PROBABILISTIC ROBOTICS:
ROBOT PERCEPTION

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1

The problem situation is represented in figures 1, 2 and 3. The frame $\mathcal{R} = (0, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ is the reference frame, the frame $\mathcal{R}' = (R, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{k})$ is linked to the robot / image plane of the camera. f is the focal length of the camera, h is the distance between the focal point and the ceiling. The ceiling is the plane defined by equation $z = h - f$. The coordinates in frames $\mathcal{R}$ and $\mathcal{R}'$ are related by equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \\ 0 \end{bmatrix}$$

or using the homogeneous coordinates:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & x_r \\ \sin\theta & \cos\theta & 0 & y_r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

The inverse of this transformation is

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & \\ -\sin\theta & \cos\theta & 0 & -R(\theta)\begin{bmatrix} x_r \\ y_r \\ 0 \end{bmatrix} \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 & -\cos\theta x_r + \sin\theta y_r \\ -\sin\theta & \cos\theta & 0 & -\sin\theta x_r - \cos\theta y_r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

where

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Moreover, simple perspective geometry shows that the coordinates (x'_m, y'_m, θ'_m) of the marker in $\mathcal{R}'$ and the coordinates of its image in the camera plane $z = 0$ (x_i, y_i, θ_i) are related by

$$\begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix} = \begin{bmatrix} \frac{f}{h} & 0 & 0 \\ 0 & \frac{f}{h} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'_m \\ y'_m \\ \theta'_m \end{bmatrix}$$

1.1.

$$\begin{bmatrix} x_m \\ y_m \\ \theta_m \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & x_r \\ \sin\theta & \cos\theta & 0 & y_r \\ 0 & 0 & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{h}{f} & 0 & 0 & 0 \\ 0 & \frac{h}{f} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ \theta_i \\ 1 \end{bmatrix}$$
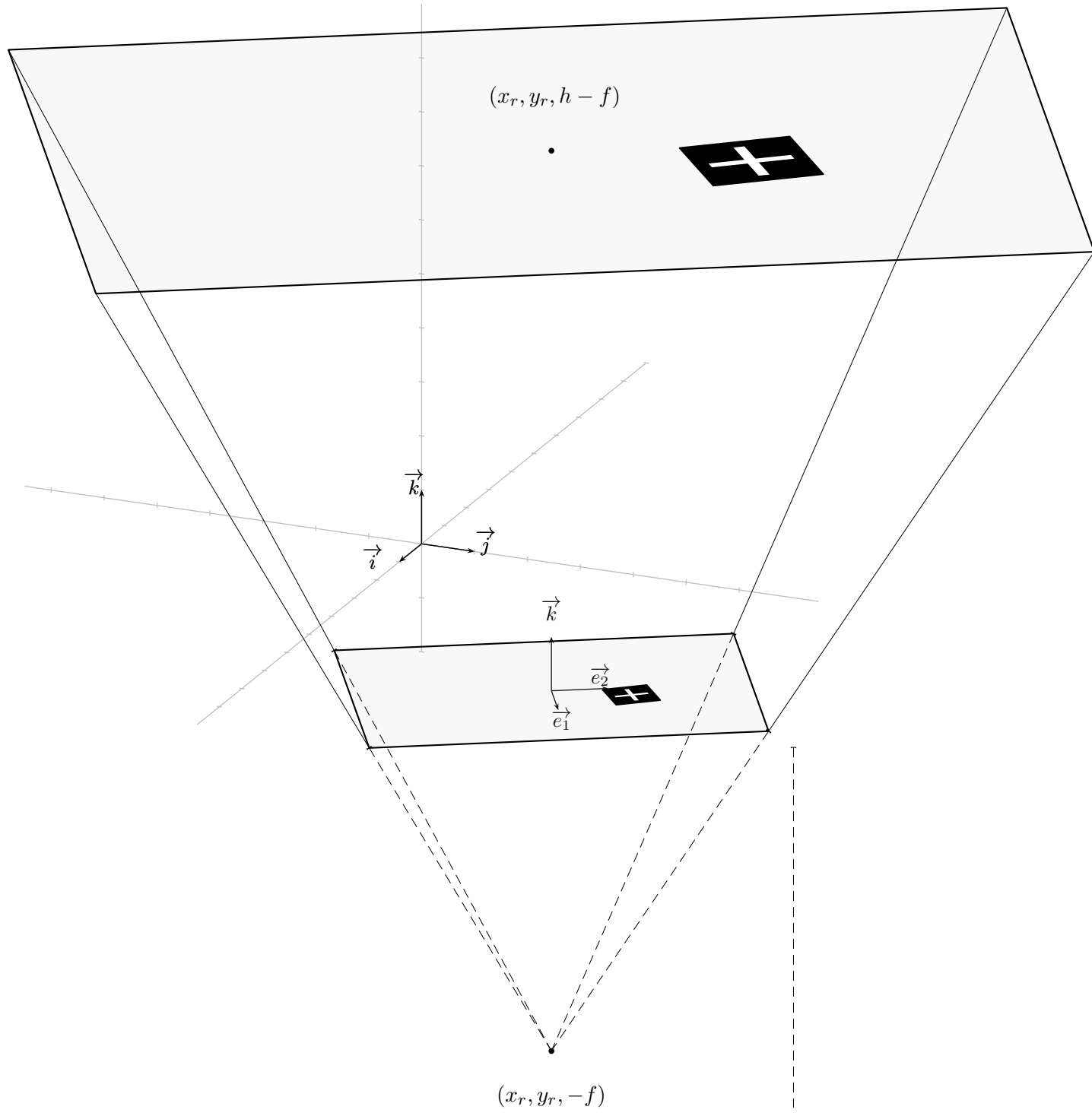
$(x_r, y_r, h - f)$

$\overrightarrow{k}$

$\overrightarrow{j}$

$\overrightarrow{i}$

$\overrightarrow{k}$

$\overrightarrow{e_2}$

$\overrightarrow{e_1}$

$(x_r, y_r, -f)$

FIGURE 1. Problem setting

or

$$
\begin{cases}
x_m = \dfrac{h}{f}(x_i \cos\theta - y_i \sin\theta) + x_r \\[2mm]
y_m = \dfrac{h}{f}(x_i \sin\theta + y_i \cos\theta) + y_r \\[2mm]
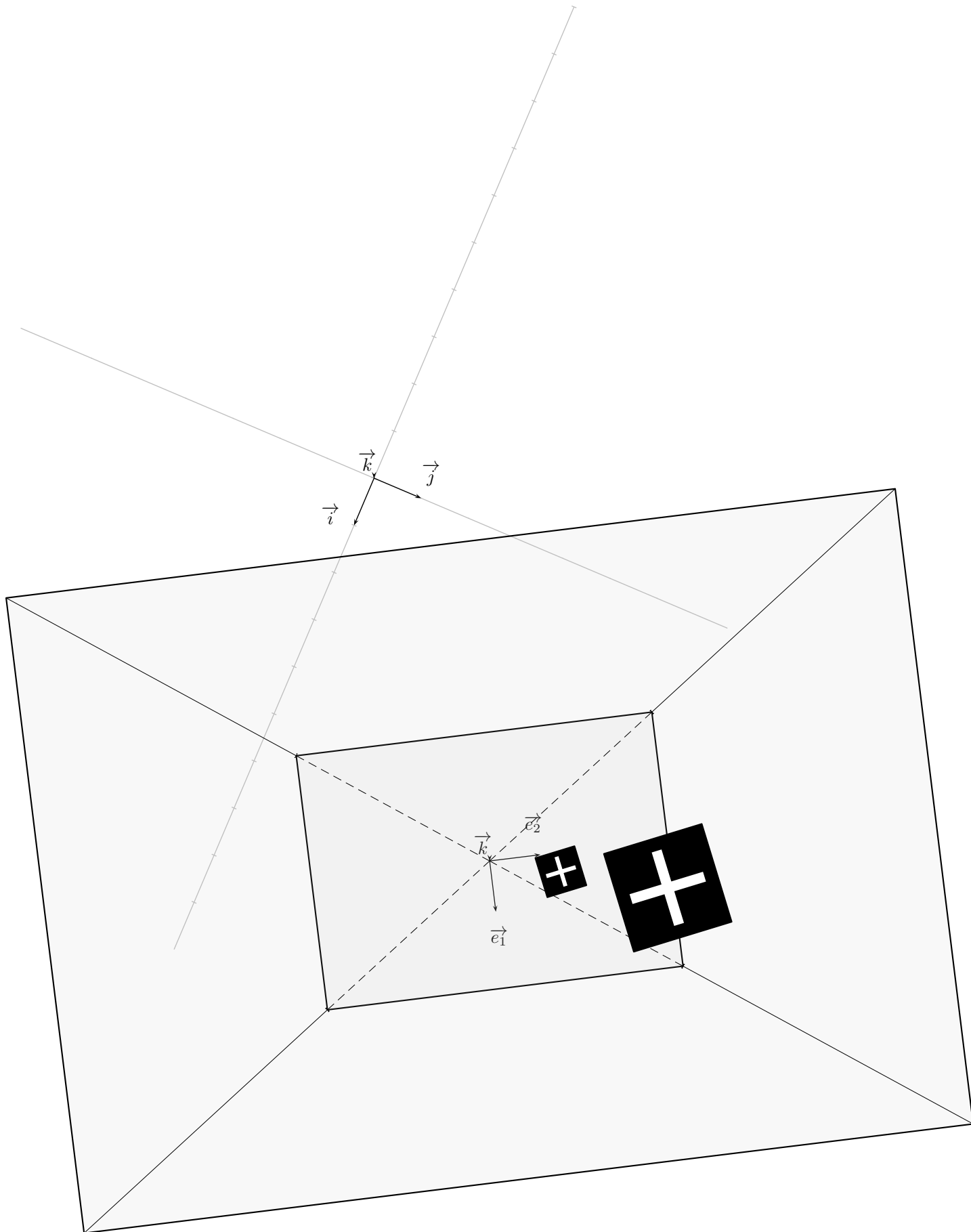\theta_m = \theta_i + \theta
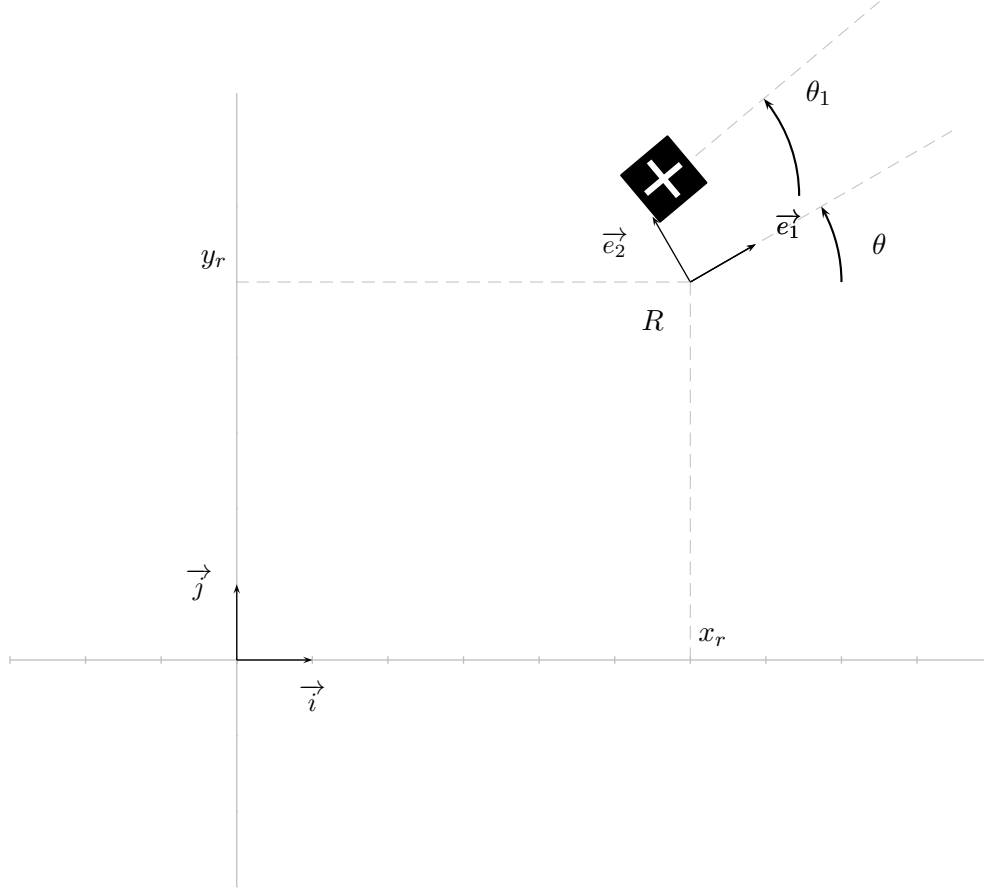\end{cases}
$$

FIGURE 2. Top view

FIGURE 3. Frames definition

1.2.

$$
\begin{bmatrix} x_i \\ y_i \\ \theta_i \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{f}{h} & 0 & 0 & 0 \\ 0 & \frac{f}{h} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & -\cos\theta x_r + \sin\theta y_r \\ -\sin\theta & \cos\theta & 0 & -\sin\theta x_r - \cos\theta y_r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ \theta_m \\ 1 \end{bmatrix}
$$

$$
\begin{cases}
x_i = \dfrac{f}{h}(x_m\cos\theta + y_m\sin\theta) - \dfrac{f}{h}(x_r\cos\theta + y_r\sin\theta) \\[2mm]
y_i = \dfrac{f}{h}(-x_m\sin\theta + y_m\cos\theta) - \dfrac{f}{h}(-x_r\sin\theta + y_r\cos\theta) \\[2mm]
\theta_i = \theta_m - \theta
\end{cases}
$$

1.3.

$$
\begin{cases}
\theta = \theta_m - \theta_i \pmod{\dfrac{\pi}{2}} \\[2mm]
x_r = x_m - \dfrac{h}{f}(x_i\cos\theta - y_i\sin\theta) \qquad ① \\[2mm]
y_r = y_m - \dfrac{h}{f}(x_i\sin\theta + y_i\cos\theta) \qquad ②
\end{cases}
$$

Because of the symetry of the cross, its orientation is defined only modulo $\frac{\pi}{2}$. This means that there is 4 distincts poses of the robot which are compatible with given coordinates of landmark (x_m, y_m, θ_m) and (x_i, y_i, θ_i).

1.4. Suppose we capture 2 indistinguishable crosses on the camera, whose coordinates $(x_{m,j}, y_{m,j}, \theta_{m,j})$, $j \in \{1, 2\}$ are known (but not the association to respective landmarks). Provided that the relative angle between the two is not 0 or $45°$ $(\theta_{m,1} - \theta_{m,2} \notin \{k\frac{\pi}{4}, \quad k \in \mathbb{Z}\})$, we can uniquely identify the landmarks by measuring the relative angle since $\theta_{i,1} - \theta_{i,2} = \theta_{m,1} - \theta_{m,2}$. The pose orientation θ is then the unique solution modulo 2π to the system:

$$
\begin{cases}
x_{m,2} - \dfrac{h}{f}(x_{i,2}\cos\theta - y_{i,2}\sin\theta) = x_{m,1} - \dfrac{h}{f}(x_{i,1}\cos\theta - y_{i,1}\sin\theta) \\[3mm]
y_{m,2} - \dfrac{h}{f}(x_{i,2}\sin\theta + y_{i,2}\cos\theta) = y_{m,1} - \dfrac{h}{f}(x_{i,1}\sin\theta + y_{i,1}\cos\theta)
\end{cases}
$$

$$
\Leftrightarrow
\begin{cases}
(x_{i,1} - x_{i,2})\cos\theta + (-y_{i,1} + y_{i,2})\sin\theta = \dfrac{f}{h}(x_{m,1} - x_{m,2}) \\[3mm]
(y_{i,1} - y_{i,2})\cos\theta + (x_{i,1} - x_{i,2})\sin\theta = \dfrac{f}{h}(y_{m,1} - y_{m,2})
\end{cases}
$$

$$
\Leftrightarrow \quad
\begin{bmatrix} x_{i,1} - x_{i,2} & -y_{i,1} + y_{i,2} \\ y_{i,1} - y_{i,2} & x_{i,1} - x_{i,2} \end{bmatrix}
\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}
= \frac{f}{h}
\begin{bmatrix} x_{m,1} - x_{m,2} \\ y_{m,1} - y_{m,2} \end{bmatrix}
$$

$$
\Leftrightarrow \quad
\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}
= \frac{f}{h((x_{i,1} - x_{i,2})^2 + (y_{i,1} - y_{i,2})^2)}
$$
$$
\times
\begin{bmatrix} x_{i,1} - x_{i,2} & y_{i,1} - y_{i,2} \\ -y_{i,1} + y_{i,2} & x_{i,1} - x_{i,2} \end{bmatrix}
\begin{bmatrix} x_{m,1} - x_{m,2} \\ y_{m,1} - y_{m,2} \end{bmatrix}
$$

Then we can find the position of the robot using ① and ②.