Private-Key Encryption and Pseudorandomness (Part I)

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Outline

Content

Idea of Computational Security

Computational security vs. Information-theoretical security

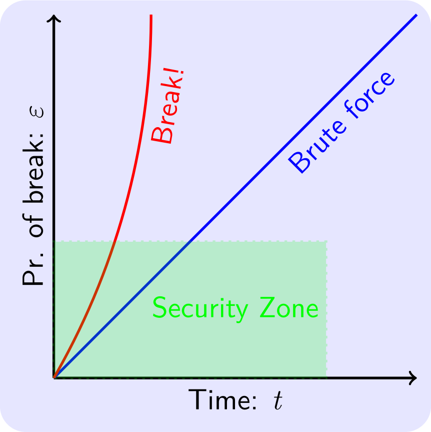
Kerckhoffs's Another Principle

A [cipher] must be practically, if not mathematically, indecipherable.

- Information-theoretical security: Perfect secrecy.
- Computational security:
  - Only preserved against adversaries that run in a **feasible amount of time**.
  - Adversaries can succeed with some **very small probability**.

Necessity of the Relaxations

Limit the power of adversary (against brute force with pr. 1 in time linear in $|\mathcal{K}|$) and allow a negligible probability (against random guess with pr. $1/|\mathcal{K}|$).



Concrete Approach

A scheme is (t, ε)-**secure** if every adversary running for time at most t succeeds in breaking the scheme with probability at most ε.

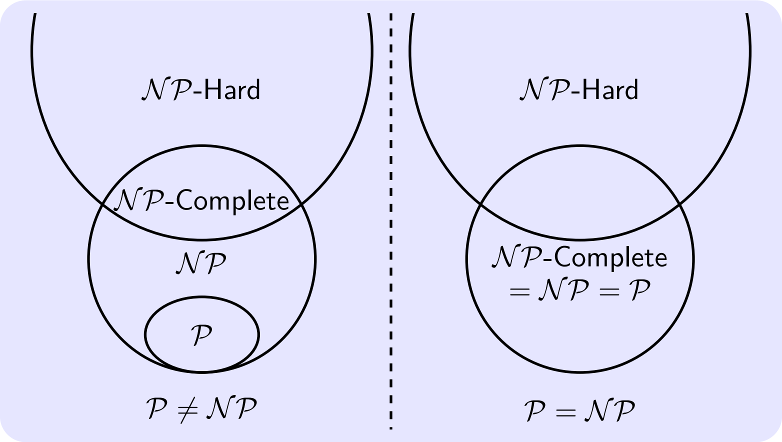
Example

Optimal security: when the key has length n, an adversary running in time t can succeed with probability at most $t/2^n$.

$t = 2^{80}$	2^{20} 1GHz CPUs run 35 years.
$n = 128$	2^{170} atoms in the planet.
$\varepsilon = 2^{-48}$	once every 100 years with probability 2^{-30}/sec.

- But one may ask: What type of computing power? What if under Moore's Law? Which kind of implementation? What is the success probability of running for 10 years? Does the key length matter?

$\mathcal{P} = \mathcal{NP}$?



The majority of computer scientists believe $\mathcal{P} \neq \mathcal{NP}$.

This is very dangerous!

Asymptotic Approach

Time t and probability ε are described as functions of **security parameter** n (usually, the length of key).

- Time is PPT (**probabilistic polynomial-time**): n^a for constant a.
- Probability is **negligible**: smaller than any inverse polynomial n^{-b} for constant b.
- Caution: ‘Security’ for large enough values of n.

Example

“Breaking the scheme” with probability $2^{40} \cdot 2^{-n}$ in n^3 minutes.

$n \leq 40$	6 weeks with probability 1.
$n = 50$	3 months with probability $1/1000$.
$n = 500$	more than 200 years with probability 2^{-500}.

Efficient Computation

- An algorithm A runs in **polynomial time** if there exists a polynomial $p(\cdot)$ such that, for every input $x \in 0, 1^*$, $A(x)$ terminates within at most $p(|x|)$ steps.
- A **probabilistic** algorithm has the capability of "tossing coins".
- A can run another $\textsc{ppt}$ A' as a sub-routine in polynomial-time.
- Random number generators should be designed for cryptographic use, not `random()` in C.
- Open question: Does probabilistic adversaries are more powerful than deterministic ones?

Negligible Success Probability

- A function f is **negligible** if for every polynomial $p(\cdot)$ there exists an N such that for all integers $n > N$ it holds that $f(n) < \frac{1}{p(n)}$.
- $\mathsf{negl}_3(n) = \mathsf{negl}_1(n) + \mathsf{negl}_2(n)$ is negligible.
- positive polynomial p, $\mathsf{negl}_4(n) = p(n) \cdot \mathsf{negl}_1(n)$ is negligible.
- Problem X is *hard* if X cannot be solved by any polynomial-time algorithm except with negligible probability.

Remarks on The Asymptotic Approach

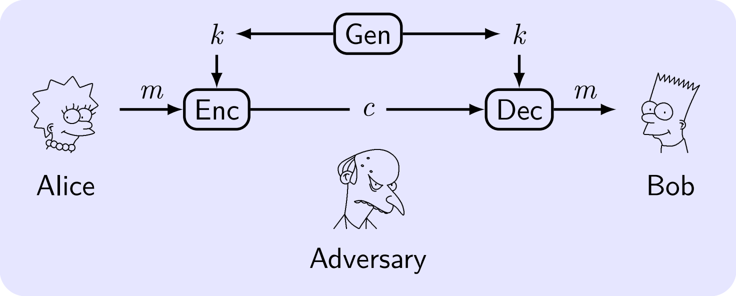
- The longer the key, the higher the security.
- Increasing n to defend against increases in computing power.
- Convention: algorithm input is 1^n. (a string of n 1's)

Example

	Honest party	Adversary	
	$10^6 \cdot n^2$	$10^8 \cdot n^4$	2^{20-n}
1GHz $n = 50$	2.5 sec.	1 week	2^{-30}
16GHz $n = 500$	0.625 sec.	16 week	2^{-80}

Content

Defining Private-key Encryption Scheme



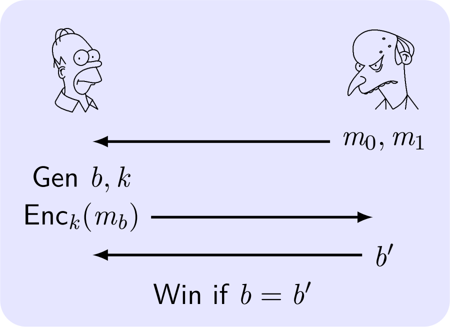
A **Private-key encryption scheme** Π is a tuple of $\textsc{ppt}$ $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$

- $k \leftarrow \mathsf{Gen}(1^n), |k| \geq n$ (security parameter).
  $\mathsf{Gen}(1^n)$ chooses $k \leftarrow \{0,1\}^n$ uniformly at random (***u.a.r***).
- $c \leftarrow \mathsf{Enc}_k(m), m \in \{0,1\}^*$ (all finite-length binary strings).
  Fixed-length if $m \in \{0,1\}^{\ell(n)}$.
- $m := \mathsf{Dec}_k(c)$.
- $\mathsf{Dec}_k(\mathsf{Enc}_k(m)) = m$.

Eavesdropping Indistinguishability Experiment

The eavesdropping indistinguishability experiment $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n)$:

1. $\mathcal{A}$ is given input 1^n, outputs m_0, m_1 of the same length.
2. $k \leftarrow \mathsf{Gen}(1^n)$, a random bit $b \leftarrow \{0,1\}$ is chosen. Then $c \leftarrow \mathsf{Enc}_k(m_b)$ (challenge ciphertext) is given to $\mathcal{A}$.
3. $\mathcal{A}$ outputs b'. If $b' = b$, $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1$, otherwise 0.



m_0, m_1

Gen b, k

$\mathsf{Enc}_k(m_b)$

b'

Win if $b = b'$

Defining Private-key Encryption Security

Definition 1

Π has **indistinguishable encryptions in the presence of an eavesdropper** if $\forall$ PPT $\mathcal{A}$, $\exists$ a negligible function negl such that

$$\Pr\left[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1\right] \leq \frac{1}{2} + \mathsf{negl}(n),$$

where the probability it taken over the random coins used by $\mathcal{A}$.

Properties of Definition of Indistinguishable

1. **No single bit** can be guessed in a randomly-chosen plaintext with probability significantly better than $1/2$.

   $$\Pr[\mathcal{A}(1^n, \mathsf{Enc}_k(m)) = m^i] \leq \frac{1}{2} + \mathsf{negl}(n).$$

   where m^i is the ith bit of m, $\mathcal{A}(\cdot)$ outputs the guess.

2. **No function of plaintext** can be learned regardless of the *a priori* distribution over the plaintext.
   $\forall$ $\mathcal{A}$, $\exists$ $\mathcal{A}'$, $\forall$ f and $\forall$ $S \in \{0,1\}^*$,

   $$\left|\Pr[\mathcal{A}(1^n, \mathsf{Enc}_k(m)) = f(m)] - \Pr[\mathcal{A}'(1^n) = f(m)]\right| \leq \mathsf{negl}(n),$$

   where $m \in S_n \overset{\text{def}}{=} S \cap \{0,1\}^n$.

Semantic Security

Intuition: No partial information leaks.

Definition 2

Π is **semantically secure in the presence of an eavesdropper** if $\forall$ PPT $\mathcal{A}$, $\exists \mathcal{A}'$ such that $\forall$ distribution $X = (X_1, \dots)$ and $\forall f, h$,

$$|\Pr[\mathcal{A}(1^n, \mathsf{Enc}_k(m), h(m)) = f(m)] - \Pr[\mathcal{A}(1^n, h(m)) = f(m)]|$$

$$\leq \mathsf{negl}(n).$$

where m is chosen according to X_n, $h(m)$ is external information.

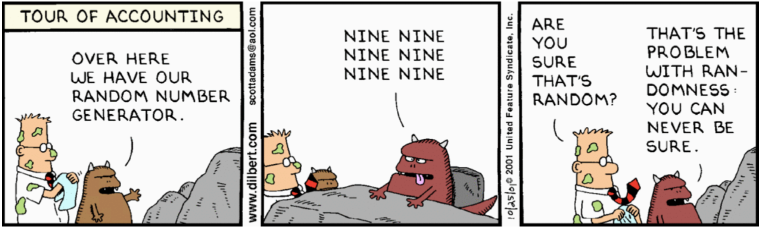
Theorem 3

*A private-key encryption scheme has **indistinguishable** encryptions in the presence of an eavesdropper $\iff$ it is **semantically secure** in the presence of an eavesdropper.*

Content

Conceptual Points of Pseudorandomness

- True randomness can not be generated by a describable mechanism.
- Pseudorandom looks truly random for the observers who don't know the mechanism.
- No fixed string can be "pseudorandom" which refers to a distribution.
- It is impossible to definitively prove randomness.


TOUR OF ACCOUNTING
OVER HERE WE HAVE OUR RANDOM NUMBER GENERATOR.
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NINE NINE NINE NINE NINE NINE
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ARE YOU SURE THAT'S RANDOM?
THAT'S THE PROBLEM WITH RANDOMNESS: YOU CAN NEVER BE SURE.

Distinguisher: Statistical Tests

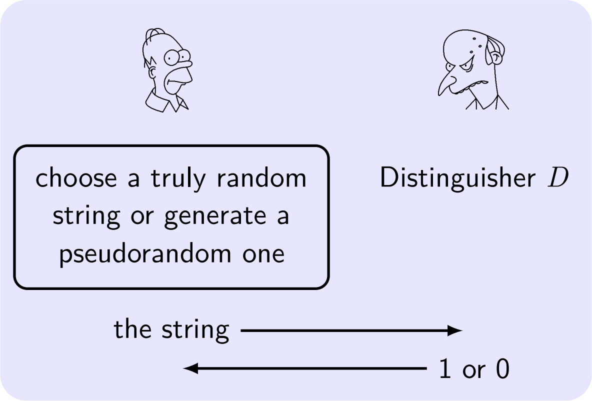
The pragmatic approach is to take many sequences of random numbers from a given generator and subject them to a battery of statistical tests.[1]

- $D(x) = 0$ if $|\#0(x) - \#1(x)| \leq 10 \cdot \sqrt{n}$
- $D(x) = 0$ if $|\#00(x) - n/4| \leq 10 \cdot \sqrt{n}$
- $D(x) = 0$ if max-run-of-$0(x) \leq 10 \cdot \log n$

[1] State-of-the-art: NIST Special Publication 800-22 "*A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic Applications*"

Intuition for Defining Pseudorandom

Intuition: Generate a long string from a short truly random seed, and the pseudorandom string is indistinguishable from truly random strings.



choose a truly random string or generate a pseudorandom one

Distinguisher D

the string →

← 1 or 0

Definition of Pseudorandom Generators

Definition 4

A deterministic polynomial-time algorithm $G : \{0,1\}^n \to \{0,1\}^{\ell(n)}$ is a **pseudorandom generator (PRG)** if

1. (Expansion:) $\forall n, \ell(n) > n$.
2. (Pseudorandomness): $\forall$ PPT distinguishers D,

$$|\Pr[D(r) = 1] - \Pr[D(G(s)) = 1]| \leq \mathsf{negl}(n),$$

where r is chosen *u.a.r* from $\{0,1\}^{\ell(n)}$, the **seed** s is chosen *u.a.r* from $\{0,1\}^n$. $\ell(\cdot)$ is the **expansion factor** of G.

Remarks on Pseudorandom Generators

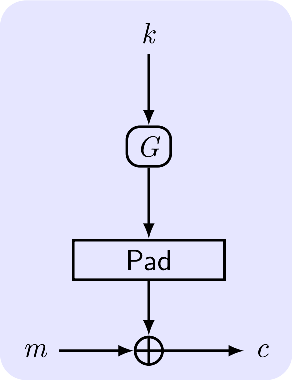
- **Sparse outputs**: In the case of $\ell(n) = 2n$, only 2^{-n} of strings of length $2n$ occurs.
- **Brute force attack**: Given an unlimited amount of time, one can distinguish $G(s)$ from r with a high probability by generating all strings with all seeds.

$$|\Pr[D(r) = 1] - \Pr[D(G(s)) = 1]| \geq 1 - 2^{-n}$$

- **Sufficient seed space**: s must be long enough against brute force attack.
- **Existence**: Under the weak assumption that *one-way functions* exists, or $\mathcal{P} \neq \mathcal{NP}$.

Content

A Secure Fixed-Length Encryption Scheme



Construction 5

- $|G(k)| = \ell(|k|)$, $m \in \{0,1\}^{\ell(n)}$.
- Gen: $k \in \{0,1\}^n$.
- Enc: $c := G(k) \oplus m$.
- Dec: $m := G(k) \oplus c$.

Theorem 6

This fixed-length encryption scheme has indistinguishable encryptions in the presence of an eavesdropper.

Reduction (Complexity)

A **reduction** is a transformation of one problem A into another problem B.

Reduction $A \leq_m B$ [2] : A is **reducible** to B if solutions to B exist and whenever given the solutions A can be solved.

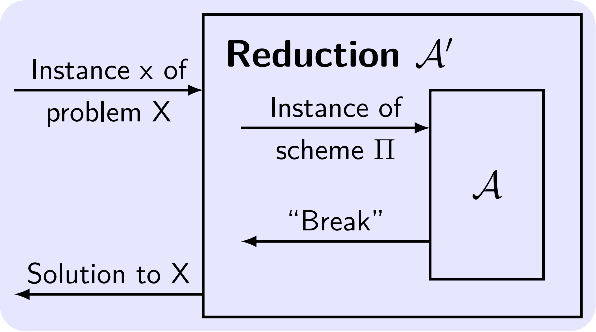
Solving A **cannot be harder** than solving B.

Example

- “measure the area of a rectangle” $\leq_m$ “measure the length and width of rectangle”
- “calculate x^2” $\leq_m$ “calculate $x \times y$”

[2] m means the mapping reduction.

Proofs of Reduction

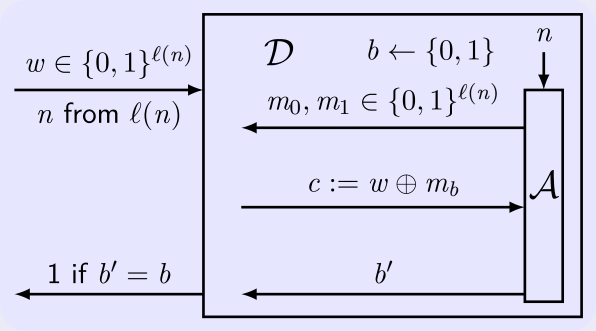


- A PPT $\mathcal{A}$ can break Π with probability $\varepsilon(n)$.
- **Assumption**: Problem X is *hard* to solve.
- **Reduction**: Reduce $\mathcal{A}'$ to $\mathcal{A}$. $\mathcal{A}'$ solves x efficiently with probability $1/p(n)$, running $\mathcal{A}$ as a sub-routine.
- **Contradiction**: If $\varepsilon(n)$ is non-negligible, then $\mathcal{A}'$ solves X efficiently with non-negligible probability $\varepsilon(n)/p(n)$.

Proof of Indistinguishable Encryptions

Idea: Use $\mathcal{A}$ to construct D for G, so that D distinguishes G when $\mathcal{A}$ breaks $\tilde{\Pi}$. Since D cannot distinguish G, so that $\mathcal{A}$ cannot break $\tilde{\Pi}$.

Proof.



$$\Pr[D(w) = 1] = \Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\tilde{\Pi}}(n) = 1]$$

□

Proof of Indistinguishable Encryptions (Cont.)

Proof.

To prove $\varepsilon(n) \overset{\text{def}}{=} \Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1] - \frac{1}{2}$ is negligible.

(1) If w is r chosen *u.a.r*, then $\tilde{\Pi}$ is OTP.

$$\Pr[D(r) = 1] = \Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\tilde{\Pi}}(n) = 1] = \frac{1}{2};$$

(2) If w is $G(k)$, then $\tilde{\Pi} = \Pi$.

$$\Pr[D(G(k)) = 1] = \Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1] = \frac{1}{2} + \varepsilon(n).$$

Use Definition **??**:

$$|\Pr[D(r) = 1] - \Pr[D(G(k)) = 1]| = \varepsilon(n) \leq \mathsf{negl}(n).$$

□

Handling Variable-Length Messages

Definition 7

A **deterministic** polynomial-time algorithm G is a **variable output-length pseudorandom generator** if

1. $G(s, 1^{\ell})$ outputs a string of length $\ell > 0$, where s is a string.
2. $G(s, 1^{\ell})$ is a prefix of $G(s, 1^{\ell'})$, $\ell' > \ell$.[3]
3. $G_{\ell}(s) \overset{\text{def}}{=} G(s, 1^{\ell(|s|)})$. Then $\forall \ell(\cdot)$, G_{ℓ} is a PRG with expansion factor ℓ.

Both Construction **??** and Theorem **??** hold here.

[3] for technical reasons to prove security.

Computational Security vs. Info.-theoretical Security

	Computational	**Info.-theoretical**
Adversary	PPT eavesdropping	no limited eavesdropping
Definition	indistinguishable $\frac{1}{2} + \text{negl}$	indistinguishable $\frac{1}{2}$
Assumption	pseudorandom	random
Key	short random str.	long random str.
Construction	XOR pad	XOR pad
Prove	reduction	-