
COMPUTATIONAL METHODS AND VERIFICATION THEOREM FOR PORTFOLIO-CONSUMPTION OPTIMIZATION UNDER EXPONENTIAL O-U DYNAMICS

Zhaoxiang Zhong*
Faculty of Science
National University of Singapore
e1349203@u.nus.edu

Haiming Song
School of Mathematics
Jilin University
songhaiming@jlu.edu.cn

August 11, 2025

ABSTRACT

In this paper, we focus on the problem of optimal portfolio-consumption policies in a multi-asset financial market, where the n risky assets follow Exponential Ornstein-Uhlenbeck processes, along with one risk-free bond. The investor's preferences are modeled using Constant Relative Risk Aversion utility with state-dependent stochastic discounting. The problem can be formulated as a high-dimensional stochastic optimal control problem, wherein the associated value function satisfies a Hamilton-Jacobi-Bellman (HJB) equation, which constitutes a necessary condition for optimality. We apply a variable separation technique to transform the HJB equation to a system of ordinary differential equations (ODEs). Then a class of hybrid numerical approaches that integrate exponential Rosenbrock-type methods with Runge-Kutta methods is proposed to solve the ODE system. More importantly, we establish a rigorous verification theorem that provides sufficient conditions for the existence of value function and admissible optimal control, which can be verified numerically. A series of experiments are performed, demonstrating that our proposed method outperforms the conventional grid-based method in both accuracy and computational cost. Furthermore, the numerically derived optimal policy achieves superior performance over all other considered admissible policies.

1 Introduction

Optimal portfolio-consumption problem is one of the most important topics in mathematical finance. Merton [23] proposes the fundamental stochastic control model to address this problem, where the stock price follows geometric Brownian motion. His work has attracted significant attention, leading to the development of numerous models and approaches aimed at adapting to the stochastic nature of real-world markets. In particular, various mean-reverting models are intro-

*This preprint has been submitted to *European Journal of Operational Research (EJOR)*. Please note that this version is the original manuscript and does not represent the final published article. Minor reformatting and figure adjustments have been made compared to the version submitted to *EJOR*.

duced to the problem, driven by the discovery of mean-reverting behavior in asset prices [5, 8, 27], such as Ornstein-Uhlenbeck (O-U) model [34], exponential O-U model [1] and Cox-Ingersoll-Ross model [16]. In this paper, we will consider exponential O-U model, which effectively describes and simulates the general mean-reverting dynamics over time of certain asset prices.

A variety of utility functions have been integrated into the portfolio-consumption optimization framework, among which the Constant Absolute Risk Aversion (CARA) utility [20, 31] and the Constant Relative Risk Aversion (CRRA) utility [15, 24] are the most prevalent ones. The CARA framework has been widely studied for its analytical tractability; however, it assumes constant absolute risk aversion, which implies a fixed allocation to risky assets regardless of wealth—an assumption that contradicts empirical evidence showing wealth-dependent risk preferences. In contrast, CRRA utility provides a more realistic framework, as it captures how investment behavior varies with changes in wealth [35]. Moreover, recent studies have emphasized CRRA utility formulations featuring stochastic discounting, including non-exponential discounting [7], state-dependent discounting [16], and regime-switching discounting [26]. The state-dependent discounting setting, in which the discount rate explicitly depends on the underlying stock price dynamics, allows the model to better capture the changing risk preferences of investors in financial markets. Overall, a model that incorporates exponential O-U dynamics and CRRA utility with state-dependent stochastic discounting provides a more realistic and flexible framework for addressing real-world financial optimization problems.

In terms of solutions, researchers have proposed a wide range of techniques to analyze optimal portfolio-consumption problems. Early contribution by Karatzas [13] utilizes martingale methods to derive weak solutions. With the introduction of Hamilton-Jacobi-Bellman (HJB) equations, many studies have reformulated the problem into a HJB-kind nonlinear partial differential equation (PDE) and derived closed-form solutions in 1-dimensional case [17, 25, 38]. To tackle more general cases, grid-based numerical schemes have been introduced, including Markov chain approximation combined with logarithmic transformation [36], fixed-point iterative methods [2], and hybrid Monte Carlo with finite difference approaches [32]. However, multi-asset problems involving CRRA utility functions give rise to nonlinear high-dimensional HJB equations that are typically intractable for closed-form solutions. Moreover, conventional grid-based numerical methods suffer from the curse of dimensionality when applied to such problems. To address this computational challenge, various numerical methods have been proposed, such as sparse grid methods [30], Hopf formulas [6], and deep learning methods [19]. These works provide a good approximation of the high-dimensional value function, however, to the best of our knowledge, their numerical implementation can often be complex and lack rigorous verification theorems to ensure optimality.

In this paper, we also adopt the framework of deriving the HJB equation for the optimal control problem. Then we apply a variable separation technique to transform the high-dimensional HJB equation into an ODE system, significantly reducing computational complexity. To solve the resulting ODEs numerically, we develop hybrid numerical approaches that integrate exponential Rosenbrock-type methods [11, 18] with Runge-Kutta methods, which demonstrate superior accuracy and efficiency. Most importantly, we derive a rigorous verification theorem to validate the optimality of our approach, which is an aspect often overlooked in previous numerical methods.

The remainder of this paper is organized as follows: In Section 2, we start with introducing the corresponding modeling framework of our paper. In Section 3, the HJB equation corresponding to the stochastic optimal control problem is derived, along with the form of the optimal policy. Then a variable separation technique is applied to transform the HJB equation into an ODE system. In

Section 4, we introduce the numerical methods to solve the ODE system and present a convergence theorem. A rigorous verification theorem is derived in Section 5. In Section 6, we conduct a series of numerical experiments to evaluate the performance of our algorithms. Finally, we summarize our findings and present conclusions in Section 7.

2 Problem formulation

In this section, we introduce the modeling framework and formulate the stochastic optimal control problem including the constraint equations and the optimization objective.

2.1 Constraint Equations

In our paper, we extend the assumptions of the financial market model in [1] to the n -dimensional setting.

Assumption 2.1. *The financial market under consideration is assumed to be arbitrage-free over a fixed time horizon $[0, T]$, where $0 \leq T < \infty$. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0 \leq t \leq T})$ be a complete filtered probability space satisfying usual conditions, and $\left\{ B_t = \left(B_t^{(i)} \right)_{1 \leq i \leq n}^\top, 0 \leq t \leq T \right\}$ denote the n -dimensional Brownian motion defined on it. The market consists of $n+1$ tradable assets: one risk-free bond and n risky stocks. Within this framework, a self-financed small investor who allocates wealth across these assets is considered.*

Assumption 2.2. *The price $P_0(t)$ of bond is governed by following ODE:*

$$\begin{cases} dP_0(t) = rP_0(t) dt, & 0 < t \leq T, \\ P_0(0) = p_0, \end{cases}$$

where $p_0 > 0$ is the initial price, $r > 0$ is the constant interest rate. The price $P_i(t)$ of i th ($1 \leq i \leq n$) stock follows exponential O-U process

$$\begin{cases} dP_i(t) = \alpha_i(\mu_i - \ln P_i(t))P_i(t) dt + \sigma_i P_i(t) dB_t, & 0 < t \leq T, \\ P_i(0) = p_i, \end{cases}$$

where $p_i > 0$ is the initial price, $\alpha_i, \mu_i > 0$ are constants, $\sigma = (\sigma_{ij})_{1 \leq i, j \leq n}$ is the non-degenerate volatility matrix whose i th row is vector σ_i and entries are all positive.

Assumption 2.3. *A portfolio-consumption pair $(\pi(t), C(t))$ is defined, where $\pi(t) = (\pi_i(t))_{1 \leq i \leq n}^\top$ denotes the allocation of wealth across the n stocks, with $\pi_i(t)$ representing the amount invested in the i th stock at time t . $C(t)$ denotes the consumption rate at time t . In addition, $\pi_0(t)$ represents the amount of wealth invested in the bond at time t . Assume $\pi(t)$ and $C(t)$ are measurable and adapted to the filtration $\{\mathcal{F}_t\}$, with $C(t) \in [0, +\infty)$, $\pi_i(t) \in \mathbb{R}$, for all $t \in [0, T]$. Note that for $i = 0, 1, \dots, n$, $\pi_i(t)$ may take negative values, indicating short-selling of the corresponding bond or stock.*

Remark 2.4. *For analytical convenience, we let $S(t) = (S_i(t))_{1 \leq i \leq n}^\top$, where $S_i(t) = \ln P_i(t)$. By Ito equation [13], we have*

$$\begin{cases} dS(t) = \text{diag}(\alpha)(w - S(t)) dt + \sigma dB_t, & 0 < t \leq T, \\ S(0) = S_0, \end{cases} \quad (2.1)$$

where

$$\text{diag}(\alpha) = \begin{pmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n \end{pmatrix}, \quad w = \left(\mu_i + \frac{\|\sigma_i\|^2}{2\alpha_i} \right)_{1 \leq i \leq n}^\top.$$

Let $X(t)$ denotes the wealth at time $t \in [0, T]$. The amount of wealth allocated to the bond is then given by $\pi_0(t) \equiv X(t) - \sum_{i=1}^n \pi_i(t)$. Building on equation (2.1) and Ito equation, we can readily derive that $X(t)$ has following dynamics:

$$\begin{cases} dX(t) = rX(t) dt + \pi^\top(t)(\text{diag}(\alpha)(\mu - S(t)) - re) dt - C(t) dt + \pi^\top(t)\sigma dB_t, & 0 < t \leq T, \\ X(0) = X_0, \end{cases} \quad (2.2)$$

where e is n -dimensional column vector of ones and $\mu = (\mu_i)_{1 \leq i \leq n}^\top$.

2.2 Optimization Objective

To facilitate, define portfolio process $\pi \triangleq \{\pi(t), 0 \leq t \leq T\}$, consumption process $C \triangleq \{C(t), 0 \leq t \leq T\}$ and corresponding wealth process $X \triangleq \{X(t), 0 \leq t \leq T\}$. An optimization objective functional is stated as

$$J(\pi, C) \triangleq \mathbb{E} \left[\int_0^T U(t, C(t)) dt + U(T, X(T)) \right]. \quad (2.3)$$

The utility function $U(t, c)$ represents the CRRA utility with a stochastic discount factor that depends on the stock price. Specifically,

$$U(t, c) \triangleq \gamma^{-1} \psi(t)^{1-\gamma} c^\gamma, \quad \gamma \in (0, 1),$$

where

$$\psi(t) \triangleq \exp \left\{ -\frac{1}{1-\gamma} \int_0^t [\rho_0 + S(u)^\top \rho + S(u)^\top \varrho S(u)] du \right\}, \quad \rho_0 \in \mathbb{R}, \quad \rho \in \mathbb{R}^n, \quad \varrho \in \mathbb{S}^n. \quad (2.4)$$

Here, \mathbb{S}^n denotes the set of all symmetric $n \times n$ matrices. Next we introduce the admissible set.

Definition 2.5. A portfolio-consumption pair process (π, C) is said to be admissible for any initial wealth X_0 , if and only if (π, C) satisfy following conditions:

- (a) $\mathbb{E} \left[\int_0^T C(t) dt \right] < +\infty$,
- (b) $\mathbb{E} \left[\int_0^T \|\pi(t)\|^2 dt \right] < +\infty$,
- (c) $\mathbb{E} [|J(\pi, C)|] < +\infty$,
- (d) $X(t) \geq 0, \quad \forall t \in [0, T]$.

Here, $\|\cdot\|$ denotes the Frobenius norm, and we will adopt this notation throughout. The class of all such pairs is denoted by \mathcal{A}_{X_0} .

Remark 2.6. Under the conditions (a)-(b) in Definition 2.5, it guarantees that SDE (2.2) has unique strong solution

$$\begin{aligned} X(t) = \exp\{rt\} \left\{ X_0 + \int_0^t \exp\{-ru\} [\pi(u)^\top (\text{diag}(\alpha)(\mu - S(u)) - re) - C(u)] du \right. \\ \left. + \int_0^t \exp\{-ru\} \pi(u)^\top \sigma dB_u \right\}. \end{aligned} \quad (2.5)$$

Our aim is to find the optimal portfolio-consumption pair process (π^*, C^*) within \mathcal{A}_{X_0} that maximizes (2.3) subject to the constraint (2.2).

3 HJB Equation and Corresponding ODE System

In this section, we derive the HJB equation corresponding to stochastic optimal control problem (2.2)-(2.3). We also present the optimal policy, which can be expressed in a feedback form through the value function. Moreover, a variable separation technique [1, 16] including power transformation and Feynman-Kac representation formula [14] is applied to transform the HJB equation to an ODE system.

3.1 HJB Equation and Optimal Policy

To solve the optimal control problem in a continuous sense, value function is defined as the maximum utility attainable starting from time t , wealth $X(t) \equiv x$, logarithm stock price $S(t) \equiv S$ and discount factor $\psi(t) \equiv \psi$. Specifically,

$$V(t, x, S, \psi) \triangleq \sup_{(\pi, C) \in \mathcal{A}_x} \mathbb{E}^{(t, x, S, \psi)} \left[\int_t^T U(u, C(u)) du + U(T, X(T)) \right]. \quad (3.1)$$

Here $\mathbb{E}^{(t, x, S, \psi)}$ denotes the expectation from initial state (t, x, S, ψ) . Suppose that $V(t, x, S, \psi) \in C^{1,2}([0, T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_+)$, where \mathbb{R}_+ denotes the set of all positive real numbers and $C^{1,2}$ denotes the class of functions that are once continuously differentiable in time and twice continuously differentiable in all state variables. Then V is a classical solution to the HJB equation (see [37, Proposition 3.5 of Chapter 4]), which, in the context of our problem, is given as

$$\frac{\partial v}{\partial t} + rx \frac{\partial v}{\partial x} + \beta \psi \frac{\partial v}{\partial \psi} + \mathcal{L}_S v + \sup_{(\pi, C) \in \mathcal{A}_x} \{ \mathcal{G}_{xS\psi} v(\pi, C) \} = 0, \quad (3.2)$$

where

$$\begin{aligned} \mathcal{L}_S v &= (w^\top - S^\top) \text{diag}(\alpha) D_S v + \frac{1}{2} \text{tr} \{ D_{SS}^2 v \sigma \sigma^\top \}, \\ \mathcal{G}_{xS\psi} v(\pi, C) &= \gamma^{-1} \psi^{1-\gamma} C^\gamma + \frac{\partial v}{\partial x} \pi^\top [\text{diag}(\alpha)(\mu - S) - re] \\ &\quad - \frac{\partial v}{\partial x} C + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} \pi^\top \sigma \sigma^\top \pi + \pi^\top \sigma \sigma^\top D_{xS}^2 v, \end{aligned}$$

and

$$D_S v = \left(\frac{\partial v}{\partial S_i} \right)_{1 \leq i \leq n}^\top, \quad D_{SS}^2 v = \left(\frac{\partial^2 v}{\partial S_i \partial S_j} \right)_{1 \leq i, j \leq n}, \quad \beta = -\frac{1}{1-\gamma} [\rho_0 + S^\top \rho + S^\top \varrho S],$$

with terminal condition

$$v(T, x, S, \psi) = U(T, x). \quad (3.3)$$

We denote the solution to the HJB equation by v to distinguish it from the value function V , as the solution may represent a broader class of functions that includes V . HJB equation (3.2)-(3.3) constitutes as a necessary condition for value function. A rigorous verification theorem, which provides sufficient conditions under which a candidate function v coincides with the value function V , will be derived in Section 4. When both the necessary and sufficient conditions are fulfilled, the candidate function v is indeed the value function V .

As demonstrated in Chapter 4 of [37], if $V \in C^{1,2}([0, T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$ and the sufficient conditions for optimality are satisfied, solving for the supremum yields optimal policy (π^*, C^*) .

Therefore, we now focus on solving PDE (3.2)-(3.3) to obtain a candidate solution. To reduce the complexity of the HJB equation, we consider applying a variable separation technique. Following the idea in [38], equation (3.3) motivates the following power transformation ansatz:

$$v(t, x, S, \psi) = \gamma^{-1} \psi^{1-\gamma} x^\gamma \varphi^{1-\gamma}(t, S), \quad (3.4)$$

where $\varphi(t, S) \in C^{1,2}([0, T] \times \mathbb{R}^n)$. By substituting formula (3.4) into equation (3.2)-(3.3) and solving the supremum in (3.2), we obtain the PDE satisfied by $\varphi(t, S)$ as

$$\begin{aligned} & \frac{\partial \varphi}{\partial t} + (w^\top - S^\top) \text{diag}(\alpha) D_S \varphi + \frac{\gamma}{1-\gamma} [(\mu^\top - S^\top) \text{diag}(\alpha) - re^\top] D_S \varphi \\ & + \left\{ \frac{r\gamma}{1-\gamma} + \beta + \frac{\gamma [(\mu^\top - S^\top) \text{diag}(\alpha) - re^\top] (\sigma \sigma^\top)^{-1} [\text{diag}(\alpha)(\mu - S) - re]}{2(1-\gamma)^2} \right\} \varphi \\ & + \frac{1}{2} \text{tr} \{ D_{SS}^2 \varphi \sigma \sigma^\top \} + 1 = 0, \end{aligned} \quad (3.5)$$

with terminal condition

$$\varphi(T, S) = 1. \quad (3.6)$$

In the next step, we only have to solve PDE (3.5)-(3.6).

Remark 3.1. *Following the approach of [1], once PDE (3.5)-(3.6) has been solved and the sufficient conditions for optimality are satisfied, one can derive the optimal policy in feedback form:*

$$\begin{aligned} \pi^*(t, X^*(t), S(t)) &= \frac{1}{1-\gamma} X^*(t) (\sigma \sigma^\top)^{-1} [\text{diag}(\alpha)(\mu - S(t)) - re] \\ &+ X^*(t) D_S \varphi(t, S(t)) \varphi^{-1}(t, S(t)), \end{aligned} \quad (3.7)$$

$$C^*(t, X^*(t), S(t)) = \varphi^{-1}(t, S(t)) X^*(t). \quad (3.8)$$

3.2 ODE System

To derive an explicit form for φ , we adopt the change of measure approach introduced in [1]. Reviewing (2.2), if Novikov condition

$$\mathbb{E} \left[\exp \left\{ \frac{1}{2} \int_0^T \frac{\gamma^2}{(1-\gamma)^2} \|\sigma^{-1}(\text{diag}(\alpha)(\mu - S(t)) - re)\|^2 dt \right\} \right] < +\infty \quad (3.9)$$

is satisfied, by Gisanov Theorem [14], we can define a new probability measure $\bar{\mathbb{P}}$ under which the process $S(t)$ satisfies

$$dS(t) = \text{diag}(\alpha)(w - S(t)) dt + \frac{\gamma}{1-\gamma} [\text{diag}(\alpha)(\mu - S(t)) - re] dt + \sigma d\bar{B}_t. \quad (3.10)$$

We note that condition (3.9) is essential as it will be incorporated into the verification theorem in Section 4. Assume condition (3.9) holds, we define

$$H(S) = \frac{r\gamma}{1-\gamma} + \beta + \frac{\gamma [(\mu^\top - S^\top) \text{diag}(\alpha) - re^\top] (\sigma \sigma^\top)^{-1} [\text{diag}(\alpha)(\mu - S) - re]}{2(1-\gamma)^2}.$$

Based on equation (3.10), by Feynman-Kac representation Formula, we obtain expectation representation of φ as

$$\begin{aligned} \varphi(t, S) &= \bar{\mathbb{E}}^{(t, x, S, \psi)} \left[\exp \left\{ \int_t^T H(S(\tau)) d\tau \right\} + \int_t^T \exp \left\{ \int_t^u H(S(\tau)) d\tau \right\} du \right] \\ &\triangleq \varphi_1(t, S) + \varphi_2(t, S). \end{aligned} \quad (3.11)$$

Subsequently, we only have to solve φ_1, φ_2 to obtain φ .

Lemma 3.2. $\varphi_2(t, S)$ can be written as

$$\varphi_2(t, S) = \int_t^T \varphi_1(u, S) du. \quad (3.12)$$

Proof. First, we define $\tilde{\varphi}_1(t, \tau, S)$ as the function $\varphi_1(t, S)$ with the fixed terminal time T replaced by a variable $\tau \in (t, T]$. We can derive that

$$\begin{aligned} \varphi_2(t, S) &= \bar{\mathbb{E}}^{(t, x, S, \psi)} \left[\int_t^T \exp \left\{ \int_t^\tau H(S(u)) du \right\} d\tau \right] \\ &= \int_t^T \bar{\mathbb{E}}^{(t, x, S, \psi)} \left[\exp \left\{ \int_t^\tau H(S(u)) du \right\} \right] d\tau \\ &= \int_t^T \tilde{\varphi}_1(t, \tau, S) d\tau. \end{aligned} \quad (3.13)$$

Since both the drift and diffusion terms in SDE (2.1), as well as the function H , are independent of time t , the system $\tilde{\varphi}_1$ is time-consistent. Hence, there exists a function $\phi(\tau - t, S)$ satisfying

$$\phi(\tau - t, S) = \tilde{\varphi}_1(t, \tau, S).$$

By equation (3.13) and setting $T - u = \tau - t$, we have

$$\varphi_2(t, S) = \int_t^T \phi(\tau - t, S) d\tau = \int_t^T \phi(T - u, S) du = \int_t^T \tilde{\varphi}_1(u, T, S) du = \int_t^T \varphi_1(u, S) du.$$

□

Now we only have to solve $\varphi_1(t, S)$. By Feynman-Kac Formula, the conditional expectation representation of φ_1 can be equivalently formulated as the following PDE:

$$\begin{aligned} \frac{\partial \varphi_1}{\partial t} &+ (w^\top - S^\top) \text{diag}(\alpha) D_S \varphi_1 + \frac{\gamma}{1 - \gamma} [(\mu^\top - S^\top) \text{diag}(\alpha) - re^\top] D_S \varphi_1 \\ &+ \left\{ \frac{\gamma [(\mu^\top - S^\top) \text{diag}(\alpha) - re^\top] (\sigma \sigma^\top)^{-1} [\text{diag}(\alpha)(\mu - S) - re]}{2(1 - \gamma)^2} \right. \\ &\left. + \frac{1}{1 - \gamma} (r\gamma - \rho_0 - S^\top \rho - S^\top \varrho S) \right\} \varphi_1 + \frac{1}{2} \text{tr} \left\{ D_{SS}^2 \varphi_1 \sigma \sigma^\top \right\} = 0, \end{aligned} \quad (3.14)$$

with terminal condition

$$\varphi_1(T, S) = 1.$$

Note that $H(S)$ is a quadratic function of S , we assume a solution ansatz for $\varphi_1(t, S)$ of this form [1]:

$$\varphi_1(t, S) = \exp \{ S^\top g(t) S + S^\top f(t) + f_0(t) \}. \quad (3.15)$$

Here

$$g(t) = (g_{ij}(t))_{n \times n} : [0, T] \rightarrow \mathbb{R}^{n \times n}, \quad f(t) = (f_i(t))_{1 \leq i \leq n}^\top : [0, T] \rightarrow \mathbb{R}^n, \quad f_0(t) : [0, T] \rightarrow \mathbb{R}.$$

We can always rewrite $g(t)$ as $g_s(t) + g_{ns}(t)$, where

$$g_s(t) = \frac{g(t) + g(t)^\top}{2}, \quad g_{ns}(t) = \frac{g(t) - g(t)^\top}{2}.$$

Since $S^\top g_{ns}(t)S = 0$, the non-symmetric part contributes nothing to quadratic form $S^\top g(t)S$. Hence it's quite natural to assume that $g(t)$ is symmetric. Combining equation (3.14) and (3.15), and with variable separation, we have

$$g'(t) = \frac{1}{1-\gamma} \text{diag}(\alpha)g(t) + \frac{1}{1-\gamma} g(t) \text{diag}(\alpha) - 2g(t)\sigma\sigma^\top g(t) - \frac{\gamma \text{diag}(\alpha)(\sigma\sigma^\top)^{-1} \text{diag}(\alpha)}{2(1-\gamma)^2} + \frac{1}{1-\gamma} \varrho, \quad (3.16a)$$

$$f'(t) = \frac{1}{1-\gamma} \text{diag}(\alpha)f(t) - 2g(t)\sigma\sigma^\top f(t) - 2g(t) \text{diag}(\alpha)w - \frac{2\gamma}{1-\gamma} g(t) [\text{diag}(\alpha)\mu - re] + \frac{\gamma}{(1-\gamma)^2} \text{diag}(\alpha)(\sigma\sigma^\top)^{-1} [\text{diag}(\alpha)\mu - re] + \frac{1}{1-\gamma} \rho, \quad (3.16b)$$

$$f_0'(t) = -w^\top \text{diag}(\alpha)f(t) - \frac{\gamma}{1-\gamma} [\mu^\top \text{diag}(\alpha) - re^\top] f(t) - \frac{1}{2} f^\top(t) \sigma\sigma^\top f(t) - \text{tr} \{g(t)\sigma\sigma^\top\} - \frac{\gamma [\mu^\top \text{diag}(\alpha) - re^\top] (\sigma\sigma^\top)^{-1} [\text{diag}(\alpha)\mu - re]}{2(1-\gamma)^2} - \frac{r\gamma - \rho_0}{1-\gamma}, \quad (3.16c)$$

with terminal conditions

$$g(T) = 0_{n \times n}, \quad f(T) = 0_{n \times 1}, \quad f_0(T) = 0.$$

This reduces the problem to solving the ODE system (3.16a)-(3.16c).

Lemma 3.3. *Under Assumptions 2.1, 2.2 and 2.3, $g(t)$, $f(t)$, $f_0(t)$ are uniformly bounded in Frobenius norm on $[0, T]$.*

It is straightforward to verify that Lemma 3.3 follows naturally. Upon substituting $t = T - t$, the function $g(t)$ is characterized by a finite, quadratically decreasing term alongside a linearly increasing term. Applying Gronwall's lemma [10], we conclude that $g(t)$ remains uniformly bounded in Frobenius norm over the compact interval $[0, T]$. Similarly, we know that the functions $f(t)$ and $f_0(t)$ are likewise uniformly bounded in Frobenius norm on $[0, T]$.

Remark 3.4. *Once g , f and f_0 are determined, φ_1 and φ can be obtained subsequently. Based on Remark 3.1, this ultimately allows us to derive the optimal policy.*

4 Numerical Methods for Solving ODE System

In this section, we focus on solving the ODE system (3.16a)-(3.16c). We present hybrid numerical approaches that integrate exponential Rosenbrock-type methods [11, 18] with Runge-Kutta methods to numerically solve (3.16a)-(3.16c). Moreover, the convergence theorem is established.

We observe that (3.16a) conforms to the class of symmetric matrix Riccati equations discussed in [18]. It presents two numerical methods: ExpEuler, which is second-order, and Erow3, which is third-order. These two methods are particularly well-suited for large stiff ODEs like (3.16a), outperforming standard Runge-Kutta methods in such scenarios. Therefore, we employ ExpEuler/Erow3 to solve (3.16a). Subsequently, we apply Runge-Kutta methods to solve equations (3.16b) and (3.16c), as they are well-posed linear ODEs. To preserve the desired order of convergence, we employ second-order and third-order Runge-Kutta schemes, respectively. For simplicity, we denote the ExpEuler method combined with the RK2 scheme by ExpEuler-RK2, and the Erow3 method combined with the RK3 scheme by Erow3-RK3.

To facilitate a clearer presentation of our algorithms, we denote right-hand side of (3.16a), (3.16b), and (3.16c) by $G(g(t))$, $F(g(t), f(t))$, and $F_0(g(t), f(t))$, respectively. And for the sake of completeness in the algorithmic framework, we first revisit the numerical schemes of ExpEuler and Erow3, as introduced in [18]. Let h be the time discretization step size and $t_k = hk$. For a given point g_k in the state space, (3.16a) is rewritten as

$$g'(t) = \mathcal{S}_k(g) + \mathcal{G}_k(g),$$

where $\mathcal{S}_k(g)$ denotes the Frechet derivative of $G(g)$ at g_k and $\mathcal{G}_k(g)$ denotes the nonlinear remainder $G(g) - \mathcal{S}_k(g)$. Furthermore, exponential integrator is defined as follows:

$$\mathcal{Y}_j(z) = \int_0^1 \exp\{(1-\theta)z\} \frac{\theta^{j-1}}{(j-1)!} d\theta.$$

Then denote the second-order and third-order approximations of $g(t_k)$ by $g_k^{(2)}$ and $g_k^{(3)}$, respectively. Li et al. [18] provide following second-order ExpEuler scheme

$$g_{k+1}^{(2)} = g_k^{(2)} + h\mathcal{Y}_1(h\mathcal{S}_k)(G(g_k^{(2)})),$$

and third-order Erow3 scheme

$$\begin{aligned} \hat{g}_{k+1} &= g_k^{(3)} + h\mathcal{Y}_1(h\mathcal{S}_k)(G(g_k^{(3)})), \\ g_{k+1}^{(3)} &= \hat{g}_{k+1} + h\mathcal{Y}_1(h\mathcal{S}_k)(G(g_k^{(3)})) + 2h\mathcal{Y}_3(h\mathcal{S}_k) \left(\mathcal{G}_k(\hat{g}_{k+1}) - \mathcal{G}_k(g_k^{(3)}) \right). \end{aligned}$$

Now we establish numerical schemes for $f(t), f_0(t)$. For $p = 2, 3$, denote the p th-order approximations of $f(t)$ and $f_0(t)$ by $f_k^{(p)}$ and $f_{0,k}^{(p)}$, respectively. The scheme of ExpEuler-RK2 and Erow3-RK3 for solving $f(t)$ takes the form of

$$f_{k+1}^{(p)} = f_k^{(p)} + h \sum_{i=1}^p d_i^{(p)} y_{k,i}^{(p)}, \quad (4.1)$$

$$y_{k,i}^{(p)} = F \left(g_{k+l_i^{(p)}}^{(p)}, f_k^{(p)} + h \sum_{j=1}^{i-1} m_{ij}^{(p)} y_{k,j}^{(p)} \right), \quad p = 2, 3. \quad (4.2)$$

where $d_i^{(p)}, l_i^{(p)}, m_{ij}^{(p)}$ represents the coefficients for Runge-Kutta schemes with order p . We adopt the coefficients of the midpoint method and trapezoidal rule for the cases $p = 2$ and $p = 3$, respectively (see [4]). Similarly, the ExpEuler-RK2 and Erow3-RK3 scheme for solving $f_0(t)$ is

$$\begin{aligned} f_{0,k+1}^{(p)} &= f_{0,k}^{(p)} + h \sum_{i=1}^p d_i^{(p)} z_{k,i}^{(p)}, \\ z_{k,i}^{(p)} &= F_0 \left(g_{k+l_i^{(p)}}^{(p)}, f_{k+l_i^{(p)}}^{(p)} \right), \quad p = 2, 3. \end{aligned}$$

To establish the convergence theorem, we give following lemma first.

Lemma 4.1. *Under Assumptions 2.1, 2.2 and 2.3, the function $F(g, f)$ is Lipschitz continuous with respect to g and f , with corresponding Lipschitz constants L^g and L^f . Moreover, the function $F_0(g, f)$ is Lipschitz continuous in g and f , with corresponding Lipschitz constants L_0^g and L_0^f .*

Proof. We present the proof for $F(g, f)$ only, as the argument for $F_0(g, f)$ proceeds in a similar manner. Rewrite $F(g, f)$ as the form of

$$F(t, g, f) = Af + gBf + gC + D,$$

with

$$A = \frac{\text{diag}(\alpha)}{1 - \gamma}, \quad B = -2\sigma\sigma^\top, \quad C = -2\text{diag}(\alpha)w - 2\gamma \frac{\text{diag}(\alpha)\mu - re}{1 - \gamma},$$

$$D = \frac{\gamma \text{diag}(\alpha)(\sigma\sigma^\top)^{-1}[\text{diag}(\alpha)\mu - re]}{(1 - \gamma)^2} + \frac{1}{1 - \gamma}\rho.$$

By the boundedness of functions $g(t)$ and $f(t)$, we have

$$\|F(\hat{g}, f) - F(g, f)\| = \|(\hat{g} - g)(B_1 f + C_1)\| \leq L^g \|\hat{g} - g\|,$$

$$\|F(g, \hat{f}) - F(g, f)\| = \|(A_1 + gB_1)(\hat{f} - f)\| \leq L^f \|\hat{f} - f\|,$$

where L^g and L^f are Lipschitz constants. □

Theorem 4.2. *Suppose Assumptions 2.1, 2.2, and 2.3 hold, and that the time discretization step size h satisfies*

$$\frac{1}{2}K(K - 1)h^{p+1} \leq C_{\mathcal{H}}, \quad h < \frac{C_p}{L^f}, \quad p = 2, 3, \quad (4.3)$$

where K is the number of time discretization steps, $C_{\mathcal{H}}$ is a sufficiently small constant and $C_p > 0$ is a method-dependent constant that may depend on the order p . Then ExpEuler-RK2 and Erow3-RK3 is globally second-order and third-order convergent.

Proof. Under the first assumption of (4.3), the ExpEuler and Erow3 methods achieve global convergence orders of two and three, respectively [11]. Therefore, for $p = 2, 3$, we have

$$\|g(t_k) - g_k^{(p)}\| \leq ch^p.$$

Let $\hat{y}_{k,i}^{(p)}$ denote the value of (4.2) by substituting $g_{k+l_i^{(p)}}^{(p)}$ with $g(t_{k+l_i^{(p)}})$. Correspondingly, let $\hat{f}_i^{(p)}$ denote the numerical solution computed from (4.1) substituting $y_{k,i}^{(p)}$ with $\hat{y}_{k,i}^{(p)}$. By Lemma 4.1, we know that function F is Lipschitz continuous with respect to f . Additionally, by the stability theorem for Runge-Kutta methods in [9, Chapter 10], we have that

$$\|f(t_k) - \hat{f}_k^{(p)}\| \leq ch^p,$$

under assumptions of Lemma 3.3 and the second assumption of (4.3). Then for $p = 2, 3$, we obtain error bound

$$\begin{aligned} \|f(t_k) - f_k^{(p)}\| &= \|f(t_k) - \hat{f}_k^{(p)} + \hat{f}_k^{(p)} - f_k^{(p)}\| \\ &\leq \|f(t_k) - \hat{f}_k^{(p)}\| + \|\hat{f}_k^{(p)} - f_k^{(p)}\| \\ &\leq ch^p + \|\hat{f}_k^{(p)} - f_k^{(p)}\|. \end{aligned} \quad (4.4)$$

where c is some positive constant determined by Lipschitz constant L^f . Throughout the paper, this c may change from one instance to another, depending on the specific terms involved. Denote

$\|\hat{f}_k^{(p)} - f_k^{(p)}\|$ by $e_k^{(p)}$, we have

$$\begin{aligned}
e_{k+1}^{(p)} &\leq e_k^{(p)} + h \sum_{i=1}^p d_i^{(p)} \|\hat{y}_{k,i}^{(p)} - y_{k,i}^{(p)}\| \\
&= e_k^{(p)} + h \sum_{i=1}^p d_i^{(p)} \left\| F \left(g(t_{k+l_i^{(p)}}), \hat{f}_k^{(p)} + h \sum_{j=1}^{i-1} m_{ij}^{(p)} \hat{y}_{k,j}^{(p)} \right) \right. \\
&\quad \left. - F \left(g_{k+l_i^{(p)}}^{(p)}, f_k^{(p)} + h \sum_{j=1}^{i-1} m_{ij}^{(p)} y_{k,j}^{(p)} \right) \right\| \\
&\leq e_k^{(p)} + h \sum_{i=1}^p d_i^{(p)} \left(cL^g h^p + L^f e_k^{(p)} + hL_f \sum_{j=1}^{i-1} m_{ij}^{(p)} \|\hat{y}_{k,j}^{(p)} - y_{k,j}^{(p)}\| \right) \\
&\leq e_k^{(p)} + ch^{p+1} + che_k^{(p)} + ch^2 \left(\sum_{i=1}^p \sum_{j=1}^{i-1} \|\hat{y}_{k,j}^{(p)} - y_{k,j}^{(p)}\| \right) \\
&\leq ce_k^{(p)} \sum_{i=0}^p h^i + ch^p \sum_{i=1}^p h^i \\
&\leq (1 + ch^2) e_k^{(p)} + ch^{p+1}.
\end{aligned} \tag{4.5}$$

Note that the transition from the third-to-last line to the penultimate line in estimate (4.5) follows from the application of a recursive estimation approach. For any $1 \leq k \leq K$, by Gronwall's Lemma, we obtain

$$\begin{aligned}
e_k^{(p)} &\leq ch^{p-1} [(1 + ch^2)^k - 1] \\
&\leq ch^{p-1} (\exp\{ckh^2\} - 1) \\
&\leq ch^{p-1} (\exp\{cTh\} - 1) \\
&\leq ch^{p-1} (cTh + O(h^2)) \\
&\leq ch^p.
\end{aligned} \tag{4.6}$$

Combining estimate (4.4) and (4.6), the error bound

$$\|f(t_k) - f_k^{(p)}\| \leq ch^p.$$

We observe that the function $F_0(g, f)$ is independent of f_0 . Similarly, we obtain error bound

$$\|f_0(t_k) - f_{0,k}^{(p)}\| \leq ch^p.$$

□

Remark 4.3. It is worth noting that when $\sigma\sigma^\top$ is diagonal and $\varrho = 0_{n \times n}$, the structure of $H(S)$ admits matrix function $g(t)$ to be diagonal as well, (3.16a) can be decoupled into n independent 1-dimensional Riccati equations [1]. Consequently, closed-form solutions become tractable. We briefly examine this special case and derive the corresponding closed-form solutions of the ODE system, which can serve as a benchmark for validating the accuracy of our numerical methods in this particular scenario. Let $(\sigma\sigma^\top)_{ii} = q_i$, and $g(t) = \text{diag}(g_{11}(t), \dots, g_{nn}(t))$. Substituting $g(t)$

back into (3.16a)-(3.16c) and after some calculations, we have

$$g_{ii}(t) = \frac{\gamma \alpha_i}{2(1-\gamma)q_i} \frac{\sinh [\alpha_i(T-t)/\sqrt{1-\gamma}]}{\sinh [\alpha_i(T-t)/\sqrt{1-\gamma}] + \sqrt{1-\gamma} \cosh [\alpha_i(T-t)/\sqrt{1-\gamma}]}, \quad (4.7a)$$

$$f_i(t) = \int_t^T \zeta_i(u) \exp \left\{ \int_t^u \kappa_i(s) ds \right\} du, \quad (4.7b)$$

$$\begin{aligned} f_0(t) = & \int_t^T \left\{ \sum_{i=1}^n \left[w_i \alpha_i + \frac{\gamma}{1-\gamma} (\mu_i \alpha_i - r) \right] f_i(u) + \frac{1}{2} \sum_{i=1}^n q_i f_i^2(u) + \sum_{i=1}^n q_i g_{ii}(u) \right\} du \\ & + \left[\sum_{i=1}^n \frac{\gamma (\mu_i \alpha_i - r)^2}{2(1-\gamma)^2 q_i} + \frac{-\rho_0 + r\gamma}{1-\gamma} \right] (T-t), \end{aligned} \quad (4.7c)$$

where

$$\begin{aligned} \kappa_i(t) &= 2q_i g_{ii}(t) - \alpha_i/(1-\gamma), \\ \zeta_i(t) &= 2[\alpha_i w_i + \gamma(\mu_i \alpha_i - r)/(1-\gamma)]g_{ii}(t) - \gamma \alpha_i (\mu_i \alpha_i - r)/[(1-\gamma)^2 q_i] - \rho_i(1-\gamma). \end{aligned}$$

By combining equations (3.4), (3.11), (3.12), and (3.15), when the functions g , f , and f_0 are obtained either analytically or numerically, the solution to the HJB equation (3.2)-(3.3) admits the following explicit form:

$$\begin{aligned} v(t, x, S, \psi) &= \gamma^{-1} \psi^{1-\gamma} x^\gamma [\varphi_1(t, S) + \varphi_2(t, S)]^{1-\gamma} \\ &= \gamma^{-1} \psi^{1-\gamma} x^\gamma \left[\exp \{ f_0(t) + f^\top(t)S + S^\top g(t)S \} \right. \\ &\quad \left. + \int_t^T \exp \{ f_0(u) + f^\top(u)S + S^\top g(u)S \} du \right]^{1-\gamma}. \end{aligned} \quad (4.8)$$

However, (4.8) remains a candidate solution for the value function of the original optimal control problem. Therefore, a verification theorem is essential to validate the optimality of (4.8).

5 Verification theorem

In this section, we establish a rigorous verification theorem that provides sufficient conditions for the candidate solutions to be optimal. Additionally, we prove that verification theorem is valid for numerical solutions as well.

To facilitate the proof of the verification theorem, we first establish the following lemmas.

Lemma 5.1. (cf. [22]) *Let $W \sim N(\eta, \Sigma)$ be a n -dimensional random variable and given $R \in \mathbb{S}^n$, then the moment-generating function of $Q = W^\top R W$ is defined as*

$$\begin{aligned} M_Q(s) &= \mathbb{E}[\exp\{sQ\}] \\ &= \begin{cases} |I - 2sR\Sigma|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \eta^\top [I - (I - 2sR\Sigma)^{-1}] \Sigma^{-1} \eta \right\}, & s \in (s_{\min}, s_{\max}), \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Here

$$s_{\min} = \begin{cases} \frac{1}{2\lambda_{\min}(R\Sigma)}, & \lambda_{\min}(R\Sigma) < 0, \\ -\infty, & \lambda_{\min}(R\Sigma) \geq 0, \end{cases} \quad s_{\max} = \begin{cases} \frac{1}{2\lambda_{\max}(R\Sigma)}, & \lambda_{\max}(R\Sigma) > 0, \\ +\infty, & \lambda_{\max}(R\Sigma) \leq 0, \end{cases}$$

where $\lambda_{\min}(R\Sigma)$ and $\lambda_{\max}(R\Sigma)$ denote the smallest and largest eigenvalue of matrix $R\Sigma$.

Lemma 5.2. (cf. [33]) For any terminal time $\tau \in [t, T]$, $S(\tau)$ has an n -dimensional normal distribution with mean vector

$$\eta(\tau) = \exp\{-(\tau - t) \operatorname{diag}(\alpha)\} S(t) + (I - \exp\{-(\tau - t) \operatorname{diag}(\alpha)\}) w,$$

and covariance matrix

$$(\Sigma(\tau))_{ij} = \frac{(\sigma_i \sigma_j^\top)(1 - \exp\{-(\tau - t)(\alpha_i + \alpha_j)\})}{\alpha_i + \alpha_j}.$$

Here $\Sigma(\tau) \in \mathbb{S}_{++}^n$, where \mathbb{S}_{++}^n denotes the set of all symmetric positive definite $n \times n$ matrices.

Lemma 5.3. (cf. [13]) If condition

$$\mathbb{E} \left[\exp \left\{ \max \left\{ \frac{1}{2}, \frac{\gamma^2}{2(1-\gamma)^2} \right\} \int_0^T \|\sigma^{-1}(\operatorname{diag}(\alpha)(\mu - S(t)) - re)\|^2 dt \right\} \right] < +\infty \quad (5.1)$$

holds for any $t \in [0, T]$, then Novikov condition (3.9) is guaranteed and we can also define a new probability measure \tilde{P} where condition (d) in Definition 2.5 is equivalent to

$$\tilde{\mathbb{E}} \left[\int_0^T C(t) \exp\{-rt\} dt \right] \leq x, \quad \tilde{\mathbb{E}} [X(T) \exp\{-rT\}] \leq x.$$

Lemma 5.4. (cf. [1]) Let function $v(t, x, S, \psi)$ be a classical solution of HJB equation (3.2)-(3.3) and the family $\{v(\tau, X(\pi^*, C^*)(\tau), S(\tau), \psi(\tau))\}_{\tau \in [0, T]}$ is uniformly integrable. Then we have

$$v(t, x, S, \psi) = V(t, x, S, \psi).$$

The 1-dimensional case of Lemma 5.4 is given in [1, Lemmas 4.1-4.2], and the extension to the n -dimensional case follows naturally. We omit the details here.

Theorem 5.5. Under Assumptions 2.1, 2.2, 2.3, for any $\tau \in [0, T]$, suppose the following conditions hold:

$$\lambda_{\max}(g(\tau)\Sigma(\tau)) < \frac{1}{4(1-\gamma)}, \quad (5.2)$$

$$-\lambda_{\min}(\varrho\Sigma(\tau)) < \frac{1}{8(1-\gamma)}, \quad (5.3)$$

$$\lambda_{\max}(\Gamma\Sigma(\tau)) < \frac{1}{8\gamma}, \quad (5.4)$$

$$\lambda_{\max}(\Pi(\tau)\Sigma(\tau)) < \frac{1}{256\gamma^2}, \quad (5.5)$$

where

$$\begin{aligned} \Gamma &= \operatorname{diag}(\alpha)(\sigma\sigma^\top)^{-1} \operatorname{diag}(\alpha), \\ \Pi(\tau) &= 4g(\tau)\sigma\sigma^\top g(\tau) + \frac{1}{1-\gamma} \operatorname{diag}(\alpha)\sigma\sigma^\top \operatorname{diag}(\alpha), \end{aligned}$$

then the candidate solution v given in (4.8) is indeed the value function (3.1). Furthermore, suppose that for any $0 \leq t \leq u \leq T$, it also has

$$\lambda_{\max}(\Gamma\Sigma(t)) < \min \left\{ \frac{1}{8}, \frac{(1-\gamma)^2}{\gamma^2} \right\}, \quad (5.6)$$

$$\lambda_{\max}(\Pi(t)\Sigma(t)) < \frac{1}{256}, \quad (5.7)$$

$$-\lambda_{\min}(g(t)\Sigma(t)) < \frac{1}{4}, \quad (5.8)$$

$$\lambda_{\max}([(g(u) - g(t))\Sigma(t)]) < \frac{1}{16}, \quad (5.9)$$

and the initial wealth x is chosen such that

$$x = \max \left\{ \tilde{\mathbb{E}} \left[\int_0^T C(t) \exp\{-rt\} dt \right], \tilde{\mathbb{E}} [X(T) \exp\{-rT\}] \right\}, \quad (5.10)$$

then optimal portfolio-consumption policies are given by equations (3.7)-(3.8).

Proof. From the statement of Lemma 5.4, to establish the optimality of the candidate solution v , it suffices to prove that $\{v(\tau, X(\tau)^{(\pi^*, C^*)}, S(\tau), \psi(\tau))\}_{\tau \in [0, T]}$ is uniformly integrable. It is notable that for any given $\epsilon > 0$ and stopping time $\tau \in [0, T]$, by equation (4.8) and Holder's inequality, we have

$$\begin{aligned} & \mathbb{E}^{(t, x, S, \psi)} [v^{1+\epsilon}(\tau, X^*(\tau), S(\tau), \psi(\tau))] \\ &= \gamma^{-(1+\epsilon)} \mathbb{E}^{(t, x, S, \psi)} \left[\left(\sum_{i=1}^2 \varphi_i(\tau, S(\tau)) \right)^{(1-\gamma)(1+\epsilon)} \psi(\tau)^{(1-\gamma)(1+\epsilon)} (X^*(\tau))^{\gamma(1+\epsilon)} \right] \\ &\leq c \mathbb{E}^{(t, x, S, \psi)} \left[\left(\sum_{i=1}^2 \varphi_i^{(1-\gamma)(1+\epsilon)}(\tau, S(\tau)) \right) \psi(\tau)^{(1-\gamma)(1+\epsilon)} (X^*(\tau))^{\gamma(1+\epsilon)} \right] \\ &= c \sum_{i=1}^2 \mathbb{E}^{(t, x, S, \psi)} \left[\varphi_i^{(1-\gamma)(1+\epsilon)}(\tau, S(\tau)) \psi(\tau)^{(1-\gamma)(1+\epsilon)} (X^*(\tau))^{\gamma(1+\epsilon)} \right] \\ &\leq c \sum_{i=1}^2 \left\{ \mathbb{E}^{(t, x, S, \psi)} \left[\varphi_i^{2(1-\gamma)(1+\epsilon)}(\tau, S(\tau)) \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E}^{(t, x, S, \psi)} [\psi(\tau)^{4(1-\gamma)(1+\epsilon)}] \right\}^{\frac{1}{4}} \\ &\quad \times \left\{ \mathbb{E}^{(t, x, S, \psi)} [(X^*(\tau))^{4\gamma(1+\epsilon)}] \right\}^{\frac{1}{4}}. \end{aligned} \quad (5.11)$$

Therefore, to demonstrate the integrability, we only need to prove that for any $\tau \in [t, T]$, the right-hand side of above equation is bounded.

We first handle the first expectation term in the right-hand side of (5.11), by equations (3.12), (3.15) and Lemma 3.3, we obtain

$$\begin{aligned} & \mathbb{E}^{(t, x, S, \psi)} \left[\varphi_1^{2(1-\gamma)(1+\epsilon)}(\tau, S(\tau)) \right] \\ &= \mathbb{E}^{(t, x, S, \psi)} [\exp\{2(1-\gamma)(1+\epsilon)(f_0(\tau) + f(\tau)^\top S(\tau) + S(\tau)^\top g(\tau)S(\tau))\}] \\ &\leq c \mathbb{E}^{(t, x, S, \psi)} [\exp\{2(1-\gamma)(1+\epsilon)(f(\tau)^\top S(\tau) + S(\tau)^\top g(\tau)S(\tau))\}]. \end{aligned}$$

Furthermore, by Young's inequality [1], it has

$$2(1-\gamma)(1+\epsilon)f^\top(\tau)S(\tau) \leq c + S(\tau)^\top \delta S(\tau), \quad \forall \tau \in [t, T],$$

where δ denotes an arbitrary positive definite diagonal matrix, and c is determined by the upper bound of $f_0(\tau)$, $f_0(\tau)$ and δ . Hereafter, each occurrence of δ corresponds to a new application of the matrix form of Young's inequality. Based on the discussion above, we have

$$\begin{aligned} & \mathbb{E}^{(t,x,S,\psi)} \left[\varphi_1^{2(1-\gamma)(1+\epsilon)}(\tau, S(\tau)) \right] \\ & \leq c \mathbb{E}^{(t,x,S,\psi)} [\exp\{S(\tau)^\top [2(1-\gamma)(1+\epsilon)g(\tau) + \delta] S(\tau)\}]. \end{aligned} \quad (5.12)$$

Let us set

$$s = 2, \quad R = 2(1-\gamma)(1+\epsilon)g(\tau) + \delta.$$

By Lemma 5.1-5.2, given the arbitrariness of ϵ and matrix δ , right-hand side of estimate (5.12) is bounded when condition (5.2) is satisfied. Moreover, by equation (3.12), the first expectation term in the right-hand side of (5.11) is also bounded under the same condition.

Next we proceed to prove that the second expectation term of the right-hand side of (5.11) is bounded. Review equation (2.4) and by Young's inequality, we obtain

$$\begin{aligned} & \mathbb{E}^{(t,x,S,\psi)} [\psi(\tau)^{4(1-\gamma)(1+\epsilon)}] \\ & = \mathbb{E}^{(t,x,S,\psi)} \left[\exp \left\{ -4(1-\gamma)(1+\epsilon) \int_0^\tau [\rho_0 + S(u)^\top \rho + S(u)^\top \varrho S(u)] du \right\} \right] \\ & \leq c \int_0^\tau \mathbb{E}^{(t,x,S,\psi)} [\exp \{ S(u)^\top [\delta - 4(1-\gamma)(1+\epsilon)\varrho] S(u) \}] du. \end{aligned} \quad (5.13)$$

Similarly, by Lemma 5.1-5.2 and the arbitrariness of ϵ and δ , the expectation on the right-hand side of estimate (5.13) is bounded if condition (5.3) is guaranteed.

Finally, let us prove that the third expectation term of the right-hand side of (5.11) is bounded. We denote the wealth when consumption is absent ($C \equiv 0$) by $Z(t)$ and the corresponding optimal wealth by $Z^*(t)$. Obviously, we have that for any $t \in [0, T]$, $X^*(t) \leq Z^*(t)$. Thus, we only need to prove $\mathbb{E}^{(t,x,S,\psi)} [(Z(\tau))^{4\gamma(1+\epsilon)}] < +\infty$. Following the proof of Theorem 4.4 of [1], by equation (2.5), we can readily derive

$$\begin{aligned} & \mathbb{E}^{(t,x,S,\psi)} [(Z(\tau))^{4\gamma(1+\epsilon)}] \\ & \leq c \int_t^\tau \mathbb{E}^{(t,x,S,\psi)} \left[\exp \left\{ 8\gamma(1+\epsilon) S(u)^\top \left(\delta + \frac{1}{2} \text{diag}(\alpha)(\sigma\sigma^\top)^{-1} \text{diag}(\alpha) \right) S(u) \right\} \right] du \\ & \quad \times \int_t^\tau \mathbb{E}^{(t,x,S,\psi)} \left[\exp \left\{ 128\gamma^2(1+\epsilon)^2 S(u)^\top [\delta + 4g(t)\sigma\sigma^\top g(t) \right. \right. \\ & \quad \left. \left. + \frac{1}{1-\gamma} \text{diag}(\alpha)\sigma\sigma^\top \text{diag}(\alpha)] S(u) \right\} \right]^{\frac{1}{4}} du. \end{aligned} \quad (5.14)$$

By Lemma 5.1-5.2 and the arbitrariness of ϵ and matrix δ , right-hand integral of estimate (5.14) is bounded by conditions (5.4)-(5.5). Consequently, the boundedness of the third expectation in (5.11) is assured by these same conditions. So far, we have proved that the candidate solution v is the value function V if conditions (5.2)-(5.5) are satisfied.

Now let us prove that conditions (5.6)-(5.9) guarantee (π^*, C^*) to be admissible. We have proved that for $\{v(\tau, X^{(\pi^*, C^*)}(\tau), S(\tau), \psi(\tau))\}_\tau$ is uniformly integrable under conditions (5.2)-(5.5). Therefore, condition (c) in Definition 2.5 is also guaranteed. Next we will prove rest of the admissibility conditions.

Firstly, to ensure the feasibility of the change of measure in Section 3.2 and Lemma 5.3, it's sufficient to verify that condition (5.1) is satisfied. By Young's and Jensen's inequalities, we derive

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ \max \left\{ \frac{1}{2}, \frac{\gamma^2}{2(1-\gamma)^2} \right\} \int_0^T \|\sigma^{-1} \text{diag}(\alpha)(\mu - S(u))\|^2 du \right\} \right] \\ & \leq c \int_0^T \mathbb{E} \left[\exp \left\{ \max \left\{ \frac{1}{2}, \frac{\gamma^2}{2(1-\gamma)^2} \right\} S(u)^\top [\delta + \text{diag}(\alpha)(\sigma\sigma^\top)^{-1} \text{diag}(\alpha)] S(u) \right\} \right] du. \end{aligned}$$

Then, by Lemmas 5.1-5.2 and the arbitrariness of δ , we can conclude that condition (5.6) ensures condition (5.1). Consequently, with x defined in equation (5.10), condition (d) in Definition 2.5 is fulfilled, according to Lemma 5.3.

Then let us verify that C^* satisfies condition (a) in Definition 2.5. From the analysis in estimate (5.14), we conclude that $\mathbb{E}[X^*(t)]^4 < +\infty$ holds for any $t \in [0, T]$ under conditions (5.6)-(5.7), which helps us to prove the L^1 integrability of C^* . By applying equations (3.8) and (3.15), together with Cauchy's inequality and Young's inequality, we obtain

$$\begin{aligned} & \mathbb{E} \left[\int_0^T C^*(t) dt \right] \\ & \leq \int_0^T \mathbb{E} [\varphi^{-1}(t, S(t)) |X^*(t)|] dt \\ & \leq \int_0^T \left\{ \mathbb{E} [\varphi^{-2}(t, S(t))] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} [(X^*(t))^2] \right\}^{\frac{1}{2}} dt \\ & \leq c \int_0^T \left\{ \mathbb{E} [\exp \{-2(f_0(t) + S(t)^\top f(t) + S(t)^\top g(t)S(t))\}] \right\}^{\frac{1}{2}} dt \\ & \leq c \int_0^T \left\{ \mathbb{E} [\exp \{S(t)^\top [\delta - 2g(t)]S(t)\}] \right\}^{\frac{1}{2}} dt. \end{aligned} \tag{5.15}$$

In the same way, by Lemma 5.1-5.2 and arbitrariness of δ , if condition (5.8) is satisfied, the right-hand side of estimate (5.15) is bounded, which directly guarantees the L^1 integrability of C^* .

At last, to establish the L^1 integrability of π^* , by equations (3.7) and (3.15), Lemma 3.3, Holder's inequality and Jensen's inequality, we obtain

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \|\pi^*(t)\|^2 dt \right] \\ & \leq c + 2c \int_0^T \mathbb{E} [\varphi^{-2}(t, S(t)) \|D_S \varphi(t, S(t))\|^2 (X^*(t))^2] dt \\ & \leq c + 2c \left\{ \int_0^T \mathbb{E} [\varphi^{-4}(t, S(t)) \|D_S \varphi(t, S(t))\|^4] dt \right\}^{\frac{1}{2}} \left\{ \int_0^T \mathbb{E} [(X^*(t))^4] dt \right\}^{\frac{1}{2}} \\ & \leq c + c \left\{ \int_0^T \mathbb{E} \left[\left\| \int_t^T \exp \{f_0(u) + S(t)^\top (f(u) - f(t)) + S(t)^\top (g(u) - g(t))S(t)\} \right. \right. \right. \\ & \quad \left. \left. \left. \times (f(u) + 2g(u)S(t)) du + f(t) + 2g(t)S(t) \right\|^4 \right] dt \right\}^{\frac{1}{2}} \\ & \leq c + c \left\{ \int_0^T \int_t^T \left\{ \mathbb{E} [\exp \{S(t)^\top [8(g(u) - g(t)) + \delta]S(t)\}] \right\}^{\frac{1}{2}} du dt \right\}^{\frac{1}{2}}. \end{aligned} \tag{5.16}$$

By Lemma 5.1-5.2, condition (5.9) guarantees the boundedness of the right-hand side of estimate (5.16), and therefore, guarantees the integrability of π^* .

By Pontryagin's Maximum Principle, we know that (π^*, C^*) maximizes the Hamiltonian $\mathcal{G}_{xS\psi}$. Therefore, it's the optimal policy. The proof of the theorem is completed. \square

Corollary 5.6. *Under the assumptions of Theorem 4.2, let $\hat{v}(t, x, S, \psi)$ denotes the numerical approximation of $v(t, x, S, \psi)$ using ExpEuler-RK2, Erow3-RK3 methods, if the discretization step h is small enough and numerical solution $\hat{g}(t)$ satisfies conditions (5.2)-(5.9), then $\hat{v}(t, x, S, \psi)$ provides a good approximation for the value function $V(t, x, S, \psi)$. Moreover, the optimal policy can be approximated by computing the feedback control given in (3.7)-(3.8) using the numerical solution $\hat{\varphi}(t, S)$.*

Proof. Combining Remark 3.4 with Theorem 4.2, for any given compact set $\mathcal{K}_1 \subset [0, T] \times \mathbb{R}^n$ and any $\varepsilon > 0$, there exists a sufficiently small discretization step h , such that

$$\begin{aligned} \|g(t) - \hat{g}(t)\| &\leq \varepsilon, & \forall t \in [0, T], \\ |\varphi(t, S) - \hat{\varphi}(t, S)| &\leq \varepsilon, & \forall (t, S) \in \mathcal{K}_1. \end{aligned}$$

When $\hat{g}(t)$ satisfies the verification conditions, it's reasonable to conclude that $g(t)$ also satisfies them, given the continuity of the functions $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$. Hence, our numerical methods offer a reliable approximation of v , and therefore, of the value function V . Moreover, we note that both φ and $\hat{\varphi}$ exhibit smoothness, then we can compute the gradient $D_S \hat{\varphi}(t, S)$ to approximate $D_S \varphi(t, S)$. By formulas (3.7)-(3.8), for any given compact set $\mathcal{K}_2 \subset [0, T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_+$ and any $\varepsilon > 0$, there also exists a sufficiently small discretization step h , such that

$$\begin{aligned} \|\pi^*(t, x, S, \psi) - \hat{\pi}^*(t, x, S, \psi)\| &\leq \varepsilon, & \forall (t, x, S, \psi) \in \mathcal{K}_2, \\ |C^*(t, x, S, \psi) - \hat{C}^*(t, x, S, \psi)| &\leq \varepsilon, & \forall (t, x, S, \psi) \in \mathcal{K}_2, \end{aligned}$$

where $(\hat{\pi}^*, \hat{C}^*)$ denotes the numerical optimal policy. Hence, it is reasonable to establish that $(\hat{\pi}^*, \hat{C}^*)$ is a good approximation of optimal policy. \square

6 Numerical Experiments

In this section, we present several numerical experiments to assess the performance of ExpEuler-RK2 and Erow3-RK3. First, we examine the accuracy of these two methods in solving the ODE system (3.16a)-(3.16c). We then test the convergence order and computational efficiency of ExpEuler-RK2 and Erow3-RK3, verifying the results presented by Theorem 4.2. Additionally, we conduct a comparison with a conventional grid-based method to demonstrate the superiority of Erow3-RK3. Finally, we apply Erow3-RK3 to a real-world market case to showcase its practical application. All experiments are performed under Windows 11 and Matlab R2024b running on a laptop with 13th Gen Intel Core i7-13650HX processor with 2.60 GHz and RAM 16 GB.

6.1 Accuracy and Efficiency

To evaluate ExpEuler-RK2 and Erow3-RK3, we consider the special case in Remark 4.3, which admits closed-form solutions. Let us set time horizon $t \in [0, 1]$ and the coefficient settings are as follows:

$$\begin{aligned} r &= 0.5, \quad \gamma = 0.5, \quad \rho_0 = 0, \quad \rho = \text{zeros}(n, 1), \quad \alpha = 0.3 + 0.4 \times \text{rand}(n, 1), \\ \mu &= 5 + 3 \times \text{rand}(n, 1) \quad \sigma = \text{orth}(0.01 \times \text{rand}(n, n)), \quad \varrho = \text{zeros}(n, n). \end{aligned}$$

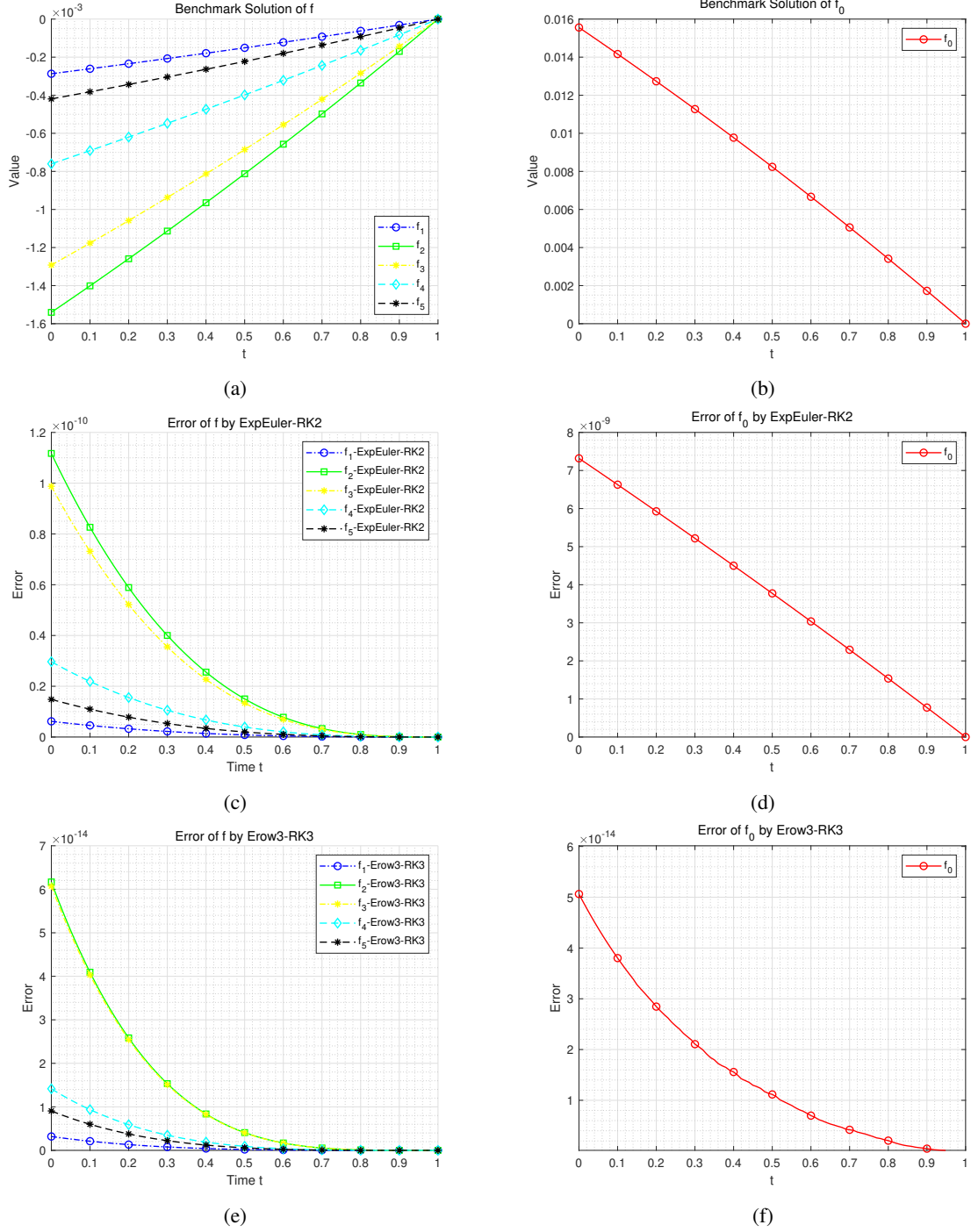


Figure 1: Benchmark solutions and Numerical Errors

Closed-form solutions of (3.16a)-(3.16c) are given in equations (4.7a)-(4.7c), we divide the interval $[0, 1]$ into 100 uniform subintervals (i.e., with a step size of $h = 0.01$), and apply the Composite Simpson's numerical integration rule to compute benchmark solutions. These benchmarks are then used to examine the accuracy and convergence order of our methods. Since ExpEuler and Erow3 are already proved to be great solvers for matrix Riccati equations like (3.16a) in [18], we

will only evaluate the performance of our methods on $f(t)$ and $f_0(t)$. We first evaluate the accuracy of ExpEuler-RK2 and Erow3-RK3 in the case of $n = 5$.

In Figure 1, panel (1a)-(1b) present the benchmark solution for each entry of $f(t)$ and $f_0(t)$. Panels (1c)-(1d) and (1e)-(1f) illustrate the numerical errors obtained using ExpEuler-RK2, Erow3-RK3 for solving $f(t)$ and $f_0(t)$, respectively. The figure demonstrates that both schemes solve the ODE system with high accuracy. And Erow3-RK3 delivers markedly superior precision compared with ExpEuler-RK2. Moreover, we observe that the numerical error decreases as time progresses, which is attributed to the terminal nature of the problem.

In addition, we test convergence order of these two algorithms in terms of solving the ODE system. We set $n = 10$ and select different step sizes: $h = 2^{-k}$, $k = 3, \dots, 7$. The relative errors are measured in the Frobenius norm for matrix and vector functions. In Figure 2, panels (2a) and (2c) show the logarithmic values of the numerical errors for ExpEuler-RK2 and Erow3-RK3 against the degree of freedom, which shows that convergence orders are precisely as established in Theorem 4.2. And panels (2b) and (2d) illustrate the error against computation time (seconds), which shows that Erow3-RK3 outperforms ExpEuler-RK2 in terms of efficiency. Therefore, we use Erow3-RK3 for the remaining experiments.

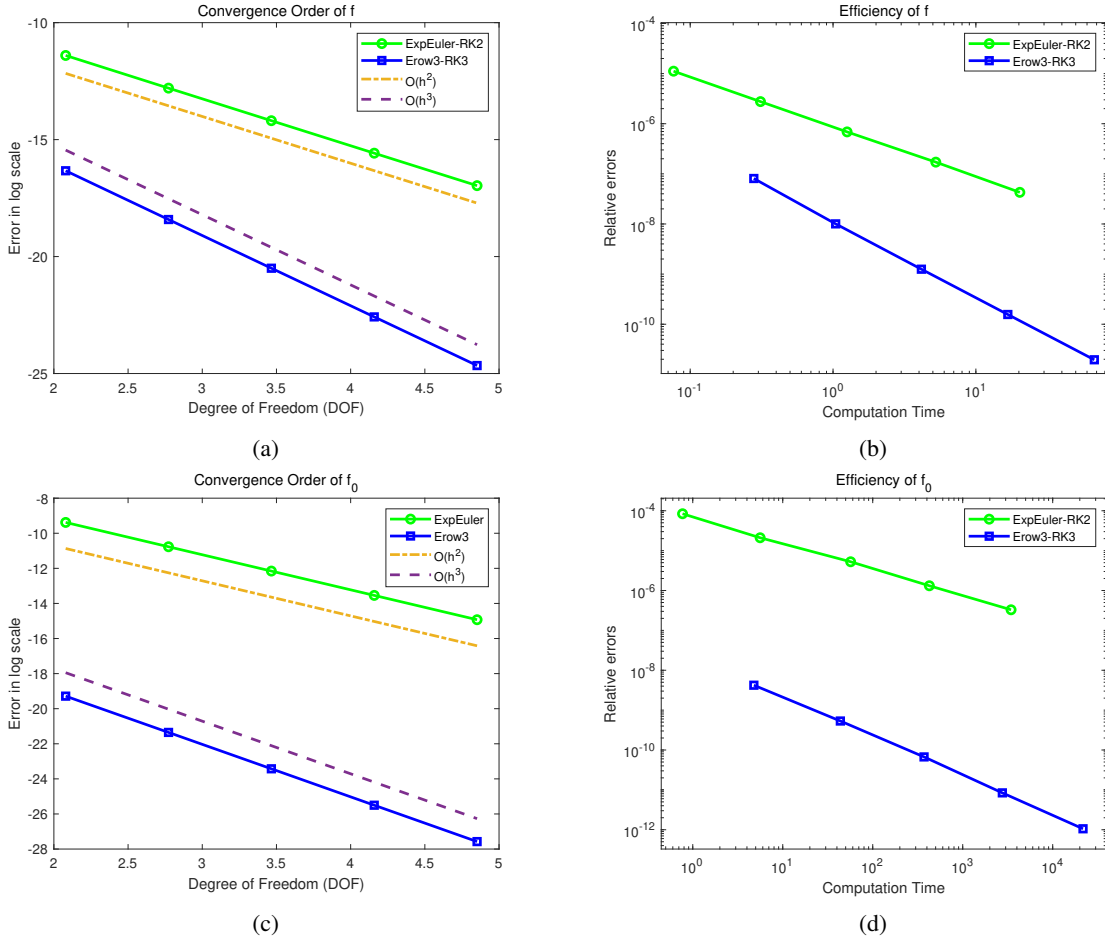


Figure 2: Order and Efficiency Test

6.2 Advantage of Erow3-RK3

To compare our method with grid-based approach, we focus on the solvers of PDE (3.5)-(3.6) in 1-dimensional case. The coefficients and intervals of the problem are selected as

$$r = 0.01, \quad \gamma = 0.5, \quad \rho_0 = 0.01, \quad \rho = 0, \quad \alpha = 0.005, \\ \mu = 3, \quad \sigma = 1, \quad t \in [0, 1], \quad S \in [-10, 10].$$

It is notable that the analytical approach is feasible in 1-dimensional case, therefore we compute the benchmark solution using the numerical integration method in Section 6.1. Then we employ Erow3-RK3 to solve $\varphi(t, S)$ with 25 time discretization steps. Additionally, we directly solve PDE (3.5)-(3.6) using the integration-difference method, which employs the Finite Difference Method (FDM) with exact boundary conditions obtained through numerical integration. This approach is less computationally demanding than the benchmark solution and more accurate than the FDM with truncated boundary conditions. We discretize both time and space with 100 uniform steps for the integration-difference method and compare its numerical error with Erow3-RK3.

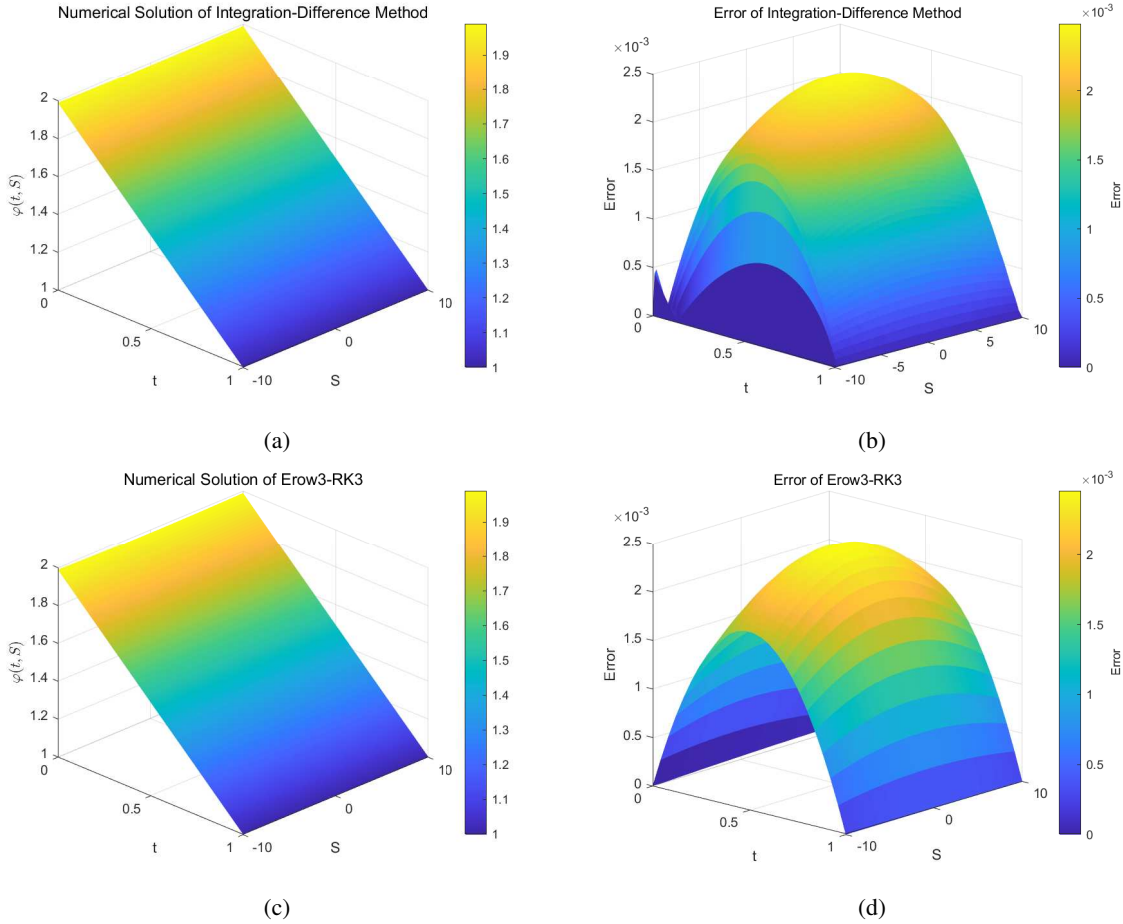


Figure 3: Numerical Solutions and Errors

In Figure 3, the panels (3a)-(3b) present the numerical solution and error of the integration-difference method, while (3c)-(3d) show those of the Erow3-RK3 Method. We observe that these two methods demonstrate similar error distribution. However, the Erow3-RK3 method achieves a computation time of 438 seconds, resulting in a 41% reduction compared to the FDM (738

seconds), thereby demonstrating its superior computational efficiency. This efficiency gain is likely due to our variable-separation strategy, which allows Errow3-RK3 to solve only the time-dependent ODE subsystem, whereas the FDM must also resolve the state domain.

6.3 Application of Errow3-RK3

Schwartz [29] estimates the parameters of exponential O-U model for oil, copper and gold price. In this subsection, we focus on the optimal control problem (2.3) subject to the constraint (2.2) in 2-dimensional case and adopt the parameter estimates for oil obtained from different contracts. Additionally, we introduce a perturbation of 0.01 to the off-diagonal entries of the volatility matrix. The coefficients and intervals are

$$\begin{aligned} r &= 0.3, \quad \gamma = 0.5, \quad \rho_0 = 0.03, \quad \rho = [0.02, 0.01]^\top, \\ \alpha &= (0.301, 0.428)^\top, \quad \mu = (3.093, 2.991)^\top, \quad \sigma = \begin{pmatrix} 0.334 & 0.01 \\ 0.01 & 0.257 \end{pmatrix}, \\ \varrho &= \begin{pmatrix} 0.002 & 0 \\ 0 & 0.002 \end{pmatrix}, \quad t \in [0, 0.25], \quad S \in [1, 3] \times [1, 3]. \end{aligned}$$

A procedural flowchart is provided in Figure 4 to illustrate the solution process for the original optimal control problem.

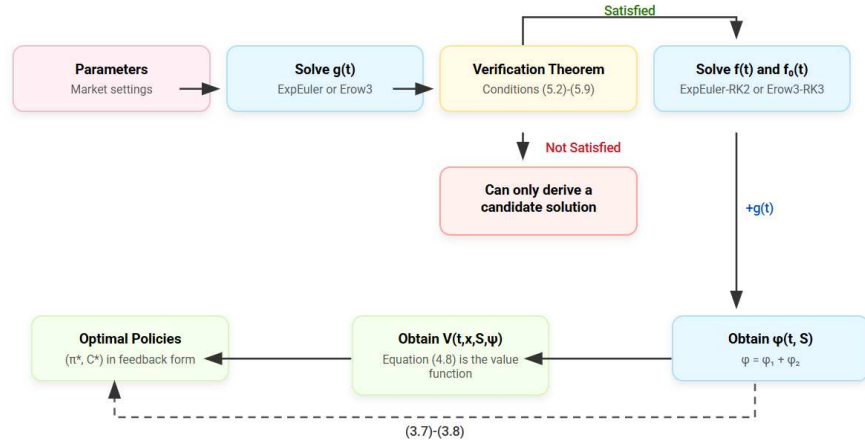


Figure 4: Procedural flowchart

Following the procedure outlined in Figure 4, we first verify whether conditions (5.2)-(5.9) are satisfied. This is done by reformulating each inequality so that all terms are on the right-hand sides, and then checking for positivity. Using Errow3 to solve for $g(t)$ with 100 time discretization steps, we numerically evaluate the values of the right-hand sides and present the results in Figure 5 to examine whether they remain positive throughout.

As shown in Figure 5, the reformulated right-hand sides of conditions (5.2)-(5.9) are all positive, which confirms that the verification conditions are fully satisfied. Then we employ Errow3-RK3 to solve $\varphi(t, S)$ with 50 time discretization steps. Subsequently, we derive the numerical optimal policy function from (3.7)-(3.8), which is a feedback control. Starting from initial state values, once $X(t_k), S(t_k)$ are obtained, they are substituted into the policy formula to determine the portfolio investment and consumption values. These control inputs are then used to drive the constrained SDE system forward to compute $X(t_{k+1})$ and $S(t_{k+1})$ dynamically. We set the initial state as $S(0) = 2$. Given the computational intractability of determining x according to (5.10),

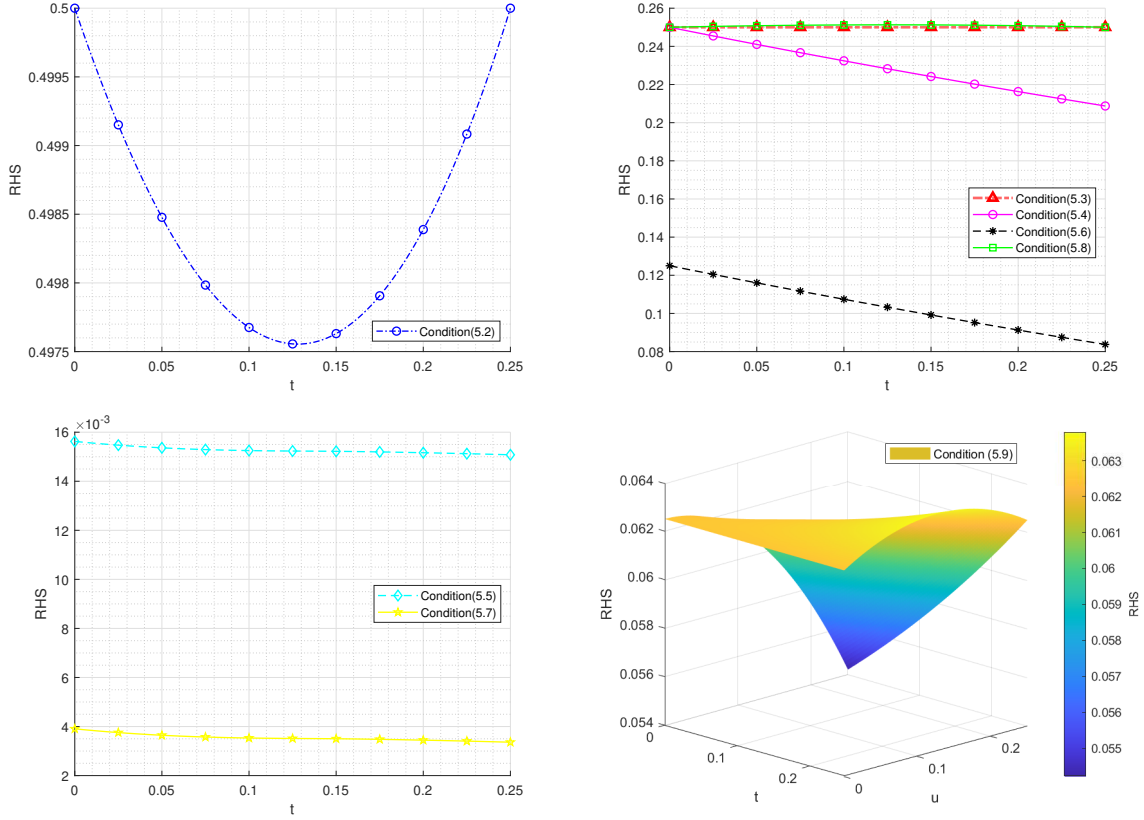


Figure 5: Verification Conditions

we opt for a conservative choice of x large enough to ensure positivity of the resulting wealth processes under all candidate policies. In our case, we set $x = 25$. We apply Euler-Maruyama Method to simulate the state process $S(t)$, the optimal wealth process $\{\hat{X}^*(t)\}$, the absolute value of the optimal portfolio process $\{|\hat{\pi}^*(t)|\}$, and the optimal consumption process $\{\hat{C}^*(t)\}$, with all values presented on a logarithmic scale. We generate 50 simulation paths and plot them in the Figure 6.

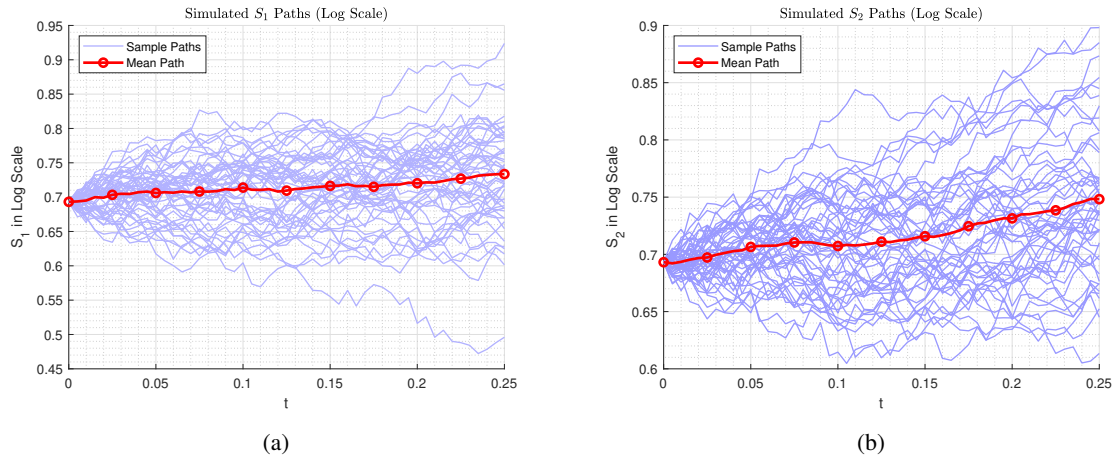


Figure 6: Simulation Results

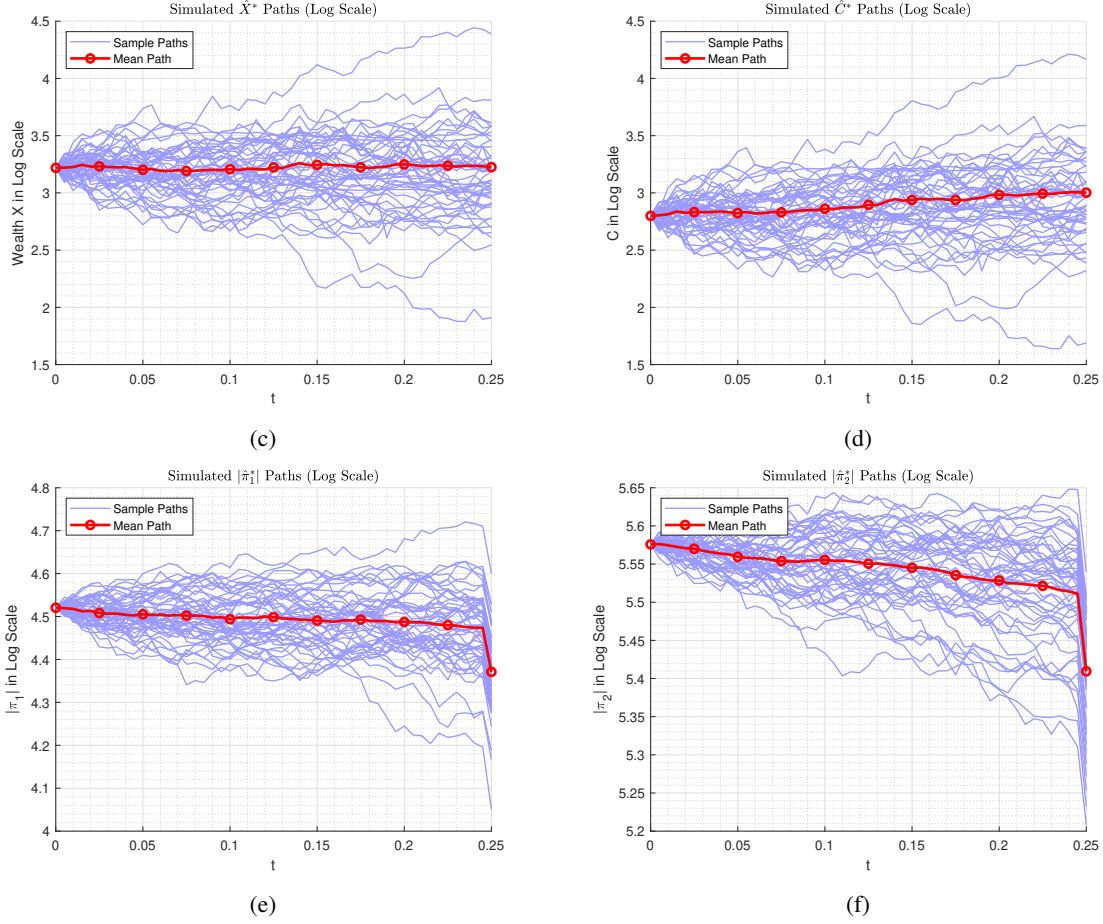


Figure 6: Simulation Results

From Figure 6, panel (6c) shows that \hat{X}^* exhibits considerable fluctuation magnitude while the mean path remains relatively stable. Panels (6e) and (6f) illustrate that both $|\hat{\pi}_1^*|$ and $|\hat{\pi}_2^*|$ exhibit initial stability, followed by a significant downward trajectory in the latter period. Furthermore, both components of $|\hat{\pi}^*|$ exhibit episodes of high leverage at certain times, which may be considered unrealistic in real-world markets. However, since our study is primarily theoretical, such outcomes are acceptable in this context. In practical applications, however, it is crucial to impose constraints on the leverage ratio.

To validate the optimality of our numerical result, we compute the mean utility across simulated paths to approximate value of objective functional (2.3) and compare it with that obtained from alternative admissible policies. However, computing the utility of every admissible policy is infeasible. We consider several commonly studied investment policies, including the riskless policy [21], the consumptionless policy [23], the bondless policy (no risk-free asset) [12], the random policy [3], leverage policies with different leverage ratio [28], and our numerical policy.

Table 1 presents a comparison of mean utilities. The first column lists the investment policies under consideration, the second to fourth columns detail the corresponding (π, C) policies, and the final column reports the associated mean utility for each policy. We note that under the random policy, ξ_1, ξ_2, ξ_3 are independent and identically distributed uniform random variables on the

Policies	$\pi_1(t, X_t)$	$\pi_2(t, X_t)$	$C(t, X_t)$	Mean Utility
Riskless	0	0	$X_t/2$	5.9252
No Consumption	$X_t/3$	$X_t/3$	0	4.6904
No Consumption (alt.)	$X_t/4$	$X_t/2$	0	4.6952
No Bonds	$X_t/2$	$X_t/2$	$X_t/4$	5.6821
No Bonds (alt.)	$2X_t/3$	$X_t/3$	$X_t/3$	5.7994
Random	$\xi_1 \cdot X_t/2$	$\xi_2 \cdot X_t/2$	$\xi_3 \cdot X_t/4$	5.3393
Balanced Leverage	X_t	$-X_t/2$	$X_t/2$	5.9368
Moderate Leverage	$3X_t$	$-2.5X_t$	$X_t/3$	5.5259
High Leverage	$-5X_t$	$5X_t$	$2X_t/3$	5.2690
Extreme Leverage	$-8X_t$	$10X_t$	$X_t/4$	3.6834
Our Numerical Policy	$\hat{\pi}_1^*(t, X_t)$	$\hat{\pi}_2^*(t, X_t)$	$\hat{C}^*(t, X_t)$	6.3022

Table 1: Mean Utility Comparison

interval $[0,1]$. Table 1 demonstrates that $(\hat{\pi}^*, \hat{C}^*)$ outperforms other admissible policies in terms of mean utility.

7 Conclusions

In this paper, we studied the problem of multi-asset portfolio-consumption optimization with exponential O-U stock dynamics and stochastic discounting CRRA utility. We developed hybrid numerical methods to address the computational challenges of high-dimensional stochastic optimal control problems. Leveraging variable separation techniques, we reformulate the HJB equation as an ODE system and develop the ExpEuler-RK2 and Erow3-RK3 schemes for efficient numerical approximation. Most importantly, we derived a rigorous verification theorem with numerically verifiable conditions that guarantee the optimality of our solutions, which distinguishes our approach from existing numerical methods that typically lack theoretical guarantees. Our methods demonstrate significant computational advantages, with Erow3-RK3 achieving 41% faster computation than conventional grid-based approach, while maintaining third-order convergence. Numerical experiments confirm that the optimal policies computed by our method outperform alternative policies by achieving higher mean utility, validating both the accuracy and practical effectiveness of our methodology.

Acknowledgements

This work was supported by the National Key Research and Development Program of China under the grant No. 2020YFA0713602 and the Graduate Innovation Fund of Jilin University under the grant Nos. 2024KC036, 2025CX097 and 2025CX090.

References

- [1] Fred Espen Benth and Kenneth Hvistendahl Karlsen. “A Note on Merton’s Portfolio Selection Problem for the Schwartz Mean-Reversion Model”. In: *Stochastic Analysis and Applications* 23.4 (2005), pp. 687–704. DOI: [10.1081/SAP-200064457](https://doi.org/10.1081/SAP-200064457).
- [2] Brahim Berdjane and Serguei Pergamenshchikov. “Optimal consumption and investment for markets with random coefficients”. In: *Finance and Stochastics* 17.3 (2013), pp. 419–446. DOI: [10.1007/s00780-012-0193-0](https://doi.org/10.1007/s00780-012-0193-0).

- [3] Patrick Burns. “Random Portfolios for Performance Measurement”. In: *Optimisation, Econometric and Financial Analysis*. Ed. by Erricos John Kontoghiorghe and Cristian Gatu. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007, pp. 227–249. ISBN: 978-3-540-36626-3. DOI: [10.1007/3-540-36626-1_11](https://doi.org/10.1007/3-540-36626-1_11).
- [4] J.C. Butcher. “A history of Runge-Kutta methods”. In: *Applied Numerical Mathematics* 20.3 (1996), pp. 247–260. ISSN: 0168-9274. DOI: [https://doi.org/10.1016/0168-9274\(95\)00108-5](https://doi.org/10.1016/0168-9274(95)00108-5).
- [5] John Y. Campbell. “Stock returns and the term structure”. In: *Journal of Financial Economics* 18.2 (1987), pp. 373–399. ISSN: 0304-405X. DOI: [https://doi.org/10.1016/0304-405X\(87\)90045-6](https://doi.org/10.1016/0304-405X(87)90045-6).
- [6] Jérôme Darbon and Stanley Osher. *Algorithms for Overcoming the Curse of Dimensionality for Certain Hamilton-Jacobi Equations Arising in Control Theory and Elsewhere*. 2016. arXiv: [1605.01799 \[math.OC\]](https://arxiv.org/abs/1605.01799). URL: <https://arxiv.org/abs/1605.01799>.
- [7] Ivar Ekeland and Traian A. Pirvu. “Investment and consumption without commitment”. In: *Mathematical Finance and Economics* 2.1 (2008), pp. 57–86. DOI: [10.1007/s11579-008-0014-6](https://doi.org/10.1007/s11579-008-0014-6).
- [8] Eugene F. Fama and Kenneth R. French. “Permanent and Temporary Components of Stock Prices”. In: *Journal of Political Economy* 96.2 (1988), pp. 246–273. ISSN: 00223808, 1537534X. URL: <http://www.jstor.org/stable/1833108> (visited on 03/06/2025).
- [9] David F. Griffiths and Desmond J. Higham. *Numerical Methods for Ordinary Differential Equations. Initial Value Problems*. 1st ed. Springer Undergraduate Mathematics Series. Published: 25 November 2010; eBook Published: 11 November 2010. Springer London, 2010, pp. XIV+271. ISBN: 978-0-85729-147-9. DOI: [10.1007/978-0-85729-148-6](https://doi.org/10.1007/978-0-85729-148-6).
- [10] T. H. Gronwall. “Note on the Derivatives with Respect to a Parameter of the Solutions of a System of Differential Equations”. In: *Annals of Mathematics* 20.4 (1919), pp. 292–296. ISSN: 0003486X, 19398980. URL: <http://www.jstor.org/stable/1967124> (visited on 05/17/2025).
- [11] Marlis Hochbruck, Alexander Ostermann, and Julia Schweitzer. “Exponential Rosenbrock-Type Methods”. In: *SIAM Journal on Numerical Analysis* 47.1 (2009), pp. 786–803. DOI: [10.1137/080717717](https://doi.org/10.1137/080717717).
- [12] Raymond Kan, Xiaolu Wang, and Guofu Zhou. “Optimal Portfolio Choice with Estimation Risk: No Risk-Free Asset Case”. In: *Management Science* 68.3 (2022), pp. 2047–2068. DOI: [10.1287/mnsc.2021.3989](https://doi.org/10.1287/mnsc.2021.3989).
- [13] Ioannis Karatzas, John P. Lehoczky, and Steven E. Shreve. “Optimal Portfolio and Consumption Decisions for a “Small Investor” on a Finite Horizon”. In: *SIAM Journal on Control and Optimization* 25.6 (1987), pp. 1557–1586. DOI: [10.1137/0325086](https://doi.org/10.1137/0325086).
- [14] Ioannis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus*. 2nd ed. Vol. 113. Graduate Texts in Mathematics. New York, NY: Springer, 1998. ISBN: 978-0-387-97655-6. DOI: [10.1007/978-1-4612-0949-2](https://doi.org/10.1007/978-1-4612-0949-2).
- [15] Konstantinos Kassimatis. “Mean-variance versus utility maximization revisited: The case of constant relative risk aversion”. In: *International Review of Financial Analysis* 78 (2021), p. 101932. ISSN: 1057-5219. DOI: <https://doi.org/10.1016/j.irfa.2021.101932>.
- [16] Holger Kraft. “OPTIMAL PORTFOLIOS WITH STOCHASTIC SHORT RATE: PITFALLS WHEN THE SHORT RATE IS NON-GAUSSIAN OR THE MARKET PRICE OF RISK IS UNBOUNDED”. In: *International Journal of Theoretical and Applied Finance* 12.06 (2009), pp. 767–796. DOI: [10.1142/S0219024909005452](https://doi.org/10.1142/S0219024909005452).
- [17] P.G.L. Leach, J.G. O’Hara, and W. Sinkala. “Symmetry-based solution of a model for a combination of a risky investment and a riskless investment”. In: *Journal of Mathematical Analysis and Applications* 334.1 (2007), pp. 368–381. ISSN: 0022-247X. DOI: <https://doi.org/10.1016/j.jmaa.2006.11.056>.
- [18] Dongping Li, Xiuying Zhang, and Renyun Liu. “Exponential integrators for large-scale stiff Riccati differential equations”. In: *Journal of Computational and Applied Mathematics* 389 (2021), p. 113360. ISSN: 0377-0427. DOI: <https://doi.org/10.1016/j.cam.2020.113360>.
- [19] Xingjian Li, Deepanshu Verma, and Lars Ruthotto. “A Neural Network Approach for Stochastic Optimal Control”. In: *SIAM Journal on Scientific Computing* 46.5 (2024), pp. C535–C556. DOI: [10.1137/23M155832X](https://doi.org/10.1137/23M155832X).
- [20] Guiyuan Ma and Song-Ping Zhu. “Optimal investment and consumption under a continuous-time coin-tegration model with exponential utility”. In: *Quantitative Finance* 19.7 (2019), pp. 1135–1149. DOI: [10.1080/14697688.2019.1570317](https://doi.org/10.1080/14697688.2019.1570317).
- [21] Harry Markowitz. “Portfolio Selection”. In: *The Journal of Finance* 7.1 (1952), pp. 77–91. ISSN: 00221082, 15406261. URL: <http://www.jstor.org/stable/2975974> (visited on 08/04/2025).
- [22] A. M. Mathai and Serge B. Provost. *Quadratic forms in random variables : theory and applications*. Statistics : textbooks and monographs. M. Dekker, 1992. URL: <https://cir.nii.ac.jp/crid/1130000794501893504>.

- [23] Robert C Merton. “Optimum consumption and portfolio rules in a continuous-time model”. In: *Journal of Economic Theory* 3.4 (1971), pp. 373–413. ISSN: 0022-0531. DOI: [https://doi.org/10.1016/0022-0531\(71\)90038-X](https://doi.org/10.1016/0022-0531(71)90038-X).
- [24] Yingting Miao and Qiang Zhang. *Optimal Investment and Consumption Strategies with General and Linear Transaction Costs under CRRA Utility*. 2023. arXiv: [2304.07672 \[q-fin.PM\]](https://arxiv.org/abs/2304.07672). URL: <https://arxiv.org/abs/2304.07672>.
- [25] Huyền Pham. “Smooth Solutions to Optimal Investment Models with Stochastic Volatilities and Portfolio Constraints”. In: *Applied Mathematics & Optimization* 46.1 (2002), pp. 55–78. DOI: [10.1007/s00245-002-0735-5](https://doi.org/10.1007/s00245-002-0735-5).
- [26] Traian A. Pirvu and Huayue Zhang. “Investment–consumption with regime-switching discount rates”. In: *Mathematical Social Sciences* 71 (2014), pp. 142–150. ISSN: 0165-4896. DOI: <https://doi.org/10.1016/j.mathsocsci.2014.07.001>.
- [27] James M. Poterba and Lawrence H. Summers. “Mean reversion in stock prices: Evidence and Implications”. In: *Journal of Financial Economics* 22.1 (1988), pp. 27–59. ISSN: 0304-405X. DOI: [https://doi.org/10.1016/0304-405X\(88\)90021-9](https://doi.org/10.1016/0304-405X(88)90021-9).
- [28] Paul van Rensburg. “Optimal portfolio leverage”. In: *Journal of Asset Management* 17.1 (2016), pp. 22–33. ISSN: 1479-179X. DOI: [10.1057/jam.2015.36](https://doi.org/10.1057/jam.2015.36).
- [29] Eduardo S. Schwartz. “The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging”. In: *The Journal of Finance* 52.3 (1997), pp. 923–973. DOI: <https://doi.org/10.1111/j.1540-6261.1997.tb02721.x>.
- [30] Jie Shen and Haijun Yu. “Efficient Spectral Sparse Grid Methods and Applications to High-Dimensional Elliptic Equations II. Unbounded Domains”. In: *SIAM Journal on Scientific Computing* 34.2 (2012), A1141–A1164. DOI: [10.1137/110834950](https://doi.org/10.1137/110834950).
- [31] Gyoocheol Shim and Yong Hyun Shin. “Portfolio Selection with Subsistence Consumption Constraints and CARA Utility”. In: *Mathematical Problems in Engineering* 2014.1 (2014), p. 153793. DOI: <https://doi.org/10.1155/2014/153793>.
- [32] Wan-Yu Tsai and Arash Fahim. “A numerical scheme for a singular control problem: Investment–consumption under proportional transaction costs”. In: *Journal of Computational and Applied Mathematics* 333 (2018), pp. 170–184. ISSN: 0377-0427. DOI: <https://doi.org/10.1016/j.cam.2017.10.035>.
- [33] P. Vatiwutipong and N. Phewchean. “Alternative way to derive the distribution of the multivariate Ornstein–Uhlenbeck process”. In: *Advances in Difference Equations* 2019.1 (2019), p. 276. DOI: [10.1186/s13662-019-2214-1](https://doi.org/10.1186/s13662-019-2214-1).
- [34] Jessica A. Wachter. “Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets”. In: *The Journal of Financial and Quantitative Analysis* 37.1 (2002), pp. 63–91. ISSN: 00221090, 17566916. URL: <http://www.jstor.org/stable/3594995> (visited on 08/04/2025).
- [35] Peter P. Wakker. “Explaining the characteristics of the power (CRRA) utility family”. In: *Health Economics* 17.12 (2008), pp. 1329–1344. DOI: <https://doi.org/10.1002/hec.1331>.
- [36] Jinchun Ye. “A numerical method for consumption-portfolio problems”. In: *Proceedings of the 2011 American Control Conference*. 2011, pp. 3868–3873. DOI: [10.1109/ACC.2011.5991218](https://doi.org/10.1109/ACC.2011.5991218).
- [37] Jiongmin Yong and Xun Yu Zhou. *Stochastic Controls: Hamiltonian Systems and HJB Equations*. 1st ed. Vol. 43. Stochastic Modelling and Applied Probability. Springer Science+Business Media New York. New York, NY: Springer, 1999. ISBN: 978-0-387-98723-1. DOI: [10.1007/978-1-4612-1466-3](https://doi.org/10.1007/978-1-4612-1466-3).
- [38] Thaleia Zariphopoulou. “Optimal Investment and Consumption Models with Non-linear Stock Dynamics”. In: *Mathematical Methods of Operations Research* 50.2 (1999), pp. 271–296. ISSN: 1432-5217. DOI: [10.1007/s001860050098](https://doi.org/10.1007/s001860050098).