UC Berkeley
Department of Electrical Engineering and Computer Sciences
EE126: Probability and Random Processes
Discussion 10
Date: Wednesday, April 12, 2017

Question:
(a) Assume that X, Y, Z are zero-mean random variables, and Y and Z are orthogonal. Show that
$$L[X|Y, Z] = L[X|Y] + L[X|Z].$$
(b) Assume that X, Y, Z are zero-mean random variables. Show that
$$L[X|Y, Z] = L[X|Y] + L[X|Z - L[Z|Y]].$$

See Section 8.1 of Walrand's book for the solution.

Jointly Gaussian Random Variables
There are three 'equivalent' definitions of a jointly Gaussian (JG) random vector.

(1) A random vector $X = (X_1, X_2, \ldots, X_k)^T$ is JG if there exists a base random vector $Z = (Z_1, Z_2, \ldots, Z_\ell)^T$ whose components are independent standard normal random variables, a transition matrix $A \in \mathbb{R}^{k\times\ell}$, and a mean vector $\mu \in \mathbb{R}^k$, such that $X = AZ + \mu$.

(2) A random vector $X = (X_1, X_2, \ldots, X_k)^T$ is JG if $Y = \sum_{i=1}^k a_i X_k$ is normally distributed for every $a = (a_1, a_2, \ldots, a_k)^T \in \mathbb{R}^k$. *Note: a point mass on a value is considered as a normal distribution with zero variance.* Thus, if $X_1 = \mathcal{N}(0, 1)$ and $X_2 = -X_1$, $X = (X_1, X_2)^T$ still is a JG random vector.

(3) (Non-degenerate case only) A random vector $X = (X_1, X_2, \ldots, X_k)^T$ is JG if
$$f_X(x) = \frac{1}{\sqrt{|\Sigma|}(2\pi)^{k/2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$$
for a positive definite matrix $\Sigma \in \mathbb{R}^{k\times k}$ and a vector $\mu \in \mathbb{R}^k$. In this case, Σ is the covariance matrix and μ is the mean vector of X.

We can find a relation between A in (1) and Σ in (3).
$$\begin{aligned}\Sigma &= E[(X-\mu)(X-\mu)^T] \\ &= E[(AZ)(AZ)^T] \\ &= AE[ZZ^T]A^T.\end{aligned}$$

Since the components of Z are independent standard Gaussian random variables, the expectation term in the middle is equal to the identity matrix. Therefore,
$$\Sigma = AA^T.$$

Note that when A is not full row-rank, elements of X are not linearly independent and Σ is not positive definite.

Question: Show that if the elements of a JG random vector are uncorrelated, they are independent.

Solution: If the elements of X are uncorrelated, then the off-diagonal elements of the covariance matrix Σ are zero:

$$\Sigma = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2).$$

Then,

$$\Sigma^{-1} = \mathrm{diag}\left(1/\sigma_1^2, 1/\sigma_2^2, \ldots, 1/\sigma_k^2\right)$$

and

$$|\Sigma| = \det(\Sigma) = \prod_{i=1}^{k} \sigma_i^2.$$

We can write the joint density of X as

$$\begin{aligned}
f_X(x) &= \frac{1}{\sqrt{|\Sigma|}(2\pi)^{k/2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right) \\
&= \frac{1}{\left(\prod_{i=1}^{k} \sigma_i\right)(2\pi)^{k/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{k}(x_i-\mu_i)\frac{1}{\sigma_i^2}(x_i-\mu_i)\right) \\
&= \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{(x_i-\mu_i)^2}{2\sigma_i^2}\right) \\
&= \prod_{i=1}^{k} f_{X_i}(x_i)
\end{aligned}$$

Since the joint distribution can be written as the product of marginal distributions, the elements of X are independent.