Given a quadratic Bezier of the following form:

$$B(t) = (1-t)^2 P_0 + 2(1-t)tP_1 + t^2 P_2$$

Where the P's are 3D position vectors (although this works in any dimension).

We know we can get the arc length of this curve by integrating:

$$Length = \int_0^t \left|\left|B'(t)\right|\right| dt = \int_0^t \sqrt{B_x^{2\prime}(t) + B_y^{2\prime}(t) + B_z^{2\prime}(t)}\, dt$$

We can note that:

$$\begin{aligned} B'(t) &= -2(1-t)P_0 + 2(1-2t)P_1 + 2tP_2 \\ &= -2P_0 + 2tP_0 + 2P_0 + 2P_1 - 4tP_1 + 2tP_2 \\ &= 2t(P_0 - 2P_1 + P_2) + 2(P_1 - P_0) \end{aligned}$$

To simplify we will group the constants:

$$\begin{aligned} a &= P_0 - 2P_1 + P_2 \\ b &= 2(P_1 - P_0) \end{aligned}$$

$$B'(t) = 2ta + b$$

Thus, our adjusted equation:

$$\begin{aligned} Length &= \int_0^t \sqrt{(2ta_x + b_x)^2 + \left(2ta_y + b_y\right)^2 + (2ta_z + b_z)^2}\, dt \\ &= \int_0^t \sqrt{4t^2a_x^2 + 4ta_xb_x + b_x^2 + 4t^2a_y^2 + 4ta_yb_y + b_y^2 + 4t^2a_z^2 + 4ta_zb_z + b_z^2}\, dt \\ &= \int_0^t \sqrt{4\left(a_x^2 + a_y^2 + a_z^2\right)t^2 + 4\left(a_xb_x + a_yb_y + a_zb_z\right)t + b_x^2 + b_y^2 + b_z^2}\, dt \end{aligned}$$

Once again to simplify and make our lives easier we will redefine the constants:

$$\begin{aligned} A &= 4(a_x^2 + a_y^2 + a_z^2) = 4(a \cdot a) \\ B &= 4(a_xb_x + a_yb_y + a_zb_z) = 4(a \cdot b) \\ C &= b_x^2 + b_y^2 + b_z^2 = b \cdot b \end{aligned}$$

$$Length = \int_0^t \sqrt{At^2 + Bt + C}\, dt = \int_0^t \sqrt{A\left(t^2 + \frac{B}{A}t + \frac{C}{A}\right)}\, dt = \sqrt{A}\int_0^t \sqrt{t^2 + \frac{B}{A}t + \frac{C}{A}}\, dt$$

Now we will complete the square:

$$\begin{aligned} Length &= \sqrt{A}\int_0^t \sqrt{t^2 + \frac{B}{A}t + \frac{C}{A} + \left(\frac{B}{2A}\right)^2 - \left(\frac{B}{2A}\right)^2}\, dt \\ &= \sqrt{A}\int_0^t \sqrt{\left(t + \frac{B}{2A}\right)^2 + \frac{C}{A} - \left(\frac{B}{2A}\right)^2}\, dt \end{aligned}$$

Once again, we will simplify anything without a t in it to make the integration easier (don't worry, there is a reason I took the square root for r):

$$q = \frac{B}{2A}$$
$$r = \sqrt{\frac{C}{A} - \left(\frac{B}{2A}\right)^2} = \sqrt{\frac{C}{A} - q^2}$$

$$Length = \sqrt{A}\int_0^t \sqrt{(t+q)^2 + r^2}\, dt$$

Next to do one more quick integration step before things get really messy we will use a simple substitution. Recall, $\int f\big(g(x)\big)g'(x)\, dx = \int f(u)\, du$ where $u = g(x)$

Where $u = t + q$ then $\int \sqrt{u^2 + r^2}\, du$

Alright, this is where things are going to get painful. To solve this one let's remember from basic trig that in a right-angle triangle, $\tan(\theta) = \frac{Opposite}{Adjacent}$

We can also remember that:

$$adjacent^2 + opposite^2 = hypotenuse^2$$
$$hypotenuse = \sqrt{adjacent^2 + opposite^2}$$
$$h = \sqrt{a^2 + b^2}$$

Hey, this looks like something we can substitute… It is!

Since we have $\int \sqrt{u^2 + r^2}\, du$ we know the interior ($\sqrt{u^2 + r^2}$) would equal the hypotenuse of a right-angle triangle with opposite and adjacent lengths of u and r. Thus, we can rearrange our trigonometric identity for u (for simplicity we will say u is the opposite, but it doesn't matter).

$$u = r\tan(\theta)$$
$$\frac{du}{d\theta} = r\sec^2(\theta)$$
$$du = r\sec^2(\theta)\, d\theta$$

So, our integral becomes:

$$\int \sqrt{u^2 + r^2}\, du = \int r\sec^2(\theta)\sqrt{r^2\tan^2(\theta) + r^2}\, d\theta = r^2 \int \sec^2(\theta)\sqrt{\tan^2(\theta) + 1}\, d\theta$$

Note that $\tan^2(\theta) + 1 = \sec^2(\theta)$

$$r^2 \int \sec^2(\theta)\sqrt{\sec^2(\theta)}\, d\theta = r^2 \int \sec^3(\theta)\, d\theta$$

That seems a lot more reasonable to integrate, yeah?

By using integration by reduction and note that

$$\frac{d(\ln|\tan(\theta)+\sec(\theta)|)}{d\theta} = \frac{1}{\tan(\theta)+\sec(\theta)}\left(\frac{-1}{\sin(\theta)-1}\right) = \frac{1}{\frac{\sin(\theta)}{\cos(\theta)}+\frac{1}{\cos(\theta)}}\left(\frac{-1}{\sin(\theta)-1}\right) = \frac{-1}{(\sin(\theta)-1)\left(\frac{\sin(\theta)+1}{\cos(\theta)}\right)} =$$
$$\frac{-\cos(\theta)}{(\sin(\theta)-1)(\sin(\theta)+1)} = \frac{-\cos(\theta)}{\sin^2(\theta)-1} = \frac{-\cos(\theta)}{-\cos^2(\theta)} = \frac{1}{\cos(\theta)} = \sec(\theta)$$

This lets us get:

$$= \frac{r^2}{2}\left(\sec^2(\theta)\sin(t) + \int \sec(\theta)\,d\theta\right)$$
$$= \frac{r^2}{2}(\sec^2(\theta)\sin(\theta) + \ln|\tan(\theta) + \sec(\theta)|) + C$$
$$= \frac{r^2}{2}\left(sec(\theta)\tan(\theta) + \ln\left|\tan(\theta) + \frac{1}{\cos(\theta)}\right|\right) + C$$
$$= \frac{r^2}{2}\left(\frac{\tan(\theta)}{\cos(\theta)} + \ln\left|\tan(\theta) + \frac{1}{\cos(\theta)}\right|\right) + C$$

We're almost there, we have the integrated function, but it is in terms of theta, we don't know theta. To get it back in terms of u and r (and then t) we need to go back to our basic trig again, SOHCAHTOA. Remember we defined opposite as u and adjacent as r and the hypotenuse as $\sqrt{u^2 + r^2}$. So, plugging in $u = t + q$ and the trig function we get:

$$\tan(\theta) = \frac{u}{r} = \frac{t+q}{r} \qquad \cos(\theta) = \frac{r}{\sqrt{u^2+r^2}} = \frac{r}{\sqrt{(t+q)^2+r^2}}$$

That's pretty much it! We just plug this all into our original length formula and remember it was a definite integral from 0 to 1.

$$Length = \sqrt{A}\int_0^t \sqrt{(t+q)^2 + r^2}\,dt = \frac{r^2\sqrt{A}}{2}\left(\frac{\tan(\theta)}{\cos(\theta)} + \ln\left|\tan(\theta) + \frac{1}{\cos(\theta)}\right|\right) + C$$

$$Length\Big|_0^t = \frac{r^2\sqrt{A}}{2}\left(\frac{\tan(\theta)}{\cos(\theta)} + \ln\left|\tan(\theta) + \frac{1}{\cos(\theta)}\right|\right)\Big|_0^t$$

Code:

```lua
local abs = math.abs;
local log = math.log;
local sqrt = math.sqrt;

local function quadraticBezierLength(t, p0, p1, p2)
    local a = p0 - 2*p1 + p2;
    local b = 2*(p1 - p0);
    local A = 4*(a:Dot(a));
    local B = 4*(a:Dot(b));
    local C = b:Dot(b);

    local q = B/(2*A);
    local r = sqrt(C/A - q*q);

    local tan0 = (0+q)/r;
    local cos0 = r/sqrt((0+q)^2 + r*r);
    local inner0 = tan0/cos0 + log(abs(tan0 + 1/cos0));

    local tant = (t+q)/r;
    local cost = r/sqrt((t+q)^2 + r*r);
    local innert = tant/cost + log(abs(tant + 1/cost));

    return r*r*sqrt(A)*0.5*(innert - inner0);
end;
```