PROBABILISTIC ROBOTICS:
GAUSSIAN FILTERS

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1

1.1. We can define the state vector to include position and velocity:

$$Y_t = \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix}$$

1.2. Evolution of the state will be approximated to the first order :

$$\begin{aligned} x_{t+1} &= x_t + \Delta t \times \dot{x}_t + \frac{1}{2}(\Delta t)^2 \times \ddot{x}_t + o(\Delta t^2) \\ \Rightarrow x_{t+1} &\approx x_t + 1 \times \dot{x}_t + \frac{1}{2} \times \ddot{x}_t \end{aligned}$$

$$\begin{aligned} \dot{x}_{t+1} &= \dot{x}_t + \Delta t \times \ddot{x}_t + o(\Delta t) \\ \Rightarrow \dot{x}_{t+1} &\approx \dot{x}_t + \ddot{x}_t \end{aligned}$$

We can modelize evolution of the state with matrix equality:

$$\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2}\ddot{x}_t \\ \ddot{x}_t \end{bmatrix}}_{\epsilon_t}$$

$$\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}}_{G} \ddot{x}_t$$

We have

$$G \times G^T = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

As a linear function of gaussian $\ddot{x}_t$, the random variable ϵ_t is known to be multivariate Gaussian $\mathcal{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \underbrace{\sigma^2 G G^T}_{R})$. The conditional law of random variable $\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix}$ given $(\begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix})$ is then known to be also Gaussian $\mathcal{N}(A \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix}, R)$.

Moreover, if we assume the random variable $\begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix}$ to be Gaussian $\mathcal{N}(\mu_t, \Sigma_t)$, then $\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix}$ is Gaussian $\mathcal{N}(\underbrace{A\mu_t}_{\overline{\mu_{t+1}}}, \underbrace{A\Sigma_t A^T + R}_{\overline{\Sigma_{t+1}}})$.

Note that since we do not incorporate measurements for now, we have using the notation of the book

$$\overline{\mu_{t+1}} = \mu_{t+1}$$
$$\overline{\Sigma_{t+1}} = \Sigma_{t+1}$$

1.3. From the previous relations, it is clear that

$$\forall t \in \mathbb{N}, \mu_t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let's t compute the first covariance matrices, using $\sigma^2 = 1$:

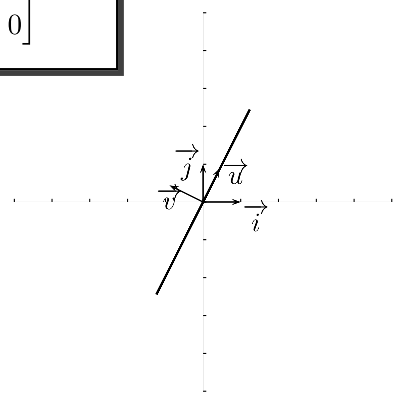
t	Σ_t
0	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
1	$\begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 1 \end{bmatrix}$
2	$\begin{bmatrix} 2.5 & 2 \\ 2 & 2 \end{bmatrix}$
3	$\begin{bmatrix} 8.75 & 4.5 \\ 4.5 & 3 \end{bmatrix}$
4	$\begin{bmatrix} 21 & 8 \\ 8 & 4 \end{bmatrix}$
5	$\begin{bmatrix} 41.25 & 12.5 \\ 12.5 & 5 \end{bmatrix}$
6	$\begin{bmatrix} 71.5 & 18 \\ 18 & 6 \end{bmatrix}$
7	$\begin{bmatrix} 113.75 & 24.5 \\ 24.5 & 7 \end{bmatrix}$
8	$\begin{bmatrix} 170 & 32 \\ 32 & 8 \end{bmatrix}$
9	$\begin{bmatrix} 242.25 & 40.5 \\ 40.5 & 9 \end{bmatrix}$
10	$\begin{bmatrix} 332.5 & 50 \\ 50 & 10 \end{bmatrix}$

The uncertainty ellipses are centered at $\mu_t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, have axes whose directions are given by the eigenvectors of Σ_t, and the semi-minor and semi-major axis are scaled by eigenvalues; for detailed derivation, cf. Σ_t is a symetric real matrix, it can be diagonalized in an orthonormal basis, with real eigenvalues

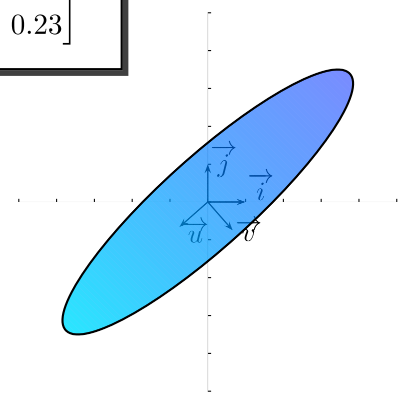
$$\Sigma_t = P_t D_t P_t^T$$

where P_t is an orthogonal matrix and D_t is diagonal. Note the first ellipse is degenerated since Σ_1 has a null eigenvalue.

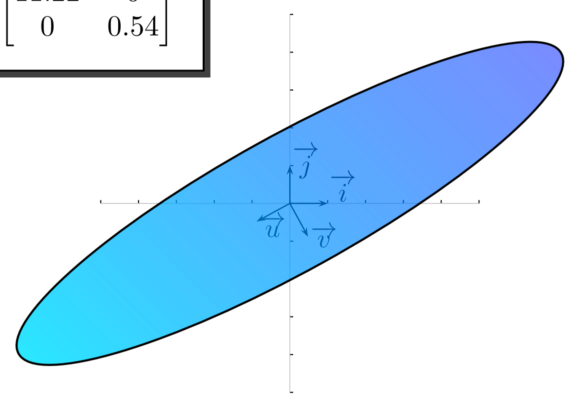
$$P = \begin{bmatrix} 0.45 & .89 \\ -0.89 & 0.45 \end{bmatrix}$$
$$D = \begin{bmatrix} 1.25 & 0 \\ 0 & 0 \end{bmatrix}$$



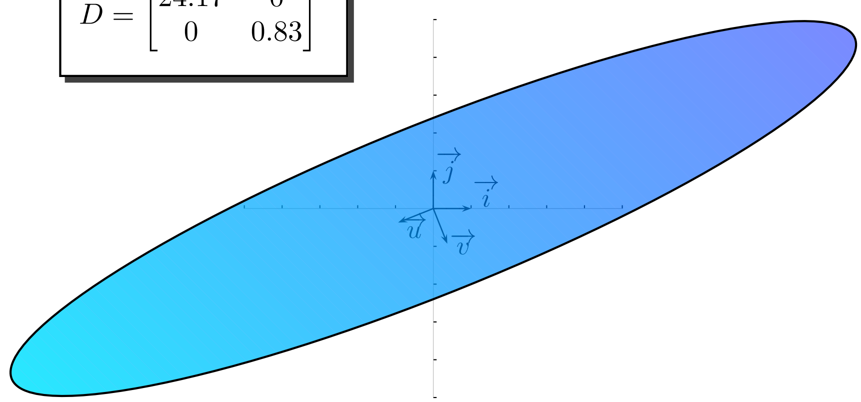
$$P = \begin{bmatrix} -0.75 & 0.66 \\ -0.66 & -0.75 \end{bmatrix}$$
$$D = \begin{bmatrix} 4.27 & 0 \\ 0 & 0.23 \end{bmatrix}$$



$$P = \begin{bmatrix} -0.88 & 0.48 \\ -0.48 & -0.88 \end{bmatrix}$$
$$D = \begin{bmatrix} 11.22 & 0 \\ 0 & 0.54 \end{bmatrix}$$



$$P = \begin{bmatrix} -0.93 & -0.37 \\ -0.37 & -0.93 \end{bmatrix}$$
$$D = \begin{bmatrix} 24.17 & 0 \\ 0 & 0.83 \end{bmatrix}$$



2

2.1. We can modelize the noisy observation of the position by:

$$\begin{aligned} z_t &= x_t + \xi_t \\ &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} + \xi_t \end{aligned}$$

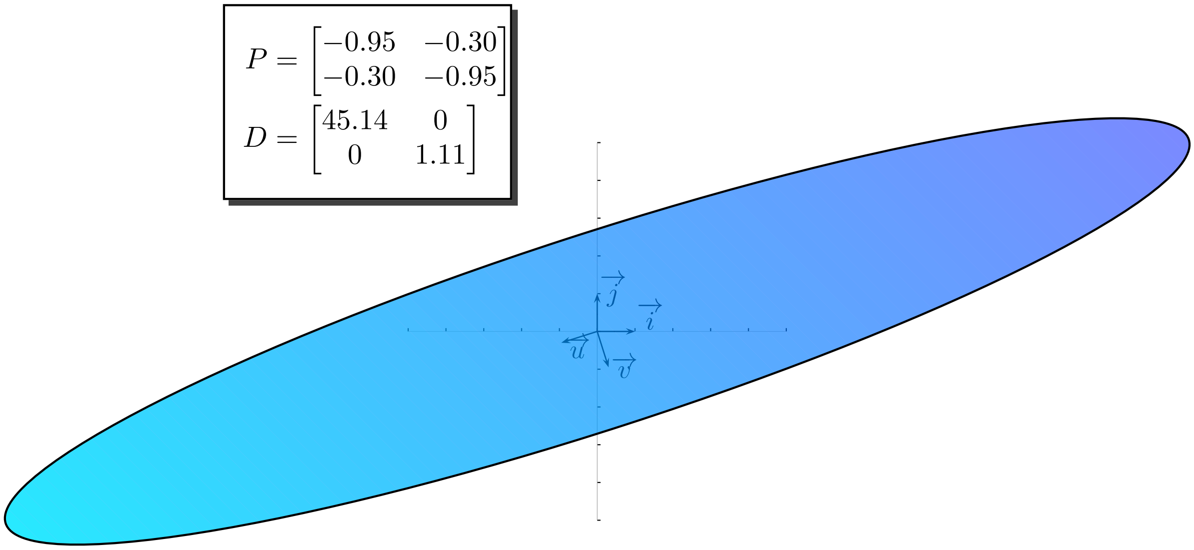


FIGURE 0. 0.95 uncertainty ellipses for the gaussian posteriors at dates $t \in [\![1, 5]\!]$

where the random variable ξ_t is gaussian $\xi_t \hookrightarrow \mathcal{N}(0, \underbrace{\sigma^2}_{Q})$.

2.2. Before the observation the moments are:

$$\bar{\mu}_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\bar{\Sigma}_5 = \begin{bmatrix} 41.25 & 12.5 \\ 12.5 & 5 \end{bmatrix}$$

The measurement update consists of

$$\begin{aligned} K_5 &= \bar{\Sigma}_5 C^T (C\bar{\Sigma}_5 C^T + Q)^{-1} \\ &= \begin{bmatrix} 0.80 \\ 0.24 \end{bmatrix} \\ \mu_5 &= \bar{\mu}_5 + K_5(z_5 - C\bar{\mu}_5) \\ &= \begin{bmatrix} 4.02 \\ 1.22 \end{bmatrix} \\ \Sigma_5 &= (I - K_5 C)\bar{\Sigma}_5 \\ &= \begin{bmatrix} 8.05 & 2.44 \\ 2.44 & 1.95 \end{bmatrix} \end{aligned}$$

We draw the corresponding ellipse:

3

Let's recall the definition of the characteristic function of a random variable: (a tad of theory of measure is needed)($\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$)

Let X be an random variable which takes it values in $\mathbb{R}^d$. The characteristic function of X is the function which goes from $\mathbb{R}^d$ to $\mathbb{C}$:

$$\begin{aligned} \forall t \in \mathbb{R}^d, \quad \varphi_X(t) &= E \exp i\langle X, t \rangle \\ &= \int_{\mathbb{R}^d} \exp i\langle x, t \rangle \, \mathrm{d}P_X(x) \end{aligned}$$

Note the characteristic function of X is the Fourier transform of its probability distribution. We

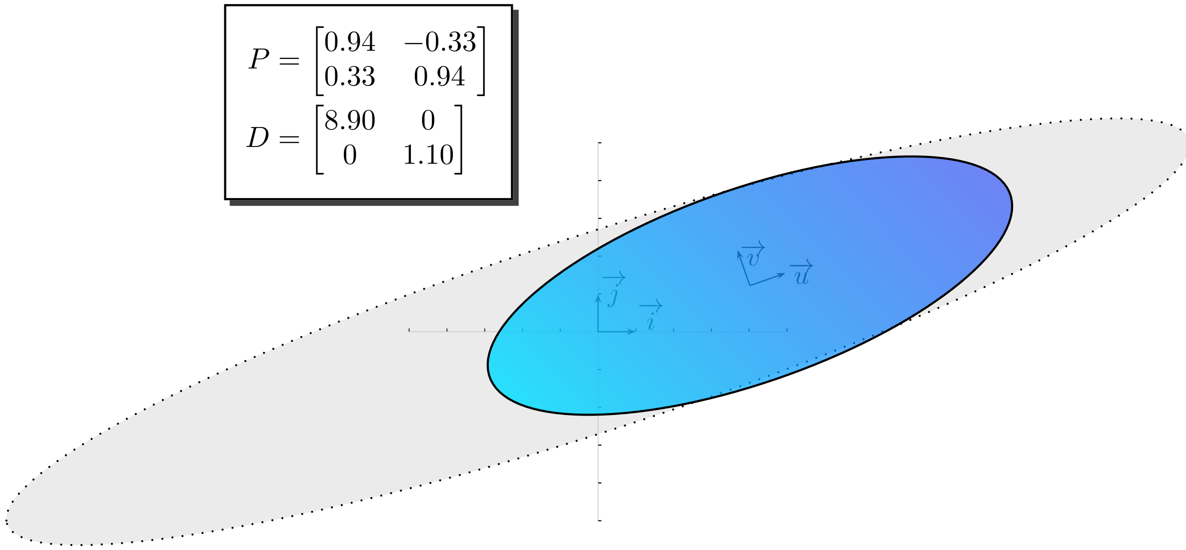


FIGURE 1. 0.95 uncertainty ellipse for the gaussian posterior after measurement of position at dates $t = 5$

will also use the following properties to derive the prediction update of Kalman filter:

Let two r.v. X_1 and X_2 be independant and take their values in $\mathbb{R}^d$. The characteristic function of their sum is given by $$\forall t \in \mathbb{R}^d, \quad \varphi_{X_1+X_2}(t) = \varphi_{X_1}(t) \times \varphi_{X_2}(t)$$
Random variable X is Gaussian $X \hookrightarrow \mathcal{N}(\mu, \Lambda)$ if and only if its characteristic function φ_X is defined by $$\forall t \in \mathbb{R}^d, \quad \varphi_X(t) = \exp i\langle \mu, t\rangle \exp[-\frac{1}{2}\langle \Lambda t, t\rangle]$$

We now restate the equation of prediction step for Kalman filter:

$$x_t = Ax_{t-1} + Bu_t + \epsilon_t$$

We assume x_{t-1} is Gaussian $x_{t-1} \hookrightarrow \mathcal{N}(\mu_{t-1}, \Sigma_{t-1})$, and ϵ_t is Gaussian $\epsilon_t \hookrightarrow \mathcal{N}(0, R_t)$. x_{t-1} and ϵ_t are assumed to be independant. u_t is deterministic. By linearity we know $y_t = Ax_{t-1} + Bu_t$ is Gaussian $y_t \hookrightarrow \mathcal{N}(A\mu_{t-1} + Bu_t, A\Sigma_{t-1}A^T)$. By the above property, we can compute the characteristic function of r.v. x_t

$$\begin{aligned} \forall u \in \mathbb{R}^d, \quad \varphi_{x_t}(u) &= \varphi_{y_t}(u) \times \varphi_{\epsilon_t}(u) \\ &= \exp i\langle A\mu_{t-1} + Bu_t, u\rangle \exp[-\frac{1}{2}\langle A\Sigma_{t-1}A^T u, u\rangle] \times \exp i\langle 0, u\rangle \exp[-\frac{1}{2}\langle R_t t, t\rangle] \\ &= \exp i\langle A\mu_{t-1} + Bu_t, u\rangle \exp[-\frac{1}{2}\langle (A\Sigma_{t-1}A^T + R_t)u, u\rangle] \end{aligned}$$

using again the property of gaussian in the other direction, this shows r.v. x_t is Gaussian $x_t \hookrightarrow \mathcal{N}(\underbrace{A\mu_{t-1} + Bu_t}_{\bar{\mu}_t}, \underbrace{A\Sigma_{t-1}A^T + R_t}_{\bar{\Sigma}_t})$.