T1.

证明 $((\neg p \to q) \wedge \neg q) \to p$ is a tautology.

(a) Truth Table

(b) equivalence

(c) Rule of Inference

(2) (5 marks) Show that the following pairs are not equivalent using a domain with two elements, and your choice of predicates $P(x)$ and $Q(x)$.

(a) (3 marks) $\forall x\,(P(x) \oplus Q(x))$ and $\forall x\, P(x) \oplus \forall x\, Q(x)$. **Solution.** Ther[illegible] a few possible answers: For the domain $\{1,2\}$, and predicates $P($ [illegible] and $Q(x) : x = 2$, the first proposition is $(P(1) \oplus Q(1)) \wedge (P(2)$ [illegible] $Q(2)) \equiv (T \oplus F) \wedge (F \oplus T) \equiv T \wedge T$ which is true. The second [illegible] $Q(1))$ [illegible] $(P(2) \wedge Q(2)) \equiv (T \wedge F) \oplus (F \wedge T) \equiv F \oplus F$ whi[illegible] equivalent.

(b) (2 marks) $\exists x\,(P(x) \oplus Q(x))$ and $\exists x\, P(x) \oplus \exists x\, Q(x)$. [illegible] domain and predicates in the previous answer also [illegible] sition is true (universal instantiation) but the sec[illegible] and $\exists x\, Q(x)$ are both true ($T \oplus T \equiv F$)

Another example that works for both (a) and (b): $P(x)$ [illegible]

The truth table of $\oplus$ is given as follows:

$$p \oplus q = T \text{ if } \{p, q\} = \{T, F\}, \quad p \oplus q = F \text{ if } p = q = T \text{ or } p = q = F$$

(3) (8 marks total) The assignment

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{2x+3}{4x-5}$$

is not a well-defined function.

(a) (4 marks) Show that we can delete one point each from the domain and the codomain of f so that the resulting function is both injective and surjective.

(b) (4 marks) Compute the inverse of the function obtained in (a).

Solution. (a) $f : \mathbb{R} \setminus \{5/4\} \to \mathbb{R} \setminus \{1/2\}$.

(b) $f^{-1} : \mathbb{R} \setminus \{1/2\} \to \mathbb{R} \setminus \{5/4\}$, $f^{-1}(y) = \frac{5y+3}{4y-2}$.

(4) (7 marks total) Let a be an integer that is not divisible by 4.

(a) (1 mark) What are all the possible remainders when a is divided by 4? Give your answers only, no additional justification required.

(b) (3 marks) Show that $(a+1)(a+2)(a+3) \equiv 0 \bmod 4$.

(c) (3 marks) Compute $3 + 3^3 + 3^5 + 3^7 + \cdots + 3^{99} \bmod 4$.

Solution. (a) 1,2,3.

(b) Separate into cases according to (a).

(c) $3 \times 50 = 150 = 2 \bmod 4$

(5) (7 marks) Use the Fast Exponentiation method to compute $106^{203} \bmod 111$

Solution. $(-5)^{1+2+8+64+128}$

(6) (8 marks total) Let $a = 791$ and $b = 1057$. Show your work to

(a) (4 marks) compute $d = \gcd(a, b)$ using the Euclidean algorithm;

(b) (4 marks) express d as a $\mathbb{Z}$-linear combination of a and b.

(7) (2 marks each, 18 marks total) Determine whether the following statements are True/False. Give a short justification or counter-example, whichever is applicable.

(1 mark) for each correct True/False; if correct, (1 mark) additionally for a sufficient justification or a correct counter-example.

(a) $P : (\neg p \vee \neg q) \wedge (q \vee p)$ and $Q : (\neg p \vee q) \wedge (\neg q \vee p)$ are equivalent.

Solution. False. P is $p \oplus q$ and Q is $p \leftrightarrow q$, and $p \oplus q \equiv \neg(p \leftrightarrow q)$. □

(b) $P : (p \to q) \wedge (q \to r)$ is logically equivalent to $Q : p \to r$.

Solution. False, when p and r are both True but q is False, P [illegible]
Q is True (another counter-example: p and r are both False but q T[illegible]

(c) The following argument is valid:

"If Arrakis is dry, then it is hot. Either Arrakis is dry, or it is spicy.

Therefore, Arrakis is hot or spicy."

Solution. True. This is a valid argument, using e.g. resolution

(d) Let $P(x, y) : x > y$ on the domain $\{(x, y) \mid x \neq y\} \subset A^2$ where A is a non-empty set of real numbers. Then, if $\forall x \exists y\, P(x, y)$ is true, then $\exists x \forall y\, P(y, x)$ must be false.

...gument, using e.g. resolution. □

... $> y$ on the domain $\{(x,y) \mid x \neq y\} \subset A^2$ where A is a non-empty set of real numbers. Then, if $\forall x \exists y\, P(x,y)$ is true, then $\exists x \forall y\, P(y,x)$ must be false.

Solution. True. Here, the two propositions are negations of each other, because the negation of $x < y$ is exactly $y > x$ when $x \neq y$.

In plain language, the first one says that "every element in A has another element that is less than it" i.e. A has no least element, whereas the second one says that "there is an element in A that is less than all other elements" i.e. A has a least element. They are exactly negations of each other. □

(e) Let S be a set, and denote by $\mathcal{P}(S)$ be its power set, then $\mathcal{P}(\{\emptyset, b, \{a, \{a\}\}\})$ has 8 elements.

$P(\{\phi, b, \{a, \{a\}\}\})$

Solution. True; $\{\emptyset, b, \{a, \{a\}\}\}$ has three elements: $\emptyset$, b, and $\{a, \{a\}\}$. □

(f) If $A \subset B$ and $C \subset D$, then

$$(B \times D) \setminus (A \times C) = (B \setminus A) \times (D \setminus C).$$

Solution. False, take $A = \{a\} \subset B = \{a, b\}$ and $C = \{c\} \subset D = \{c, d\}$ then e.g. (b, c) is contained in the set on the LHS but not the RHS.

$\{a, \{a\}\}\}$ has three elements: $\varnothing$, b, and $\{a, \{a\}\}$. □

$A \subset B$ and $C \subset D$, then

$$(B \times D) \setminus (A \times C) = (B \setminus A) \times (D \setminus C).$$

Solution. False, take $A = \{a\} \subset B = \{a, b\}$ and $C = \{c\} \subset D = \{c, d\}$, then e.g. (b, c) is contained in the set on the LHS but not the RHS. Correct arguments using cardinality may also be accepted. □

(g) There exists a bijection from the open interval $(-23, 23)$ to the open half line $(2025, \infty)$.

Solution. True, e.g. $f : (-23, 23) \to (2025, \infty)$ where $f(x) = 2025 \frac{46}{23 - x}$ is strictly increasing on $(-23, 23)$ and its range is $(2025, \infty)$ so it is a bijection. Other examples also exist e.g. by shifting and dilating $\tan x$. □

(h) For any three sets A, B, C, we must have

$$(A \cup B) \cap (B \cup C) \cap (C \cup A) = (A \cap B) \cup (B \cap C) \cup (C \cap A)$$

Solution. True, both sides are exactly the set of elements that are in two or more out of the sets A, B, and C. You can also apply the distributive and/or

(i) Let a, m be non-zero integer, $b.c$ are multiple of a. If $b \equiv c \mod m$,

then $b/a \equiv c/a \mod m$.