Diffie-Hellman Problem and Elgamal Encryption Scheme

Yu Zhang

HIT/CST/NIS

Cryptography, Autumn, 2014

Outline

1. Cyclic Groups and Discrete Logrithms
2. Diffie-Hellman Assumptions and Applications
3. The Elgamal Encryption Scheme

Content

Cyclic Groups and Generators

$\mathbb{G}$ is finite and $g \in \mathbb{G}$, $\langle g \rangle \stackrel{\text{def}}{=} \{g^0, g^1, \ldots,\} = \{g^0, g^1, \ldots, g^{i-1}\}$.

- The **order** of g is the smallest positive integer i with $g^i = 1$.
- $\mathbb{G}$ is a **cyclic group** if $\exists$ g has order $m = |\mathbb{G}|$. $\langle g \rangle = \mathbb{G}$, g is a **generator** of $\mathbb{G}$.

- Is $\mathbb{Z}_6^*$, $\mathbb{Z}_7^*$, or $\mathbb{Z}_8^*$ with '$\cdot$' cyclic?

- $\langle g \rangle$ is a subgroup of $\mathbb{G}$, and $|\langle g \rangle| \mid |\mathbb{G}|$.
- If p is prime, then $\mathbb{Z}_p^*$ is cyclic.

Using Prime-Order Groups

Theorem 1

If $\mathbb{G}$ is of prime order, then $\mathbb{G}$ is cyclic. All $g \in \mathbb{G}$ except the identity are generators.

- The discrete logarithm problem is hardest in such groups.
- Finding a generator in such groups is trivial.
- Any non-zero exponent will be invertible modulo the order.
- A necessary condition for the DDH problem to be hard is that $\mathrm{DH}_g(h_1, h_2)$ by itself should be indistinguishable from a random group element. This is (almost) true for such groups.

Generating Prime-Order (Sub)Groups in $\mathbb{Z}_p^*$

- $y \in \mathbb{Z}_p^*$ is a **quadratic residue modulo** p if $\exists x \in \mathbb{Z}_p^*$ such that $x^2 \equiv y \pmod{p}$. (Q: show QRs in $\mathbb{Z}_7^*$)
- The set of QR is a subgroup with order $(p-1)/2$ ($x^2 \equiv (p-x)^2 \pmod{p}$).
- p is a **strong prime** if $p = 2q+1$ with q prime.

Algorithm 1: A group generation algorithm $\mathcal{G}$

input : Security parameter 1^n

output: Cyclic group $\mathbb{G}$, its order q, and a generator g

generate a random $(n+1)$-bit strong prime p

$q := (p-1)/2$

choose an arbitrary $x \in \mathbb{Z}_p^*$ with $x \neq \pm 1 \bmod p$

$g := x^2 \bmod p$

return p, q, g

Discrete Logarithm

If $\mathbb{G}$ is a cyclic group of order q, then $\exists$ a generator $g \in \mathbb{G}$ such that $\{g^0, g^1, \ldots, g^{q-1}\} = \mathbb{G}$.

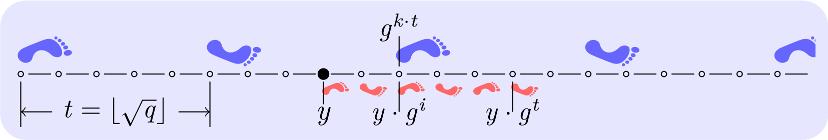
- $\forall h \in \mathbb{G}$, $\exists$ a unique $x \in \mathbb{Z}_q$ such that $g^x = h$.
- $x = \log_g h$ is the **discrete logarithm of** h **with respect to** g.
- If $g^{x'} = h$, then $\log_g h = [x' \bmod q]$.
- $\log_g 1 = 0$ and $\log_g(h_1 \cdot h_2) = [(\log_g h_1 + \log_g h_2) \bmod q]$.

Show an instance of DL problem in $\mathbb{Z}_7^*$

Overview of Discrete Logarithm Algorithms

- Given a generator $g \in \mathbb{G}$ and $y \in \langle g \rangle$, find x such that $g^x = y$.
- **Brute force**: $\mathcal{O}(q)$, $q = \text{ord}(g)$ is the order of $\langle g \rangle$.
- **Baby-step/giant-step** method [Shanks]: $\mathcal{O}(\sqrt{q} \cdot \text{polylog}(q))$.
- **Pohlig-Hellman** algorithm: when q has small factors.
- **Index calculus** method: $\mathcal{O}(\exp(\sqrt{n \cdot \log n}))$.
- The best-known algorithm is the **general number field sieve** with time $\mathcal{O}(\exp(n^{1/3} \cdot (\log n)^{2/3}))$.
- Elliptic curve groups vs. $\mathbb{Z}_p^*$: more efficient for the honest parties, but that are equally hard for an adversary to break. (Both 1024-bit $\mathbb{Z}_p^*$ and 132-bit elliptic curve need 2^{66} steps.)

The Baby-Step/Giant-Step Algorithm



Algorithm 2: The baby-step/giant-step algorithm

input : $g \in \mathbb{G}$ and $y \in \langle g \rangle$; $q =$ ord(g) ($t := \lfloor \sqrt{q} \rfloor$)

output: $\log_g y$

for $i = 0$ **to** $\lfloor q/t \rfloor$ **do compute** $g_i := g^{i \cdot t}$ `/* giant steps */`
sort the pairs (i, g_i) by g_i
for $i = 0$ **to** t **do**
  compute $y_i := y \cdot g^i$ `/* baby steps */`
  if $y_i = g_k$ *for some* k **then return** $[kt - i \bmod q]$

The time complexity is $\mathcal{O}(\sqrt{q} \cdot \text{polylog}(q))$.

Example of Baby-Step/Giant-Step Algorithm

In $\mathbb{Z}_{29}^*$, $q = 28$, $g = 2$, $y = 17$.

$t = 5$, compute the giant steps:

$$2^0 = 1,\ 2^5 = 3,\ 2^{10} = 9,\ 2^{15} = 27,\ 2^{20} = 23,\ 2^{25} = 11.$$

compute the baby steps:

$$17 \cdot 2^0 = 17,\ 17 \cdot 2^1 = 5,\ 17 \cdot 2^2 = 10,$$

$$17 \cdot 2^3 = 20,\ 17 \cdot 2^4 = 11,\ 17 \cdot 2^5 = 22.$$

$2^{25} = 11 = 17 \cdot 2^4$. So $\log_2 17 = 25 - 4 = 21$.

The Discrete Logarithm Assumption

The discrete logarithm experiment $\mathsf{DLog}_{\mathcal{A},\mathcal{G}}(n)$:

1. Run a group-generating algorithm $\mathcal{G}(1^n)$ to obtain $(\mathbb{G}, q, g)$, where $\mathbb{G}$ is a cyclic group of order q (with $\|q\| = n$), and g is a generator of $\mathbb{G}$.
2. Choose $h \leftarrow \mathbb{G}$. $\left(x' \leftarrow \mathbb{Z}_q \text{ and } h := g^{x'}\right)$
3. $\mathcal{A}$ is given $\mathbb{G}, q, g, h$, and outputs $x \in \mathbb{Z}_q$.
4. $\mathsf{DLog}_{\mathcal{A},\mathcal{G}}(n) = 1$ if $g^x = h$, and 0 otherwise.

Definition 2

The discrete logarithm problem is hard relative to $\mathcal{G}$ if $\forall$ PPT algorithm $\mathcal{A}$, $\exists$ negl such that

$$\Pr[\mathsf{DLog}_{\mathcal{A},\mathcal{G}}(n) = 1] \le \mathsf{negl}(n).$$

Content

Diffie-Hellman Assumptions

- **Computational Diffie-Hellman (CDH)** problem:

$$\mathsf{DH}_g(h_1, h_2) \stackrel{\mathsf{def}}{=} g^{\log_g h_1 \cdot \log_g h_2}$$

- **Decisional Diffie-Hellman (DDH)** problem:
  Distinguish $\mathsf{DH}_g(h_1, h_2)$ from a random group element h'.

Definition 3

DDH problem is hard relative to $\mathcal{G}$ if $\forall$ PPT $\mathcal{A}$, $\exists$ negl such that

$$|\Pr[\mathcal{A}(\mathbb{G}, q, g, g^x, g^y, g^z) = 1] - \Pr[\mathcal{A}(\mathbb{G}, q, g, g^x, g^y, g^{xy}) = 1]|$$

$$\leq \mathsf{negl}(n).$$

Intractability of DL, CDH and DDH

DDH is easier than CDH and DL.

Secure Key-Exchange Experiment

The key-exchange experiment $\mathsf{KE}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n)$:

1. Two parties holding 1^n execute protocol Π. Π results in a **transcript** trans containing all the messages sent by the parties, and a **key** k that is output by each of the parties.
2. A random bit $b \leftarrow \{0,1\}$ is chosen. If $b = 0$ then choose $\hat{k} \leftarrow \{0,1\}^n$ *u.a.r*, and if $b = 1$ then set $\hat{k} := k$.
3. $\mathcal{A}$ is given trans and $\hat{k}$, and outputs a bit b'.
4. $\mathsf{KE}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1$ if $b' = b$, and 0 otherwise.

Definition 4

A key-exchange protocol Π is secure in the presence of an eavesdropper if $\forall$ PPT $\mathcal{A}$, $\exists$ negl such that

$$\Pr[\mathsf{KE}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1] < \frac{1}{2} + \mathsf{negl}(n).$$

Diffie-Hellman Key-Exchange Protocol

$(\mathbb{G}, q, g) \leftarrow \mathcal{G}$

$x \leftarrow \mathbb{Z}_q$

$h_1 := g^x \xrightarrow{\mathbb{G}, q, g, h_1}$

$y \leftarrow \mathbb{Z}_q$

$\xleftarrow{h_2} h_2 := g^y$

$k_A := h_2^x$ $\quad$ $k_B := h_1^y$

$$k_A = k_B = k = g^{xy}.$$

$\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}$ denote an experiment where if $b = 0$ the adversary is given $\hat{k} \leftarrow \mathbb{G}$.

Theorem 5

If DDH problem is hard relative to $\mathcal{G}$, then DH key-exchange protocol Π is secure in the presence of an eavesdropper (with respect to the modified experiment $\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}$).

Security

Insecurity against active adversaries (Man-In-The-Middle).

Proof.

$$
\begin{aligned}
&\Pr\left[\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1\right] \\
&\quad = \frac{1}{2} \cdot \Pr\left[\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 | b = 1\right] + \frac{1}{2} \cdot \Pr\left[\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 | b = 0\right]
\end{aligned}
$$

If $b = 1$, then give true key; otherwise give random g^z.

$$
\begin{aligned}
&= \frac{1}{2} \cdot \Pr\left[\mathcal{A}(g^x, g^y, g^{xy}) = 1\right] + \frac{1}{2} \cdot \Pr\left[\mathcal{A}(g^x, g^y, g^z) = 0\right] \\
&= \frac{1}{2} \cdot \Pr\left[\mathcal{A}(g^x, g^y, g^{xy}) = 1\right] + \frac{1}{2} \cdot \left(1 - \Pr\left[\mathcal{A}(g^x, g^y, g^z) = 1\right]\right) \\
&= \frac{1}{2} + \frac{1}{2} \cdot \left(\Pr\left[\mathcal{A}(g^x, g^y, g^{xy}) = 1\right] - \Pr\left[\mathcal{A}(g^x, g^y, g^z) = 1\right]\right) \\
&\leq \frac{1}{2} + \frac{1}{2} \cdot \mathsf{negl}(n)
\end{aligned}
$$

□

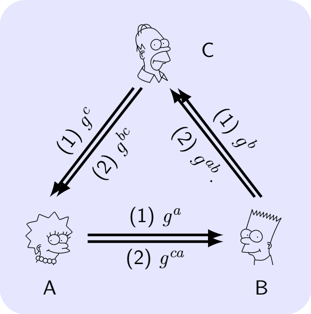
Example of DHKE

$\mathbb{G} = \mathbb{Z}_{11}^{*}$

The order $q = ?$
The set of quadratic residues ?
Is $g = 3$ a generator?
If $x > 2$ and $y = x + 1$, what is x and y ?
What's the message from Bob to Alice?
How does Alice compute the key?
How does Bob compute the key?

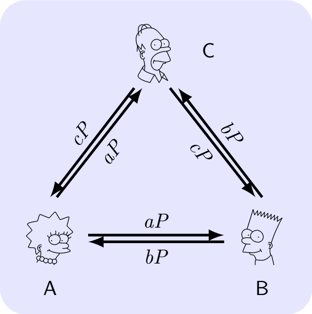
Triparties Key Exchange

DH-based KE in 2 rounds:



Key= g^{abc}.

Joux's KE in 1 round:



Key= $e(P, P)^{abc}$ in bilinear map.

Open Problem

How to exchange keys between 4 parties in one round?

Content

Lemma on Perfectly-secret Private-key Encryption

Lemma 6

$\mathbb{G}$ is a finite group and $m \in \mathbb{G}$ is an arbitrary element. Then choosing random $g \leftarrow \mathbb{G}$ and setting $g' := m \cdot g$ gives the same distribution for g' as choosing random $g' \leftarrow \mathbb{G}$. I.e, $\forall \hat{g} \in \mathbb{G}$:

$$\Pr[m \cdot g = \hat{g}] = 1/|\mathbb{G}|.$$

Proof.

Let $\hat{g} \in \mathbb{G}$ be arbitrary, then

$$\Pr[m \cdot g = \hat{g}] = \Pr[g = m^{-1} \cdot \hat{g}].$$

Since g is chosen *u.a.r*, the probability that g is equal to the fixed element $m^{-1} \cdot \hat{g}$ is exactly $1/|\mathbb{G}|$. $\square$

The Elgamal Encryption Scheme

An algorithm $\mathcal{G}$, on input 1^n, outputs a description of a cyclic group $\mathbb{G}$, its order q (with $\|q\| = n$), and a generator g.

Construction 7

- Gen: *on input* 1^n *run* $\mathcal{G}(1^n)$ *to obtain* $(\mathbb{G}, q, g)$. *Choose a random* $x \leftarrow \mathbb{Z}_q$ *and compute* $h := g^x$. $pk = \langle \mathbb{G}, q, g, h \rangle$ *and* $sk = \langle \mathbb{G}, q, g, x \rangle$.
- Enc: *on input* pk *and* $m \in \mathbb{G}$, *choose a random* $y \leftarrow \mathbb{Z}_q$ *and output* $\langle c_1, c_2 \rangle = \langle g^y, h^y \cdot m \rangle$.
- Dec: *on input* sk *and* $\langle c_1, c_2 \rangle$, *output* $m := c_2 / c_1^x$.

Theorem 8

If the DDH problem is hard relative to $\mathcal{G}$, *then the Elgamal encryption scheme is CPA-secure.*

Example of Elgamal Encryption

Encoding binary strings:

- the subgroup of quadratic residues modulo a strong prime $p = (2q + 1)$.
- a string $\hat{m} \in \{0,1\}^{n-1}$, $n = \|q\|$.
- map $\hat{m}$ to the plaintext $m = [(\hat{m} + 1)^2 \bmod p]$.
- The mapping is one-to-one and efficiently invertible.

$q = 83$, $p = 2q + 1 = 167$, $g = 2^2 = 4 \pmod{167}$, $\hat{m} = 011101$

The receiver chooses secrete key $37 \in \mathbb{Z}_{83}$.
The public key is $pk = \langle 167, 83, 4, [4^{37} \bmod 167] = 76 \rangle$.
$\hat{m} = 011101 = 29$, $m = [(29 + 1)^2 \bmod 167] = 65$.
Choose $y = 71$, the ciphertext is
$\langle [4^{71} \bmod 167], [76^{71} \cdot 65 \bmod 167] \rangle = \langle 132, 44 \rangle$.

Decryption: $m = [44 \cdot (132^{37})^{-1}] \equiv [44 \cdot 66] \equiv 65 \pmod{167}$.
65 has the two square roots 30 and 137, and $30 < q$, so $\hat{m} = 29$.

Proof of Security of Elgamal Encryption Scheme

Proof.

Idea: Prove that Π is secure in the presence of an eavesdropper by reducing an algorithm D for DDH problem to the eavesdropper $\mathcal{A}$.

Modify Π to $\tilde{\Pi}$: the encryption is done by choosing random $y \leftarrow \mathbb{Z}_q$ and $z \leftarrow \mathbb{Z}_q$ and outputting the ciphertext:

$$\langle g^y, g^z \cdot m \rangle.$$

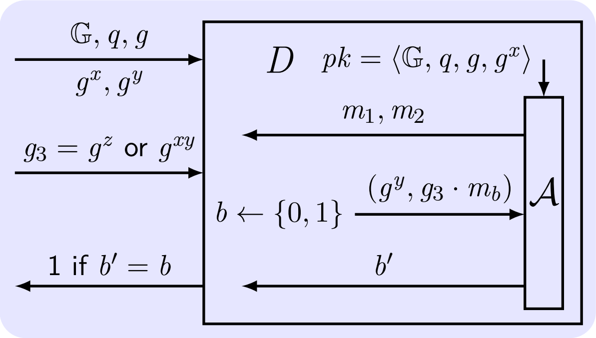
- $\tilde{\Pi}$ is not an encryption scheme.
- g^y is independent of m.
- $g^z \cdot m$ is a random element independent of m (Lemma 6).

$$\Pr\left[\mathsf{PubK}^{\mathsf{eav}}_{\mathcal{A},\tilde{\Pi}}(n) = 1\right] = \frac{1}{2}.$$

$\square$

Proof (Cont.)

D receives $(\mathbb{G}, q, g, g^x, g^y, g_3)$ where g_3 equals either g^{xy} or g^z, for random x, y, z:



Proof (Cont.)

Case I: $g_3 = g^z$, ciphertext is $\langle g^y, g^z \cdot m_b \rangle$.

$$\Pr[D(g^x, g^y, g^z) = 1] = \Pr\left[\mathsf{PubK}^{\mathsf{eav}}_{\mathcal{A},\tilde{\Pi}}(n) = 1\right] = \frac{1}{2}.$$

Case II: $g_3 = g^{xy}$, ciphertext is $\langle g^y, g^{xy} \cdot m_b \rangle$.

$$\Pr[D(g^x, g^y, g^{xy}) = 1] = \Pr\left[\mathsf{PubK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1\right] = \varepsilon(n).$$

Since the DDH problem is hard,

$$\begin{aligned} \mathsf{negl}(n) &\geq |\Pr[D(g^x, g^y, g^z) = 1] - \Pr[D(g^x, g^y, g^{xy}) = 1]| \\ &= |\frac{1}{2} - \varepsilon(n)|. \end{aligned}$$

CCA in Elgamal Encryption

Constructing the ciphertext of the message $m \cdot m'$.

Given $pk = \langle g, h \rangle$, $c = \langle c_1, c_2 \rangle$, $c_1 = g^y$, $c_2 = h^y \cdot m$,
Method I: compute $c_2' := c_2 \cdot m'$, and $c' = \langle c_1, c_2' \rangle$.

$$\frac{c_2'}{c_1^x} = ?$$

Method II: compute $c_1'' := c_1 \cdot g^{y''}$, and $c_2'' := c_2 \cdot h^{y''} \cdot m'$.

$$c_1'' = g^y \cdot g^{y''} = g^{y+y''} \text{ and } c_2'' = ?$$

so $c'' = \langle c_1'', c_2'' \rangle$ is an encryption of $m \cdot m'$.

Elgamal Implementation Issues

- **Sharing public parameters**: $\mathcal{G}$ generates parameters $\mathbb{G}, q, g$.
  - generated "once-and-for-all".
  - used by multiple receivers.
  - each receiver must choose their own secrete values x and publish their own public key containing $h = g^x$.

Parameter sharing

In the case of Elgamal, the public parameters can be shared. In the case of RSA, can parameters be shared?

Summary

- cyclic group, discrete log., baby-step/giant-step
- CDH, DDH, DHKE protocol.
- Elgamal encryption, sharing public parameters.