BREAKING DOWN THE BARRIERS

Though similarly path-dependent, barrier options are easier to value than standard American options, as Mark Rubinstein **and** Eric Reiner **explain**

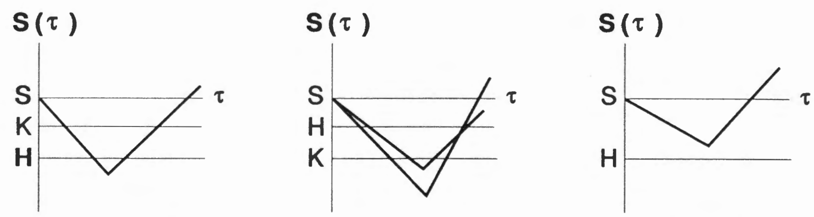
The pay-off of a standard European option depends only on the price of the underlying asset on the expiration date. In particular, given the final price of the underlying asset, the pay-off will be the same regardless of the path taken by the underlying asset during the life of the option to reach that final price. Whether the underlying asset price reaches a given price by moving down and then up, or up and then down, matters not to the buyer or seller of the option. It is as if it did not matter whether you travelled from Paris to London by air or by the Channel Tunnel, as long as you arrived on time.

The terminology used to describe this feature is path independence. More generally, the pay-off from an option may depend on some aspect of the price path. For example, the pay-off of a lookback option depends on the minimum or maximum price of the underlying asset attained during the life of the option, and the pay-off of an Asian option depends on the average price. In this article, we will examine a simpler type of path-dependent option in which the pay-off depends not only on the final price of the underlying asset but also on whether or not the underlying asset has reached some other barrier price during the life of the option.

Our objective is to value a variety of these options in a Black-Scholes environment;[1] that is, one in which the underlying asset can be assumed to follow a lognormal random walk, and in which arbitrage arguments allow the use of a risk-neutral valuation approach – discount the expected pay-off of the option at expiration by the riskless rate, where the underlying asset price is expected to appreciate at the same riskless rate less payouts.

These options are in a sense intermediate between standard European and American options. They resemble American options since their value depends on how the underlying asset price behaves through time. But they are simpler to value than American options since the critical boundary of the underlying asset price is determined in advance and specified in the contract. As a result, unlike American options, it will be possible to state "closed form" valuation solutions.

Figure 1



Call pay-off (K>H)
prob[S*>K: S(τ)≤H]

Call pay-off (K<H)
prob[H≥S*>K]
+ prob[S*>H: S(τ)≤H]

Rebate pay-off
prob[S*>H]
− prob[S*>H: S(τ)≤H]

To do this we will need the density of the natural logarithm of the risk-neutral underlying asset return, u:

$$f(u) \equiv (1/\sigma\sqrt{2\pi t})e^{-\frac{1}{2}v^2}$$
$$\text{with } v \equiv (u-\mu t)/\sigma\sqrt{t},\ \mu \equiv \log(r/d) - \tfrac{1}{2}\sigma^2$$

This is just a normal density function in which r is one plus the rate of interest, d is one plus the pay-out rate of the underlying asset, σ is the volatility of the underlying asset, and t is the time-to-expiration of the option.[2]

We also need another density. Given that the underlying asset price first starts at S *above* the barrier H, the density of the natural logarithm of the underlying asset return when the underlying asset price breaches the barrier but ends up below the barrier at expiration is:

$$g(u) \equiv e^{2\mu\alpha/\sigma^2}(1/\sigma\sqrt{2\pi t})e^{-\frac{1}{2}v^2}$$
$$\text{with}$$
$$v \equiv (u - 2\eta\alpha - \eta\mu t)/\sigma\sqrt{t},\quad \alpha \equiv \log(H/S)$$

This is a normal density premultiplied by $e^{2\mu\alpha/\sigma^2}$. Here $\eta = 1$. Alternatively, given that the underlying asset price first starts *below* the barrier, the density of the natural logarithm of the underlying asset return when the underlying asset price breaches the barrier but ends up at expiration below the barrier is the same expression but in which $\eta = -1$.

"In" barrier options

Our first example is a down-and-in call. Although you pay for this option up front, you do not receive the call until the underlying asset price reaches a prespecified level termed the barrier or knock-in boundary, H.[3] If, after elapsed time $\tau \le t$, the underlying asset price hits the barrier, you receive a standard European call with strike price K and time-to-expiration $t-\tau$. On the other hand, if through

[1] To our knowledge, the only published solution for the options covered in this article has been for the down-and-out option (without adjustment for payouts). See John Cox and Mark Rubinstein, *Options Markets*, page 410, Prentice-Hall, 1985

[2] Since Black and Scholes are only interested in the price of the underlying asset at expiration, they can allow r, d, and σ to be known functions of time. However, since the options discussed in this article depend in complex ways on the time paths of these variables, to keep matters simple we assume here that these variables are constant through time

[3] In this case, the standard call is received conditional on the behaviour of a random variable (underlying asset price). See, Mark Rubinstein, *Pay now, choose later*, *RISK*, February 1991, for an analysis of a standard call that is received unconditionally but at some prespecified future date

elapsed time t, the barrier is never hit, then instead you receive a rebate R at that time (at expiration). Expressed concisely, the pay-off from this option is:

down-and-in call ($S > H$)	$\max[0, S^* - K]$ if for some $\tau \le t$, $S(\tau) \le H$
	R (at expiry) if for all $\tau \le t$, $S(\tau) > H$

where $S(\tau)$ is the price of the underlying asset after elapsed time τ, and $S^* \equiv S(t)$ is the price of the underlying asset at expiration. To be interesting, the initial price S of the underlying asset must be greater than the barrier.

The graphs in Figure 1 below help illustrate what can happen. The lines in the first two graphs show paths where the barrier is crossed and the call is received. The line in the third graph shows a path that leads to a rebate pay-off at expiry. The statements below state the probability that each outcome will occur. It is necessary to distinguish between two cases: one in which $K>H$ and one in which $K<H$. In the first case, to receive a positive pay-off from the call the underlying asset price must end up above the strike price while having first touched the barrier. On the other hand, the rebate is received only if the underlying price ends up above the barrier without ever having hit the barrier before expiration. For this first case, there are thus three types of outcome:

$$S^*>K \text{ conditional on } S(\tau) \le H \text{ for some } \tau \le t \rightarrow \text{pay-off} = S^* - K$$

$$S^* \le K \text{ conditional on } S(\tau) \le H \text{ for some } \tau \le t \rightarrow \text{pay-off} = 0$$

$$S(\tau) > H \text{ for all } \tau \le t \rightarrow \text{pay-off} = R$$

In the second case, to receive a positive pay-off from the call the underlying asset price must end up above the barrier while having first touched the barrier, or end up below the barrier but above the strike price. Again, the rebate is only received if the underlying price ends up above the barrier without ever having hit the barrier prior to expiration. Here there are four types of outcome:

$$S^*>H \text{ conditional on } S(\tau) \le H \text{ for some } \tau \le t \rightarrow \text{pay-off} = S^* - K$$

Barrier options are simpler to value than American options since the critical boundary of the underlying asset price is determined in advance and specified in the contract. It is therefore possible to state "closed form" valuation solutions

$$S^* \le H \text{ and } S^* > K \rightarrow \text{pay-off} = S^* - K$$

$$S^* \le K \rightarrow \text{pay-off} = 0$$

$$S(\tau) > H \text{ for all } \tau \le t \rightarrow \text{pay-off} = R$$

The second and third outcomes are simplified by the fact that since the underlying asset price starts out above the barrier, if the underlying asset price then finishes below the barrier it must necessarily have breached the barrier at some time.

Consider first the $K>H$ case. The value of the option is the sum of two terms, the first (call pay-off) corresponding to $\text{prob}[S^*>K\text{: } S(\tau) \le H]$ and the second (rebate) corresponding to $\text{prob}[S^*>H] - \text{prob}[S^*>H\text{: } S(\tau) \le H]$:

$$[3] \equiv r^{-t} \int \phi(Se^u - K) g(u) du = \phi S d^{-t} (H/S)^{2\lambda} N(\eta y) - \phi K r^{-t} (H/S)^{2\lambda - 2} N(\eta y - \eta\sigma\sqrt{t})$$

$$[5] \equiv R r^{-t} \int [f(u) - g(u)] du = R r^{-t} [N(\eta x_1 - \eta\sigma\sqrt{t}) - (H/S)^{2\lambda - 2} N(\eta y_1 - \eta\sigma\sqrt{t})]$$

where the first integral is taken over the region $\log(K/S)$ to $\eta\infty$, the second integral is taken from $\log(H/S)$ to $\eta\infty$,

$$x_1 \equiv [\log(S/H \div \sigma\sqrt{t}] + \lambda\sigma\sqrt{t}$$
$$y \equiv [\log(H^2/SK) \div \sigma\sqrt{t}] + \lambda\sigma\sqrt{t}$$
$$y_1 \equiv [\log(H/S) \div \sigma\sqrt{t}] + \lambda\sigma\sqrt{t}$$
$$\lambda \equiv 1 + (\mu/\sigma^2)$$

$N(\cdot)$ is the standard normal distribution function and the binary variables, η and ϕ, are currently both set equal to 1.

The current value of the down-and-in call can then be expressed as

$$C_{di(K>H)} = [3] + [5]\ \{\eta = 1, \phi = 1\}$$

If instead $K<H$, we will need terms corresponding to $\text{prob}[H \ge S^* > K]$ and $\text{prob}[S^*>H\text{: } S(\tau) \le H]$, as well as the rebate term. Since

$$\text{prob}[H \ge S^* > K] = \text{prob}[S^* > K] - \text{prob}[S^* > H]$$

we have the three corresponding integrals:

$$[1] \equiv r^{-t} \int \phi(Se^u - K) f(u) du = \phi S d^{-t} N(\phi x) - \phi K r^{-t} N(\phi x - \phi\sigma\sqrt{t})$$

$$[2] \equiv r^{-t} \int \phi(Se^u - K) f(u) du = \phi S d^{-t} N(\phi x_1) - \phi K r^{-t} N(\phi x_1 - \phi\sigma\sqrt{t})$$

$$[4] \equiv r^{-t} \int \phi(Se^u - K) g(u) du = \phi S d^{-t} (H/S)^{2\lambda} N(\eta y_1) - \phi K r^{-t} (H/S)^{2\lambda - 2} N(\eta y_1 - \eta\sigma\sqrt{t})$$

where the first integral is taken over the region $\log(K/S)$ to $\phi\infty$, the second integral is taken over the region $\log(H/S)$ to $\phi\infty$, the third integral is taken over the region $\log(H/S)$ to $\eta\infty$,

$$x \equiv [\log(S/K) \div \sigma\sqrt{t}] + \lambda\sigma\sqrt{t}$$

and the binary variables, η and ϕ, are currently both set equal to 1.

Using this we can write the current value of the down-and-in call as:

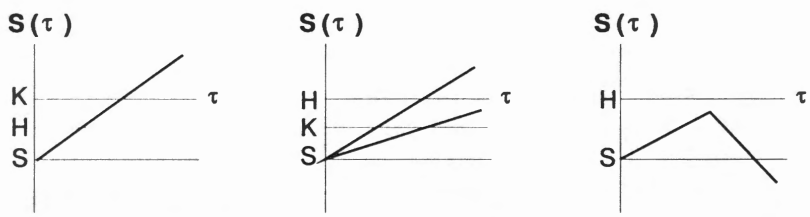
$$C_{di(K<H)} = [1] - [2] + [4] + [5]\ \{\eta = 1, \phi = 1\}$$

Our next barrier option is an up-and-in call. This option is identical to a down-and-in call except that the underlying asset price starts out *below* instead of above the barrier. Expressed concisely, the pay-off from this option is:

up-and-in call ($S < H$)	$\max[0, S^* - K]$ if for some $\tau \le t$, $S(\tau) \ge H$
	R (at expiry) if for all $\tau \le t$, $S(\tau) < H$

The graphs in Figure 2 help illustrate what can happen.

Figure 2



Call pay-off (K>H)	Call pay-off (K<H)	Rebate pay-off
prob[S*>K]	prob[S*≥H]	prob[S*<H]
	+ prob[H>S*>K: S(τ)≥H]	− prob[S*<H: S(τ)≥H]

For the K>H case, the new quantities are prob[S*<H] and prob[S*<H: S(τ)≥H]. The density corresponding to the former is of course f(u) and the density corresponding to the latter is identical to g(u), but with $\eta=-1$. Therefore,

$$C_{ui(K>H)} = [1] + [5]\,\{\eta=-1, \phi=1\}$$

For the K<H case, we first restate

$$\text{prob}[H>S^*>K: S(\tau)\geq H] = \text{prob}[S^*<H: S(\tau)\geq H] - \text{prob}[S^*\leq K: S(\tau)\geq H]$$

Then, we can write immediately that

$$C_{ui(K<H)} = [2] - [3] + [4] + [5]\,\{\eta=-1, \phi=1\}$$

For our next options, down-and-in puts and up-and-in puts, we simply provide graphs (figures 3 and 4) from which the stated results can easily be inferred.

"Out" barrier options

Corresponding to each of these four "in" barrier options are four "out" options. For example, in a down-and-out call, a standard call comes into existence when the down-and-out is issued, but the standard call is extinguished prior to expiration if the underlying asset price ever drops below the knock-out boundary, H. In that case the buyer of the option may be paid a fixed rebate, R. Otherwise, if the underlying asset price never drops below H, the down-and-out call will have the same pay-off as a standard call. Expressed concisely, the pay-off from this option is:

down-and-out call	max[0,S*−K] if for all τ≤t, S(τ)> H
(S>H)	R (at hit) if for some τ≤t, S(τ) ≤ H

Here is one possible use of a down-and-out call. Suppose you are holding a covered call but will be forced to liquidate the underlying asset if its price falls sharply. If you sold a down-and-out call in place of a standard call, you could arrange to have the call liquidated automatically at the same time.

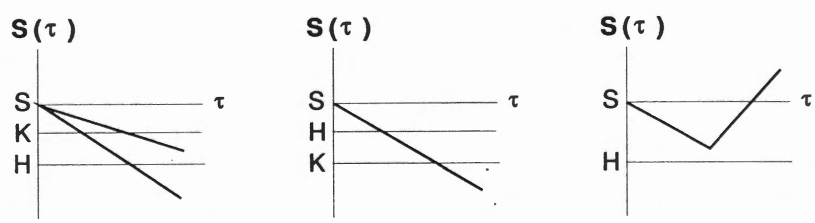
If the rebate R=0, the following parity relationship makes it easy to write down the values of down-and-out calls:

pay-off from standard option = pay-off from down-and-out option + pay-off from down-and-in option

To see this, suppose you own otherwise identical down-and-out and down-and-in options with no rebates. If the common barrier is never hit, then you receive the

Figure 3

down-and-in put:	max[0, K − S*] if for some τ ≤ t, S(τ) ≤ H
(S > H)	R (at expiry) if for all τ ≤ t, S(τ) > H



Put pay-off (K>H)	Put pay-off (K<H)	Rebate pay-off
prob[S*≤H]	prob[S*<K]	prob[S*>H]
+ prob[H<S*<K: S(τ)≤H]		− prob[S*>H: S(τ)≤H]

$$P_{di(K>H)} = [2] - [3] + [4] + [5]\,\{\eta = 1, \phi = -1\}$$

$$P_{di(K<H)} = [1] + [5]\,\{\eta = 1, \phi = -1\}$$

[4] This is not actually a new density since it can be derived by differentiating the integral of g(u) with respect to t

pay-off from a standard option; if the common barrier is hit, as the down-and-out option is extinguished, the down-and-in option delivers you a standard option identical to the one you lost when the down-and-out option was cancelled. Thus, even in this case, you end up receiving the pay-off from a standard option.

The only difficulty comes from the rebate. For "in" options, it is not possible to receive the rebate prior to expiration, since one continues to remain in doubt about whether or not the barrier will never be hit. However, for an "out" option, it is possible as well as customary for the rebate to be paid the moment the barrier is hit. This complicates the risk-neutral valuation problem since the rebate may now be received at a random rather than prespecified time. Thus, we need an additional density of the first passage time (τ) for the underlying asset price to hit the barrier:[4]

$$h(\tau) \equiv (-\eta\alpha/\sigma\tau\sqrt{2\pi\tau})e^{-\frac{1}{2}v^2}$$
$$\text{with } v \equiv (-\eta\alpha + \eta\mu\tau)/\sigma\sqrt{\tau}$$

Here, $\eta = 1$ if the barrier is being approached from above and $\eta = -1$ if the barrier is being approached from below. The present value of the rebate is then the expected rebate discounted by the interest rate raised to the power of the first passage time:

$$[6] \equiv R\int r^{-\tau}h(\tau)d\tau = R[(H/S)^{a+b}N(\eta z) + (H/S)^{a-b}N(\eta z - 2\eta b\sigma\sqrt{t})]$$

where the region of integration is from 0 to t, and

$$z \equiv [\log(H/S) \div \sigma\sqrt{t}] + b\sigma\sqrt{t}$$

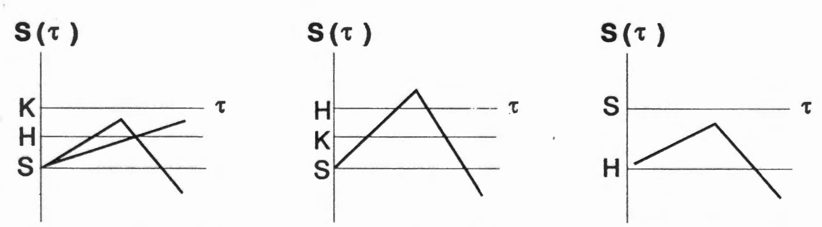
$$a \equiv \mu/\sigma^2,\ b \equiv [\sqrt{(\mu^2 + 2(\log r)\sigma^2)}]/\sigma^2$$

Using these relationships, we can now write down the valuation solutions for the down-and-out call and the three remaining "out" options:

$$C_{do(K>H)} = [1] - [3] + [6]\ \{\eta=1, \phi=1\}$$
$$C_{do(K<H)} = [2] - [4] + [6]\ \{\eta=1, \phi=1\}$$

Figure 4

up-and-in put: (S < H) — max[0, K − S*] if for some $\tau \le t$, $S(\tau) \ge H$; R (at expiry) if for all $\tau \le t$, $S(\tau) < H$



Put pay-off (K>H)
prob[H≤S*<K] + prob[S*<H: S(τ)≥H]

Put pay-off (K<H)
prob[S*<K: S(τ)≥H]

Rebate pay-off
prob[S*<H] − prob[S*<H: S(τ)≥H]

$$P_{ui(K>H)} = [1] - [2] + [4] + [5]\ \{\eta = -1, \phi = -1\}$$
$$P_{ui(K<H)} = [3] + [5]\ \{\eta = -1, \phi = -1\}$$

up-and-out call (S < H) — max[0, S* − K] if for all $\tau \le t$, $S(\tau) < H$; R (at hit) if for some $\tau \le t$, $S(\tau) \ge H$

$$C_{uo(K>H)} = [6[\ \{\eta=-1, \phi=1\}$$

$$C_{uo(K<H)} = [1] - [2] + [3] - [4] + [6]\ \{\eta=-1, \phi=1\}$$

down-and-out put (S > H) — max[0, K − S*] if for all $\tau \le t$, $S(\tau) > H$; R (at hit) if for some $\tau \le t$, $S(\tau) \le H$

$$P_{do(K>H)} = [1] - [2] + [3] - [4] + [6]\ \{\eta=1, \phi=-1\}$$
$$P_{do(K<H)} = [6]\ \{\eta=1, \phi=-1\}$$

up-and-out put (S < H) — max[0, K − S*] if for all $\tau \le t$, $S(\tau) < H$; R (at hit) if for some $\tau \le t$, $S(\tau) \ge H$

$$P_{uo(K>H)} = [2] - [4] + [6]\ \{\eta=-1, \phi=-1\}$$
$$P_{uo(K<H)} = [1] - [3] + [6]\ \{\eta=-1, \phi=-1\}$$

At first it may be surprising that the rebate provides the only contribution to the value of an up-and-out call when the strike price is greater than the barrier. But it is easy to see why. Since S < H < K, in order for the underlying asset price to end up above the strike price it must first breach the barrier, but in this event, the call is extinguished. Similarly, a down-and-out put will also only be valued for the rebate when the strike price is less than the barrier. ■

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