

More on Public-Key Cryptography

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- 1** Information-Theoretic MACs
- 2** Primes and Factoring
- 3** Discrete Logarithm Algorithms
- 4** More Public-key Schemes

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Definition of Information-Theoretic MAC Security

It is impossible to achieve "perfect" MAC, as the adversary can output a valid tag with probability $1/2^{|t|}$ at least.

The one-time MAC experiment $\text{Macforge}_{\mathcal{A}, \Pi}^{1\text{-time}}$:

- 1 $k \leftarrow \text{Gen}$.
- 2 \mathcal{A} outputs a message m' , and is given a tag $t' \leftarrow \text{Mac}_k(m')$, and outputs (m, t) .
- 3 $\text{Macforge}_{\mathcal{A}, \Pi}^{1\text{-time}} = 1 \iff \text{Vrfy}_k(m, t) = 1 \wedge m \neq m'$.

Definition 1

A MAC Π is **one-time ϵ -secure** if \forall PPT \mathcal{A} :

$$\Pr[\text{Macforge}_{\mathcal{A}, \Pi}^{1\text{-time}} = 1] \leq \epsilon.$$

Construction of Information-Theoretic MACs

Definition 2

A function $h: \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{T}$ is a **Strongly Universal Function (SUF)** if for all distinct $m, m' \in \mathcal{M}$ and all $t, t' \in \mathcal{T}$, it holds that:

$$\Pr[h_k(m) = t \wedge h_k(m') = t'] = 1/|\mathcal{T}|^2.$$

where the probability is taken over uniform choice of $k \in \mathcal{K}$.

Construction 3

- Let $h: \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{T}$ be an SUF.
- Gen: $k \leftarrow \{0, 1\}^n$ u.a.r.
- Mac_k(m): $t := h_k(m)$.
- Vrfy_k(m, t): $1 \iff t \stackrel{?}{=} h_k(m)$. (If $m \in \mathcal{M}$, then output 0.)

Theorem 4

If h is an SUF, Construction is a $1/|\mathcal{T}|$ -secure MAC.

Construction of An SUF

Theorem 5

For any prime P , the function h is an SUF:

$$h_{a,b}(m) \stackrel{\text{def}}{=} [a \cdot m + b \mod p]$$

Proof.

$h_{a,b}(m) = t$ and $h_{a,b}(m') = t'$, only if $a \cdot m + b = t \mod p$ and $a \cdot m' + b = t' \mod p$. We have $a = [(t - t') \cdot (m - m')^{-1} \mod p]$ and $b = [t - a \cdot m \mod p]$, which means there is a unique key (a, b) . Since there are $|\mathcal{T}|^2$ keys,

$$\Pr[h_k(m) = t \wedge h_k(m') = t'] = \frac{1}{|\mathcal{T}|^2}.$$



Limitations on Information-Theoretic MACs

Any ℓ -time 2^{-n} -secure MAC requires keys of length at least $(\ell + 1) \cdot n$.

Theorem 6

Let Π be a 1-time 2^{-n} -secure MAC where all keys are the same length. Then the keys must have length at least $2n$.

Proof.

Let $\mathcal{K}(t) \stackrel{\text{def}}{=} \{k \mid \text{Vrfy}_k(m, t) = 1\}$. For any t , $|\mathcal{K}(t)| \leq 2^{-n} \cdot |\mathcal{K}|$. Otherwise, (m, t) would be a valid forgery with probability at least $|\mathcal{K}(t)|/|\mathcal{K}| > 2^{-n}$. The probability that \mathcal{A} outputs a valid forgery is at least

$$\sum_t \Pr[\text{Mac}_k(m) = t] \cdot \frac{1}{|\mathcal{K}(m, t)|} \geq \sum_t \Pr[\text{Mac}_k(m) = t] \cdot \frac{2^n}{|\mathcal{K}|} = \frac{2^n}{|\mathcal{K}|}$$

As the probability is at most 2^{-n} , $|\mathcal{K}| \geq 2^{2n}$. Since all keys have the same length, each key must have length at least $2n$. \square

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Integer Factorization/Factoring

“The problem of distinguishing prime numbers from composite numbers and of resolving the latter into their prime factors is known to be one of the most important and useful in arithmetic.” – Gauss (1805)

The “hardest” numbers to factor seem to be those having only large prime factors.

- The best-known algorithm is the **general number field sieve** [Pollard] with time $\mathcal{O}(\exp(n^{1/3} \cdot (\log n)^{2/3}))$.
- RSA Factoring Challenge: RSA-768 (232 digits)
 - Two years on hundreds of machines (2.2GHz/2GB, 1500 years)
 - Factoring a 1024-bit integer: about 1000 times harder.

Generating Random Primes

Algorithm 1: Generating a random prime

input : Length n ; parameter t

output: A random n -bit prime

```
1 for  $i = 1$  to  $t$  do
2    $p' \leftarrow \{0, 1\}^{n-1}$ 
3    $p := 1 \parallel p'$ 
4   if  $p$  is prime then return  $p$ 
5
6 return fail
```

To show its efficiency, we need understand two issues:

- the probability that a randomly-selected n -bit integer is prime.
- how to efficiently test whether a given integer p is prime.

The Distribution of Prime

Theorem 7 (Prime number theorem)

\exists a constant c such that, $\forall n > 1$, a randomly selected n -bit number is prime with probability at least c/n .

The probability that a prime is *not* chosen in $t = n^2/c$ iterations is

$$\left(1 - \frac{c}{n}\right)^t = \left(\left(1 - \frac{c}{n}\right)^{n/c}\right)^n \leq \left(e^{-1}\right)^n = e^{-n}.$$

The algorithm will fail with a negligible probability.

Testing Primality

- **Trial division:** Divide N by $a = 2, 3, \dots, \sqrt{N}$.
- **Probabilistic algorithm for approximately computing:**
 - Atlantic City algorithm with two-sided error.
 - Monte Carlo algorithm with one-sided error.
 - Las Vegas algorithm with zero-sided error.
- **Fermat primality test:** $a^{N-1} \equiv 1 \pmod{N}$.
- a is a **witness** that N is composite if $a^{N-1} \not\equiv 1 \pmod{N}$.
- a is a **liar** if N is composite and $a^{N-1} \equiv 1 \pmod{N}$.
- **Carmichael numbers:** composite numbers without witnesses.

Theorem 8

If \exists a witness, then at least half the elements of \mathbb{Z}_N^ are witnesses.*

The Miller-Rabin Primality Test

$N - 1 = 2^r u$, u is odd. $a \in \mathbb{Z}_N^*$ is a **strong witness** if

- 1 $a^u \neq \pm 1$, and
- 2 $a^{2^i u} \neq -1$ for $i \in \{1, \dots, r-1\}$.

Lemma 9

$x \in \mathbb{Z}^*$ is a **square root of 1 modulo N** if $x^2 \equiv 1 \pmod{N}$. If N is an odd prime then the only x are $[\pm 1 \pmod{N}]$.

Theorem 10

N is an odd, composite number that is not a prime power. Then at least half the elements of \mathbb{Z}_N^* are strong witnesses.

Theorem 11

If N is prime, then the Miller-Rabin test always outputs “prime”. If N is composite, then the algorithm outputs “prime” with probability at most 2^{-t} ¹.

¹Actually, it is at most 4^{-t} .

Describing The Algorithm

Algorithm 2: The Miller-Rabin primality test

input : Integer $N > 2$ and parameter t

output: A decision as to whether N is prime or composite

```
1 if  $N$  is a perfect power then return "composite"  
2 compute  $r \geq 1$  and  $u$  odd such that  $N - 1 = 2^r u$   
3 LOOP: for  $s = 1$  to  $t$  do  
4    $a \leftarrow \{2, \dots, N - 2\}$   
5    $x = a^u \bmod N$   
6   if  $x = \pm 1$  then do next LOOP  
7   for  $i = 1$  to  $r$  do  
8      $x = x^2 \bmod N$   
9     if  $x = -1$  then do next LOOP  
10  return "composite"  
11 return "prime"
```

Examples of Primality Tests

Liars in Fermat primality test

$2^{340} \equiv 1 \pmod{341}$, but $341 = 11 \cdot 31$.

$5^{560} \equiv 1 \pmod{561}$, but $561 = 3 \cdot 11 \cdot 17$.

Carmichael numbers < 10000 :

561, 1105, 1729, 2465, 2821, 6601, 8911.

Examples of Miller-Rabin test

Carmichael number $1729 = 7 \cdot 13 \cdot 19$.

$1729 - 1 = 1728 = 2^6 \cdot 27$. So $r = 6, u = 27$. $a = 671$.

$$671^{27} \equiv 1084 \pmod{1729}$$

$$671^{27 \cdot 2} \equiv 1065 \pmod{1729}$$

$$671^{27 \cdot 2^2} \equiv 1 \pmod{1729}$$

Algorithms for Factoring

- **Factoring** $N = pq$. p, q are of the same length n .
- **Trial division**: $\mathcal{O}(\sqrt{N} \cdot \text{polylog}(N))$.
- **Pollard's $p - 1$ method**: effective when $p - 1$ has “small” prime factors.
- **Pollard's rho method**: $\mathcal{O}(N^{1/4} \cdot \text{polylog}(N))$.
- **Quadratic sieve algorithm** [Carl Pomerance]: sub-exponential time $\mathcal{O}(\exp(\sqrt{n \cdot \log n}))$.
- The best-known algorithm is the **general number field sieve** [Pollard] with time $\mathcal{O}(\exp(n^{1/3} \cdot (\log n)^{2/3}))$.

Pollard's $p - 1$ Method

Idea: Fermat's little theorem: $y = x^{(p-1) \cdot k} \equiv 1 \pmod{p}$. Then $(y - 1) \equiv 0 \pmod{p}$ and $p \mid (y - 1)$. So $p = \gcd(y - 1, N)$. To make the exponent a large multiple of $(p - 1)$:

$$M = \text{lcm}(\{i \mid i \leq B\}) = \prod_{\text{prime } i \leq B} i^{\lfloor \log_i B \rfloor}.$$

If $p - 1$ has only “small” factors, then the bound B will be small.

Algorithm 3: Pollard's $p - 1$ algorithm for factoring

input : Integer N

output: A non-trivial factor of N

- 1 $x \leftarrow \mathbb{Z}_N^*$
 - 2 $y := [x^M \bmod N]$
 - 3 $p := \gcd(y - 1, N)$
 - 4 **if** $p \notin \{1, N\}$ **then return** p
-

Pollard's Rho (ρ) Method

Idea: Using the improved birthday attack² to find x, x' such that $x \neq x' \wedge x \equiv x' \pmod{p}$. Then $p \mid (x - x')$, $p = \gcd(x - x', N)$. $F(x) = x^2 + b$, where $b \not\equiv 0, -2 \pmod{N}$.

Algorithm 4: Pollard's rho algorithm for factoring

input : Integer N

output: A non-trivial factor of N

```
1  $x_0 \leftarrow \mathbb{Z}_N^*$ 
2 for  $i = 1$  to  $2^{n/2}$  do
3    $x_i := [F(x_{i-1}) \bmod N]$ 
4    $x_{2i} := [F(F(x_{2i-2})) \bmod N]$ 
5    $p := \gcd(x_{2i} - x_i, N)$ 
6   if  $p \notin \{1, N\}$  then return  $p$ 
```

²Floyd's cycle-finding algorithm (the "tortoise and the hare" algorithm).

Proof of Pollard's ρ Method

Lemma 12

Let x_1, \dots be a sequence with $x_m \equiv F(x_{m-1}) \pmod{N}$. F satisfies that $x \equiv x' \pmod{N} \implies F(x) \equiv F(x') \pmod{N}$. If $x_I \equiv x_J \pmod{p}$ with $I < J$, then $\exists i < J$ such that $x_i \equiv x_{2i} \pmod{p}$.



Proof.

See the proof of improved birthday attack. □

According to the lemma of birthday problem, given a sequence of length $O(N^{1/4})$, find such pair with probability $1/4$.

Example of Pollard's $p - 1$ and ρ methods

Factorizing $N = 5917$ with Pollard's $p - 1$ method

Choose $B = 5$, $M = \text{lcm}(1, 2, 3, 4, 5) = 60$.

For $x = 2$, $y \equiv x^M \equiv 2^{60} \equiv 3417 \pmod{5917}$.

$p = \text{gcd}(y - 1, N) = \text{gcd}(3416, 5917) = 61$.

Factorizing $N = 8051$ with Pollard's ρ method

$f(x) = x^2 + 1$, $x_0 = 2$.

i	x_i	x_{2i}	$\text{gcd}(x_{2i} - x_i, N)$
1	5	26	1
2	26	7474	1
3	677	871	97

The Quadratic Sieve Algorithm

Idea: Find x, y with $x^2 \equiv y^2 \pmod{N}$ and $x \not\equiv \pm y \pmod{N}$.
 $x^2 - y^2 \equiv 0 \pmod{N} \implies (x + y)(x - y) \equiv 0 \pmod{N}$.

$\gcd(x + y, N)$ and $\gcd(x - y, N)$ will give p .

Finding congruence of squares:

- 1 Choose a factor base $B = \{p_1, \dots, p_k\}$ of prime numbers.
- 2 Use '**sieve theory**' to find $\ell = k + 1$ distinct x_1, \dots, x_ℓ for which $[x_i^2 \bmod N]$ decompose into the elements of B :
$$x_i^2 \equiv \prod_{j=1}^k p_j^{e_j} \pmod{N}.$$
- 3 Write x_i^2 as an exponent vector $\langle e_{i,1}, \dots, e_{i,k} \rangle \pmod{2}$.
- 4 Find the addition of vectors = the zero vector $\pmod{2}$.
 $X = \{x_{\ell_1}, \dots, x_{\ell_n}\}. \forall i, E_i = \sum_{j=1}^n e_{\ell_j, i} \equiv 0 \pmod{2}.$
- 5 Find a pair: $x = \prod_{i=1}^n x_{\ell_i} \not\equiv y = \prod_{i=1}^k p_i^{E_i/2} \pmod{N}.$

Example of Quadratic Sieve Algorithm

Factorizing $N = 377753$ with quadratic sieve algorithm

$$B = \{2, 13, 17, 23, 29\}.$$

$$620^2 \equiv 17^2 \cdot 23 \pmod{N}$$

$$621^2 \equiv 2^4 \cdot 17 \cdot 29 \pmod{N}$$

$$645^2 \equiv 2^7 \cdot 13 \cdot 23 \pmod{N}$$

$$655^2 \equiv 2^3 \cdot 13 \cdot 17 \cdot 29 \pmod{N}$$

$$[620 \cdot 621 \cdot 645 \cdot 655 \bmod N]^2 \equiv [2^7 \cdot 13 \cdot 17^2 \cdot 23 \cdot 29 \bmod N]^2$$

$$\implies 127194^2 \equiv 45335^2 \pmod{N},$$

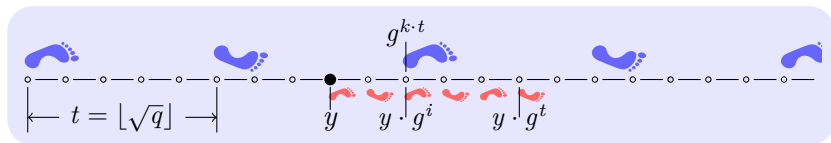
Computing $\gcd(127194 - 45335, 377753) = 751$.

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Overview of Discrete Logarithm Algorithms

- Given a generator $g \in \mathbb{G}$ and $y \in \langle g \rangle$, find x such that $g^x = y$.
- **Brute force:** $\mathcal{O}(q)$, $q = \text{ord}(g)$ is the order of $\langle g \rangle$.
- **Baby-step/giant-step** method [Shanks]: $\mathcal{O}(\sqrt{q} \cdot \text{polylog}(q))$.
- **Pohlig-Hellman** algorithm: when q has small factors.
- **Index calculus** method: $\mathcal{O}(\exp(\sqrt{n \cdot \log n}))$.
- The best-known algorithm is the **general number field sieve** with time $\mathcal{O}(\exp(n^{1/3} \cdot (\log n)^{2/3}))$.
- Elliptic curve groups vs. \mathbb{Z}_p^* : more efficient for the honest parties, but that are equally hard for an adversary to break. (Both 1024-bit \mathbb{Z}_p^* and 132-bit elliptic curve need 2^{66} steps.)

The Baby-Step/Giant-Step Algorithm



Algorithm 5: The baby-step/giant-step algorithm

input : $g \in \mathbb{G}$ and $y \in \langle g \rangle$; $q = \text{ord}(g)$ ($t := \lfloor \sqrt{q} \rfloor$)

output: $\log_g y$

- 1 **for** $i = 0$ **to** $\lfloor q/t \rfloor$ **do compute** $g_i := g^{i \cdot t}$ /* giant steps */
- 2 **sort** the pairs (i, g_i) by g_i
- 3 **for** $i = 0$ **to** t **do**
- 4 **compute** $y_i := y \cdot g^i$ /* baby steps */
- 5 **if** $y_i = g_k$ **for some** k **then return** $[kt - i \bmod q]$

The time complexity is $\mathcal{O}(\sqrt{q} \cdot \text{polylog}(q))$.

Example of Baby-Step/Giant-Step Algorithm

In \mathbb{Z}_{29}^* , $q = 28$, $g = 2$, $y = 17$.

$t = 5$, compute the giant steps:

$$2^0 = 1, 2^5 = 3, 2^{10} = 9, 2^{15} = 27, 2^{20} = 23, 2^{25} = 11.$$

compute the baby steps:

$$17 \cdot 2^0 = 17, 17 \cdot 2^1 = 5, 17 \cdot 2^2 = 10,$$

$$17 \cdot 2^3 = 20, 17 \cdot 2^4 = 11, 17 \cdot 2^5 = 22.$$

$$2^{25} = 11 = 17 \cdot 2^4. \text{ So } \log_2 17 = 25 - 4 = 21.$$

The Pohlig-Hellman Algorithm

Idea: when q is known and has small factors, reduces the discrete logarithm instance to multiple instances in groups of smaller order.

According to CRT: If $q = \prod_{i=1}^k q_i$ and $\forall i \neq j, \gcd(q_i, q_j) = 1$, then

$$\mathbb{Z}_q \simeq \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k} \text{ and } \mathbb{Z}_q^* \simeq \mathbb{Z}_{q_1}^* \times \cdots \times \mathbb{Z}_{q_k}^*$$

$$(g_i)^x \stackrel{\text{def}}{=} \left(g^{q/q_i}\right)^x = (g^x)^{q/q_i} = y^{q/q_i} \text{ for } i = 1, \dots, k.$$

We have k instances in k smaller groups, $\text{ord}(g_i) = q_i$.³

Use any other algorithm to solve $\log_{g_i}(y^{q/q_i})$.

Answers are $\{x_i\}_{i=1}^k$ for which $g_i^{x_i} \equiv y^{q/q_i} \equiv g_i^x$.

$\forall i, x \equiv x_i \pmod{q_i}$. $x \bmod q$ is uniquely determined (CRT).

The time complexity is $\mathcal{O}(\max_i \{\sqrt{q_i}\} \cdot \text{polylog}(q))$.

³If $p \mid q$, then $\text{ord}(g^p) = q/p$.

Example of Pohlig-Hellman Algorithm

In \mathbb{Z}_{31}^* , $q = 30 = 5 \cdot 3 \cdot 2$, $g = 3$, $y = 26 = g^x$.

$$(g^{30/5})^x = y^{30/5} \implies (3^6)^x = 26^6 \implies 16^x \equiv 1$$

$$(g^{30/3})^x = y^{30/3} \implies (3^{10})^x = 26^{10} \implies 25^x \equiv 5$$

$$(g^{30/2})^x = y^{30/2} \implies (3^{15})^x = 26^{15} \implies 30^x \equiv 30$$

$$x \equiv 0 \pmod{5}, x \equiv 2 \pmod{3}, x \equiv 1 \pmod{2},$$

so $x \equiv 5 \pmod{30}$.

The Index Calculus Method

Idea: find a relatively small factor base and build a system of ℓ linear equations related to g ; find a linear equation related to y ; solve $\ell + 1$ linear equations to give $\log_g y$.

- 1 for \mathbb{Z}_p^* , choose a base $B = \{p_1, \dots, p_k\}$ of prime numbers.
- 2 find $\ell \geq k$ distinct x_1, \dots, x_ℓ for which $[g^{x_i} \bmod p]$ decompose into the elements of B : $g^{x_i} \equiv \prod_{j=1}^k p_j^{e_{ij}} \pmod{p}$.
- 3 ℓ equations: $x_i = \sum_{j=1}^k e_{ij} \cdot \log_g(p_j) \pmod{p-1}$.
- 4 find x^* for which $[g^{x^*} \cdot y \bmod p]$ can be factored.
- 5 new equation: $x^* + \log_g y = \sum_{j=1}^k e_j^* \cdot \log_g(p_j) \pmod{p-1}$.
- 6 Use linear algebra to solve equations and give $\log_g y$.

The time complexity is identical to that of the quadratic sieve.

Example of Index Calculus Method

$p = 101$, $g = 3$ and $y = 87$. $B = \{2, 5, 13\}$.

$3^{10} \equiv 65 \pmod{101}$ and $65 = 5 \cdot 13$. Similarly, $3^{12} \equiv 80 = 2^4 \cdot 5 \pmod{101}$ and $3^{14} \equiv 13 \pmod{101}$. The linear equations:

$$x_1 = 10 \equiv \log_3 5 + \log_3 13 \pmod{100}$$

$$x_2 = 12 \equiv 4 \cdot \log_3 2 + \log_3 5 \pmod{100}$$

$$x_3 = 14 \equiv \log_3 13 \pmod{100}.$$

We also have $x^* = 5$, $3^5 \cdot 87 \equiv 32 \equiv 2^5 \pmod{101}$, or

$$5 + \log_3 87 \equiv 5 \cdot \log_3 2 \pmod{100}.$$

Adding the 2nd and 3rd equations and subtracting the 1st, we derive $4 \cdot \log_3 2 \equiv 16 \pmod{100}$. So $\log_3 2$ is 4, 29, 54, or 79. Trying all shows that $\log_3 2 = 29$. The last equation gives $\log_3 87 = 40$.

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Additional Public-key Schemes

- **Goldwasser-Micali** based on deciding quadratic residuosity problem. (first scheme proven to be CPA-secure)
- **Rabin**: based on the computing square root problem. (security equivalent to the hardness of factoring)
- **Paillier**: based on the decisional composite residuosity problem. (efficient and homomorphic)
- **Elliptic curve**: forms a cyclic group with DH problem (efficient).

Quadratic Residues Modulo a Prime

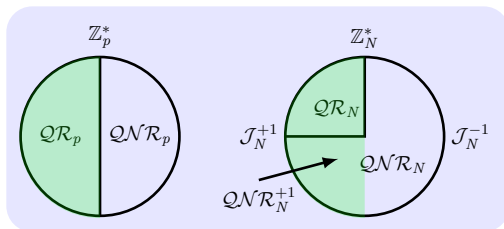
- $y \in \mathbb{G}$ is a **quadratic residue (qr)** if $\exists x \in \mathbb{G}$ with $x^2 = y$. Otherwise, y is a **quadratic non-residue (qnr)**.
- In an abelian group, the set of qr forms a subgroup.
- In \mathbb{Z}_p^* , $p > 2$ is prime, every qr has two square roots.
- The set of qr/qnr is $\mathcal{QR}_p/\mathcal{QNR}_p$, $|\mathcal{QR}_p| = |\mathcal{QNR}_p| = \frac{p-1}{2}$.
- $\mathcal{J}_p(x)$ is **Jacobi symbol** of x modulo p :

$$\mathcal{J}_p(x) \stackrel{\text{def}}{=} \begin{cases} +1 & \text{if } x \text{ is a qr} \\ -1 & \text{if } x \text{ is not a qr} \end{cases}$$

- $\mathcal{J}_p(x) = x^{\frac{p-1}{2}} \bmod p$.
- $\mathcal{J}_p(xy) = \mathcal{J}_p(x) \cdot \mathcal{J}_p(y)$.

Quadratic Residues Modulo a Composite

- $N = pq$, p, q distinct primes, in Chinese Remainder Theorem:
 $x \in \mathbb{Z}_N^*$ with $x \leftrightarrow (x_p, x_q) = ([x \bmod p], [x \bmod q])$.
- x is a qr mod $N \iff x_p/x_q$ are qr mod p/q .
- x is a qr mod $N \iff \mathcal{J}_p(x) = \mathcal{J}_q(x) = +1$.
- Qr x has 4 roots: $(\pm x_p, \pm x_q)$, so $\frac{|\mathcal{QR}_N|}{|\mathbb{Z}_N^*|} = \frac{|\mathcal{QR}_p| |\mathcal{QR}_q|}{|\mathbb{Z}_N^*|} = \frac{1}{4}$.
- $\mathcal{J}_N(x) \stackrel{\text{def}}{=} \mathcal{J}_p(x) \cdot \mathcal{J}_q(x)$. $\mathcal{J}_N(xy) = \mathcal{J}_N(x) \cdot \mathcal{J}_N(y)$.
- $\mathcal{QNR}_N^{+1}(x) \stackrel{\text{def}}{=} \{x | x \text{ is qnr, but } \mathcal{J}_N(x) = +1\}$.



Goldwasser-Micali Scheme

- **Deciding quadratic residuosity (DQR)** of x , where x is randomly chosen from \mathcal{J}_N^{+1} (\mathcal{QR}_N and \mathcal{QNR}_N^{+1}).
- For DQR, no solution is better than factoring N .

Construction 13

- Gen: (N, p, q) , $z \leftarrow \mathcal{QNR}_N^{+1}$. $pk = \langle N, z \rangle$ and $sk = \langle p, q \rangle$.
- Enc: $m \in \{0, 1\}$, $x \leftarrow \mathbb{Z}_N^*$, output $c := [z^m \cdot x^2 \bmod N]$.
- Dec: If c is a qr, output 0; otherwise 1.

Goldwasser-Micali scheme is CPA-secure if DQR problem is hard.

Computing Square Roots mod a Prime

Algorithm 6: computing square root of a prime

input : Prime p ; quadratic residue $a \in \mathbb{Z}_p^*$

output: A square root of a

```
1 case  $p = 3 \bmod 4$ : return  $[a^{\frac{p+1}{4}} \bmod p]$ 
2 case  $p = 1 \bmod 4$ : let  $b$  be a qnr modulo  $p$ 
3 compute  $\ell$  and  $m$  odd with  $2^\ell \cdot m = \frac{p-1}{2}$ 
4  $r := 2^\ell, r' := 0$ 
5 for  $i = \ell$  to 1 do
6    $r := r/2, r' := r'/2$  /* maintain  $a^r \cdot b^{r'} = 1 \bmod p$  */
7   if  $a^r \cdot b^{r'} = -1 \bmod p$  then  $r' := r' + 2^\ell \cdot m$ 
   /* now  $r = m$ ,  $r'$  is even, and  $a^r \cdot b^{r'} = 1 \bmod p$  */
8 return  $[a^{\frac{r+1}{2}} \cdot b^{\frac{r'}{2}} \bmod p]$ 
```

- **Computing square roots (CSR)** of $qr \bmod N$ is **proven to be hard** if factoring N is hard.
- $N = pq$ is a **Blum integer** if $p \neq q$ and $p \equiv q \equiv 3 \bmod 4$.
- \mathcal{QR} for Blum integer can form TDP.

Construction 14

- Gen: *Blum integer* $N = pq$, $pk = N$ and $sk = \langle p, q \rangle$.
- Enc: $m \in \{0, 1\}$, $x \leftarrow \mathcal{QR}_N$, output $c := \langle [x^2 \bmod N], \text{lsb}(x) \oplus m \rangle$.
- Dec: Input $\langle c, c' \rangle$. $x = c^{1/2}$, output $\text{lsb}(x) \oplus c'$.

Rabin scheme is CPA-secure if factoring problem is hard.

- $\mathbb{Z}_N \times \mathbb{Z}_N^* \simeq \mathbb{Z}_{N^2}^*$ with $f(a, b) = [(1 + N)^a \cdot b^N \bmod N^2]$.
- $\text{Res}(N^2)$ is the set of N th residue mod N^2 : $\{(0, b) | b \in \mathbb{Z}_N^*\}$.
- **Decisional composite residuosity (DCR)** problem is to distinguish a random element of $\mathbb{Z}_{N^2}^*$ from one of $\text{Res}(N^2)$.

Construction 15

- Gen: (N, p, q) , $pk = N$ and $sk = \langle N, \phi(N) \rangle$.
- Enc: $m \in \mathbb{Z}_N$, $r \leftarrow \mathbb{Z}_N^*$, *output* $c := [(1 + N)^m \cdot r^N \bmod N^2]$.
- Dec: *output* $\left[\frac{[c^{\phi(N)} \bmod N^2] - 1}{N} \cdot \phi(N)^{-1} \bmod N \right]$.

$$c^{\phi(N)} \bmod N^2 \leftrightarrow (m, r)^{\phi(N)} = (m \cdot \phi(N), r^{\phi(N)}).$$

Paillier scheme is CPA-secure if DCR problem is hard.

Homomorphic Encryption

- **Homomorphic Encryption** with \circ : $\text{Dec}_{sk}(c_1 \circ c_2) = m_1 \circ m_2$.
- Elgamal encryption is homomorphic with \times :
 $\langle g^{y_1}, h^{y_1} \cdot m_1 \rangle \cdot \langle g^{y_2}, h^{y_2} \cdot m_2 \rangle = \langle g^{y_1+y_2}, h^{y_1+y_2} \cdot m_1 m_2 \rangle$
- Paillier scheme is homomorphic with $+$:
 $\text{Enc}_N(m_1) \cdot \text{Enc}_N(m_2) = \text{Enc}_N([m_1 + m_2 \bmod N])$.
- **Application**: voting without learning any individual votes.

$$c_i := [(1 + N)^{v_i} \cdot r^N \bmod N^2], v_i \in \{0, 1\}$$

$$c^* := [\prod_i c_i \bmod N^2], v^* = \sum_i v_i$$

- First **Fully** homomorphic with \times and $+$ by Craig Gentry in 2009.

Elliptic curve group: points with “addition” operation.

Any **elliptic curve** is a plane algebraic curve:

$$y^2 \equiv x^3 + Ax + B \pmod{p}$$

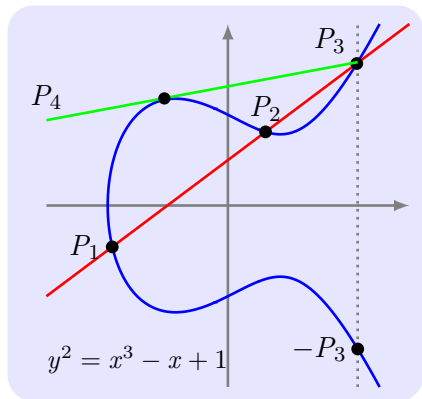
where $A, B \in \mathbb{Z}_p$ are constants with $4A^3 + 27B^2 \not\equiv 0 \pmod{p}$.

$\hat{E}(\mathbb{Z}_p)$ is the set of pairs $(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p$:

$$\hat{E}(\mathbb{Z}_p) \stackrel{\text{def}}{=} \{(x, y) \mid x, y \in \mathbb{Z}_p \wedge y^2 \equiv x^3 + Ax + B \pmod{p}\}$$

$E(\mathbb{Z}_p) \stackrel{\text{def}}{=} \hat{E}(\mathbb{Z}_p) \cup \{\mathcal{O}\}$, \mathcal{O} is identity, “**point at infinity**”.

"Addition" on Points of Elliptic Curves



Every line intersects the curve in 3 points:

- count twice if tangent.
- count \mathcal{O} at the vertical infinity of y -axis.

"Addition" on points:

- $P + \mathcal{O} = \mathcal{O} + P = P$.
- If P_1, P_2, P_3 are co-linear, then $P_1 + P_2 + P_3 = \mathcal{O}$.

$$-P = (x, -y)$$

$$P_1 + P_2 = -P_3$$

$$2P_4 = -P_3$$

$$dP = P + (d-1)P$$

$$sk = (P, d); pk = (P, Q = dP)$$

Key Size Comparison

Key lengths (in bits) with comparable security

Symmetric	RSA/DH	ECC
56	512	112
80	1024	160
112	2048	224
128	3072	256
192	7680	384
256	15360	521