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Exotic Options pricing using Monte Carlo methods

Introduction:

The aim of this research project is to explore the application of Monte Carlo (MC) simulation in pricing exotic options, specifically Asian options and Barrier options. This study will also make use of the Black-Scholes model, a widely-used pricing model in the financial industry to model the underlying asset's behaviour.

The problem at hand is that pricing exotic options, which have non-linear payoffs and are path-dependent, can be extremely challenging for financial institutions and investors. Traditional models, such as the Black-Scholes model, may not accurately price these options, because more often than not there are no closed-form mathematical solutions, which can lead to significant financial losses. Therefore, the use of MC simulation, a powerful numerical method, has become increasingly popular in the financial industry to price exotic options. Hence the objective of this research is to investigate the effectiveness of MC simulation in pricing Asian options and Barrier options. The technical specifications of this research project include the use of Python to implement MC simulation and analyse the resulting data.

Background:

Before getting into exotic options it is best to understand what are financial derivatives. As the term suggests, derivatives are financial contracts negotiated and agreed upon between two or more parties, where its values are derived from the price movements of one or more underlying assets or some benchmarks (Fernando, 2023). Derivatives are vital to the financial industry for several reasons with the most important one being risk management. Investors often trade derivatives, either through the exchange or over-the-counter, to hedge against risks in the market (Fernando, 2023).

One particular category of financial derivatives is the Options contract, which has two subclasses: call option and put option. Ever since 1973, organised exchanges have facilitated the trading of modern call and put options using stocks as their underlying asset (Weert, 2008). The purchase of a call option on a stock grants the buyer the right to buy the stock at a predetermined price, called the strike price, and before a predetermined expiration date, but it does not impose an obligation to do so. Conversely, the purchase of a put option grants the buyer the right to sell the stock at a predetermined strike price and before an expiration date, but again, it does not impose an obligation to do so. When the buyers do decide to use their right to perform the action of buying or selling, it is said that the option is exercised.

Plain vanilla options are often used to refer to basic call and put options that lack any special or complex features. There are two general classes of vanilla options, one is the European Option, where the buyer can only exercise their right to buy or sell on the expiration date; the other one is the American Option, where the buyers can exercise their right anytime before the expiration date (Kwok, 2008). However, there are many limitations to these vanilla options due to its simplicity and fixed payoff structure, which limits their flexibility to meet the needs of different investors. In particular, when investors want to use options to hedge against certain risks, they must pay a premium upfront in order to acquire the option, which increases cost. Furthermore, vanilla options can be limited in their effectiveness as a hedging tool in volatile markets. One reason is that they have fixed strike prices, which means that if the market moves significantly in one direction, the option may become out of the money and lose much of its value. Additionally, vanilla options are often priced using models that assume constant volatility, such as the original Black-Scholes model, which “often fail when compared to the real market data” (Janková, 2018, p.3). Therefore, in order to address these limitations, a new category of options called Exotic options has been developed.

Unlike the fixed and straightforward framework of vanilla options, exotic options are designed to offer more customised or tailored risk management solutions to investors. They can provide more flexibility in terms of the payoff structure and can be tuned to meet the specific needs of an investor. Moreover exotic options can also be used to hedge against more complex risks, which cannot be hedged using standard vanilla options, and enhance the returns on an investment portfolio by taking on more sophisticated and complex trading strategies.

Aside from the nuanced details in payoff structure between the vanilla and exotic options, there is also a fundamental difference between the two that we must emphasise. That is, vanilla options are path-independent (Zhang, 1995). The payoff of the American and European options depends only on the price of the underlying asset at the time of exercise, not on the path that the asset price took to get there. Exotic options on the other hand are path-dependent (Wilmott, 2007), which means their value is not only determined by the underlying asset's price at expiration but also the path it takes to get there. As Wilmott (2007) has pointed out, there are two types of path dependency, namely strong and weak, with the difference being strong path-dependence options work in higher dimensions that require us to introduce an additional variable to deal with path dependency, whereas the weak path-dependence ones do not. In order to understand the effectiveness of MC simulation methods in pricing both weakly and strongly path-dependent options, two exotic options will be considered in this thesis: Barrier Options (weakly path-dependent) and Asian Options (strongly path-dependent).

Black-Schole Model for pricing options:
One of the very popular models used in industry to price options is the Black-Scholes model developed and made publicly available by Fischer Black and Myron Scholes in 1973 (Henderson, 2014). The model is a mathematical formula used to determine the fair price for a European call or put option by considering factors like the underlying asset price, the option's exercise price, time until expiration, interest rate, and volatility (Black and Schole, 1973). The foundation of the model is based on many assumptions, to name a few, the underlying asset price follows a lognormal random walk, interest rate and volatility of the market are known constants, and no transaction costs. Under these assumptions, the model is able to construct a delta-hedged portfolio consisting of a long position in the stock and a short position in the option. By continuous hedging, the fluctuations in value of a long stock position can be balanced out to some extent by the corresponding changes in value of a short option position, and the return on the hedged position becomes certain (Black and Schole, 1973). Although there have been many criticisms of the unrealistic assumptions made by the model, nonetheless, the easy to use closed form solution for pricing the European option proposed by the model, together with its framework for understanding the relationship between the price of an option and the underlying asset were invaluable to the development of quantitative finance. In this thesis, the mathematical concepts used to price the Barrier and Asian options will also be based on the Black-Scholes framework.

Barrier Options:
As previously mentioned, Barrier options are path-dependent options. The payoff of these contracts is linked to the level of the realised asset path, and specific conditions in the contract are activated if the asset price reaches a certain “barrier”, whether it is too high or too low (Wilmott, 2007). The barrier can be specified as either "in" or "out," and it can be set at different levels, such as "up" or "down." In general, an "in" barrier option only becomes active or "knocks in" if the price of the underlying asset reaches the barrier level during the life of the option. Conversely, an "out" barrier option becomes void or "knocks out" if the price of the underlying asset reaches the barrier level during the life of the option. The barrier level can be set at different points relative to the current price of the underlying asset. For example, an "up" barrier option has a barrier level set above the current price of the underlying asset, while a "down" barrier option has a barrier level set below the current price of the underlying asset.

The mix and match of these features provides a set of Barrier options listed below:

- up-and-in call option

- up-and-out call option
- down-and-in call option
- down-and-out call option
- up-and-in put option
- up-and-out put option
- down-and-in put option
- down-and-out put option

Furthermore, Barrier option is also weakly path-dependent, because it works in low dimensionality. In other words, under the Black-Scholes framework, the value of the contract only depends on two independent variables: the current price level of the underlying asset (S), and time to expiry (t), which makes it a two dimensional option (Wilmott, 2007).

Asian Options:
For Asian options the payoff is based on the average price of the underlying asset over a period of time before expiry, rather than the price at a specific point in time. The average can be calculated over different time intervals, such as the entire life of the option or a specific subset of the option's life (Weert, 2008). The Asian option is said to be strongly path-dependent, because it is a high dimensional option. In order to price such option using the Black-Scholes framework, on top of the asset price (S) and time to expiry (t), it needs to keep track of a third independent variable, such as the average price of the underlying asset over a certain period (Wilmott, 2007).

The reason why it is important to keep track of the average price is because it ties in strongly with the way the payoff is calculated for Asian options. If we define the average price as A, E as the strike price and S as the price at expiry, we can form two different categories of Asian options depending on how the payoff function is defined. In the payoff function, if we replace the strike price E with the average price A, then the payoff at expiration for an Asian Call strike option is equal to the difference between the asset price and the average price of the underlying asset. If the asset price is lower than the average price, the payoff is zero. Similarly, in an Asian put strike option, the payoff is the difference between the average price and the asset price if the asset price is lower than the average price, and zero if the asset price is higher than the average price (Wilmott, 2013).

Average Strike call payoff: max(S - A, 0)
Average Strike put payoff: max(A - S, 0)

On the other hand, if we replace the asset price S with the average price A instead, we get what is called the rate options with their payoffs defined below (Wilmott, 2013):

Average Rate call payoff: max(A - E, 0)
Average Rate put payoff: max(E - A, 0)

Aside from the technical details, Asian options are popular in the market due to their ability to lower the likelihood of market manipulation close to the expiry date, and provide more hedging opportunities for businesses with a series of exposures (Fusai and Meucci, 2008). For example businesses involving thinly traded commodities such as rare earth metals and agricultural products.

Monte Carlo Simulations:
The Monte Carlo Method was created by John von Neumann and Stanislaw Ulam during World War II as a means of enhancing decision making in situations where outcomes were uncertain. They have named this method after a well-known casino destination, Monaco, as probability plays a central role in the simulation process, much like in games of roulette (Kenton, 2023).

MC simulation is based on the idea of using random numbers to solve complex mathematical problems. The method involves simulating a large number of possible outcomes, each with a different set of randomly generated inputs. By repeating this process many times, the simulation produces a distribution of possible outcomes, allowing for the calculation of probabilities and expected values. Based on the law of large numbers, the accuracy of MC methods can be improved by simply increasing the number of simulations because the estimated results should converge to the real value (Glasserman, 2004).

The MC method is simple and yet very powerful for pricing European style options. The concept readily carries over to exotic and path-dependent contracts. The key steps required are simply:

1) simulate the random walk starting at today's value over the required timeframe, which gives one realisation of the price path of the underlying asset.
2) Calculate the payoff of the option for this realisation.
3) Repeat many more realisations of the asset's price movement over the same timeframe.
4) Estimate the average payoff of all the realisations
5) Calculate the present value of the average payoff and that will be the option's value today.

The implementation of the MC method is very straightforward to code, and we can make adjustments to how the payoffs should be calculated for different exotic options. For example when pricing Barrier Options, we can examine whether each realisation has hit the barrier and decide whether the contract should be nullified or not. For Asian options on the other hand, we can keep track of the full path of each realisation and calculate the required averages to be used in the final payoffs. However the disadvantages of the MC method are also very apparent. In order to get something close to the accurate answer, tens of thousands of realisations need to be run, which is time consuming (Wilmott, 2007). Furthermore, the Greeks in the Black-Scholes equation are calculated by taking the partial derivatives of the option price with respect to some parameters, such as asset price, time, volatility and interest rate. In MC simulations, the partial derivatives are estimated by making small changes to the simulated inputs and taking the average, which can be computationally expensive and inaccurate (Gracianti, 2018). This makes it harder to get accurate estimates of the Greeks.

Project Schedule:

In this thesis, the method of MC Simulations will be used to try and price the two exotic options, Barrier option and Asian option. In terms of data acquisition, all data will be generated from running the MC simulations, hence no real world data will be sourced.

The workflow of this project can be broken down into three main stages: understanding the maths, running MC simulations, analysing the results and performance of the MC method. To be more specific, in the first stage, we need to first understand how to price Barrier Options and Asian Options in the partial differential equation framework, which does not have a closed form solution. Then in the second stage, where the majority of the coding takes place, we apply the MC method to find the numerical solutions to the partial differential equations, which will involve generating many realisations of asset prices for the options. Different variations of the MC algorithm, such as the Euler-Maruyama and Milstein methods will also be used. Then in the last stage, we will analyse the data generated from stage two and analyse the performance of the different MC methods, and measure their level of convergence.

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