Heuristics

Anuj Pathania

Lesson Objectives (Learning Goals)

1. Define and differentiate between different optimization problems.
2. Calculate and appreciate a problem’s state space.
3. Recognize the vital role of Heuristics in exploring the state space.
4. Abstract out the shape (properties) of a state space.

Optimization Fundamentals

Company produces two types of Chocolate A and B.

- A requires 1 unit of milk and 3 units of choco.
  - A sells for 6 Euro.
- B requires 1 unit of milk and 2 units of choco.
  - B sells for 5 Euro.

Company has 5 units of milk and 12 units of choco.

How many units of A and B should be produce to maximize sale?

Optimization Fundamentals

Maximize Profit $P = 6A + 5B$

Such That,

$A + B \leq 5$ *[Limited Milk Supply]*

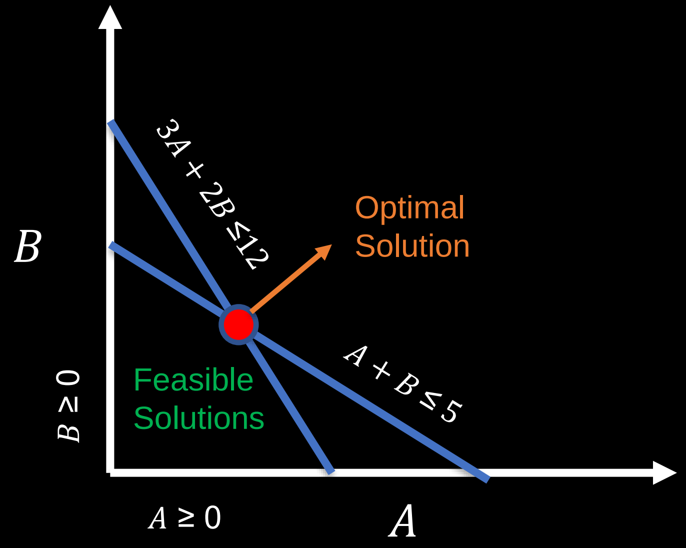
$3A + 2B \leq 12$ *[Limited Coco Supply]*

$A \geq 0$ & $B \geq 0$ *[Non-Negative Chocolate Production]*

Linear Programming

Optimization Fundamentals

- What is an optimization problem?
  - The problem of finding the best solution among all the feasible solutions.
  - Can be continuous or discrete.



Optimization Fundamentals

- What is an optimization variable?
  - Variable whose value can be optimized to solve the optimization problem.
  - Also known as decision variable.

$$A \,\&\, B$$

Optimization Fundamentals

- What is an objective function?
  - A mathematical function which quantizes the quality of a solution.
  - Must be satisfied, maximized, or minimized.

Maximize Profit $P = 6A + 5B$

Optimization Fundamentals

- What is a constraint?
  - Mathematical equalities and inequalities that defines the domain of optimization variables.
  - Feasible solution must satisfy the constraints.

Such That,

$A + B \leq 5$ *[Limited Milk Supply]*

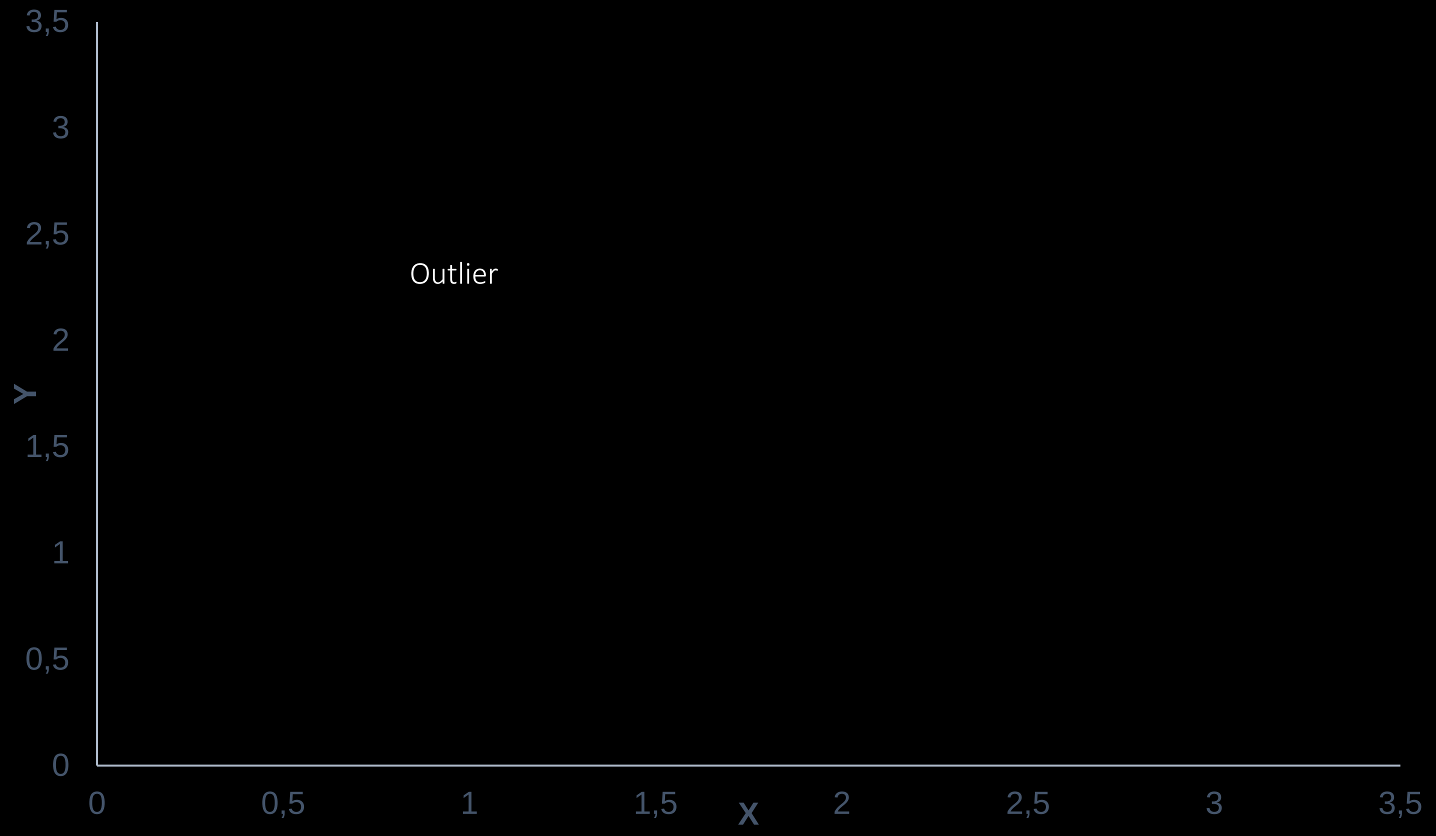
$3A + 2B \leq 12$ *[Limited Coco Supply]*

$A \geq 0$ & $B \geq 0$ *[Non-Negative Chocolate Production]*

Three Types of Optimization Problems

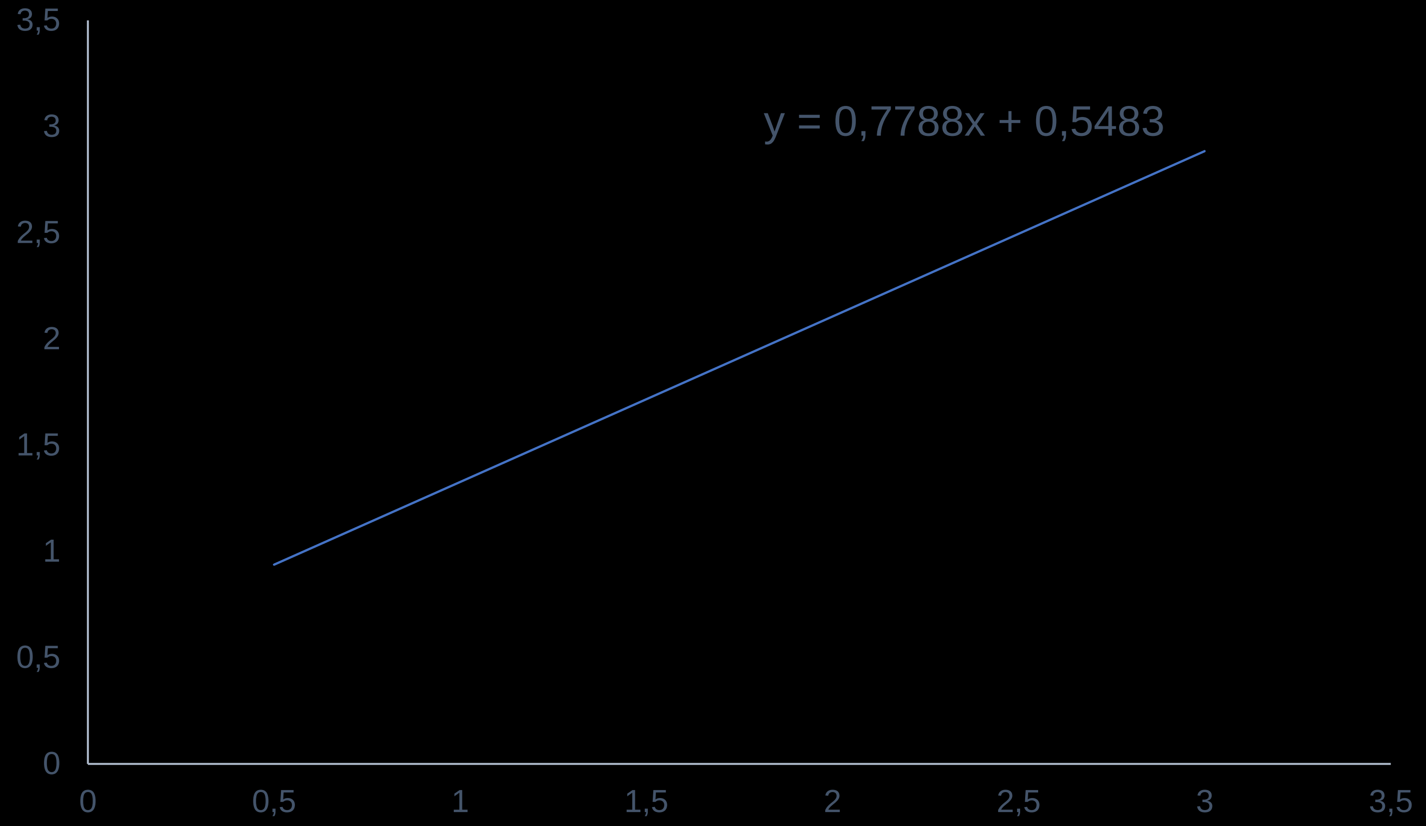
1. Free Optimization Problem (FOP)
2. Constraint Optimization Problem (COP)
3. Constraint Satisfaction Problem (CSP)

Free Optimization Problem (FOP)



Bunch of (X,Y) coordinates. Can you find the linear equation that best captures relationship between X & Y?

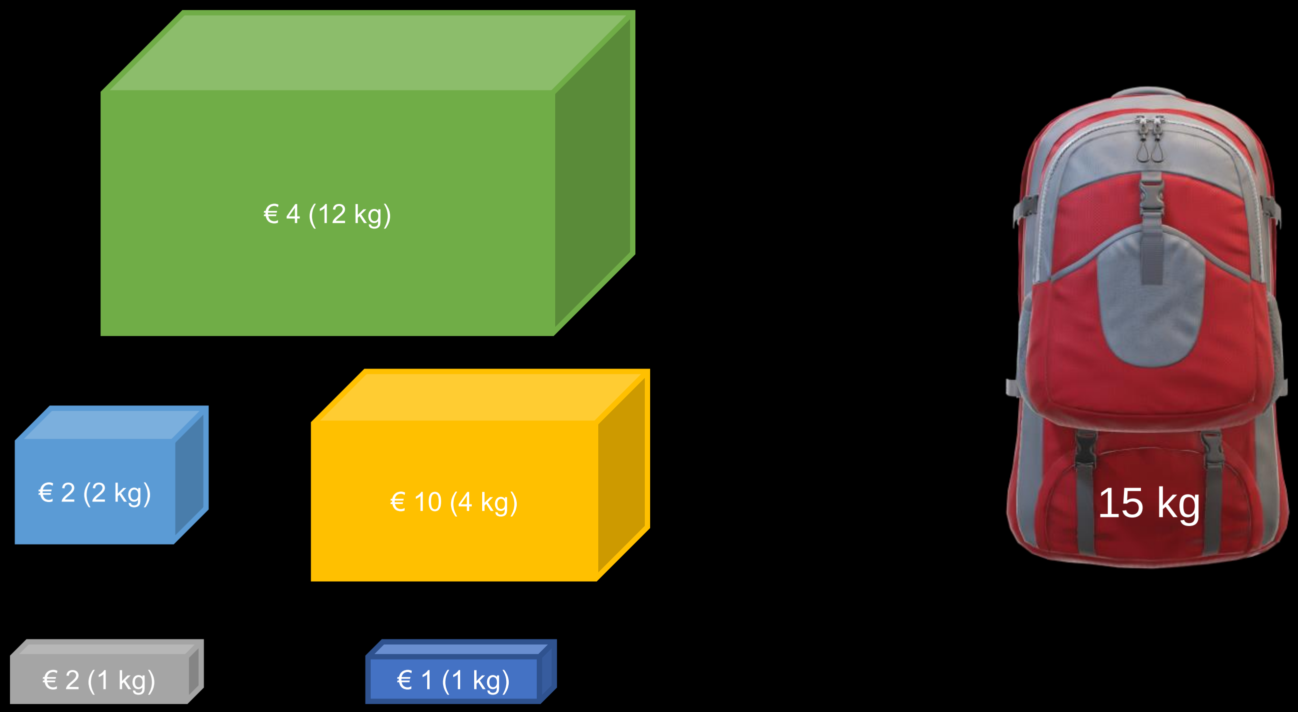
Free Optimization Problem (FOP)



$$Minimize\ \frac{\sum (\text{yactual} - \text{y}_{\text{predicted}})^2}{n}$$

Easily solved using linear regression by minimizing Mean Square Error.

Constraint Optimization Problem (COP)



Objective: Maximize €
Constraint: Weight < 15 kg.

Choose boxes that maximize the € value while weight is less than capacity of bag (One Dimensional Knapsack).

Constraint Satisfaction Problem (CSP)

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

→

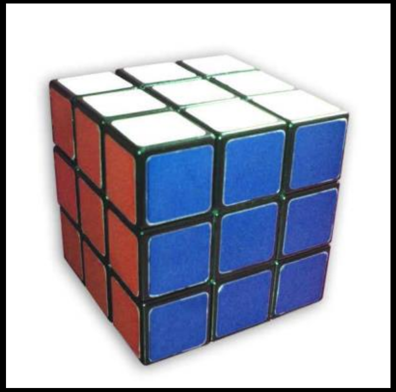
5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Solve Sudoku exactly without violating Sudoku row, column, and box constraint.

Summary: Three Types of Optimization Problems

	No Constraint	Constraint
No Satisfaction	Free Optimization Problem (FOP)	Constraint Optimization Problem (COP)
Satisfaction		Constraint Satisfaction Problem (CSP)

POP Quiz



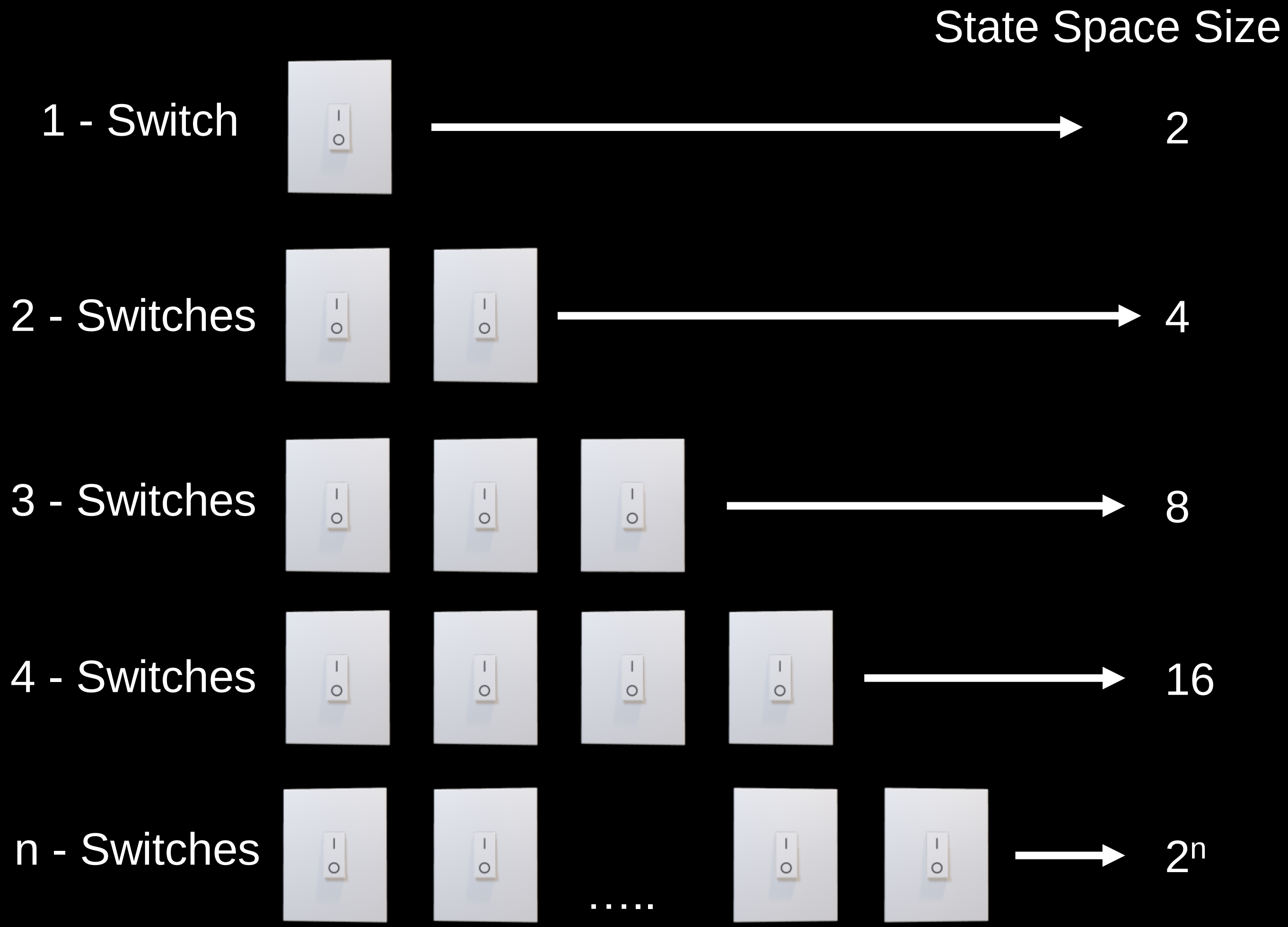
What kind of problem is a Rubik's Cube?

1. Free Optimization Problem (FOP)
2. Constraint Optimization Problem (COP)
3. Constraint Satisfaction Problem

Questions?

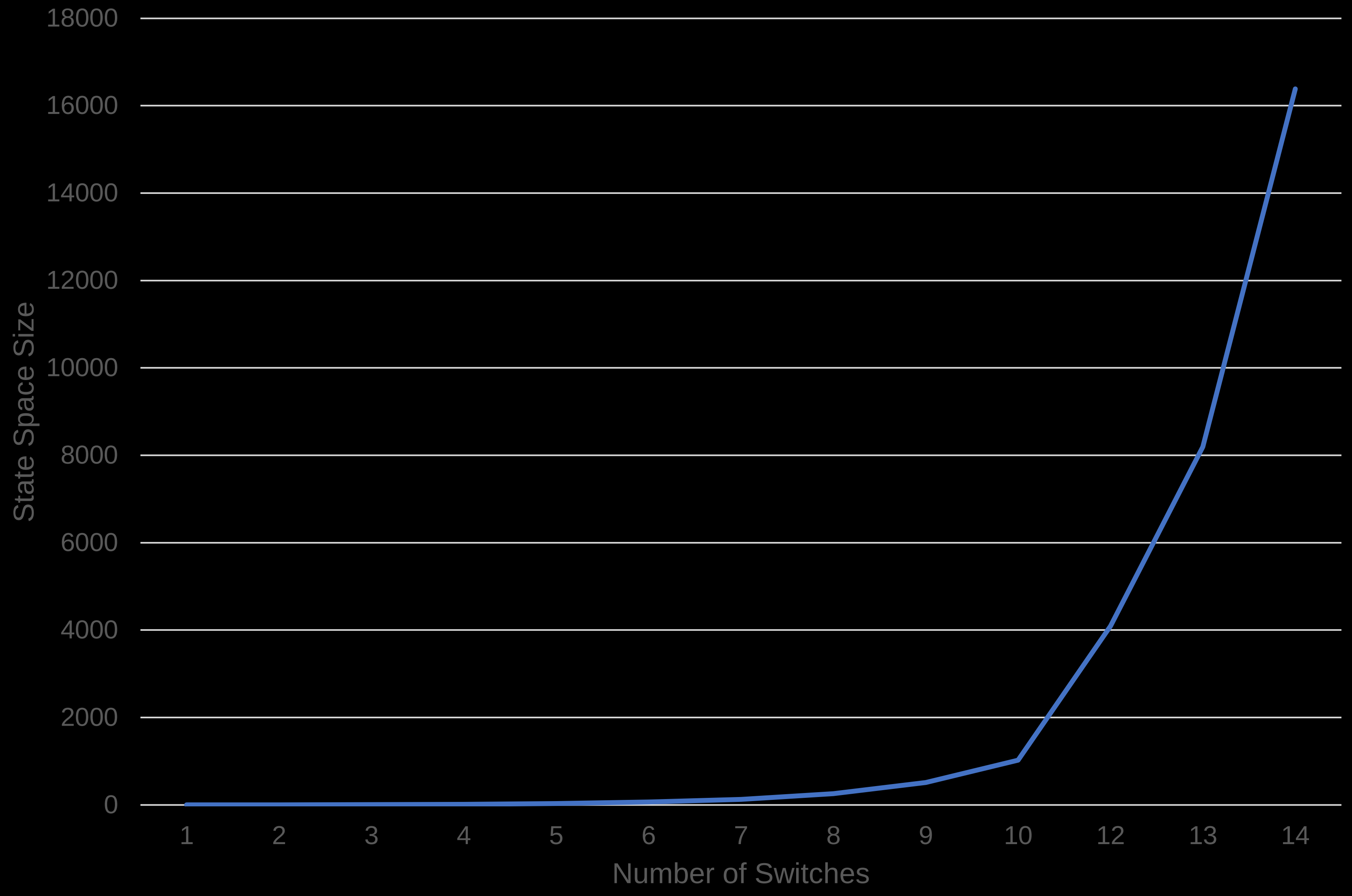
State Space

On or Off

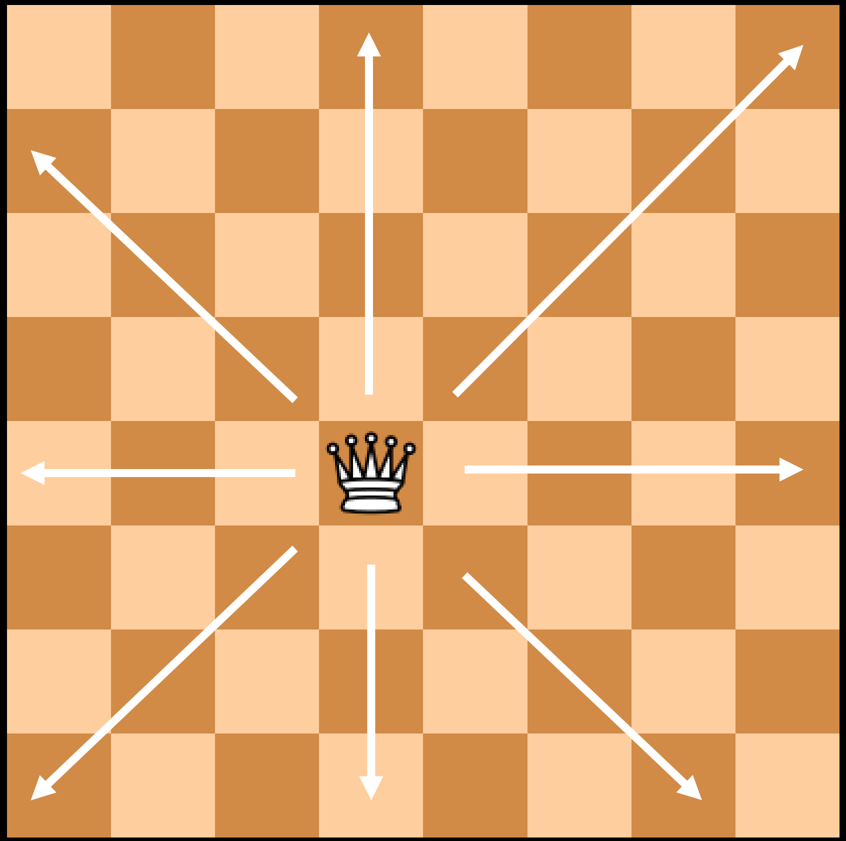


Set of all possible configurations of a problem.

Exponential State Space

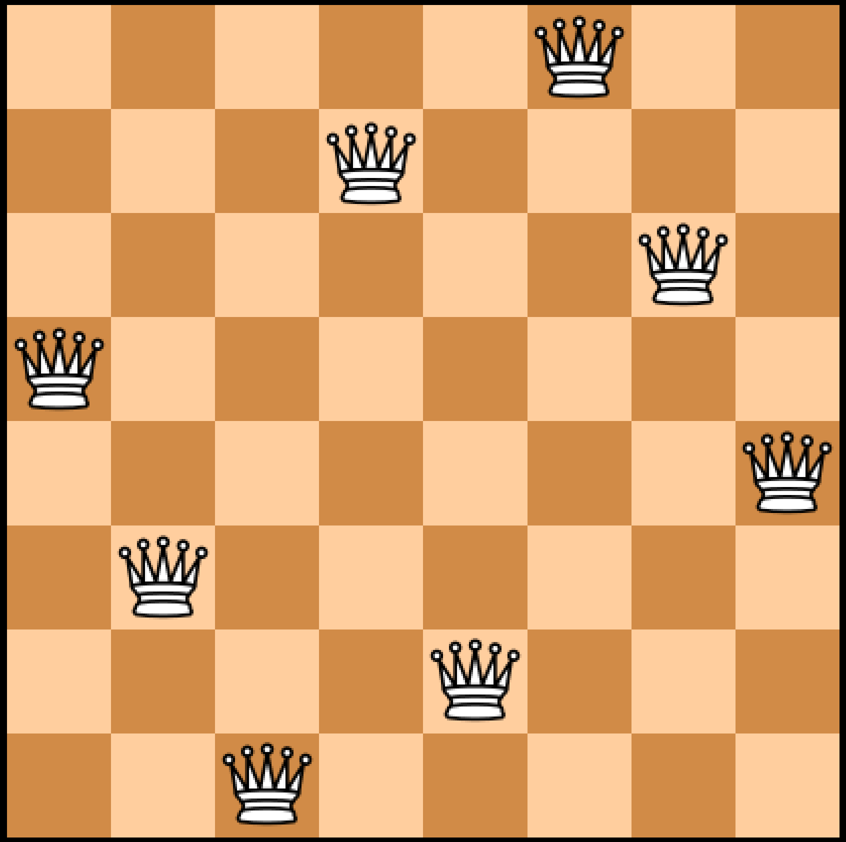


State Space



N-Queen Puzzle.

State Space



8-Queen Puzzle.

1^{st} Queen: 64 Squares

2^{nd} Queen: 63 Squares

3^{rd} Queen: 62 Squares

4^{th} Queen: 61 Squares

5^{th} Queen: 60 Squares

6^{th} Queen: 59 Squares

7^{th} Queen: 58 Squares

8^{th} Queen: 57 Squares

Total Ways = 64 * 63 * 62 * 61 * 60 * 59 * 58 * 57 = 4426165368 = $\binom{64}{8}$

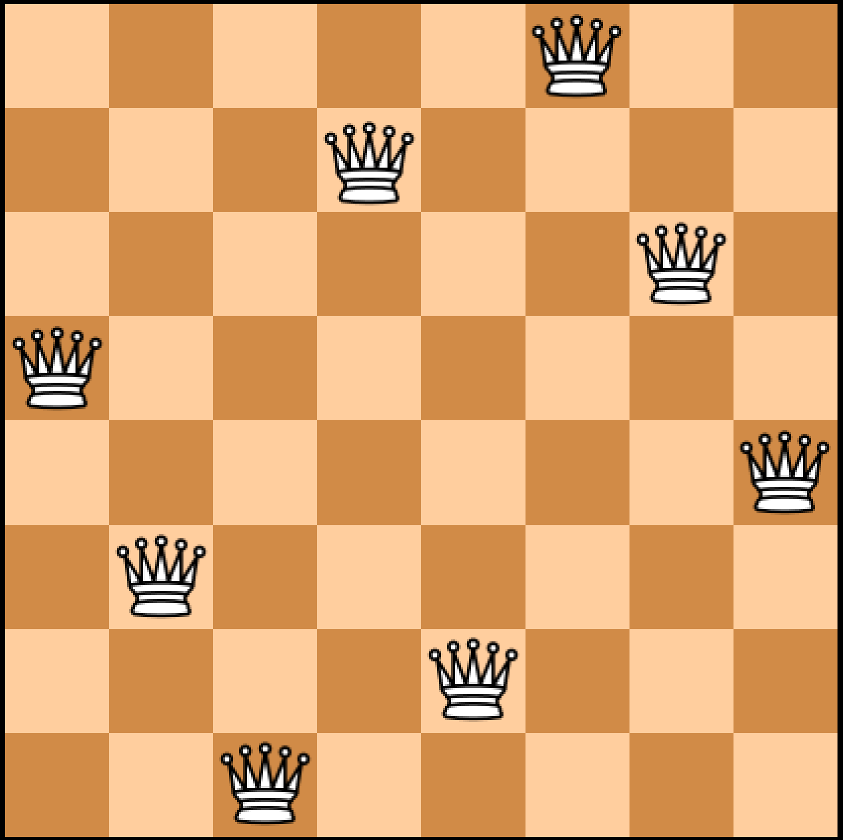
$$\binom{n}{k} = \frac{n!}{(n-k)!}$$

Where,

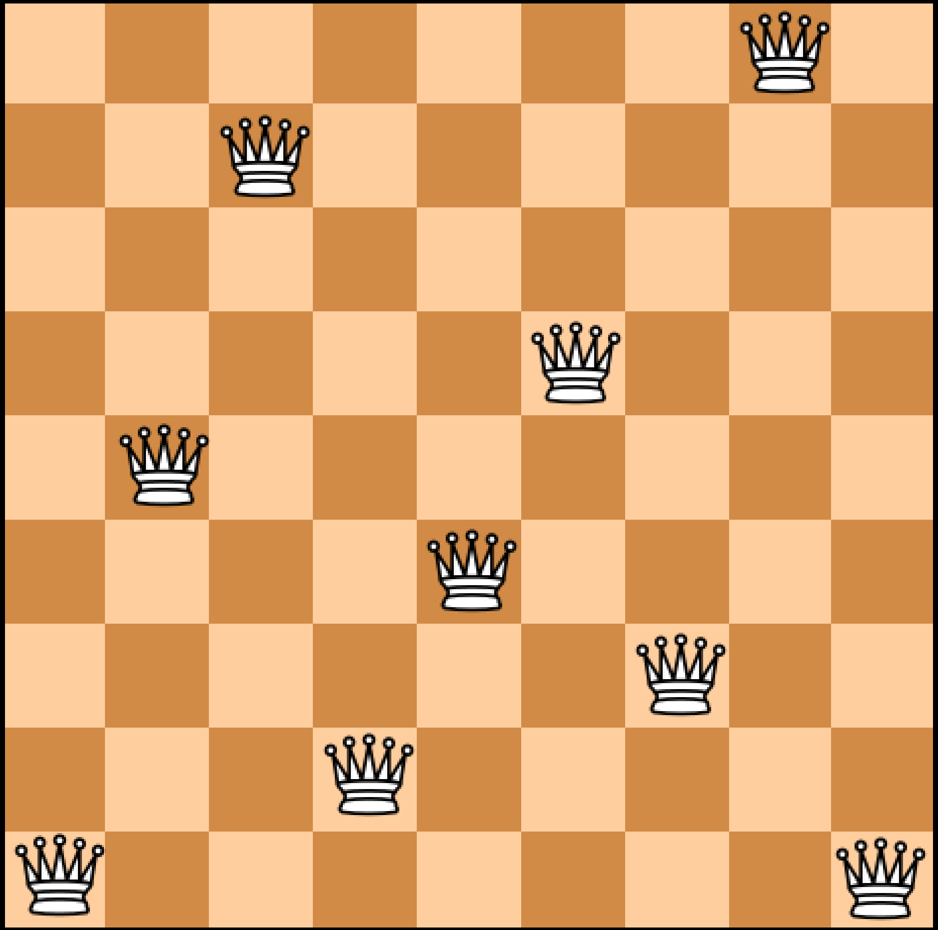
n is number of squares

k is number of queens

State Space



8-Queen Problem: $\binom{64}{8} = 4426165368$

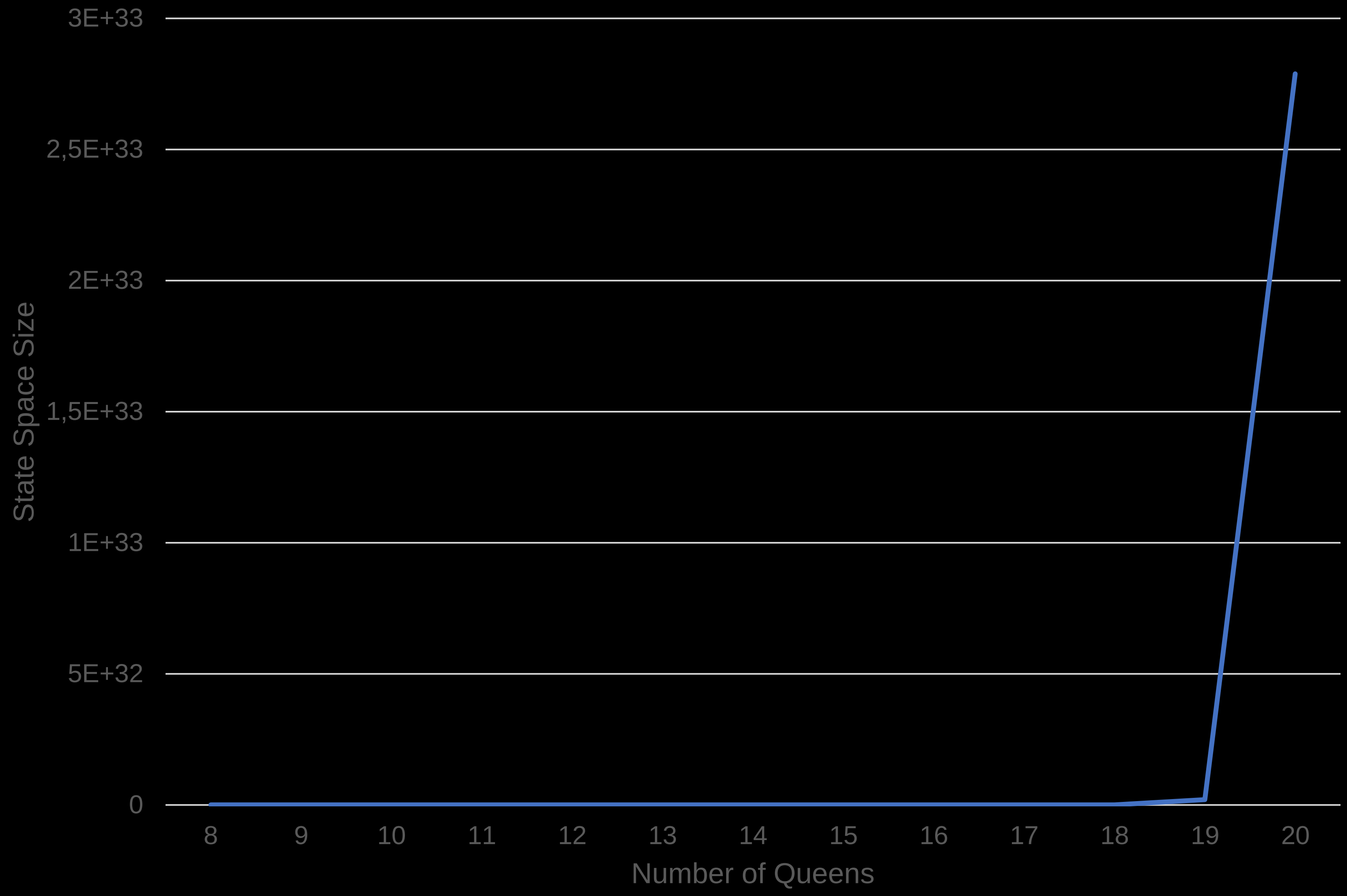


9-Queen Problem: $\binom{81}{9} = 260887834350$

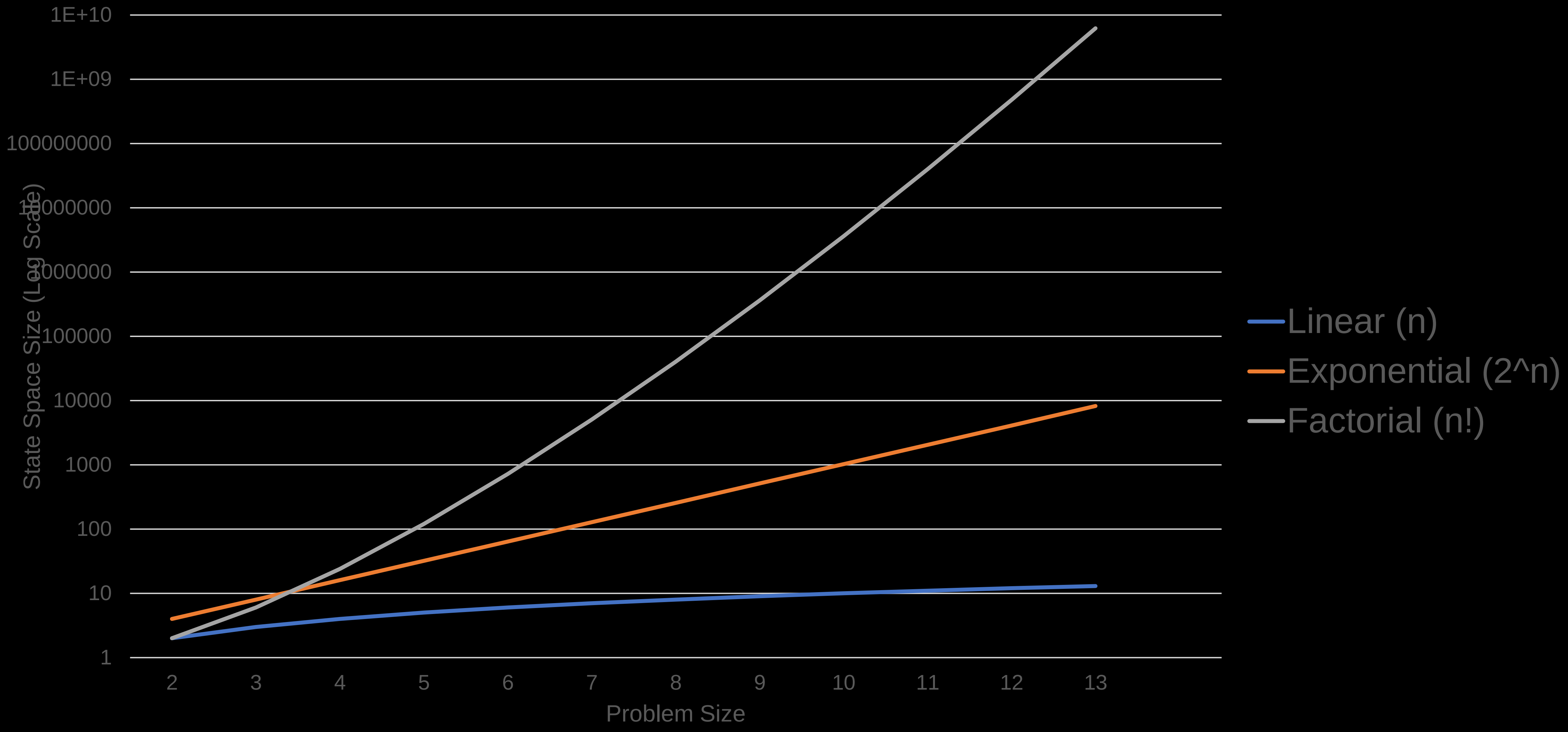
State Space

Queens	Squares	State Space	Approximate Time to Solve
8	64	4426165368	4.42 Seconds
9	81	260887834350	4.34 Minutes
10	100	17310309456440	4.8 Hours
11	121	1276749965026540	14.77 Days
12	144	103619293824707000	39.40 Months
13	169	9176358300744340000	290.78 Years
14	196	880530516383349000000	27 902.92 Years
15	225	91005567811177500000000	2 883 854.02 Years
16	256	10078751602022300000000000	319 383 187 Years
17	289	1190739044344490000000000000	$3.77330493 \times 10^{10}$ Years
18	324	149482492334195000000000000000	$4.73691552 \times 10^{12}$ Years
19	361	19870867053543800000000000000000	$6.29683229 \times 10^{14}$ Years
20	400	2788360983670900000000000000000000	$8.83597149 \times 10^{16}$ Years

Factorial (N-Queen) State Space



Summary: State Space Comparison



POP Quiz

What is upper bound for state-space for Tic-Tac-Toe?

1. 3^9
2. 9^3
3. $9!$

Questions?

Coffee Break!
(10 Mins)

Computer Algorithm

Algorithm is a sequence of well-defined steps a computer can follow in order to solve a problem.

Bubble Sort: An algorithm to sort a set of number.

- Step 1: Iteratively compare adjacent pair of numbers.
- Step 2: Swap numbers if not in correct order.
- Step 3: Repeat until there is no more swapping.

Computational Complexity

2,1,3,4,5,6,7,8,9,10

Best-Case

Bubble Sort (1 Step)

1,2,3,4,5,6,7,8,9,10

Bubble Sort (<=100 Steps)

10,9,8,7,6,5,4,3,2,1

Worst-Case

Primary Focus on Worst-Case

Problem Complexity

Worst Case Complexity: Big O Notation → O(…)

$O\ (n^{\cdots})$

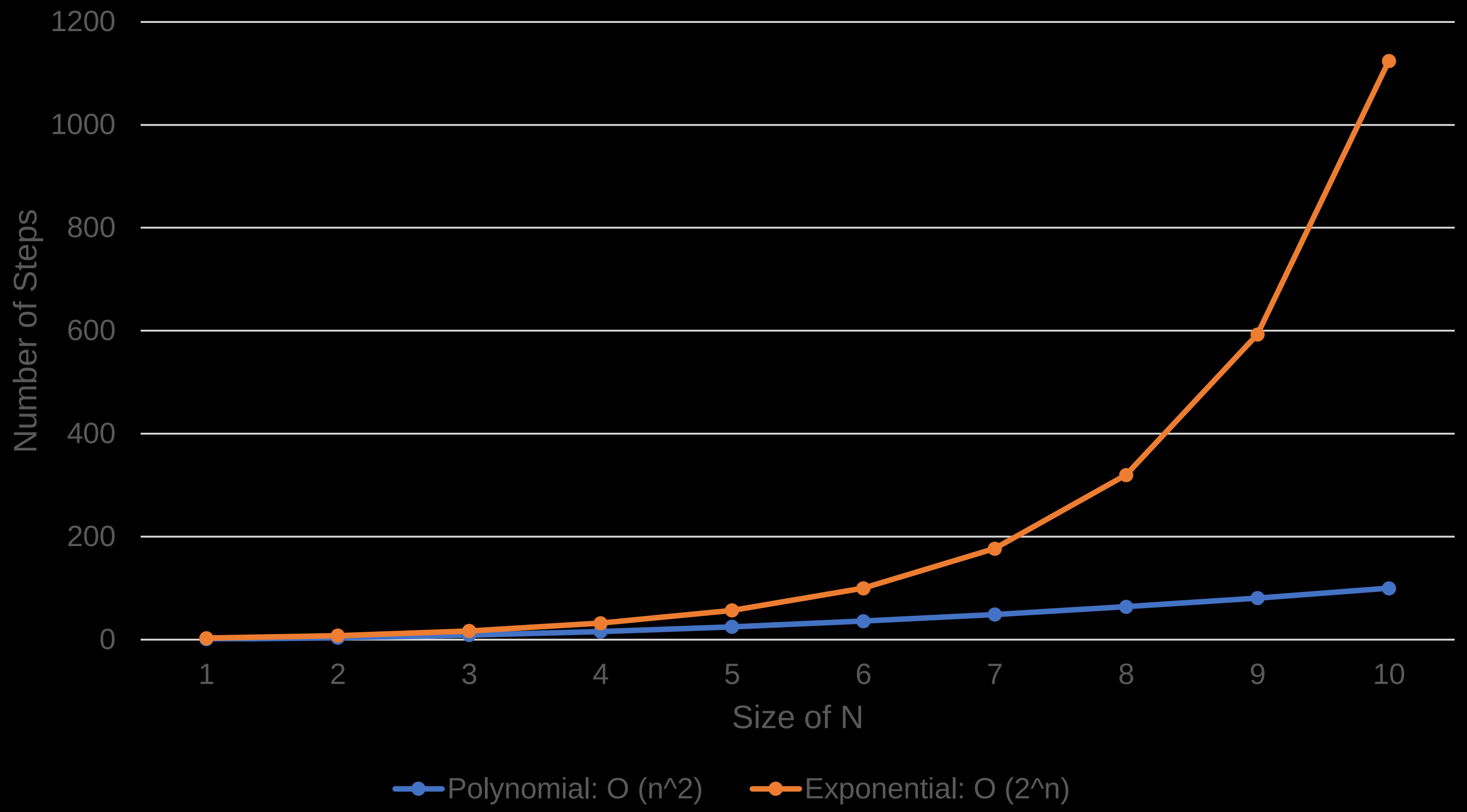
Polynomial

$O(\ldots^{n})$

Exponential

“n” is the length of the input.

Polynomial vs Exponential



As computers get faster, even very large polynomial problems solve quickly. However, exponential problems remain intractable even with small inputs.

Fast Algorithms

Sorting $\xrightarrow{\text{Fast Algorithm}}$ Bubble Sort $\xrightarrow{\text{Faster Algorithm}}$ Quick Sort

Bubble Sort: $O\ (n^{\wedge}2)$

Quick Sort: $O\ (n \ln n)$

Sudoko $\xrightarrow{\text{Fast Algorithm}}$???

(And Lots of Other Very Important Problems)

What’s the Solution?

Heuristics

Smart and fast algorithms that provide a near-optimal solution in a reasonable amount of time for COP problems.

Sudoko is Special!

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

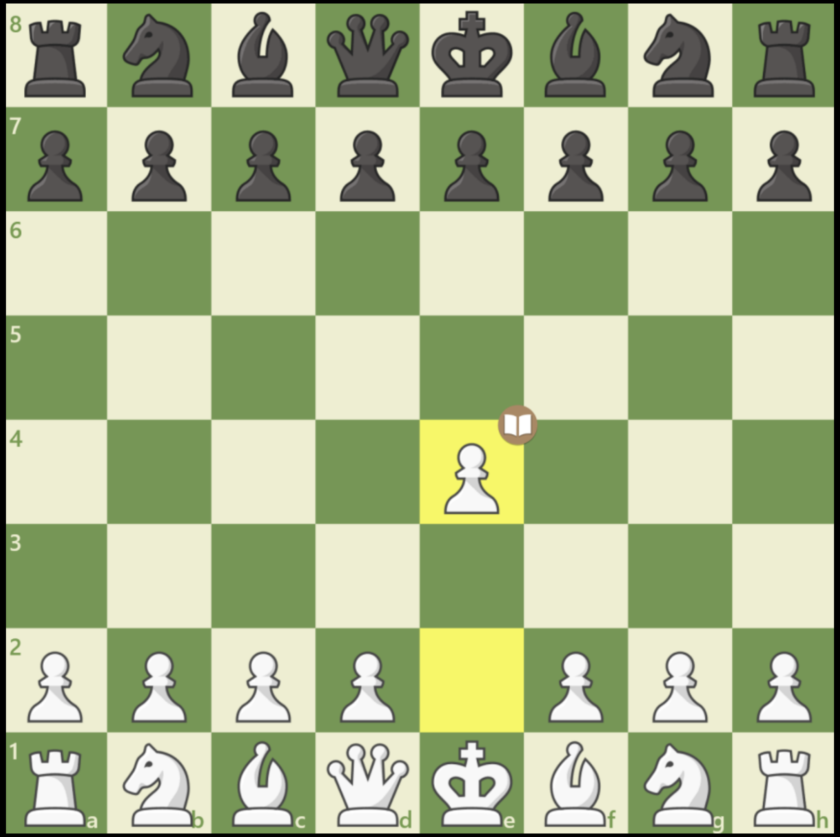
Difficult To Solve

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Easy to Check

Chess is Harder!

Difficult To Solve

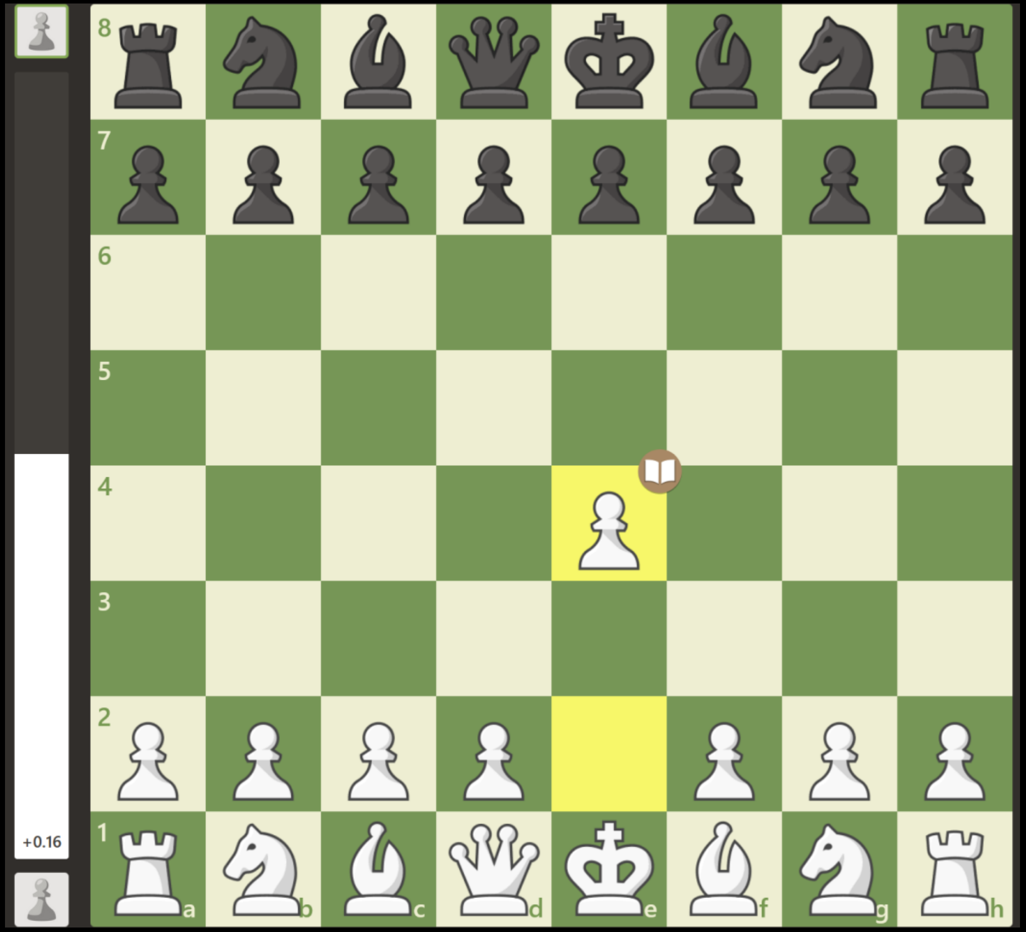


Difficult To Check

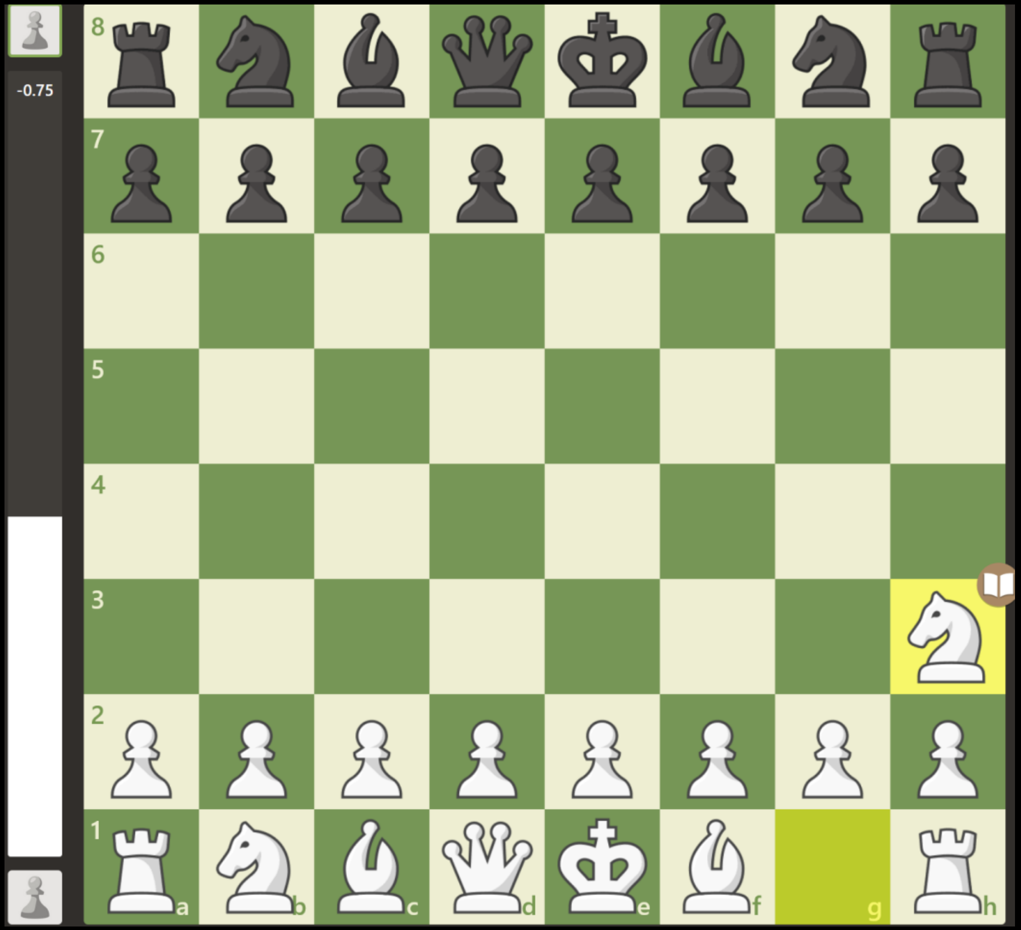
Is this (E4) the best chess opening for White?

Heuristics in Chess

Good Opening (+0.16)



Bad Opening (-0.75)



Chess engines use heuristic objective functions to score positions.

Summary: Problem Difficulty and Heurisitics

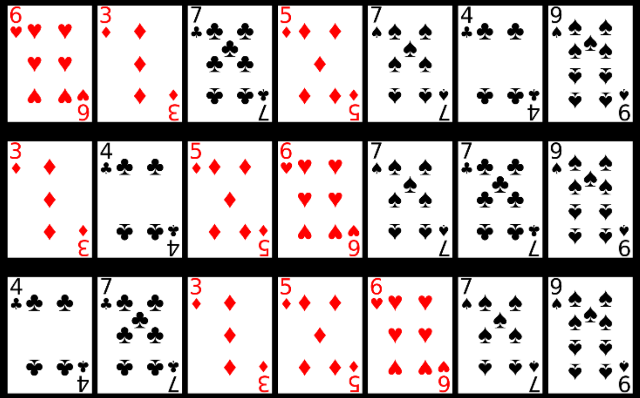
Chess

Hard

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Sudoku

???



Sorting

Easy

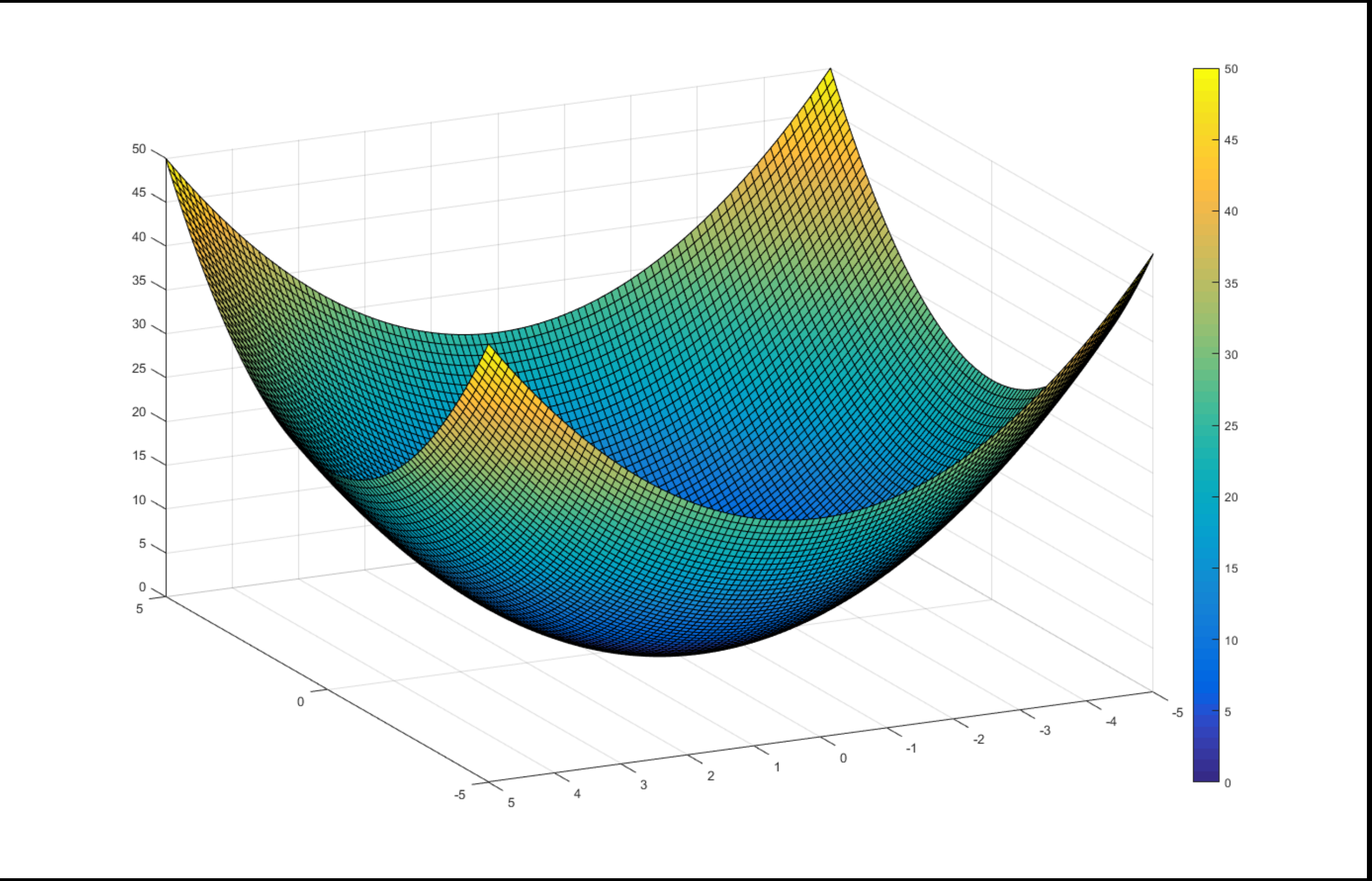
POP Quiz

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

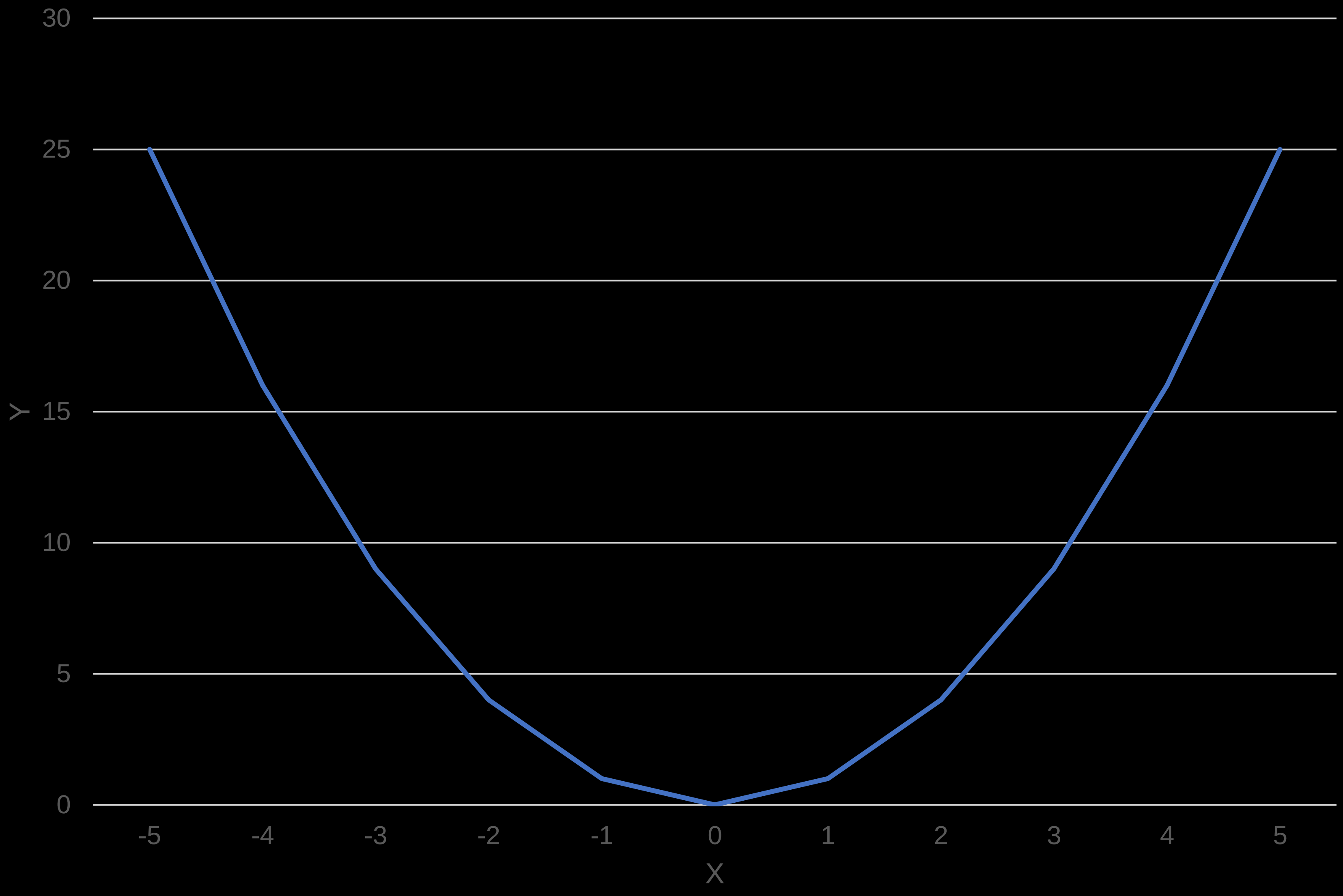
How difficult you think sudoku really is?

1. Easy as Sorting
2. Hard as Chess
3. We will never know!

Convex State-Space (Global Minima)

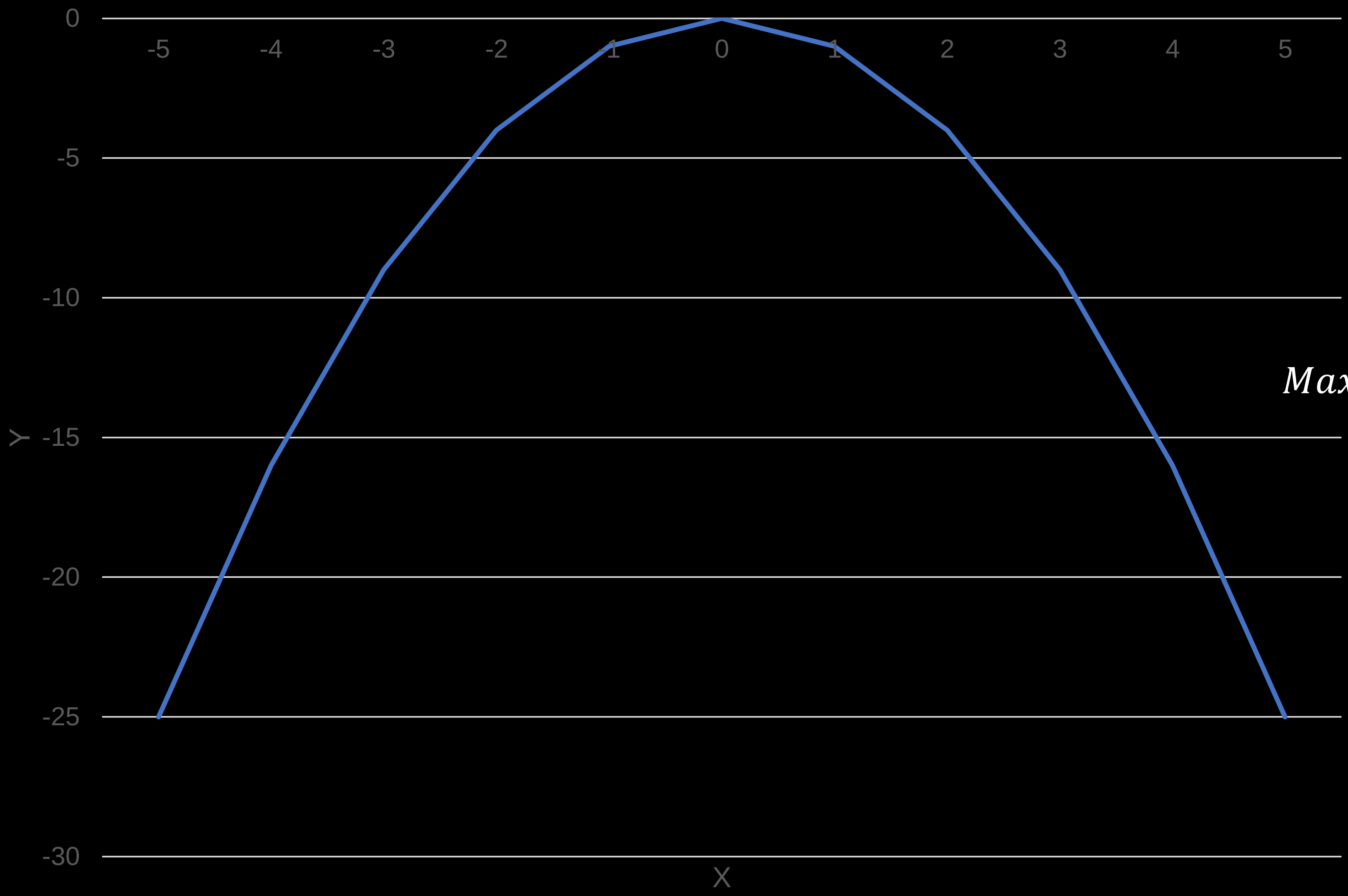


Gradient Descent to Global Minima



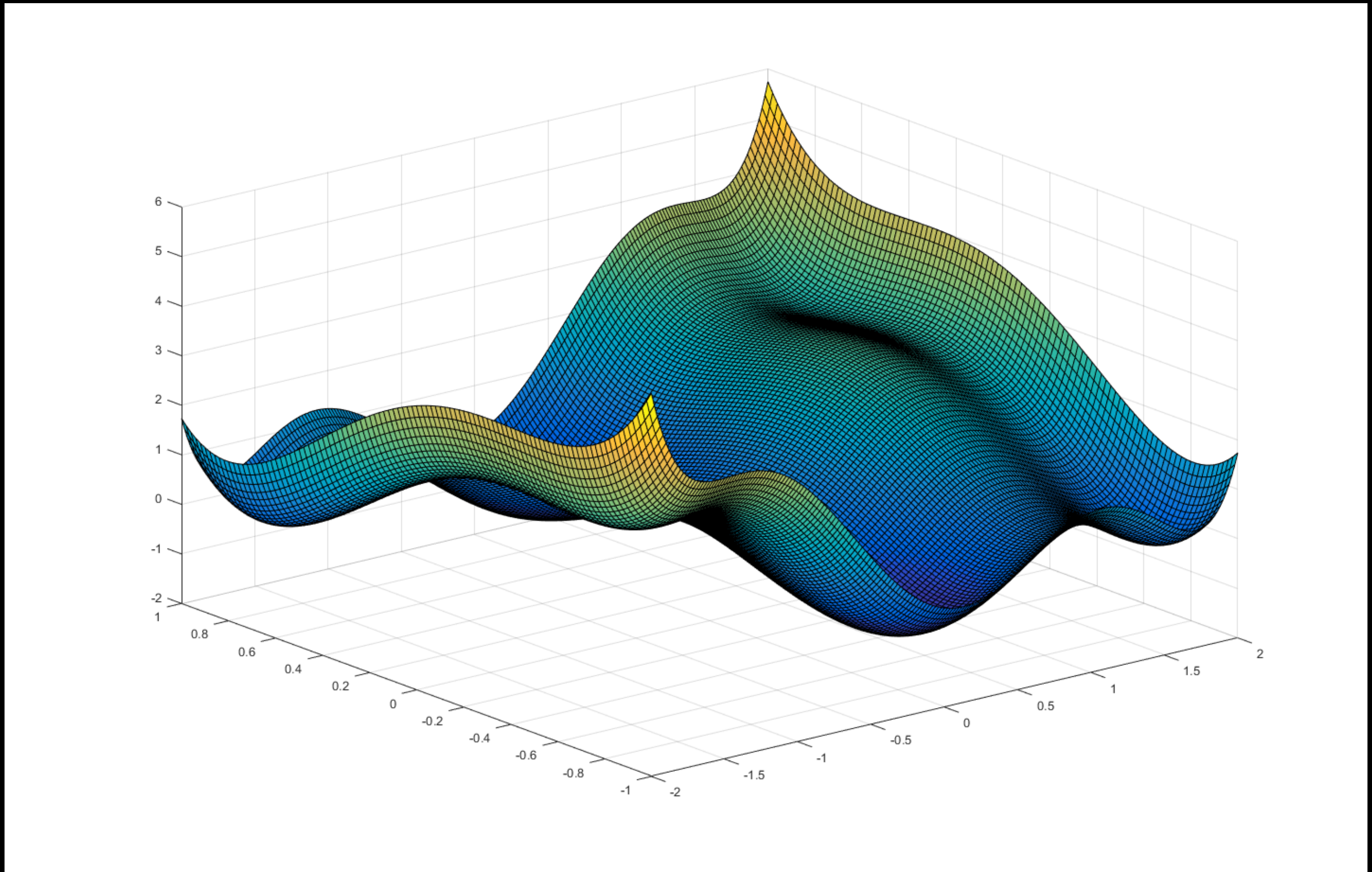
$Minimize\ y = x^2$

Gradient Ascent to Global Maxima

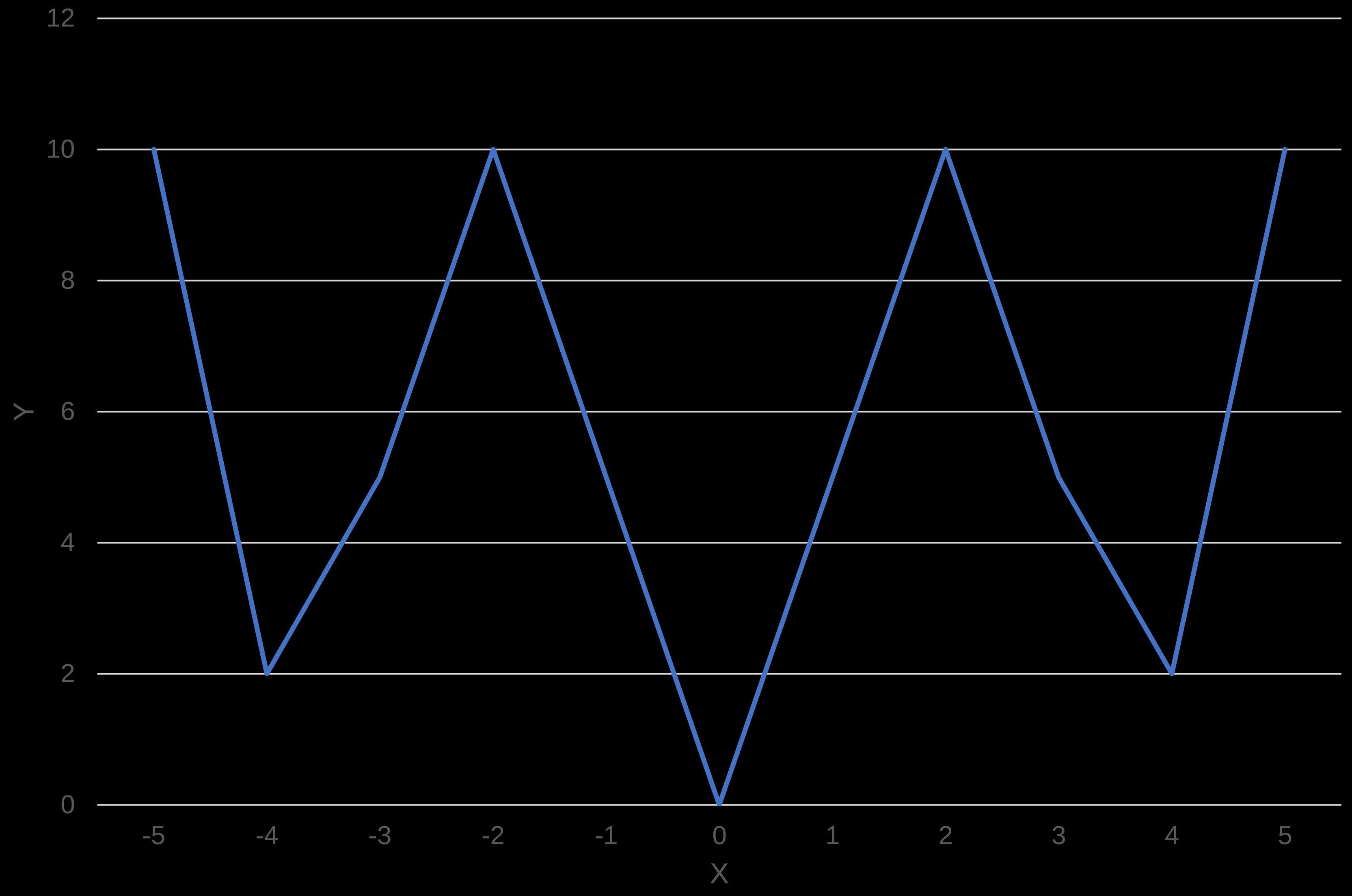


$Maximize\ y = -x^2$

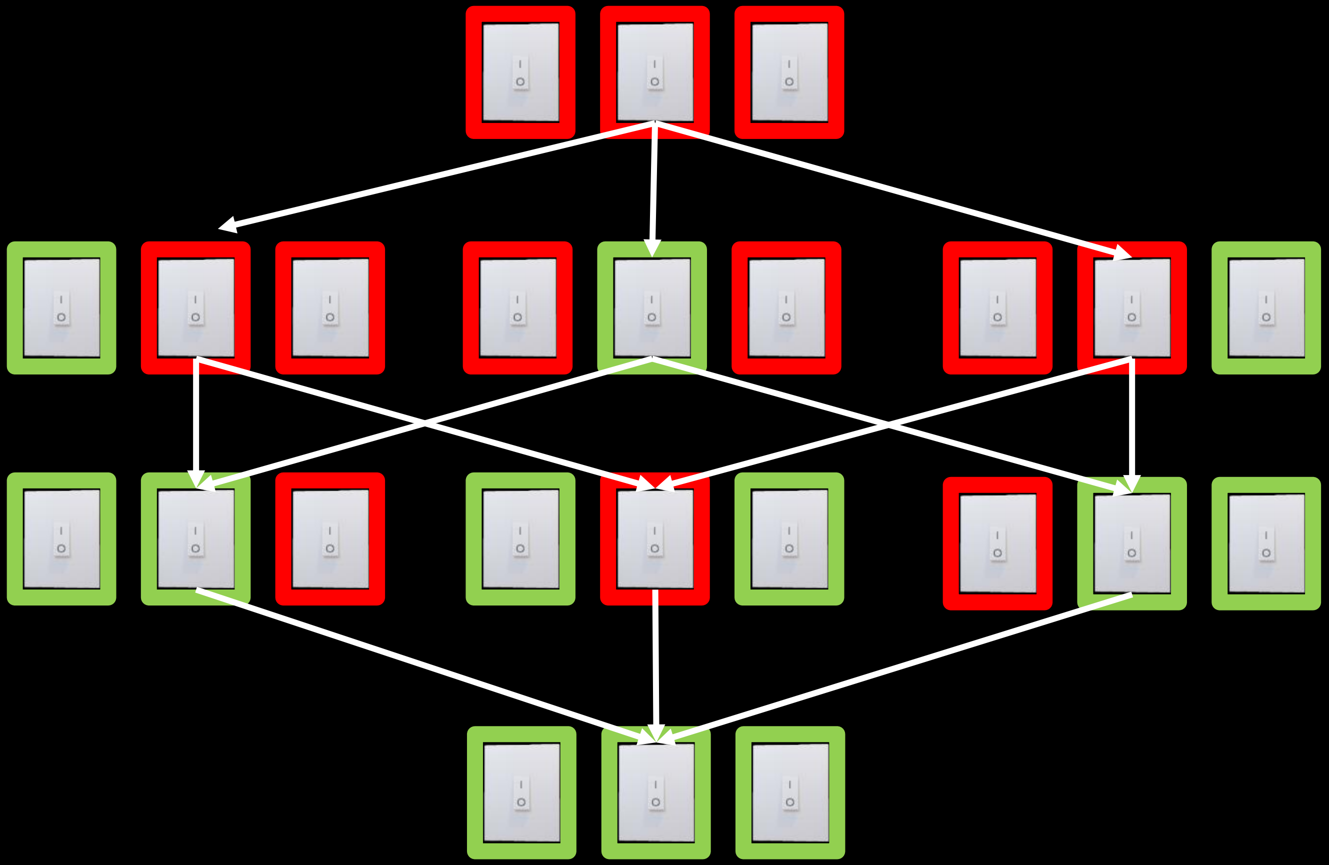
Non-Convex State-Space (Local Minima)



Gradient Descent to Local Minima



Turn On All Switches



3 Switches → 2^3 or 8 States

Summary: Global Minima vs Local Minima

	Exploration Difficulty	Minima	Solution Quality
Convex State-Space	Easy	Global Minima	Optimal
Non-Convex State-Space	Hard	Local Minima	Sub-Optimal

Questions?