Exercise Sheet 5

1 Convolutional Kernel

a) $$\sum_{i,j}^{N} c_i c_j k(x_i, x_j) \geq 0$$

$$= \sum_{i,j} c_i c_j \sum_{t=-\infty}^{\infty} (x_i * x_j)_t (x_j * x_i)_t$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} c_i c_j \sum_{-\infty}^{N} \left(\sum_{t=-\infty}^{\infty} x_i(\tau) \cdot x_j(t-\tau)\right)\left(\sum_{-\infty}^{\infty} x_i(\tau') x_i(t-\tau')\right)$$

$$= \sum_{s=-\infty}^{\infty} \left(\sum^{N} c_i \sum_{\tau=-\infty}^{N} x_i(\tau) x_i(s+\tau)\right)\left(\sum_{j=1}^{N} c_j \sum_{\tau'=-\infty}^{N} x_j(\tau') x_j(s+\tau')\right)$$

$$= \sum_{s=-\infty}^{\infty}\left(\sum_{i=1}^{N} c_i \sum_{\tau=-\infty}^{\infty} x_i(\tau) x_i(s+\tau)\right)^2 \geq 0$$

b) $$k(x,x') = \sum_s \sum_\tau x(\tau) x(s+\tau) \sum_{\tau'} x'(\tau') x'(s+\tau')$$

$$= \langle \underbrace{(x * x)}_{\phi(x)}, \underset{\phi(x')}{(x' * x')} \rangle$$

2 Weighted Degree Kernels

a) $$\sum_{i,j} c_i c_j \sum_m \beta_m \sum_l \mathbb{1}(u_{l,m}(x) = u_{l,m}(z)) \geq 0$$

$$\Leftrightarrow \sum_{i,j} c_i c_j \sum_{m,l} \beta_m \sum_{s\in A^m} \mathbb{1}(u_{l,m}(x)=s)\,\mathbb{1}(u_{l,m}(z)=s)$$

$$\Leftrightarrow \sum_{l,m,s} \beta_m \sum_i c_i \mathbb{1}(u_{l,m}(x)=s) \sum_j c_j \mathbb{1}(u_{l,m}(z)=s)$$

$$\Leftrightarrow \sum_{l,m,s} \underbrace{\beta_m}_{\geq 0} \underbrace{\left(\sum_i c_i \mathbb{1}(u_{l,m}(x=s))\right)^2}_{\geq 0} \geq 0$$

b)
$$\sum_{m,l} \beta_m \sum_{s \in A^m} \mathbb{1}(u_{l,m}(x) = s)\, \mathbb{1}(u_{l,m}(z) = s)$$

$$\Leftrightarrow \sum_{l,s} \sqrt{\beta}\, \mathbb{1}(u_l(x) = s)\, \sqrt{\beta}\, \mathbb{1}(u_l(z) = s)$$

$$\Leftrightarrow \langle \underbrace{\sqrt{\beta}\, \mathbb{1}(u_l(x) = s)_{l,s}}_{\phi(x)},\ \underbrace{\sqrt{\beta}\, \mathbb{1}(u_l(z) = s))_{l,s}}_{\phi(z)} \rangle$$

c)
$$k(x,x') = \sum_l \underbrace{\beta_2}_{=1} \sum_{s \in A^2} \mathbb{1}(u_{l2}(x) = s)\, \mathbb{1}(u_{l2}(x') = s)$$

$$= \langle (\mathbb{1}(u_{l2}(x) = s))_{l,s}\, (\mathbb{1}(u_{l2}(x') = s))_{l,s} \rangle$$

3 Fisher Kernel

a)
$$k(x,x') = G_x^T\, E_z[G_z G_z^T]^{-1}\, G_{x'}$$
$$= (\Sigma^{-1}(x-\mu))^T E_z[(\Sigma^{-1}(x-z))(\Sigma^{-1}(x-z))^T]^{-1}$$
$$\Sigma^{-1}(x'-\mu)$$
$$= (x-\mu)^T \Sigma^{-1} (\Sigma^{-1} \underbrace{\Sigma \Sigma^{-1}}_{\mathbb{1}})^{-1} \Sigma^{-1} (x'-\mu)$$
$$= (x-\mu)^T \Sigma^{-1} \underbrace{(\Sigma^{-1})^{-1} \Sigma^{-1}}_{= \mathbb{1}} (x'-\mu)$$
$$= (x-\mu)^T \Sigma^{-1} (x'-\mu)$$

b)
$$k(x,x') = (x-\mu)^T \Sigma^{-1} (x'-\mu)$$
$$= (x-\mu)^T L L^T (x'-\mu)$$
$$= (L^T(x-\mu))^T (L^T(x'-\mu)) = \langle \underbrace{L^T(x-\mu)}_{\phi(x)}, \underbrace{L^T(x'-\mu)}_{\phi(x')} \rangle$$