UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Process

Solution 10
Spring 2017

Issued: Thursday, April 13, 2017

Self-graded Scores Due: 5pm, Monday, April 24, 2017
Submit your self-graded scores via the google form:
https://goo.gl/forms/JMeMWFTMaSl6tOaB3. Make sure that you use your **SORTABLE NAME** on bCourses.

Problem 1. We first compute the LLSE: We need $E[\Theta]$, σ_X^2, and $\text{cov}(X, \Theta)$. First, $E[\Theta] = 50$. The variance of X is

$$\text{var}(X) = E[\text{var}(X|\Theta)] + \text{var}(E[X|\Theta]).$$

The first term can be found as follows.

$$\text{var}(X|\Theta) = \Theta^2/12 \Rightarrow E[\text{var}(X|\Theta)] = \int_0^{100} \frac{1}{100}\frac{\theta^2}{12} d\theta = \frac{10000}{36}$$

By noting $E[X|\Theta] = \Theta/2$, the second term is $\frac{1}{4}\frac{10000}{12} = \frac{10000}{48}$. Thus, $\text{var}(X) = \frac{70000}{144}$.

Now, the covariance can be found as follows.

$$\text{cov}(X, \Theta) = E[\Theta X] - E[\Theta]E[X]$$

We found $E[\Theta]$ above, and $E[X] = E[E[X|\Theta]] = E[\Theta/2] = 25$. Also,

$$E[\Theta X] = E[E[\Theta X|\Theta]] = E[\Theta^2/2] = (\text{var}(\Theta)+E[\Theta]^2)/2 = (\frac{10000}{12}+2500)/2 = \frac{5000}{3}.$$

Thus, $\text{cov}(X, \Theta) = \frac{1250}{3}$. Then, the LLSE of Θ is

$$L[\Theta|X] = E[\Theta] + \frac{\text{cov}(X, \Theta)}{\sigma_X^2}(X - E[X]) = 50 + \frac{6}{7}(X - 25).$$

Now, we compute the MAP rule: Given X, $X \leq \Theta \leq 100$. In order to maximize $f(X|\Theta) = \frac{1}{\Theta}$, one should choose $\hat{\Theta} = X$.

Finally, the MMSE is given by: The MMSE of Θ given X is $E[\Theta|X]$. We know that $f(\theta|x) = \frac{f(x|\theta)f(\theta)}{f(x)}$. We can easily see that $f(\theta) = \frac{1}{100}\mathbf{1}\{0 \leq \theta \leq 100\}$, and that $f(x|\theta) = \frac{1}{\theta}\mathbf{1}\{0 \leq x \leq \theta\}$. To find $f(x)$ we integrate the joint distribution $f(x, \theta) = f(x|\theta)f(\theta)$ with respect to θ.

$$f(x) = \int_{\theta=x}^{100} \frac{1}{100\theta} d\theta = \frac{1}{100} \log \frac{100}{x}.$$

Thus,

$$f(\theta|x) = \frac{1}{\theta \log \frac{100}{x}} \mathbf{1}\{0 \leq x \leq \theta \leq 100\}.$$

$$E[\Theta|X] = \int_{\theta=x}^{100} \frac{\theta}{\theta \log \frac{100}{x}} d\theta = \frac{100-x}{\log \frac{100}{x}}.$$

(a) The different estimators are as shown in Figure 1.

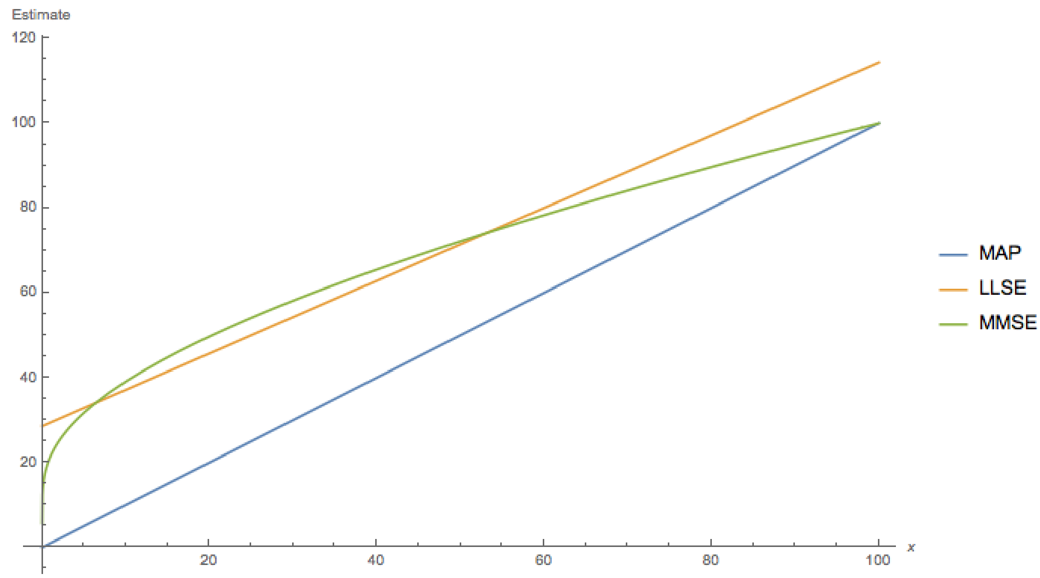

Figure 1: Comparison of estimators

(b) The mean squared error of the MMSE as a function of X is shown in Figure 2.

Problem 2. Let $X_1 = E[X|Y]$ and $X_2 = E[X|Y, Z]$. Note that

$$E((X - X_2)(X_2 - X_1)) = E(E[(X - X_2)(X_2 - X_1)|Y, Z]).$$

Also,

$$E[(X - X_2)(X_2 - X_1)|Y, Z] = (X_2 - X_1)E[X - X_2|Y, Z] = (X_2 - X_1)(X_2 - X_2) = 0.$$

Hence,

$$E((X - X_1)^2) = E((X - X_2 + X_2 - X_1)^2) = E((X - X_2)^2) + E((X_2 - X_1)^2) \geq E((X - X_2)^2).$$

The intuition of this inequality is that as we gain more information, the MMSE becomes more precise, i.e., the variance of the estimator decreases.

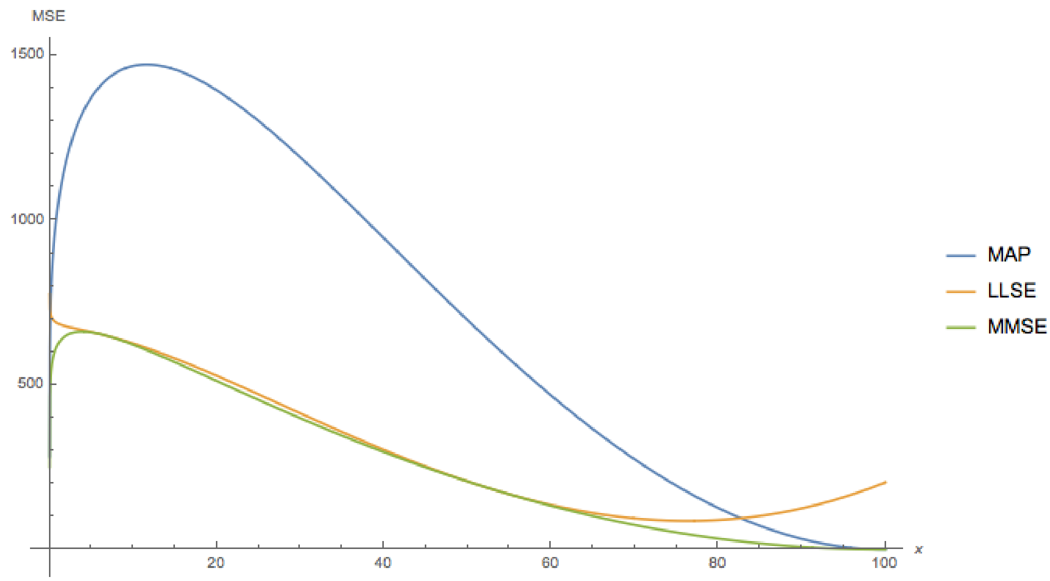

Figure 2: Mean squared error

Problem 3. (a) Note that $f_{Y|X}(y|x) = f_{Z|X}(y-x^2) = f_{Y|X}(y|-x)$. Thus, one can see that for any estimator $\hat{X}$, $E[\hat{X}|X=x] = E[\hat{X}|X=-x]$. Let this quantity be b. Note that in order for $\hat{X}$ to be unbiased, one needs $E[\hat{X}|X=x] - x = 0$ as well as $E[\hat{X}|X=-x] + x = 0$. This happens only when $x = 0$, so there is no unbiased estimator based on the observations Y.

(b) Note that $X+Y$ and $X-Y$ are orthogonal, so $L[X|X+Y, X-Y] = L[X|X+Y] + L[X|X-Y]$. Note that $L[X|X+Y] = \frac{1}{2}(X+Y)$ and $L[X|X-Y] = \frac{1}{2}(X-Y)$. Thus, $L[X|X+Y, X-Y] = \frac{1}{2}(X+Y) + \frac{1}{2}(X-Y)$.

Problem 4. The LLSE of U given V is $L[U|V] = \mu_U + \frac{\rho}{\sigma_V^2}(V - \mu_V)$. We also know that for jointly Gaussian random variables, $Z' = U - L[U|V]$ is independent of V, we have:

$$U = \mu_U + \frac{\rho}{\sigma_V^2}(V - \mu_V) + Z'$$

Let $a = \frac{\rho}{\sigma_V^2}, Z = \mu_U - \frac{\rho}{\sigma_V^2}\mu_V + Z'$. Then $U = aV + Z$ and Z is independent of V. Thus we can see that $E[Z] = \mu_U - \frac{\rho}{\sigma_V^2}\mu_V$, and $\text{var}(Z) = \sigma_U^2 - \frac{\rho^2}{\sigma_V^2}$. Then, we know that $Z \sim \mathcal{N}(\mu_U - \frac{\rho}{\sigma_V^2}\mu_V, \sigma_U^2 - \frac{\rho^2}{\sigma_V^2})$. Thus, we know that $a = 0.5$ and $Z \sim \mathcal{N}(-1, 2)$.

Problem 5. First note that since $(X, X+Y_1, \ldots, X+Y_n)$ are jointly Gaussian, the MMSE is linear: $E[X|Y] = \sum_i c_i(X+Y_i) + c_0$. Note that by symmetry, all the coefficients are equal and $c_0 = 0$ as the estimator is unbiased. Thus, we can see that $\hat{X}_n$ takes the form $a_n(nX + \sum_{i=1}^n Y_i)$. Now, note that $E[(X - a_n(nX+Y_1))(X+Y_1)] = 0$ by orthogonality. Thus, $E[X^2] = a_n n E[X^2] - a_n E[Y^2] = 1 - a_n(n+1) = 0$, so $a_n = \frac{1}{n+1}$. Thus, one has $X - \hat{X} = \frac{1}{n+1}X - \frac{\sum_i Y_i}{n+1} = \mathcal{N}(0, \frac{1}{(n+1)^2} + \frac{n}{(n+1)^2}) = \mathcal{N}(0, \frac{1}{n+1}) = \frac{1}{\sqrt{n+1}}Z$, where Z is standard normal. Thus, $P(|\frac{1}{\sqrt{n+1}}Z| > 0.1) =$

$P(|Z| > 0.1\sqrt{n+1}) = 0.05$. Using the table, this holds if $0.1\sqrt{n+1} = 1.96$, so $n = \lceil 383.16 \rceil = 384$.

Problem 6. (a) $\hat{X}(0) = \frac{1}{2}Y(0)$

(b) We use the scalar Kalman filter. Note that $\hat{X}(10) = \frac{1}{2}\hat{X}(9) + k_{10}(Y(10) - \frac{1}{2}\hat{X}(9))$. All that is left to compute is the gain k_{10}. We compute $k_{10} = 0.531$ so that $\hat{X}(10) = 3.32$.