UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Discussion 7

Date: Wednesday, March 8, 2017

Problem 1. (Final Sp06) Consider a particle moving according to the following Markov Chain:

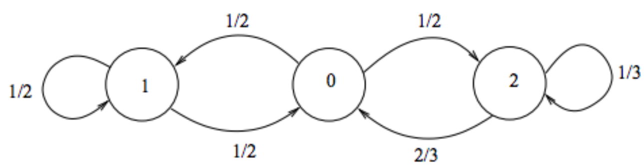

Figure 1: Markov Chain for Problem 1

(a.) Classify the states in the Markov Chain. Is it periodic?

Solution: It is irreducible and aperiodic.

(b.) In the long run, what fraction of time does it spend in state 1?

Solution: One needs to find the invariant distribution: $\pi(0) = \frac{4}{11}, \pi(1) = \frac{4}{11}, \pi(2) = \frac{3}{11}$.

(c.) Suppose X_0 is selected according to the steady state distribution. Conditioned on $X_2 = 2$, what is the probability that $X_0 = 0$?

Solution: We want:

$$
\begin{aligned}
P(X_0 = 0 | X_2 = 2) &= \frac{P(X_0 = 0, X_2 = 2)}{P(X_2 = 2)} \\
&= \frac{P(X_0 = 0, X_1 = 2, X_2 = 2)}{P(X_2 = 2)} \\
&= \frac{2}{9}
\end{aligned}
$$

(d.) Suppose $X_0 = 0$, find the expected amount of time until the particle has visited all the states.

Solution: Let T denote the amount of time until the particle has visited all the states. We have:

$$
\begin{aligned}
E[T] &= \frac{1}{2}(E[T|X_1 = 1] + E[T|X_1 = 2]) \\
&= \frac{13}{2}
\end{aligned}
$$

(e.) Suppose there are two particles moving according to the Markov Chain. One particle starts in state 1 and the other starts in state 2. What is the expected amount of time before at least one of the particles is at state 0?

Solution: Let T_i be the amount of time it takes for the particle starting in state i to reach 0. We are interested in $E[T] = E[\min T_1, T_2]$. Note that $T_1 \sim \text{Geom}(\frac{1}{2})$ and $T_2 \sim \text{Geom}(\frac{2}{3})$. Thus, using the tail sum formula, we can see that $E[T] = \frac{6}{5}$.

Problem 2. (Final Fa14) An online dating website tries to match couples. Let X_n be the number of members of this site at time slot n. We want to analyze the discrete-time process $\{X_n, n \geq 0\}$. At each time slot, exactly one of the following events happens: (i) Two persons are happily matched and leave the website forever with probability p, (ii) A single frustrated person leaves the system individually with probability q , and (iii) a new member joins the system with probability $r = 1-p-q$. If there is only one member in the system, that member leaves with probability $p + q = 1 - r$. Suppose that $r - q - 2p > 0$: is the Markov Chain $\{X_n, n \geq 0\}$ positive recurrent, null recurrent, or transient? Prove your answer.

Solution: Consider the iid random variables Z_i which have value 1 with probability r, -1 with probability q and -2 with probability p. Now, note that $X_n = \max X_{n-1} + Z_n, 0$. Thus, $X_n \geq X_{n-1}+Z_n$ and we have $X_n \geq X_0+Z_1+Z_2+\cdots+Z_n$. Thus, $\frac{X_n}{n} \geq \frac{X_0}{n} + \frac{Z_1+Z_2+\cdots+Z_n}{n} \to E[Z]$. Now, note that under the given condition, $E[Z] > 0$, and thus, $X(n) \to \infty$. This implies that each state is visited only finitely many times, and the chain is transient.