Theoretical Constructions of Pseudorandom Objects

Yu Zhang

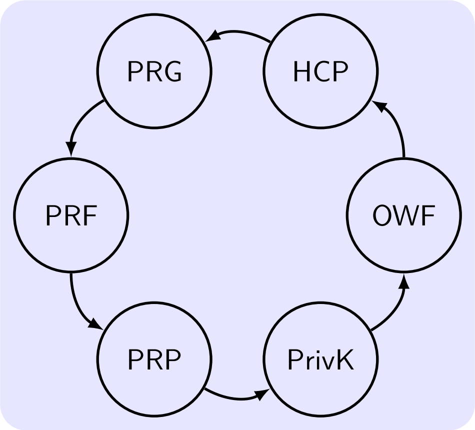
HIT/CST/NIS

Cryptography, Spring, 2014

Outline

Content

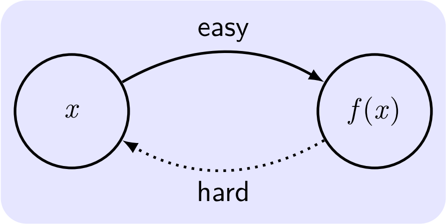
Overview



One of contributions of modern cryptography

The existence of one-way functions is equivalent to the existence of all (non-trivial) private-key cryptography.

One-Way Functions (OWF)



The inverting experiment $\mathsf{Invert}_{\mathcal{A},f}(n)$:

1. Choose input $x \leftarrow \{0,1\}^n$. Compute $y := f(x)$.
2. $\mathcal{A}$ is given 1^n and y as input, and outputs x'.
3. $\mathsf{Invert}_{\mathcal{A},f}(n) = 1$ if $f(x') = y$, otherwise 0.

Definitions of OWF/OWP

For polynomial-time algorithm M_f and $\mathcal{A}$.

Definition 1

A function $f : \{0,1\}^* \to \{0,1\}^*$ is **one-way** if:

1. (Easy to compute): $\exists\ M_f$: $\forall x, M_f(x) = f(x)$.
2. (Hard to invert): $\forall\ \mathcal{A}$, $\exists$ negl such that

$$\Pr[\mathsf{Invert}_{\mathcal{A},f}(n) = 1] \leq \mathsf{negl}(n).$$

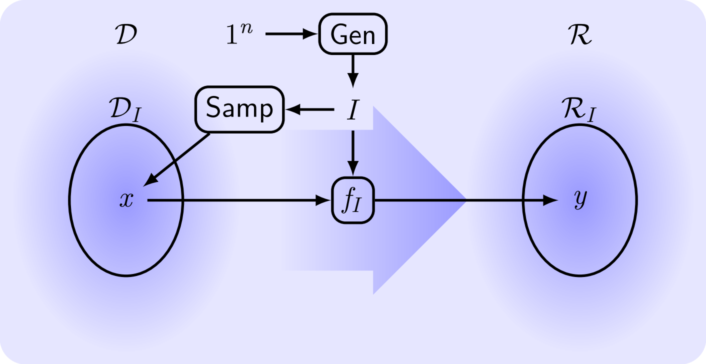
or

$$\Pr_{x \leftarrow \{0,1\}^n}[\mathcal{A}(f(x)) \in f^{-1}(f(x))] \leq \mathsf{negl}(n).$$

Definition 2

Let $f : \{0,1\}^* \to \{0,1\}^*$ be length-preserving, and f_n be the restriction of f to the domain $\{0,1\}^n$. A OWP f is a **one-way permutation** if $\forall n$, f_n is a bijection.

Families of Functions



Definition 3

$\Pi = (\mathsf{Gen}, \mathsf{Samp}, \mathsf{f})$ is a **family of functions** if:

1. **Parameter-generation** algorithm: $I \leftarrow \mathsf{Gen}(1^n)$.
2. **sampling** algorithm: $x \leftarrow \mathsf{Samp}(I)$.
3. The deterministic **evaluation** algorithm: $y := f_I(x)$.

Families of OWF and OWP

The inverting experiment $\mathsf{Invert}_{\mathcal{A},\Pi}(n)$:

1. $\mathsf{Gen}(1^n)$ obtains I, $\mathsf{Samp}(I)$ obtains a random $x \leftarrow \mathcal{D}_I$. $y := f_I(x)$.
2. $\mathcal{A}$ is given I and y as input, and outputs x'.
3. $\mathsf{Invert}_{\mathcal{A},\Pi}(n) = 1$ if $f_I(x') = y$, and 0 otherwise.

Definition 4

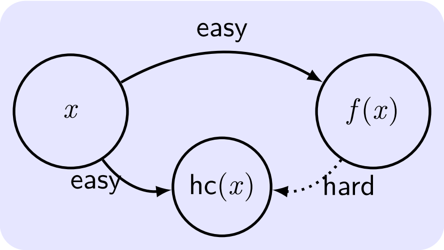
a function/permutation family Π is **one-way** if $\forall$ PPT $\mathcal{A}$, $\exists$ negl such that

$$\Pr[\mathsf{Invert}_{\mathcal{A},\Pi}(n) = 1] \leq \mathsf{negl}(n).$$

Candidate One-Way Function

- **Multiplication and factoring**:
  $f_{\mathsf{mult}}(x, y) = (xy, \|x\|, \|y\|)$, x and y are equal-length primes.
- **Modular squaring and square roots**:
  $f_{\mathsf{square}}(x) = x^2 \bmod N$.
- **Discrete exponential and logarithm**:
  $f_{g,p}(x) = g^x \bmod p$.
- **Subset sum problem**:
  $f(x_1, \ldots, x_n, J) = (x_1, \ldots, x_n, \sum_{j \in J} x_j)$.
- **Cryptographically secure hash functions**:
  Practical solutions for one-way computation.

Hard-Core Predicates (HCP)



Definition 5

A function hc : $\{0,1\}^* \to \{0,1\}$ is a **hard-core predicate of a function** f if (1) hc can be computed in polynomial time, and (2) $\forall$ PPT $\mathcal{A}$, $\exists$ negl such that

$$\Pr_{x \leftarrow \{0,1\}^n}[\mathcal{A}(f(x)) = \mathsf{hc}(x)] \leq \frac{1}{2} + \mathsf{negl}(n).$$

Content

A HCP for Any OWF

Theorem 6

f is OWF. Then $\exists$ an OWF g along with an HCP gl *for g. If f is a permutation then so is g.*

$g(x, r) \stackrel{\text{def}}{=} (f(x), r)$, for $|x| = |r|$, and define

$$\mathsf{gl}(x, r) \stackrel{\text{def}}{=} \bigoplus_{i=1}^{n} x_i \cdot r_i.$$

r is a random subset of $\{1, \ldots, n\}$. [Goldreich and Levin]

Constructing PRG from OWP

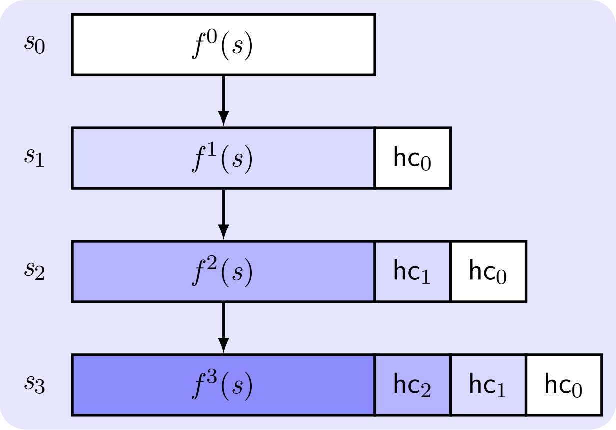
Theorem 7

f is an OWP and hc *is an HCP of f. Then $G(s) \stackrel{\text{def}}{=} (f(s), \mathsf{hc}(s))$ constitutes a PRG with expansion factor $\ell(n) = n + 1$.*

Theorem 8

If $\exists$ a PRG with expansion factor $\hat{\ell}(n) = n + 1$, then $\forall$ polynomial $p(n) > n$, $\exists$ a PRG with expansion factor $\ell(n) = p(n)$.

Blum-Micali Generator



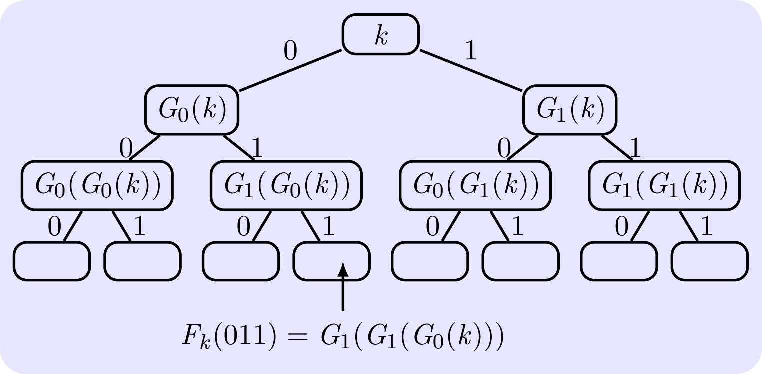
$$G(s) = (f^{p'(n)}(s), \mathsf{hc}_{[p'(n)-1]}(f^{[p'(n)-1]}(s)), \ldots, \mathsf{hc}_0(s)),$$

is a PRG with expansion factor $p(n) = n + p'(n)$.

Constructing PRF from PRG

Theorem 9

If ∃ a PRG with expansion factor $\ell(n) = 2n$, then ∃ a PRF.



$$F_k(x_1 x_2 \cdots x_n) = G_{x_n}(\cdots(G_{x_2}(G_{x_1}(k)))\cdots),\ G(s) = (G_0(s), G_1(s)).$$

Constructing PRP from PRF

$F^{(r)}$ is an r-round Feistel network with the mangler function F.

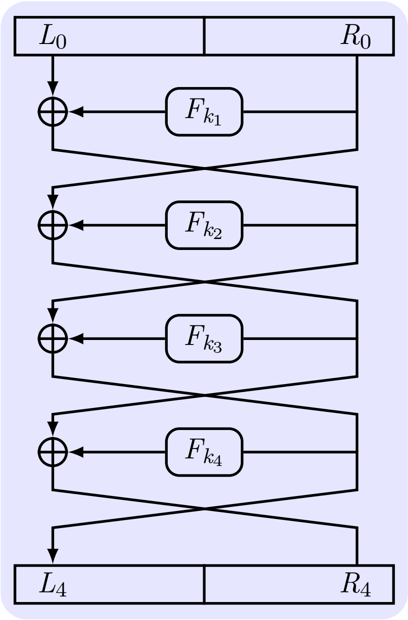
Theorem 10

If F is a length-preserving PRF, then $F^{(3)}$ is a PRP that maps $2n$-bit strings to $2n$-bit strings (and uses a key of length $3n$).

Theorem 11

If F is a length-preserving PRF, then $F^{(4)}$ is a strong PRP that maps $2n$-bit strings to $2n$-bit strings (and uses a key of length $4n$).

A Four-Round Feistel Network



Necessary Assumptions

Theorem 12

Assume that $\exists$ OWP. Then $\exists$ PRG, PRF, strong PRP, CCA-secure private-key encryption schemes, and secure MAC.

Proposition 13

If $\exists$ a private-key encryption scheme that has indistinguishable encryptions in the presence of an eavesdropper, then $\exists$ an OWF.

Proof.

$f(k, m, r) \stackrel{\text{def}}{=} (\mathsf{Enc}_k(m, r), m)$. $\square$

Summary

- OWF implies secure private-key encryption scheme and MAC.
- Secure private-key encryption scheme/MAC implies OWF.