PROBABILISTIC ROBOTICS: MOBILE ROBOT LOCALIZATION: GRID AND MONTE CARLO

Pierre-Paul TACHER

1

Let d the dimension of state space. For simplicity we assume the dimension d_1 of the measurement space satisfy $d_1 = O(d)$.

The EKF algorithm implies the computation of Jacobians and evaluations of functions whose complexities depend on the particular model. Thus we will ignore it here (we assume complexity is $O(1)$). The time complexity of EKF localization is dominated by the constant number of order d matrix multiplications and inversions; it can be shown [1] the time complexity of such operation is the same, and current state of the art [2] guarantees complexity $O(d^\omega)$, where $\omega < 2.3728639$. Thus this is also the time complexity of EKF algorithm. The memory complexity is $O(d^2)$.

Let's consider the grid algorithm; for now we use a common discretization resolution h for all axes and not dependant on d so that the number of grid cells is $O(h^d)$. The loop over the grid cells contains $O(h^d)$ additions / multiplications, hence a time complexity of $O(h^{2d})$. The memory complexity is $O(h^d)$.

The time or memory complexity of MCL is $O(M)$, where M is the number of particles. This number have to be scaled with d to represent accurately the probability distribution. We will roughly estimate this dependency: Let M_d an estimate of the number of particles needed to represent gaussian r.v. $\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \end{bmatrix} \hookrightarrow \mathcal{N}(0, I_d)$. In dimension $d+1$, let's estimate how many particles are needed to represent gaussian r.v. $\begin{bmatrix} X_1 \\ \vdots \\ X_d \\ X_{d+1} \end{bmatrix} \hookrightarrow \mathcal{N}(0, I_{d+1})$; for a fixed $x_{d+1} \in \mathbb{R}$, the d dimensional gaussian r.v. $\begin{bmatrix} X_1 \\ \vdots \\ X_d \\ x_{d+1} \end{bmatrix}$ conditioned on $(X_{d+1} = x_{d+1})$ needs approximately M_d particles with n+1$^{\text{th}}$ coordinate x_{d+1} to be represented. And we need to choose at least M_1 values of x_{d+1} to hope to represent the gaussian marginals $\begin{bmatrix} x_1 \\ \vdots \\ x_d \\ X_{d+1} \end{bmatrix}$. So we have $M_{d+1} \approx M_d \times M_1$ so that we end in exponential dependency $M_d \approx M_1^d$.

---

[1] Cormen, Thomas H.; Leiserson, Charles E.; Rivest, Ronald L.; Stein, Clifford: *Introduction to Algorithms (3rd ed.)*, MIT Press and McGraw-Hill, Theorems 28.1 & 28.2(2009)

[2] Le Gall, François: *Powers of tensors and fast matrix multiplication*, Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation (2014)

2

The key point is that the r.v. $Z_1, Z_2, \dots, Z_k$ conditioned on x_t are assumed to be independant. The loop over the k measurements in EKF localization algorithm can be interpreted as simple iterations of the basic EKF where there is no motion between the measurement updates. We have,

$$p(x_t \mid z_1) = \eta p(z_1 \mid x_t) \times p(x_t)$$

and

$$\begin{aligned}\forall i \in [\![2, k]\!], \quad p(x_t \mid z_{1:i+1}) &= \eta p(z_{i+1} \mid x_t, z_{1:i}) \times p(x_t \mid z_{1:i}) \\ &= \eta p(z_{i+1} \mid x_t) \times p(x_t \mid z_{1:i})\end{aligned}$$

A straighforward induction shows that $\forall i \in [\![1, k]\!]$ the law of x_t conditioned on $z_{1:i}$ is gaussian; since it is known that the law of Z_i conditioned on x_t is gaussian $\hookrightarrow \mathcal{N}(h(\bar{\mu}, j(i)) + H_{j(i)}(x_t - \bar{\mu}), Q_t)$,the mean and covariance matrices are computed by the relations already derived in the original Kalmann filter

$$\begin{aligned}\forall i \in [\![1, k]\!], \quad K_i &= \Sigma_{i-1} H_{j(i)}^T (H_{j(i)} \Sigma_{i-1} H_{j(i)}^T + Q)^{-1} \\ \mu_i &= \mu_{i-1} + K_i(z_i - h(\bar{\mu}, j(i))) \\ \Sigma_i &= (1 - K_i H_{j(i)}) \Sigma_{i-1}\end{aligned}$$

where $\mu_0 = \bar{\mu}$, $\Sigma_0 = \bar{\Sigma}$, $\mu_k = \mu$ and $\Sigma_k = \Sigma$. These are precisely the updates in the loop over measurements in the EKF localization algorithm.

3

Each $w_t^{[m]} = p(z_t \mid x_t^{[m]})$ can be seen as independant realization of the random variable $p(z_t \mid X_t)$, where X_t is the random variable giving the position of a particule at time t, conditioned on past actions and observations $z_{1:t-1}, u_{1:t}$. This r.v. has a finite expectancy given by

$$\begin{aligned}E(p(z_t \mid X_t)) &= \int_\Omega p(z_t \mid X_t) \,\mathrm{d}P \\ &= \int_\Omega p(z_t \mid X_t, z_{1:t-1}, u_{1:t}) \,\mathrm{d}P \\ &= \int_\Omega \frac{p(X_t \mid z_{1:t}, u_{1:t}) p(z_t \mid z_{1:t-1}, u_{1:t})}{p(X_t \mid z_{1:t-1}, u_{1:t})} \,\mathrm{d}P \\ &= p(z_t \mid z_{1:t-1}, u_{1:t}) \int_\mathbb{R} \frac{p(x_t \mid z_{1:t}, u_{1:t})}{p(x_t \mid z_{1:t-1}, u_{1:t})} p(x_t \mid z_{1:t-1}, u_{1:t}) \,\mathrm{d}x_t \\ &= p(z_t \mid z_{1:t-1}, u_{1:t}) \underbrace{\int_\mathbb{R} p(x_t \mid z_{1:t}, u_{1:t}) \,\mathrm{d}x_t}_{=1} \\ &= p(z_t \mid z_{1:t-1}, u_{1:t})\end{aligned}$$

where we assumed $p(X_t \mid z_{1:t-1}, u_{1:t}) > 0$ almost surely. The strong law of large numbers[3] shows that, almost surely,

$$\lim_{M \to +\infty} \frac{1}{M} \sum_{m=0}^{M} w_t^{[m]} = p(z_t \mid z_{1:t-1}, u_{1:t})$$

[3]Durrett, Rick: *Probability: Theory and Examples (4th ed.)*, Cambridge University Press, Theorems 2.4.1(2013)