Private-Key Encryption and Pseudorandomness (Part I)

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Outline

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Idea of Computational Security

Computational security vs. Information-theoretical security

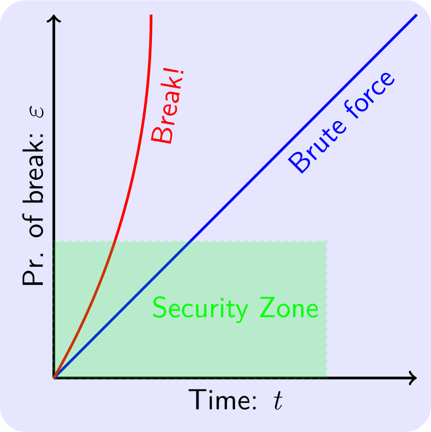
Kerckhoffs's Another Principle

A [cipher] must be practically, if not mathematically, indecipherable.

- Information-theoretical security: Perfect secrecy.
- Computational security:
  - Only preserved against adversaries that run in a **feasible amount of time**.
  - Adversaries can succeed with some **very small probability**.

Necessity of the Relaxations

Limit the power of adversary (against brute force with pr. 1 in time linear in $|\mathcal{K}|$) and allow a negligible probability (against random guess with pr. $1/|\mathcal{K}|$).



Concrete Approach

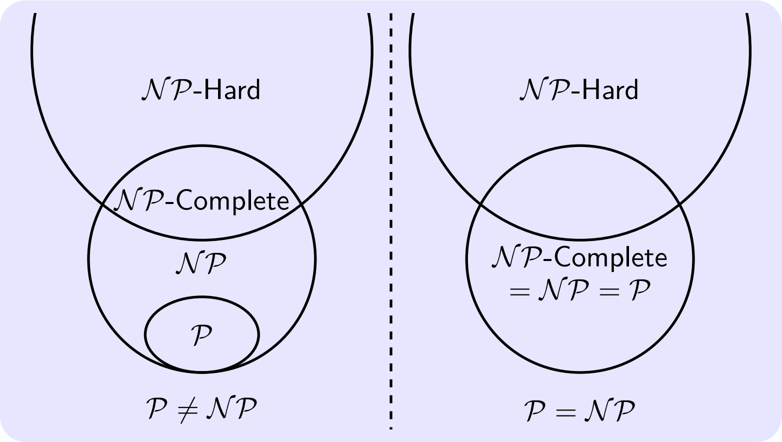
A scheme is (t, ε)-**secure** if every adversary running for time at most t succeeds in breaking the scheme with probability at most ε.

Example

Optimal security: when the key has length n, an adversary running in time t can succeed with probability at most $t/2^n$.

$t = 2^{80}$	2^{20} 1GHz CPUs run 35 years.
$n = 128$	2^{170} atoms in the planet.
$\varepsilon = 2^{-48}$	once every 100 years with probability 2^{-30}/sec.

- But one may ask: What type of computing power? What if under Moore's Law? Which kind of implementation? What is the success probability of running for 10 years? Does the key length matter?



The majority of computer scientists believe $\mathcal{P} \neq \mathcal{NP}$.

This is very dangerous!

Asymptotic Approach

Time t and probability ε are described as functions of **security parameter** n (usually, the length of key).

- Time is PPT (**probabilistic polynomial-time**): n^a for constant a.
- Probability is **negligible**: smaller than any inverse polynomial n^{-b} for constant b.
- Caution: 'Security' for large enough values of n.

Example

"Breaking the scheme" with probability $2^{40} \cdot 2^{-n}$ in n^3 minutes.

$n \leq 40$	6 weeks with probability 1.
$n = 50$	3 months with probability $1/1000$.
$n = 500$	more than 200 years with probability 2^{-500}.

Efficient Computation

- An algorithm A runs in **polynomial time** if there exists a polynomial $p(\cdot)$ such that, for every input $x \in 0, 1^*$, $A(x)$ terminates within at most $p(|x|)$ steps.
- A **probabilistic** algorithm has the capability of "tossing coins".
- A can run another $\textsc{ppt}$ A' as a sub-routine in polynomial-time.
- Random number generators should be designed for cryptographic use, not `random()` in C.
- Open question: Does probabilistic adversaries are more powerful than deterministic ones?

Negligible Success Probability

- A function f is **negligible** if for every polynomial $p(\cdot)$ there exists an N such that for all integers $n > N$ it holds that $f(n) < \frac{1}{p(n)}$.
- $\mathsf{negl}_3(n) = \mathsf{negl}_1(n) + \mathsf{negl}_2(n)$ is negligible.
- positive polynomial p, $\mathsf{negl}_4(n) = p(n) \cdot \mathsf{negl}_1(n)$ is negligible.
- Problem X is *hard* if X cannot be solved by any polynomial-time algorithm except with negligible probability.

Remarks on The Asymptotic Approach

- The longer the key, the higher the security.
- Increasing n to defend against increases in computing power.
- Convention: algorithm input is 1^n. (a string of n 1's)

Example

	Honest party	Adversary	
	$10^6 \cdot n^2$	$10^8 \cdot n^4$	2^{20-n}
1GHz $n = 50$	2.5 sec.	1 week	2^{-30}
16GHz $n = 500$	0.625 sec.	16 week	2^{-80}

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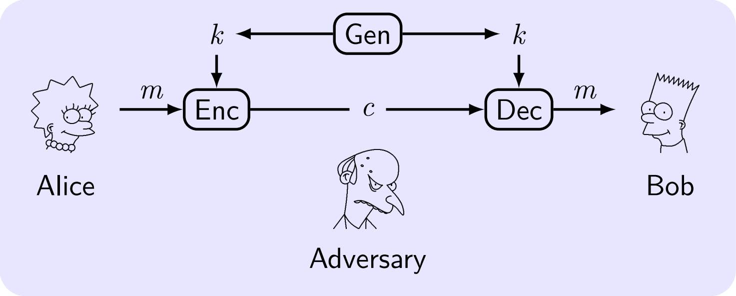
Defining Private-key Encryption Scheme



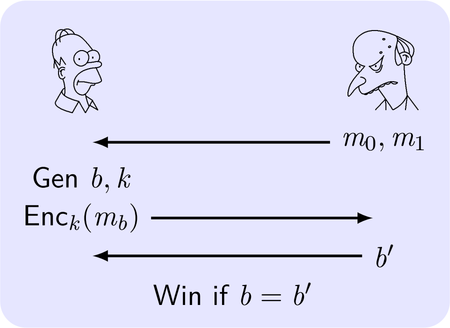
A **Private-key encryption scheme** Π is a tuple of $\textsc{ppt}$ $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$

- $k \leftarrow \mathsf{Gen}(1^n), |k| \geq n$ (security parameter).
  $\mathsf{Gen}(1^n)$ chooses $k \leftarrow \{0,1\}^n$ uniformly at random (***u.a.r***).
- $c \leftarrow \mathsf{Enc}_k(m), m \in \{0,1\}^*$ (all finite-length binary strings).
  Fixed-length if $m \in \{0,1\}^{\ell(n)}$.
- $m := \mathsf{Dec}_k(c)$.
- $\mathsf{Dec}_k(\mathsf{Enc}_k(m)) = m$.

Eavesdropping Indistinguishability Experiment

The eavesdropping indistinguishability experiment $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n)$:

1. $\mathcal{A}$ is given input 1^n, outputs m_0, m_1 of the same length.
2. $k \leftarrow \mathsf{Gen}(1^n)$, a random bit $b \leftarrow \{0,1\}$ is chosen. Then $c \leftarrow \mathsf{Enc}_k(m_b)$ (challenge ciphertext) is given to $\mathcal{A}$.
3. $\mathcal{A}$ outputs b'. If $b' = b$, $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1$, otherwise 0.



Defining Private-key Encryption Security

Definition 1

Π has **indistinguishable encryptions in the presence of an eavesdropper** if $\forall$ PPT $\mathcal{A}$, $\exists$ a negligible function negl such that

$$\Pr\left[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1\right] \leq \frac{1}{2} + \mathsf{negl}(n),$$

where the probability it taken over the random coins used by $\mathcal{A}$.

Properties of Definition of Indistinguishable

1. **No single bit** can be guessed in a randomly-chosen plaintext with probability significantly better than $1/2$.

$$\Pr[\mathcal{A}(1^n, \mathsf{Enc}_k(m)) = m^i] \leq \frac{1}{2} + \mathsf{negl}(n).$$

where m^i is the ith bit of m, $\mathcal{A}(\cdot)$ outputs the guess.

2. **No function of plaintext** can be learned regardless of the *a priori* distribution over the plaintext.
$\forall\ \mathcal{A}$, $\exists\ \mathcal{A}'$, $\forall\ f$ and $\forall\ S \in \{0,1\}^*$,

$$\left|\Pr[\mathcal{A}(1^n, \mathsf{Enc}_k(m)) = f(m)] - \Pr[\mathcal{A}'(1^n) = f(m)]\right| \leq \mathsf{negl}(n),$$

where $m \in S_n \stackrel{\mathsf{def}}{=} S \cap \{0,1\}^n$.

Semantic Security

Intuition: No partial information leaks.

Definition 2

Π is **semantically secure in the presence of an eavesdropper** if $\forall$ PPT $\mathcal{A}$, $\exists \mathcal{A}'$ such that $\forall$ distribution $X = (X_1, \dots)$ and $\forall f, h$,

$$|\Pr[\mathcal{A}(1^n, \mathsf{Enc}_k(m), h(m)) = f(m)] - \Pr[\mathcal{A}(1^n, h(m)) = f(m)]|$$

$$\leq \mathsf{negl}(n).$$

where m is chosen according to X_n, $h(m)$ is external information.

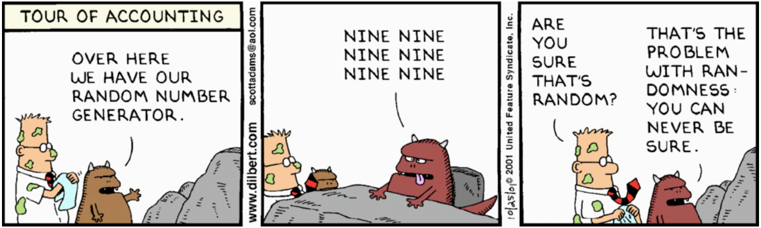
Theorem 3

*A private-key encryption scheme has **indistinguishable** encryptions in the presence of an eavesdropper $\iff$ it is **semantically secure** in the presence of an eavesdropper.*

Content

Conceptual Points of Pseudorandomness

- True randomness can not be generated by a describable mechanism.
- Pseudorandom looks truly random for the observers who don't know the mechanism.
- No fixed string can be "pseudorandom" which refers to a distribution.
- It is impossible to definitively prove randomness.



Distinguisher: Statistical Tests

The pragmatic approach is to take many sequences of random numbers from a given generator and subject them to a battery of statistical tests.[1]

- $D(x) = 0$ if $|\#0(x) - \#1(x)| \leq 10 \cdot \sqrt{n}$
- $D(x) = 0$ if $|\#00(x) - n/4| \leq 10 \cdot \sqrt{n}$
- $D(x) = 0$ if max-run-of-$0(x) \leq 10 \cdot \log n$

[1] State-of-the-art: NIST Special Publication 800-22 "*A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic Applications*"

Intuition for Defining Pseudorandom

Intuition: Generate a long string from a short truly random seed, and the pseudorandom string is indistinguishable from truly random strings.

choose a truly random string or generate a pseudorandom one

Distinguisher D

the string $\longrightarrow$

$\longleftarrow$ 1 or 0

Definition of Pseudorandom Generators

Definition 4

A deterministic polynomial-time algorithm $G : \{0,1\}^n \to \{0,1\}^{\ell(n)}$ is a **pseudorandom generator (PRG)** if

1. (Expansion:) $\forall n, \ell(n) > n$.
2. (Pseudorandomness): $\forall$ PPT distinguishers D,

$$|\Pr[D(r) = 1] - \Pr[D(G(s)) = 1]| \leq \mathsf{negl}(n),$$

where r is chosen *u.a.r* from $\{0,1\}^{\ell(n)}$, the **seed** s is chosen *u.a.r* from $\{0,1\}^n$. $\ell(\cdot)$ is the **expansion factor** of G.

Remarks on Pseudorandom Generators

- **Sparse outputs**: In the case of $\ell(n) = 2n$, only 2^{-n} of strings of length $2n$ occurs.
- **Brute force attack**: Given an unlimited amount of time, one can distinguish $G(s)$ from r with a high probability by generating all strings with all seeds.

$$|\Pr[D(r) = 1] - \Pr[D(G(s)) = 1]| \geq 1 - 2^{-n}$$

- **Sufficient seed space**: s must be long enough against brute force attack.
- **Existence**: Under the weak assumption that *one-way functions* exists, or $\mathcal{P} \neq \mathcal{NP}$.

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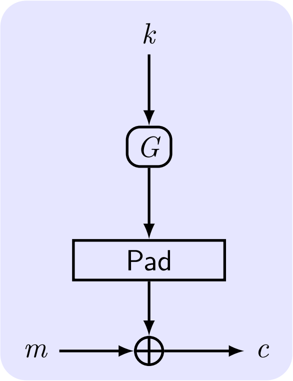
A Secure Fixed-Length Encryption Scheme



Construction 5

- $|G(k)| = \ell(|k|)$, $m \in \{0,1\}^{\ell(n)}$.
- Gen: $k \in \{0,1\}^n$.
- Enc: $c := G(k) \oplus m$.
- Dec: $m := G(k) \oplus c$.

Theorem 6

This fixed-length encryption scheme has indistinguishable encryptions in the presence of an eavesdropper.

Reduction (Complexity)

A **reduction** is a transformation of one problem A into another problem B.

Reduction $A \leq_m B$ [2] : A is **reducible** to B if solutions to B exist and whenever given the solutions A can be solved.

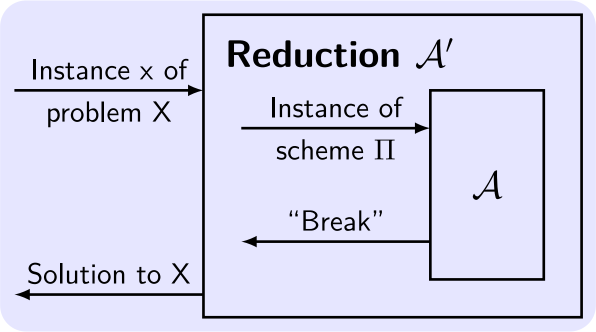
Solving A **cannot be harder** than solving B.

Example

- "measure the area of a rectangle" $\leq_m$ "measure the length and width of rectangle"
- "calculate x^2" $\leq_m$ "calculate $x \times y$"

[2] m means the mapping reduction.

Proofs of Reduction

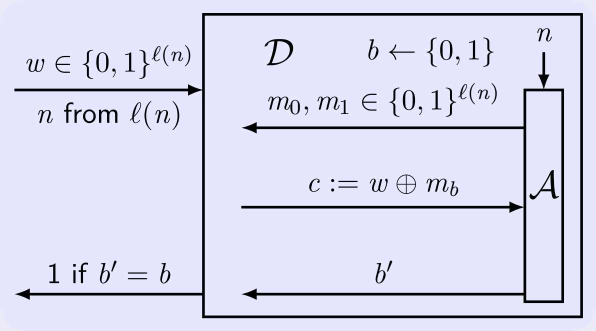


- A PPT $\mathcal{A}$ can break Π with probability $\varepsilon(n)$.
- **Assumption**: Problem X is *hard* to solve.
- **Reduction**: Reduce $\mathcal{A}'$ to $\mathcal{A}$. $\mathcal{A}'$ solves x efficiently with probability $1/p(n)$, running $\mathcal{A}$ as a sub-routine.
- **Contradiction**: If $\varepsilon(n)$ is non-negligible, then $\mathcal{A}'$ solves X efficiently with non-negligible probability $\varepsilon(n)/p(n)$.

Proof of Indistinguishable Encryptions

Idea: Use $\mathcal{A}$ to construct D for G, so that D distinguishes G when $\mathcal{A}$ breaks $\tilde{\Pi}$. Since D cannot distinguish G, so that $\mathcal{A}$ cannot break $\tilde{\Pi}$.

Proof.



$$\Pr[D(w) = 1] = \Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\tilde{\Pi}}(n) = 1]$$

□

Proof of Indistinguishable Encryptions (Cont.)

Proof.

To prove $\varepsilon(n) \stackrel{\text{def}}{=} \Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1] - \frac{1}{2}$ is negligible.

(1) If w is r chosen *u.a.r*, then $\tilde{\Pi}$ is OTP.

$$\Pr[D(r) = 1] = \Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\tilde{\Pi}}(n) = 1] = \frac{1}{2};$$

(2) If w is $G(k)$, then $\tilde{\Pi} = \Pi$.

$$\Pr[D(G(k)) = 1] = \Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1] = \frac{1}{2} + \varepsilon(n).$$

Use Definition 4:

$$|\Pr[D(r) = 1] - \Pr[D(G(k)) = 1]| = \varepsilon(n) \leq \mathsf{negl}(n).$$

$\square$

Handling Variable-Length Messages

Definition 7

A **deterministic** polynomial-time algorithm G is a **variable output-length pseudorandom generator** if

1. $G(s, 1^\ell)$ outputs a string of length $\ell > 0$, where s is a string.
2. $G(s, 1^\ell)$ is a prefix of $G(s, 1^{\ell'})$, $\ell' > \ell$.[3]
3. $G_\ell(s) \stackrel{\text{def}}{=} G(s, 1^{\ell(|s|)})$. Then $\forall \ell(\cdot)$, G_ℓ is a PRG with expansion factor ℓ.

Both Construction 5 and Theorem 6 hold here.

[3] for technical reasons to prove security.

Computational Security vs. Info.-theoretical Security

	Computational	**Info.-theoretical**
Adversary	PPT eavesdropping	no limited eavesdropping
Definition	indistinguishable $\frac{1}{2} + \mathsf{negl}$	indistinguishable $\frac{1}{2}$
Assumption	pseudorandom	random
Key	short random str.	long random str.
Construction	XOR pad	XOR pad
Prove	reduction	-