Supplementary Material of "Convex Optimization-Based Model Predictive Control for Mars Ascent Vehicle Guidance System"

I. List of Materials

This PDF file includes the following materials.
i) Proof of algorithmic stability.
ii) Appendix.

II. Proof of Algorithmic Stability

Proof: The actual dynamic system described in this letter can be expressed as:

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{f}(\hat{\boldsymbol{x}}(t), \boldsymbol{u}(t)) + \boldsymbol{a}_p(t) \tag{II. 1}$$

where $\hat{\boldsymbol{x}}$ denotes the actual states, $\boldsymbol{a}_p(t)$ is the equivalent disturbance acceleration.

The nominal dynamic system is:

$$\dot{\bar{\boldsymbol{x}}}(t) = \boldsymbol{f}(\bar{\boldsymbol{x}}(t), \boldsymbol{u}(t)) \tag{II. 2}$$

where $\bar{\boldsymbol{x}}$ denotes the nominal states.

We made the following assumptions:

Assumption 1: The solution obtained from each optimization is always feasible.

Assumption 2: The disturbance to the Mars ascent vehicle (MAV) is bounded. There exists $\epsilon_{\boldsymbol{a}_p} > 0$ such that $\|\boldsymbol{a}_p\|_{L\infty} \leq \epsilon_{\boldsymbol{a}_p}$.

Assumption 3: The dynamics equation $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is Lipschitz continuous, i.e., there exist Lipschitz constants $\mathrm{Lip}f_{\boldsymbol{x}}$ and $\mathrm{Lip}f_{\boldsymbol{u}}$ such that

$$\|\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{u}_1) - \boldsymbol{f}(\boldsymbol{x}_2, \boldsymbol{u}_2)\| \leq \mathrm{Lip}f_{\boldsymbol{x}} \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| + \mathrm{Lip}f_{\boldsymbol{u}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|. \tag{II. 3}$$

The analysis is based on *Assumption* 1, so we need to discuss the feasibility of the optimization algorithm. Although the method proposed in this letter eliminates the dynamic constraints and expands the size of the feasible domain, there is still the possibility of infeasibility. The convergence performance of the algorithm is improved through relaxation variables and trust-region constraints, and infeasibility is avoided to a great extent. Strict MCPI convergence cannot be guaranteed [13], but a large number of numerical tests have shown that its convergence boundary is quite large in the vast majority of nonlinear problems, even with randomly generated initial guesses [14]. Therefore, within the scope of this study, *Assumption* 1 is reasonable.

The computation time domain of the optimization algorithm during the i^{th} model predictive control implementation is $t \in [t_i, \Gamma_i]$, where t_i is the update sampling time, Γ_i is the terminal flight time.

The nominal state at t_{i+1} during the i^{th} update cycle under the guidance command $[t_i, t_{i+1}] \to \boldsymbol{u}^*(t, \hat{\boldsymbol{x}}(t_i))$ is given by:

$$\bar{\boldsymbol{x}}^k(t_{i+1}) = \hat{\boldsymbol{x}}(t_i) + \int_{t_i}^{t_{i+1}} \boldsymbol{f}(\bar{\boldsymbol{x}}^{k-1}(t), \boldsymbol{u}^*(t, \hat{\boldsymbol{x}}(t_i)))\mathrm{d}t \tag{II. 4}$$

where $\bar{\boldsymbol{x}}^k$ is the nominal trajectory obtained by k^{th} optimization.

The actual state at t_{i+1} is as follows:

$$\begin{aligned}\hat{\boldsymbol{x}}(t_{i+1}) = \hat{\boldsymbol{x}}(t_i) &+ \int_{t_i}^{t_{i+1}} \boldsymbol{f}(\hat{\boldsymbol{x}}(t), \boldsymbol{u}^*(t, \hat{\boldsymbol{x}}(t_{i-1})))\mathrm{d}t \\ &+ \int_{t_i}^{t_{i+1}} \boldsymbol{a}_p(t)\mathrm{d}t.\end{aligned} \tag{II. 5}$$

According to (II. 4), (II. 5) and *Assumption* 3, the state deviation at t_{i+1} can be expressed as:

$$\begin{aligned}&\left\|\hat{\boldsymbol{x}}(t_{i+1}) - \bar{\boldsymbol{x}}^k(t_{i+1})\right\| \leq \mathrm{Lip}f_{\boldsymbol{x}} \cdot \int_{t_i}^{t_{i+1}} \left\|\hat{\boldsymbol{x}}(t) - \bar{\boldsymbol{x}}^{k-1}(t)\right\| \mathrm{d}t \\ &+ \mathrm{Lip}f_{\boldsymbol{u}} \cdot \int_{t_i}^{t_{i+1}} \|\boldsymbol{u}^*(t, \hat{\boldsymbol{x}}(t_{i-1})) - \boldsymbol{u}^*(t, \hat{\boldsymbol{x}}(t_i))\| \,\mathrm{d}t \\ &+ \|\boldsymbol{a}_p\|_{L\infty} \cdot \Delta t_i.\end{aligned} \tag{II. 6}$$

According to the Bellman Optimality Principle, if there are no disturbances within time domain $[t_{i-1}, t_i]$, then we have:

$$\boldsymbol{u}^*(t, \hat{\boldsymbol{x}}(t_{i-1})) = \boldsymbol{u}^*(t, \hat{\boldsymbol{x}}(t_i)),\ t \in [t_i, \Gamma_i]. \tag{II. 7}$$

If considering disturbances, according to *Assumption* 2, we can get that the optimal control variables in two consecutive updates satisfy $\|\boldsymbol{u}^*(t, \hat{\boldsymbol{x}}(t_{i-1})) - \boldsymbol{u}^*(t, \hat{\boldsymbol{x}}(t_i))\|_{L\infty} \leq \epsilon_{u,i}$, where $\epsilon_{u,i}$ represents a measure of the effect of disturbances on the system within the time domain $[t_{i-1}, t_i]$. According to (II. 6), it yields that:

$$\begin{aligned}\left\|\hat{\boldsymbol{x}}(t_{i+1}) - \bar{\boldsymbol{x}}^k(t_{i+1})\right\| &\leq \mathrm{Lip}f_{\boldsymbol{x}} \cdot \int_{t_i}^{t_{i+1}} \left\|\hat{\boldsymbol{x}}(t) - \bar{\boldsymbol{x}}^{k-1}(t)\right\| \mathrm{d}t \\ &+ \mathrm{Lip}f_{\boldsymbol{u}} \cdot \epsilon_{u,i} \cdot \Delta t_i + \epsilon_{\boldsymbol{a}_p} \cdot \Delta t_i.\end{aligned} \tag{II. 8}$$

Since the criterion for determining the convergence of the optimization is that the deviation between two consecutive iterations is smaller than a small value, thus $\bar{\boldsymbol{x}}^k(t_{i+1}) \approx \bar{\boldsymbol{x}}^{k-1}(t_{i+1})$. From Gronwall's Lemma (see the Appendix), (II. 8) reduces to:

$$\left\|\hat{\boldsymbol{x}}(t_{i+1}) - \bar{\boldsymbol{x}}^k(t_{i+1})\right\| \leq (\mathrm{Lip}f_{\boldsymbol{u}} \cdot \epsilon_{u,i} \cdot \Delta t_i + \epsilon_{\boldsymbol{a}_p} \cdot \Delta t_i) \cdot e^{\mathrm{Lip}f_{\boldsymbol{x}} \cdot \Delta t_i}. \tag{II. 9}$$

Let $K = \mathrm{Lip}f_{\boldsymbol{u}} \cdot \epsilon_{u,i} + \epsilon_{\boldsymbol{a}_p}$, we have:

$$\left\|\hat{\boldsymbol{x}}(t_{i+1}) - \bar{\boldsymbol{x}}^k(t_{i+1})\right\| \leq K \cdot \Delta t_i \cdot e^{\mathrm{Lip}f_{\boldsymbol{x}} \cdot \Delta t_i}. \tag{II. 10}$$

We can conclude that the deviation between the nominal state and the actual state of the system is bounded within any update cycle, and the execution strategy of MPC ensures that guidance errors do not propagate between trajectory optimization update cycles. Since the solutions obtained from the optimization are always feasible, it can be inferred that regardless of whether the prediction horizon reaches the terminal time, the actual state trajectory will always converge to a bounded neighborhood of the expected terminal state, thereby ensuring the stability of the algorithm's implicit closed-loop control. The proof is completed. ■

Appendix: Mathematical Foundations

Gronwall's Lemma:

Let $[t_0, t_f] \mapsto y(t) \in \mathbb{R}_+$ be an integrable function that satisfies Gronwall's inequality:

$$y(t) \leq a(t) + \int_{t_0}^{t} b(s)y(s)\mathrm{d}s$$

where a and b are continuous, nonnegative, bounded functions, with $t \mapsto a(t)$ nondecreasing over the interval $[t_0, t_f]$; then

$$y(t) \leq a(t)e^{B(t)}$$

where

$$B(t) := \int_{t_0}^{t} b(s)\mathrm{d}s$$