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THE PRICING FOR A CLASS OF BARRIER OPTIONS*

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Abstract. In this paper, we derive some Black-Scholes type formulas for a class of barrier options by using the riskless hedging principle due to Black and Scholes. Furthermore, the put-call parities are proposed.

1. Introduction

Barrier options are a well-established and widely used class of path-dependent derivative securities (see [5, 6]). The first mention of barrier options in the published literature appears to have been by Snyder [17]. Because adding barriers is a convenient method for reducing an option's cost, a rapid growth of the barrier option market is reported (see [11]).

Barrier options are path-dependent options which come in various flavours and forms, but their key characteristic is that these types of options are either initiated or exterminated upon reaching a certain barrier level; that is, they are either knocked in or knocked out. In this paper, we consider a class of barrier options—the European-style single barrier. These options come in eight flavours (see [9]). The knock-out options are four types:

(1) an up-and-out call option;
(2) a down-and-out call option;
(3) an up-and-out put option;
(4) a down-and-out put option.

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The knock-in options are similar four types:

(1) an up-and-in call option;
(2) a down-and-in call option;
(3) an up-and-in put option;
(4) a down-and-in put option.

There are relationships between the prices of barrier options and regular options as follows:

$$V_{regular}(S,t) = V_{up-and-out}(S,t) + V_{up-and-in}(S,t) \tag{1.1}$$
$$= V_{down-and-out}(S,t) + V_{down-and-in}(S,t). \tag{1.2}$$

Following the first valuation of a barrier option which has been derived by Merton [12] in 1973, many pricing methodologies for barrier options come up. Rubinstein and Reiner [16] and Rich [14] utilized the expectations method to obtain the closed form solutions for all barrier options. And the academic literature presents an array of analytical and numerical techniques for pricing barrier options, such as the Boyle and Lau's binomial model [2], the Ritchken's trinomial model [15], the Boyle and Tian's finite difference approach [3] and so on (see [7, 20]). Very recently, some methods by using neural networks have been proposed (see [19]). In this paper, we continue to develop the pricing methodologies for barrier options with dividend yield q by applying the riskless hedging principle which was pioneered by Black and Scholes [1], where q is a constant. And, some Black-Scholes type formulas are obtained, which are the explicit valuation formulas for European barrier options. Using some transformations, we easily obtain the relationships between regular options and barrier options from the Black-Scholes type formulas. Furthermore, we establish the put-call parities for barrier options.

2. Models

Let S be a random walk of the underlying asset price, and S_B be a barrier value. We denote by V the value of a barrier option, which is a function of S and time t. The value of the barrier option on the expiration date T is the payoff, namely,

$$V(S,T) = \begin{cases} (S-K)^+, & \text{knock-out call option;} \\ 0, & \text{knock-in calloption;} \\ (K-S)^+, & \text{knock-out put option;} \\ 0, & \text{knock-in put option,} \end{cases} \tag{2.1}$$

where K is the strike price.

The value of the barrier option on the barrier value S_B is as follows

$$V(S_B,t) = \begin{cases} 0, & \text{knock-out option;} \\ V_{regular}(S_B,t), & \text{knock-in option.} \end{cases} \tag{2.2}$$

We can find the value of a barrier option during its lifetime $V(S,t)$ by riskless hedging principle. In the risk neutral world, the underlying asset price S is assumed to follow the geometric Brownian motion

$$dS = (\mu - q)Sdt + \sigma S dB_t,$$

where B_t is a standard Brownian motion with a mean of zero and a variance of dt, and μ and σ represent the expected return rate and volatility respectively, where μ and σ are constants. We now use the Δ-hedging principle to derive the governing partial differential equation (PDE) for the price of a barrier option. Let the portfolio Π be composed by a short position of Δ units of the underlying asset S and a long position of barrier option V, namely,

$$\Pi = V - \Delta S.$$

We will choose suitable value of Δ such that Π is riskfree during the period $[t, t+dt]$, where dt is one time step and Δ is held fixed. Since we consider the dividend yield q, the increment of the value of this portfolio in one time step dt is

$$d\Pi = dV - \Delta dS - \Delta S q dt. \tag{2.3}$$

On the other hand, Π is riskfree, then

$$d\Pi = r\Pi dt, \tag{2.4}$$

where r represents the riskless interest rate (r is a constant).

According to (2.3) and (2.4), we have

$$\begin{aligned} dV - \Delta dS &= r\Pi dt + \Delta S q dt \\ &= r(V - \Delta S)dt + \Delta S q dt. \end{aligned} \tag{2.5}$$

Applying the *Itô* formula [13] to $V = V(S,t)$ gives

$$dV = \left[\frac{\partial V}{\partial t} + (\mu - q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt + \sigma S \frac{\partial V}{\partial S} dB_t. \tag{2.6}$$

Combining (2.5) and (2.6), we have

$$\begin{aligned} &\left[\frac{\partial V}{\partial t} + (\mu - q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta(\mu - q)S\right] dt + \left(\sigma S \frac{\partial V}{\partial S} - \Delta \sigma S\right) dB_t \\ = &\ (rV - \Delta r S + \Delta S q)\, dt. \end{aligned}$$

We eliminate the random component by choosing

$$\Delta = \frac{\partial V}{\partial S}$$

and find the Black-Scholes equation as follows

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S\frac{\partial V}{\partial S} - rV = 0. \tag{2.7}$$

We now establish the valuation model of barrier option. Thus the valuation problem for barrier option is to solve the PDE (2.7) in different domains D,

$$D = \begin{cases} \{(S,t) | 0 \le S \le S_B, 0 \le t \le T\}, \text{ up option;} \\ \{(S,t) | S_B \le S < \infty, 0 \le t \le T\}, \text{ down option.} \end{cases} \tag{2.8}$$

3. Black-Scholes formula and put-call parity

We can derive the pricing formulas for barrier options by solving the PDE (2.7) with the final condition (2.1) and the boundary condition (2.2) in domain D defined by (2.8). To illustrate the method, we take the European up-and-out call option for example. Thus the valuation problem for European up-and-out call option is to solve a PDE final-boundary value problem as follows

$$(I)\quad \begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \quad (S,t) \in D, \\ V(S,T) = (S-K)^+, \\ V(S_B,t) = 0, \end{cases}$$

where $D = \{(S,t) | 0 \le S \le S_B, 0 \le t \le T\}$.

As demonstrated by Brennan and Schwartz [4], Geske and Shastri [8], and Hull and White [10], among others, it is more efficient to make the following transformations

$$x = \ln \frac{S}{S_B}$$

and

$$V = S_B u.$$

Given these transformations, the problem for determining solution (I) simplifies to

$$(II)\quad \begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + (r-q-\frac{\sigma^2}{2})\frac{\partial u}{\partial x} - ru = 0, \quad (x,t) \in D', \\ u(x,T) = (e^x - K_B)^+, \\ u(0,t) = 0, \end{cases}$$

where $D' = \{(x,t) | -\infty < x < 0, 0 \le t \le T\}$ and $K_B = \frac{K}{S_B}$.

Using a transformation defined as

$$u = e^{\alpha x + \beta(T-t)} W,$$

where

$$\alpha = -\frac{1}{\sigma^2}(r-q-\frac{\sigma^2}{2}), \quad \beta = -r - \frac{1}{2\sigma^2}(r-q-\frac{\sigma^2}{2})^2,$$

the problem for determining solution (II) simplifies to

$$(III)\quad \begin{cases} \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial x^2} = 0, \quad (x,t) \in D'', \\ W(x,T) = e^{-\alpha x}(e^x - K_B)^+, \\ W(0,t) = 0, \end{cases}$$

where $D'' = \{(x,t) | -\infty < x < 0, 0 \le t \le T\}$.

By applying the image method, we define $\varphi(x)$ as follows

$$\varphi(x) = \begin{cases} e^{-\alpha x}(e^x - K_B)^+, \ x < 0; \\ -e^{\alpha x}(e^{-x} - K_B)^+, \ x > 0. \end{cases}$$

It is easy to see that $\varphi(x)$ is an odd function, that is, $\varphi(x) = -\varphi(-x)$. Given this definition, we establish a Cauchy problem of thermal-conductance equation as follows

$$(IV) \quad \begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 W}{\partial x^2} = 0, \quad (x,t) \in D''', \\ W(x,T) = \varphi(x), \end{cases}$$

where $D''' = \{(x,t) | -\infty < x < \infty, 0 \le t \le T\}$.

Because the solution of the problem for determining solution (IV) must be an odd function, the solution which is restricted within $\{(x,t) | -\infty < x < 0, 0 \le t \le T\}$ is sure to satisfy the problem for determining solution (III). Thus solving the problem for determining solution (III) substitute by solving the problem for determining solution (IV).

It is well known that the solution of the Cauchy problem of thermal-conductance equation (IV) can be expressed by Poisson formula as follows

$$\begin{aligned} W(x,t) &= \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{2\sigma^2(T-t)}} \varphi(\xi) d\xi \\ &= \frac{1}{\sigma\sqrt{2\pi(T-t)}} \left[\int_0^{\infty} e^{-\frac{(x-\xi)^2}{2\sigma^2(T-t)}} \varphi(\xi) d\xi - \int_0^{\infty} e^{-\frac{(x+\xi)^2}{2\sigma^2(T-t)}} \varphi(\xi) d\xi \right] \\ &= \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_0^{\infty} \left[e^{-\frac{(x+\xi)^2}{2\sigma^2(T-t)}} - e^{-\frac{(x-\xi)^2}{2\sigma^2(T-t)}} \right] e^{\alpha\xi}(e^{-\xi} - K_B)^+ d\xi. \end{aligned}$$

Since $u = e^{\alpha x + \beta(T-t)} W$ and

$$\alpha = -\frac{1}{\sigma^2}(r - q - \frac{\sigma^2}{2}), \quad \beta = -r - \frac{1}{2\sigma^2}(r - q - \frac{\sigma^2}{2})^2,$$

we have

$$\begin{aligned} & u(x,t) \\ &= e^{\alpha x + \beta(T-t)} \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_0^{\infty} \left[e^{-\frac{(x+\xi)^2}{2\sigma^2(T-t)}} - e^{-\frac{(x-\xi)^2}{2\sigma^2(T-t)}} \right] e^{\alpha\xi}(e^{-\xi} - K_B)^+ d\xi \\ &= e^{\alpha x + \beta(T-t)} \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_0^{-\ln K_B} \left[e^{-\frac{(x+\xi)^2}{2\sigma^2(T-t)}} - e^{-\frac{(x-\xi)^2}{2\sigma^2(T-t)}} \right] e^{\alpha\xi}(e^{-\xi} - K_B) d\xi \\ &= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^{-\ln K_B} e^{-\frac{[(x+\xi)+(r-q-\frac{\sigma^2}{2})(T-t)]^2}{2\sigma^2(T-t)}} (e^{-\xi} - K_B) d\xi \\ &\quad - \frac{e^{-r(T-t) - \frac{2}{\sigma^2}(r-q-\frac{\sigma^2}{2})x}}{\sigma\sqrt{2\pi(T-t)}} \int_0^{-\ln K_B} e^{-\frac{[(x-\xi)-(r-q-\frac{\sigma^2}{2})(T-t)]^2}{2\sigma^2(T-t)}} (e^{-\xi} - K_B) d\xi \end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{-q(T-t)}}{\sqrt{2\pi}} e^{x} \int_{-\infty}^{\frac{x-\ln K_B+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega - \frac{e^{-q(T-t)}}{\sqrt{2\pi}} e^{x} \int_{-\infty}^{\frac{x+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \\
&\quad - \left[\frac{K_B e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\ln K_B+(r-q-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega - \frac{K_B e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x+(r-q-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \right] \\
&\quad - \left[\frac{e^{-q(T-t)-\frac{2}{\sigma^2}(r-q)x}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-x-\ln K_B+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \right. \\
&\qquad \left. - \frac{e^{-q(T-t)-\frac{2}{\sigma^2}(r-q)x}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-x+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \right] \\
&\quad + \left[\frac{K_B}{\sqrt{2\pi}} e^{-r(T-t)-\frac{2}{\sigma^2}(r-q-\frac{\sigma^2}{2})x} \int_{-\infty}^{\frac{-x-\ln K_B+(r-q-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \right. \\
&\qquad \left. - \frac{K_B}{\sqrt{2\pi}} e^{-r(T-t)-\frac{2}{\sigma^2}(r-q-\frac{\sigma^2}{2})x} \int_{-\infty}^{\frac{-x+(r-q-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \right] \\
&= \left[\frac{e^{-q(T-t)}}{\sqrt{2\pi}} e^{x} \int_{-\infty}^{\frac{x-\ln K_B+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega - \frac{K_B e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\ln K_B+(r-q-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \right. \\
&\qquad - \frac{e^{-q(T-t)-\frac{2}{\sigma^2}(r-q)x}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-x-\ln K_B+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \\
&\qquad \left. + \frac{K_B}{\sqrt{2\pi}} e^{-r(T-t)-\frac{2}{\sigma^2}(r-q-\frac{\sigma^2}{2})x} \int_{-\infty}^{\frac{-x-\ln K_B+(r-q-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \right] \\
&\quad - \left[\frac{e^{-q(T-t)}}{\sqrt{2\pi}} e^{x} \int_{-\infty}^{\frac{x+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega - \frac{K_B e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x+(r-q-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \right. \\
&\qquad - \frac{e^{-q(T-t)-\frac{2}{\sigma^2}(r-q)x}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-x+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \\
&\qquad \left. + \frac{K_B}{\sqrt{2\pi}} e^{-r(T-t)-\frac{2}{\sigma^2}(r-q-\frac{\sigma^2}{2})x} \int_{-\infty}^{\frac{-x+(r-q-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \right].
\end{aligned}
$$

Since $V=S_B u, x=\ln\frac{S}{S_B}$ and $K_B=\frac{K}{S_B}$, we have

$$\begin{aligned}
&V(S,t)\\
=&\Big[e^{-q(T-t)}SN(d_1)-Ke^{-r(T-t)}N(d_2)-e^{-q(T-t)}(\frac{S}{S_B})^{-\frac{2}{\sigma^2}(r-q)}S_B N(d_3)\\
&+Ke^{-r(T-t)}(\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)}N(d_4)\Big]-\Big[e^{-q(T-t)}SN(d_5)-Ke^{-r(T-t)}N(d_6)\\
&-e^{-q(T-t)}(\frac{S}{S_B})^{-\frac{2}{\sigma^2}(r-q)}S_B N(d_7)+Ke^{-r(T-t)}(\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)}N(d_8)\Big],
\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}
d_1&=\frac{\ln\frac{S}{K}+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},\quad d_2=d_1-\sigma\sqrt{T-t},\\
d_3&=\frac{\ln\frac{S_B^2}{SK}+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},\quad d_4=d_3-\sigma\sqrt{T-t},\\
d_5&=\frac{\ln\frac{S}{S_B}+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},\quad d_6=d_5-\sigma\sqrt{T-t},\\
d_7&=\frac{\ln\frac{S_B}{S}+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},\quad d_8=d_7-\sigma\sqrt{T-t}
\end{aligned}$$

and the function $N(x)$ is the cumulative probability function for a standardized normal variable. By using the Black-Scholes formula [18] for a regular European call option (with the same strike price, expiration date and dividend yield), (3.1) simplifies to

$$\begin{aligned}
V(S,t)=&V_{regular}(S,t)-(\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)}V_{regular}(\frac{S_B^2}{S},t)\\
&-\Big[V'_{regular}(S,t)-(\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)}V'_{regular}(\frac{S_B^2}{S},t)\Big],
\end{aligned}\tag{3.2}$$

where $V'_{regular}(S,t)$ and $V'_{regular}(\frac{S_B^2}{S},t)$ represent the pricing formulas for regular European call option when $K=S_B$. Thus we obtain the Black-Scholes formula (3.2) for European up-and-out call option, that is,

$$\begin{aligned}
V_{up-and-out}(S,t)=&V_{regular}(S,t)-(\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)}V_{regular}(\frac{S_B^2}{S},t)\\
&-\Big[V'_{regular}(S,t)-(\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)}V'_{regular}(\frac{S_B^2}{S},t)\Big].
\end{aligned}\tag{3.3}$$

Combining (1.1) and (3.3), the pricing formula for European up-and-in call option (with the same barrier, strike price, expiration date and dividend yield) soon obtain as follows

$$\begin{aligned}
V_{up-and-in}(S,t)=&(\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)}V_{regular}(\frac{S_B^2}{S},t)\\
&+\Big[V'_{regular}(S,t)-(\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)}V'_{regular}(\frac{S_B^2}{S},t)\Big].
\end{aligned}\tag{3.4}$$

Similarly, we can derive the pricing formulas for European down-and-out call option and European down-and-in call option respectively as follows

$$V_{down-and-out}(S,t) = V_{regular}(S,t) - (\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)} V_{regular}(\frac{S_B^2}{S}, t) \tag{3.5}$$

and

$$V_{down-and-in}(S,t) = (\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)} V_{regular}(\frac{S_B^2}{S}, t). \tag{3.6}$$

To acquire the pricing formulas for put options in barrier options, we establish the put-call parities for barrier options instead of by applying the riskless hedging principle.We denote the values of put and call options by $p(S,t)$ and $c(S,t)$, respectively.

Theorem 3.1. *The put-call parity relationship between European up-and-out put option and European up-and-out call option is*

$$\begin{aligned} & p_{up-and-out}(S,t) + Se^{-q(T-t)}[1 - N(d_5)] \\ = & c_{up-and-out}(S,t) + Ke^{-r(T-t)}[1 - N(d_6)] \\ & + \left(\frac{S}{S_B}\right)^{1-\frac{2}{\sigma^2}(r-q)} \left\{\frac{S_B^2}{S} e^{-q(T-t)}[1 - N(d_7)] - Ke^{-r(T-t)}[1 - N(d_8)]\right\}. \end{aligned}$$

Proof. Suppose that

$$W(S,t) = c_{up-and-out}(S,t) - p_{up-and-out}(S,t).$$

Since the Black-Scholes equation, final conditions and boundary conditions are linear, we know that W must satisfy the PDE final-boundary value problem as follows

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 W}{\partial S^2} + (r-q)S\dfrac{\partial W}{\partial S} - rW = 0, \quad (S,t) \in D, \\ W(S,T) = S - K, \\ W(S_B, t) = 0, \end{cases}$$

where $D = \{(S,t) | 0 \le S \le S_B, 0 \le t \le T\}$.

Using the following transformations,

$$x = \ln \frac{S}{S_B}$$

and

$$W = S_B u \ ,$$

we have

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 u}{\partial x^2} + (r - q - \dfrac{\sigma^2}{2})\dfrac{\partial u}{\partial x} - ru = 0, \quad (x,t) \in D', \\ u(x,T) = e^x - K_B, \\ u(0,t) = 0, \end{cases}$$

where $D' = \{(x,t) | -\infty < x < 0, 0 \le t \le T\}$, and $K_B = \frac{K}{S_B}$.

By using the same method as before, we can get the solution of this problem as follows

$$
\begin{aligned}
& u(x,t) \\
= & \frac{e^{-q(T-t)}}{\sqrt{2\pi}} e^x \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega - \frac{e^{-q(T-t)}}{\sqrt{2\pi}} e^x \int_{-\infty}^{\frac{x+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \\
& - \left[\frac{K_B e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega - \frac{K_B e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x+(r-q-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \right] \\
& - \left[\frac{e^{-q(T-t)-\frac{2}{\sigma^2}(r-q)x}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega - \frac{e^{-q(T-t)-\frac{2}{\sigma^2}(r-q)x}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-x+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \right] \\
& + \left[\frac{K_B}{\sqrt{2\pi}} e^{-r(T-t)-\frac{2}{\sigma^2}(r-q-\frac{\sigma^2}{2})x} \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega \right. \\
& \left. \quad - \frac{K_B}{\sqrt{2\pi}} e^{-r(T-t)-\frac{2}{\sigma^2}(r-q-\frac{\sigma^2}{2})x} \int_{-\infty}^{\frac{-x+(r-q-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\omega^2}{2}} d\omega \right].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
& W(S,t) \\
= & S e^{-q(T-t)}[1 - N(d_5)] - K e^{-r(T-t)}[1 - N(d_6)] \\
& - S_B e^{-q(T-t)} (\frac{S}{S_B})^{-\frac{2}{\sigma^2}(r-q)} [1 - N(d_7)] + K e^{-r(T-t)} (\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)} [1 - N(d_8)].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& p_{up-and-out}(S,t) + S e^{-q(T-t)}[1 - N(d_5)] \\
= & c_{up-and-out}(S,t) + K e^{-r(T-t)}[1 - N(d_6)] \\
& + \left(\frac{S}{S_B}\right)^{1-\frac{2}{\sigma^2}(r-q)} \left\{ \frac{S_B^2}{S} e^{-q(T-t)}[1 - N(d_7)] - K e^{-r(T-t)}[1 - N(d_8)] \right\}.
\end{aligned}
$$

This completes the proof.

By using the similar proof of Theorem 3.1, it is easy to get the following Theorems 3.2–3.4.

Theorem 3.2. *The put-call parity relationship between European down-and-out put option and European down-and-out call option is*

$$
\begin{aligned}
& p_{down-and-out}(S,t) + S e^{-q(T-t)} N(d_5) \\
= & c_{down-and-out}(S,t) + K e^{-r(T-t)} N(d_6) \\
& + (\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)} \left[\frac{S_B^2}{S} e^{-q(T-t)} N(d_7) - K e^{-r(T-t)} N(d_8) \right].
\end{aligned}
$$

Theorem 3.3. *The put-call parity relationship between European up-and-in put option and European up-and-in call option is*

$$p_{up-and-in}(S,t) = c_{up-and-in}(S,t).$$

Theorem 3.4. *The put-call parity relationship between European down-and-in put option and European down-and-in call option is*

$$p_{down-and-in}(S,t) = c_{down-and-in}(S,t).$$

From (3.3) and Theorem 3.1, we can get the pricing formula for European up-and-out put option as follows

$$\begin{aligned} p_{up-and-out}(S,t) = {} & Se^{-q(T-t)}[N(d_1)-1] - Ke^{-r(T-t)}[N(d_2)-1] \\ & - S_B e^{-q(T-t)}(\frac{S}{S_B})^{-\frac{2}{\sigma^2}(r-q)}[N(d_3)-1] \\ & + Ke^{-r(T-t)}(\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)}[N(d_4)-1]. \end{aligned}$$

By (3.4) and Theorem 3.3, we have the pricing formula for European up-and-in put option as follows

$$\begin{aligned} p_{up-and-in}(S,t) = {} & Se^{-q(T-t)}N(d_5) - Ke^{-r(T-t)}N(d_6) \\ & + S_B e^{-q(T-t)}(\frac{S}{S_B})^{-\frac{2}{\sigma^2}(r-q)}[N(d_3)-N(d_7)] \\ & - Ke^{-r(T-t)}(\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)}[N(d_4)-N(d_8)]. \end{aligned}$$

From (3.5) and Theorem 3.2, we obtain the pricing formula for European down-and-out put option as follows

$$\begin{aligned} p_{down-and-out}(S,t) = {} & Se^{-q(T-t)}[N(d_1)-N(d_5)] - Ke^{-r(T-t)}[N(d_2)-N(d_6)] \\ & - S_B e^{-q(T-t)}(\frac{S}{S_B})^{-\frac{2}{\sigma^2}(r-q)}[N(d_3)-N(d_7)] \\ & + Ke^{-r(T-t)}(\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)}[N(d_4)-N(d_8)]. \end{aligned}$$

By (3.6) and Theorem 3.4, we get the pricing formula for European down-and-in put option as follows

$$p_{down-and-in}(S,t) = S_B e^{-q(T-t)}(\frac{S}{S_B})^{-\frac{2}{\sigma^2}(r-q)}N(d_3) - Ke^{-r(T-t)}(\frac{S}{S_B})^{1-\frac{2}{\sigma^2}(r-q)}N(d_4).$$

Remark. If $q = 0$, then all the results presented in this paper are suitable for the cases of barrier options without dividend yield.

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