# 1 Introduction

A group X is *factorizable* if X contains two subgroups $H$ and $K$ such that $X = HK$. The factorization is *exact* if $H \cap K = 1_X$.

# 2 Prepare

Throughout the paper, we use notation X to denote an exact product of two dihedral subgroups $H = \langle a \rangle \rtimes \langle b \rangle \cong D_{2n}$ and $K = \langle c \rangle \rtimes \langle d \rangle \cong D_{2m}$ where m,n $\geq 3$.

Choose a maximal subgroup G of H such the $G \geq H$ then two situations happen (i) $G \cap K = \langle c_1 \rangle$ (ii) $G \cap K = \langle c_1, d \rangle$ where $c_1 \in \langle c \rangle$.

First suppose that $G \cap K = \langle c_1 \rangle$.It was classified by Yu Hao and Kan Hu [1] .Then exactly one of the following holds:

$$(i) G = G_X = H\langle c \rangle, X = (H\langle c \rangle) \rtimes \langle d \rangle, X/G_X \cong C_2. \tag{1}$$

$$(ii) G = H\langle c^2 \rangle, G_X = \langle a^3, c^2 \rangle, X = (K\langle a \rangle) \rtimes \langle b \rangle \, and \, X/G_X \cong S_4. \tag{2}$$

Since exact product of a dihedral group and a cyclic group was classified by Yu Hao, the structure of X in this case is clear.

Now we assumed that $G \cap K = \langle c_1, d \rangle$. Since

$$\langle c_1 \rangle = \bigcap_{k,t \in \mathbb{Z}} \langle c_1 \rangle^{c^k d^t} \leq \bigcap_{k,t \in \mathbb{Z}} G^{c^k d^t} = \bigcap_{i,j,k,t \in \mathbb{Z}} G^{a^i b^j c^k d^t}, \tag{3}$$

we have $\langle c_1 \rangle \leq G_X$. Consider now the quotient group $\overline{X} := X/G_X$ .We have $\langle \overline{c} \rangle$ is cyclic, while $\overline{G}$ is a maximal stabilizer of G in $[X : G]$, so $\langle \overline{c} \rangle \cap \overline{G} = \langle \overline{c_1} \rangle = \langle \overline{1} \rangle$. $\overline{X}$ is a primitive permutation group on the coset $[X : G]$ which contain a cyclic regular subgroup $\langle \overline{c} \rangle$.By checking the result of Li [2], we have $\mathbb{Z}_p \cong \langle \overline{c} \rangle \leq \overline{X} \leq AGL(1, p)$ with p prime.

If p = 2, then $\overline{X} \leq \mathbb{Z}_2$, $\langle \overline{c} \rangle \cong \mathbb{Z}_2$,so $\overline{X} = \langle \overline{c} \rangle \cong \mathbb{Z}_2$ and $\langle \overline{c_1} \rangle = \langle \overline{c^2} \rangle$ with m even. Since $|X : G| = 2$,$G \lhd X$ and $X = G \rtimes \langle c \rangle$ with $G = H\langle c^2, d \rangle$. Let R denote the maximal subgroup of G contain H. (1) If R = H, $G/R_G$ is a primitive permutation group on the coset $[G : R]$ containing a dihedral group $\langle c^2, d \rangle$ regular. By checking the result of Li [2],we have$(H, \langle c^2, d \rangle) = (A_4, D_4), (S_4, D_4) or (PGL(2, 1)C_f, D_4)$. (2) If R contain an element of Klein group $\langle c^2, d \rangle$ with order 2.We may assume $c^2$ in R, then $c^2 \in R_G$ ,$G/R_G$ is a primitive permutation group on the coset $[G : R]$ containing a cyclic group $\langle \overline{d} \rangle$ regular.Then $\mathbb{Z}_2 \cong \langle \overline{d} \rangle \leq \overline{G} \leq AGL(1, 2)$.Thus, $R_G = R$, and so $G = (H\langle c^2 \rangle) \rtimes \langle d \rangle = (H \rtimes \langle c^2 \rangle) \rtimes \langle d \rangle$.

Now suppose $p \neq 2$. If $d \in G_X$, then $\langle c_1, d\rangle \leq G_X$, thus $\langle c_1, d\rangle \leq G_X \trianglelefteq \langle c, d\rangle$, and so $c_1 = c^2$ which is contradict with $p \neq 2$. So we have $d \notin G_X$.From $\overline{X} \leq AGL(1,p)$, we have $\overline{X} \cong \mathbb{Z}_p \rtimes \mathbb{Z}_r$ with $r|p-1$ .Thus the Sylow-p subgroup of $\overline{X}$ is unique. From $\overline{X} = \overline{G}\langle\overline{c}\rangle$ with the order of $\langle\overline{c}\rangle$ is p, we have $\langle\overline{c}\rangle \trianglelefteq \overline{X}$, and so $\overline{X} = \langle\overline{c}\rangle \rtimes \overline{G}$. Therefore, $\overline{G} \cong \mathbb{Z}_r$ with $r|p-1$. $\overline{G} = \langle\overline{a}, \overline{b}\rangle\langle\overline{d}\rangle \cong \mathbb{Z}_r$. So $\overline{G}$ is cyclic, and thus is abelian.

(i) $a \in G_X$, $b \in G_X$, then we have $\overline{G} = \langle\overline{d}\rangle \cong \mathbb{Z}_2$, $G_X = \langle a, b\rangle\langle c^p\rangle$ and $G = \langle a, b\rangle\langle c^p, d\rangle$.

(ii) $a \in G_X$, $b \notin G_X$, then we have $\overline{G} = \langle\overline{b}\rangle\langle\overline{d}\rangle \cong K_4$ which is contradicted with $\overline{G}$ is cyclic.

(iii) $a^2 \in G_X$, $b \in G_X$, then we have $\overline{G} = \langle\overline{a}\rangle\langle\overline{d}\rangle \cong K_4$ which is contradicted with $\overline{G}$ is cyclic.

(iv) $a^2 \in G_X$, $b \notin G_X$, then we have $\overline{G} = \langle\overline{a}, \overline{b}\rangle\langle\overline{d}\rangle \cong K_4 \rtimes \mathbb{Z}_2$ which is contradicted with $\overline{G}$ is abelian.

In conclusion, only the case $\overline{G} = \langle\overline{d}\rangle \cong \mathbb{Z}_2$ is possible. So $G_X = \langle a, b\rangle\langle c^p\rangle$ and $G = \langle a, b\rangle\langle c^p, d\rangle = G_X \rtimes \langle d\rangle$.

Remain work to classify the exact production of two dihedral group is to determine structure of the situation (i),(ii) and (iii).

# References


[1] Kan Hu and Hao Yu and. On exact products of two dihedral groups. *Communications in Algebra*, 0(0):1–9, 2025.

[2] Cai Li. Finite edge-transitive cayley graphs and rotary cayley maps. *Transactions of the American Mathematical Society*, 358(10):4605–4635, 2006.