UCB-TYPE ALGORITHM FOR BUDGET-CONSTRAINED EXPERT LEARNING

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ABSTRACT

In many modern applications, a system must dynamically choose between several adaptive learning algorithms that are trained online. Examples include model selection in streaming environments, switching between trading strategies in finance, and orchestrating multiple contextual bandit or reinforcement learning agents. At each round, a learner must select one predictor among K adaptive experts to make a prediction, while being able to update at most $M \leq K$ of them under a fixed training budget.

We address this problem in the *stochastic setting* and introduce M-LCB, a computationally efficient UCB-style meta-algorithm that provides *anytime regret guarantees*. Its confidence intervals are built directly from realized losses, require no additional optimization, and seamlessly reflect the convergence properties of the underlying experts. If each expert achieves internal regret $\tilde{O}(T^\alpha)$, then M-LCB ensures overall regret bounded by $\tilde{O}\left(\sqrt{\frac{KT}{M}} + (K/M)^{1-\alpha} T^\alpha\right)$.

To our knowledge, this is the first result establishing regret guarantees when multiple adaptive experts are trained simultaneously under per-round budget constraints. We illustrate the framework with two representative cases: (i) parametric models trained online with stochastic losses, and (ii) experts that are themselves multi-armed bandit algorithms. These examples highlight how M-LCB extends the classical bandit paradigm to the more realistic scenario of coordinating stateful, self-learning experts under limited resources.

Keywords expert algorithms, budget-constrained learning, multi-armed bandits

1 Introduction

In many applications, one must dynamically choose between multiple models. Recommendation systems may run several predictors in parallel, updating them on incoming user feedback. Financial platforms rely on switching between trading strategies as market regimes evolve. Large-scale online services manage a portfolio of contextual bandits or reinforcement learning algorithms.

These scenarios' objective is to dynamically select the most accurate model at each step, while managing a limited computational budget for training. This setup falls within the framework of sequential decision-making.

Classical multi-armed bandit (MAB) algorithms [1, 2, 3], when addressing this problem, usually assume a static or adversarial reward distribution for each arm. Expert algorithms [4, 5] usually require full feedback and do not account for how experts' learning rate. Neither approach fully addresses the challenge of managing multiple simultaneously-learning experts within a per-round training budget.

We bridge this gap by proposing a procedure that unifies prediction with selective training, accounting for a fixed per-round computational budget. Specifically, the contributions of this work are as follows:

- **Novel UCB-Type Meta-Algorithm (M-LCB)**: we propose M-LCB, a novel Upper Confidence Bound (UCB)-type meta-algorithm. It manages a pool of K self-learning experts in a stochastic environment while accounting for a limited per-round learning budget $M(M \leq K)$.

- **Computational Efficiency**: we provide a method for constructing confidence bounds directly from realized losses. It is computationally efficient and sidesteps the need for expensive auxiliary optimization.

- **Theoretical analysis**: we estimate the meta-algorithm's performance in terms of the experts' individual convergence rates. For instance, when the experts' regrets are $\tilde{O}(n^\alpha)$, the overall regret scales as $\tilde{O}(\sqrt{KT/M} + (K/M)^{1-\alpha}T^\alpha)$.

- **Extension to Multi-Play Bandits:** we demonstrate that M-LCB extends to the multiple-play bandit setting.

1.1 Related works

Self-learning experts (arms).   The work [6] introduces self-learning arms in the MAB setting: each arm is a black-box parametric function that generates a reward, and its parameter is updated after the arm is played. At each round, the learner selects an arm using a UCB-type index, observes the reward, and then updates the corresponding parameter.

Model selection at the meta-level.   The work [7] introduces a parameter-free aggregation of multiple online learners within the full information framework.

The procedure CORRAL [8] corrals a pool of bandit algorithms via log-barrier online-mirror descent (OMD) with importance-weighted feedback. The authors derive distribution-free guarantees in stochastic and adversarial settings.

The work [9] proposes a dynamic balancing meta-algorithm based on known regret rate expressions for the base learners. In their setup, the regret is defined with respect to the globally optimal action, and only one learner is updated per round. In contrast, our formulation uses *per-expert, prefix-hindsight* guarantees $U_k(T, \delta)$ defined with respect to each expert's local optimum.

The work [10] removes the need for known regret rates by estimating per-learner coefficients online, obtaining high-probability, data-dependent model-selection guarantees for stochastic bandits (again, with a single learner updated per round).

The closest setting to ours is [11]. It considers model selection using a *smoothing wrapper*. The authors show that the CORRAL meta-algorithm combined with their wrapper achieves regret $\tilde{O}(\sqrt{TK} + K^\alpha T^{1-\alpha} + K^{1-\alpha}T^\alpha c(\delta))$ when the regret of base learners satisfies $O(T^\alpha c(\delta))$. The dynamic balancing approach [9] yields a similar general bound $\tilde{O}\left(\sqrt{KT} + M^{1-\alpha}T^\alpha c(\delta)\right)$. Both regret bounds coincides with what we get when training one expert.

Our algorithm achieves the same order of dependency on T and α, while additionally supporting simultaneous training of up to M adaptive experts with confidence intervals computed directly from realized per-arm losses.

MABs with updates of multiple arms.   In this setting, the meta-procedure can update or observe several arms per round. Several works consider the *adversarial* case. [12] study prediction with limited advice (query at most M arms). The authors obtain the regret bound $\tilde{O}\left(\sqrt{\frac{KT \log K}{M}}\right)$. It smoothly bridges the full-information case and the bandit

setting. [13] presents the minimax-optimal regret $\tilde{O}\left(\max\{\sqrt{KT/M}, \sqrt{T\log K}\}\right)$. Specifically, it matches the lower bounds from [12]. However, these works assume non-learning arms (experts).

In the *stochastic* case, [13] considers the multi-armed bandit with additional observations: the learner plays one arm and may observe up to M extra arms per round. They propose the *KL-UCB-AO* algorithm that achieves asymptotically optimal *logarithmic* regret. However, it has a limited applicability scope due to the properties of the Kullback-Leibler-based selection rule.

Multiple-play multi-armed bandits.   In the *multiple-play* setting (see [14, 15]), a meta-procedure selects M arms per round and observes semi-bandit feedback. UCB-based algorithms for combinatorial bandits [16] achieve $\tilde{O}(\sqrt{KT/M})$ regret under stochastic rewards, providing a baseline for subset-level performance analysis. These results serve as a benchmark for multiple-play MAB extension of M-LCB.

Structure of the paper.   Section 2 formalizes the problem setup. Section 3 presents the M-LCB algorithm. Section 4 contains the theoretical analysis. Section 2.5 illustrates the framework on parametric arms and summarizes inner-to-global rates. We conclude with a discussion of open directions.

2 Problem Setup

This section formalizes the setting. The meta-procedure $\mathcal{P}$ manages a pool of K experts. Each expert is capable of learning and providing advice. At each round t, $\mathcal{P}$ selects an advisor—the expert whose advice will be used for that round—and allocates a limited training budget across the experts to support their learning. The environment then reveals the truth (the random true outcome or label), $\mathcal{P}$ incurs the loss based on the advisor's advice and the truth, and the experts selected for learning update their models based on the truth. The objective is to minimize the overall regret of P relative to the best expected expert choice.

Section 2.1 describes the meta-procedure $\mathcal{P}$. Section 2.2 introduces the regret. Section 2.3 discusses the self-learning experts. Section 2.4 introduces the advice. Section 2.5 illustrates the framework with specific examples.

2.1 Procedure protocol

Let $\mathbf{U}$ be a decision space and let $\mathbf{E}$ be the space of random outcomes generated by the environment. A loss function ℓ is

$$\ell : \mathbf{U} \times \mathbf{E} \to \mathbb{R}_+.$$

Each expert $k \in [K]$ is specified by a tuple $(\mathbf{W}_k, \mathbf{H}_k, \mathscr{A}_k, g_k, \upsilon_k)$. Here, $\mathbf{W}_k$ is the **state space** (or **parameter space**) of the expert. $\mathbf{H}_k$ is the **history space**: the expert maintains its **state history** $\mathcal{H}_k^t \in \mathbf{H}_k$ at each time step t, i.e., $\mathcal{H}_k^t$ records the evolution of the expert's internal state and all training data received up to time t. $\mathscr{A}_k$ is the (possibly) black-box **online learning algorithm** updating the state of the expert $\mathbf{w}_k^{t+1} := \mathcal{A}_k(\mathcal{H}_k^t) \in \mathbf{W}_k$ based on its history $\mathcal{H}_k^t$ (see Section 2.3 for more detail). $g_k : \mathbf{W}_k \to \mathbf{U}$ maps the expert's current state $\mathbf{w}_k$ to its **advice** $\mathbf{u} \in \mathbf{U}$. Finally, $\upsilon_k : \mathbf{H}_k \to \mathbf{U}$ produces **safe advice** (see Section 2.4).

At each round t, the meta-procedure $\mathcal{P}$ selects a training set $S_t \subseteq [K]$ taking into account the per-round budget M, i.e., $|S_t| \leq M$. Further, $\mathcal{P}$ selects the advisor $i_t \in S_t$. Then it acts in two stages: prediction and learning.

Prediction.   The advisor i_t provides a **safe advice** $u^t := v_{i_t}(\mathcal{H}_{i_t}^t) \in \mathbf{U}$. Subsequently, the environment reveals an i.i.d. outcome $\xi^t \sim D$ in $\mathbf{E}$. $\mathcal{P}$ then incurs loss $\ell(u^t, \xi^t)$.

Learning.   Each expert $k \in S_t$ incurs loss

$$\ell_k^t(w_k^t) := \ell\left(g_k(w_k^t), \xi^t\right), \quad w_k^t \in \mathbf{W}_k.$$

Using the new information, i.e., $\ell_k^t(w_k^t)$, the experts update their learning history $\mathcal{H}_k^t$ and current state via algorithm $\mathscr{A}_k$.

The box below summarizes the meta-procedure's protocol inspired by the "prediction with limited advice" game [12].

Protocol: Self-learning experts with limited advice

For $t = 1, 2, \ldots$:

1. The meta-procedure selects an advisor $i_t \in [K]$ and a training subset $S_t \subseteq [K]$ with $|S_t| \leq M$ and $i_t \in S_t$.

2. Expert i_t produces a safe advice $u^t = v_{i_t}(\mathcal{H}_{i_t}^t) \in \mathbf{U}$.

3. The environment samples $\xi^t \sim D$
   - $\mathcal{P}$ incurs loss $\ell(u^t, \xi^t)$
   - Experts $k \in S_t$ incur loss $\ell_k^t(w_k^t)$.

4. Experts $k \in S_t$ update history $\mathcal{H}_k^t$ and internal state $w_k^{t+1} = \mathscr{A}_k(\mathcal{H}_k^t)$.

2.2 Regret

For each expert k, we define its expected parametrized loss as

$$L_k(w) := \mathbb{E}_{\xi \sim D}[\ell(g_k(w), \xi)], \quad w \in \mathbf{W}_k.$$

The smallest loss across all experts is

$$L^\star := \min_{k \in [K]} L_k^\star, \quad L_k^\star := \min_{w \in \mathcal{W}_k} L_k(w). \tag{1}$$

We define the regret of $\mathcal{P}$ after T rounds as

$$\text{Reg}(T) := \sum_{t=1}^{T} \ell(u^t, \xi^t) - T \cdot L^\star.$$

This choice of regret is similar to the regret in the classic stochastic MAB setting, but it is extended to the functional setup. It also matches the standard objective in stochastic learning. In the rest of the text, we assume the loss function is bounded.

Assumption 1 (Stochastic bounded losses). *At each round, the environment draws i.i.d. $\xi \sim D$ from an unknown distribution D supported on $\mathbf{E}$. The loss is $\ell : \mathbf{U} \times \mathbf{E} \to [0, 1]$.*

Remark 1 (On the bounded loss). *This study focuses on the case of bounded loss. However, it can be extended to unbounded loss (e.g., sub-Gaussian or heavy-tailed). Specifically, the proofs require a different choice of concentration inequalities. In this case, the regret guarantees hold up to a log-term.*

2.3 Self-learning experts

Recall that S_τ is a set of experts selected for learning at round τ. We define the set of time steps at which the expert has been trained up to time t as

$$I_k(t) := \{\tau \leq t : k \in S_\tau\}, \quad n_k^t := |I_k(t)|.$$

In other words, n_k^t is the number of training sessions up to the time moment t. Denoting as w_k^τ the state of k-th expert at round τ, we write it's learning history up to round t as

$$\mathcal{H}_k^t := \{(w_k^\tau, \ell_k^\tau(w_k^\tau)) : \tau \in I_k(t)\}.$$

For all $k \in S_t$ the learning algorithm $\mathscr{A}_k$ maps the history to a new state,

$$w_k^{t+1} := \mathscr{A}_k(\mathcal{H}_k^t).$$

Expert regret. At step t, the prefix-hindsight regret of $\mathscr{A}_k$ is

$$R_{\mathscr{A}_k}(t) := \sum_{\tau \in I_k(t)} \left(\ell_k^\tau(w_k^\tau) - \ell_k^\tau(w_k^\star) \right),$$

where $w_k^\star \in \arg\min_{w \in \mathcal{W}_k} \sum_{\tau \in I_k(t)} \ell_k^\tau(w)$.

Such a choice of regret is typical for Online Convex Optimization and Online Learning [5, 17, 18]. Moreover, it is common for both stochastic and deterministic settings. The works [19, 20] discussed stochastic extensions and the bandit case.

We assume that each expert admits a high-probability regret bound:

Assumption 2 (Anytime (U_k, δ)-bound)**.** *For any confidence level $\delta \in (0,1)$, the algorithm $\mathscr{A}_k$ satisfies*

$$\mathbb{P}\left\{\forall t \geq 1 : R_{\mathscr{A}_k}(t) \leq U_k(t, \delta)\right\} \geq 1 - \delta,$$

where $U_k(t, \delta)$ is a non-negative non-decreasing function in t.

Remark 2 (Example)**.** *For Online Gradient Descent (OGD) on convex G-Lipschitz losses over a domain of diameter R, it holds deterministically that $R_n(\mathscr{A}_k) = O(GR\sqrt{n})$ [5]. Thus, OGD satisfies (U_k, δ)-boundedness with $U_k(n, \delta) = O(GR\sqrt{n})$, independently of δ.*

2.4 Safe advice

Many stochastic algorithms (e.g., gradient methods, bandit algorithms) guarantee convergence only in *average* or in distribution rather than pointwise in the last iterate; see [4, 21, 11]. So, inspired by the idea of online-to-batch conversion [17], we introduce smoothing wrappers. They aggregate past states into a *safe advice*. Let the expected loss related to an advice $u \in \mathbf{U}$ be

$$L(u) := \mathbb{E}_{\xi \sim D}[\ell(u, \xi)]. \tag{2}$$

Assumption 3 (Smoothing wrapper)**.** *Each expert k admits a wrapper producing a safe advice $\upsilon_k : \mathbf{H}_k \to \mathbf{U}$ producing an advice $u_k^t := \upsilon_k(\mathcal{H}_k^{t-1})$ such that*

$$L(u_k^t) \leq \frac{1}{n_k^t} \sum_{\tau \in I_k(t)} L_k(w_k^\tau).$$

Examples.   If loss ℓ is convex w.r.t. $u \in \mathbf{U}$ and $\mathbf{U}$ is a convex set, a natural choice is the average

$$u_k^t = \frac{1}{n_k^t} \sum_{\tau \in I_k(t)} g_k(w_k^\tau). \tag{3}$$

One can also uniformly sample u^t from $\{g_k(w_k^\tau) : \tau \in I_k(t)\}$ [11].

2.5 Examples

We illustrate the framework's applicability with two cases: (i) parametric models trained online, and (ii) multi-armed bandit algorithms treated as experts.

2.5.1 Parametric models trained online.

This case is related to statistical model selection [22]. At each round t, the environment generates data $\xi^t := (x_t, y_t) \sim D$, with D supported on some instance-label space $\mathcal{X} \times \mathcal{Y}$. An expert k is a parametric predictor with state space $\mathbf{W}_k \subseteq \mathbb{R}^{p_k}$ and prediction function $g_k : \mathcal{X} \times \mathbf{W_k} \to \mathcal{Y}$. The corresponding loss is

$$\ell_k^t(w) = \ell\big(g_k(x_t; w), y_t\big), \qquad w \in \mathbf{W_k}.$$

To provide theoretical guarantees, we assume that $\ell_k^t(\cdot)$ is convex and G-Lipschitz in w.

Learning algorithm $\mathscr{A}_k$.   If $\mathscr{A}_k$ is OGD and $k \in S_t$, the state update is

$$w_k^{t+1} = w_k^t - \eta_t \nabla \ell_k^t\left(w_k^t\right),$$

with η_t being the step size. OGD is (U_k, δ)-bounded (see Remark 2) with $U_k(n, \delta) = O(GR\sqrt{n})$, where R is the diameter of the feasible set.

Safe advice.   Since the loss function is convex, safe advice is (3).

2.5.2 Multi-armed bandit algorithms.

In this example, each expert $k \in [K]$ is a *stochastic bandit algorithm* allocating probabilities over a finite set of base actions. At round t, the state of expert k is a probability vector

$$w_k^t \in \Delta^{d_k},$$

where d_k is the number of available base actions. The global decision space $\mathbf{U}$ corresponds to degenerate distributions that select a single action per round.

The *realized loss* is obtained by sampling $a_t \sim w_k^t$ and observing $\ell(a_t, \xi^t)$:

$$\ell_k^t(w_k^t) = \ell(a_t, \xi^t), \qquad a_t \sim w_k^t,$$

while the *expected loss* $\mathbb{E}_{a \sim w_k^t}[\ell(a, \xi^t)]$ is used in the regret analysis. In this setup, the expected loss coincides with the standard notion of stochastic bandit loss, so our definition of regret recovers the classical stochastic bandit formulation. After observing the outcome, the expert updates its internal history with $(a_t, \ell(a_t, \xi^t))$.

Safe advice. A natural smoothing option is the average of past distribution vectors of the bandits. If bandit's outputs are full probability vectors and the loss function is convex, the safe advice is (3). If a bandit only produces realized actions, the marginal distribution over previous samples can be used instead, as suggested in [11].

Remark 3 (On (U_k, δ)-bounds). *In stochastic bandits, algorithms typically provide anytime, high-probability regret bounds against the best arm, such as* UCB *[1], Thompson Sampling [23], and more recent variants like* Anytime-UCB *[24] and data-driven UCB methods for heavy-tailed rewards [25]. In this case, our (U_k, δ)-boundedness assumption becomes stronger, which simplifies part of the analysis.*

This demonstrates that our framework covers both online optimization and stochastic bandit setups, treating learning algorithms and adaptive bandit procedures within a single formulation. In the latter case, where experts are themselves bandit learners, we refer to [11] for a detailed overview of practical applications.

3 The M-LCB algorithm

We begin with introducing the key ingredient—LCB_k and UCB_k—the lower and the upper bounds bracketing with high probability the unknown optimal loss $L_k^\star$ of the k-th expert (see (1)).

Definition 1 (UCB and LCB). *Fix an expert $k \in [K]$. Its normalized running loss incurred at training sessions up to t is*

$$L_{\mathscr{A}_k}(t) := \frac{1}{n_k^t} \sum_{\tau \in I_k(t)} \ell_k^\tau(w_k^\tau), \quad n_k^t = |I_k(t)|.$$

The associated confidence bounds bracketing $L_k^\star$ are

$$\text{LCB}_k(t, \delta) := L_{\mathscr{A}_k}(t) - \frac{U_k\left(n_k^t, \delta_{\text{arm}}\right)}{n_k^t} - G(n_k^t, \delta_{n_k^t}),$$

$$\text{UCB}_k(t, \delta) := L_{\mathscr{A}_k}(t) + H(n_k^t, \delta_{n_k^t}),$$

where $\delta_{\text{arm}} = \frac{\delta}{2K}$, $\delta_n = \frac{\delta}{7Kn^2}$ and

$$G(n, \delta) = \sqrt{\frac{2\log(1/\delta)}{n}} + \frac{2\log(1/\delta)}{3n}, \qquad H(n, \delta) = \sqrt{\frac{2\log(1/\delta)}{n}},$$

with $U_k(\cdot, \cdot)$ being the regret bound from Assumption 2.

At round t, for each expert k we compute $\text{LCB}_k(t, \delta)$ and $\text{UCB}_k(t, \delta)$, and use the following rules to select the training subset S_t and the advisor (predicting expert) i_t,

$$S_t := \arg \min_{\substack{S \subseteq [K], \\ |S| \leq M}} \sum_{k \in S} \text{LCB}_k(t, \delta), \quad i_t := \arg \min_{k \in S_t} \text{UCB}_k(t, \delta)$$

Algorithm 1 presents M-LCB.

3.1 Alternative confidence bounds

A proof technique based on self-normalized processes ensures different LCB and UCB bounds.

Lower bound. Let $\mathrm{x}_n(\delta) := \log \frac{1}{\delta} - \frac{2}{3} + 2\log(1 + \log n)$ for any $n \geq 1$. Denote $G(n, \delta) := \frac{2\mathrm{x}_n(\delta)}{3n}$, for any $t \geq 1$

$$\text{LCB}_k(t, \delta) := L_{\mathscr{A}_k}(t) - \sqrt{3G(n_k^t, \delta)L_{\mathscr{A}_k}(t)} - G(n_k^t, \delta) - \frac{U_k(t, \delta)}{n_k^t}.$$

Lemma 6 proves the reult.

Upper bound. For any $t \geq 1$ that

$$\text{UCB}_k(t, \delta) := L_{\mathscr{A}_k}(t) + \frac{9 \log \frac{1}{\delta}}{2 n_k^t} \left(6 + \log \log \frac{1}{\delta} + \log(1 + 4 n_k^t L_{\mathscr{A}_k}(t)) \right)$$
$$+ \frac{1}{n_k^t} \sqrt{\log \frac{1}{\delta} (1 + 4 n_k^t L_{\mathscr{A}_k}(t)) \left(1 + \frac{1}{2} \log \left(1 + 4 n_k^t L_{\mathscr{A}_k}(t) \right) \right)}.$$

Lemma 7 proves the result.

Algorithm 1 M-LCB

1: **Input:** experts $\{(\mathbf{W}_k, \mathscr{A}_k, g_k, \upsilon_k)\}_{k=1}^K$, per-round budget M, confidence parameter δ
2: **Output:** (i) sequence of advices $\{u^t\}_{t=1}^T$; (ii) experts trained at each round $\{S_t\}_{t=1}^T$
3: Initialize each expert with $\delta_{\text{arm}} = \delta/(2K)$ (see Def. 1)
4: **for** $t = 1, 2, \ldots, T$ **do**
5:   **for** each $k \in [K]$ **do**
6:     Compute $\text{LCB}_k(t, \delta), \text{UCB}_k(t, \delta)$
7:   **end for**
8:   $S_t \leftarrow \arg \min_{S \subseteq [K], |S| \leq M} \sum_{k \in S} \text{LCB}_k(t, \delta)$
9:   $i_t \leftarrow \arg \min_{k \in S_t} \text{UCB}_k(t, \delta)$
10:   $u^t \leftarrow \upsilon_{i_t}(\mathcal{H}_{i_t}^{t-1})$
11:   Play u^t, suffer $\ell(u^t, \xi^t)$
12:   **for** each $k \in S_t$ **do**
13:     Observe $\ell_k^t(w_k^t)$.
14:     Update history $\mathcal{H}_k^t \leftarrow \mathcal{H}_k^{t-1} \cup \{w_k^t, \ell_k^t(w_k^t)\}$
15:     $w_k^{t+1} \leftarrow \mathscr{A}_k(\mathcal{H}_k^t)$
16:     $n_k^{t+1} \leftarrow n_k^t + 1$
17:   **end for**
18:   **for** each $k \notin S_t$ **do**
19:     $\mathcal{H}_k^t \leftarrow \mathcal{H}_k^{t-1}$
20:     $w_k^{t+1} \leftarrow w_k^t$ // unchanged, no update
21:     $n_k^{t+1} \leftarrow n_k^t$
22:   **end for**
23: **end for**

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4 Regret bounds

4.1 Lower bounds on the regret

Alongside the upper bounds, we also derive a minimax lower bound.

Definition 2 (Stochastic tasks with α–regret lower bound). *Let $\alpha \in [0, 1]$. We say that a family $\mathcal{F}_\alpha$ of stochastic online learning problems admits an α–regret lower bound if for every learning algorithm $\mathscr{A}$ and every horizon $T \geq 1$*

$$\sup_{f \in \mathcal{F}_\alpha} \mathbb{E}\big[R_{\mathscr{A}}(T)\big] \geq c\, T^\alpha,$$

for some constant $c > 0$ independent of T and $\mathscr{A}$.

Theorem 1 (Lower bound). *Consider K experts, horizon T, and a per-round budget M. Fix $\alpha \in [0.5, 1]$. There exists a class $\mathcal{F}_\alpha$ satisfying Definition 2, such that if each expert $k \in [K]$ solves a problem $f_k \in \mathcal{F}_\alpha$, then, for sufficiently small $\sqrt{\frac{K \log K}{MT}}$ and for any learning algorithm $\mathscr{A}_k$ and meta-procedure $\mathcal{P}$*

$$\sup_{f_k \in \mathcal{F}_\alpha,\, k \in [K]} \mathbb{E}\, \text{Reg}(T) \geq c_1 \sqrt{\frac{K\, T}{M}} \;+\; c_2\, T^\alpha \left(\frac{K}{M} \right)^{1-\alpha},$$

where $c_1, c_2 > 0$ are absolute constants.

This result establishes a fundamental performance limit for managing multiple learnable experts under a per-round budget. We propose the M-LCB algorithm matching the lower bound up to a logarithmic factor in the case of bounded loss. The proof of Theorem 4 is in the Supplementary Material in Section D.

Proof idea. The proof uses heavy-tailed multi-armed bandits [26] to construct $\mathcal{F}_\alpha$. The key ingredients are from [27, 12]. The two terms in the bound arise from exploration complexity and expert learning complexity, respectively.

4.2   M-LCB regret bounds

This section establishes regret guarantees for M-LCB. First, we define the *concentration event* $\mathcal{E}_\delta$ ensuring that all confidence bounds hold simultaneously for every expert and every time step:

$$\mathcal{E}_\delta := \left\{ \forall k \in [K], \ \forall t \geq 1 \quad \begin{array}{l} \mathtt{LCB}_k(t,\delta) \leq L_k^\star \\ L_k^\star \leq \mathtt{UCB}_k(t,\delta) \\ L(u_k^t) \leq \mathtt{UCB}_k(t,\delta) \end{array} \right\}, \tag{4}$$

We show that these bounds hold with high probability.

Lemma 1 (Anytime confidence bounds). *Under Assumptions 1-3 for any $\delta \in (0,1)$ it holds $\mathbb{P}(\mathcal{E}_\delta) \geq 1 - \delta$.*

Next, we establish high-probability bounds on the pseudo-regret and the realized regret.

Lemma 2 (Regret bounds). *Let the pseudo-regret be*

$$\overline{\mathrm{Reg}}(T) := \sum_{t=1}^{T} L(u^t) - T \cdot L^\star,$$

with $L(\cdot)$ defined in (2). Fix confidence level $\delta \in (0,1)$ and let the Assumptions 1-3 hold. Then for the pseudo-regret of Algorithm 1 it holds on the concentration event $\mathcal{E}_\delta$ that for all $T \geq 1$

$$\overline{\mathrm{Reg}}(T) \leq \Delta(T) := \sum_{\tau \in I_{k^\star}(T)} \left[\mathit{UCB}_{k^\star}(\tau,\delta) - \mathit{LCB}_{k^\star}(\tau,\delta) \right]$$

$$+ \frac{1}{M} \sum_{k=1}^{K} \sum_{\tau \in I_k(T)} \left[\mathit{UCB}_k(\tau,\delta) - \mathit{LCB}_k(\tau,\delta) \right],$$

where $k^\star$ is the index of the best expert. Moreover, with probability at least $1 - \delta$, it holds that for all $T \geq 1$

$$\mathrm{Reg}(T) \leq \Delta(T) + O(\sqrt{T}).$$

We specify this result for the case when the experts' regrets are $\tilde{O}(n^\alpha)$.

Theorem 2 (Convergence rates). *Let $\alpha, \delta \in (0,1)$. Suppose each expert k satisfies Assumption 2 with $U_k(t,\delta) = O\big(t^\alpha c(\delta)\big)$. Then, with probability at least $1 - \delta$ the regret of Algorithm 1 is bounded for all $T \geq 1$ as*

$$\mathrm{Reg}(T) = O\left(\sqrt{\frac{KT}{M} \log(\frac{KT}{\delta})} + \left(\frac{K}{M}\right)^{1-\alpha} T^\alpha c(\delta) \right).$$

4.3   A connection to Multiple-Play Bandits

M-LCB also extends to the multiple-play bandit setting [14, 15]. Specifically, the performance of a procedure can also be measured by how close the selected subsets S_t are to the best possible subset of experts. Specifically, the performance of a procedure can also be assessed by how close the average loss of the selected subsets S_t is to the average loss of the optimal subset of M experts.

In this setting, UCB-based algorithms (e.g., [16]) achieve $\tilde{O}(\sqrt{KT/M})$ convergence rates under fixed stochastic rewards distribution.

Lemma 3 (Top-M experts regret). *Fix a confidence level $\delta \in (0,1)$, and assume the same conditions as in Lemma 2. Let $\overline{L}^\star = \min_{S \subseteq [K], |S| \leq M} \frac{1}{M} \sum_{k \in S} L_k^\star$ denote the mean optimal loss among the M best experts.*

$$\mathrm{Reg}_M(T) := \sum_{t=1}^{T} \left[\frac{1}{M} \sum_{k \in S_t} L_k^\star - \overline{L}^\star \right],$$

Table 1: Comparison with related results. We assume that $U_k(T, \delta) = O(T^\alpha c(\delta))$ with $\alpha \in [0, 1]$, where $c(\delta)$ is typically poly-logarithmic. For *multiple-play bandits* the regret as in Section 4.3. For [12, 13] regret is defined as for expert algorithms (See papers). For other methods the regret as in Section 2.2. All rates are up to logarithmic factors. A mark ✓ marks supported properties. **Our M-LCB algorithm attains optimal rates for both regret definitions.**

Algorithm / Reference	Learnable	Multi-arm	Multiple-play	Regret rate (up to logs)
CORRAL + smoothing wrapper [11]	✓	✗	✗	$\tilde{O}\left(\sqrt{KT} + K^\alpha T^{1-\alpha} + K^{1-\alpha} T^\alpha c(\delta)\right)$
EXP3.P + smoothing wrapper [11]	✓	✗	✗	$\tilde{O}\left(\sqrt{KT} + K^{\frac{1-\alpha}{2-\alpha}} T^{\frac{1}{2-\alpha}} c(\delta)^{\frac{1}{2-\alpha}}\right)$
Dynamic Balancing [9]	✓	✗	✗	$\tilde{O}\left(\sqrt{KT} + K^{1-\alpha} T^\alpha c(\delta)\right)$
Prediction with Limited Advice [12]	✗	✓	✗	$\tilde{O}\left(\sqrt{\frac{KT \log K}{M}}\right)$
H-INF [13]	✗	✓	✗	$\tilde{O}\left(\max\{\sqrt{KT/M}, \sqrt{T \log K}\}\right)$
CombUCB1 [16]	✗	✗	✓	$\tilde{O}\left(\sqrt{KT/M}\right) \alpha = \frac{1}{2}$
M-LCB [this work]	✓	✓	✓	$\tilde{O}\left(\sqrt{\frac{KT}{M}} + (K/M)^{1-\alpha} T^\alpha c(\delta)\right)$

The following bound holds with probability at least $1 - \delta$

$$\mathrm{Reg}_M(T) \leq \frac{1}{M} \sum_{k=1}^{K} \sum_{\tau \in I_k(T)} \left[UCB_k(\tau, \delta) - LCB_k(\tau, \delta)\right].$$

This immediately yields the convergence rate for top-M mean regret.

Theorem 3 (Convergence rate for Top-M mean regret). *Let $\alpha \in (0, 1]$ and $\delta \in (0, 1)$. Suppose each expert k satisfies Assumption 2 with $U_k(t, \delta) = O(t^\alpha c(\delta))$. Run Algorithm 1 with the confidence bounds of Definition 1. Then, with probability at least $1 - \delta$ the mean regret of selected arms $R(T)$ (see Proposition 3) is, for all $T \geq 1$,*

$$\mathrm{Reg}_M(T) = O\left(\sqrt{\frac{KT}{M} \log \frac{KT}{\delta}} + \left(\frac{K}{M}\right)^{1-\alpha} T^\alpha c(\delta)\right).$$

Remark 4 (On constants). *If the constants β_k in the inner bounds, $U_k(t, \delta) = O(\beta_k t^\alpha c(\delta))$ are significant (e.g., depend on the dimension or Lipschitz constant of the task), then the regret bound refines to*

$$O\left(\sqrt{\frac{KT}{M} \log(\frac{KT}{\delta})} + T^\alpha c(\delta)\left[\beta_{k^\star} + \left(\frac{1}{M} \sum_{k=1}^{K} \beta_k^{\frac{1}{1-\alpha}}\right)^{1-\alpha}\right]\right).$$

providing a more precise characterization of the dependence on individual expert complexities. .

4.4 Comparison with existing results

Table 1 lists meta-algorithms used for model selection and budgeted multi-arm training. It indicates whether each method supports *learnable experts*, *multi-arm updates*, and guarantees on the *multi-play regret* (average loss of selected subsets).

Model-selection algorithms such as [11, 9] achieve order-optimal rates in T and K for single-arm updates, assuming known or estimated expert regret bounds $U_k(T, \delta)$. However, they do not take into account per-round training budgets and subset-level performance. Multi-play bandit methods [12, 13, 16] handle budgeted updates but do not train arms.

M-LCB bridges these tasks: it manages *learnable experts* via *multiple expert updates per round*. Moreover, it extends to the *multi-play* setting and achieves the same order-optimal rate for the corresponding regrets.

Some examples of tasks and base algorithms that can be handled within this framework are provided in the Supplementary materials, Section B.

5   Proofs

Lemma 4 (Empirical loss concentration (Lemma B.10 in [22])). *Let $\mathbf{W}$ be a state space and $\mathcal{D}$ a distribution on $\mathbf{E}$. Let $\ell : \mathbf{W} \times \mathbf{E} \to [0, 1]$ be a bounded loss function. Fix a predictor $w \in \mathbf{W}$ and define its expected loss $L(w) := \mathbb{E}_{\xi \sim \mathcal{D}}[\ell(w, \xi)]$.*

Then, for any $n > 0$ and $\delta \in (0, 1)$, with probability at least $1 - \delta$ over the draw of $\{\xi_t\}_{t=1}^n$ i.i.d. from $\mathcal{D}$,

$$\frac{1}{n} \sum_{t=1}^{n} \ell(w, \xi_t) - L(w) \leq \sqrt{\frac{2L(w) \log(1/\delta)}{n}} + \frac{2 \log(1/\delta)}{3n}.$$

Proof. Follows directly from Bernstein's inequality. □

Lemma 5 (Azuma's inequality (from Theorem D.2 [28])). *Let $\{\mathcal{F}_t\}$ be a filtration, and let X_t be a sequence of random variables adapted to $\mathcal{F}_t$ with $\mathbb{E}[X_t \mid \mathcal{F}_{t-1}] = 0$ and $|X_t| \leq 1$. Then, for any fixed $n \geq 1$ and any $\delta \in (0, 1)$, with probability at least $1 - \delta$,*

$$\left| \frac{1}{n} \sum_{t=1}^{n} X_t \right| \leq \sqrt{\frac{2 \log(1/\delta)}{n}}.$$

Proof of Lemma 1. Fix an arm k and update count $n \geq 1$. We prove the inequalities in (4), and then apply a union bound over all (k, n) and arms.

Step 1: Lower Confidence Bound. By Assumption 2 (anytime (U_k, δ)-boundedness), with probability at least $1 - \delta_{\mathrm{arm}}$, simultaneously for all n,

$$\frac{1}{n} \sum_{\tau=1}^{n} \ell_k^\tau(w_k^\tau) - \frac{U_k(n, \delta_{\mathrm{arm}})}{n} \leq \min_{w \in \mathcal{W}_k} \frac{1}{n} \sum_{\tau=1}^{n} \ell_k^\tau(w). \tag{5}$$

Let $w_k^\star \in \arg\min_w L_k(w)$, so $L_k^\star = L_k(w_k^\star)$. By i.i.d. stochastic losses and boundedness, Lemma 4, applied to state space $\mathcal{W}_k$, loss function $\ell_k(\cdot, \cdot)$ and predictor $w_k^\star \in \mathcal{W}_k$, with number of items n and confidence δ_n gives, with probability at least $1 - \delta_n$:

$$\frac{1}{n} \sum_{\tau=1}^{n} \ell_k^\tau(w_k^\star) \leq L_k^\star + G(n, \delta_n). \tag{6}$$

Since $\min_w \frac{1}{n} \sum_{\tau=1}^{n} \ell_k^\tau(w) \leq \frac{1}{n} \sum_{\tau=1}^{n} \ell_k^\tau(w_k^\star)$, combining (5) and (6) yields

$$L_{\mathscr{A}_k}(n) - \frac{U_k(n, \delta_{\mathrm{arm}})}{n} - G(n, \delta_n) \leq L_k^\star.$$

Evaluating this expression at $n = n_k(t)$ and using the definition of $\mathtt{LCB}_k(t, \delta)$ (Definition 1), we obtain $\mathtt{LCB}_k(t, \delta) \leq L_k^\star$.

Step 2: Upper Confidence Bound. Condition on the history $\mathcal{H}_k^\tau$ (in our definition history at time τ includes all up to time $\tau - 1$): then w_k^τ is measurable while ξ^τ is independent, so $\mathbb{E}[\ell_k^\tau(w_k^\tau) \mid \mathcal{H}_k^\tau] = L_k(w_k^\tau)$. Thus $\{\ell_k^\tau(w_k^\tau) - L_k(w_k^\tau)\}_{\tau=1}^n$ is a bounded martingale difference sequence. By Lemma 5 applied with sample size n and confidence level δ_n, with probability at least $1 - \delta_n$,

$$\left| \frac{1}{n} \sum_{\tau=1}^{n} \ell_k^\tau(w_k^\tau) - \frac{1}{n} \sum_{\tau=1}^{n} L_k(w_k^\tau) \right| \leq H(n, \delta_n). \tag{7}$$

By Assumption 3, for safe advice u_k^t built on $\mathcal{H}_k^t$ after n update steps : $L(u_k^t) \leq \frac{1}{n} \sum_{\tau=1}^{n} L_k(w_k^\tau)$. Combining this with (7) gives

$$L(u_k^t) \leq L_{\mathscr{A}_k}(n) + H(n, \delta_n).$$

Evaluating right hand side expression at $n = n_k(t)$ and using the definition of $\mathtt{UCB}_k(t, \delta)$ (Definition 1), we obtain $L(u_k^t) \leq \mathtt{UCB}_k(t, \delta)$.

Step 3: Union Bound.   Finally, we bound the probability of the concentration event $\mathcal{E}_\delta$. There are three types of events: (i) anytime arm guarantees (5), (ii) fixed-predictor concentration (6), and (iii) martingale concentration (7). For (i), each arm contributes at most δ_{arm}, so over K arms the total failure probability is $\leq K\delta_{\text{arm}} = \delta/2$. For (ii) and (iii), we allocated δ_n per event. Since

$$\sum_{k=1}^{K}\sum_{n=1}^{\infty} 2\delta_n = \sum_{k=1}^{K}\sum_{n=1}^{\infty} \frac{2\delta}{7Kn^2} \leq \frac{2\delta}{7}\cdot\frac{\pi^2}{6} < \delta/2,$$

both concentration bounds hold simultaneously for all (k, n) with probability at least $1 - \delta/2$. Thus the overall failure probability is at most δ, and the concentration event $\mathcal{E}_\delta$ holds with probability at least $1 - \delta$. $\qquad\square$

Remark 5. *In the proof of Lemma 1 at step* (6) *for bounded losses one may use* $L_k^* \leq Z_k(t,\delta) := \min(1, \text{UCB}_k(t,\delta))$ *to get a tighter bound* $G_k(t,\delta) := \sqrt{\frac{2Z_k(t,\delta)\log(1/\delta)}{3t}} + \frac{2\log(1/\delta)}{t}$.

Proof of Lemma 2. **Pseudo-regret bound**. S_t and i_t denote, respectively, the training subset and the prediction arm selected at round t by Algorithm 1. Let $k^\star \in \arg\min_{k\in[K]} L_k^\star$ be the index of the best arm in terms of expected loss.

Algorithm runs using the confidence bounds defined in Proposition 1, so concentration event $\mathcal{E}_\delta$ (4) holds with probability at least $1 - \delta$. In the sequel, we condition on $\mathcal{E}_\delta$ and prove the regret bounds under this event.

Under event $\mathcal{E}_\delta$, for safe advice u_t at time step t provided by i_t: $u^t = v_{i_t}(\mathcal{H}_{i_t}^t)$ (See Assumption 2) expected loss is bounded by UCB:

$$\overline{\text{Reg}}(T) = \sum_{t=1}^{T}\big[L(u_t) - L^\star\big] \leq \sum_{t=1}^{T}\big[\text{UCB}_{i_t}(t,\delta) - L^\star\big],$$

splitting the sum depending on whether $k^\star$ is trained at t gives

$$\overline{\text{Reg}}(T) \leq \underbrace{\sum_{t:\, k^\star \in S_t}\big[\text{UCB}_{i_t}(t,\delta) - L^\star\big]}_{A} + \underbrace{\sum_{t:\, k^\star \notin S_t}\big[\text{UCB}_{i_t}(t,\delta) - L^\star\big]}_{B}. \qquad (8)$$

Term A. Since $i_t = \arg\min_{k\in S_t} \text{UCB}_k(t,\delta)$, for rounds with $k^\star \in S_t$, :$\text{UCB}_{i_t}(t,\delta) \leq \text{UCB}_{k^\star}(t,\delta)$. Under $\mathcal{E}_\delta$, $L^\star \geq \text{LCB}_{k^\star}(t,\delta)$. Therefore,

$$A \leq \sum_{t:\, k^\star \in S_t}\Big[\text{UCB}_{k^\star}(t,\delta) - \text{LCB}_{k^\star}(t,\delta)\Big] = \sum_{\tau\in I_{k^\star}(T)}\Big[\text{UCB}_{k^\star}(t,\delta) - \text{LCB}_{k^\star}(t,\delta)\Big] \qquad (9)$$

The last equality is from fact that k^* is updated in such and only such terms of sum.

Term B. By construction of i_t, we have $\text{UCB}_{i_t}(t,\delta) \leq \frac{1}{M}\sum_{k\in S_t} \text{UCB}_k(t,\delta)$. Substituting this into term B in (8) and applying the standard add–subtract trick with $\frac{1}{M}\sum_{k\in S_t} \text{LCB}_k(t,\delta)$ for each t in the sum, we obtain:

$$B \leq \underbrace{\sum_{t:\, k^\star \notin S_t}\Big[\frac{1}{M}\sum_{k\in S_t}\text{LCB}_k(t,\delta) - L^\star\Big]}_{C} + \underbrace{\sum_{t:\, k^\star \notin S_t}\frac{1}{M}\sum_{k\in S_t}\Big[\text{UCB}_k(t,\delta) - \text{LCB}_k(t,\delta)\Big]}_{D}.$$

For the Term C, by the selection rule of S_t and the fact that $k^\star \notin S_t$, we must have $\frac{1}{M}\sum_{k\in S_t}\text{LCB}_k(t,\delta) \leq L^\star$, because otherwise replacing some $k \in S_t$ by $k^\star$ would strictly decrease the sum of LCBs (under $\mathcal{E}_\delta$ we have $\text{LCB}_{k^\star}(t,\delta) \leq L^\star$). Thus that bracket is non-positive. Therefore,

$$B \leq \sum_{t:\, k^\star \notin S_t}\frac{1}{M}\sum_{k\in S_t}\Big[\text{UCB}_k(t,\delta) - \text{LCB}_k(t,\delta)\Big] \leq \frac{1}{M}\sum_{t=1}^{T}\sum_{k\in S_t}\Big[\text{UCB}_k(t,\delta) - \text{LCB}_k(t,\delta)\Big] =$$

$$= \frac{1}{M}\sum_{k=1}^{K}\sum_{\tau\in I_k(T)}\Big[\text{UCB}_k(\tau,\delta) - \text{LCB}_k(\tau,\delta)\Big]. \qquad (10)$$

Combining (9) and (10) with (8) proves the best-arm bound stated in the proposition.

Regret bound The proof is follows from decomposition of regret into stochastic part and pseudo regret:

$$\text{Reg}(T) = \sum_{t=1}^{T} \left(\ell(u^t, \xi^t) - L(u_t) \right) + \sum_{t=1}^{T} \left(L(u_t) - L^\star \right).$$

Since generated data is independent from advice u_t at time t, the first term is bounded by concentration inequality (e.g. Lemma 5). The second term is a pseudo regret, which was bounded above. $\qquad\square$

Proof of Theorem 2. The regret bound in Lemma 2 considers the differences $\text{UCB}_k(t, \delta) - \text{LCB}_k(t, \delta)$, which by Definition 1 depend only on the number of updates n_k^t and the confidence level δ. We denote this quantity by $\Delta_k(n, \delta) := H(n, \delta_n) + G(n, \delta_n) + \frac{U_k(n, \delta_{\text{arm}})}{n}$.

One can see, that $\Delta_k(n_k^t, \delta) = \text{UCB}(t, \delta) - \text{LCB}(t, \delta)$. Substituting Δ_k into the pseudo–regret bound of Proposition 2, we obtain a sum over update indices that can be rewritten as a sequential sum over the number of updates of each expert,

$$\Delta(T) = \sum_{\tau=1}^{n_{k^\star}^T} \Delta_{k^\star}(\tau, \delta) + \frac{1}{M} \sum_{k=1}^{K} \sum_{\tau=1}^{n_k^T} \Delta_k(\tau, \delta),$$

which can be directly estimated using the known forms of H, G, and U_k.

The rest of the proof proceeds by bounding the concentration and inner–learning terms separately. Logarithmic factors $\log(\delta_n)$ slowly increase, so we upper–bound them by $\log(\frac{KT}{\delta})$. This only affects the constants in $O(\cdot)$. Using the standard summation bounds $\sum_{\tau \leq n} \tau^{-1/2} = O(\sqrt{n})$ and $\sum_{\tau \leq n} \tau^{\alpha-1} = O(n^\alpha)$, together with the concavity inequality $\sum_k (n_k^t)^\alpha \leq K^{1-\alpha}(MT)^\alpha$, we obtain the result.

$\qquad\square$

Lemma 6 (LCB). *Let Assumptions 1-3 be true. Fix $w_k \in \mathbf{W}_k$. The following bound holds with probability at least $1 - e^{-x}$*

$$L_k(w_k) \geq L_{\mathscr{A}_k}(t) - \sqrt{\frac{2x_{n_k^t}}{n_k^t} L_{\mathscr{A}_k}(t)} - \frac{2x_{n_k^t}}{3n_k^t} - \frac{U_k(t, e^{-x})}{n_k^t}$$

where $x_n := x - \frac{2}{3} + 2 \log(1 + \log n)$ for any $n \geq 1$.

Proof. Denote $\ell_k(w_k) := \ell(w_k, \xi)$, $\sigma_k^2 := \text{Var}(\ell_k(w_k))$ Fix $k \in [K]$ and $w_k \in \mathbf{W}_k$. Set $X_k^{n_t^k} := \ell_k^t(w_k) - L_k(w_k)$. By Freedman's inequality, it holds with probability at least $1 - e^{-x}$ for any $n \geq 1$

$$\max_{s \leq n} \sum_{i=1}^{s} X_k^i = \max_{t:\, n_k^t \leq n} \sum_{\tau \in I_k(t)} (\ell_k^\tau(w_k) - L_k(w_k))$$

$$\leq \sigma_k \sqrt{2nx} + \frac{2}{3}x \leq \sqrt{2nL(w_k)x} + \frac{2}{3}x,$$

where the last inequality holds since $\sigma_k^2 \leq \mathbb{E}\ell_k(w_k) = L_k(w_k)$. Consequently, for any $m \geq 1$ and $x > 0$ with probability at least $1 - e^{-x}$

$$\max_{2^{m-1} \leq s < 2^m} \sum_{i=1}^{s} X_k^i \leq \sqrt{2^{m+1}L_k(w_k)x} + \frac{2}{3}x.$$

Setting $x_s := x - \frac{2}{3} + 2 \log(1 + \log s)$, we use union bound over m and get for any $s \geq 1$

$$\frac{1}{s} \sum_{i=1}^{s} X_k^i \leq 2\sqrt{\frac{L_k(w_k)}{s} x_s} + \frac{2}{3s} x_s.$$

In other words, for any $t \geq 1$

$$\frac{1}{n_k^t} \sum_{\tau \in I_k(t)}^{s} \ell_k^\tau(w_k) \leq L_k(w_k) + 2\sqrt{\frac{L_k(w_k)}{n_k^t} x_{n_k^t}} + \frac{2}{3n_k^t} x_{n_k^t}.$$

Combining this result with Assumption 2, we get

$$L_{\mathscr{A}_k}(t) \leq L_k(w_k) + 2\sqrt{\frac{L_k(w_k)}{n_k^t}\mathrm{x}_{n_k^t}} + \frac{2}{3n_k^t}\mathrm{x}_{n_k^t} + \frac{U_k(t,e^{-\mathrm{x}})}{n_k^t}.$$

Note that

$$L_k(w_k) \geq L_{\mathscr{A}_k}(t) - \left(\frac{2}{3n_k^t}\mathrm{x}_{n_k^t} + \frac{U_k(t,e^{-\mathrm{x}})}{n_k^t}\right).$$

It is easy to see that

$$L_{\mathscr{A}_k}(t) - L_k(w_k) \leq \sqrt{\frac{2\mathrm{x}_{n_k^t}}{n_k^t}}.$$

The claim follows $\qquad\square$

Lemma 7 (UCB). *With probability at least $1 - e^{-\mathrm{x}}$ for any $t \geq 1$*

$$\frac{1}{n_k^t}\sum_{\tau \in I_k(\tau)} L_k(w_k^\tau) \leq L_{\mathscr{A}_k}(t) + \frac{9\mathrm{x}}{2n_k^t}\left(6 + \log \mathrm{x} + \log(1 + 4n_k^t L_{\mathscr{A}_k}(t))\right) +$$

$$+ \frac{1}{n_k^t}\sqrt{2\mathrm{x}(1 + 4n_k^t L_{\mathscr{A}_k}(t))\left(1 + \frac{1}{2}\log\left(1 + 4n_k^t L_{\mathscr{A}_k}(t)\right)\right)}.$$

Proof. Now let $w_k^t \in \mathbf{W}_k$ be the state of the expert at time t and set $Y_k^{n_k^t} := \ell_k^t(w_k^t)$. Applying Lemma 8, we get the result. $\qquad\square$

Lemma 8. *Let $X_t \in [0,1]$ be a process adapted to a filtration $\mathcal{F}_t$. Set $\mu_t := \mathbb{E}[X_t|\mathcal{F}_{t-1}]$ Let $S_t = \sum_{i=1}^t X_i$ and $U_t = \sum_{i=1}^t \mu_i$. Then for any $\mathrm{x} \geq 1$ with probability at least $1 - e^{-\mathrm{x}}$ and for all t simultaneously it holds*

$$|S_t - U_t| \leq \sqrt{2\mathrm{x}(S_t + 3U_t + 1)\left(1 + \frac{1}{2}\log(S_t + 3U_t + 1)\right)}. \tag{11}$$

Moreover,

$$U_t \leq S_t + \frac{9\mathrm{x}}{2}\left(6 + \log \mathrm{x} + \log(1 + 4S_t)\right) + \sqrt{2\mathrm{x}(1 + 4S_t)\left(1 + \frac{1}{2}\log\left(1 + 4S_t\right)\right)}.$$

Proof. First, we note that $|X_i - \mu_i|^2 \leq X_i + \mu_i$ and $\mathbb{E}[|X_i - \mu_i|^2|\mathcal{F}_{i-1}] \leq 2\mu_i$. Thus, by Theorem 9.21 from [29], the process $\exp\left\{\lambda(S_t - U_t) - \lambda^2/2(S_t + 3U_t)\right\}$ is a super-martingale for all $\lambda \in \mathbb{R}$. Define the stopping time τ as the first moment when (11) is violated. Then Corollary 12.5 from [29] with $A = S_\tau - U_\tau$, $B^2 = S_\tau + 3U_\tau$ and $y = 1$ ensures that with probability at least $1 - e^{-\mathrm{x}}$

$$A \leq \sqrt{2\mathrm{x}(B^2 + 1)\left(1 + \frac{1}{2}\log(B^2 + 1)\right)}.$$

Using the definition of A, B, and τ, we get the first result. The second result follows from Lemma 9 $\qquad\square$

6 Limitations and Future Work

Our analysis assumes that the confidence scaling function $c(\delta)$ is known. The case of unknown $c(\delta)$ is considered by Pacchiano et al. [11], where the resulting regret bounds exhibit a weaker dependence on $c(\delta)$, while Dann et al. [10] estimate it online, yielding adaptive but less interpretable guarantees.

Another important direction concerns algorithms with additional observations per round, such as limited-advice and multi-play bandits [12, 13, 16]. Extending their analysis to the setting of *learnable experts* would unify these frameworks.

Finally, a promising extension is the *contextual* regime, where experts specialize based on observed features or domains, connecting our framework to contextual bandits [30].

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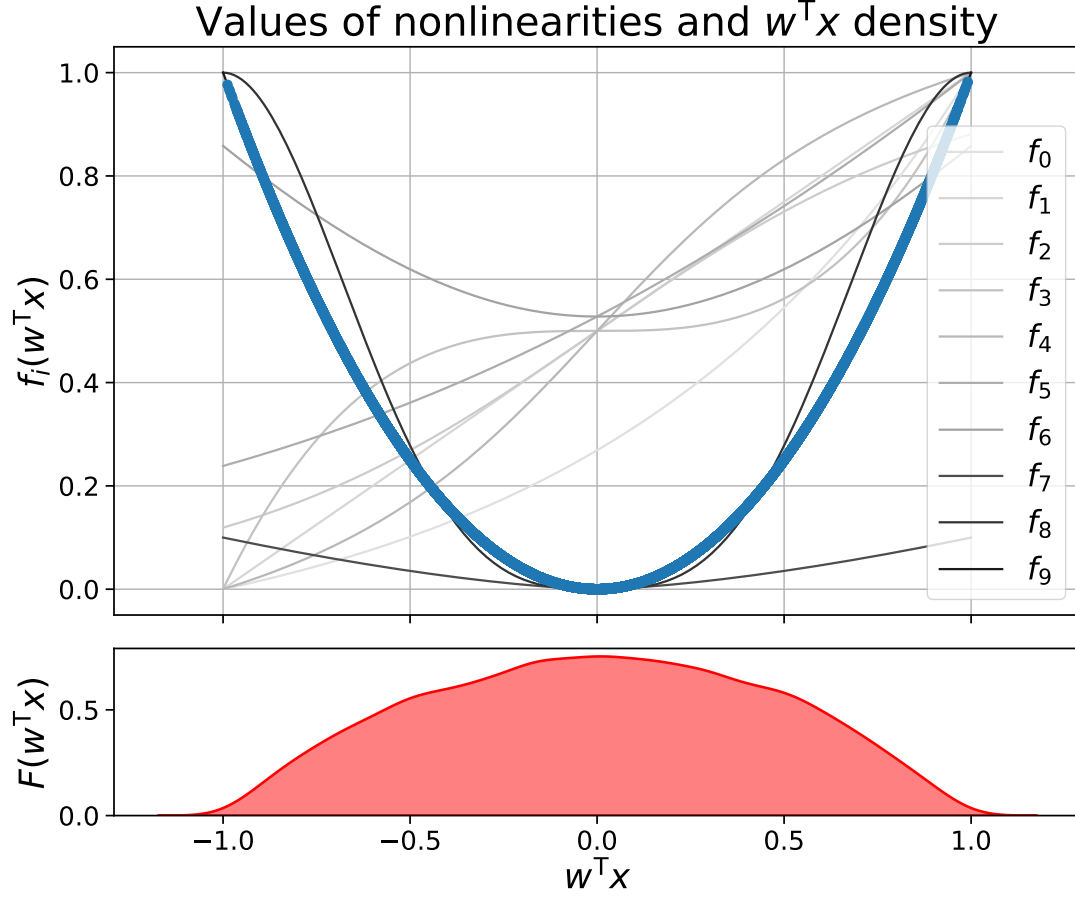
Figure 1: Nonlinear link functions associated with the arms (top) and the density of generated data points (bottom). One can see that the last three functions are highly similar where the data is concentrated, making it hard to distinguish the optimal arm.

A   Numerical Experiments

We evaluate the performance of our proposed algorithm, M-FLCB, on synthetic problems designed to test its ability to manage adaptive arms under a computational budget. We compare against two baselines: ED^2RB[10], an algorithm with guarantees for learnable arms, and LimitedAdvice [12], an expert algorithm capable of handling multiple arm updates per round. In all experiments we consider update limits $M \in \{1, 2, 3\}$, average results over 30 independent runs, and display ± 0.5 standard deviations as shaded regions.

A.1   Model Selection among Generalized Linear Models

We consider a model selection problem with $K = 10$ arms. Each arm k represents a generalized linear model (GLM) with a distinct, fixed link function $f_k : \mathbb{R} \to \mathbb{R}$. At each round t, a feature vector $x_t \in \mathbb{R}^d$ is drawn uniformly from the unit sphere $\mathbb{S}^{d-1}$, and the label is generated by the optimal arm $k^\star = 9$ as

$$r_t = f_{k^\star}(w^\top x_t),$$

As illustrated in Figure 1, the link functions f_7, f_8, and f_9 are highly similar in regions where the data is dense, presenting a nontrivial exploration challenge.
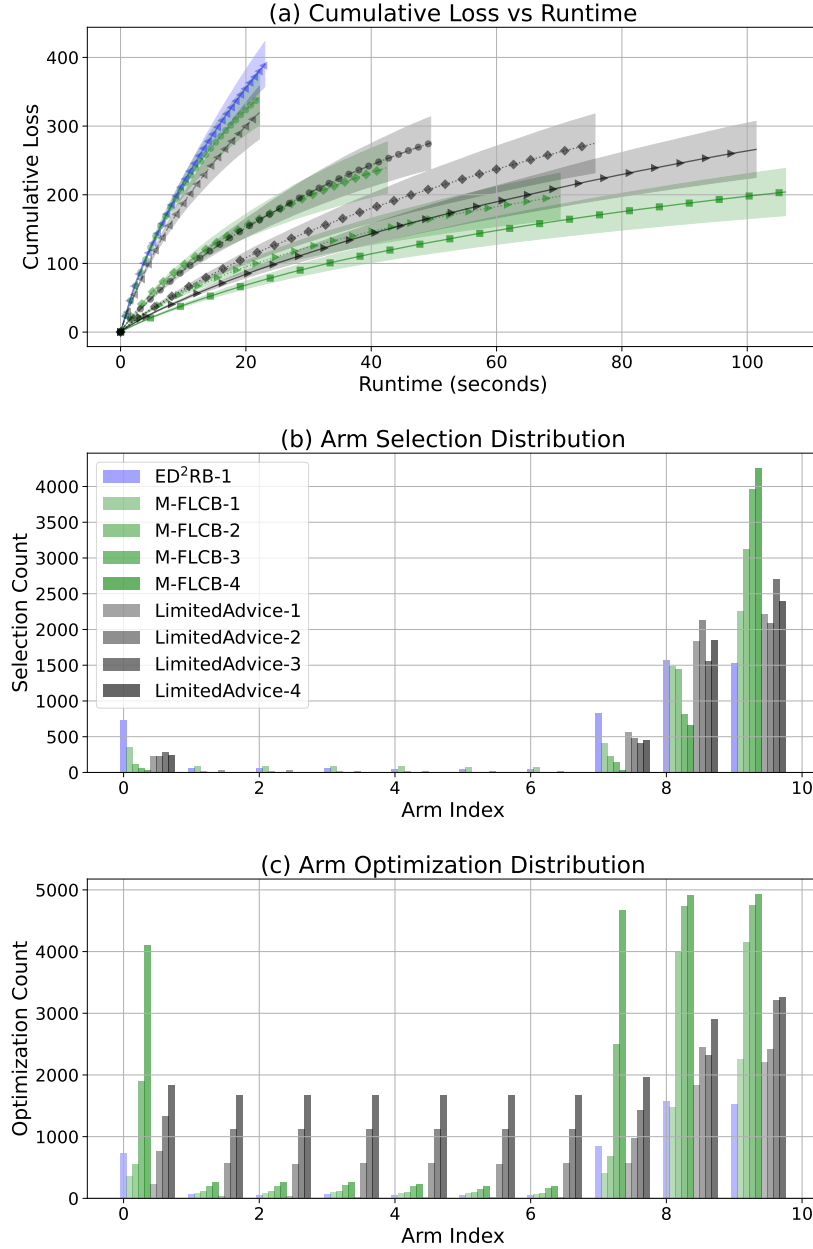
16

Figure 2: Performance comparison on the GLM model selection problem. (a) Cumulative regret. (b) Final distribution of arm selection. (c) Allocation of computational budget across arms.

Results.   Figure 2 summarizes the results. Panel (a) shows that M-FLCB achieves sublinear regret and is competitive with both baselines. Panel (b) reports the final arm selection distribution: M-FLCB successfully identifies the optimal arm ($k = 9$). Panel (c) presents the distribution of the computational budget across arms. M-FLCB allocates updates primarily to top-performing arms, while LimitedAdvice spreads updates more evenly, leading to less efficient use of the training budget.

Hyperparameters

For ED^{2}RB, the exploration parameter was tuned, with $c = 0.1$ giving the best results. For M-FLCB, concentration terms were scaled by a factor of 0.3. Parameters for LimitedAdvice were set according to its theoretical analysis [12].

Table 2: Examples of inner-arm convergence rates $U_k(n, \delta)$ and resulting global regret (up to logarithmic factors and an additive term $O(\sqrt{(K/M)T})$). For each expert k, the inner algorithm satisfies $U_k(t, \delta) = O(\beta_k t^\alpha c(\delta))$, and the corresponding global regret scales as $O(T^\alpha c(\delta) \|\boldsymbol{\beta}\|_{M,1-\alpha})$, where $\|\boldsymbol{\beta}\|_{M,\gamma} = \left(\frac{1}{M} \sum_{k=1}^{K} \beta_k^{\frac{1}{\gamma}} \right)^\gamma$. Parameter conventions: K — # experts; M — per-round training budget; T — horizon; N_k — # base arms/actions; d_k — feature dimension; ε — heavy-tail moment exponent ($\mathbb{E}|X|^{1+\varepsilon} \leq \sigma^{1+\varepsilon}$); L, R, C, G — Lipschitz, diameter, range, and gradient constants.

Inner algorithm / problem	Inner rate $U_k(n, \delta)$	Global regret (up to logs)
OGD / OMD (convex Lipschitz)	$O(G_k R_k \sqrt{n}), \alpha = \frac{1}{2},$	$O(T^{1/2} \|GD\|_{M,1/2})$
Bandit Convex Optimization (bounded f_t) [31]	$O(C_k d_k n^{5/6}), \alpha = \frac{5}{6}$	$O(T^{5/6} \|Cd\|_{M,1/6})$
Bandit Convex Optimization (L–Lipschitz f_t) [31]	$O(\sqrt{C_k L_k R_k} d_k n^{3/4}), \alpha = \frac{3}{4}$	$O(T^{3/4} \|\sqrt{CLR}d\|_{M,1/4})$
Heavy–tailed stochastic bandits [26]	$\tilde{O}(n^\alpha N_k^{1-\alpha}), \alpha = \frac{1}{1+\varepsilon}$	$O(T^\alpha [\frac{1}{M} \sum_{k=1} N_k]^{1-\alpha})$
Heavy–tailed stochastic bandits (Symmetric noise) [32]	$O(\sqrt{N_k n}\log n^{\frac{3}{2}}), \alpha = \frac{1}{2}$	$O(T^{1/2} [\frac{1}{M} \sum_{k=1} N_k]^{\frac{1}{2}})$
Heavy–tailed linear bandits [33]	$O(d_k^{\frac{3}{2}-\alpha} n^\alpha), \alpha = \frac{1}{1+\varepsilon}$	$O(T^\alpha \|d^{\frac{3}{2}-\alpha}\|_{M,1-\alpha})$
Hedge / Exponential Weights (over N_k experts)	$O(\sqrt{n \log N_k}), \alpha = \frac{1}{2},$	$O(T^{1/2} \|\sqrt{\log N}\|_{M,1/2})$

B Usage Examples

Table 2 summarizes several representative base algorithms, their inner convergence rates, and the resulting global regret when combined with M-FLCB. Alongside standard convex and exponential-weighted learners, we include "hard" stochastic problems with heavy-tailed rewards, which exhibit slower convergence characterized by larger α.

Different experts may operate over distinct action spaces. For instance, in bandit-based experts, each learner may control its own set of arms, while in parametric or linear models, the dimensionality of the feature space may vary. Such heterogeneity is reflected in parameters like N_k or d_k in Table 2, and is naturally handled within the M-FLCB framework.

C Auxiliary results

Lemma 9. *Under conditions of Lemma 8 the following inequality holds*

$$U_t \leq S_t + \frac{9\mathrm{x}}{2}\Big(6 + \log \mathrm{x} + \log(1 + 4S_t)\Big)$$
$$+ \sqrt{2\mathrm{x}(1 + 4S_t)\Big(1 + \tfrac{1}{2}\log(1 + 4S_t)\Big)}$$

Proof. Consider

$$U_t \leq S_t + \sqrt{2\mathrm{x}\big(1 + S_t + 3U_t\big)\Big(1 + \tfrac{1}{2}\log\big(1 + S_t + 2U_t\big)\Big)}.$$

Set $\Delta := U_t - S_t$. If $\Delta \leq 0$, the bound holds.
Case $\Delta \geq 0$. Note that

$$\Delta^2 \leq 2\mathrm{x}\big(1 + 4S_t + 3\Delta\big)\Big(1 + \tfrac{1}{2}\log\big(1 + 4S_t + 2\Delta\big)\Big).$$

Denote $a := 2\mathrm{x} \geq 2, b := 1 + 4S_t \geq 1$. Thus,

$$c := \log(b + 3\Delta) \leq \log b + \frac{3\Delta}{b} \qquad \text{(by concavity)}.$$

Consequently,

$$\Delta^2 \leq a(b + 3\Delta)\left(1 + \frac{c}{2}\right)$$

$$\leq 3a\left(1 + \frac{c}{2}\right)\Delta + ab + ab\frac{\log b}{2} + \frac{3a}{2}\Delta$$

$$= \frac{3a}{2}(3 + c)\Delta + ab\left(1 + \frac{\log b}{2}\right).$$

By the quadratic, inequality we get

$$\Delta \leq \frac{3a}{2}(3 + c) + \sqrt{ab\left(1 + \frac{\log b}{2}\right)}.$$

Thus,

$$b + 3\Delta \leq b + \frac{9a}{2}(3 + c) + 3\sqrt{ab\left(1 + \frac{\log b}{2}\right)}$$

$$\leq b + \frac{27a}{2}\left(1 + \frac{c}{3}\right) + 3\sqrt{ab\left(1 + \frac{c}{2}\right)}$$

$$\leq \left(\sqrt{b} + \sqrt{\frac{27}{2}a\left(1 + \frac{c}{3}\right)}\right)^2.$$

$$c = \log(b + 3\Delta) \leq 2\log\left(\sqrt{b} + \sqrt{\frac{27}{2}a\left(1 + \frac{c}{3}\right)}\right)$$

$$\leq 2\log\sqrt{b} + 2\log\left(1 + \sqrt{\frac{27}{2}\frac{a}{b}\left(1 + \frac{c}{3}\right)}\right)$$

$$\leq \log b + \log(20a) + \frac{c}{3}.$$

Consequently, $c \leq \frac{3}{2}\left(\log b + \log(20a)\right)$. Thus

$$\Delta \leq \frac{9a}{4}\left(2 + \log(20a) + \log b\right) + \sqrt{ab\left(1 + \frac{\log b}{2}\right)}.$$

The claim follows.

$\square$

D   Lower Bounds

We establish minimax lower bounds for our problem setup. The proof combines information-theoretic arguments (as in [12]) with the internal hardness of heavy-tailed bandits [26]. The main idea is to construct a family of perturbed games, relate the probability of identifying the optimal expert to KL divergences via Pinsker's inequality, and transfer internal regret bounds from the null game (where all experts are identical) to the perturbed games.

Bandit setting and regret conventions.   For concreteness, we consider experts represented by independent two–armed stochastic bandit problems. Each expert $h \in [K]$ has two arms with losses with expectations are $\ell_{h,1}$ and $\ell_{h,2}$, respectively. We note $\ell_h^\star = \min\{\mu_{h,1}, \mu_{h,2}\}$ At each round $t = 1, \ldots, T$, expert h selects an arm $I_t \in \{1, 2\}$ and receives the corresponding loss $\ell_{h,I_t,t}$. The (expected) cumulative regret of expert h within its own subproblem is defined as

$$R_h^{\text{in}}(T) = \sum_{t=1}^T \mathbb{E}[\ell_{h,I_t}] - T \cdot \ell_h^\star,$$

where the subscript "in" emphasizes that this regret is *internal* to the metaprocedure. That is, each expert h acts as an independent learning agent whose own regret $R_h^{\text{in}}(T)$ contributes to the overall regret of the meta–learner.

D.1 Class of α–hard stochastic tasks

Theorem 4 (Lower bound). *Consider K experts, horizon T, and a per-round budget M. Fix $\alpha \in [0.5, 1]$. There exists a class $\mathcal{F}_\alpha$ satisfying Definition 2, such that if each expert $k \in [K]$ solves a problem $f_k \in \mathcal{F}_\alpha$, then, for sufficiently small $\sqrt{\frac{K \log K}{MT}}$ and for any learning algorithm $\mathscr{A}_k$ and meta-procedure $\mathcal{P}$*

$$
\sup_{f_k \in \mathcal{F}_\alpha, \, k \in [K]} \mathbb{E} \operatorname{Reg}(T) \geq c_1 \sqrt{\frac{KT}{M}} \; + \; c_2 \, T^\alpha \left(\frac{K}{M} \right)^{1-\alpha},
$$

where $c_1, c_2 > 0$ are absolute constants.

We consider heavy-tailed multi-armed bandits [26] as base to build $\mathcal{F}_\alpha$. Each arm i provides rewards with mean μ_i and $(1 + \beta)$-moment bounded noise:

$$
\mathbb{E}_{X \sim \nu_i} |X - \mu_i|^{1+\beta} \leq u, \tag{12}
$$

for some $u > 0$ and $\beta \in (0, 1]$.

In our proof, as the canonical class $\mathcal{F}_\alpha$, we consider the *heavy-tailed multi-armed bandits* introduced by [26]. From their analysis, the following corollary holds:

Corollary 1 (from Bubeck et al., 2013, Thm. 2). *For a two-armed heavy-tailed bandit satisfying* (12)*, there exist distributions ν_1, ν_2 with $u = 1$ and gap $\ell_{h,1} - \ell_{h,2} = \Delta$ such that, for any algorithm and any horizon n,*

$$
R^{in}(n) \geq n\Delta \left(1 - c_\beta \sqrt{n \Delta^{\frac{1+\beta}{\beta}}} \right), \tag{13}
$$

where $c_\beta > 0$ depends only on β. For fixed n, optimizing Δ as

$$
\Delta = c_0 \, n^{-\frac{\beta}{1+\beta}}
$$

with sufficiently small $c_0 > 0$ yields

$$
R^{in}(n) \geq c' \, n^{\frac{1}{1+\beta}}, \tag{14}
$$

for some absolute constant $c' > 0$.

Hence each subproblem is α–regret lower bound with $R_{\text{in}}(n) \geq c \, n^\alpha$, $\alpha = \frac{1}{1+\beta} \in [0.5, 1)$.

D.2 Construction of the composite game

Let K be the number of experts, M the per-round optimization budget, and T the horizon. We construct K perturbed games together with one symmetric *null game*. The setup depends on two small parameters: $\Delta > 0$ (internal hardness) and $\varepsilon > 0$ (cross-expert separation), and we assume $\Delta \leq \varepsilon$.

Perturbed games. In the h-th perturbed game, expert h faces a two-armed bandit with means $\ell_{h,1} = \frac{1}{2} - \frac{\varepsilon}{2}$ and $\ell_{h,2} = \frac{1}{2} - \frac{\varepsilon}{2} - \Delta$, while any expert $h' \neq h$ faces $\ell_{h,1} = \frac{1}{2} + \frac{\varepsilon}{2}$ and $\ell_{h,2} = \frac{1}{2} + \frac{\varepsilon}{2} + \Delta$.

Thus the h-th expert is uniquely optimal in game h.

Null game. All experts face identical subproblems with $\ell_{h,1} = \frac{1}{2} + \frac{\varepsilon}{2}$ and $\ell_{h,2} = \frac{1}{2} + \frac{\varepsilon}{2} + \Delta$. Each subproblem is hard, i.e. with internal regret characterized by Corollary 1.

D.3 Step 1: Regret decomposition

At each round t, the learner selects a subset $S_t \subseteq [K]$ of at most M experts to update, and then chooses one expert $H_t \in S_t$ for prediction. Then algorithm suffer (pseudo) loss $\ell_t^{H_t} \in \{\ell_{H_t,1}, \ell_{H_t,2}\}$. Define the empirical frequencies

$$
\hat{q}_h = \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}\{H_t = h\}, \qquad J \sim \hat{q},
$$

where J is a random variable representing the expert index sampled according to $\hat{q}$. Denote by $\mathbb{P}_h$ the law of J under the h-th game, and let $\mathbb{E}_h[\cdot]$ denote expectations in that game. Then

$$
\mathbb{P}_h(J = h) \; = \; \mathbb{E}_h \left[\frac{1}{T} \sum_{t=1}^{T} \mathbf{1}\{H_t = h\} \right].
$$

Then the expected regret in game h can be bounded as follows.

$$R_h(T) := \mathbb{E}_h\left[\sum_{t=1}^{T}(\ell_t^{H_t} - (\tfrac{1}{2} - \tfrac{\varepsilon}{2} - \Delta))\right]$$

$$= \mathbb{E}_h\left[\sum_{t=1}^{T}\left[\mathbf{1}\{H_t = h\}\left(\ell_t^{H_t} - (\tfrac{1}{2} - \tfrac{\varepsilon}{2} - \Delta)\right) + \right.\right.$$

$$\left.\left. + \mathbf{1}\{H_t \neq h\}\left(\ell_t^{H_t} - (\tfrac{1}{2} - \tfrac{\varepsilon}{2} - \Delta)\right)\right]\right] \geq$$

$$= \varepsilon T \sum_{h' \neq h}\mathbb{P}_i(J = h') + \mathbb{E}_h\left[\sum_{t=1}^{T}\left(\ell_t^{H_t} - \ell_{H_t}^\star\right)\right]$$

$$= \varepsilon(1 - \mathbb{P}_h(J = h)) + \mathbb{E}_h\left[\sum_{t=1}^{T}\left(\ell_t^{H_t} - \ell_{H_t}^\star\right)\right],$$

The inequality is obtained using the add-subtract trick with $(\varepsilon + \Delta)$ in the second term.

Taking the supremum over games gives

$$\mathrm{Reg}(T) \geq \sup_h R_h(T) \geq T\left(1 - \frac{1}{K}\sum_{h=1}^{K}\mathbb{P}_h(J = h)\right) + \frac{1}{K}\sum_{h=1}^{K}\mathbb{E}_h\sum_{t=1}^{T}(\ell_t^{H_t} - \ell_{H_t}^\star). \tag{15}$$

We refer to the first term as the *identification term*, and to the second as the *internal regret term*, since grouping by arms reveals it as the sum of internal regrets of experts over their prediction rounds.

D.4   Step 2: Pinsker and null-game internal bound

Lemma 10 (Pinsker's inequality)**.** *For any h and event A,*

$$|\mathbb{P}_h(A) - \mathbb{P}_\emptyset(A)| \leq \sqrt{\tfrac{1}{2}\mathrm{KL}(\mathbb{P}_\emptyset\|\mathbb{P}_h)}.$$

In particular:

$$\mathbb{P}_h[J = h] \leq \mathbb{P}_\emptyset[J = h] + \sqrt{\tfrac{1}{2}\mathrm{KL}(\mathbb{P}_\emptyset\|\mathbb{P}_h)}$$

To bound the *identification term*, by the concavity of the square root we get:

$$\frac{1}{K}\sum_{h=1}^{K}\mathbb{P}_h[J = h] \leq \frac{1}{K} + \sqrt{\frac{1}{2K}\sum_{h=1}^{K}\mathrm{KL}(\mathbb{P}_\emptyset\|\mathbb{P}_h)}. \tag{16}$$

To bound the *internal regret term* we form the following Lemma:

Lemma 11 (Internal regret in perturbed games)**.** *Let $\alpha \in (0, 1]$ and suppose each expert's subproblem satisfies* (13). *Choose Δ as in* (21)*, i.e. $\Delta = c_0\left(\frac{K}{MT}\right)^{1-\alpha}$, with c_0 sufficiently small. If the parameters K, M, T are such that $\sqrt{\frac{K\log(8K)}{MT}}$ is sufficiently small, and for a perturbed game h the divergence satisfies $\mathrm{KL}(P_\emptyset\|P_h) \leq \frac{1}{2}$, then*

$$\mathbb{E}_h\left[\sum_{t=1}^{T}\left(\ell_t^{H_t} - \ell_{H_t}^\star\right)\right] \geq c''T^\alpha\left(\frac{K}{M}\right)^{1-\alpha},$$

for some constant $c'' > 0$ independent from M, K, T.

D.5   Step 3: KL computation for identification term

Lemma 12 (KL for T rounds)**.** *For each $h \in [K]$,*

$$\mathrm{KL}(P_\emptyset\|P_h) \leq \frac{36\,\varepsilon^2}{1 - 9\varepsilon^2}\,T.$$

Moreover,

$$\sum_{h=1}^{K}\mathrm{KL}(P_\emptyset\|P_h) \leq \frac{36\,\varepsilon^2}{1 - 9\varepsilon^2}\,MT.$$

D.6 Step 4: Putting all together

From (15) the regret splits into the *identification* and *internal* terms. For the identification term, (16) substituted into (15) and the KL bounds of Lemma 12 give

$$\sup_h R_h(T) \ \geq \ \varepsilon\, T \left(1 - \tfrac{1}{K} - c_{\mathrm{id}}\, \varepsilon \sqrt{\tfrac{M}{T}}\right),$$

with some $c_{\mathrm{id}} > 0$. Hence with the choice with sufficiently small $\gamma > 0$

$$\varepsilon \ = \ \gamma \sqrt{\tfrac{K}{MT}} \quad (\gamma > 0 \text{ small}), \tag{17}$$

we obtain

$$\sup_h R_h(T) \ \geq \ c_1 \sqrt{\tfrac{KT}{M}}. \tag{18}$$

For the internal term, by Lemma 11, if each subproblem is α–hard (Definition 2) and we choose

$$\Delta \ = \ c_0 \left(\tfrac{K}{MT}\right)^{1-\alpha} \quad (c_0 > 0 \text{ small}), \tag{19}$$

then

$$\frac{1}{K} \sum_{h=1}^{K} \mathbb{E}_h \left[\sum_{t=1}^{T} \left(\ell_t^{H_t} - \ell_{H_t}^\star\right)\right] \ \geq \ c_2\, T^\alpha \left(\tfrac{K}{M}\right)^{1-\alpha}. \tag{20}$$

Summing (18) and (20) inside (15) yields the final bound:

$$\mathrm{Reg}(T) \ \geq \ c_1 \sqrt{\tfrac{KT}{M}} \ + \ c_2\, T^\alpha \left(\tfrac{K}{M}\right)^{1-\alpha}.$$

Parameter check.   The choices (17)–(19) satisfy all required side conditions: (i) *concentration* holds as soon as $\sqrt{\tfrac{K \log(8K)}{MT}}$ is sufficiently small (Lemma 13); (ii) *KL control* follows from Lemma 12 with (17), yielding $\mathrm{KL}(P_\emptyset \| P_h) \leq \tfrac{1}{2}$ for all h when γ is small; (iii) the construction assumes $\Delta \leq \varepsilon$, which holds for large enough MT since

$$\Delta/\varepsilon = \tfrac{c_0}{\gamma}\, (K/MT)^{\frac{1}{2}-\alpha} \leq 1$$

whenever $\alpha \geq \tfrac{1}{2}$ and $c_0 \leq \gamma$. All constants c_1, c_2 depend only on α and the universal constants from the cited lemmas, and not on K, M, T.

D.7 Proofs of Lemmas for Lower Bounds

Corollary 1. The construction follows the proof of Theorem 2 in [26]. Let ν_1, ν_2 be the two heavy-tailed distributions defined therein, which satisfy the moment condition (12) with $u = 1$. By reduction to the Bernoulli case (see also [27, Theorem 2.6]), the expected regret satisfies

$$R_n \geq n\Delta \left(1 - \sqrt{n\mathrm{KL}(\nu_2\|\nu_1)}\right).$$

Using the bound $\mathrm{KL}(\nu_2\|\nu_1) \leq C_\beta \Delta^{\frac{1+\beta}{\beta}}$ for a constant $C_\beta > 0$ gives (13). Optimizing over Δ by setting $\Delta = c_0 n^{-\frac{\beta}{1+\beta}}$ and taking c_0 small enough makes the parenthesis positive, yielding (14). $\square$

D.7.1 Zero Game analysis

Lemma 13 (Concentration of update counts). *Consider null game. For any $\eta \in (0,1)$ there exists an absolute constant $C > 0$ such that, with*

$$\delta = C \sqrt{\frac{K \log(2K/\eta)}{MT}} \in (0,1),$$

the following holds with probability at least $1 - \eta$:

$$\left|n_i(T) - \tfrac{MT}{K}\right| \ \leq \ \delta \tfrac{MT}{K} \qquad \text{simultaneously for all } i \in [K].$$

Denote this event $\mathcal{E}_{conc}$.

Lemma 14 (Internal regret lower bound under concentration). *Assume that for each expert $i \in [K]$, the internal subproblem satisfies Equation 13 for some constants $c_\beta > 0$, $\beta \in (0,1]$, and any $\Delta > 0$. Let $\alpha = \frac{1}{1+\beta}$ and let $\mathcal{E}_{\text{conc}}$ denote the concentration event from Lemma 13. Then, on $\mathcal{E}_{\text{conc}}$, choosing*

$$\Delta = c_0\big((1+\delta)\tfrac{MT}{K}\big)^{-(1-\alpha)}, \qquad c_0 \leq \tfrac{1}{4c_\beta}, \tag{21}$$

we have

$$\sum_{t=1}^{T} \big(\ell_t^{H_t} - \ell_{H_t}^\star\big) \geq c' T^\alpha \big(\tfrac{K}{M}\big)^{1-\alpha}, \tag{22}$$

for some constant $c' > 0$ depending only on c_0 and α.

Lemma 13. We work under the probability distribution $\mathbb{P}_\emptyset$ induced by the randomization of the learner in the null game. To enforce symmetry even for deterministic algorithms, we assume that before the game begins, the K expert indices are randomly permuted. Hence, by symmetry, for every t and i,

$$p_t^{(i)} := \mathbb{E}_\emptyset[\mathbf{1}\{i \in S_t\} \mid \mathcal{F}_{t-1}] = \Pr_\emptyset(i \in S_t \mid \mathcal{F}_{t-1}) = \frac{M}{K},$$

and therefore $\sum_{t=1}^{T} p_t^{(i)} = MT/K$.

Define the martingale-difference sequence

$$X_t^{(i)} := \mathbf{1}\{i \in S_t\} - p_t^{(i)}, \qquad \mathbb{E}_\emptyset[X_t^{(i)} \mid \mathcal{F}_{t-1}] = 0, \quad |X_t^{(i)}| \leq 1.$$

Then

$$n_i(T) - \frac{MT}{K} = \sum_{t=1}^{T} X_t^{(i)} =: S_T^{(i)}.$$

Let $V_T^{(i)}$ be the predictable quadratic variation:

$$V_T^{(i)} = \sum_{t=1}^{T} \text{Var}_\emptyset(\mathbf{1}\{i \in S_t\} \mid \mathcal{F}_{t-1}) = \sum_{t=1}^{T} p_t^{(i)}(1 - p_t^{(i)}) \leq \sum_{t=1}^{T} p_t^{(i)} = \frac{MT}{K}.$$

Applying Freedman's inequality (martingale Bernstein bound), for any $u > 0$,

$$\mathbb{P}_\emptyset\Big(|S_T^{(i)}| \geq u\Big) \leq 2\exp\left(-\frac{u^2}{2(V_T^{(i)} + u/3)}\right).$$

Set $u = \delta\frac{MT}{K}$ with $\delta \in (0,1)$. Using $V_T^{(i)} \leq (M/K)T$ and $u \leq (MT/K)$ gives

$$\mathbb{P}_\emptyset\big(|n_i(T) - \tfrac{MT}{K}| \geq \delta\tfrac{MT}{K}\big) \ \leq \ 2\exp\big(-c\delta^2 \tfrac{M}{K}T\big)$$

for some absolute constant $c \in (0,1)$.

Finally, applying a union bound over all $i \in [K]$ yields

$$\mathbb{P}_\emptyset\Big(\max_i |n_i(T) - \tfrac{MT}{K}| \geq \delta\tfrac{MT}{K}\Big) \ \leq \ 2K\exp\big(-c\delta^2 \tfrac{M}{K}T\big).$$

Choosing

$$\delta = C\sqrt{\tfrac{K\log(2K/\eta)}{MT}}$$

with sufficiently large C ensures that the right-hand side is at most η. Hence the stated event $\mathcal{E}_{\text{conc}}$ holds with probability at least $1 - \eta$. $\qquad\square$

Lemma 14. Summing internal regret across experts and substituting this into (13),

$$\sum_{t=1}^{T}\sum_{h \in S_t}(\ell_t^h - \ell_h^\star) = \sum_{h=1}^{K} R_h^{\text{in}}\big(n_h(T)\big) \geq \Delta\sum_{h=1}^{K} n_h(T)\Big(1 - \sqrt{n_h(T)\Delta^{\frac{1+\beta}{\beta}}}\Big).$$

Since the played expert is uniform in S_t in the null game,

$$\sum_{t=1}^{T}(\ell_t^{H_t} - \ell_{H_t}^{\star}) = \frac{1}{M}\sum_{h=1}^{K} R_h^{\mathrm{in}}(n_h(T)) \geq \frac{1}{M}\Delta\sum_{h=1}^{K} n_h(T)\Big(1 - \sqrt{n_h(T)\Delta^{\frac{1+\beta}{\beta}}}\Big).$$

Under $\mathcal{E}_{\mathrm{conc}}$, all counts satisfy $n_h(T) \in [(1-\delta)\frac{MT}{K}, (1+\delta)\frac{MT}{K}]$ and $\sum_h n_h(T) = MT$, hence

$$\sum_{t=1}^{T}(\ell_t^{H_t} - \ell_{H_t}^{\star}) \geq \sum_{h=1}^{K} R_h^{\mathrm{in}}(n_h(T)) \geq T\Delta\Big(1 - c_\beta\sqrt{(1+\delta)\frac{MT}{K}\Delta^{\frac{1+\beta}{\beta}}}\Big).$$

Choosing Δ as in (21) ensures that $c_\beta\sqrt{(1+\delta)\frac{MT}{K}\Delta^{\frac{1+\beta}{\beta}}} \leq \frac{1}{2}$, hence

$$\sum_{t=1}^{T}(\ell_t^{H_t} - \ell_{H_t}^{\star}) \geq \tfrac{1}{2}T\Delta = c'T^\alpha\big(\tfrac{K}{M}\big)^{1-\alpha},$$

which yields (22). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 11. Specify parameters for Lemma 13. Take $\eta = \frac{1}{4}$ and let K, M, T such that $\delta = C\sqrt{\frac{K\log(2K/\eta)}{MT}} \leq 1/2$. Then in zero game $\mathcal{E}_{\mathrm{conc}} = \{\forall\, h \in [K] : |n_h(T) - \frac{MT}{K}| \leq \frac{1}{2}\frac{MT}{K}\}$ is satisfied with probability $\geq \frac{3}{4}$.

Since $\mathrm{KL}(P_\emptyset\|P_h) \leq 1/2$, By Pinsker's inequality,

$$\big|\mathbb{P}_h(\mathcal{E}_{\mathrm{conc}}) - \mathbb{P}_\emptyset(\mathcal{E}_{\mathrm{conc}})\big| \leq \sqrt{\tfrac{1}{2}\mathrm{KL}(P_\emptyset\|P_h)} \leq \sqrt{\kappa/2} = 1/2,$$

hence $\mathbb{P}_h(\mathcal{E}_{\mathrm{conc}}) \geq 1 - 1/4 - 1/2 = 1/4$. On $\mathcal{E}_{\mathrm{conc}}$, Lemma 14 yields the realized regret bound

$$\sum_{t=1}^{T}(\ell_t^{H_t} - \ell_{H_t}^{\star}) \geq c'T^\alpha\big(\tfrac{K}{M}\big)^{1-\alpha}.$$

Taking expectations under $\mathbb{P}_h$ gives

$$\mathbb{E}_h\Bigg[\sum_{t=1}^{T}(\ell_t^{H_t} - \ell_{H_t}^{\star})\Bigg] \geq c'T^\alpha\big(\tfrac{K}{M}\big)^{1-\alpha}\mathbb{P}_h(\mathcal{E}_{\mathrm{conc}}) \geq (1/4)c'T^\alpha\big(\tfrac{K}{M}\big)^{1-\alpha}.$$

And constants adsorm into c'' $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

D.7.2   KL computation

Lemma 12. The proof follows [12], and provided here for completeness. The only change is the KL bounding, since in our setup on each arm the not a fixed Bernoulli distribution is specified, but a mixture of distributions. By the data-processing inequality, $\mathrm{KL}(P_\emptyset\|P_h) \leq \mathrm{KL}(\tilde{P}_\emptyset^T\|\tilde{P}_h^T)$, so it suffices to bound the latter. Using the chain rule for KL divergence,

$$\mathrm{KL}(\tilde{\mathbb{P}}_\emptyset^T\|\tilde{\mathbb{P}}_h^T) = \sum_{t=1}^{T}\sum_{o_1^{t-1}} \tilde{\mathbb{P}}_\emptyset^{t-1}(o_1^{t-1})\mathrm{KL}\Big(\tilde{\mathbb{P}}_\emptyset^t(\cdot \mid o_1^{t-1})\Big\|\tilde{\mathbb{P}}_h^t(\cdot \mid o_1^{t-1})\Big) =$$

$$= \sum_{t=1}^{T}\sum_{o_1^{t-1}} \tilde{\mathbb{P}}_\emptyset^{t-1}(o_1^{t-1})\mathbf{1}\{h \in O_t \mid o_1^{t-1}\}\mathrm{KL}\Big(\tilde{\mathbb{P}}_\emptyset^t(\cdot \mid o_1^{t-1})\Big\|\tilde{\mathbb{P}}_h^t(\cdot \mid o_1^{t-1})\Big) \leq$$

$$\leq \frac{6\varepsilon^2}{1-\varepsilon^2}E_\emptyset\Bigg[\sum_{t=1}^{T}\mathbf{1}\{h \in O_t\}\Bigg]$$

The Inequality is from fact, that each arm at moment t has a bernoulli distribution. h in h game is with parameter $p_1 \in \left[\frac{1}{2} + \frac{\varepsilon}{2}, \frac{1}{2} + \frac{3\varepsilon}{2}\right]$. And h in zero game with parameter $p \in \left[\frac{1}{2} - \frac{3\varepsilon}{2}, \frac{1}{2} - \frac{\varepsilon}{2}\right]$. Then, by the standard quadratic upper bound on Bernoulli KL divergence,

$$\mathrm{KL}(\mathrm{Ber}(p)\|\mathrm{Ber}(p_1)) \leq \frac{(p-p_1)^2}{p_1(1-p_1)} \leq \frac{36\varepsilon^2}{1-9\varepsilon^2}.$$

To obtain the total bound, we sum over $h \in [K]$:

$$\sum_{h=1}^{K} \mathrm{KL}(P_\emptyset \| P_h) \leq \frac{36\varepsilon^2}{1 - 9\varepsilon^2} \mathbb{E}_\emptyset \left[\sum_{t=1}^{T} \sum_{h=1}^{K} \mathbf{1}\{h \in O_t\} \right].$$

The first inequality then follows from the fact, that each of the arm is selected no more than T times. At each round t, at most M experts are observed, i.e. $\sum_{h=1}^{K} \mathbf{1}\{h \in O_t\} \leq M$. Hence

$$\sum_{h=1}^{K} \mathrm{KL}(P_\emptyset \| P_h) \leq \frac{36\varepsilon^2}{1 - 9\varepsilon^2} \varepsilon^2 M T,$$

which completes the proof. $\square$