UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Process

Solution 1
Spring 2017

Issued: Tuesday, January 17, 2017

Self-graded Scores Due: 5pm, Monday, January 30, 2017
Submit your self-graded scores via the google form:
https://goo.gl/forms/iRA5Uy9Jk35uWZHu2. Make sure that you use your **SORTABLE NAME** on bCourses.

Problem 1. Let A be the event that 1 or 2 appears in a die roll, B be the event that 1 or 3 appears, and C be the event that 1 or 4 appears. Then, $\Pr(A) = \Pr(B) = \Pr(C) = 1/2$. Furthermore,

$$\Pr(A \cap B) = \Pr(\{1 \text{ appears}\}) = 1/4 = \Pr(A)\Pr(B).$$

So A and B are pairwise independent. Similarly (A, C) and (B, C) are pairwise independent. However,

$$\Pr(A \cap B \cap C) = \Pr(\{1 \text{ appears}\}) = 1/4 \neq \Pr(A)\Pr(B)\Pr(C) = 1/8.$$

So these 3 events are not mutually independent.
The answer is not unique; any other valid answer is acceptable.

Problem 2. Let A be the event the batch is accepted. Then $A = A_1 \cap A_2 \cap A_3 \cap A_4$, where A_i is the event that the ith item is not defective. We thus have:

$$\begin{aligned} P(A) &= P(A_1)P(A_2|A_1)P(A_3|A_2, A_1)P(A_4|A_3, A_2, A_1) \\ &= \frac{45}{50} \cdot \frac{44}{49} \cdot \frac{43}{48} \cdot \frac{42}{47} \end{aligned}$$

Problem 3. Let n be the amount in the larger envelope and m the amount in the smaller envelope. Also, let $X = \text{Number of tosses} + \frac{1}{2}$. Now, define the 3 events:

$$A = X < m, B = m < X < n, C = n < X$$

Let W be the event that you end up with the larger envelope when you use your friend's strategy. Thus, we have:

$$P(W|A) = \frac{1}{2}, P(W|B) = 1, P(W|C) = \frac{1}{2}$$

By the law of total probability, we thus have that:

$$P(W) = P(W|A)P(A) + P(W|B)P(B) + P(W|C)P(C) = \frac{1}{2}(1 + P(B)) > \frac{1}{2}$$

so your friend is correct.

Problem 4. (a.) Let F be the event that the coin is fair, B the event that the coin is biased, and S the event that there are 2 heads in 4 flips. We have:

$$\begin{aligned}
P(F|S) &= \frac{P(S|F)P(F)}{P(S)} \\
&= \frac{P(S|F)P(F)}{P(S|F)P(F) + P(S|B)P(B)} \\
&= \frac{\binom{4}{2}(\frac{1}{2})^4(\frac{1}{2})}{\binom{4}{2}(\frac{1}{2})^4(\frac{1}{2}) + \binom{4}{2}(\frac{3}{4})^2(\frac{1}{4})^2(\frac{1}{2})} \\
&= 0.64
\end{aligned}$$

(b.) Flip the coin twice. If the sequence is heads-tails, declare a head and if the sequence is tails-heads, declare a tails. If either heads-heads or tails-tails were flipped, flip the coin twice again and repeat the procedure until a decision is made. Suppose that A_k is the event that a decision was made at the kth round. Then, we have:

$$\begin{aligned}
P(H) &= \sum_{k=1}^{\infty} P(H|A_k)P(A_k) \\
&= \sum_{k=1}^{\infty} \frac{1}{2}P(A_k) \\
&= \frac{1}{2}
\end{aligned}$$

Problem 5. Let A_i be the event that in the ith round, the ball is passed from Bob. Then we have $\Pr(A_1) = 1$, and

$$\begin{aligned}
\Pr(A_i) &= \Pr(A_i|A_{i-1})\Pr(A_{i-1}) + \Pr(A_i|\bar{A}_{i-1})\Pr(\bar{A}_{i-1}) \\
&= 0 + \frac{1}{m-1}(1 - \Pr(A_{i-1})),
\end{aligned}$$

which gives us

$$\Pr(A_i) - \frac{1}{m} = -\frac{1}{m-1}[\Pr(A_{i-1}) - \frac{1}{m}],$$

and then

$$\Pr(A_i) - \frac{1}{m} = (-\frac{1}{m-1})^{n-1}[\Pr(A_1) - \frac{1}{m}].$$

By $\Pr(A_1) = 1$, we can get $\Pr(A_n) = \frac{1}{m}[1 - (-\frac{1}{m-1})^{n-2}]$.

Problem 6. Enumerate the vertices of the cube and let B_i be the event that vertex

i is blue. Note that:

$$\begin{aligned} P(B_1 \cup B_2 \cup B_3 \cdots \cup B_8) &\leq \sum_{i=1}^{8} P(B_i) && (1) \\ &= \sum_{i=1}^{8} \frac{1}{10} && (2) \\ &= \frac{8}{10} && (3) \\ &< 1 && (4) \end{aligned}$$

In other words, the probability of at least one vertex being blue is *less* than 1, so there must exist a coloring where each vertex is red.

Note: This is an example of a powerful tool known as the probabilistic method.

Problem 7. Note that in order for Hillary to have been strictly ahead of The Donald, there can never have been a tie over the course of the game. Thus, letting T be the event that they are eventually tied and A the event that Hillary is always strictly ahead, we can see that $P(A) = 1 - P(T)$. Now, to compute $P(T)$, we make a small observation. Let D_i be the event that the ith point went to the Donald, and H_i the analagous event for Hillary. Then, we have $P(T, D_1) = P(T, H_1)$. To see this, consider any sequence of points in which The Donald scored the first point and they were eventually tied. Now, flip all of the points until they are tied, and keep the remainder of the sequence of the same. From this one to one correspondence, we can see that $P(T, D_1) = P(T, H_1)$. Now, note that since Hillary won, $P(T, D_1) = P(D_1) = \frac{m}{m+n}$. Thus, $P(T) = \frac{2m}{m+n}$ and $P(A) = \frac{n-m}{n+m}$.

Problem 8. First, we define some events. Let C be the event that the the best city of the N is chosen. Now, let C_i be the event that the ith city is the best city *and* is selected. Additionally, let S_i be the event that out of the i cities interviewed so far, the second best city was in the first m interviewed. Lastly, we define B_i as the event that the ith city is the best city. We are looking for:

$$\begin{aligned} P(C) &= \sum_{i=1}^{N} P(C_i) \\ &= \sum_{i=1}^{N} P(S_i \cap B_i) \\ &= \sum_{i=1}^{N} P(S_i | B_i) P(B_i) \end{aligned}$$

Where the first equality comes from the law of total probability. Now, we note that $P(B_i) = \frac{1}{N}$. Additionally, since the second best of the i candidates must be in the first m of the other $i-1$ candidates, we have $P(S_i|B_i) = \frac{m}{i-1}$. Thus:

$$P(C) = \frac{m}{N} \sum_{i=m+1}^{N} \frac{1}{i-1}$$

Arriving at this point is sufficient for full credit. We now note:

$$\begin{aligned} P(C) &= \frac{m}{N} \sum_{i=m+1}^{N} \frac{1}{i-1} \\ &= \frac{m}{N} \sum_{i=m+1}^{N} \frac{1}{N} \frac{N}{i-1} \end{aligned}$$

We can thus let $x = \frac{m}{N}$, $t = \frac{i}{N}$. By noting the Riemann sum, we see that this last sum corresponds to a Riemann sum of the function $\frac{1}{t}$ on the interval $[x, 1]$. Thus, we see that $P(C)$ converges to $-x \ln x$. Taking the derivative and setting to 0, we find that the optimal value of x is $\frac{1}{e}$, and thus the optimal cutoff is $m = \frac{N}{e}$ and the probability of selecting the best candidate given that cutoff is $\frac{1}{e}$.

Note: This problem is a famous example from optimal stopping theory and is commonly known as the secretary problem (a boss is interviewing secretaries instead of a commissioner interviewing city representatives). In fact, one may use a dynamic programming approach to see why the policy outlined here is in fact the optimal policy. For those interested, the details of such an approach can be found in *Dynamic Programming and the Secretary Problem* by M.J. Beckmann.