UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Problem Set 7
Spring 2017

Issued: Thursday, March 9, 2017 **Due:** Thursday, March 16, 2017

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1. **Inventory Management**

Consider a Markov chain (X_n), where X_n represents the quantity of an item in stock at time n. We will assume that the changes in stock are modeled by a simple random walk, that is,

$$\Pr(X_{n+1} = i + 1 \mid X_n = i) = \Pr(X_{n+1} = i - 1 \mid X_n = i) = \frac{1}{2}.$$

A (s, S) policy (for $S > s$) is given as follows:

- If the stock ever drops to 0, then buy enough items until we have replenished our stock to s.
- If the stock ever reaches S, then sell enough items until our stock is back down to s.

In other words, s is the *baseline*, and we return the baseline as soon as the stock drops to 0 or increases to S.

(a) Let V_k denote the number of visits to k starting from i, before we reach 0 or S. Calculate $E_i[V_k] = E[V_k \mid X_0 = i]$.

(b) $E_s[V] = \sum_{k=1}^{S-1} kE_s[V_k]$ represents the expected total quantity of the item that we will have in stock, starting from the baseline s, until we reach 0 or S. Calculate $E_s[V]$.

(c) A *cycle* starts at the baseline and ends when we reach 0 or S (and then the next cycle starts). Let T_i denote the length of the ith cycle. Associated with each cycle is a *transaction cost* c_t, which represents the need to buy or sell items to meet our policy. Also, there is a *holding cost* c_h which is assumed to be proportional to V: $c_h = hV$. Therefore, the long-run average cost incurred by the policy is

$$\text{LRAC} = \frac{c_t + hE_s[V]}{E[T_i]}.$$

(The LRAC is an abbreviation for long-run average cost.) Find the optimal policy (s^*, S^*) which minimizes the LRAC.

2. **Poisson Process Warm-Up**

Consider a Poisson process $\{N_t, t \geq 0\}$ with rate $\lambda = 1$. Let random variable S_i denote the time of the ith arrival.

(a) Given $S_3 = s$, find the joint distribution of S_1 and S_2.

(b) Find $E[S_2 \mid S_3 = s]$.

(c) Find $E[S_3 \mid N_1 = 2]$.

(d) Give an interpretation, in terms of a Poisson process with rate λ, of the following fact:

If N is a geometric random variable with parameter p, and X_i are IID exponential random variables with parameter λ, then $X_1 + \cdots + X_N$ has the exponential distribution with parameter λp.

3. Bus Arrivals at Cory Hall

Starting at time 0, the F line makes stops at Cory Hall according to a Poisson process of rate λ. Students arrive at the stop according to an independent Poisson process of rate μ. Every time the bus arrives, all students waiting get on.

(a) Given that the interarrival time between bus $i-1$ and bus i is x, find the distribution for the number of students entering the ith bus.

(b) Given that a bus arrived at 9:30 AM, find the distribution for the number of students that will get on the next bus.

(c) Find the distribution of the number of students getting on the next bus to arrive after 11:00 AM. (You can assume that time 0 was infinitely far in the past.)

4. Sum-Quota Sampling

Consider the problem of estimating the mean inter-arrival time of a Poisson process. In what follows, recall that N_t denotes the number of arrivals by time t.

Sum-quota sampling is a form of sampling in which the number of samples is not fixed in advance; instead, we wait until a fixed *time* t, and take the average of the interarrival times seen so far. If we let X_i denote the ith inter-arrival time, then

$$\bar{X} = \frac{X_1 + \cdots + X_{N_t}}{N_t}.$$

Of course, the above quantity is not defined when $N_t = 0$, so instead we must condition on the event $\{N_t > 0\}$. Compute $E[\bar{X} \mid N_t > 0]$, assuming that N_t is a Poisson process of rate λ.

5. Taxi Queue

Empty taxis pass by a street corner at a Poisson rate of two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner at a Poisson rate of one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. John arrives at the street corner at a given time. Find his expected waiting time, given that he joins the queue. Assume that the process is in steady state.

6. Poisson Queues

A continuous-time queue has Poisson arrivals with rate λ, and it is equipped with infinitely many servers. The servers can work in parallel on multiple customers, but they are non-cooperative in the sense that a single customer can only be served by one server. Thus, when there are k customers in the queue, k servers are active. Suppose that the service time of each customer is exponentially distributed with rate μ and they are i.i.d.

(a) Argue that the queue-length is a Markov chain. Draw the transition diagram of the Markov chain.

(b) Prove that for all finite values of λ and μ the Markov chain is positive-recurrent and find the invariant distribution.