178
7500	3.0014	0.04049	0.1621	3.0010	0.04046	0.1169	3.0011	0.04048	0.1643
10000	3.0205	0.03581	0.1971	3.0198	0.03578	0.1433	3.0201	0.03581	0.2113
25000	3.0035	0.02251	0.6134	3.0030	0.02249	0.4824	3.0032	0.02251	0.6539
50000	2.9923	0.01597	1.9151	2.9917	0.01596	1.3537	2.9919	0.01597	1.9797
75000	2.9971	0.01305	3.1236	2.9967	0.01304	2.2420	2.9967	0.01305	3.9595
100000	2.9837	0.01127	3.7420	2.9830	0.01126	2.6156	2.9833	0.01127	5.3264

Table 7: Pricing of geometric average rate call option with antithetic variate

The plots in Figure 10, Figure 10 and Figure 6 are the confidence intervals of different random walk models. Take Figure 6 as an example, initial estimates deviates from the exact value with large margin. As the number of simulations increase, the estimates quickly converges to the exact value, which is always within the bounds of the confidence intervals. This observation suggest that Monte Carlo method is indeed a reliable method for estimating this type of Asian option.

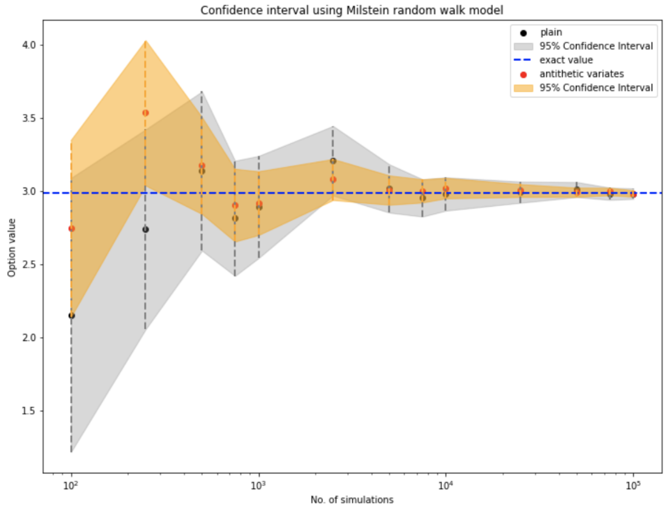


Figure 6: Confidence Interval of Milstein model for geometric average rate call option

Example: geometric average rate Put option

This example shows the pricing of an Geometric average rate Put option with the following parameters: $S_0 = 100, \sigma = 0.2, r = 0.03, E = 105$, $T_0 = 01/01/2024$, $T_1 = 01/01/2024$, $T_2 = 31/12/2024$ and $N = 365$.

Exact solution calculated by Equation 5.19 is:

Exact value
6.6983

Table 8: Exact value of geometric average rate put option

Table 9 and Table 10 are the estimated option prices by crude Monte Carlo and antithetic variates methods.

Figure 7 is the confidence interval of using Milstein model, which shows the convergence to exact value as number of simulations increase. Confidence interval plots of Euler scheme and Euler-Maruyama models can be found in Figure 12 and Figure 13 in subsection A.2.

M	Euler scheme	Error	Time	Euler Maruyama	Error	Time	Milstein	Error	Time
100	5.6733	0.64791	0.0027	5.6710	0.64796	0.0009	5.6731	0.64787	0.0011
250	6.9465	0.50376	0.0271	6.9492	0.50385	0.0025	6.9462	0.50373	0.0035
500	6.4207	0.33145	0.0224	6.4168	0.33143	0.0110	6.4204	0.33143	0.0057
750	6.5147	0.26809	0.0321	6.5148	0.26824	0.0073	6.5143	0.26808	0.0070
1000	6.7221	0.23106	0.0210	6.7233	0.23116	0.0089	6.7218	0.23104	0.0087
2500	6.6769	0.14695	0.0650	6.6769	0.14701	0.0217	6.6765	0.14694	0.0252
5000	6.8575	0.10632	0.0968	6.8573	0.10636	0.0449	6.8572	0.10631	0.0566
7500	6.5878	0.08612	0.1227	6.5872	0.08614	0.0583	6.5875	0.08612	0.0692
10000	6.6474	0.07378	0.1597	6.6463	0.07380	0.0747	6.6470	0.07377	0.0937
25000	6.7159	0.04724	0.8527	6.7155	0.04725	0.2997	6.7156	0.04724	0.4143
50000	6.7602	0.03358	1.2758	6.7596	0.03359	0.7112	6.7598	0.03358	0.8786
75000	6.7255	0.02715	1.4123	6.7250	0.02716	0.9016	6.7252	0.02715	1.1551
100000	6.6538	0.02346	1.7243	6.6534	0.02347	1.1644	6.6534	0.02346	1.6403

Table 9: Pricing of geometric average rate put option with crude Monte Carlo

M	Euler scheme	Error	Time	Euler Maruyama	Error	Time	Milstein	Error	Time
100	6.0838	0.20477	0.0521	6.0832	0.20495	0.0092	6.0835	0.20475	0.0048
250	7.0345	0.17655	0.0077	7.0364	0.17658	0.0052	7.0341	0.17654	0.0103
500	6.6462	0.11633	0.0104	6.6434	0.11648	0.0137	6.6458	0.11632	0.0108
750	6.7672	0.09497	0.0147	6.7667	0.09517	0.0108	6.7668	0.09496	0.0137
1000	6.6142	0.07983	0.0195	6.6140	0.07994	0.0141	6.6138	0.07983	0.0186
2500	6.7092	0.05398	0.0493	6.7094	0.05405	0.0384	6.7089	0.05398	0.0507
5000	6.7875	0.03806	0.1176	6.7871	0.03811	0.1197	6.7872	0.03806	0.2304
7500	6.7351	0.03079	0.1316	6.7349	0.03082	0.1117	6.7348	0.03078	0.1612
10000	6.6799	0.02630	0.2098	6.6789	0.02633	0.1605	6.6795	0.02629	0.2248
25000	6.7151	0.01686	0.6279	6.7144	0.01688	0.5142	6.7147	0.01686	0.9874
50000	6.7286	0.01201	1.9494	6.7280	0.01203	1.4980	6.7282	0.01201	2.7467
75000	6.7042	0.00973	3.2659	6.7036	0.00974	2.0900	6.7038	0.00972	4.2171
100000	6.7102	0.00844	3.7849	6.7098	0.00845	4.4285	6.7099	0.00844	4.0936

Table 10: Pricing of geometric average rate put option with antithetic variates

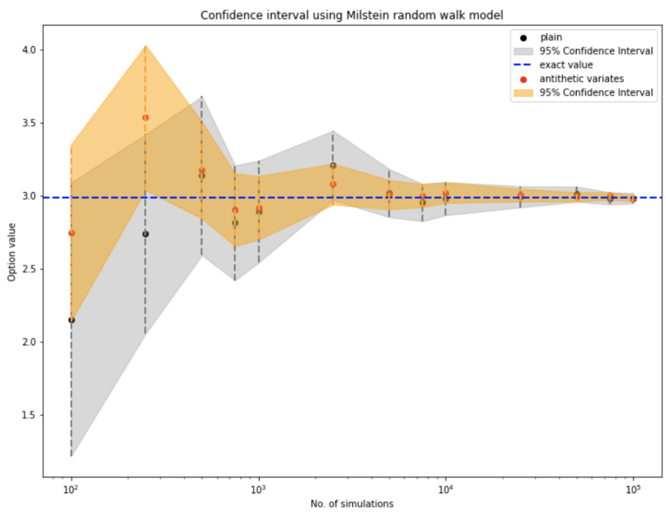


Figure 7: Confidence Interval of Milstein model for geometric average rate put option

Example: geometric average rate Call option - Average of 30 days before expiry

This example shows the pricing of an Geometric average rate Put option with the following parameters: $S_0 = 100, \sigma = 0.2, r = 0.03, E = 105$, $T_0 = 01/01/2024$, $T_1 = 01/12/2024$, $T_2 = 31/12/2024$ and $N = 365$. In this case, the average price is calculated with prices in the last 31 days of the contract.

This example presents an option price identical to the one showcased in example Table 5 when employing the exact closed-form solution as described in Equation 5.18. This is because the closed-form solution does not consider a changing averaging period other than averaging throughout the entire contract duration. Thus we can make use of Equation 3.15, the closed form solution for calculating vanilla European Call option to obtain another price for comparison.

Exact value	vanilla European Call
2.9849	7.1281

Table 11: Exact value of geometric average rate call option and vanilla European call option

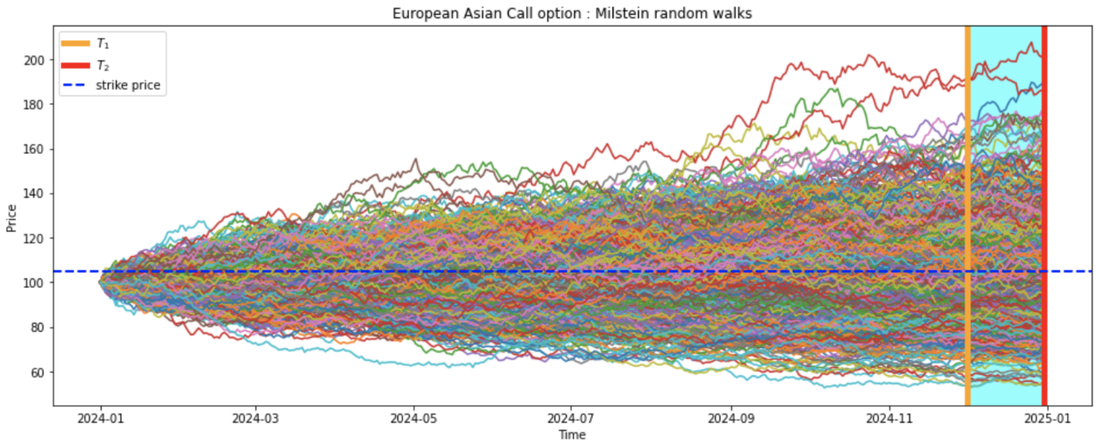


Figure 8: Monte Carlo simulation of geometric average rate Call option with average price of 30 days before expiry

Based on the findings presented in Table 13, the Monte Carlo estimation yields an approximate price of 3.43. This value is nearer to the precise figure derived from the analytical solution, 2.9849, and is notably distant from the value for a standard European Call option, 7.1281. This observation aligns with the common understanding that exotic options tend to have lower prices compared to vanilla options.

M	Euler scheme	Error	Time	Euler Maruyama	Error	Time	Milstein	Error	Time
100	3.3111	0.60469	0.0055	3.3087	0.60497	0.0025	3.3106	0.60463	0.0042
250	3.4521	0.38269	0.0060	3.4488	0.38235	0.0017	3.4517	0.38266	0.0050
500	3.4757	0.30365	0.0094	3.4791	0.30392	0.0042	3.4753	0.30362	0.0055
750	3.5669	0.23297	0.0155	3.5637	0.23271	0.0086	3.5665	0.23295	0.0069
1000	3.6138	0.20926	0.0176	3.6137	0.20927	0.0082	3.6133	0.20923	0.0097
2500	3.6480	0.13795	0.0578	3.6460	0.13780	0.0236	3.6476	0.13793	0.0285
5000	3.3937	0.09225	0.1607	3.3928	0.09218	0.0550	3.3933	0.09224	0.1288
7500	3.4827	0.07682	0.1470	3.4828	0.07678	0.0641	3.4823	0.07681	0.0902
10000	3.4615	0.06576	0.2071	3.4613	0.06573	0.0973	3.4610	0.06575	0.1554
25000	3.4226	0.04154	0.6607	3.4226	0.04152	0.2606	3.4222	0.04154	0.3408
50000	3.4382	0.02944	1.6105	3.4379	0.02943	0.7043	3.4378	0.02944	0.9080
75000	3.3881	0.02384	1.6569	3.3873	0.02383	0.9250	3.3877	0.02384	1.2925
100000	3.4418	0.02083	2.2045	3.4413	0.02082	1.3594	3.4414	0.02083	1.6672

Table 12: Pricing of geometric average rate call option with crude Monte Carlo, average price of last 30 days

M	Euler scheme	Error	Time	Euler Maruyama	Error	Time	Milstein	Error	Time
100	3.2191	0.35522	0.0044	3.2202	0.35568	0.0050	3.2187	0.35518	0.0044
250	3.3450	0.23496	0.0114	3.3442	0.23508	0.0047	3.3447	0.23493	0.0042
500	3.5288	0.18092	0.0097	3.5328	0.18093	0.0071	3.5284	0.18091	0.0085
750	3.3029	0.13480	0.0138	3.3038	0.13473	0.0099	3.3025	0.13479	0.0121
1000	3.6747	0.13013	0.0186	3.6757	0.13008	0.0137	3.6743	0.13011	0.0183
2500	3.3472	0.08001	0.0525	3.3467	0.07992	0.0404	3.3468	0.08001	0.0736
5000	3.3857	0.05542	0.1734	3.3851	0.05538	0.0861	3.3853	0.05542	0.1306
7500	3.4527	0.04593	0.1716	3.4525	0.04590	0.1707	3.4523	0.04593	0.2257
10000	3.4252	0.03954	0.2145	3.4248	0.03951	0.1617	3.4248	0.03953	0.3015
25000	3.4389	0.02511	0.6157	3.4384	0.02510	0.4586	3.4385	0.02511	0.7828
50000	3.4287	0.01777	1.8512	3.4280	0.01776	1.2390	3.4283	0.01777	2.0998
75000	3.4222	0.01442	3.2032	3.4214	0.01441	3.4427	3.4217	0.01442	4.9393
100000	3.4336	0.01252	3.4040	3.4332	0.01252	2.6169	3.4332	0.01252	6.5162

Table 13: Pricing of geometric average rate call option with antithetic variates, average price of last 30 days

Figure 9 illustrates the confidence interval obtained through the Monte Carlo approach for this dynamic averaging timeframe. The values tend to converge towards an estimation that lies above the exact value derived from the analytic solution (represented by the blue dashed line), yet below the value of a vanilla call option (represented by the green dashed line). Notably, none of the exact solutions fell within this interval. Consequently, when dealing with Asian options characterized by varying user-defined averaging periods, relying on the Monte Carlo method is considered a more dependable pricing strategy.

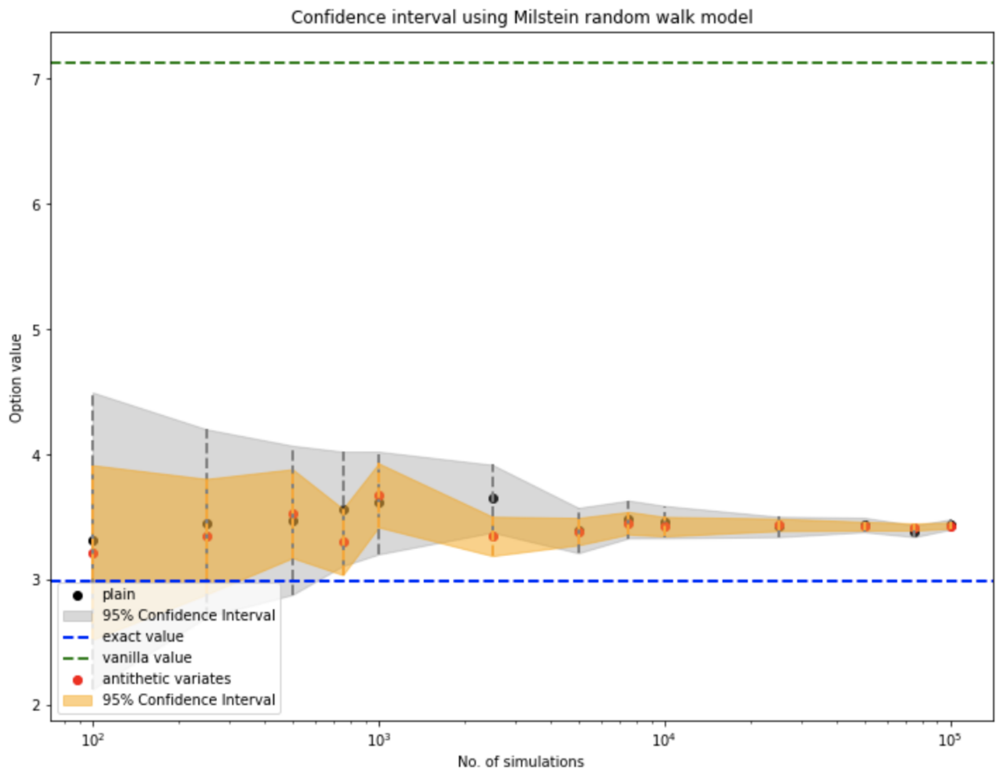


Figure 9: Confidence Interval of Milstein model for geometric average rate call option with average price of 30 days before expiry

8 Conclusion

References

[1] J. Fernando, "Derivatives: Types, considerations, and pros and cons, investopedia." https://www.investopedia.com/terms/d/derivative.asp, January 2023. Accessed: April 16, 2023.

[2] F. Weert, *Exotic Options Trading.* Chichester, England: John Wiley & Sons, 2008.

[3] P. Wilmott, *Paul Wilmott introduces quantitative finance.* John Wiley & Sons, 2007.

[4] Y.-K. Kwok, *Mathematical models of Financial Derivatives.* Springer Berlin, Heidelberg, 2008.

[5] J. Hull, *Options futures and other derivatives.* Pearson Education India, 2003.

[6] Z. Janková, "Drawbacks and limitations of black-scholes model for options pricing.," *Journal of Financial Studies & Research*, pp. 1–7, 2018.

[7] P. Zhang, *An introduction to exotic options.* European Financial Management, 1995.

[8] V. Henderson, "Black–scholes model.," *Wiley StatsRef: Statistics Reference Online.*, 2014.

[9] F. Black and M. Scholes, "Ethe pricing of options and corporate liabilities.," *Journal of political economy*, vol. 81, no. 3, pp. 637–654, 1973.

[10] P. Wilmott, S. Howison, and J. Dewynne, *The mathematics of financial derivatives: a student introduction.* Cambridge university press, 1995.

[11] R. Hu, N. Huang, C. Zhao, and B. Lee, "The pricing for a class of barrier options.," *NONLINEAR ANALYSIS FORUM*, vol. 11, no. 1, p. 33, 2006.

[12] S. G. Kou, "On pricing of discrete barrier options.," *Statistica Sinica*, pp. 955–964, 2003.

[13] E. Reiner and M. Rubinstein, "Breaking down the barriers.," *Risk*, vol. 4, no. 8, pp. 28–35, 1991.

[14] P. Wilmott, *Paul Wilmott on quantitative finance.* John Wiley & Sons, 2013.

[15] R. Merton, "Theory of rational option pricing," *The Bell Journal of economics and management science*, pp. 141–183, 1973.

[16] N. Kunitomo and M. Ikeda, "Pricing options with curved boundaries," *Mathematical finance*, pp. 275–298, 1992.

[17] H. Kat and L. Verdonk, "Tree surgery.," *Cass Business School Research Paper.*, 1995.

[18] M. Broadie, P. Glasserman, and S. Kou, "A continuity correction for discrete barrier options," *Mathematical Finance*, pp. 325–349, 1997.

[19] E. W., "Asian option pricing and volatility.," 2012.

[20] A. Kemna and A. Vorst, "A pricing method for options based on average asset values," *Journal of Banking & Finance*, pp. 113–129, 1990.

[21] G. Fusai and A. Meucci, "Pricing discretely monitored asian options under lévy processes.," *Journal of Banking & Finance*, vol. 32, no. 10, pp. 2076–2088, 2008.

[22] J. F. EP., *An Introduction to Numerical Methods and Analysis.* Wiley, 2013.

[23] W. Kenton, "Monte carlo simulation: History, how it works, and 4 key steps, investopedia." `https://www.investopedia.com/terms/m/montecarlosimulation.asp`, March 2023. Accessed: April 17, 2023.

[24] P. Glasserman, *Monte Carlo methods in financial engineering.* New York: springer, 2004.

[25] D. J. HIGHAM, *An Introduction to Financial Option Valuation.* Cambridge University Press, 2004.

[26] G. Gracianti, "Computing greeks by finite difference using monte carlo simulation and variance reduction techniques.," *BIMIPA*, vol. 25, no. 1, pp. 80–93, 2018.

[27] F. Rouah, "Euler and milstein discretization.," *Documento de trabajo, Sapient Global Markets, Estados Unidos.*, 2011.

A Appendix

Supplementary materials (such as source code, user menu, etc) could be included. Each appendix must be labelled (for example, Appendix A, Appendix A.1, Appendix A.2, Appendix B, Appendix B.1, etc.) and with heading. All Appendices must be referred in the text.

A.1 Appendix A

A.1.1 up-and-out call option

A.1.2 down-and-in call option

A.1.3 down-and-out call option

A.1.4 up-and-in put option

A.1.5 up-and-out put option

A.1.6 down-and-in put option

A.1.7 down-and-ioutput option

A.2 Appendix B

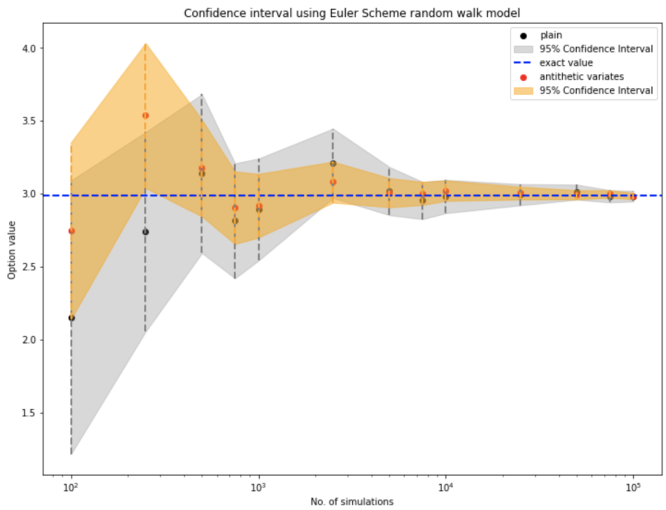


Figure 10: Confidence Interval of Euler Scheme model

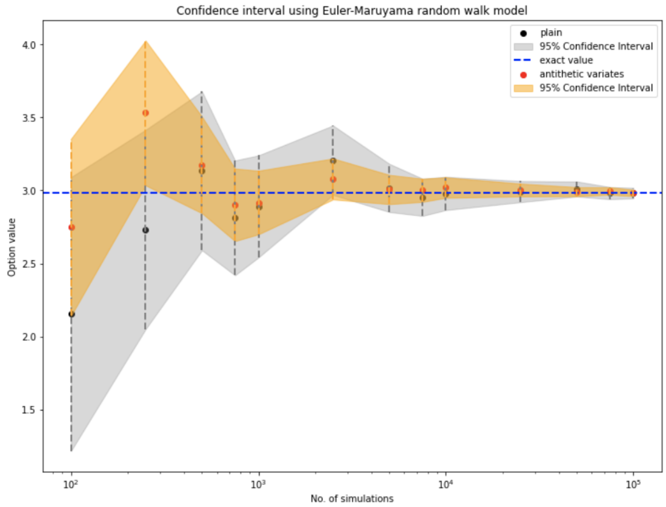


Figure 11: Confidence Interval of Euler-Maruyama model for geometric average rate call option

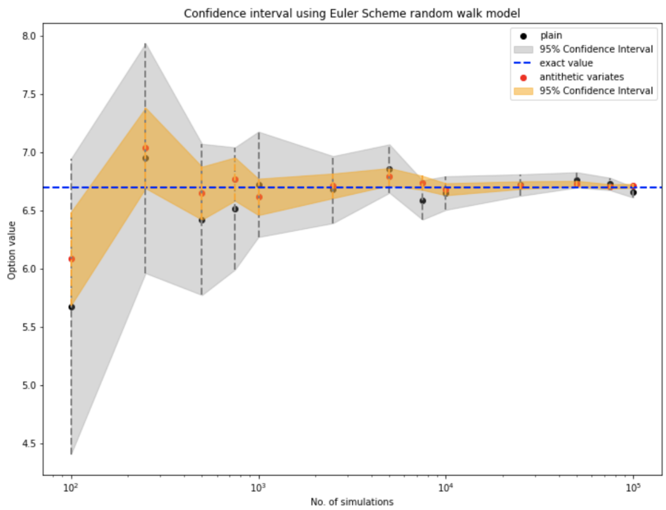


Figure 12: Confidence Interval of Euler Scheme model for geometric average rate put option

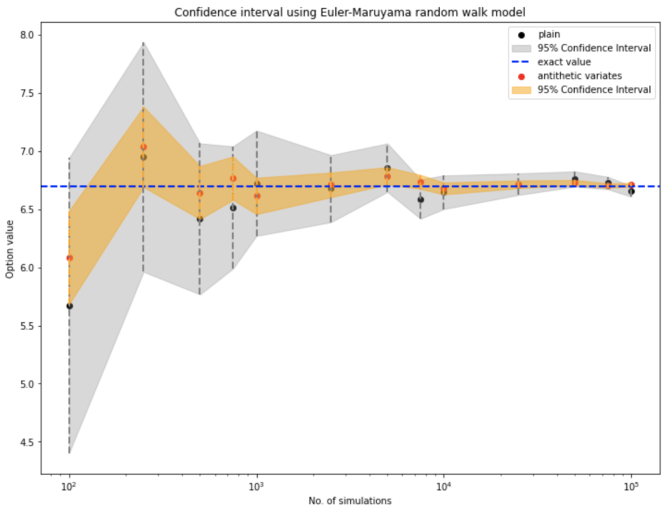


Figure 13: Confidence Interval of Euler-Maruyama model for geometric average rate put option