UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Discussion 1
GSI : Kabir Chandrasekher

Date: Wednesday, January 18, 2017

Problem 1.

(a.) $\frac{3}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{2}$

(b.) $\frac{\frac{3}{4} \cdot \frac{1}{2}}{\frac{1}{2}}$

(c.) $\frac{\frac{1}{2} \cdot (\frac{3}{4})^3 + \frac{1}{2} \cdot (\frac{1}{4})^3}{\frac{1}{2} \cdot (\frac{3}{4})^2 + \frac{1}{2} \cdot (\frac{1}{4})^2}$

Problem 2. We introduce the indicator X_i which takes the value 1 if the ith attendee receives his or her own hat and 0 otherwise. Additionally, let N be the event that none of the attendees receive their own hats. We are interested in the following:

$$\begin{aligned} P(N) &= 1 - P(X_1 \cup X_2 \cup \cdots \cup X_n) && (1) \\ &= 1 - \big(\sum_{i_1} P(X_{i_1}) - \sum_{i_1 < i_2} P(X_{i_1}, X_{i_2}) + \cdots + (-1)^{n+1} P(X_1, X_2, \ldots, X_n)\big) && (2) \end{aligned}$$

where the second equality follows from the principle of inclusion-exclusion. Now, note that:

$$\begin{aligned} \sum_{i_1 < i_2 < \cdots < i_k} P(X_{i_1}, X_{i_2}, \ldots, X_{i_k}) &= \sum_{i_1 < i_2 < \cdots < i_k} P(X_{i_1}) P(X_{i_2} | X_{i_1}) \cdots P(X_{i_k} | X_{i_1}, X_{i_2}, \ldots, X_{i_k}) \\ &= \frac{1}{k!} \end{aligned}$$

Thus:

$$\begin{aligned} P(N) &= 1 - \big(1 - \frac{1}{2!} + \cdots + (-1)^{n+1} \frac{1}{n!}\big) \\ &= \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \end{aligned}$$

Note that as $n \to \infty$, $P(N) \to \frac{1}{e}$.

Problem 3. **Solution 1:** We use the law of total probability. Let W be the event that Bob wins. Now suppose that $2n$ coins have been tossed (n by Alice and n by Bob). Let E be the event they have the same number of heads, B the event Bob has more heads than Alice and A the event that Alice has more heads. Then we have:

$$\begin{aligned} P(W) &= P(W|E)P(E) + P(W|B)P(B) + P(W|A)P(A) \\ &= \frac{1}{2}P(E) + P(B) \\ &= \frac{1}{2} \end{aligned}$$

The send equality follows since $P(W|A) = 0$, $P(W|B) = 1$, and $P(W|E) = \frac{1}{2}$. The last equality follows since $P(B) = P(A)$ (by symmetry).

Solution 2: Note that since Bob tosses 1 more coin than Alice, Bob can either have more tails than Alice or more heads than Alice. Notice that there is a $1:1$ correspondence between these events, so the probability that he has more heads is $\frac{1}{2}$.