Exercise Sheet 12

1 Deep SVDD

a) Minimum at $\vec{w} = \vec{0}$, $b = 1$

b)
$$J = \min_W \frac{1}{N} \sum_{i=1}^{N} \| w^T x_i - 1 \|^2 = \min_W \frac{1}{N} \sum_{i=1}^{N} (w^T x_i - 1)^T (w^T x_i - 1)$$
$$= \min_W \frac{1}{N} \sum_{i=1}^{N} w^T x_i x_i^T w - 2 w^T x_i + 1$$
$$= \min_W w^T \frac{\sum_{i=1}^{N} x_i x_i^T}{N} w - \frac{2}{N} \sum_{i=1}^{N} x_i^T w + \frac{1}{N} \sum_{i=1}^{N} 1$$

$$\frac{\partial J}{\partial w} = (\Sigma + \Sigma^T) w - 2 \frac{1}{N} \sum_{i=1}^{N} x_i = 0$$

$$\Leftrightarrow 2 \Sigma w - 2 \bar{x} = 0 \quad \Leftrightarrow \quad w = \Sigma^{-1} \bar{x}$$

2 Restricted Boltzmann Machine

a)
$$p(h_k = 1 | x) = \frac{p(x, h_k = 1)}{p(x)} = \frac{\sum_{h_{-k}} p(x, h_k = 1, h_{-k})}{p(x)}$$
$$= \frac{\sum_{h_{-k}} p(x, h_k = 1, h_{-k})}{\sum_{q \in \{0,1\}} \sum_{h_{-k}} p(x, h_k = q, h_{-k})}$$
$$= \frac{\sum_{h_{-k}} \frac{1}{Z} \exp(x^T W_{:,k} + b_k - E(x, h_{-k}))}{\sum_{q \in \{0,1\}} \sum_{h_{-k}} \frac{1}{Z} \exp(x^T W_{:,k} q + b_k q - E(x, h_{-k}))}$$
$$= \frac{\exp(x^T W_{:,k} + b_k) \sum_{h_{-k}} \exp(-E(x, h_{-k}))}{\sum_{q \in \{0,1\}} \exp(x^T W_{:,k} q + b_k q) \sum_{h_{-k}} \exp(-E(x, h_{-k}))}$$
$$= \frac{\exp(x^T W_{:,k} + b_k)}{1 + \exp(x^T W_{:,k} + b_k)} = \mathrm{sigm}(x^T W_{:,k} + b_k)$$

b) $$p(x_j = 1 \mid h) = \frac{p(x_j = 1, h)}{p(h)} = \frac{\sum_{x_{-j}} p(x_j = 1, x_{-j}, h)}{\sum_{q \in \{0,1\}} \sum_{x_{-j}} p(x_j = q, x_{-j}, h)}$$

$$= \frac{\sum_{x_{-j}} \frac{1}{Z} \exp(W_{j,:}^T h - E(x_{-j}, h))}{\sum_{q \in \{0,1\}} \sum_{x_{-j}} \frac{1}{Z} \exp(W_{j,:}^T h q - E(x_{-j}, h))}$$

$$= \frac{\exp(W_{j,:}^T h)}{1 + \exp(W_{j,:}^T h)} = \operatorname{sigm}(W_{j,:}^T h)$$

c) $$p(x) = \sum_h p(x, h) = \sum_h \frac{1}{Z} \exp(-E(x, h))$$

$$= \sum_h \frac{1}{Z} \exp(x^T W h - b^T h) = \sum_h \frac{1}{Z} \exp(\sum_k x^T W_{:,k} h_k - b_k^T h_k)$$

$$= \frac{1}{Z} \sum_h \prod_k \exp(x^T W_{:,k} h_k - b_k^T h_k)$$

$$= \frac{1}{Z} \prod_k (1 + \exp(x^T W_{:,k} - b_k))$$

$$= \frac{1}{Z} \exp(\log(\prod_k (1 + \exp(x^T W_{:,k} - b_k))))$$