UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Discussion 6
Spring 2017

Date: Wednesday, March 1, 2017

Problem 1. Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y \sim \text{Poi}(\lambda)$. Find the Chernoff bounds for

(a) $P(X - \mu \geq \epsilon)$

(b) $P(Y - \lambda \geq \epsilon)$

Solution:

(a) For every $s > 0$,

$$\begin{aligned} P(X-\mu \geq \epsilon) &= P(e^{s(X-\mu)} \geq e^{s\epsilon}) \\ &\leq \frac{E[e^{s(X-\mu)}]}{e^{s\epsilon}} \end{aligned}$$

Since $X - \mu \sim \mathcal{N}(0, \sigma^2)$, we can write

$$\begin{aligned} E[e^{s(X-\mu)}] &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{sz} e^{-z^2/(2\sigma^2)} dz \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-(z-s\sigma^2)^2 + s^2\sigma^4}{2\sigma^2}} dz \\ &= e^{\frac{s^2\sigma^2}{2}} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-(z-s\sigma^2)^2}{2\sigma^2}} dz}_{1} \\ &= e^{s^2\sigma^2/2}. \end{aligned}$$

Then,

$$\begin{aligned} P(X-\mu \geq \epsilon) &\leq e^{\frac{s^2\sigma^2}{2} - s\epsilon} \quad \forall s > 0 \\ \implies P(X-\mu \geq \epsilon) &\leq \inf_{s>0} e^{\frac{s^2\sigma^2}{2} - s\epsilon} = e^{-\frac{\epsilon^2}{2\sigma^2}}. \end{aligned}$$

(b) For every $s > 0$,

$$\begin{aligned} P(Y-\lambda \geq \epsilon) &= P(e^{sY} \geq e^{s(\lambda+\epsilon)}) \\ &\leq E[e^{sY}] e^{-s(\lambda+\epsilon)}. \end{aligned}$$

To find the transform of Y:

$$
\begin{aligned}
E[e^{sY}] &= \sum_{k=0}^{\infty} e^{sk} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^s)^k}{k!} \\
&= e^{-\lambda + \lambda e^s}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
P(Y - \lambda \geq \epsilon) &\leq \inf_{s>0} \exp\{-\lambda + \lambda e^s - s(\lambda + \epsilon)\} \\
&= \exp\{-\lambda + \lambda e^s - s(\lambda + \epsilon)\}|_{s=\log((\lambda+\epsilon)/\lambda)} \\
&= \exp\left\{\epsilon - (\lambda + \epsilon)\log\left(\frac{\lambda + \epsilon}{\lambda}\right)\right\}.
\end{aligned}
$$

Problem 2. A discrete-time Markov chain with seven states has the following transition probabilities:

$$
p_{ij} = \begin{cases} 0.5, & (i,j) = (3,2), (3,4), (5,6) \text{ and } (5,7) \\ 1, & (i,j) = (1,3), (2,1), (4,5), (6,7) \text{ and } (7,5) \\ 0, & \text{otherwise} \end{cases}.
$$

In the questions below, let X_k be the state of the Markov chain at time k.

(a) Give a pictorial representation of the discrete-time Markov chain.

(b) For what values of n is $\Pr(X_n = 5 \mid X_0 = 1) > 0$?

(c) What is the set of states $A(i)$ that are accessible from state i, for each $i = 1, 2, \ldots, 7$? Is the Markov chain irreducible?

Solution:

(a) See the pictorial representation in Figure 1.

(b) State 5 is reachable from state 1 of three transitions. Paths from state 1 to state 5 also include paths with a loop from 1 back to 1 (of length 3) and/or a loop from 5 back to 5 by way of state 7 (either length 2 or length 3). Therefore potential path lengths are $3 + 2m + 3n$, for $m, n \geq 0$. Therefore, $\Pr(X_n = 5 \mid X_0 = 1) > 0$ for $n = 3$ or $n \geq 5$.

(c) From states 1, 2, and 3, all states are accessible because there is a non-zero probability path from these states by way of state 3 to any other state. From states, 4, 5, 6, and 7, paths only exist to states 5, 6, and 7. Therefore, the chain is not irreducible.

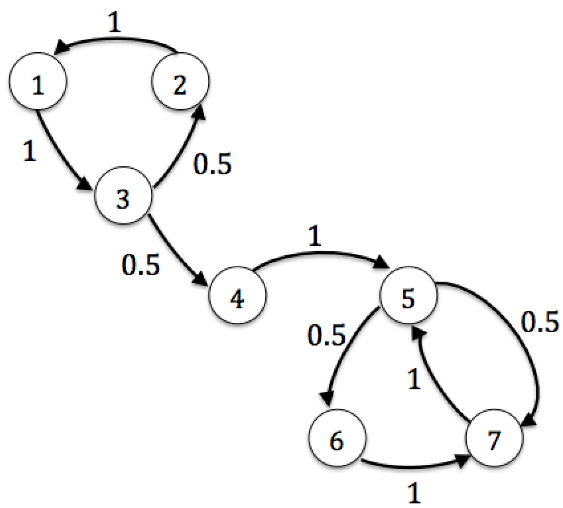

Figure 1: Pictorial representation of the discrete-time Markov chain.