UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESS

Solution 11
Spring 2017

Issued: Thursday, April 20, 2017

Self-graded Scores Due: 5pm, Monday, May 1, 2017
Submit your self-graded scores via the google form:
https://goo.gl/forms/IDtqh2GLgORuEyiE3. Make sure that you use your
SORTABLE NAME on bCourses.

Problem 1. **Solution 1:** We note that X, Y, Z are jointly Gaussian, so $E[X|Y,Z] = L[X|Y,Z]$. Now, we can see that $L[X|Y,Z] = L[X|Y] + L[X|Z - L[Z|Y]]$. We can see that $L[X|Y] = \frac{1}{3}Y$ and $L[Z|Y] = \frac{1}{3}Y$. Also, we can see that $L[X|Z - L[Z|Y]] = 0$, so $E[X|Y,Z] = \frac{1}{3}Y$.

Solution 2: Since $\mu = [0 \ \ 0 \ \ 0]^T$, we have

$$\begin{aligned}
E[X|Y,Z] &= [E[XY] \ \ E[XZ]] \begin{bmatrix} E[Y^2] & E[YZ] \\ E[YZ] & E[Z^2] \end{bmatrix}^{-1} \begin{bmatrix} Y \\ Z \end{bmatrix} \\
&= [3 \ \ 1] \begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} Y \\ Z \end{bmatrix} \\
&= \frac{1}{3}Y
\end{aligned}$$

Problem 2. We are interested in $Q[X|Y] = aY^2 + bY + c$. By orthogonality, one has:

$$\begin{aligned}
&E[X - aY^2 - bY - c] = 0 \to \frac{1}{3} - \frac{a}{3} - c = 0 \\
&E[XY - aY^3 - bY^2 - cY] = 0 \to \frac{1}{3} - \frac{b}{3} = 0 \\
&E[XY^2 - aY^4 - bY^3 - cY^2] = 0 \to \frac{2(1-a)}{5} - \frac{c}{3} = 0
\end{aligned}$$

Where we note that $E[Y^3] = 0$ because y^3 is an odd function and $E[Y^4] = \frac{1}{5}$. Thus, we can see $Q[X|Y] = Y^2 + Y$.

Problem 3. (a) **Solution 1:** Consider a Poisson process with rate λ that is split into two indpendent Poisson processes with rates $p\lambda$ and $(1-p)\lambda$ by flipping a coin with probability $1-p$ at each arrival. We condition on the time of the first arrival to the Poisson process with rate $(1-p)\lambda$. Let N_1 be the number of arrivals to the Poisson process with rate $p\lambda$ and note that $N = N_1 + 1$. Thus, we are interested in $E[N_1 + 1|T] = 1 + E[N_1|T] = 1 + \lambda pT$, since the number of arrivals in a Poisson process during a fixed time interval is Poisson.

Solution 2:

First, we calculate $\Pr(N = n \mid T = t)$.

$$\begin{aligned}\Pr(N = n \mid T = t) &= \frac{\Pr(N = n) f_{T|N}(t \mid n)}{\sum_{k=1}^{\infty} \Pr(N = k) f_{T|N}(t \mid k)} \\ &= \frac{(1-p)p^{n-1}\lambda^n t^{n-1} e^{-\lambda t}/(n-1)!}{\sum_{k=1}^{\infty}(1-p)p^{k-1}\lambda^k t^{k-1} e^{-\lambda t}/(k-1)!} \\ &= \frac{\lambda(\lambda pt)^{n-1}/(n-1)!}{\lambda \sum_{k=1}^{\infty}(\lambda pt)^{k-1}/(k-1)!} = \frac{(\lambda pt)^{n-1}}{e^{\lambda pt}(n-1)!}, \qquad n \in \mathbb{Z}_+.\end{aligned}$$

Next, we calculate $E[N \mid T = t]$.

$$\begin{aligned}E[N \mid T = t] &= \sum_{n=1}^{\infty} n \frac{(\lambda pt)^{n-1}}{e^{\lambda pt}(n-1)!} \\ &= \sum_{n=1}^{\infty} \frac{(\lambda pt)^{n-1}}{e^{\lambda pt}(n-1)!} + \sum_{n=1}^{\infty}(n-1)\frac{(\lambda pt)^{n-1}}{e^{\lambda pt}(n-1)!} \\ &= 1 + \frac{\lambda pt}{e^{\lambda pt}} \sum_{n=2}^{\infty} \frac{(\lambda pt)^{n-2}}{(n-2)!} = 1 + \frac{\lambda pt}{e^{\lambda pt}} e^{\lambda pt} = 1 + \lambda pt.\end{aligned}$$

Hence, the MMSE is $E[N \mid T] = 1 + \lambda pT$. The MMSE is linear, so it is also the LLSE.

(b) Suppose you fix $Y = 1$, then $E[X|Y = 1] = 0.75$ as X is uniform on the line segment between $(0.5, 1), (1, 1)$. In fact, $E[X|Y = y]$ is simply the midpoint of the line segment which is the intersection of the shaded region and the line horizontal line going through the point $(0, y)$. One can thus see that the MMSE $E[X|Y] = \frac{1}{2}Y + \frac{1}{4}$ as shown in the figure below. As the MMSE is linear, it coincides with the LLSE.

Problem 4. (a) We run through the Kalman filter equations and see $\sigma^2_{1|0} = 2$, $k_1 = 0.444$, $\sigma^2_{1|1} = 0.222$, $\sigma^2_{2|1} = 1.11$, $k_2 = 0.408$, $\sigma^2_{2|2} = 0.204$, $\sigma^2_{3|2} = 1.102$, $k_3 = 0.408$, $\sigma^2_{3|3} = 0.204$.

(b) Since we know that $\lim_{n\to\infty} \sigma^2_{n|n}$ converges to a constant, let $\lim_{n\to\infty} \sigma^2_{n|n} = \lim_{n\to\infty} \sigma^2_{n-1|n-1} = x$. Noting that $\sigma^2_{n|n} = \sigma^2_{n|n-1}(1 - k_n c)$, we can see that $\sigma^2_{n|n} = \frac{\sigma^2_{n|n-1}}{4\sigma^2_{n|n-1}+1}$. Additionally, we have $\sigma^2_{n|n-1} = \frac{1}{2}\sigma^2_{n-1|n-1} + 1$. Thus, we can see:

$$\sigma^2_{n|n} = \frac{\frac{1}{2}\sigma^2_{n-1|n-1} + 1}{2\sigma^2_{n-1|n-1} + 5}$$

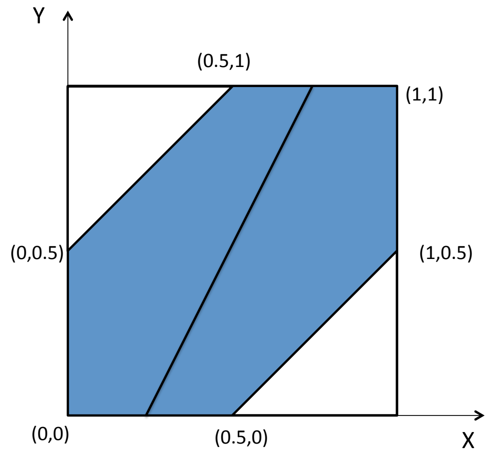

Figure 1: The black line indicates the LLSE

Thus, $X = \frac{\frac{1}{2}X+1}{2X+5}$. Solving the quadratic gives $X = 0.2037$, thus we have $\lim_{n\to\infty} \sigma^2_{n|n} = 0.2037, \lim_{n\to\infty} \sigma^2_{n|n-1} = 1.1018, \lim_{n\to\infty} k_n = 4.075$. Thus, we can see that after only three iterations of the Kalman filter, it seems to converge to steady state!

Problem 5. (a) We have:

$$
\begin{aligned}
& MAP[W_1, W_2, \ldots, W_n | X_1, X_2, \ldots, X_n, Y] \\
& = \text{argmax}_{w_1,\ldots,w_n} f_{W_1,\ldots,W_n|X_1,\ldots,X_n,Y}(w_1, \ldots, w_n | x_1, \ldots, x_n, y_n) \\
& = \text{argmax}_{w_1,\ldots,w_n} f_{W_1,W_2,\ldots,W_n}(w_1, \ldots, w_n) f_{Y|X_1,\ldots,X_n,W_1,\ldots,W_n}(x_1, \ldots, x_n, w_1, \ldots, w_n) \\
& = \text{argmax}_{w_1,\ldots,w_n} \exp(-\frac{1}{2\sigma^2}(y - \sum_i w_n x_n)^2) \exp(-\lambda \sum_i |w_i|) \\
& = \text{argmin}_{w_1,w_2,\ldots,w_n} (y - \sum_i w_n x_n)^2 + \mu \sum_i |w_i|
\end{aligned}
$$

where $\mu = 2\lambda$. You may recognize this as ℓ_1-regularized least squares. This is also the Lagrange multiplier formulation of the LASSO problem, where the constraint on the ℓ_1 norm is replaced by a penalty.

(b),(c) See p. 164-165 in Walrand.

Problem 6. By running the Viterbi algorithm, we can find that the result is $(0, 0, 1, 1, 1)$.