Multivariate Gaussians

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Multivariate Gaussian Density

$Z_1, \dots, Z_n$ are i.i.d. $\mathcal{N}(0,1)$. Density?

$$f_Z(z) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z_i^2\right) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2} z^\top z\right).$$

$A \in \mathbb{R}^{n \times n}$ is an invertible matrix.

$\mu \in \mathbb{R}^n$ is a vector of means.

Change of variables: $X = AZ + \mu$.

$$f_X(x)\,\mathrm{d}x = f_Z(z)\,\mathrm{d}z.$$

Multivariate calculus: $\mathrm{d}x = \det(A)\,\mathrm{d}z$.

$$\begin{aligned} f_X(x) &= \frac{1}{\det(A)} f_Z(A^{-1}(x-\mu)) \\ &= \frac{1}{(2\pi)^{n/2}\det(A)} \exp\left(-\frac{1}{2}(x-\mu)^\top (A^{-1})^\top A^{-1}(x-\mu)\right). \end{aligned}$$

Multivariate Gaussian Density

$$f_X(x) = \frac{1}{(2\pi)^{n/2}\det(A)} \exp\left(-\frac{1}{2}(x-\mu)^\top (AA^\top)^{-1}(x-\mu)\right).$$

Let $\Sigma = AA^\top$.

$\det(A) = \det(\Sigma)^{1/2}$.

$$f_X(x) = \frac{1}{(2\pi)^{n/2}(\det(\Sigma))^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right).$$

We assumed A is invertible (non-degenerate case).

Σ is the covariance matrix: $\Sigma_{i,j} = \mathrm{cov}(X_i, X_j)$.

Proof: $\mathrm{cov}((AZ)_i, (AZ)_j) = \mathrm{cov}(\sum_{k=1}^n A_{i,k}Z_k, \sum_{l=1}^n A_{j,l}Z_l)$
$= \sum_{k=1}^n A_{i,k}A_{j,k} = (AA^\top)_{i,j}$.

Linear Combinations of Joint Gaussians

Linear combinations of jointly Gaussian random variables are jointly Gaussian.

Write $X = AZ + \mu$.

Then, $BX = (BA)Z + B\mu$.

$\implies$ BX is jointly Gaussian.

Conversely, if any linear combination of $X_1, \ldots, X_n$ is Gaussian, then $X_1, \ldots, X_n$ are jointly Gaussian.

Example: Let $X \sim \mathcal{N}(0, 1)$ and $\Pr(Z = \pm 1) = 1/2$ be independent.

X and $Y = XZ$ are marginally Gaussian.

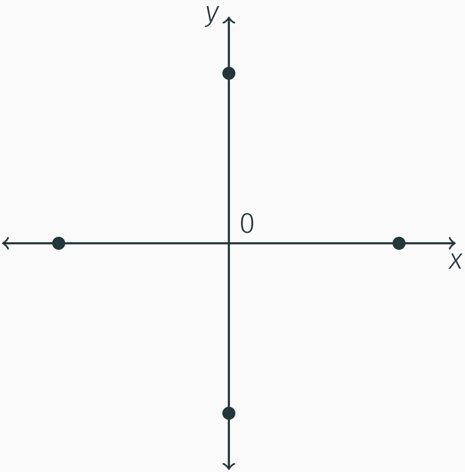
X and $Y = XZ$ are not jointly Gaussian.

$X - Y = (1 - Z)X$.

$\Pr(X - Y = 0) = 1/2$.

Uncorrelated Random Variables

Uncorrelated random variables are not necessarily independent.



Uncorrelated Gaussians

If $X_1, \ldots, X_n$ are uncorrelated, jointly Gaussian, they are independent.

uncorrelated $\implies$ Σ is diagonal

Let $\Sigma^{-1} = \operatorname{diag}(\sigma_1^{-2}, \ldots, \sigma_n^{-2})$.

$$\begin{aligned} f_X(x) &= \frac{1}{(2\pi)^{n/2}(\det(\Sigma))^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right) \\ &= \frac{1}{(2\pi)^{n/2}\sigma_1 \cdots \sigma_n} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} \frac{1}{\sigma_i^2}(x_i-\mu_i)^2\right) \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{1}{2\sigma_i^2}(x_i-\mu_i)^2\right). \end{aligned}$$

Joint density factors into product of marginal densities.

Lemma (Independent Random Variables)

If X and Y are independent, then so are X and $\phi(Y)$.

Proof:

$$\begin{aligned}\Pr(X \in A, \phi(Y) \in B) &= \Pr(X \in A, Y \in \phi^{-1}(B)) \\ &= \Pr(X \in A) \Pr(Y \in \phi^{-1}(B)) \\ &= \Pr(X \in A) \Pr(\phi(Y) \in B).\end{aligned}$$

Also recall:

If X and Y are independent, zero-mean, they are orthogonal.

$Y - L[Y \mid X]$ is orthogonal to any linear function of X.

$Y - E[Y \mid X]$ is orthogonal to any function of x.

For jointly Gaussian X and Y, $L[Y \mid X]$ is Gaussian.

Jointly Gaussian LLSE = MMSE

For jointly Gaussian X, Y, the LLSE and the MMSE coincide.

$Y - L[Y \mid X]$ and X are orthogonal.

$Y - L[Y \mid X]$ and X are uncorrelated.

$Y - L[Y \mid X]$ and X are independent.

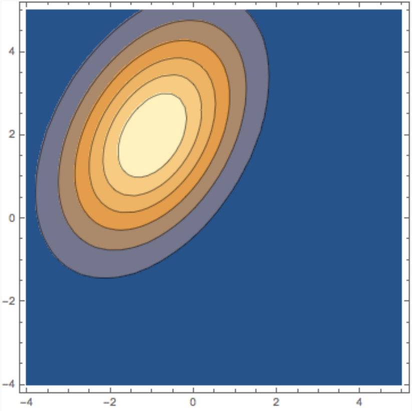
$Y - L[Y \mid X]$ and $\phi(X)$ are independent.

$Y - L[Y \mid X]$ and $\phi(X)$ are orthogonal.

$\implies$ $Y - L[Y \mid X]$ is orthogonal to any $\phi(X)$, so $L[Y \mid X] = E[Y \mid X]$.

Visualization

$$X \sim \mathcal{N}\left(\begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}2 & 1\\1 & 3\end{bmatrix}\right).$$



Eigenvectors of $\boldsymbol{\Sigma}$:

$$\lambda_1 = 3.62, \qquad v_1 = \begin{bmatrix}0.526\\0.851\end{bmatrix}$$

$$\lambda_2 = 1.38, \qquad v_2 = \begin{bmatrix}-0.851\\0.525\end{bmatrix}$$