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Mathematical models for Multi Container Loading Problems with practical constraints

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A R T I C L E I N F O

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A B S T R A C T

We address the multi container loading problem of a company that serves its customers' orders by building pallets with the required products and loading them into trucks. The problem is solved by using integer linear models. To be useful in practice, our models consider three types of constraints: geometric constraints, so that pallets lie completely inside the trucks and do not overlap; weight constraints, defining the maximum weights supported by a truck and by each axle, as well as the position of the centre of gravity of the cargo; and dynamic stability constraints. These last constraints forbid empty spaces between pallets to avoid cargo displacement when the truck is moving, and limit differences between the heights of adjacent pallets to prevent tall pallets tipping over short ones. We also consider extensions of the models to the case of heavy loads, requiring a special configuration of the pallets in the truck, and to the case in which the orders must be served over a set of time periods to meet delivery dates. The computational study that we performed on a large number of real instances with up to 44 trucks shows that the proposed models are able to obtain optimal solutions in most cases and very small gaps when optimality could not be proven.

1. Introduction

The Multi Container Loading Problem (MCLP) consists in loading a set of products into the minimum number of containers, while satisfying different types of constraints. Among the many variants of the MCLP, the real case inspiring this study is the problem of a distribution company that serves its customers' orders by first building pallets with the required products and then putting them into trucks. Achieving an optimal solution to this problem, that is, minimizing the number of trucks, has many economic and environmental benefits.

In the first phase of the loading process, the products are put on pallets. Items of the same product are arranged together forming a rectangle, called a layer, covering the pallet base horizontally. The composition and dimensions of the layers of each product type have been previously decided, so the problem is to stack layers for building pallets. These pallets are then placed in the trucks. The company uses a set of identical trucks, large enough to fill all the orders.

Road transportation is severely regulated for safety reasons, introducing specific constraints into the problem. The total weight a truck can carry and the weight each axle can support are strictly limited. Excesses put traffic safety at risk and damage the roads and are therefore controlled and penalized. Moreover, the weight of the cargo needs to be evenly distributed over the truck floor so that its center of gravity is as close as possible to the center of the truck and never behind the rear axle.

To be useful in practice, solutions to the MCLP have to be stable. On the one hand, static (or vertical) stability, concerning the equilibrium of the items when the vehicle is not moving, must be guaranteed to avoid risks in the loading and unloading processes. On the other hand, dynamic (or horizontal) stability, preventing the items being displaced during the journey when the vehicle accelerates, brakes, and turns must also be ensured.

In recent years, an increasing number of studies have considered some of these practical constraints when solving Single and Multi Container Loading Problems. Bortfeldt and Wäscher's review (Bortfeldt & Wäscher, 2013) points out that constraints related to weight and static stability have been included in many studies on single container loading problems, but dynamic stability has received less attention so far. The multi container case has been less studied and more research is needed on problems with different types of containers and realistic

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constraints, as concluded in the survey by Zhao, Bennell, Betkas, and Dowsland (2016). In most cases, heuristic and metaheuristic approaches have been followed, while mathematical models and exact algorithms have usually addressed only basic problems.

Our proposal is to solve practical multicontainer loading problems by means of Integer Linear Programming (ILP) models. They are flexible for adding and removing constraints in order to meet the specific conditions of the problem being solved, they are easy to implement and in the last few years the available solvers have shown very good computational performance, as also confirmed by our own results in this study. Several integer linear models have been proposed for single container loading problems (Huang, Hwang, & Lu, 2016; Junqueira, Morabito, & Yamashita, 2012), but results for the multi-container case are still very scarce.

Starting from a model we developed in a previous study (Alonso, Alvarez-Valdes, Iori, Parreño, & Tamarit, 2017), already including constraints on the weight, on the axle support, and on the position of the center of gravity, we now focus on dynamic stability constraints, which have not been considered in previous integer models for container loading problems. First, we add constraints preventing empty spaces between pallets, to avoid cargo displacement when the truck is moving. Then, we consider several alternatives to limit excessive differences between the heights of adjacent pallets and prevent a tall pallet tipping over a short one, and discuss their relative advantages. We also study extensions of the models to the case of heavy cargoes, which requires a specific configuration of the pallets in the truck, and to the case, which arises frequently in practice in companies like the one in this study, in which the orders have to be served over a set of time periods to meet delivery dates. For this extension, two alternatives have been studied and compared in terms of flexibility and computational complexity. These extensions have not previously been addressed in container loading problems.

A distribution company has provided us with 111 real instances on which the models have been tested. Optimal solutions are achieved for most of the instances with short computing times, as the results show. When optimal solutions are not reached, there are usually very small gaps, and therefore we can conclude that the models we have developed are able to obtain high quality solutions for this problem.

The remainder of the paper is structured as follows. Related existing research is reviewed in Section 2. The MCLP is formally described in Section 3. The test instances are presented in Section 4 and for each instance upper and lower bounds are calculated. In Section 5, the initial model is presented, including the main characteristics of the problem as well as several types of weight constraints. In Section 6, we consider dynamic stability and propose alternative ways of dealing with this issue. Section 7 studies the special case of heavy loads in which a specific distribution of pallets is needed, while Section 8 extends the model to the case of orders to be served over a set of periods. Section 9 contains the conclusions.

2. Previous work

The Single Container Loading Problem (SCLP) is a classical problem in Cutting and Packing and has been studied from many different perspectives. Besides the geometric constraints, realistic constraints, like those considered in this study, have received increasing attention in recent years. In 1995, Bischoff and Ratcliff (1995) listed twelve conditions to be considered when feasible loading plans are to be constructed. More recently, Bortfeldt and Wäscher (2013) have written a complete review of the container loading problem and its practical constraints. Zhao et al. (2016) have also reviewed the use of these constraints, basically in the single container problem.

Some of these practical constraints are related to weight. There is a maximum weight that can be loaded into the container (Bortfeldt, Gehring, & Mack, 2003; Egeblad, Garavelli, Lisi, & Pisinger, 2010), and in trucks with several axles the maximum weight that each axle can bear is also limited (Lim, Ma, Qiu, & Zhu, 2013; Pollaris, Braekers, Caris, Janssens, & Limbourg, 2016). For heavy cargoes, weight is the most restrictive constraint, even more than the space occupied. Moreover, the weight of the cargo has to be evenly spread over the container floor. A good weight distribution requires the center of gravity of the load to be as close as possible to the geometric center of the container floor and never behind the rear axle (Bortfeldt & Gehring, 2001).

Another very common constraint concerns the orientation of the items. Although in some cases the orientation is not restricted (Parreño, Alvarez-Valdes, Tamarit, & Oliveira, 2008), it is more usual that only one vertical orientation is permitted, while 90° rotations of items in the horizontal plane are allowed (Iori & Martello, 2010). Sometimes, items cannot be rotated at all (Junqueira et al., 2012).

Stackability or load-bearing constraints are used to prevent items located below other items being damaged due to the excessive weight above them. They can be imposed by putting a limit on the number of items on top of any other item (Bischoff & Ratcliff, 1995), by prohibiting other items being placed on top of a fragile item (Terno, Scheithauer, Sommerweiss, & Riehme, 2000), or by limiting the maximum weight an item can support per unit area (Alonso, Alvarez-Valdes, Parreño, & Tamarit, 2014; Junqueira et al., 2012). Load-bearing constraints, as well as other problem-specific conditions, have also been considered by Toffolo, Esprit, Wauters, and Vanden Berghe (2017) and Correcher, Alonso, Parreño, and Alvarez-Valdes (2017) in the solution of a multi container problem arising in the distribution centers of a large automotive company.

Another family of constraints that is receiving increasing attention due to their practical importance is related to the stability of the cargo. In order to be useful in practice, the cargo distribution has first to be vertically or statically stable, so the loaded items do not fall when the truck is not moving (Ramos, Oliveira, Gonçalves, & Lopes, 2016). This condition is usually considered by imposing full base support, whereby the base of each item is completely supported by other items (Araujo & Armentano, 2007; Fanslau & Bortfeldt, 2010), or partial base support, when only a given percentage of the base has to be supported (Jin, Ohno, & Du, 2004; Junqueira et al., 2012). Then, the cargo also has to be horizontally or dynamically stable, so that the items do not move when the vehicle is moving. Displacements of items inside the truck during the journey can damage the products being transported (Ramos, Oliveira, Gonçalves, & Lopes, 2015).

However, the literature on problems similar to that studied here, considering pallet building and truck loading together and including practical constraints, is relatively scarce. According to the typology introduced by Wäscher, Haußner, and Schumann (2007) for cutting and packing problems, the two problems can be classified as Single Stock Size Cutting Stock Problems. Morabito, Morales, and Widmer (2000) solve a two-dimensional pallet and truck loading problem in which items cannot be stacked in a two-phase approach, first building pallets and then loading trucks using Morabito and Morales (1998) 5-block algorithm. Takahara (2005) solves the problem using two lists: an ordered list of items and an ordered list of pallets and containers into which the items have to be loaded. These lists are handled by applying several metaheuristic procedures. Doerner, Fuellerer, Gronalt, Hartl, and Iori (2007) study a vehicle routing problem in which the products to be served are first put on pallets and then the pallets are stacked, producing piles, and develop a Tabu Search and an Ant Colony Optimization algorithm. In the vehicle routing problem studied by Pollaris et al. (2016) the items are loaded forming homogeneous pallets that are placed in the truck in two rows but cannot be stacked. They propose an MIP formulation to minimize transportation costs, considering axle weight and multidrop constraints. The problem solved by Moura and Bortfeldt (2017) also combines vehicle routing and container loading. First, homogeneous pallets are built by adapting the Moura and Oliveira GRASP algorithm Moura and Oliveira, 2005 and then pallets are placed in the trucks using a tree search procedure. Sheng, Hongxia, Xisong, and Changjian (2016) load pallets into containers but allow the container to be filled up with individual boxes, using a tree search procedure to load the pallets and then a greedy algorithm for filling the residual spaces.

3. Description of problem

The order sent by a customer to the distribution center is composed of a list of products $j \in J$ that have to be completely served. Each product has a predetermined layer layout, with dimensions (l_j, w_j, h_j), so the demand of each product is expressed as a number of required layers n_j, each with a weight q_j. The company uses just one type of pallet with dimensions (l^p, w^p, h^p) and weight q^p. Layers are piled up on pallet bases.

The pallets formed by the demanded layers are then loaded into trucks. The distribution company has a set K of identical trucks, large enough to load all the pallets. Each truck has a load space of dimensions (L, W, H) and a weight Q^e. The total weight a truck can support is Q and the maximum weights supported by the front and rear axles are Q_1 and Q_2, respectively, while the distances from the front to each axle are δ_1 and δ_2.

According to the dimensions of the truck base (L, W) and the pallet base (l^p, w^p), the possible positions of the pallets on the truck floor define a set I. As we want to pack as many pallets as possible on each truck, we first check whether $\left\lfloor \frac{L}{w^p} \right\rfloor * \left\lfloor \frac{W}{l^p} \right\rfloor \geqslant \left\lfloor \frac{L}{l^p} \right\rfloor * \left\lfloor \frac{W}{w^p} \right\rfloor$. If this is the case, $|I| = \left\lfloor \frac{L}{w^p} \right\rfloor * \left\lfloor \frac{W}{l^p} \right\rfloor$, composing a grid with $\left\lfloor \frac{L}{w^p} \right\rfloor$ positions along the truck's length and $\left\lfloor \frac{W}{l^p} \right\rfloor$ positions across the truck's width. The positions are defined starting from the front and going to the back of the truck. The other case would be treated in a similar way. An example of a common grid with two pallets filling the truck width can be seen in Fig. 1, taken from Alonso et al. (2017). In the following, let (G_x, G_y) denote the coordinates of the center of the truck and let (p_i^x, p_i^y) denote the coordinates of the center of position $i \in I$.

4. A real world benchmark dataset

The benchmark set used in this study is composed of 111 real instances taken from the daily activity of a distribution company. The instances were provided by ORTEC (ORTEC, 2018), a consultancy company working on the optimization of production and logistic problems. They are available at page 3D https://www.euro-online.org/websites/esicup/data-sets/#1535975694118-eedb4714-39e4, labelled as "'Demand over the time"'. The instances, which were already used by Alonso et al. (2017) in a previous paper, show high variability. The number of product types varies from 1 to 142, and the customers' demands from 241 to 9537 layers.

For each instance a lower bound on the number of trucks required to send all the products is given by the expression:

$$L_{init} = max\left\{ \left\lceil \frac{\sum_{j\in J} q_j n_j}{Q} \right\rceil, \left\lceil \frac{\sum_{j\in J} h_j n_j}{H\,|I|} \right\rceil \right\} \tag{1}$$

The first term considers the sum of all demanded layers and divides it by the total weight supported by the truck. The second term adds up the height of the layers and divides it by the height of the truck and the number of positions on the truck floor. For the instances in the dataset, the bound based on weight tends to be larger, indicating a majority of heavy products, but this tendency is not uniform, and, in fact, there are instances with light products in which the bound based on the number of pallets is the larger.

We also computed an upper bound on the number of required trucks, called U_{init}, by using a heuristic constructive algorithm developed in Alonso et al. (2017) and adapted here to consider all the constraints described in the following sections, so that it always returns a feasible solution for the corresponding configuration. For the initial model, the upper bound ranges between 5 and 44 trucks. On the basis of the difference between these lower and upper bounds, the instances are classified into four classes (A when the difference is 0, B when it is 1, C when it is 2, and D when it is 3 or more).

5. The initial MCLP model

In this section, we review the model we developed in Alonso et al. (2017), because it will be the starting point of our new models to which we will progressively add realistic constraints.

We define variables:

$$x_{kij} = \text{number of layers of product } j \text{ packed in position } i \text{ of truck } k$$

$$y_k = \begin{cases} 1, & \text{if truck } k \text{ is used} \\ 0, & \text{otherwise} \end{cases}$$

$$z_{ki} = \begin{cases} 1, & \text{if a pallet is packed in position } i \text{ of truck } k \\ 0, & \text{otherwise} \end{cases}$$

The initial model is:

$$\text{(Initial)}\ \min \sum_{k\in K} y_k \tag{2}$$

$$\sum_{k\in K} \sum_{i\in I} x_{kij} = n_j \qquad j \in J \tag{3}$$

$$\sum_{j\in J} h_j x_{kij} + h^p z_{ki} \leqslant H' y_k \qquad k \in K,\ i \in I \tag{4}$$

$$\sum_{i\in I} (q^p z_{ki} + \sum_{j\in J} q_j x_{kij}) \leqslant Q y_k \qquad k \in K \tag{5}$$

$$\sum_{i\in I} (q^p z_{ki} + \sum_{j\in J} q_j x_{kij})(\delta_2 - p_i^x) \leqslant Q_1(\delta_2 - \delta_1) y_k \qquad k \in K \tag{6}$$

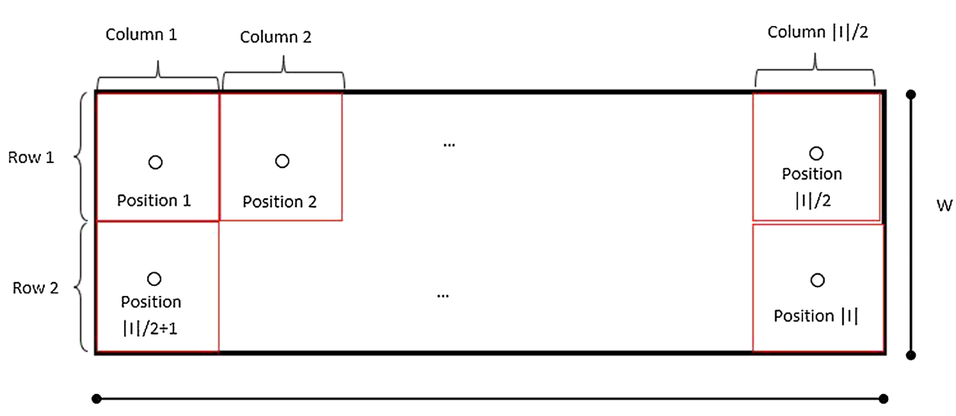


Fig. 1. Pallet positions on the truck floor.

$$\sum_{i\in I}(q^p z_{ki} + \sum_{j\in J} q_j x_{kij})(p_i^x - \delta_1) \leqslant Q_2(\delta_2 - \delta_1)y_k \qquad k \in K \tag{7}$$

$$\sum_{j\in J} h_j x_{kij} \leqslant (H' - h^p)z_{ki} \qquad k \in K,\ i \in I \tag{8}$$

$$z_{ki} \leqslant \sum_{j\in J} x_{kij} \qquad k \in K,\ i \in I \tag{9}$$

$$Q^e G_x + \sum_{i\in I} p_i^x q^p z_{ki} + \sum_{i\in I}\sum_{j\in J} p_i^x q_j x_{kij} \leqslant \left(\sum_{i\in I} q^p z_{ki} + \sum_{i\in I}\sum_{j\in J} q_j x_{kij} + Q^e\right)(G_x + \tau_1^x) \quad k \in K \tag{10}$$

$$Q^e G_x + \sum_{i\in I} p_i^x q^p z_{ki} + \sum_{i\in I}\sum_{j\in J} p_i^x q_j x_{kij} \geqslant \left(\sum_{i\in I} q^p z_{ki} + \sum_{i\in I}\sum_{j\in J} q_j x_{kij} + Q^e\right)(G_x - \tau_2^x) \quad k \in K \tag{11}$$

$$Q^e G_y + \sum_{i\in I} p_i^y q^p z_{ki} + \sum_{i\in I}\sum_{j\in J} p_i^y q_j x_{kij} \leqslant \left(\sum_{i\in I} q^p z_{ki} + \sum_{i\in I}\sum_{j\in J} q_j x_{kij} + Q^e\right)(G_y + \tau_1^y) \quad k \in K \tag{12}$$

$$Q^e G_y + \sum_{i\in I} p_i^y q^p z_{ki} + \sum_{i\in I}\sum_{j\in J} p_i^y q_j x_{kij} \geqslant \left(\sum_{i\in I} q^p z_{ki} + \sum_{i\in I}\sum_{j\in J} q_j x_{kij} + Q^e\right)(G_y - \tau_2^y) \quad k \in K \tag{13}$$

$$y_k \geqslant y_{k+1} \qquad k \in K\colon k < |K| \tag{14}$$

$$y_k = 1 \qquad k \in K\colon k \leqslant L_{init} \tag{15}$$

$$x_{kij} \geqslant 0, \text{ integer} \qquad k \in K,\ i \in I, j \in J \tag{16}$$

$$y_k \in \{0, 1\} \qquad k \in K \tag{17}$$

$$z_{ki} \in \{0, 1\} \qquad k \in K,\ i \in I \tag{18}$$

The objective function (2) minimizes the number of trucks. Constraints (3) ensure that the demand for each product is met. The height of each pallet cannot exceed the truck height, according to constraints (4). On the right-hand side we could have put H, the truck inner height, but this value can be adjusted to H' by solving the following Subset Sum Problem (SSP):

$$H' = h^p + \max\{\sum_{j\in J} h_j \xi_j, \text{ subject to } \sum_{j\in J} h_j \xi_j \leqslant (H - h^p),$$
$$0 \leqslant \xi_j \leqslant n_j \text{ and integer for } j \in J\}$$

which gives the maximum height that a subset of layers, plus the pallet base, can attain without exceeding H. Similarly, constraints (5) do not allow the sum of the weights of the pallets in a truck to exceed the total weight Q. Constraints (6) and (7) limit the weight on the front and rear axles, respectively. In these cases, the load supported by each axle depends on the position of the layer in the truck. Constraints (8) and (9) link layers and pallets bases: at each position of each truck, if there are some layers, then there must be a pallet base too, and viceversa.

For safety reasons, the center of gravity of the loaded truck has to be located as near as possible to its geometric center and never behind the rear axle. Recall that (G_x, G_y) are the coordinates of the center of the truck. We consider that a load is safely distributed if its center of gravity lies in the x-interval $[G_x - \tau_1^x, G_x + \tau_2^x]$ and in the y-interval $[G_y - \tau_1^y, G_y + \tau_2^y]$, where τ are input tolerance parameters fixed by the user. Constraints (10)–(13) ensure that the load is feasible with respect to these tolerances. Constraints (14) sort the trucks, so truck $k + 1$ can only be used if truck k is also used, whereas constraints (15) set to 1 variables associated with trucks whose index is lower than or equal to the initial lower bound. Finally, constraints (16)–(18) define the domain of the variables.

The model was tested on the benchmark previously described. In our tests, we set τ_1^x and τ_2^x so that lengthwise the center of gravity had to be between the front of the truck and the rear axle. We then set τ_1^y and τ_2^y to $\frac{W}{8}$, and Q^e to 3500 kg. The results obtained by the ILP model (2)–(18) are given in Table 1.

The table gives information on the performance of the model. Each row contains average or total values for a class of instances. The first column indicates the name of the class and the second the number of instances in this class (*inst.*). L and U are the average lower and upper bounds obtained by the model for each instance. If the model obtains the optimal solutions for all instances, as in classes A and C, these values match. Otherwise, there is a difference in the instances in which optimality could not be proven, as in classes B and D. In these cases, *missed* is the number of instances for which optimality is not proven, *gap* is the total difference between U and L for all the instances in the class, *nodes* is the average number of nodes explored by the model's enumeration tree, and *sec* the average computational time in seconds; *nodes*$_{opt}$ and *sec*$_{opt}$ give the same information as the two previous columns, but referring only to the instances solved to proven optimality. The model was coded in C++ and solved on a computer Intel Core i7-4790CPU (3.6 GHz, 16 GB) using CPLEX 12.51 with 4 threads. The time limit per instance was set to 3600 CPU seconds.

Classes A and C seem easier and the model is able to solve all their instances in less than one second on average. Classes B and D are more challenging. The five instances not solved to optimality are very different, with numbers of products ranging from 1 to 25 and numbers of layers between 1929 and 4043, and they are not the ones requiring the largest numbers of trucks. The difference between the upper and the lower bound is just one truck in all cases in which optimality is not proven. In summary, the initial model can be considered satisfactorily solved.

6. Dynamic stability constraints

Stability is considered one of the most important issues when solving a container loading problem (CLP), because unstable loads would cause damage to the products being sent and could be dangerous for the people handling them. Loading strategies that do not take stability into account cannot be used in practice. This is why an increasing number of studies in the CLP literature consider stability constraints (37.3% of the papers reviewed by Bortfeldt & Wäscher (2013)). Two types of stability can be distinguished: static and dynamic stability.

Static stability (also known as vertical stability) concerns the capacity of the loaded boxes to withstand the force of gravity acting on them, and deals with situations in which the truck is not moving. In all our models, static stability is always ensured because all pallets are placed on the floor of the truck and every layer is supported by the

Table 1
Computational results of the initial model (2)–(18).

Class	Inst.	Missed	Gap	L	U	Nodes	Sec	Nodes$_{opt}$	Sec$_{opt}$
A	27	0	0	8.19	8.19	0	0.2	0	0.2
B	53	2	2	9.11	9.15	1169969	196.2	388138	62.8
C	20	0	0	9.45	9.45	198	1.0	198	1.0
D	11	3	3	18.27	18.55	2224105	988.8	721	9.4
Avg/sum	111	5	5	9.86	9.90	779076	191.9	186837	31.1

pallet base or by another layer.

Dynamic stability (also known as horizontal stability) is related to the capacity of the loaded boxes to withstand the inertia of their own bodies and not be displaced with respect to the x and y axes. This kind of stability encompasses situations where the truck is displaced horizontally and is exposed to speed variations during its journey. Bischoff and Ratcliff (1995) proposed two metrics for evaluating dynamic cargo stability. The first metric (M1) is the average number of supporting boxes for each box that is not located on the truck floor, with an alternative (M1a) that does not consider contact areas of less than 5% of the base area of a box. The last metric (M2) is related to lateral support and measures the percentage of boxes that do not have at least three of their four sides in contact with other boxes or the walls of the truck. In our case, all pallets are located on the truck floor and all layers are supported by other layers or pallet bases, so M1 (and M1a) is always 1. It is more interesting to consider M2 for the pallets in a truck, to know whether a pallet has at least three of its sides in contact with other pallets or the walls of the truck, because that would be a good indicator of the stability of the cargo.

In the remaining part of this section, we consider two conditions that would increase the dynamic stability of the cargo, on the one hand, forbidding empty spaces between pallets to prevent horizontal displacements, and on the other hand, avoiding excessive differences in height between adjacent pallets, which could result in the taller pallet tipping over the shorter one.

6.1. Compactness

A pallet placed in a truck is dynamically unstable if it is displaced when the truck is moving due to acceleration and braking forces or when the truck turns to the right or to the left. Longitudinal stability is required when the truck is subjected to speeding-up and braking forces, applied in the lengthwise direction, whereas lateral stability is required when the truck turns and the forces are applied in a widthwise direction. If pallets are loaded as they appear in Fig. 2, those numbered 1, 5, and 6 could be displaced towards the back of the truck due to acceleration. Pallets 2 and 6 could be displaced instead towards the front if the truck brakes. Similarly, if the pallets are placed in the truck as shown in the top view in Fig. 3, pallets 2, 3, 4, and 6 could be displaced when the truck turns right or left.

To prevent longitudinal movements during the journey, we impose placing the pallets in adjacent positions along the truck, thus avoiding gaps among them. To achieve this, we make use of two new families of variables:

$$f_{ki} = \begin{cases} 1, & \text{if there is an empty space to the left of position } i \\ & \quad \text{of truck } k \\ 0, & \text{otherwise} \end{cases}$$

$$b_{ki} = \begin{cases} 1, & \text{if there is an empty space to the right of position } i \\ & \quad \text{of truck } k \\ 0, & \text{otherwise} \end{cases}$$

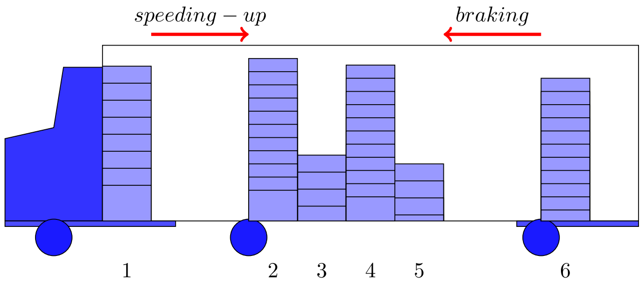


Fig. 2. Longitudinal stability forces.

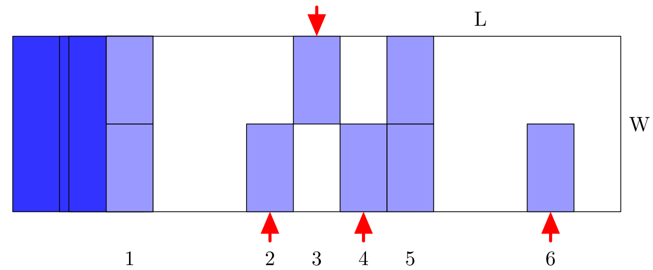


Fig. 3. Lateral stability forces.

Variables f_{ki} take value 1 when there is a pallet in position i but not in position $(i-1)$. Conversely, variables b_{ki} take value 1 if there is a pallet in position i but not in position $(i+1)$.

To compact the load longitudinally, we add the following constraints to the initial model:

$$f_{ki} \geqslant z_{ki} - z_{k,i-1} \quad k \in K, \;\; i \in \{2, ..., |I|\} \backslash \{|I|/2 + 1\} \tag{19}$$

$$f_{ki} \geqslant z_{ki} \qquad k \in K, \;\; i \in \{1, |I|/2 + 1\} \tag{20}$$

$$b_{ki} \geqslant z_{ki} - z_{k,i+1} \quad k \in K, \;\; i \in \{1, ..., |I| - 1\} \backslash \{|I|/2\} \tag{21}$$

$$b_{ki} \geqslant z_{ki} \quad k \in K, \;\; i \in \{|I|/2, |I|\} \tag{22}$$

$$\sum_{1 \leqslant i \leqslant |I|/2} f_{ki} \leqslant 1 \qquad k \in K \tag{23}$$

$$\sum_{|I|/2+1 \leqslant i \leqslant |I|} f_{ki} \leqslant 1 \qquad k \in K \tag{24}$$

$$\sum_{1 \leqslant i \leqslant |I|/2} b_{ki} \leqslant 1 \qquad k \in K \tag{25}$$

$$\sum_{|I|/2+1 \leqslant i \leqslant |I|} b_{ki} \leqslant 1 \qquad k \in K \tag{26}$$

$$f_{ki}, b_{ki} \in \{0, 1\} \quad k \in K, \;\; i \in I \tag{27}$$

Constraints (19) force variables f_{ki} to take value 1 if there is a pallet in position i ($z_{ki} = 1$) but not in position $i - 1$ ($z_{ki-1} = 0$). Positions 1 and $|I|/2 + 1$ are in the front column of the truck, so no pallet can be placed to their left. In these cases, considered in constraints (20), variables f_{ki} simply take the value of the corresponding variables z_{ki}. This is done to make a proper count of the number of empty spaces to the left on each row, which is then limited to 1 by constraints (23) and (24). In this way, there can be no more than one empty space to the left of all the pallets placed in each row and no empty space between two pallets. The same reasoning applies for variables b_{ki}: the special cases correspond to the last column of the truck, corresponding to positions $|I|/2$ and $|I|$, that are considered in constraints (22), whereas the number of empty spaces is limited by constraints (25) and (26).

Similarly, we enforce lateral stability by adding new variables and constraints to the model to avoid situations in which there are columns with a pallet in one position and an empty space in the other position. The new variables that we invoke are:

$$s_{ki} = \begin{cases} 1, & \text{if there is only one pallet in the column defined by position } i \\ & \quad \text{in truck } k, \, i \in \{1, ..., |I|/2\} \\ 0, & \text{otherwise} \end{cases}$$

To prevent lateral displacements, the following constraints are added to the initial model:

$$s_{ki} \geqslant z_{ki} - z_{k,i+|I|/2} \qquad k \in K, \;\; i \in \{1, ..., |I|/2\} \tag{28}$$

$$s_{ki} \geqslant z_{k,i+|I|/2} - z_{ki} \qquad k \in K, \;\; i \in \{1, ..., |I|/2\} \tag{29}$$

$$\sum_{1 \leqslant i \leqslant |I|/2} s_{ki} \leqslant 1 \qquad k \in K \tag{30}$$

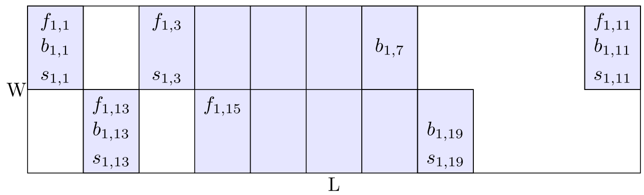


Fig. 4. A solution that does not accomplish with constraints (19)–(31).

$$s_{ki} \in \{0, 1\} \qquad k \in K, \ \ i \in \{1, \dots, |I|/2\} \tag{31}$$

Constraints (28) force variables s_{ki} to take value 1 if there is a pallet in a position i in the first row and not in the same column of the second row (that is, in position $i + |I|/2$). Conversely, constraints (29) force variables s_{ik} to be equal to 1 when there is a pallet in a position of the second row, $i \in |I|/2 + 1, \dots, |I|$, but not in the same column of the first row (position $i - |I|/2$). Constraints (30) limit the number of lateral empty spaces to 1, so as to allow solutions where an odd number of pallets is loaded into a truck.

In Fig. 4 we can see an overhead view of the first truck in a solution with $|I| = 22$, showing variables f_{ki}, b_{ki}, and s_{ki} taking value 1. This solution does not satisfy the longitudinal and lateral stability constraints because there is more than one empty space to the left, to the right, and to the side of the pallets loaded. Fig. 5 shows a solution in which these longitudinal and lateral stability constraints are satisfied.

The computational results obtained by adding both longitudinal and lateral stability constraints to the initial model are shown in Table 2. It can be observed that the resulting model is harder to solve than the previous one, and that the number of instances for which optimality is not proven increases from 5 to 13. However, the gap for each instance is at most one truck, and the computational effort required is still quite low on average. The effect of these stability constraints on the lengthwise and widthwise compactness of the load can be observed in Fig. 6, which depicts a truck load taken from different solutions of the same benchmark instance. Fig. 6(a) shows part of the solution obtained by the initial model, whereas Fig. 6(b) takes into account longitudinal constraints and Fig. 6(c) longitudinal and lateral constraints. An undesired effect that can be seen in Fig. 6(c) is that empty spaces can be filled with short pallets with very few layers. This produces an increment in the number of pallets in the solution and also large differences in the heights of adjacent pallets that can affect stability. The next subsection will address this question, by looking for pallets with similar heights.

6.2. Pallet heights

There are several ways in which the difference between the heights of the pallets can be addressed. In this section we consider three alternatives and compare the results obtained by each of them in terms of the solution difficulty of the resulting model, the number of pallets obtained, and their usefulness in practice.

- *Limiting the difference in height between consecutive pallets*
  A direct way of addressing the question is to limit the difference in height between consecutive adjacent pallets in the same row or column. We define three new parameters:
  – $\lambda_{speed-up}$, the maximum height difference between pallets in positions i and $i+1$,
  – λ_{brake}, the maximum height difference between pallets in positions $i+1$ and i;
  – λ_{side}, the maximum height difference between two pallets in the same column.
  By looking at Fig. 2, we can see that $\lambda_{speed-up}$ aims at controlling the height difference between pallets 2 and 3, to prevent pallet 2 from tipping over pallet 3 when the truck accelerates, and λ_{brake} controls the difference between pallets 4 and 3 to prevent pallet 4 from tipping over pallet 3 when the truck brakes.
  The following constraints are added to the model:
  In order to include the height of the pallet we have to add to the model the following equations.

$$\sum_{j \in J} h_j x_{kij} - \sum_{j \in J} h_j x_{k,i+1,j} \leqslant \lambda_{speed-up} + H b_{ki}$$
$$k \in K, \ \ i \in I \setminus \{|I|/2, |I|\} \tag{32}$$

$$\sum_{j \in J} h_j x_{k,i+1,j} - \sum_{j \in J} h_j x_{kij} \leqslant \lambda_{brake} + H f_{k,i+1}$$
$$k \in K, \ \ i \in I \setminus \{|I|/2, |I|\} \tag{33}$$

$$\sum_{j \in J} h_j x_{kij} - \sum_{j \in J} h_j x_{k,i+|I|/2,j} \leqslant \lambda_{side} + H s_{ki}$$
$$k \in K, \ \ i \in \{1, \dots, |I|/2\} \tag{34}$$

$$\sum_{j \in J} h_j x_{k,i+|I|/2,j} - \sum_{j \in J} h_j x_{kij} \leqslant \lambda_{side} + H s_{ki}$$
$$k \in K, \ \ i \in \{1, \dots, |I|/2\} \tag{35}$$

  Constraints (32)–(35) limit the difference between consecutive pallets in all directions, except for the cases in which there is an empty space, identified by variables f_{ki}, b_{ki}, and s_{ki}. As the heights of pallet bases are the same at all positions, it is not necessary to include them when calculating the difference in height between two pallets.
  The results obtained by the resulting model are shown in Table 3. The parameters chosen for this run were $\lambda_{speed-up} = \lambda_{brake} = \lambda_{side} = \frac{H}{2}$. It can be observed that adding these constraints makes the model harder, increasing the number of instances not solved to optimality and including six large instances for which not even a feasible integer solution was found within the time limit. In these cases, the solution provided is the one obtained by the heuristic algorithm described in Section 4. This situation in which not even a feasible solution was found within the time limit also appeared in the other models in this section. In order to reduce these cases to a minimum, the CPLEX option of putting emphasis on feasibility was activated. In Fig. 7 the effect of constraints (32)–(35) can be observed. They control the difference in height between adjacent pallets, but can produce a staircase effect.
- *Uniform height of the pallets*
  Another way of avoiding consecutive pallets with an excessive difference in height is to impose an overall balance between all the pallets in the solution or among the pallets in each truck. This can be done by invoking a continuous variable γ, changing the objective function from (2)–(36), and adding a new constraint (37) to the model with dynamic stability. In this way, the maximum pallet height is minimized and therefore all pallets have a similar height, although very short pallets are allowed.

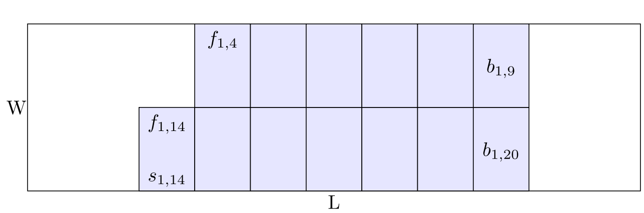


Fig. 5. A solution satisfying constraints (19)–(31).

Table 2
Computational results of the model with longitudinal and lateral stability constraints (2)–(31).

Class	Inst.	Missed	Gap	L	U	Nodes	Sec	Nodes_{opt}	Sec_{opt}
A	27	0	0	8.19	8.19	9	0.4	9	0.4
B	53	9	9	9.11	9.28	1755607	613.4	2977	2.5
C	20	1	1	9.45	9.50	131262	183.3	5208	3.5
D	11	3	3	18.27	18.55	2433382	1078.0	5298	132.1
Avg/sum	111	13	13	9.86	9.97	1103062	432.8	2781	12.7

$$\min \quad \sum_{k \in K} H y_k + \gamma \tag{36}$$

$$\sum_{j \in J} h_j x_{kij} + h^p z_{ki} \leqslant \gamma \qquad k \in K, \ i \in I \tag{37}$$

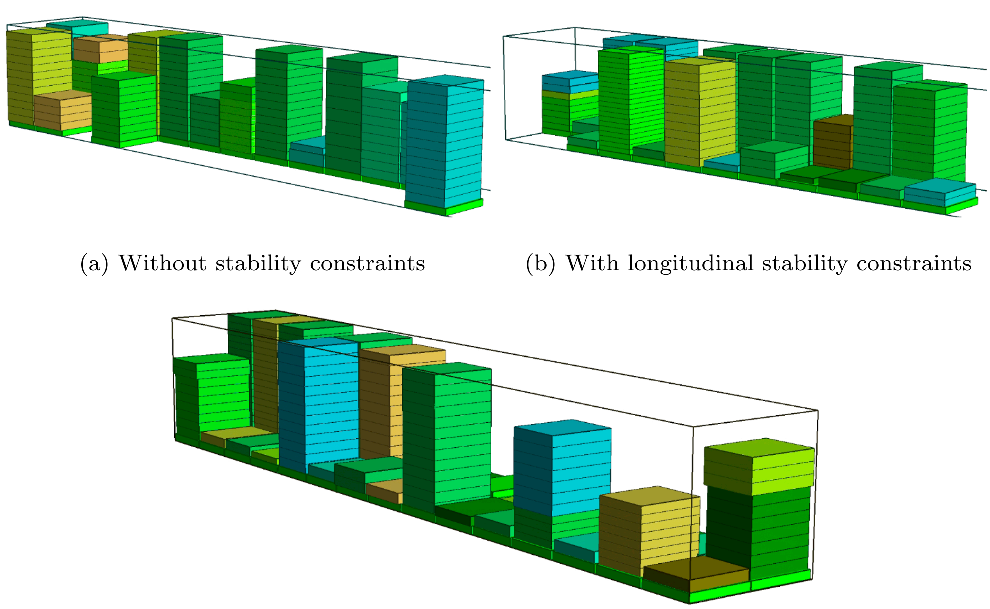
The results of this configuration appear in Table 4. Although the objective function has been changed, in order to compare with the other tables the *gap* column still shows the difference in the number of trucks between upper and lower bounds. There are 16 instances for which the number of trucks in the best solution obtained did not match the lower bound, as reported in the table, and for one of them no integer feasible solution was found. In addition, the objective function contains a continuous variable γ and its optimal value could only be found in 12 out of 111 instances, thus producing long average running times. The effect of this change in the model can be observed in Fig. 8. The heights of the pallets are quite similar, but the balance is sometimes obtained by making short pallets and therefore wasting much of the truck's volume.

- *Imposing a minimum pallet height*
  The two alternatives just described try to control the difference in height between pallets, but they do not control the number of pallets in the solution. Sometimes, the constraints introduced for control produce the undesired effect of increasing the number of pallets, building some very short pallets that are stable but very inefficient in handling and transporting products. An alternative is to impose a minimum height for all the pallets, for instance $H/2$. If all pallets are taller than $H/2$, the differences in height cannot be very large and, at the same time, no short pallets are allowed, thus reducing the total number of pallets.
  To impose this condition, a single new constraint must be added to the model:

$$\sum_{j \in J} h_j x_{kij} \geqslant (H/2) z_{ki} \quad k \in K, \ i \in I \tag{38}$$

The results, shown in Table 5, are slightly better than those in Table 4, both in terms of instances non-optimally solved, 9, and instances for which a feasible solution is not found, 4. It is worth noting that there is an instance ($D31$) in which an odd number of layers of a unique product has to be sent, but, in order to satisfy (38), pallets should be composed of at least two layers. In this quite singular case, the model cannot produce a feasible solution. The solution reported in Table 5 for this case was obtained by the constructive heuristic, and contains one pallet with just one layer whose height is lower than $H/2$. In Fig. 9, the effect of constraints (38) can be observed. Each truck contains a set of tall pallets placed without gaps. If the products are heavy, the weight limits can be reached with a number of pallets lower than the number of allowed positions in the truck, as can be seen in the figure.
A possible way of introducing some flexibility into the last model is to allow at most one pallet shorter than $H/2$ in every truck. This can be done by defining a new variable:

$$u_{ki} = \begin{cases} 1, & \text{if the pallet at position } i \text{ of truck } k \text{ is shorter than } H/2 \\ 0, & \text{otherwise} \end{cases}$$



(a) Without stability constraints (b) With longitudinal stability constraints

(c) With lateral and longitudinal stability constraints

Fig. 6. Solution without and with stability constraints.

Table 3
Computational results when limiting the height difference of consecutive pallets (model (2)–(31) + (32)–(35)).

Class	Inst.	Missed	Gap	L	U	Nodes	Sec	$Nodes_{opt}$	Sec_{opt}
A	27	0	0	8.19	8.19	108	1.0	108	1.0
B	53	10	10	9.13	9.32	1152683	658.6	23223	42.9
C	20	2	3	9.40	9.55	46748	114.2	51943	114.2
D	11	6	10	18.00	18.91	1904291	1365.5	719	10.2
Avg/sum	111	18	23	9.83	10.04	747543	449.3	20861	42.8

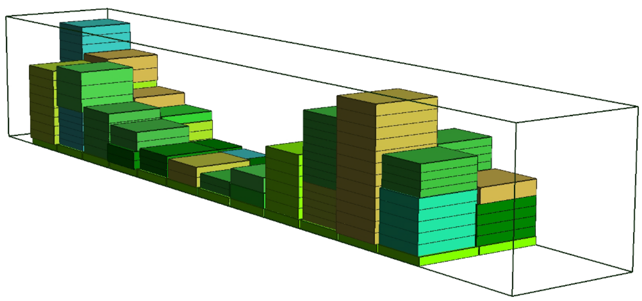


Fig. 7. Limiting the difference between consecutive pallets.

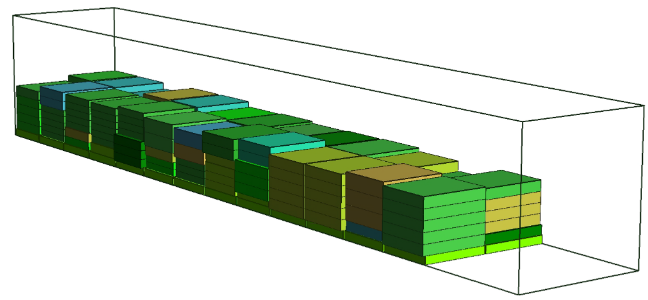


Fig. 8. Balancing the height of the pallets.

and replacing constraints (38) with:

$$\sum_{j\in J} h_j x_{kij} \geqslant (H/2)z_{ki} - (H/2)u_{ki} \quad k \in K,\ i \in I \tag{39}$$

$$u_{ki} \leqslant \sum_{j\in J} h_j x_{kij} \quad k \in K,\ i \in I \tag{40}$$

$$\sum_{i\in I} u_{ki} \leqslant 1 \quad k \in K \tag{41}$$

thus ensuring that at most one pallet per truck can be shorter than $H/2$.

The addition of these new variables and constraints makes the model more difficult to solve, as can be observed in Table 6. Instance $D31$ is now solved, but for other two instances no integer solution is found within the time limit, and the average computational effort is doubled.

The three alternatives considered in this section are compared in Table 7. For each model, including the initial one, the table shows the number of instances optimally and non-optimally solved, the number of instances for which a feasible solution was not found, the sum of the lower bounds, the total number of trucks corresponding to the sum of the upper bounds, the total number of pallets, and the value of metric $M2$ (percentage of pallets not supported by other pallets or by the walls of the truck on at least three sides).

The results in Table 7 show that imposing stability constraints only increases the number of trucks needed in a few instances. Models are harder to solve but can still produce optimal solutions for most instances. Most of the solutions found that are not proven to be optimal are only one truck away from the lower bound. For the very few cases in which models did not provide a feasible output, the solution provided by the heuristic can be considered acceptable.

Among the different alternatives that we tested, the one imposing a minimum pallet height seems to be the best, as it dramatically reduces the number of pallets built, at the expense of just a minor increase in M2. Indeed, for the initial model, in which stability was not included, M2 is very high, indicating very unstable loads. For all the other models, in contrast, M2 is very low. Considering, for instance, the model in which a minimum pallet height is imposed, the total number of pallets not laterally supported on at least three sides is 506 in the 1108 trucks, corresponding to the cases in which a truck contains an odd number of pallets. The alternative of imposing a minimum height also appears to be the most useful in practice, because it ensures high occupancy of the truck volume, avoiding the undesired situations that could arise in the other two cases (Figs. 7 and 8).

7. Middle positions

The grid proposed in Section 3 fixes the possible positions for the pallets on the truck floor. It is determined by dividing the truck width into as many rows as the dimensions of pallet and truck allow, but that may not be adequate in some cases, especially if there are very heavy products. A simple example would be locating a pallet of very large weight, say M, in a truck of capacity $Q = M$. The truck could support the pallet, but the pallet could not be placed in any of the positions defined by the grid, because this would cause the load to be too skewed to one side of the truck and hence would not satisfy the center of gravity constraint. The right position for such a pallet would be in the center of the truck floor. In order to make this option possible, we decided to extend our grid and include a new central row of "middle positions", overlapping with the other two existing rows, as shown in Fig. 10.

Table 4
Computational results when imposing uniform pallets heights (model (3)–(31) + (36) and (37)).

Class	Inst.	Missed	Gap	L	U	Nodes	Sec	$Nodes_{opt}$	Sec_{opt}
A	27	0	0	8.19	8.19	1109117	3466.8	1109117	3466.8
B	53	9	9	9.08	9.25	1342454	3396.4	1095052	3354.7
C	20	1	1	9.45	9.50	1027569	2701.5	1081432	2654.2
D	11	6	9	18.18	19.00	6607229	3273.0	650374	2880.4
Avg/sum	111	16	19	11.22	11.48	1750695	3276.1	1072921	3221.5

Table 5
Computational results when imposing a minimum pallet height (model (2)–(31) + (38)).

Class	Inst.	Missed	Gap	L	U	Nodes	Sec	Nodes$_{opt}$	Sec$_{opt}$
A	27	0	0	8.19	8.19	489	150.5	489	150.5
B	53	6	6	9.11	9.23	340083	569.8	4696	183.0
C	20	0	0	9.45	9.45	905	302.6	905	302.6
D	11	7	11	18.00	19.00	50347	2439.8	1571	409.4
Avg/sum	111	13	17	9.83	9.98	167653	605.01	2636	207.7

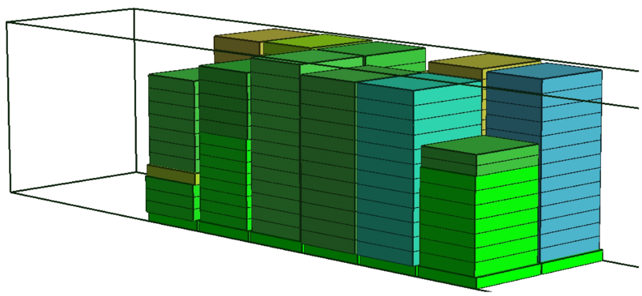


Fig. 9. Imposing a minimum height for the pallets.

Now there are three rows:

- Row 1, where positions go from 1 to $|I|/2$.
- Row 2, where positions go from $|I|/2 + 1$ to $|I|$.
- Middle row, whose positions go from $|I| + 1$ to $|I| + |I|/2$.

If there is a pallet in the middle row, the positions in the same column in the other two rows cannot be occupied. This can be ensured by the following constraints:

$$z_{ki} + z_{k,i+|I|} \leqslant 1 \qquad k \in K,\ \ i \in \{1,\dots,|I|/2\} \tag{42}$$

$$z_{ki} + z_{k,i+|I|/2} \leqslant 1 \qquad k \in K,\ \ i \in \{|I|/2,\dots,|I|\} \tag{43}$$

The lateral and longitudinal stability constraints must also be adapted to the new positions. Constraints (44)–(51) adapt constraints (19)–(27). Constraints (52) ensure that if there is only one pallet in a column, it is in the middle row.

$$f_{ki} \geqslant (z_{ki} + z_{k,i+|I|}) - (z_{k,i-1} + z_{k,i-1+|I|}) \qquad k \in K,\ \ i \in \{2,\dots,|I|/2\} \tag{44}$$

$$f_{ki} \geqslant (z_{ki} + z_{k,i+|I|/2}) - (z_{k,i-1} + z_{k,i-1+|I|/2}) \qquad k \in K,\ \ i \in \{|I|/2 + 2,\dots,|I|\} \tag{45}$$

$$f_{k1} \geqslant (z_{k1} + z_{k,1+|I|}) \qquad k \in K, \tag{46}$$

$$f_{k,|I|/2+1} \geqslant (z_{k,|I|/2+1} + z_{k,|I|+1}) \qquad k \in K \tag{47}$$

$$b_{ki} \geqslant (z_{ki} + z_{k,i+|I|}) - (z_{k,i+1} + z_{k,i+1+|I|}) \qquad k \in K,\ \ i \in \{1,\dots,|I|/2 - 1\} \tag{48}$$

$$b_{ki} \geqslant (z_{ki} + z_{k,i+|I|/2}) - (z_{k,i+1} + z_{k,i+1+|I|/2}) \qquad k \in K,\ \ i \in \{|I|/2 + 1,\dots,|I| - 1\} \tag{49}$$

$$b_{k,|I|/2} \geqslant z_{k,|I|/2} + z_{k,|I|/2+|I|} \qquad k \in K \tag{50}$$

$$b_{k,|I|} \geqslant z_{k,|I|} + z_{k,|I|+|I|/2} \qquad k \in K \tag{51}$$

$$\sum_{1 \leqslant i \leqslant |I|/2} s_{ki} = 0 \qquad k \in K \tag{52}$$

The results of the initial model of Section 5 modified by including these middle positions appear in Table 8. All instances but eleven are solved to optimality. For instances in which optimality is not proven, the gaps are always of one truck. Middle positions are occupied mainly when heavy products are involved, but also in trucks with an odd number of pallets in which one pallet tends to occupy a middle position. Fig. 11 shows the solution obtained using middle positions in an instance with 20 layers each weighing 1000 kg.

8. Demand over time

In some practical situations, the demand that a distribution company receives from a customer may involve different delivery dates. Some products have to be received on day 1, some others on day 2, and so on, within a planning horizon that usually corresponds to the working days of a week. In our study case, products can be sent before their delivery date, but not later. Therefore, a solution for the whole shipping problem, that is, for the entire time horizon, can be better than the combination of the solutions of each single day, because unused space in the trucks sent on a certain day can be used to ship in advance products required later on.

The delivery dates impose the order in which products are loaded into trucks: products for day 1 must be in the first trucks (1 to K_1), some products for day 2 fill up some of these trucks and the remaining products for day 2 must be in the next trucks ($K_1 + 1$ to K_2), and so on for the remaining periods of the planning horizon. Using this type of solution, the company can send the first K_1 trucks on the first day. On the second day, it can send trucks from $K_1 + 1$ to K_2, but, if necessary, it can adjust its data to accommodate last minute orders or cancelations and solve the problem again for the remaining days. Besides being able to adjust to changes, the company can use the trucks in a more regular way over the days of the planning horizon.

We have developed two ways of taking account of the delivery dates:

Table 6
Computational results when imposing minimum height on all pallets but one (model (2)–(31) + (39)–(41)).

Class	Inst.	Missed	Gap	L	U	Nodes	Sec	Nodes$_{opt}$	Sec$_{opt}$
A	27	0	0	8.19	8.19	966	23.7	966	23.7
B	53	8	8	9.09	9.25	233809	581.1	2983	111.5
C	20	0	0	9.45	9.45	1866	286.6	1866	286.6
D	11	7	12	18.00	19.09	43249	1546.7	1897	6.5
Avg/sum	111	15	20	9.82	10.00	116496	447.3	2138	118.9

Table 7
Comparing alternative models for controlling height differences between pallets.

Model				Missed			
	Optimal	Non-opt	No Sol	LB	Trucks	Pallets	M2 (%)
Initial model (Section 5, Table 1)	106	5	0	1094	1099	18459	31.0
Compactness (Section 6.1, Table 2)	98	13	0	1094	1107	29949	0.6
Limiting difference (Section 6.2.a, Table 3)	93	12	6	1091	1114	26573	0.3
Uniform height (Section 6.2.b, Table 4)	95	15	1	1091	1110	29488	0.7
Minimum height (Section 6.2.c, Table 5)	98	9	4	1091	1108	18153	2.8
One short pallet (Section 6.2.d, Table 6)	96	10	5	1090	1110	18204	2.6

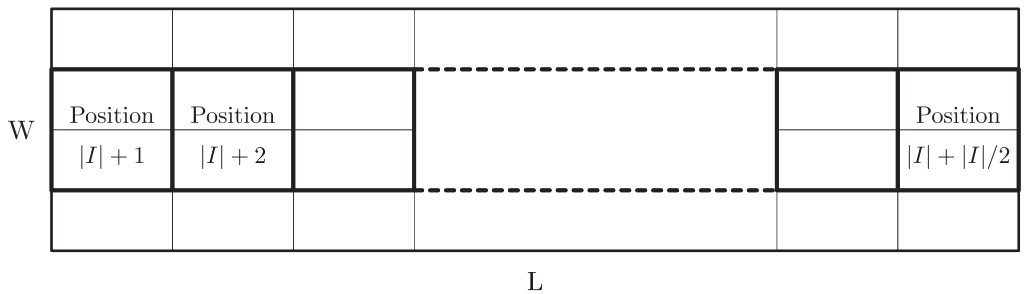


Fig. 10. A grid with middle positions.

Table 8
Computational results of the model with middle positions (model (2)–(18) + (42)–(51)).

Class	# Inst.	Missed	Gap	L	U	Nodes	Sec	Nodes$_{opt}$	Sec$_{opt}$
A	27	0	0	8.19	8.19	0	0.4	0	0.4
B	53	8	8	9.11	9.26	598685	544.7	125	1.4
C	20	0	0	9.45	9.45	38	2.1	38	2.1
D	11	3	3	18.27	18.55	1262418	1017.8	3275	49.3
Avg/sum	111	11	11	9.89	9.95	410970	361.4	326	5.1

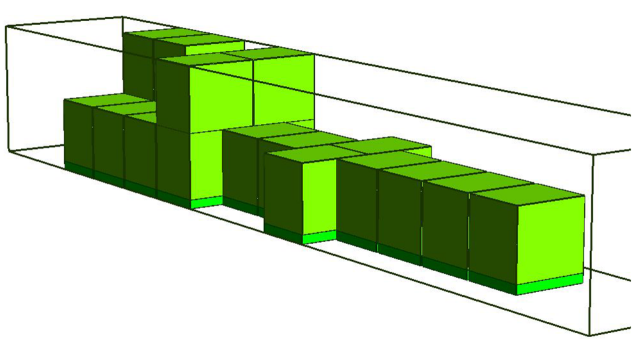


Fig. 11. Solution for an instance with 20 layers of 1000 kg.

- *Solving a single model with time constraints*
  We introduce the delivery dates into the model described in Section 5. Let D be the time horizon and recall that n_j is the total demand for product j. We denote the number of layers of product j with delivery date d by n_{dj}, for $d \in D$, so that $\sum_{d \in D} n_{dj} = n_j$. Let $N_d = \sum_j n_{dj}$ be the number of layers to be delivered on day d, and P_d be the set of products whose demand must be partially or entirely delivered on day d. The original variables x_{kij} are replaced by:

$$x_{dkij} = \text{number of layers of product } j \text{ packed in position } i \text{ of truck } k \text{ and delivered on day } d,$$

and a new set of variables is needed:

$$y_{dk} = \begin{cases} 1, & \text{if truck k contains products with delivery date } d \\ 0, & \text{otherwise} \end{cases}$$

The modified model is:

$$\text{(M-days)} \min \sum_{k \in K} y_k \tag{53}$$

$$\sum_{k \in K} \sum_{i \in I} x_{dkij} = n_{dj} \quad j \in J,\, d \in D \tag{54}$$

$$\sum_{d \in D} \sum_{j \in J} h_j x_{dkij} + h^p z_{ki} \leqslant H' y_k \quad k \in K,\ i \in I \tag{55}$$

$$\sum_{i \in I} \left(q^p z_{ki} + \sum_{d \in D} \sum_{j \in J} q_j x_{dkij} \right) \leqslant Q y_k \quad k \in K \tag{56}$$

$$\sum_{i \in I} \left(q^p z_{ki} + \sum_{d \in D} \sum_{j \in J} q_j x_{dkij} \right) (\delta_2 - p_i^x) \leqslant Q_1 (\delta_2 - \delta_1) y_k \quad k \in K \tag{57}$$

$$\sum_{i \in I} \left(q^p z_{ki} + \sum_{d \in D} \sum_{j \in J} q_j x_{dkij} \right) (p_i^x - \delta_1) \leqslant Q_2 (\delta_2 - \delta_1) y_k \quad k \in K \tag{58}$$

$$\sum_{d\in D}\sum_{j\in J} h_j x_{dkij} \leqslant (H' - h^p) z_{ki} \quad k \in K, \ i \in I \tag{59}$$

$$z_{ki} \leqslant \sum_{d\in D}\sum_{j\in J} x_{dkij} \quad k \in K, \ i \in I \tag{60}$$

$$\sum_{i\in I}\sum_{j\in P_d} x_{dkij} \leqslant N_d y_{kd} \quad k \in K, d \in D \tag{61}$$

$$\sum_{k>k'}\sum_{d<d'} y_{kd} \leqslant (|K| - k')(1 - y_{k'd'}) \quad k' \in K, d' \in D, d' > 1 \tag{62}$$

$$Q^e G_x + \sum_{i\in I} p_i^x q^p z_{ki} + \sum_{d\in D}\sum_{i\in I}\sum_{j\in J} p_i^x q_j x_{dkij} \leqslant \left(\sum_{i\in I} q^p z_{ki} + \sum_{d\in D}\sum_{i\in I}\sum_{j\in J} q_j x_{dkij} + Q^e\right)(G_x + \tau_1^x) \quad k \in K \tag{63}$$

$$Q^e G_x + \sum_{i\in I} p_i^x q^p z_{ki} + \sum_{d\in D}\sum_{i\in I}\sum_{j\in J} p_i^x q_j x_{dkij} \geqslant \left(\sum_{i\in I} q^p z_{ki} + \sum_{d\in D}\sum_{i\in I}\sum_{j\in J} q_j x_{dkij} + Q^e\right)(G_x - \tau_2^x) \quad k \in K \tag{64}$$

$$Q^e G_y + \sum_{i\in I} p_i^y q^p z_{ki} + \sum_{d\in D}\sum_{i\in I}\sum_{j\in J} p_i^y q_j x_{dkij} \leqslant \left(\sum_{i\in I} q^p z_{ki} + \sum_{d\in D}\sum_{i\in I}\sum_{j\in J} q_j x_{dkij} + Q^e\right)(G_y + \tau_1^y) \quad k \in K \tag{65}$$

$$Q^e G_y + \sum_{i\in I} p_i^y q^p z_{ki} + \sum_{d\in D}\sum_{i\in I}\sum_{j\in J} p_i^y q_j x_{dkij} \geqslant \left(\sum_{i\in I} q^p z_{ki} + \sum_{d\in D}\sum_{i\in I}\sum_{j\in J} q_j x_{dkij} + Q^e\right)(G_y - \tau_2^y) \quad k \in K \tag{66}$$

$$y_k \geqslant y_{k+1} \quad k \in K: k < |K| \tag{67}$$

$$y_k = 1 \quad k \leqslant LB \tag{68}$$

$$\sum_{d\in D} y_{kd} \leqslant |D| y_k \quad k \in K \tag{69}$$

$$x_{dkij} \geqslant 0, \text{ integer} \quad d \in D, k \in K, \ i \in I, j \in J \tag{70}$$

$$y_k \in \{0, 1\} \quad k \in K \tag{71}$$

$$y_{dk} \in \{0, 1\} \quad d \in D, k \in K \tag{72}$$

$$z_{ki} \in \{0, 1\} \quad k \in K, \ i \in I \tag{73}$$

The following main differences from the previous model can be noted. Constraints (61) link variables x_{dkij} with variables y_{dk}, so that if there are some products in truck k with delivery date d, then $y_{dk} = 1$. Constraints (62) impose an order of the products loaded into the trucks. Products with different delivery dates can be together in a truck, but if in truck k' there are products with delivery date d', in the following trucks $k > k'$ there cannot be products with earlier delivery dates, $d < d'$. For example, if products with delivery date $d = 1$ need more than two trucks, but they do not fill up a third truck, constraints (62) impose the condition that trucks 1 and 2 will only contain products with delivery date 1, and in truck 3 there will be the remaining products from day 1 plus some products from day 2. A truck with products with different delivery dates will only appear when the remaining products for a delivery date do not completely fill the truck and products for the next delivery date are used to fill it up. Then only trucks containing products for this delivery date have to be sent each day, and the part of the solution corresponding to the following days can be adjusted to accommodate last-minute changes. Constraints (69) link the new variables y_{kd} with the original variables y_k.

- *Solving a model for each day in the planning horizon*
  An alternative, more flexible, way of taking the delivery dates into account consists in solving a sequence of models: the first for products with delivery date $d = 1$, the second for products with delivery dates $d \in \{1, 2\}$, and so on. Each model, except the first, takes the solution of the previous model as a reference. The process is described in Algorithm 1.

Algorithm 1. Solving a model for each day

for $i = 1$ to $|D|$ **do**
  Consider $d = \{1, \ldots, i\}$
  Solve the Model M-Days with time horizon d and constraints (63) deactivated
  K_i = Number of trucks in the solution
  If $i > 1$, add constraints $\sum_{k > K_{i-1}} y_{i-1k} = 0$ to Model M-Days
end for

The results of these two alternative ways of addressing the demand over time are reported in Table 9. Similarly to Table 7, for each model, including the initial one, the table shows the number of instances optimally and non-optimally solved, the number of instances for which a feasible solution was not found, the sum of the lower bounds, and the total number of trucks corresponding to the sum of the upper bounds. In the cases in which a feasible solution was not found, the lower bound is obtained using expression (1) and the upper bound is provided by our heuristic constructive procedure. It can be observed that both alternative ways of including delivery dates are slightly harder than the initial model, as was to be expected because of the increased numbers of variables and constraints. The second alternative is more flexible and could in theory have produced solutions with fewer trucks, but as it is harder to solve, its theoretical advantage is not achieved in practice, as shown by the results.

9. Conclusions and future work

The increasing need to produce high-quality solutions for complex loading problems, involving many containers or trucks and considering many realistic constraints, can be addressed in different ways. In this paper we have chosen to use mathematical models and explore the possibility of first modeling the relevant constraints and then solving

Table 9
Comparing alternative models for the case of demands over time.

Model	Optimal	Non-opt	No Sol	LB	Trucks
Initial model	106	5	0	1094	1099
Solving a single model with time constraints	103	6	2	1093	1103
Solving a model for each day in the planning horizon	100	10	1	1093	1106

the models in reasonable times to produce optimal or quasi-optimal solutions. Starting from an initial model, already including constraints related to total weight, maximum weight supported by the axles, and position of the center of gravity, we have studied in detail the dynamic stability conditions required for a loading plan to be useful in practice. We have proposed several models and the results show that optimal solutions were obtained in most cases. For the instances for which optimality was not proven but a feasible solution was found by the mathematical model, the distance to a known lower bound is at most one truck. For the very few cases in which the model did not find a feasible solution, the solution built by our heuristic algorithm was provided. In one of these cases, the distance from the heuristic solution to the lower bound attained a maximum of three trucks. Although heuristic and metaheuristic procedures can be faster, the quality obtained by the solution of the mathematical models is difficult to match and therefore this approach appears to be useful for solving complex multi container problems.

Some interesting extensions of our models have also been considered, namely, the possibility of using positions on the longitudinal axis of the truck, an especially useful option in the case of heavy loads, and the case of splitting demands over a time horizon according to their expected delivery dates. A future line of research concerns the study of other interesting extensions that appear frequently in practice. An example of a possible extension appears when considering the truck with the smallest load in a solution. Usually, managers do not want to send a truck containing just a few pallets, having a low percentage of its volume or weight filled. To deal with this, two options can be considered. If the truck is not sent, then constraints and/or penalties in the objective function could be considered to find the best subset of shipped products. If the truck is sent, then similar reasoning could be used to find the best subset of products whose delivery could be brought forward. The models that we developed could be adapted to address both options.

Another interesting extension would be to consider more than one type of pallet being used simultaneously for delivering products to customers, as is the case in some applications. It would be necessary to consider an extended grid including all positions for the pallets on the truck floor. That would increase the number of variables associated with this grid, and new constraints, forbidding the simultaneous use of overlapping positions, would be needed. The study of this more complex model would be an interesting future line of research.

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