TRANSVERSALITY METHODS FOR HOMOTOPY GROUPS OF STABLE LOCI IN AFFINE GIT QUOTIENTS

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ABSTRACT. We investigate the homotopy groups of stable loci in affine Geometric Invariant Theory (GIT), arising from linear actions of complex reductive algebraic groups on complex affine spaces. Our approach extends the infinite-dimensional transversality framework of Daskalopoulos–Uhlenbeck and Wilkin to this general GIT setting. Central to our method is the construction of a G-equivariant holomorphic vector bundle over the conjugation orbit of a one-parameter subgroup (1-PS), whose fibres are precisely the negative weight spaces determining instability.

A key proposition establishes that a naturally defined evaluation map is transverse to the zero section of this bundle, implying that generic homotopies avoid all unstable and strictly semistable strata under certain dimensional inequalities. Consequently, our main theorem shows that the stable locus $V^{st}(\rho)$ is $(d_{\min}-2)$ -connected. The connectivity bound is defined as $d_{\min}=\min_j(2m_j-2\dim_{\mathbb{C}}(G\cdot\lambda_j))$, where m_j are the ranks of the negative weight spaces and $\dim_{\mathbb{C}}(G\cdot\lambda_j)$ are the dimensions of the relevant 1-PS orbits.

Our result also covers cases where semistability does not coincide with stability. The applicability of this framework is illustrated on several examples. For the 2-Kronecker quiver, our conclusions precisely match the known S^3 topology of the stable locus. In linear control theory, where GIT stability corresponds to the notion of controllability, our results determine the connectivity of the space of controllable systems. In statistical modelling, where stability for star-shaped Gaussian model corresponds to the existence of a unique Maximum Likelihood Estimate, we compute the connectivity of the space of data samples that yield such a unique estimate, providing topological insight into the problem of parameter non-identifiability.

1. Introduction

Geometric Invariant Theory (GIT) provides a powerful framework for constructing moduli spaces in terms of quotients by group actions [10]. This paper utilises GIT techniques to investigate the homotopy groups of stable loci arising from linear actions of complex reductive algebraic groups on complex affine spaces.

Let G be a complex reductive algebraic group acting linearly on a complex affine space $V \cong \mathbb{C}^N$ via a linear representation $\sigma: G \to \operatorname{GL}(V)$. Given a character $\rho: G \to \mathbb{C}^\times$, one can define stability conditions on V using semi-invariant polynomials transforming under the action of G with weights determined by ρ . V can then be stratified into ρ -stable, ρ -semistable, and ρ -unstable loci. Hoskins [8] showed that the Hesselink's stratification by adapted one-parameter subgroups (1-PS) coincides with the Morse-theoretic stratification induced by the norm square of the moment map associated with a maximal compact subgroup $K \subset G$.

Daskalopoulos and Uhlenbeck [3] employed a method based on infinite-dimensional transversality arguments to compute low-dimensional homotopy and cohomology groups of the moduli spaces of stable vector bundles on a Riemann surface, particularly in the non-coprime rank and degree case where the stable and semistable loci do not coincide. Their method demonstrated that the space of stable holomorphic structures, \mathfrak{A}_s , is 2(g-1)(n-1)-connected, where $g \geq 2$ is the genus of the Riemann surface. Wilkin [11] adapted this transversality framework to the more general setting of α -stable quiver representations. He established that the space of α -stable representations, $\operatorname{Rep}(Q,v)^{\alpha-st}$, has trivial homotopy group up to a dimension $d_{min}(Q,v,\alpha)-1$. Wilkin [11, Theorem 3.1 & Corollary 3.2] showed that generic homotopies avoid unstable representations, thus proving the triviality of these homotopy groups and subsequently determining those of the moduli space $\mathcal{M}_{\alpha}(Q,v)^{st}$.

This paper extends and generalises the infinite-dimensional transversality methods to consider the topology of stable loci in GIT quotients of affine spaces by complex reductive groups. In Section 2, we construct a G-equivariant holomorphic vector bundle $W \to G \cdot \lambda$ over the conjugation orbit of a one-parameter subgroup λ , whose fibres $W_{\lambda'}$ are the negative weight spaces $V(\lambda')_-$ corresponding to each conjugate λ' . Proposition 2.3 establishes the result that a map D defined from the product of a space $\mathcal F$ of homotopies and $G \cdot \lambda$ into the bundle W is transverse to the zero section $\mathcal O_W$. In particular, $D^{-1}(\mathcal O_W)$ precisely consists of points $(f(s,t),\lambda')$ where f(s,t) is destabilised by the 1-PS λ' , i.e., it lies entirely within non-negative weight spaces.

As in previous works by Daskalopoulos & Uhlenbeck and Wilkin, this transversality condition implies that, for generic homotopies $f \in \mathcal{F}$ satisfying the dimension inequality $n+1+dim_{\mathbb{R}}(G \cdot \lambda) < 2 \cdot \mathrm{rank}_{\mathbb{C}}(W)$, the image f(s,t) does not intersect all unstable and strictly semistable strata associated with $\lambda' \in G \cdot \lambda$. Consequently, the main theorem Theorem 2.5 proves that the stable locus $V^{st}(\rho)$ is $(d_{min}-2)$ -connected, where d_{min} is defined in terms of the minimal rank of the negative weight spaces. Moreover, this result encompasses scenarios where semistability does not coincide with stability, and the transversality techniques employed in this paper thus provide

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a method for determining the connectivity and calculating the homotopy groups of stable loci in general affine GIT settings.

Section 3 contains examples illustrating the applicability of our general results. We first consider the stable locus associated with a 2-Kronecker quiver with dimension vector (1,1). The stable locus in this case is known to be homotopy equivalent to S^3 . Our main theorem successfully recovers this fact, showing that the relevant homotopy groups are trivial, which confirms the validity of our approach.

Next, we examine the space of linear control systems $\dot{x} = Ax + Bu$ involving the action of $\mathrm{GL}_n(\mathbb{C})$ on the space of systems (A,B), for which the GIT notion of stability coincides exactly with controllability [6]. Applying our transversality results to this setting enables us to deduce homotopy properties of the space of controllable systems. The resulting path-connectedness, for instance, implies that any controllable system (A,B) can be continuously deformed into any other controllable system (A',B') through a path consisting entirely of controllable systems.

The third example connects our framework to statistical modelling by considering the space of samples for star-shaped Directed Acyclic Graph (DAG) models, which represents a linear regression problem. In this context, GIT stability corresponds to the existence and uniqueness of the Maximum Likelihood Estimation (MLE). Applying our transversality results allows us to determine the connectivity of the space V^{st} of samples with a unique MLE [2]. For instance, the path-connectedness implies that any sample f with a unique MLE can be continuously deformed into any other such sample through a continuous path that lies entirely in V^{st} . This demonstrates that the property of having a unique MLE is robust to small perturbations in the data and that the space of well-behaved data forms a single, connected space, which thus provides a foundation for methods that resolve the non-identifiability of degenerate data by perturbing the data towards this stable locus.

The paper is organised as follows. In Section 2, we lay out the general theoretical framework, introducing the necessary GIT background, and proving the key transversality result and its implications for homotopy groups. Section 3 provides detailed examples that demonstrate the application of this framework to specific problems in quiver representations, control theory, and statistical modelling, illustrating how the general results can be used to obtain concrete information about the topology of stable loci.

2. Geometric Invariant Theory and Transversality

Let G be a complex reductive algebraic group acting linearly on a complex affine space $V \cong \mathbb{C}^N$ for some $N = \dim_{\mathbb{C}} V$, via a faithful linear representation $\sigma: G \to \operatorname{GL}(V)$. The G-action on V induces an action on its ring or regular functions, $\mathbb{C}[V]$, defined by $(g \cdot f)(v) = f(g^{-1} \cdot v)$ for $g \in G, v \in V, f \in \mathbb{C}[V]$. We denote the subring of G-invariant functions by $\mathbb{C}[V]^G$, and then the affine GIT quotient is defined as $V//G := \operatorname{Spec} \mathbb{C}[V]^G$. Note that $\mathbb{C}[V]^G$ being finitely generated \mathbb{C} -algebra [5] ensures that the quotient V//G is an affine variety.

Let $\rho:G\to\mathbb{C}^\times$ be a character of G. A regular function $f\in\mathbb{C}[V]$ is a semi-invariant of weight ρ^n if it transforms under the G-action associated to $f(g\cdot v)=\rho(g)^nf(v)$ for all $g\in G, v\in V$. The set of such functions for a fixed n is denoted $\mathbb{C}[V]_{\rho^n}^G$, and the collection of all such semi-invariants forms a graded ring $\bigoplus_{n\in\mathbb{Z}}\mathbb{C}[V]_{\rho^n}^G$. With a chosen character ρ , we can define notions of stability for points in V.

Definition 2.1. Let $v \in V$.

- v is ρ -semistable if there exists a non-constant ρ^n -semi-invariant polynomial $f \in \mathbb{C}[V]_{\rho^n}^G$ for some integer n > 0 such that $f(v) \neq 0$. The set of all semi-stable points form the ρ -semistable locus $V^{ss}(\rho)$.
- A ρ -semistable point v is ρ -stable if its G-orbit $G \cdot v$ is closed in $V^{ss}(\rho)$, and its stabiliser subgroup $G_v = \{g \in G \mid g \cdot v = v\}$ is finite. The set of such points is denoted $V^{st}(\rho)$.
- v is ρ -unstable if it is not ρ -semistable. This means f(v) = 0 for all non-constant $f \in \mathbb{C}[V]_{\rho^n}^G$ with n > 0, and we have $V^{us}(\rho) = V \setminus V^{ss}(\rho)$.

A one-parameter subgroup of G is a group homomorphism $\lambda: \mathbb{C}^{\times} \to G$. The Hilbert-Mumford Criterion provides a numerical way to check these stability conditions by examining the behaviour of points under the action of one-parameter subgroups (1-PSs) of G.

Let $K \subset G$ be the maximal compact subgroup of G, and V be endowed with a K-invariant Hermitian inner product. Fix a non-trivial one-parameter subgroup $\lambda: S^1 \to G$. The conjugation orbit of λ is

$$G \cdot \lambda := \{g\lambda g^{-1} \mid g \in G\} \cong G/C_G(\lambda),$$

where $C_G(\lambda)$ is the centraliser of $\lambda(S^1)$ in G. In particular, $G \cdot \lambda$ is a smooth projective variety. For each $\lambda' \in G \cdot \lambda$, the representation V decomposes into

$$V = \bigoplus_{i \in \mathbb{Z}} V_i(\lambda'), \quad V_i(\lambda') = \{ v \in V \mid \lambda'(t) \cdot v = t^i v \},$$

and so we can denote

$$V(\lambda')_- = \bigoplus_{i < 0} V_i(\lambda'), \quad V(\lambda')_+ = \bigoplus_{i \ge 0} V_i(\lambda').$$

Set

(1)
$$W := \{ (\lambda', v) \in (G \cdot \lambda) \times V \mid v \in V(\lambda')_{-} \}.$$

The projection $\pi: W \to G \cdot \lambda$ defined by $(\lambda', v) \mapsto \lambda'$ is then a vector bundle. Then the *zero section* of W is $\mathcal{O}_W := \{(\lambda', 0) \mid \lambda' \in G \cdot \lambda\}.$

Proposition 2.2. $\pi: W \to G \cdot \lambda$ is a holomorphic G-equivariant vector bundle of rank $m := \dim_{\mathbb{C}} V(\lambda)_{-}$, which is isomorphic to the associated bundle $G \times_{C_G(\lambda)} V(\lambda)_{-} \to G/C_G(\lambda)$.

Proof. Consider the map $\Phi: G \times V(\lambda)_- \to W$ defined by

$$\Phi(g, v) = (g\lambda g^{-1}, g \cdot v).$$

To check that the codomain is W, let $\lambda'=g\lambda g^{-1}$, and consider the action of $\lambda'(t)$ on $g\cdot v_i$, which is given by $\lambda'(t)(g\cdot v_i)=(g\lambda(t)g^{-1})(g\cdot v_i)=g\lambda(t)(g^{-1}g)v_i=g\lambda(t)v_i=g(t^iv_i)=t^i(g\cdot v_i)$. Hence for any $(g,v)\in G\times V(\lambda)_-$, the vector $g\cdot v$ lies in $V(g\lambda g^{-1})_-$, so $\Phi(g,v)\in W$. Moreover, let $n\in C_G(\lambda)$, then

$$\Phi(gn^{-1}, n \cdot v) = (gn^{-1}\lambda ng^{-1}, gn^{-1}n \cdot v) = (g\lambda g^{-1}, g \cdot v) = \Phi(g, v),$$

which implies that Φ is invariant under the $C_G(\lambda)$ -action.

Note that $G \times_{C_G(\lambda)} V(\lambda)_- := (G \times V(\lambda)_-)/\sim$, where $(g_1, v_1) \sim (g_2, v_2)$ if and only if there exists an $n \in C_G(\lambda)$ such that $(g_2, v_2) = (g_1 n^{-1}, n \cdot v_1)$, therefore Φ descends to a well-defined map

$$\widetilde{\Phi}: G \times_{C_G(\lambda)} V(\lambda)_- \to W, \quad \widetilde{\Phi}([g,v]) = \Phi(g,v) = (g\lambda g^{-1}, g \cdot v),$$

and we shall prove that $\widetilde{\Phi}$ is bijective.

For the surjectivity, let $(\lambda',w)\in W$ be an arbitrarily chosen element, then there exists some $g\in G$ such that $\lambda'=g\lambda g^{-1}$. Define $v:=g^{-1}\cdot w$, then for $v_j=g^{-1}w_j$ we have $\lambda(t)v_j=\lambda(t)(g^{-1}w_j)=g^{-1}(g\lambda(t)g^{-1})w_j=g^{-1}(t^jw_j)=t^j(g^{-1}w_j)=t^jv_j$ Thus $v_j\in V(\lambda)_-$, which implies that $v=g^{-1}w\in V(\lambda)_-$. Now consider $[g,v]\in G\times_{C_G(\lambda)}V(\lambda)_-$, observe that $\widetilde{\Phi}([g,v])=(g\lambda g^{-1},g\cdot v)=(\lambda',g\cdot (g^{-1}w))=(\lambda',w)$. Hence $\widetilde{\Phi}$ is surjective.

To prove the injectivity, assume $\widetilde{\Phi}([g_1,v_1]) = \widetilde{\Phi}([g_2,v_2])$. This means $(g_1\lambda g_1^{-1},g_1\cdot v_1) = (g_2\lambda g_2^{-1},g_2\cdot v_2)$, then $g_1\lambda g_1^{-1} = g_2\lambda g_2^{-1}$ implies that $(g_2^{-1}g_1)\lambda(g_2^{-1}g_1)^{-1} = \lambda$. Let $h = g_2^{-1}g_1 \in C_G(\lambda)$, which gives $g_1 = g_2h$. Furthermore, note that from $g_1\cdot v_1 = g_2\cdot v_2$ we can get $(g_2h)\cdot v_1 = g_2\cdot v_2$, implying $h\cdot v_1 = v_2$. Thus we have $[g_1,v_1]=[g_2h,v_1]$, which is equivalent to $((g_2h)h^{-1},h\cdot v_1)=(g_2,h\cdot v_1)=(g_2,v_2)$. Therefore, we have shown that $[g_1,v_1]=[g_2,v_2]$, establishing the injectivity, and we can thus conclude that $\widetilde{\Phi}$ is bijective.

Now we show the holomorphicity of Φ . Note that G is complex Lie group, of which the group multiplication, inversion, and the conjugation are holomorphic maps. And the G-action on V is a holomorphic representation, thus Φ is a holomorphic map. Since $C_G(\lambda)$ is a closed complex subgroup of G, the $C_G(\lambda)$ -action is also holomorphic, which is in addition free and proper, hence the quotient $G \times_{C_G(\lambda)} V(\lambda)_-$ is a complex manifold and the quotient map $q:G\times V(\lambda)_- \to G\times_{C_G(\lambda)} V(\lambda)_-$ is a holomorphic submersion. Because $\Phi=\widetilde{\Phi}\circ q$, it follows that $\widetilde{\Phi}$ is holomorphic. It is furthermore a biholomorphism by the Inverse Function theorem, where it suffices to show the complex differential $d\widetilde{\Phi}_{[g,v]}$ is a \mathbb{C} -linear isomorphism at every point $[g,v]\in G\times_{C_G(\lambda)} V(\lambda)_-$, and thus W can be endowed with a complex manifold structure making $\pi:W\to G\cdot\lambda$ a holomorphic map.

Since $\widetilde{\Phi}$ is a biholomorphism that is also linear on the fibres and commutes with projections, it is an isomorphism of holomorphic vector bundles. Thus W inherits the structure of a holomorphic vector bundle from $G \times_{C_G(\lambda)} V(\lambda)_-$. To see the G-equivariance of $\widetilde{\Phi}$, let $g_0 \in G$ be arbitrary. Then we have $\widetilde{\Phi}(g_0 \cdot [g, v]) = ((g_0 g) \lambda (g_0 g)^{-1}, (g_0 g) \cdot v) = (g_0 (g \lambda g^{-1}) g_0^{-1}, g_0 (g \cdot v)) = g_0 \cdot (g \lambda g^{-1}, g \cdot v) = g_0 \cdot \widetilde{\Phi}([g, v]).$

 $(g_0(g\lambda g^{-1})g_0^{-1},g_0(g\cdot v))=g_0\cdot (g\lambda g^{-1},g\cdot v)=g_0\cdot \widetilde{\Phi}([g,v]).$ Note that the fibre $W_{\lambda'}=V_{i<0}(\lambda').$ If $\lambda'=g\lambda g^{-1},$ the map $v\mapsto g\cdot v$ restricts to an isomorphism from $V(\lambda)_-$ to $V(\lambda')_-.$ Hence $\dim_{\mathbb{C}}V(\lambda')_-=\dim_{\mathbb{C}}V(\lambda)_-$ for all $\lambda'\in G\cdot \lambda.$ Therefore, the rank of the vector bundle W is $m=\dim_{\mathbb{C}}V(\lambda)_-.$

Denote $E = G \times_{C_G(\lambda)} V(\lambda)_-$, $p_E : E \to G/C_G(\lambda)$, then we have the following commutative diagram

$$E \xrightarrow{\widetilde{\Phi}} W \qquad \downarrow_{\pi} ,$$

$$G/C_G(\lambda) \xrightarrow{\phi} G \cdot \lambda$$

where ϕ is the canonical isomorphism. So $\pi = \phi \circ p_E \circ \widetilde{\Phi}^{-1}$, and thus we can conclude that π is a holomorphic G-equivariant vector bundle that is isomorphic to the associated bundle, as stated.

Fix $n \geq 0$, a based map $f_0: S^n \to V^{st}$ with $f_0(s_0) = x_0$, and let

$$\mathcal{F} := \{ f : S^n \times I \to V \mid f(s,0) = f_0(s), f(s_0,t) = x_0, f(s,1) = x_0 \},$$

where $s \in S^n$, and $t \in I$, then its tangent space is

$$T_f \mathcal{F} = \{ \dot{f} \in C^r(S^n \times I, V) \mid \dot{f}(s, 0) = 0, \dot{f}(s_0, t) = 0, \dot{f}(s, 1) = 0 \}.$$

For $\lambda' \in G \cdot \lambda$ let $\pi_{\lambda'}^{\perp} : V \to V(\lambda')_{\perp}$ denote the orthogonal projection. Define

$$D: \mathcal{F} \times S^n \times I \times (G \cdot \lambda) \to W, \quad D(f, s, t, \lambda') = (\lambda', \pi_{\lambda'}^{\perp}(f(s, t))).$$

Let $p=(f,s,t,\lambda')\in \mathcal{F}\times S^n\times I\times (G\cdot\lambda)$, where $x:=f(s,t)\in V$. Then a tangent vector at p in $T_p(\mathcal{F}\times S^n\times I\times G\cdot\lambda)$ can be written as (\dot{f},v_s,v_t,ξ) . Then

$$dD_p(\dot{f}, v_s, v_t, \xi) = (\xi, \pi_{\lambda'}^{\perp}(\dot{f}(s, t) + v_s\partial_s f(s, t) + v_t\partial_t f(s, t)) + (d\pi_{\bullet}^{\perp})_{(\lambda', x)}(\xi)),$$

where $(d\pi^{\perp}_{\bullet})_{(\lambda',x)}(\xi) \in V(\lambda')_{-}$ denotes the derivative, with respect to λ' , of $\pi^{\perp}_{\lambda'}$, evaluated at $x \in V$ and at direction $\xi \in T_{\lambda'}(G \cdot \lambda)$. To deduce the above formula, we start by writing the evaluation map

$$\operatorname{ev}: \mathcal{F} \times S^n \times I \to V, \quad (f, s, t) \mapsto f(s, t).$$

Its differential at (f, s, t) is

$$d(\operatorname{ev}_{(f,s,t)})(\dot{f},v_s,v_t) = \dot{f}(s,t) + v_s \partial_s f(s,t) + v_t \partial_t f(s,t) \in V.$$

Next define $\Phi: (G \cdot \lambda) \times V \to W$ by $\Phi(\lambda', v) := (\lambda', \pi_{\lambda'}^{\perp}(v))$, and thus by definition $D = \Phi \circ (\mathrm{id}_{G \cdot \lambda}, \mathrm{ev})$. Pick an arbitrary (λ', x) , we have

$$d\Phi_{(\lambda',x)}(\xi,u) = (\xi, \pi_{\lambda'}^{\perp}(u) + (d\pi_{\bullet}^{\perp})_{(\lambda',x)}(\xi)),$$

where u is a tangent vector of V. Then at the point $p=(f,s,t,\lambda')$ we have $dD_p=d\Phi_{(\lambda',x)}\circ(\mathrm{id},d(\mathrm{ev})_{(f,s,t)})$. Substituting (\dot{f},v_s,v_t,ξ) and use the above equations yields

$$dD_p(\dot{f}, v_s, v_t, \xi) = d\Phi_{(\lambda', x)}(\xi, \dot{f}(s, t) + v_s \partial_s f(s, t) + v_t \partial_t f(s, t))$$

$$= (\xi, \pi_{\lambda'}^{\perp}(\dot{f}(s, t) + v_s \partial_s f(s, t) + v_t \partial_t f(s, t)) + (d\pi_{\bullet}^{\perp})_{(\lambda', x)}(\xi)),$$
(2)

as desired.

Proposition 2.3. The map D is transverse to the zero section $\mathcal{O}_W \subset W$; i.e. for every $p = (f, s, t, \lambda')$ satisfying $D(p) \in \mathcal{O}_W$ we have

$$dD_p(T_pM) + T_{D(p)}\mathcal{O}_W = T_{D(p)}W,$$

where $M = \mathcal{F} \times S^n \times I \times (G \cdot \lambda)$. Moreover, D_f is transverse to the zero section \mathcal{O}_W for f in a residual subset of \mathcal{F} .

Proof. Choose $p=(f,s,t,\lambda')$ with $D(p)=(\lambda',0)\in\mathcal{O}_W$. Since $W\to G\cdot\lambda$ is a vector bundle with fibre $V(\lambda')_-$, we have splittings

$$T_{(\lambda',0)}W = T_{\lambda'}(G \cdot \lambda) \oplus V(\lambda')_-, \quad T_{(\lambda',0)}\mathcal{O}_W = T_{\lambda'}(G \cdot \lambda) \oplus \{0\}.$$

Recall that the previous computations (2) imply that the differential at p, where x = f(s,t) and $\pi_{\lambda'}^{\perp}(x) = 0$, is given by

$$dD_p(\dot{f}, v_s, v_t, \xi) = (\xi, \pi_{\lambda'}^{\perp}(\dot{f}(s, t) + v_s \,\partial_s f(s, t) + v_t \,\partial_t f(s, t)) + (d\pi_{\bullet}^{\perp})_{(\lambda', x)}(\xi)).$$

Now fix an arbitrary vector $v \in V(\lambda')_-$, and choose a bump function $\beta: S^n \times I \to \mathbb{R}$ such that

- $\beta(s,t) = 1$,
- supp β is contained in a small ball that does not intersect the boundary $(S^n \times \{0,1\}) \cup (\{s_0\} \times I)$.

Consequently, to show surjectivity onto $V(\lambda')_-$, we consider variations with $\xi = 0$. In this case, $(d\pi^{\perp}_{\bullet})_{(\lambda',x)}(0) = 0$ because this term is linear in ξ . Thus, the differential simplifies to

$$dD_p(\dot{f}, v_s, v_t, 0) = (0, \pi_{\lambda'}^{\perp}(\dot{f}(s, t) + v_s \partial_s f(s, t) + v_t \partial_t f(s, t))).$$

Now setting $\dot{f}=\beta v, v_s=v_t=0$ gives $\pi^\perp_{\lambda'}(\dot{f}(s,t))=\pi^\perp_{\lambda'}(v)=v.$ Therefore the differential of D at p becomes

$$dD_p(\dot{f}, 0, 0, 0) = (0, v).$$

Since v is arbitrary, we have $V(\lambda')_- \subset \operatorname{Im}(dD_p)$. Hence we compute

$$dD_p(T_pM) + T_{(\lambda',0)}\mathcal{O}_W = \operatorname{Im}(dD_p) + (T_{\lambda'}(G \cdot \lambda) \oplus \{0\}) = T_{\lambda'}(G \cdot \lambda) \oplus V(\lambda')_- = T_{(\lambda',0)}W,$$

which then implies that D is transverse to \mathcal{O}_W at p. And since p was arbitrarily chosen, we conclude that D is transverse to \mathcal{O}_W globally.

Consider $X = S^n \times I \times G \cdot \lambda$, and define

$$D_f := D|_{\{f\} \times X} : X \to W, \quad (s, t, \lambda') \mapsto (\lambda', \pi_{\lambda'}^{\perp}(f(s, t))).$$

Then the subset $\mathcal{F}^{\pitchfork} = \{ f \in \mathcal{F} \mid D_f \pitchfork \mathcal{O}_W \}$ is residual in \mathcal{F} . Therefore D_f is transverse to the zero section \mathcal{O}_W for f in a residual subset by Thom transversality theorem [7, Theorem 2.1].

Corollary 2.4. Let $f \in \mathcal{F}$ be a generic homotopy. If the rank $m = \dim_{\mathbb{C}} V(\lambda)_{-}$ of the vector bundle $W \to G \cdot \lambda$ satisfies the condition

$$(3) n+1+2\dim_{\mathbb{C}}(G\cdot\lambda)<2m,$$

then the preimage $D_f^{-1}(\mathcal{O}_W)$ is empty.

Proof. Assume the preimage $D_f^{-1}(\mathcal{O}_W)$ is non-empty. Take any point $x \in D_f^{-1}(\mathcal{O}_W) \subset X = S^n \times I \times (G \cdot \lambda)$, and set $y = D_f(x) \in \mathcal{O}_W$. Because D_f is transverse to \mathcal{O}_W , we have the equality of tangent spaces

$$T_y W = dD_f(T_x X) + T_y \mathcal{O}_W.$$

Hence

$$\dim_{\mathbb{R}} T_y W = \dim_{\mathbb{R}} (dD_f(T_x X)) + \dim_{\mathbb{R}} T_y \mathcal{O}_W - \dim_{\mathbb{R}} (dD_f(T_x X)) \cap T_y \mathcal{O}_W.$$

Note that $\dim_{\mathbb{R}} T_y W = 2m + 2 \dim_{\mathbb{C}} (G \cdot \lambda)$, $\dim_{\mathbb{R}} T_y \mathcal{O}_W = 2 \dim_{\mathbb{C}} (G \cdot \lambda)$, and since the dimension of the intersection $dD_f(T_x X) \cap T_y \mathcal{O}_W$ is non-negative, we obtain the inequality

$$2m \le \dim_{\mathbb{R}}(dD_f(T_xX)) \le \dim_{\mathbb{R}} T_xX = n + 1 + 2\dim_{\mathbb{C}}(G \cdot \lambda),$$

where the second inequality follows from the rank-nullity theorem as the differential dD_f is a linear map.

The original hypothesis of the corollary is the negation of the inequality obtained above, therefore we have established that $D_f^{-1}(\mathcal{O}_W)$ must be empty.

 $D_f^{-1}(\mathcal{O}_W)$ being empty for the chosen $G\cdot\lambda$ means that for any point (s,t) in the homotopy and any $\lambda'\in G\cdot\lambda$, the point f(s,t) is non-zero. Consequently, the Hilbert-Mumford weight $\mu(f(s,t),\lambda')>0$, which implies that no λ' from this particular class can be used to show that f(s,t) is unstable or strictly semistable.

To ensure a homotopy lies entirely within the stable locus, this must hold for every class of 1-PS that can characterise non-stable points. Note that the Hesselink stratification of the non-stable locus $V^{us}(\rho) \cup (V^{ss}(\rho) \setminus V^{st}(\rho))$ is a finite union of strata, each associated with a particular conjugacy class of 1-PSs that can cause instability [8].

Let $\{[\lambda_1], [\lambda_2], ..., [\lambda_k]\}$ be the finite set of conjugacy classes of 1-PSs that characterise all possible ways a point can be ρ -unstable or strictly ρ -semistable. For each class $[\lambda_j]$, let W_j be the corresponding vector bundle defined in (1) whose fibre rank $m_j = \dim_{\mathbb{C}} V(\lambda_j)_-$ satisfies inequality (3). If the condition

$$n+1+2\dim_{\mathbb{C}}(G\cdot\lambda_j)<2m_j$$

holds for each class $[\lambda_j]$ with j=1,...,k, then Corollary 2.4 implies that there is a residual set $\mathcal{F}_j^{\pitchfork} \subset \mathcal{F}$ of homotopies f such that $D_f^{-1}(\mathcal{O}_{W_j})$ is empty. The intersection $\bigcap_{j=1}^k \mathcal{F}_j^{\pitchfork}$ is also a residual subset of \mathcal{F} because a countable intersection of residual sets is residual. Any homotopy f in the intersection avoids every Hesselink stratum, which implies that its image lies entirely in the stable locus $V^{st}(\rho)$.

Theorem 2.5. Define $d_{min} = \min_{j \in \{1,...,k\}} \{2m_j - 2\dim_{\mathbb{C}}(G \cdot \lambda_j)\}$. Let n be a non-negative integer such that $n+1 < d_{min}$, then the n-th homotopy group of the stable locus is trivial, in other words,

$$\pi_n(V^{st}(\rho)) = 0.$$

Equivalently, $V^{st}(\rho)$ is $(d_{min}-2)$ -connected.

Note that the principal G-bundle $G \to V^{st}(\rho) \to \mathcal{M}^{st}(\rho)$ induces a long exact sequence in homotopy groups

$$\cdots \to \pi_n(G) \to \pi_n(V^{st}(\rho)) \to \pi_n(\mathcal{M}^{st}(\rho)) \to \pi_{n-1}(G) \to \pi_{n-1}(V^{st}(\rho)) \to \cdots,$$

we have also assumed that $\pi_k(V^{st}(\rho)) = 0$ for $k \leq d_{min} - 2$. Then if k = 0, which means $0 \leq d_{min} - 2$, then $\pi_0(V^{st}(\rho)) = 0$. Thus we have $0 \to \pi_0(\mathcal{M}^{st}(\rho)) \to 0$, which then implies that $\pi_0(\mathcal{M}^{st}(\rho)) = 0$. Hence $\mathcal{M}^{st}(\rho)$ is path-connected.

For higher homotopy groups, consider $1 \le k < d_{min} - 2$. We observe from the long exact sequence that

- if $k \leq d_{min} 2$, then $\pi_k(V^{st}(\rho)) = 0$;
- if $i-1 \leq d_{min}-3 < d_{min}-2$, then $\pi_{k-1}(V^{st}(\rho))=0$. The corresponding segment of the long exact sequence $\pi_k(V^{st}(\rho)) \to \pi_k(\mathcal{M}^{st}(\rho)) \to \pi_{k-1}(G) \to \pi_{k-1}(V^{st}(\rho))$ becomes $0 \to \pi_k(\mathcal{M}^{st}(\rho)) \to \pi_{k-1}(G) \to 0$. The exactness implies that the connecting homomorphism $\partial_k : \pi_k((\mathcal{M}^{st}(\rho)) \to \pi_{k-1}(G))$ is an isomorphism.

This leads to the following corollary:

Corollary 2.6. Assume G acts freely on $V^{st}(\rho)$ with quotient $\mathcal{M}^{st}(\rho) := V^{st}(\rho)/G$, and let d_{min} be defined as in the preceding theorem. If n is a non-negative integer such that $n+1 < d_{min}$, then the homotopy groups of the quotient $\mathcal{M}^{st}(\rho)$ are related to the homotopy groups of G by an isomorphism

$$\pi_n(\mathcal{M}^{st}(\rho)) \cong \pi_{n-1}(G)$$

for $1 \le n < d_{min} - 1$. Furthermore, if $d_{min} > 1$, then $\mathcal{M}^{st}(\rho)$ is path-connected.

3. Examples

3.1. **Stable Locus of the 2-Kronecker Quiver.** Consider the 2-Kronecker quiver with vertices 1, 2 and two arrows $a, b: 1 \to 2$. Let the dimension vector $\mathbf{v} := (\dim V_1, \dim V_2) = (1, 1)$, and choose the stability parameter $\alpha = (1, -1)$. Then this gives

$$\alpha_1 \dim V_1 + \alpha_2 \dim V_2 = 1 + (-1) = 0,$$

which means α is admissible.

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A representation in this case is a pair $(A,B) \cong \operatorname{Hom}(\mathbb{C},\mathbb{C})^2 \cong \mathbb{C}^2$, and the non-trivial subrepresentations are π supported at vertex 1, and π' supported at vertex 2, whose slopes are $\mu_{\alpha}(\pi) = \alpha_1 \cdot 1 = 1 > 0$ and $\mu_{\alpha}(\pi') = \alpha_2 \cdot 1 = -1 < 0$, respectively. Therefore, the subrepresentation π of type (1,0) is destabilising [11]. Moreover, note that given a representation (A,B), such a representation exists if and only if

$$A = 0$$
 and $B = 0$,

which implies that the unstable locus is $\{(0,0)\}$, and the stable locus is thus $\mathbb{C}^2 \setminus \{(0,0)\} \cong \mathbb{R}^4 \setminus \{0\} \simeq S^3$.

We now compute the spaces Hom^0 and Hom^1 for the destabilising subrepresentation π with dimension vector (1,0) with complement π^{\perp} with dimension vector (0,1). Note that any morphism $\pi \to \pi^{\perp}$ vanishes since π has $V_1 = \mathbb{C}$ and $V_2 = 0$ while π^{\perp} has $V_1 = 0$ and $V_2 = \mathbb{C}$. Hence

$$\dim_{\mathbb{C}} \operatorname{Hom}^0(\pi, \pi^{\perp}) = 0.$$

Let d = (1,0) denote the dimension vector of π , and e = (0,1) denote the dimension vector of π^{\perp} . Then the Euler form χ is given by

$$\chi(\pi, \pi^{\perp}) = \dim \operatorname{Hom}^{0}(\pi, \pi^{\perp}) - \dim \operatorname{Hom}^{1}(\pi, \pi^{\perp})$$
$$= d(I - A)e^{T}$$
$$= -2,$$

where I is the identity matrix and $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ is the corresponding adjacency matrix. It follows that $\dim_{\mathbb{C}} \operatorname{Hom}^1(\pi, \pi^{\perp}) = 2$. The *minimal dimension* of the quiver representation is thus

$$d_{min} = (-2)\chi(\pi, \pi^{\perp}) = 4.$$

The transversality then implies that every basepoint-preserving map $f: S^n \to V^{\rm st}$ is null-homotopic if $n < d_{min} - 1 = 3$. In particular, we have $\pi_n(V^{\rm st}) = 0$ for $n \le 2$, and this is consistent with the stable locus being homotopy equivalent to S^3 .

3.2. Stability and Controllability of Linear Control Systems. This example arises from the action of the complex reductive group $G = \mathrm{GL}_n(\mathbb{C})$ on the affine space $V = M_{n \times n}(\mathbb{C}) \times M_{n \times m}(\mathbb{C})$ by conjugation, where GIT stability condition coincides with the classical controllability condition for the linear control system $\dot{x} = Ax + Bu$.

Let $(A, B) \in V$, then the action is $g \cdot (A, B) = (gAg^{-1}, gB)$ for all $g \in G$. The pair (A, B) is said to be *controllable* if and only if the *controllability matrix*

$$C(A, B) = [B, AB, A^2B, ..., A^{n-1}B] \in M_{n \times nm}(\mathbb{C})$$

has rank n [6]. Controllability is the fundamental property of a control system that ensures it can be driven from any initial state to any desired final state within a finite time.

Choose the character $\chi: \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}^\times$ defined by $\chi(g) = \det(g)$, and V can be endowed with the χ -linearisation, in other words, the G-action can be lifted from V to $V \times \mathbb{C}$ by defining $g \cdot (v, \zeta) := (g \cdot v, \chi(g)^{-1}\zeta)$ for any $(v, \zeta) \in V \times \mathbb{C}$. Then a point $v \in V$ is χ -semistable if

$$\overline{G \cdot (v, 1)} \cap V \times \{0\} = \emptyset.$$

Further, a point $v \in V$ is χ -stable if it is χ -semistable, its orbit $G \cdot v$ is closed within $V^{ss} \subset V$, and its stabiliser subgroup $G_v = \{g \in G \mid g \cdot v = v\}$ is finite. Equivalently, if considering the lifted space, χ -stability then requires that $G \cdot (v, 1)$ is closed in $V^{ss} \times \mathbb{C}^{\times}$, and G_v is finite.

Proposition 4.1 in [6] gives the criterion that links the GIT stability with the controllability of linear control systems. A pair (A, B) is stable in the sense of GIT if and only if it is controllable, which is equivalent to the condition that no proper A-invariant subspace contains $\operatorname{Im} B \subset \mathbb{C}^n$. This can be seen by considering the

contrapositive. If (A, B) is not controllable, there exists a proper A-invariant subspace $0 \neq W \subset \mathbb{C}^n$ containing Im B. Write $r = \dim W$, where $1 \le r \le n - 1$. Then we can choose a basis of \mathbb{C}^n so that $W = \operatorname{Span}\{e_1, ..., e_r\}$, $W^{\perp} = \operatorname{Span}\{e_{r+1},...,e_n\}$. In this adapted basis, the matrices A and B take the block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

where $A_{11} \in M_{r \times r}(\mathbb{C})$, $A_{12} \in M_{r \times (n-r)}(\mathbb{C})$, $B_1 \in M_{r \times m}(\mathbb{C})$, and the zero block in B comes from $\text{Im } B \subseteq W$. Consider the one-parameter subgroup $\lambda: \mathbb{C}^{\times} \to \mathrm{GL}_n(\mathbb{C})$ with $\lambda(t) = \mathrm{diag}(1,...,1,\zeta^{-1},...,\zeta^{-1})$, where 1 appears r times and ζ^{-1} appears n-r times. Clearly λ preserves the block decomposition. Then under conjugation, we have $\lambda(\zeta) \cdot (A, B) = (\lambda(\zeta)A\lambda(\zeta)^{-1}, \lambda(\zeta)B)$, where in blocks we yield

$$\lambda(\zeta)A\lambda(\zeta)^{-1} = \begin{pmatrix} A_{11} & \zeta A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad \lambda(\zeta)B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}.$$

Then the limit $\lim_{\zeta \to 0} \lambda(\zeta) \cdot (A, B)$ exists and the numerical invariant $\mu^{\chi}((A, B), \lambda) = \langle \chi, \lambda \rangle$ is -(n - r), which is negative since r < n. Thus, by the Hilbert-Mumford criterion, (A, B) is not χ -semistable and thus not χ -stable. Therefore, we can define the unstable stratum associated to the subspace W as

$$S_W := \{ (A, B) \in V \mid A(W) \subset W, \operatorname{Im} B \subset W \}.$$

Conversely, assume there is no proper A-invariant subspace W containing Im B. Proposition 2.5 in [9] gives two conditions for stability. Firstly, note that the kernel of the G-action is the subgroup of $GL_n(\mathbb{C})$ that acts trivially on V. For $n \geq 1$, if $g \cdot B = B$ for all $B \in M_{n \times m}(\mathbb{C})$, then g must be I_n . So the kernel of the action is trivial, which is $\{I_n\}$. And so $\chi(I_n) = \det(I_n) = 1$, the first condition holds.

Secondly, consider an arbitrary 1-parameter subgroup λ for which the limit $\lim_{\zeta \to 0} \lambda(\zeta) \cdot (A, B)$ exists in V. Similarly, we can choose a basis $\{e_1,...,e_n\}$ of \mathbb{C}^n such that $\lambda(\zeta)$ is diagonal with respect to this basis, i.e., $\lambda(\zeta)e_j=\zeta^{w_j}e_j$ for integer weights w_j , which can be ordered as $w_1\geq w_2\geq ...\geq w_n$.

Then the existence of limit implies that $V_{\geq 0} = \operatorname{Span}\{e_j \mid w_j \geq 0\}$ is an A-invariant subspace that contains Im B, which contradicts our initial assumption. Thus $V_{\geq 0}$ cannot be a proper subspace of \mathbb{C}^n , so $V_{\geq 0} = \mathbb{C}^n$, which means all basis vectors e_j are in $V_{\geq 0}$. Consequently, all weights $w_j \geq 0$. Since for the character $\chi(g) = \det(g)$, the pair $\langle \chi, \lambda \rangle = \sum_{j=1}^n w_j$, which is clearly always non-negative. Moreover, if $\langle \chi, \lambda \rangle = 0$, then every w_j must be 0, which means $\lambda(\zeta)e_j=\zeta^0e_j=e_j$ for all j. Hence the 1-PS $\lambda(\zeta)=I_n$ for all $\zeta\in\mathbb{C}^\times$, thus its image $\lambda(\mathbb{C}^{\times}) = \{I_n\}$ coincides with the kernel of the G-action on V. Therefore, we can conclude that (A, B) is χ -stable. We define the stable locus as

$$V^{st}(\chi) = \{(A, B) \in V \mid (A, B) \text{ controllable}\}.$$

Consequently, the complement $V^{us}(\chi) = V \setminus V^{st}(\chi)$ is the locus of uncontrollable systems. Fix an (n-1)-dimensional subspace $W \subset \mathbb{C}^n$, and pick the 1-PS $\mu_W(\zeta) = \operatorname{diag}(I_{n-1}, \zeta^{-1})$ in a basis $(w_1,...,w_{n-1},v)$, where $W=\operatorname{Span}\{w_1,...,w_{n-1}\}$. Then conjugating μ_W by G yields

$$[\lambda] = \{\mu_W \mid W \in \operatorname{Gr}(n-1,n)\} \cong \mathbb{CP}^{n-1}, \text{ with } \dim_{\mathbb{C}}[\lambda] = n-1.$$

For any $\mu = \mu_W \in [\lambda]$, the vector space $V = M_{n \times n}(\mathbb{C}) \times M_{n \times m}(\mathbb{C})$ decomposes under the action of $\mu(\zeta)$ into weight spaces $V_i(\mu)$. We define the negative weight space as

$$V(\mu)_{-} = \bigoplus_{i<0} V_i(\mu).$$

To determine its dimension, we start by writing an arbitrary element $(A, B) \in V$ in block form corresponding to the decomposition $\mathbb{C}^n = W \oplus W^{\perp}$, namely

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

We observe that for $\mu_W(\zeta) = \operatorname{diag}(I_{n-1}, \zeta^{-1})$, the components of (A, B) that transform with negative powers of ζ are those entries in A that map from W to W^{\perp} , and entries in B that map from the input space to W^{\perp} , which are $1 \times (n-1)$ block and $1 \times m$ block, respectively. The complex dimension of the negative weight space is therefore

$$\dim_{\mathbb{C}} V(\mu)_{-} = \dim_{\mathbb{C}} A_{21} + \dim_{\mathbb{C}} B_2 = (n-1) + m.$$

Let $W \to G \cdot \lambda$ be the G-equivariant holomorphic vector bundle whose fibre over $\mu \in G \cdot \lambda$ is $V(\mu)_-$ with rank $\dim_{\mathbb{C}} V(\mu)_{-}$. Fix $k \geq 0$ as the dimension of the sphere S^{k} . We consider the space of homotopies \mathcal{F} and map $D: \mathcal{F} \times S^k \times I \times (G \cdot \lambda) \to \mathcal{W}$ as defined generally in Section 2. The conditions for Proposition 2.3 are satisfied by this setup, and hence the map D is transverse to the zero section $\mathcal{O}_{\mathcal{W}}$ of \mathcal{W} .

Recall that $D(F, s, t, \mu) \in \mathcal{O}_{\mathcal{W}}$ if and only if $\pi_{\mu}^{\perp}(F(s, t)) = 0$, which means $F(s, t) \in \bigoplus_{i>0} V_i(\mu)$. This is the necessary condition for the limit $\lim_{\zeta \to 0} \mu(\zeta) \cdot F(s,t)$ to exist in V.

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We can now apply Corollary 2.4. For a generic homotopy $F \in \mathcal{F}$, the preimage $D_F^{-1}(\mathcal{O}_{\mathcal{W}})$ is empty if $\dim_{\mathbb{R}} S^k + 1 + 2(\dim_{\mathbb{C}} (G \cdot \lambda)) < 2\dim_{\mathbb{C}} V(\mu)_-$, rearranging gives

$$\dim_{\mathbb{R}} S^k + 1 = k + 1 < 2m.$$

This implies that F(s,t) has a non-zero component in $V(\mu)_- = \bigoplus_{i < 0} V_i(\mu)$ for any (s,t) in the homotopy and any $\mu \in G \cdot \lambda$. Consequently, the limit $\lim_{\zeta \to 0} \mu(\zeta) \cdot F(s,t)$ does not exist for any $\mu \in G \cdot \lambda$.

More generally, note that the unstable locus $V^{us}(\chi)$ is a finite union of strata, each associated with a conjugacy class of 1-PSs $[\lambda_j]$ that can achieve χ -instability. For each such class $[\lambda_j]$, we have a corresponding rank $\dim_{\mathbb{C}} V(\lambda_j)_-$ and dimension $\dim_{\mathbb{C}} (G \cdot \lambda_j)$. By applying the transversality argument to each of these finitely many classes, and since d_{min} is the minimum of $2m_j - 2\dim_{\mathbb{C}} (G \cdot \lambda_j)$ over all such destabilising conjugacy classes, if $k+1 < d_{min}$, then for a generic F, we have $F(S^k \times I) \subset V^{st}(\chi)$. This indicates that the k-th homotopy group $\pi_k(V^{st}(\chi))$ would then be trivial.

For controllable linear systems, the unstable systems are precisely those for which a proper A-invariant subspace W contains $\operatorname{Im} B$. The stratification of the unstable locus $V^{us}(\chi)$ is by the dimension r of the smallest such subspaces W. The most generic unstable systems correspond to r=n-1. This indicates that the space of controllable systems has trivial homotopy groups up to dimension n-1 under this condition.

3.3. **Directed Gaussian Graphical Models and MLE.** The work of Améndola et al. [1] provides a novel connection between Maximum Likelihood Estimation (MLE) in statistics and the principles of Geometric Invariant Theory (GIT). Their idea is to translate statistical properties, such as the existence and uniqueness of the MLE, into the language of GIT stability by using the Kempf-Ness functional. They apply this to Gaussian group models, where maximising the log-likelihood is shown to be equivalent to a norm minimisation problem over a group orbit [1, Proposition 3.4, Proposition 3.13]. This relates the boundedness of the likelihood to semistability, the existence of an MLE to polystability, and the uniqueness of the MLE to stability [1, Theorem 3.10, Theorem 3.15].

Building upon this framework, Derksen and Makam [4] employed the machinery of quiver representations to solve the MLE threshold problem for matrix normal models, which is a key example of a Gaussian group model. They provided exact formulae for the number of samples required to ensure an MLE exists and is unique [4, Theorem 1.2, Theorem 1.3], and thus proving a conjecture of Drton, Kuriki, and Hoff [4, Corollary 1.4]. The following example applies this GIT perspective to a specific Directed Acyclic Graph (DAG) model, demonstrating the topological properties of the GIT-stable space of data samples for which a unique MLE exists.

We consider a directed Gaussian graphical model on a star-shaped Directed Acyclic Graph (DAG), which is a connected DAG that has a unique child vertex, which is a vertex that has a parent, meaning there is an edge pointing to it from another vertex. This graph has k parent vertices $\{1,...,k\}$ and a single child vertex k+1, representing a linear regression problem [2].

Suppose that sample data are collected into a matrix $Y = [Y^{(1)} \cdots Y^{(k)} \mid Y^{(k+1)}]$ of size $n \times (k+1)$, where n denotes the number of observations. The Maximum Likelihood Estimation (MLE) for the regression coefficients exists and unique if and only if the matrix formed by the parent columns, $[Y^{(1)} \cdots Y^{(k)}]$, has full column rank [2, Theorem 3.5 (c)]. We consider a degenerate sample $f = [Y^{(1)} \cdots Y^{(k)} \mid Y^{(k+1)}]$, where the parent block $X := [Y^{(1)} \cdots Y^{(k)}]$ is rank-deficient. Thus there are infinitely many regression coefficients β satisfying $X^{\top}(Y^{(k+1)} - X\beta) = 0$ [2, Section 3.2 & Section 7]. We refer to Section 5 of [2] for the complete construction of the f-stabilisation, where Lemma 5.3 shows $\tilde{f}(\epsilon)$ has full column rank, hence lies in V^{st} and has a unique MLE.

Let $V = M_{n \times (k+1)}(\mathbb{C})$ be the sample space, in which a point is a block $[Y^{(1)} \cdots Y^{(k)} \mid Y^{(k+1)}]$ with each column $Y^{(i)} \in \mathbb{C}^n$. Let the reductive group be $G := \mathrm{GL}_k(\mathbb{C}) \times \mathbb{C}^{\times}$, which is embedded in $\mathrm{GL}_{k+1}(\mathbb{C})$ via

$$g = (A, t) \mapsto \begin{pmatrix} A & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

We define the right action of G on the sample space V by matrix multiplication as $Y\mapsto Yg$. The character $\rho:G\to\mathbb{C}^\times$ is then defined by $\rho(g)=t$. Note that the 1-PS λ in G takes the general form

$$\lambda(t) = \operatorname{diag}(t^{w_1}, t^{w_2}, \dots, t^{w_k}, t^{-\sum_{i=1}^k w_i}),$$

where the integers $\{w_1,\dots,w_k,-\sum_{i=1}^k w_i\}$ are the weights of the action.

There exist three equivalences between the notions of GIT stability and the existence and uniqueness of the MLE in the setting of DAG models [2, Remark 3.6], which can be stated as follows

- A sample Y is unstable if no MLE exists;
- a sample Y is polystable if an MLE exists, but possible not unique;
- a sample Y is stable if the MLE is unique.

A 1-PS is destabilising if the limit $\lim_{t\to 0} Y \cdot \lambda(t)$ exists for a non-zero Y. The minimal destabilisation corresponds to making a single parent column redundant, namely the corresponding data of this parent column

is zero for the limit to exist. We consider the specific 1-PS λ_1 defined as $\lambda_1(t) := \operatorname{diag}(t^{-1}, 1, \dots, t)$, where the only negative weight is -1, this means that $\lambda_1(t)$ acts on the first parent column $Y^{(1)}$. The corresponding negative weight space is

$$V(\lambda_1)_- = \{ [Y^{(1)} 0 \cdots 0] \mid Y^{(1)} \in \mathbb{C}^n \} \cong \mathbb{C}^n,$$

as each sample Y has n observations. Therefore, the dimension of the negative weight space is $\dim_{\mathbb{C}} V(\lambda_1)_- = n$. Observe that the 1-PS λ_1 gives the weight decomposition as

$$V_{-} = \operatorname{span}\{e_1\}, \quad V_0 = \operatorname{span}\{e_2, \dots, e_k\}, \quad V_{+} = \operatorname{span}\{e_{k+1}\},$$

and any matrix commuting with λ_1 preserves each weight space. Let $g=(g_{ij})_{1\leq i,j\leq k+1}$ be an arbitrary element in G, then the commutator relation $t^{w_i}g_{ij}=t^{w_j}g_{ij}$ for all $t\in\mathbb{C}^\times$ implies that the matrix A is of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix},$$

where $a \in \mathbb{C}^{\times}$ and $B \in \mathrm{GL}_{k-1}(\mathbb{C})$. Hence the centraliser $C_G(\lambda_1)$ is isomorphic to the product $\mathbb{C}^{\times} \times \mathrm{GL}_{k-1}(\mathbb{C}) \times \mathbb{C}^{\times}$. Thus we have $\dim_{\mathbb{C}} C_G(\lambda_1) = 1 + (k-1)^2 + 1 = (k-1)^2 + 2$. Furthermore, since $G = \mathrm{GL}_k(\mathbb{C}) \times \mathbb{C}^{\times}$, its dimension is the sum of the dimensions of the two Lie groups, namely $\dim_{\mathbb{C}} G = k^2 + 1$. Note that the quotient $G/C_G(\lambda_1) \cong G \cdot \lambda_1$ is irreducible, it follows that $\dim_{\mathbb{C}} (G \cdot \lambda_1) = \dim_{\mathbb{C}} G - \dim_{\mathbb{C}} C_G(\lambda_1) = 2k - 2$.

Recall that Proposition 2.3 establishes that $D_f: S^\ell \times I \times G \cdot \lambda_1$ is transverse to the zero section \mathcal{O}_W for generic homotopies f, where $W \cong (G \cdot \lambda_1) \times \mathbb{C}^n$. Moreover, if $\ell + 1 + 2 \dim_{\mathbb{C}}(G \cdot \lambda_1) = \ell + 4k - 3 < 2n$, then Corollary 2.4 implies that $D_f^{-1}(\mathcal{O}_W)$ is empty, and so f can be deformed such that its image remains in the stable locus V^{st} .

By Theorem 2.5, we have $d_{min}=2\dim_{\mathbb{C}}V(\lambda_1)_--2\dim_{\mathbb{C}}(G\cdot\lambda_1)=2n-4k+4$. Thus to ensure $d_{min}>0$, one requires that n>2k-2. Consequently, we have $\pi_q(V^{st})=0$ for every $0\leq q< d_{min}-2$, which implies that V^{st} is (2n-4k+2)-connected. In particular, the stable locus V^{st} is path-connected if $n\geq 2k-1$, and it is simply connected if $n\geq 2k$. These results imply that the space of samples with a unique MLE is connected, which means that any sample with a unique MLE can be continuously deformed into any other such sample through a path consisting entirely of samples with unique MLEs. Furthermore, the algebraic parameter space X_f of all f-stabilisations [2, Definition 5.8] lies in a connected ambient space and inherits the connectedness properties.

Consequently, observe from Corollary 2.6, we obtain $\pi_q(V^{st}/G) \cong \pi_{q-1}(G)$ for all $1 \leq q < d_{min} - 1$. Note also that $G = \operatorname{GL}_k(\mathbb{C}) \times \mathbb{C}^{\times}$ is homotopy equivalent to $\operatorname{U}(k) \times S^1$, we can therefore conclude that for $1 \leq q < d_{min} - 1$,

$$\pi_q(V^{st}/G) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & q = 2, \\ \mathbb{Z} & q \text{ is even, } 4 \leq q < 2k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have obtained the homotopy groups of the moduli space of stable samples.

In particular, $\pi_2(V^{st}/G) \cong \mathbb{Z} \oplus \mathbb{Z}$ implies that there can be non-trivial families of f-stabilisations parametrised by a 2-sphere, S^2 , that cannot be continuously contracted to a single model within the moduli space. Moreover, the non-trivial homotopy group $\pi_q \cong \mathbb{Z}$ implies that for each even q with $4 \leq q < 2k$, one can construct a family of f-stabilisations parametrised by a q-sphere, S^q , in a non-trivial way, i.e. they are not null-homotopic. Therefore, while the f-stabilisation method of Bérczi et al. [2] guarantees a solution with a unique MLE, the topological properties of the solution space are intricate. The existence of these non-trivial homotopy groups proves that there are entire families of such stabilisations, parametrised by spheres, which are topologically distinct and cannot be continuously deformed into one another.

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