PROBABILISTIC ROBOTICS:
ROBOT MOTION

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1

Evolution of the state will be approximated to the second and first order, for x and $\dot{x}$ respectively:

$$
\begin{aligned}
x_{t+1} &= x_t + \Delta t \times \dot{x}_t + \frac{1}{2}(\Delta t)^2 \times \hat{\ddot{x}}_t + o(\Delta t^2) \\
\Rightarrow x_{t+1} &\approx x_t + \Delta t \times \dot{x}_t + \frac{1}{2}(\Delta t)^2 \times \hat{\ddot{x}}_t
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_{t+1} &= \dot{x}_t + \Delta t \times \hat{\ddot{x}}_t + o(\Delta t) \\
\Rightarrow \dot{x}_{t+1} &\approx \dot{x}_t + \Delta t \times \hat{\ddot{x}}_t
\end{aligned}
$$

where $\hat{\ddot{x}}_t$ is the actual acceleration. If we note $\ddot{x}_t$ the commanded acceleration,

$$\hat{\ddot{x}}_t = \ddot{x}_t + \epsilon_t$$

where ϵ_t is gaussian $\epsilon_t \hookrightarrow \mathcal{N}(0, \sigma^2)$ We can modelize evolution of the state with matrix equality:

$$
\begin{aligned}
\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} &= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2}(\Delta t)^2 \\ \Delta t \end{bmatrix}}_{G} \hat{\ddot{x}}_t \\
\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} &= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(\Delta t)^2 \\ \Delta t \end{bmatrix} \ddot{x}_t + \underbrace{\begin{bmatrix} \frac{1}{2}(\Delta t)^2 \\ \Delta t \end{bmatrix} \epsilon_t}_{\epsilon_t^{'}}
\end{aligned}
$$

We have

$$G \times G^T = \begin{bmatrix} \frac{1}{4}(\Delta t)^4 & \frac{1}{2}(\Delta t)^3 \\ \frac{1}{2}(\Delta t)^3 & (\Delta t)^2 \end{bmatrix}$$

As a linear function of gaussian ϵ_t, the random variable $\epsilon_t^{'}$ is known to be multivariate Gaussian $\mathcal{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \underbrace{\sigma^2 G G^T}_{R})$. The conditional law of random variable $\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix}$ given $x_t, \dot{x}_t, \ddot{x}_t$ is then known to be also Gaussian $\mathcal{N}(\underbrace{A \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(\Delta t)^2 \\ \Delta t \end{bmatrix} \ddot{x}_t}_{\mu}, R)$. The corresponding density is

$$p(x_{t+1}, \dot{x}_{t+1} \mid x_t, \dot{x}_t, \ddot{x}_t) = \frac{1}{2\pi\sqrt{|R|}} \times e^{-\frac{1}{2}(\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} - \mu)^T R^{-1} (\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} - \mu)}$$

The fact that the covariance matrix R is not diagonal shows that x_{t+1} and $\dot{x}_{t+1}$ are correlated. The coefficient of correlation is

$$\begin{aligned}\rho &= \frac{\frac{1}{2}(\Delta t)^3}{\sqrt{\frac{1}{4}(\Delta t)^4 \times \Delta t)^2}} \\ \rho &= 1\end{aligned}$$

showing position and speed are fully correlated.

2

Conditioned on $x_t, \dot{x}_t, \ddot{x}_t$, the r.v. $\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix}$ is gaussian; we can then show (cf. A.1) that the r.v. $\dot{x}_{t+1}$ conditioned on $x_{t+1}, x_t, \dot{x}_t, \ddot{x}_t$ is also gaussian $\dot{x}_{t+1} \hookrightarrow \mathcal{N}(\nu, \Lambda)$, where

$$\begin{aligned}\nu &= E(\dot{x}_{t+1}) + \frac{\frac{1}{2}(\Delta t)^3}{\frac{1}{4}(\Delta t)^4}(x_{t+1} - E(x_{t+1})) \\ &= E(\dot{x}_{t+1}) + \frac{2}{\Delta t}(x_{t+1} - E(x_{t+1})) \\ \Lambda &= (\Delta t)^2 - \frac{(\frac{1}{2}(\Delta t)^3)^2}{\frac{1}{4}(\Delta t)^4} \\ \Lambda &= 0\end{aligned}$$

showing again the speed is deterministic function of the position.

3

We have the model

$$\begin{bmatrix} x_{n+1} \\ \dot{x}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ \dot{x}_n \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2}(\Delta t)^2 \\ \Delta t \end{bmatrix} \epsilon_t}_{\epsilon_t^{'}}$$

with $\epsilon_t^{'} \hookrightarrow \mathcal{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \underbrace{\sigma^2 G G^T}_{R})$. If we assume the random variables $\begin{bmatrix} x_n \\ \dot{x}_n \end{bmatrix}$ and $\epsilon_n^{'}$ to be independant and $\begin{bmatrix} x_n \\ \dot{x}_n \end{bmatrix} \hookrightarrow \mathcal{N}(\mu_n, \Sigma_n)$, then $\begin{bmatrix} x_{n+1} \\ \dot{x}_{n+1} \end{bmatrix}$ is Gaussian $\mathcal{N}(\underbrace{A\mu_n}_{\mu_{n+1}}, \underbrace{A\Sigma_n A^T + R}_{\Sigma_{n+1}})$. We will study the sequence of covariance matrices defined by the recurrence relation

$$\forall n \in \mathbb{N}, \quad \Sigma_{n+1} = A\Sigma_n A^T + R$$

It is convenient to write it in vectorized form

$$\begin{aligned}\begin{bmatrix} \sigma^2_{X,n+1} \\ \sigma_{XY,n+1} \\ \sigma_{XY,n+1} \\ \sigma^2_{Y,n+1} \end{bmatrix} &= \begin{bmatrix} 1 & \Delta t & \Delta t & (\Delta t)^2 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma^2_{X,n} \\ \sigma_{XY,n} \\ \sigma_{XY,n} \\ \sigma^2_{Y,n} \end{bmatrix} + \sigma^2 \begin{bmatrix} \frac{1}{4}(\Delta t)^4 \\ \frac{1}{2}(\Delta t)^3 \\ \frac{1}{2}(\Delta t)^3 \\ (\Delta t)^2 \end{bmatrix} \\ &= \begin{bmatrix} a_{11}A & a_{12}A \\ a_{21}A & a_{22}A \end{bmatrix} \begin{bmatrix} \sigma^2_{X,n} \\ \sigma_{XY,n} \\ \sigma_{XY,n} \\ \sigma^2_{Y,n} \end{bmatrix} + \sigma^2 \begin{bmatrix} \frac{1}{4}(\Delta t)^4 \\ \frac{1}{2}(\Delta t)^3 \\ \frac{1}{2}(\Delta t)^3 \\ (\Delta t)^2 \end{bmatrix} \\ \Leftrightarrow \quad \text{vec}(\Sigma_{n+1}) &= A \otimes A \times \text{vec}(\Sigma_n) + \text{vec}(R)\end{aligned}$$

where $\otimes$ is the Kronecker product. A little observation shows that

$$\forall n \in \mathbb{N}, \quad (A \otimes A)^n = \begin{bmatrix} 1 & n\Delta t & n\Delta t & n^2(\Delta t)^2 \\ 0 & 1 & 0 & n\Delta t \\ 0 & 0 & 1 & n\Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\forall n \in \mathbb{N}, \quad \mathrm{vec}(\Sigma_n) = (A \otimes A)^n \,\mathrm{vec}(\Sigma_0) + (\sum_{i=0}^{n-1} (A \otimes A)^i)\,\mathrm{vec}(R)$$

Then, using the formulas of a sum of integers and sum of squared integers,

$$\sum_{i=0}^{n-1} (A \otimes A)^i = \begin{bmatrix} n & \frac{n(n-1)}{2}\Delta t & \frac{n(n-1)}{2}\Delta t & \frac{n(n-1)(2n-1)}{6}(\Delta t)^2 \\ 0 & n & 0 & \frac{n(n-1)}{2}\Delta t \\ 0 & 0 & n & \frac{n(n-1)}{2}\Delta t \\ 0 & 0 & 0 & n \end{bmatrix}$$

from which we can build a closed formula for Σ_n. What is of interest for us here is the asymptotic behavior,

$$\begin{aligned} \sigma^2_{X,n} &\underset{n\to\infty}{\sim} \frac{n(n-1)(2n-1)}{6}(\Delta t)^2 \times (\Delta t)^2\sigma^2 \\ &\underset{n\to\infty}{\sim} \frac{n^3}{3}(\Delta t)^4\sigma^2 \\ \sigma_{XY,n} &\underset{n\to\infty}{\sim} \frac{n(n-1)}{2}\Delta t \times (\Delta t)^2\sigma^2 \\ &\underset{n\to\infty}{\sim} \frac{n^2}{2}(\Delta t)^3\sigma^2 \\ \sigma^2_{Y,n} &\underset{n\to\infty}{\sim} n(\Delta t)^2\sigma^2 \end{aligned}$$

leading to

$$\begin{aligned} \rho_{XY,n} &\underset{n\to\infty}{\sim} \frac{\frac{n^2}{2}(\Delta t)^3\sigma^2}{\sqrt{\frac{n^3}{3}(\Delta t)^4\sigma^2 \times n(\Delta t)^2\sigma^2}} \\ &= \frac{\sqrt{3}}{2} < 1 \end{aligned}$$

Asymptotically position and speed do not get fully correlated.

4

The conventions used for this exercice and the following 2 are represented in figure 1. Let's study the movement of the center of front wheel B in frame $\mathcal{R} = (O, \overrightarrow{i}, \overrightarrow{j})$. We let $\mathcal{R}' = (A, \overrightarrow{i}, \overrightarrow{j})$ the frame with origin the center of back wheel, translating with regards to $\mathcal{R}$. We can write

$$\begin{aligned} \overrightarrow{v_B}_{/\mathcal{R}} &= \overrightarrow{v_B}_{/\mathcal{R}'} + \overrightarrow{v}_{\mathcal{R}'/\mathcal{R}} \\ &= \overrightarrow{v_B}_{/\mathcal{R}'} + \overrightarrow{v_A}_{/\mathcal{R}} \qquad ① \end{aligned}$$

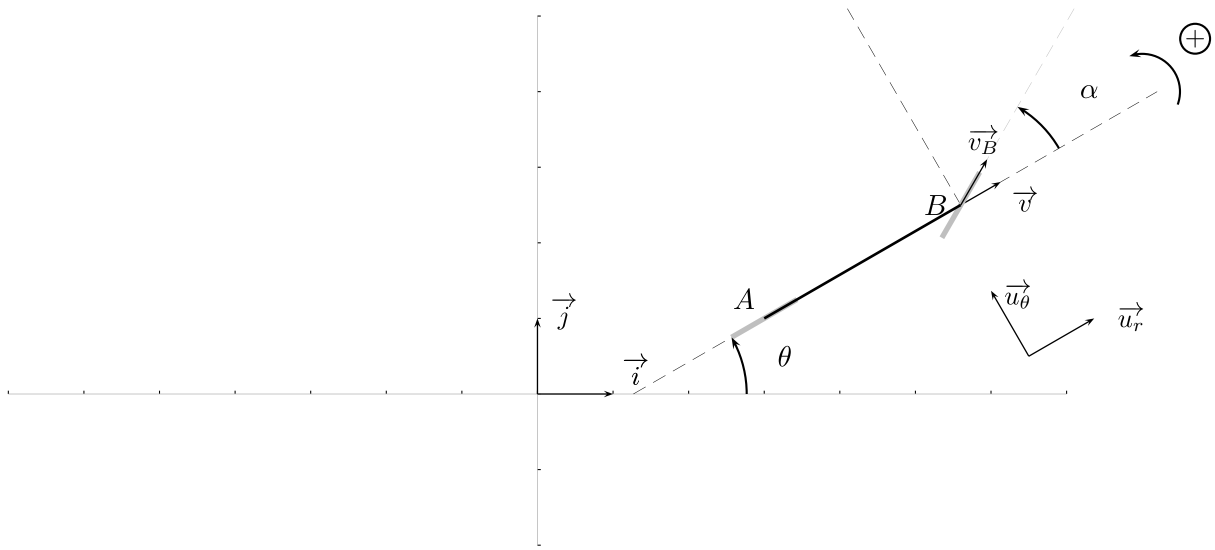

FIGURE 1. Ideal bicycle model

In frame $\mathcal{R}'$, the length AB being constant, B is in rotation

$$\begin{aligned} &\forall t \geqslant 0, \quad \left\|\overrightarrow{AB}(t)\right\|^2 = l \\ \Rightarrow\ &\forall t \geqslant 0, \quad \frac{d\overrightarrow{AB}}{dt}(t) \cdot \overrightarrow{AB}(t) = 0 \\ \Leftrightarrow &\forall t \geqslant 0, \quad \overrightarrow{v_B}_{/\mathcal{R}'}(t) \cdot \overrightarrow{AB}(t) = 0 \end{aligned}$$

so

$$\overrightarrow{v_B}_{/\mathcal{R}'} = l\dot{\theta}\overrightarrow{u_\theta}$$

A is restrained to move in the direction of the bicycle axis:

$$\overrightarrow{v_A}_{/\mathcal{R}} = vu_r$$

$$① \Rightarrow \overrightarrow{v_B}_{/\mathcal{R}} = l\dot{\theta}\overrightarrow{u_\theta} + vu_r$$

B is restrained to move in the direction of front wheel, so if we project the previous relation we get

$$\begin{aligned} &\begin{cases} v_B \sin\alpha = l\dot{\theta} \\ v_B \cos\alpha = v \end{cases} \\ \Rightarrow\ & v\tan\alpha = l\dot{\theta} \qquad ② \end{aligned}$$

Projecting now $\overrightarrow{v_B}$ on $(\overrightarrow{i}, \overrightarrow{j})$, we obtain

$$\begin{cases} v_B \cos(\alpha+\theta) = \dot{x} \\ v_B \sin(\alpha+\theta) = \dot{y} \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{v}{\cos\alpha}\cos(\alpha+\theta) = \dot{x} \\ \dfrac{v}{\cos\alpha}\sin(\alpha+\theta) = \dot{y} \end{cases}$$

$$\Leftrightarrow \begin{cases} v(\cos\theta - \tan\alpha\sin\theta) = \dot{x} & ③ \\ v(\tan\alpha\cos\theta + \sin\theta) = \dot{y} & ④ \end{cases}$$

We discretize time in ②,③ and ④ to obtain approximate motion equations:

$$\begin{cases} x_{t+1} = x_t + \hat{v}(\cos\theta_t - \tan\hat{\alpha}\sin\theta_t)\Delta t \\ y_{t+1} = y_t + \hat{v}(\tan\hat{\alpha}\cos\theta_t + \sin\theta_t)\Delta t \\ \theta_{t+1} = \theta_t + \dfrac{\hat{v}}{l}\tan\hat{\alpha} \times \Delta t \end{cases}$$

with

$$\hat{\alpha} = \alpha + \epsilon_{\sigma_\alpha^2}$$
$$\hat{v} = v + \epsilon_{\sigma_v^2}$$

where $\epsilon_{\sigma^2} \hookrightarrow \mathcal{N}(0,\sigma^2)$.

5

cf. file `bicyclespl.m` for the matlab code. In figure 2, I plotted one step of a simulation starting from state $\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \pi/4 \end{bmatrix}$.

6

This exercise consists in inversing formulas ②,③ and ④ for α, v and θ_{t+1}.

$$\Leftrightarrow \begin{cases} v(\cos\theta - \tan\alpha\sin\theta) = \dot{x} \\ v(\tan\alpha\cos\theta + \sin\theta) = \dot{y} \end{cases}$$

$$\Leftrightarrow \begin{cases} \cos\theta - \tan\alpha\sin\theta = \dfrac{\dot{x}}{v} \\ \tan\alpha\cos\theta + \sin\theta = \dfrac{\dot{y}}{v} \end{cases}$$

$$\Leftrightarrow \begin{cases} \dot{y}(\cos\theta - \tan\alpha\sin\theta) = \dot{y}\dfrac{\dot{x}}{v} \\ \dot{x}(\tan\alpha\cos\theta + \sin\theta) = \dot{x}\dfrac{\dot{y}}{v} \end{cases}$$

$$\Rightarrow \quad \tan\alpha(\dot{x}\cos\theta + \dot{y}\sin\theta) = \dot{y}\cos\theta - \dot{x}\sin\theta$$

$$\Leftrightarrow \quad \tan\alpha = \frac{\dot{y}\cos\theta - \dot{x}\sin\theta}{\dot{x}\cos\theta + \dot{y}\sin\theta}$$

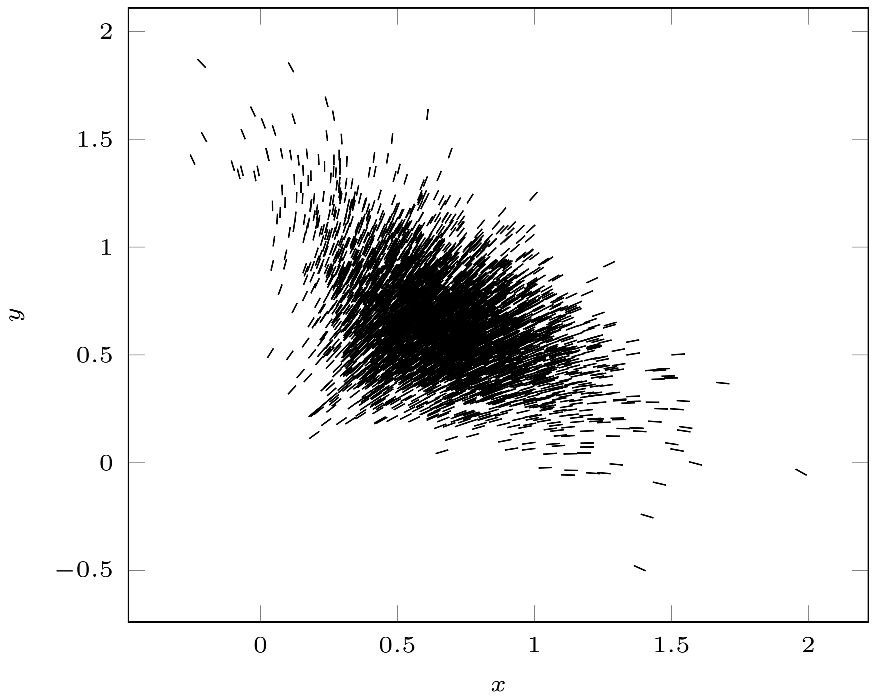

FIGURE 2. 5000 samples from motion model, $\alpha = 0$, $v = 90 \times 10^{-2}$, $\sigma_\alpha^2 = 15^{\circ 2}$, $\sigma_v^2 = 0.04v^2$.

In order to compute the most likely value of steering angle α, it suffices to insert in previous relation $\dot{x} = \frac{x_{t+1}-x_t}{\Delta t}$, $\dot{y} = \frac{y_{t+1}-y_t}{\Delta t}$ and $\theta = \theta_t$. From there we can compute the speed

$$\begin{aligned} \frac{v^2}{\cos^2\alpha} &= \dot{x}^2 + \dot{y}^2 \\ \Leftrightarrow \qquad v^2 &= \cos^2\alpha(\dot{x}^2 + \dot{y}^2) \end{aligned}$$

and finally the most likely final orientation will be given by

$$\theta_{t+1} = \theta_t + \frac{v}{l}\tan\alpha \times \Delta t$$

7

The model is represented in figure 3. The forward and sideward speeds are designed by v_1 and v_2 respectively.

$$\begin{aligned} \dot{x} &= v_1\cos\theta - v_2\sin\theta \\ \dot{y} &= v_1\sin\theta + v_2\cos\theta \\ \dot{\theta} &= \omega \end{aligned}$$

We discretize to get

$$\begin{aligned} x_{t+1} &= x_t + (v_1\cos\theta_t - v_2\sin\theta_t)\Delta t \\ y_{t+1} &= y_t + (v_1\sin\theta_t + v_2\cos\theta_t)\Delta t \\ \theta_{t+1} &= \theta_t + \omega\Delta t \end{aligned}$$

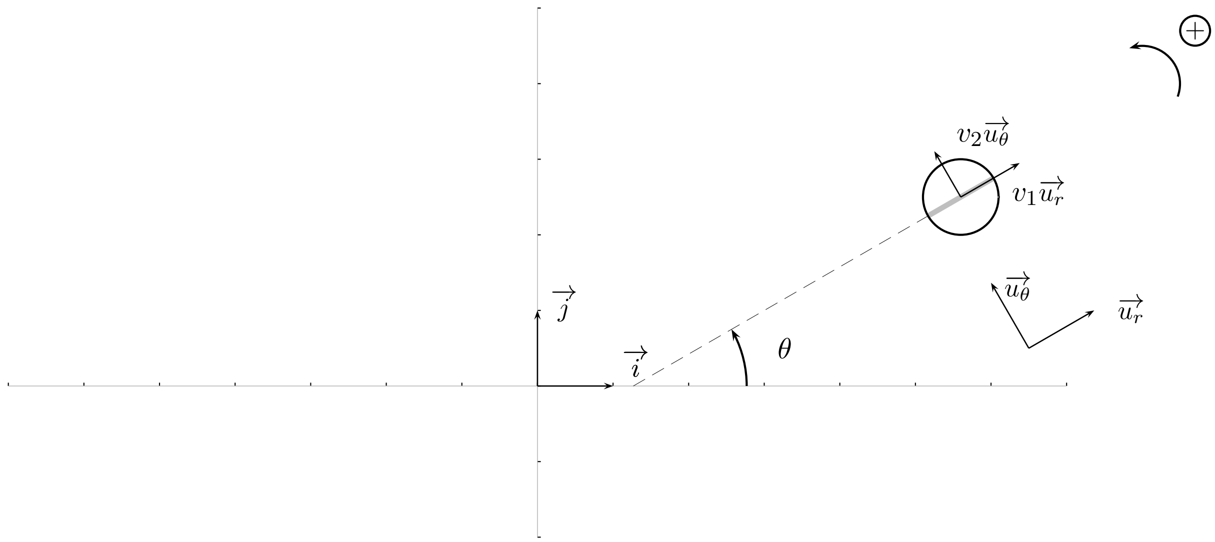

FIGURE 3. holonomic robot model

To compute the transition probability density, one needs to invert the previous system for the controls v_1, v_2 and ω;for that it suffices to notice that:

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}^{-1} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

so that

$$\begin{aligned} v_1 &= ((x_{t+1}-x_t)\cos\theta_t + (y_{t+1}-y_t)\sin\theta_t)\frac{1}{\Delta t}\\ v_2 &= (-(x_{t+1}-x_t)\sin\theta_t + (y_{t+1}-y_t)\cos\theta_t)\frac{1}{\Delta t}\\ \omega &= \frac{\theta_{t+1}-\theta_t}{\Delta t} \end{aligned}$$

8

$$\mathrm{Var}(\epsilon_{b^2}) = E[X^2] - (E[X])^2$$

$$\begin{aligned} \int_\Omega X^2\,\mathrm{d}P &= \int_{-\sqrt{6}b}^{\sqrt{6}b} x^2(\frac{1}{\sqrt{6}b} - \frac{|x|}{6b^2})\,\mathrm{d}x\\ &= 2\int_0^{\sqrt{6}b} x^2(\frac{1}{\sqrt{6}b} - \frac{|x|}{6b^2})\,\mathrm{d}x\\ &= 2(\frac{1}{\sqrt{6}b}\int_0^{\sqrt{6}b} x^2\,\mathrm{d}x - \frac{1}{6b^2}\int_0^{\sqrt{6}b} x^3\,\mathrm{d}x)\\ &= 2(\frac{1}{\sqrt{6}b}\left[\frac{x^3}{3}\right]_0^{\sqrt{6}b} - \frac{1}{6b^2}\left[\frac{x^4}{4}\right]_0^{\sqrt{6}b})\\ &= 2(\frac{1}{\sqrt{6}b}\frac{6\sqrt{6}b^3}{3} - \frac{1}{6b^2}\frac{36b^4}{4})\\ &= b^2 \end{aligned}$$

so,

$$\mathrm{Var}(\epsilon_{b^2}) = b^2$$

Let now X and Y 2 independant uniform r.v. $\hookrightarrow \mathcal{U}([-b, b])$, and let $Z = \frac{\sqrt{6}}{2}[X + Y]$. We have

$$\mathrm{Var}(Z) = \frac{3}{2}[\mathrm{Var}(X) + \mathrm{Var}(Y)]$$

and

$$\begin{aligned}
\mathrm{Var}(X) = \mathrm{Var}(Y) &= E[X^2] - (E[X])^2 \\
&= E[X^2] \\
&= \int_{-b}^{b} x^2 \frac{1}{2b} \, \mathrm{d}x \\
&= 2 \int_{0}^{b} x^2 \frac{1}{2b} \, \mathrm{d}x \\
&= 2\frac{1}{2b} \left[\frac{x^3}{3}\right]_0^b \\
&= \frac{1}{b}\frac{b^3}{3} \\
&= \frac{b^2}{3}
\end{aligned}$$

so that

$$\mathrm{Var}(Z) = b^2$$

Appendix A

A.1.

lemma. Let $Z = (X, Y)$ be a gaussian r.v. which takes its values in $\mathbb{R}^n \times \mathbb{R}^p$, $Z \hookrightarrow \mathcal{N}(\mu, \Lambda)$. We note

$$\Lambda = \begin{bmatrix} \Lambda_X & \Lambda_{XY}^T \\ \Lambda_{XY} & \Lambda_Y \end{bmatrix}$$

Then, the law of r.v. Y conditionned on $(X = x)$ is gaussian $\hookrightarrow \mathcal{N}(m, \Sigma)$, where

$$m = EY + \Lambda_{XY}\Lambda_X^{-1}(x - EX)$$
$$\Sigma = \Lambda_Y - \Lambda_{XY}\Lambda_X^{-1}\Lambda_{XY}^T$$

Proof. We already know that X is gaussian, as linear transfomation (namely projection) of gaussian Z. Let $Y' = Y - CX$. The r.v. (X, Y') is also gaussian and we can verify that

$$\Lambda_{XY'} = \Lambda_{XY} - CX$$

We can choose $C = \Lambda_{XY}\Lambda_X^{-1}$ so that X and Y' are decorrelated, but this is the same as independance since (X, Y') is gaussian. The law of $Y = Y' + CX$ conditionned on $(X = x)$ is then simply the law of $Y' + Cx$. This shows it is gaussian and

$$\begin{aligned} m &= EY' + Cx \\ &= (EY - \Lambda_{XY}\Lambda_X^{-1}EX) + \Lambda_{XY}\Lambda_X^{-1}x \\ &= EY + \Lambda_{XY}\Lambda_X^{-1}(x - EX) \\ \Sigma &= \Lambda_Y - C\Lambda_X C^T \\ &= \Lambda_Y - \Lambda_{XY}\Lambda_X^{-1}\Lambda_X\Lambda_X^{-1}\Lambda_{XY}^T \\ &= \Lambda_Y - \Lambda_{XY}\Lambda_X^{-1}\Lambda_{XY}^T \end{aligned}$$

where we have used the fact that Λ_X is symetric and Y' and CX are independant. □