**PROBABILISTIC ROBOTICS:**
**GAUSSIAN FILTERS**

Pierre-Paul TACHER

## 1

**1.1.** We can define the state vector to include position and velocity:

$$Y_t = \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix}$$

**1.2.** Evolution of the state will be approximated to the second and first order, for $x$ and $\dot{x}$ respectively :

$$\begin{aligned} x_{t+1} &= x_t + \Delta t \times \dot{x}_t + \frac{1}{2}(\Delta t)^2 \times \ddot{x}_t + o(\Delta t^2) \\ \Rightarrow x_{t+1} &\approx x_t + 1 \times \dot{x}_t + \frac{1}{2} \times \ddot{x}_t \end{aligned}$$

$$\begin{aligned} \dot{x}_{t+1} &= \dot{x}_t + \Delta t \times \ddot{x}_t + o(\Delta t) \\ \Rightarrow \dot{x}_{t+1} &\approx \dot{x}_t + \ddot{x}_t \end{aligned}$$

We can modelize evolution of the state with matrix equality:

$$\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2}\ddot{x}_t \\ \ddot{x}_t \end{bmatrix}}_{\epsilon_t}$$

$$\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}}_{G} \ddot{x}_t$$

We have

$$G \times G^T = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

As a linear function of gaussian $\ddot{x}_t$, the random variable $\epsilon_t$ is known to be multivariate Gaussian $\mathcal{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \underbrace{\sigma^2 GG^T}_{R})$. The conditional law of random variable $\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix}$ given $(\begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix})$ is then known to be also Gaussian $\mathcal{N}(A \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix}, R)$.

Moreover, if we assume the random variable $\begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix}$ to be Gaussian $\mathcal{N}(\mu_t, \Sigma_t)$, then $\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix}$ is Gaussian $\mathcal{N}(\underbrace{A\mu_t}_{\overline{\mu_{t+1}}}, \underbrace{A\Sigma_t A^T + R}_{\overline{\Sigma_{t+1}}})$.

Note that since we do not incorporate measurements for now, we have using the notation of the book

$$\overline{\mu_{t+1}} = \mu_{t+1}$$
$$\overline{\Sigma_{t+1}} = \Sigma_{t+1}$$

**1.3.** From the previous relations, it is clear that

$$\forall t \in \mathbb{N}, \mu_t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let's t compute the first covariance matrices, using $\sigma^2 = 1$:

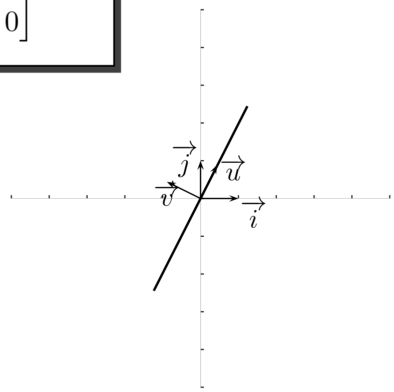
| $t$ | $\Sigma_t$ |
|---|---|
| 0 | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ |
| 1 | $\begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ |
| 2 | $\begin{bmatrix} 2.5 & 2 \\ 2 & 2 \end{bmatrix}$ |
| 3 | $\begin{bmatrix} 8.75 & 4.5 \\ 4.5 & 3 \end{bmatrix}$ |
| 4 | $\begin{bmatrix} 21 & 8 \\ 8 & 4 \end{bmatrix}$ |
| 5 | $\begin{bmatrix} 41.25 & 12.5 \\ 12.5 & 5 \end{bmatrix}$ |
| 6 | $\begin{bmatrix} 71.5 & 18 \\ 18 & 6 \end{bmatrix}$ |
| 7 | $\begin{bmatrix} 113.75 & 24.5 \\ 24.5 & 7 \end{bmatrix}$ |
| 8 | $\begin{bmatrix} 170 & 32 \\ 32 & 8 \end{bmatrix}$ |
| 9 | $\begin{bmatrix} 242.25 & 40.5 \\ 40.5 & 9 \end{bmatrix}$ |
| 10 | $\begin{bmatrix} 332.5 & 50 \\ 50 & 10 \end{bmatrix}$ |

The uncertainty ellipses of figure 0 are centered at $\mu_t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, have axes whose directions are given by the eigenvectors of $\Sigma_t$, and the semi-minor and semi-major axis are scaled by eigenvalues; for detailed derivation, cf. A.1. $\Sigma_t$ is a symetric real matrix, it can be diagonalized in an orthonormal basis, with real eigenvalues

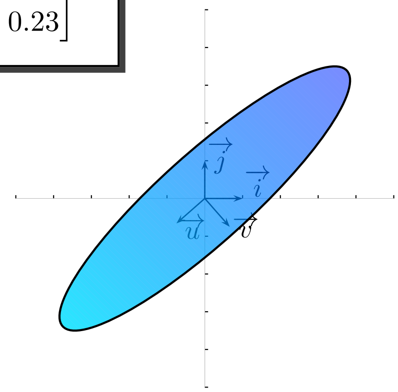
$$\Sigma_t = P_t D_t P_t^T$$

where $P_t$ is an orthogonal matrix and $D_t$ is diagonal. Note the first ellipse is degenerated since $\Sigma_1$ has a null eigenvalue.

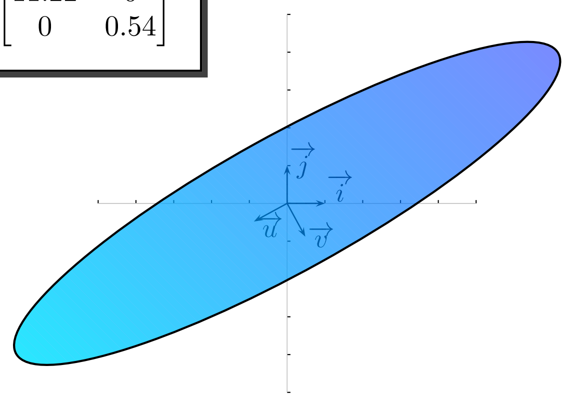
$$P = \begin{bmatrix} 0.45 & .89 \\ -0.89 & 0.45 \end{bmatrix} \quad D = \begin{bmatrix} 1.25 & 0 \\ 0 & 0 \end{bmatrix}$$



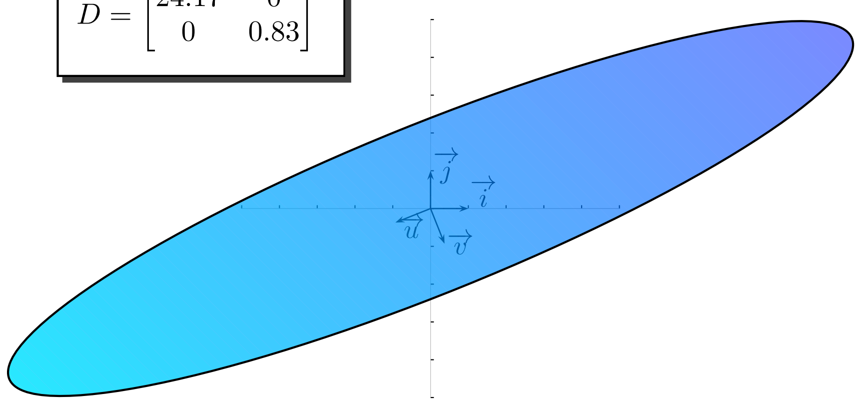

$$P = \begin{bmatrix} -0.75 & 0.66 \\ -0.66 & -0.75 \end{bmatrix} \quad D = \begin{bmatrix} 4.27 & 0 \\ 0 & 0.23 \end{bmatrix}$$




$$P = \begin{bmatrix} -0.88 & 0.48 \\ -0.48 & -0.88 \end{bmatrix} \quad D = \begin{bmatrix} 11.22 & 0 \\ 0 & 0.54 \end{bmatrix}$$




$$P = \begin{bmatrix} -0.93 & -0.37 \\ -0.37 & -0.93 \end{bmatrix} \quad D = \begin{bmatrix} 24.17 & 0 \\ 0 & 0.83 \end{bmatrix}$$




## 2

**2.1.** We can modelize the noisy observation of the position by:

$$\begin{aligned} z_t &= x_t + \xi_t \\ &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} + \xi_t \end{aligned}$$

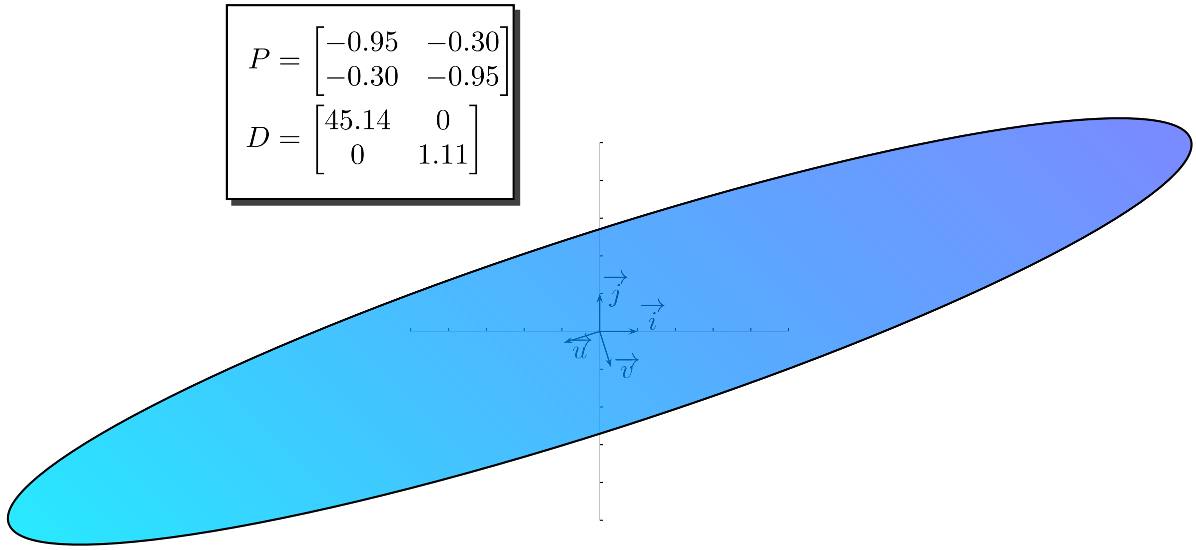



FIGURE 0. 0.95 uncertainty ellipses for the gaussian posteriors at dates $t \in [\![1, 5]\!]$

where the random variable $\xi_t$ is gaussian $\xi_t \hookrightarrow \mathcal{N}(0, \underbrace{\sigma^2}_{Q})$.

**2.2.** Before the observation the moments are:

$$\bar{\mu}_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\bar{\Sigma}_5 = \begin{bmatrix} 41.25 & 12.5 \\ 12.5 & 5 \end{bmatrix}$$

The measurement update consists of

$$\begin{aligned} K_5 &= \bar{\Sigma}_5 C^T (C \bar{\Sigma}_5 C^T + Q)^{-1} \\ &= \begin{bmatrix} 0.80 \\ 0.24 \end{bmatrix} \\ \mu_5 &= \bar{\mu}_5 + K_5 (z_5 - C \bar{\mu}_5) \\ &= \begin{bmatrix} 4.02 \\ 1.22 \end{bmatrix} \\ \Sigma_5 &= (I - K_5 C) \bar{\Sigma}_5 \\ &= \begin{bmatrix} 8.05 & 2.44 \\ 2.44 & 1.95 \end{bmatrix} \end{aligned}$$

The corresponding ellipse is drawn in figure 1.

## 3

Let's recall the definition of the characteristic function of a random variable: (a tad of theory of measure is needed)($\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$)

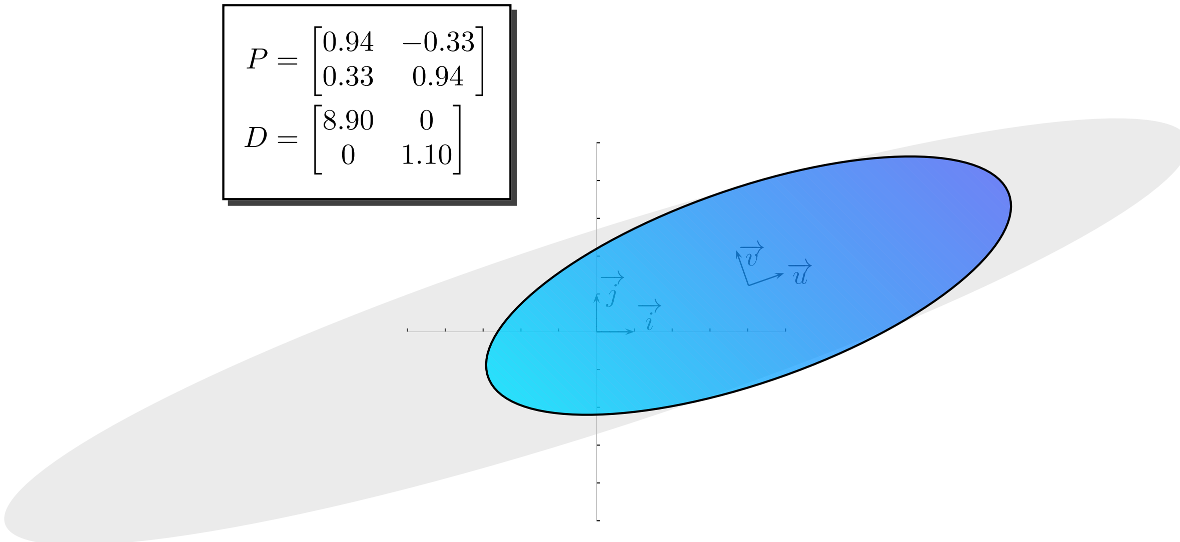



FIGURE 1. 0.95 uncertainty ellipse for the gaussian posterior after measurement of position at dates $t = 5$

**Definition.** Let $X$ be an random variable which takes it values in $\mathbb{R}^d$. The characteristic function of $X$ is the function which goes from $\mathbb{R}^d$ to $\mathbb{C}$:

$$\begin{aligned} \forall t \in \mathbb{R}^d, \quad \varphi_X(t) &= E \exp i\langle X, t\rangle \\ &= \int_{\mathbb{R}^d} \exp i\langle x, t\rangle \, \mathrm{d}P_X(x) \end{aligned}$$

Note the characteristic function of $X$ is the Fourier transform of its probability distribution. We will also use the following properties to derive the prediction update of Kalman filter:

**Theorem.** Let two r.v. $X_1$ and $X_2$ be independant and take their values in $\mathbb{R}^d$. The characteristic function of their sum is given by

$$\forall t \in \mathbb{R}^d, \quad \varphi_{X_1+X_2}(t) = \varphi_{X_1}(t) \times \varphi_{X_2}(t)$$

**Theorem.** Random variable $X$ is Gaussian $X \hookrightarrow \mathcal{N}(\mu, \Lambda)$ if and only if its characteristic function $\varphi_X$ is defined by

$$\forall t \in \mathbb{R}^d, \quad \varphi_X(t) = \exp i\langle \mu, t\rangle \exp[-\frac{1}{2}\langle \Lambda t, t\rangle]$$

We now restate the equation of prediction step for Kalman filter:

$$x_t = Ax_{t-1} + Bu_t + \epsilon_t$$

We assume $x_{t-1}$ is Gaussian $x_{t-1} \hookrightarrow \mathcal{N}(\mu_{t-1}, \Sigma_{t-1})$, and $\epsilon_t$ is Gaussian $\epsilon_t \hookrightarrow \mathcal{N}(0, R_t)$. $x_{t-1}$ and $\epsilon_t$ are assumed to be independant. $u_t$ is deterministic. By linearity we know $y_t = Ax_{t-1} + Bu_t$ is Gaussian $y_t \hookrightarrow \mathcal{N}(A\mu_{t-1} + Bu_t, A\Sigma_{t-1}A^T)$. By the above property, we can compute the

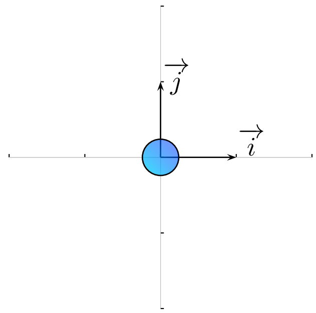



FIGURE 2. 0.95 uncertainty ellipses for the gaussian $\mathcal{N}(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0.01 & 0\\0 & 0.01\end{bmatrix})$

characteristic function of r.v. $x_t$

$$\begin{aligned}\forall u \in \mathbb{R}^d, \quad \varphi_{x_t}(u) &= \varphi_{y_t}(u) \times \varphi_{\epsilon_t}(u)\\ &= \exp i\langle A\mu_{t-1} + Bu_t, u\rangle \exp[-\frac{1}{2}\langle A\Sigma_{t-1}A^T u, u\rangle] \times \exp i\langle 0, u\rangle \exp[-\frac{1}{2}\langle R_t t, t\rangle]\\ &= \exp i\langle A\mu_{t-1} + Bu_t, u\rangle \exp[-\frac{1}{2}\langle (A\Sigma_{t-1}A^T + R_t)u, u\rangle]\end{aligned}$$

using again the property of gaussian in the other direction, this shows r.v. $x_t$ is Gaussian $x_t \hookrightarrow \mathcal{N}(\underbrace{A\mu_{t-1} + Bu_t}_{\bar{\mu}_t}, \underbrace{A\Sigma_{t-1}A^T + R_t}_{\bar{\Sigma}_t})$.

## 4

**4.1.** The system evolves according to

$$\begin{bmatrix}x'\\y'\\\theta\end{bmatrix} = \begin{bmatrix}x + \cos\theta\\y + \sin\theta\\\theta\end{bmatrix}$$

We will assume $x$, $y$ and $\theta$ are gaussian r.v.:$x \hookrightarrow \mathcal{N}(0, \sigma_1^2)$ $y \hookrightarrow \mathcal{N}(0, \sigma_1^2)$ $\theta \hookrightarrow \mathcal{N}(0, \sigma_2^2)$, where $\sigma_1^2 = 0.01$ and $\sigma_2^2 = 10000$. We could compute the expectancy and variance of $\cos\theta$ and $\sin\theta$ (cf. A.2) ,but since the variability of $\theta$ is high with regards to $2\pi$, and

$$\begin{aligned}\forall\theta \in \mathbb{R}, \quad \cos\theta &= \cos(\theta - \left\lfloor\frac{\theta}{2\pi}\right\rfloor \times 2\pi)\\ \sin\theta &= \sin(\theta - \left\lfloor\frac{\theta}{2\pi}\right\rfloor \times 2\pi)\end{aligned}$$

it seems reasonable to assume $\alpha = \theta - \lfloor\frac{\theta}{2\pi}\rfloor \times 2\pi$ is an uniform r.v. $\alpha \hookrightarrow \mathcal{U}([0, 2\pi])$. $\begin{bmatrix}x\\y\end{bmatrix}$ is gaussian $\begin{bmatrix}x\\y\end{bmatrix} \hookrightarrow \mathcal{N}(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\sigma_1^2 & 0\\0 & \sigma_1^2\end{bmatrix})$. To draw the posterior, we will first draw the corresponding 0.95 uncertainty ellipse (cf. figure 2). We will use the following notation for this disk :

$$\mathcal{D} = \{z \in \mathbb{C}, |z| \leqslant r\}$$

Next we consider the function:

$$\begin{aligned}f : \mathcal{D} \times [0, 2\pi] &\to \mathbb{C}\\ (z, \theta) &\mapsto z + e^{i\theta}\end{aligned}$$

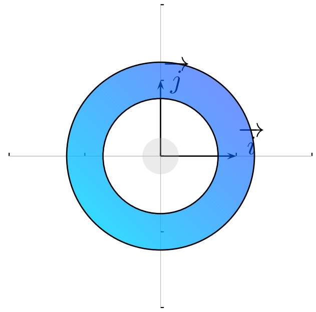


FIGURE 3. 0.95 uncertainty ring for the random variable $\begin{bmatrix} x' \\ y' \end{bmatrix}$

We can show (cf. A.3) that $f(\mathcal{D} \times [0, 2\pi]) = \underbrace{\{z \in \mathbb{C}, 1 - r \leqslant |z| \leqslant 1 + r\}}_{\mathcal{E}}$. It can then be asserted r.v. $\begin{bmatrix} x' \\ y' \end{bmatrix}$ belongs to $\mathcal{E}$ with a probability not lesser than 0.95.We represent $\mathcal{E}$ in figure 3.

**4.2.** In order to linearize the state evolution equation, we have to linearize $\cos\theta$ and $\sin\theta$ about $E\theta = 0$, which we know already makes no sense because of the high variability of $\theta$.

$$\begin{array}{rcl} \cos\theta & \underset{x \to 0}{=} & 1 + o(\theta) \\ \Rightarrow \cos\theta & \approx & 1 \\ \sin\theta & \underset{x \to 0}{=} & \theta + o(\theta) \\ \Rightarrow \sin\theta & \approx & \theta \end{array}$$

The motion model becomes

$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B}$$

If we suppose $\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$ is Gaussian $\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \hookrightarrow \mathcal{N}(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \underbrace{\begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 10000 \end{bmatrix}}_{\Sigma})$, by linearity $\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix}$ is Gaussian $\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} \hookrightarrow \mathcal{N}(B, A\Sigma A^T)$. The marginal $\begin{bmatrix} x' \\ y' \end{bmatrix}$ is Gaussian (it results from the application of linear transformation, namely projection of the previous r.v.) $\begin{bmatrix} x \\ y \end{bmatrix} \hookrightarrow \mathcal{N}(\begin{bmatrix} 01 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.01 & 0 \\ 0 & 10000 \end{bmatrix})$. From this follows the uncertainty ellipse in figure 4, which has been cropped for obvious reasons and unsurprisingly cannot capture the posterior correctly. Linearization makes sense only when we restrain the uncertainty on $\theta$. Let's take $\sigma_2^2 = 1$, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} \hookrightarrow \mathcal{N}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.01 & 0 \\ 0 & 1.01 \end{bmatrix})$$

If $\sigma_2^2 = 0.5$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} \hookrightarrow \mathcal{N}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.01 & 0 \\ 0 & 0.51 \end{bmatrix})$$

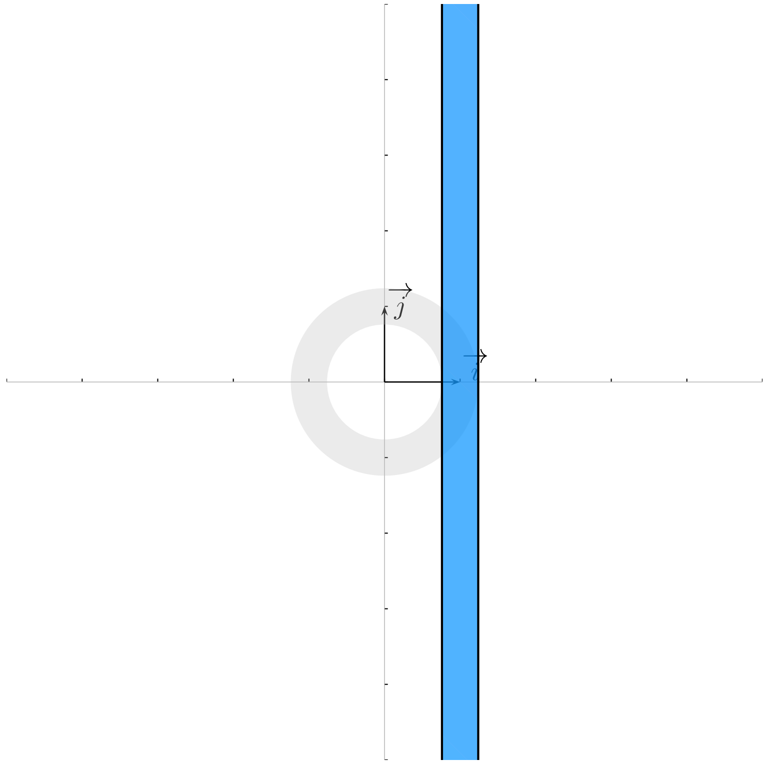


FIGURE 4. 0.95 uncertainty area for the random variable $\begin{bmatrix} x' \\ y' \end{bmatrix}$ after linearization of the system about $\theta = 0$, $\sigma_2^2 = 10000$

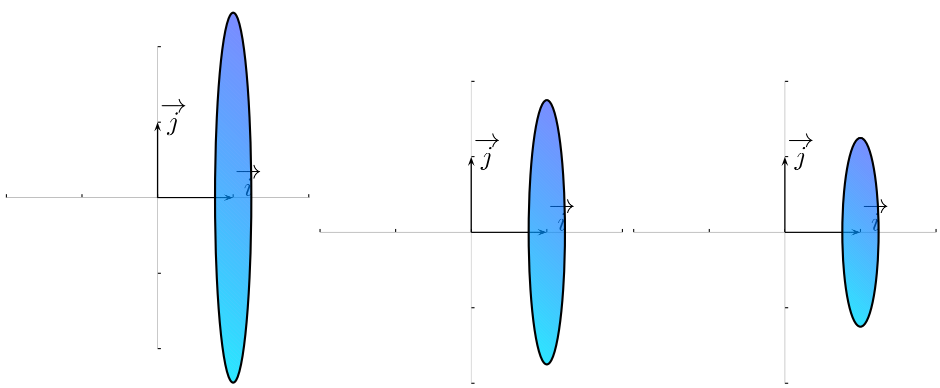


FIGURE 5. 0.95 uncertainty area for the random variable $\begin{bmatrix} x' \\ y' \end{bmatrix}$ after linearization of the system about $\theta = 0$, and using respectively $\sigma_2^2 = 1$,$\sigma_2^2 = 0.5$ and $\sigma_2^2 = 0.25$

If $\sigma_2^2 = 0.25$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} \hookrightarrow \mathcal{N}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.01 & 0 \\ 0 & 0.26 \end{bmatrix})$$

In figure 6 and 7, Matlab simulation of 1000 realisation of random variable $\begin{bmatrix} x + \cos\theta \\ y + \sin\theta \end{bmatrix}$ are superposed on relevant uncertainty area.

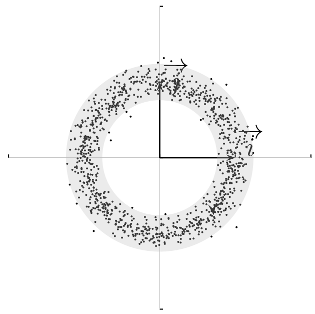


FIGURE 6. Matlab simulation of the random variable $\begin{bmatrix} x' \\ y' \end{bmatrix}$, with $\sigma_2^2 = 10000$

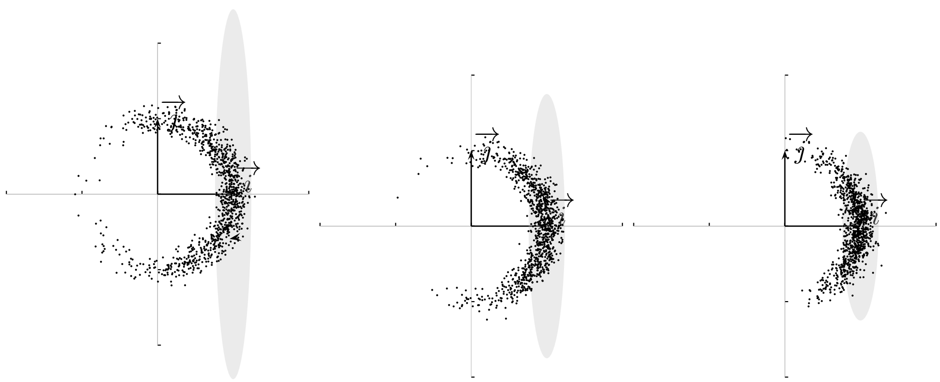


FIGURE 7. Matlab simulation of the random variable $\begin{bmatrix} x' \\ y' \end{bmatrix}$, with $\sigma_2^2 = 1$, $\sigma_2^2 = 0.5$ and $\sigma_2^2 = 0.25$

## 5

Handling an additive constant $D_1$ in motion model is straightforward.

$$x_t = Ax_{t-1} + \underbrace{Bu_t + D_1}_{\text{deterministic constant}} + \epsilon_t$$

Since $Bu_t$ is already deterministic in our model, it suffices to replace in the derivation of the motion step $Bu_t$ by $Bu_t + D_1$ to get

$$\bar{\mu}_t = Ax_{t-1} + Bu_t + D_1 + \epsilon_t$$
$$\bar{\Sigma}_t = A\Sigma_{t-1}A^T + R$$

If the measurement model is now

$$z_t = Cx_t + D_2 + \xi_t$$

Suppose we observe instead $z'_t = z_t - D_2$, that is we systematically substract $D_2$ to actual measurement. We can use the original derivation to get the moments of the posterior:

$$\mu_t = \bar{\mu}_t + K_t(z'_t - C\bar{\mu}_t)$$
$$\Sigma_t = (I - K_tC)\bar{\mu}_t$$

Now replace $z'_t$ by actual measurement.

$$\mu_t = \bar{\mu}_t + K_t(z_t - D_2 - C\bar{\mu}_t)$$
$$\Sigma_t = (I - K_tC)\bar{\mu}_t$$

## 6

Not sure to understand what is expected exactly in this exercise.

## Appendix A

### A.1.

**Uncertainty ellipse.** Let $X : \Omega \to \mathbb{R}^d$ a Gaussian random variable $X \hookrightarrow \mathcal{N}(\mu, \Sigma)$. We suppose $\Sigma$ is positive definite. Let $Y$ a random variable having a chi squared distribution with k degrees of freedom $Y \hookrightarrow \chi^2(d)$, and $c \in \mathbb{R}^+$ such that $P(Y \leqslant c) = 0.95$. We have

$$P(X \in \underbrace{\{x \in \mathbb{R}^d, (x-\mu)^T \Sigma^{-1}(x-\mu) \leqslant c)\}}_{\mathcal{E}}) = 0.95$$

*Proof.* Let

$$\begin{aligned} T : \mathbb{R}^d &\to \mathbb{R} \\ x &\mapsto (x-\mu)^T \Sigma^{-1}(x-\mu) \end{aligned}$$

and the random variable $Y' = T(X)$. Since $\Sigma$ is symetric positive definite real matrix, we write

$$\Sigma = U \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_d \end{bmatrix} U^T$$

where $U = \begin{bmatrix} U_1 \Big| U_2 \Big| \dots \Big| U_d \end{bmatrix}$ is orthogonal and $(\lambda_1, \lambda_2, \dots \lambda_d) \in (\mathbb{R}^{+*})^d$. We consider the random variable

$$Z = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{\lambda 2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\sqrt{\lambda_d}} \end{bmatrix}}_{D_1} U^T(X-\mu)$$

$Z$ is Gaussian as linear transform of a Gaussian, it is centered and the covariance matrix is

$$\begin{aligned} \Sigma_1 &= D_1 U^T \Sigma U D_1 \\ &= D_1 U^T (U D U^T) U D_1 \\ &= I_d \end{aligned}$$

This shows marginals $Z_i$ are uncorrelated, which is equivalent to independance since $Z$ is multivariate Gaussian. Also,

$$\begin{aligned} \forall x \in \mathbb{R}^d, \quad T(x) &= (x-\mu)^T U D_1^2 U^T (x-\mu) \\ &= (x-\mu)^T U D_1^2 U^T (x-\mu) \\ &= z^T z \\ &= \sum_{i=1}^d z_i^2 \end{aligned}$$

from where,

$$\begin{aligned} Y' &= T(X) \\ &= \sum_{i=1}^d Z_i^2 \end{aligned}$$

shows that $Y'$ is the sum of d independant squared standard centered normal r.v., it is known $Y' \hookrightarrow \chi^2(d)$. But then

$$\begin{aligned}
P(X \in \mathcal{E}) &= \int_\Omega 1_{X \in \mathcal{E}} \,\mathrm{d}P \\
&= \int_\Omega 1_{X \in T^{-1}([0,c])} \,\mathrm{d}P \\
&= \int_\Omega 1_{Y' \in [0,c]} \,\mathrm{d}P \\
&= P(Y' \leqslant c) \\
&= 0.95
\end{aligned}$$

□

For $d = 2$, equation of $\mathcal{E}$ in orthonormal frame $(\mu, \overrightarrow{u_1}, \overrightarrow{u_2})$ becomes

$$\begin{aligned}
\mathcal{E}: \quad & (x-\mu)^T U D_1^2 U^T (x-\mu) \leqslant c \\
\Leftrightarrow\ & \frac{(U_1^T(x-\mu))^2}{\lambda_1} + \frac{(U_2^T(x-\mu))^2}{\lambda_2} \leqslant c \\
\Leftrightarrow\ & \frac{x_1'^2}{\lambda_1} + \frac{x_2'^2}{\lambda_2} \leqslant c \\
\Leftrightarrow\ & \frac{x_1'^2}{(\sqrt{\lambda_1 c})^2} + \frac{x_2'^2}{(\sqrt{\lambda_2 c})^2} \leqslant 1
\end{aligned}$$

The frontier of $\mathcal{E}$ is an ellipse centered on $\mu$ whose directions of axes are $(\overrightarrow{u_1}, \overrightarrow{u_2})$ and semi-minor and semi-major axes are $(\sqrt{\lambda_1 c}, \sqrt{\lambda_2 c})$. For $d = 2$, $c \approx 5.99$.

### A.2. moments of $\cos\theta$ and $\sin\theta$.

**Lemma.** Let $\theta$ be a centered gaussian $\theta \hookrightarrow \mathcal{N}(0, \sigma^2)$. then,

$$\begin{aligned}
E\cos\theta &= e^{-\frac{\sigma^2}{2}} \\
E\sin\theta &= 0 \\
\operatorname{Var}\cos\theta &= \frac{1}{2}(1 - e^{-\sigma^2})^2 \\
\operatorname{Var}\sin\theta &= \frac{1}{2}(1 - e^{-2\sigma^2})
\end{aligned}$$

*Proof.* Let us consider the expectancy of r.v. $e^{i\theta}$.

$$\begin{aligned}
Ee^{i\theta} &= \int_{\mathbb{R}} e^{it} \times \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} \,\mathrm{d}t \\
&= \int_{\mathbb{R}} e^{i\langle t,1\rangle} \times \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} \,\mathrm{d}t \\
&= \varphi_\theta(1) \\
&= e^{i\langle 0,1\rangle} e^{-\frac{1}{2}\langle \sigma^2 \times 1, 1\rangle} \\
Ee^{i\theta} &= e^{-\frac{\sigma^2}{2}}
\end{aligned}$$

From which we get

$$\begin{aligned}
E\cos\theta &= \operatorname{Re} Ee^{i\theta} \\
&= e^{-\frac{\sigma^2}{2}} \\
E\sin\theta &= \operatorname{Im} Ee^{i\theta} \\
&= 0
\end{aligned}$$

Then,

$$\begin{aligned}
E\cos^2\theta &= \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} \cos^2 t e^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} (\frac{1}{2}+\frac{\cos 2t}{2}) e^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}\sigma}[\frac{1}{2}\int_{\mathbb{R}} \cos 2t e^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t + \frac{1}{2}\int_{\mathbb{R}} e^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t] \\
&= \frac{1}{2\sqrt{2\pi}\sigma}\frac{1}{2}\int_{\mathbb{R}} \cos u e^{-\frac{u^2}{8\sigma^2}}\,\mathrm{d}u + \frac{1}{2} \\
&= \frac{1}{2}\int_{\mathbb{R}} \cos u \frac{1}{\sqrt{2\pi}(2\sigma)} e^{-\frac{u^2}{2\times(2\sigma)^2}}\,\mathrm{d}u + \frac{1}{2} \\
&= \frac{1}{2}e^{-\frac{(2\sigma)^2}{2}} + \frac{1}{2} \\
E\cos^2\theta &= \frac{1}{2}(1+e^{-2\sigma^2}) \\
E\sin^2\theta &= 1 - E\cos^2\theta \\
E\sin^2\theta &= \frac{1}{2}(1-e^{-2\sigma^2})
\end{aligned}$$

The second order moments are therefore

$$\begin{aligned}
\operatorname{Var}\cos\theta &= E\cos^2\theta - (E\cos\theta)^2 \\
&= \frac{1}{2}(1+e^{-2\sigma^2}) - e^{-\sigma^2} \\
\operatorname{Var}\cos\theta &= \frac{1}{2}(1-e^{-\sigma^2})^2 \\
\operatorname{Var}\sin\theta &= E\sin^2\theta \\
&= \frac{1}{2}(1-e^{-2\sigma^2})
\end{aligned}$$

□

**A.3. image of a disk by $f : (z,\theta) \mapsto z + e^{i\theta}$.**

**Lemma.** Let

$$\mathcal{D} = \{z \in \mathbb{C}, |z| \leqslant r\}$$

and

$$\begin{aligned}
f : \mathcal{D}\times[0,2\pi] &\to \mathbb{C} \\
(z,\theta) &\mapsto z + e^{i\theta}
\end{aligned}$$

$$f(\mathcal{D}\times[0,2\pi]) = \underbrace{\{z\in\mathbb{C}, 1-r \leqslant |z| \leqslant 1+r\}}_{\mathcal{E}}$$

*Proof.* The triangular inequalities gives

$$\begin{aligned}
\forall (z,\theta)\in\mathcal{D}\times[0,2\pi],\quad 1-|z| &\leqslant |z+e^{i\theta}| \leqslant 1+|z| \\
\Rightarrow\ 1-r &\leqslant |z+e^{i\theta}| \leqslant 1+r
\end{aligned}$$

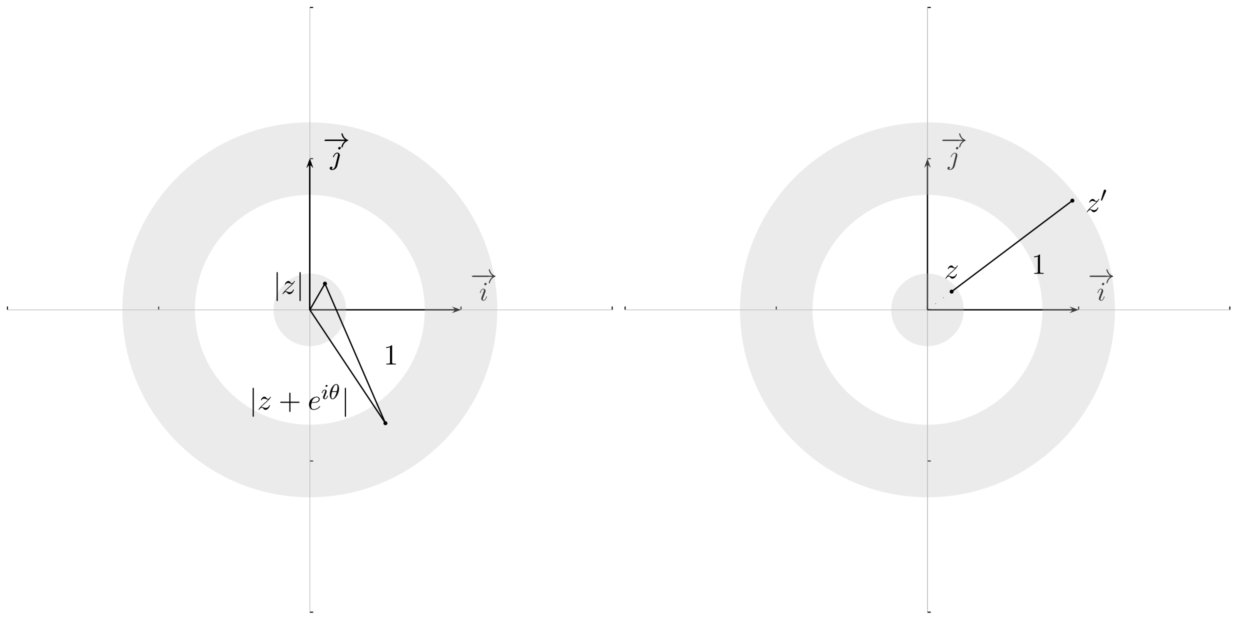



FIGURE 8. Illustration of the proof

showing that $f(\mathcal{D}\times[0,2\pi])\subset\mathcal{E}$. Reciprocally, let $z'\in\mathcal{E}^*$and

$$z = z' - \frac{1}{|z'|}z'$$
$$z = \frac{|z'|-1}{|z'|}z'$$
$$|z| \leqslant \frac{|1-|z'||}{|z'|}|z'| \leqslant r$$

so $z\in\mathcal{D}$ and we can write

$$z' = z + \underbrace{\frac{1}{|z'|}z'}_{e^{i\alpha}} \in f(\mathcal{D}\times[0,2\pi])$$
$$\Rightarrow \mathcal{E}\subset f(\mathcal{D}\times[0,2\pi])$$

$\square$