UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Discussion 9
Spring 2017

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Problem 1. (a) The MLE of the source is as follows.

$$\hat{u} = \arg\max_u P(G|u)$$

Note that the number of people that can be chosen is always 2. Thus,

$$P(G|\pm 5) = \frac{10!}{10!0!} \times \frac{1}{2^{10}}$$
$$P(G|\pm 4) = \frac{10!}{9!1!} \times \frac{1}{2^{10}}$$
$$P(G|\pm 3) = \frac{10!}{8!2!} \times \frac{1}{2^{10}}$$
$$P(G|\pm 2) = \frac{10!}{7!3!} \times \frac{1}{2^{10}}$$
$$P(G|\pm 1) = \frac{10!}{6!4!} \times \frac{1}{2^{10}}$$
$$P(G|0) = \frac{10!}{5!5!} \times \frac{1}{2^{10}}$$

Thus, the MLE of the source is node 0.

(b) Note that the number of people that can be chosen is always $3, 4, 5, 6$ as the number of infected nodes increase. Thus,

$$P(G|1) = \frac{3!}{2!1!} \times \frac{1}{360} = \frac{3}{360}$$
$$P(G|2) = \frac{4!}{2!1!1!} \times \frac{1}{360} = \frac{12}{360}$$
$$P(G|3) = \frac{3!}{2!1!} \times \frac{1}{360} = \frac{3}{360}$$
$$P(G|4) = \frac{4!}{3!1!}\frac{2!}{1!1!} \times \frac{1}{360} = \frac{8}{360}$$
$$P(G|5) = \frac{2!}{1!1!} \times \frac{1}{360} = \frac{2}{360}$$

Thus, the MLE of the source is node 2.

(c) Note that $\pi(4) = \frac{1}{3}$ and $\pi(i) = \frac{1}{6}$ for $i \neq 4$. The MLE of the source is as follows.

$$\hat{u} = \pi(u) \arg\max_u P(G|u)$$

Thus,

$$\begin{aligned}
\pi(1)P(G|1) &= \frac{1}{6}\frac{3!}{2!1!} \times \frac{1}{360} = \frac{3}{2160} \\
\pi(2)P(G|2) &= \frac{1}{6}\frac{4!}{2!1!1!} \times \frac{1}{360} = \frac{12}{2160} \\
\pi(3)P(G|3) &= \frac{1}{6}\frac{3!}{2!1!} \times \frac{1}{360} = \frac{3}{2160} \\
\pi(4)P(G|4) &= \frac{2}{6}\frac{4!}{3!1!}\frac{2!}{1!1!} \times \frac{1}{360} = \frac{16}{2160} \\
\pi(5)P(G|5) &= \frac{1}{6}\frac{2!}{1!1!} \times \frac{1}{360} = \frac{2}{2160}
\end{aligned}$$

Thus, the MAP estimate of the source is node 2.

(d) As the size of the set of neighbors depends on path, one cannot just count the number of paths.

$$\begin{aligned}
P(G|(0,0)) &= 4 \cdot \frac{1}{4} \cdot \frac{1}{6} \cdot \frac{1}{7} + 2 \cdot \frac{1}{4} \cdot \frac{1}{6} \cdot \frac{1}{8} = \frac{1}{42} + \frac{1}{96} \\
P(G|(1,0)) = P(G|(-1,0)) &= \frac{1}{4} \cdot \frac{1}{6} \cdot \frac{1}{7} + \frac{1}{4} \cdot \frac{1}{6} \cdot \frac{1}{8} = \frac{1}{168} + \frac{1}{192} \\
P(G|(0,1)) &= 2 \cdot \frac{1}{4} \cdot \frac{1}{6} \cdot \frac{1}{7} = \frac{2}{168}
\end{aligned}$$

Thus, the MLE of the source is node $(0,0)$. Also, node $(0,1)$ is more likely to be the source than node $(1,0)$ or node $(-1,0)$.

Problem 2. (Fall 2008, MT2) Given $X \in \{0,1\}$, the random variable Y is exponentially distributed with rate $3X+1$.

(a) Assume $P(X=1)=p$ and $P(X=0)=1-p$. Find the MAP estimate of X given Y.

Solution: $\text{MAP}[X|Y] = 1\{Y < \frac{1}{3}\ln\frac{4p}{1-p}\}$

(b) Find the MLE of X given Y.

Solution: $\text{MLE}[X|Y] = \frac{1}{3}\ln 4$

(c) Solve the hypothesis testing problem of X given Y with a probability of false alarm at most 0.1. That is, find $\hat{X}$ as a function of Y that maximizes $P[\hat{X} = 1|X=1]$ subject to $P[\hat{X}=1|X=0] \le 0.1$.

Solution: Declare 1 if $Y < -ln 0.9$ and 0 otherwise.

(d) For what value of p does one have the same solution for (a) and (c)?

Solution: $p = \frac{1}{1+4(0.9)^3}$