UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 10
Spring 2017

Issued: Thursday, April 13, 2017 **Due:** 8:00am Thursday, April 20, 2017

Problem 1. The difficulty of an EE126 exam, Θ, is uniformly distributed on $[0, 100]$, and Alice gets a score X that is uniformly distributed on $[0, \Theta]$. Alice gets her score back and wants to estimate the difficulty of the exam.

(a) On the same axis, plot the MAP, LLSE, and MMSE of Θ as a function of the score X.

(b) On the same axis, plot the Mean Squared Error (MSE) of the MAP, LLSE, and MMSE of Θ as a function of X.

Problem 2. Let X, Y, Z be three random variables. Prove formally that

$$E[|X - E[X|Y]|^2] \geq E[|X - E[X|Y, Z]|^2].$$

What is the intuition behind the inequality?

Problem 3. (a.) Consider $Y = X^2 + Z$ where $Z \sim \mathcal{N}(0, 1)$ and Z is independent of X. Does there exist an unbiased estimator $\hat{X}$ based on observation Y?

(b.) Suppose X and Y are independent random variables such that $X \sim U[-1, 1]$ and $Y \sim \mathcal{N}(0, \frac{1}{3})$. Find $L[X|X + Y, X - Y]$.

Problem 4. Let U, V be jointly Gaussian random variables with means $\mu_U = 1$, $\mu_V = 4$ and $\sigma_U^2 = 2.5, \sigma_V^2 = 2$ and covariance $\rho = 1$. Can we write $U = aV + Z$ where a is a scalar and Z is independent of V? If so, find a and Z, if not explain why.

Problem 5. Assume that $(X, Y_n, n \geq 0)$ are mutually independent random variables with $X \sim N(0, 1)$, $Y_n \sim N(0, 1)$. Let $\hat{X}_n$ be the MMSE of X given $X + Y_1, X + Y_2, \ldots, X + Y_n$. Find the smallest value of n such that:

$$P(|X - \hat{X}_n| > 0.1) \leq 0.05$$

Problem 6. Suppose you have a particle which begins at some state $X(0) \sim N(0,1)$. The particle moves according to the following dynamics:

$$X(n) = \frac{1}{2}X(n-1) + V(n)$$

where $V(n) \sim \mathcal{N}(0,1)$ (and independent of the past). You make observations of the following form:

$$Y(n) = X(n) + W(n)$$

where $W(n) \sim \mathcal{N}(0,1)$ (and independent of the past)

(a) Suppose that you observe $Y(0)$. Find the MMSE of the initial position $X(0)$ given this observation.

(b) Now, the process has gone for 10 time steps. At each time step, a genie has given you the MMSE $\hat{X}(n)$. However, for the final step, the genie disappears and you would like to find the estimate yourself. Using only the predictor $\hat{X}(9) = 4$ and the observation $Y(10) = 4.5$, find $\hat{X}(10)$, the MMSE of the position of the particle at time step 10.