UC Berkeley
Department of Electrical Engineering and Computer Sciences

Electrical Engineering 126: Probability and Random Processes

Discussion 9
Spring 2017

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1. **Illegal U-Turns**

   Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal and police cars drive by according to a Poisson process with rate λ. You decide to make a U-turn once you see that the road has been clear of police cars for τ units of time. Let N be the number of police cars you see before you make a U-turn.

   (a) Find $E[N]$.

   (b) Find the conditional expectation of the time elapsed between police cars $n-1$ and n, given that $N \geq n$.

   (c) Find the expected time that you wait until you make a U-turn.

   Solution:

   (a) The random variable N is equal to the number of successive interarrival intervals that are smaller than τ. Interarrival intervals are independent and each one is smaller than τ with probability $1 - e^{-\lambda\tau}$. So $\Pr(N = k) = e^{-\lambda\tau}(1-e^{-\lambda\tau})^k$, and N is a shifted geometric random variable with parameter $p = e^{-\lambda\tau}$ that starts from 0. Thus, $E[N] = 1/p - 1 = e^{\lambda\tau} - 1$.

   (b) Let T_n be the nth interarrival time. The event $\{N \geq n\}$ indicates that the time between cars $n-1$ and n is less than or equal to τ. So we want to compute

   $$E[T_n \mid T_n < \tau] = \frac{\int_0^\tau t\lambda e^{-\lambda t}\,\mathrm{d}t}{\int_0^\tau \lambda e^{-\lambda t}\,\mathrm{d}t}.$$

   Using integration by part for the integral in the numerator, we find that the answer is

   $$= \frac{1/\lambda - (\tau + 1/\lambda)e^{-\lambda\tau}}{1 - e^{-\lambda\tau}}.$$

   (c) You make the U-turn at time $T = T_1 + T_2 + \cdots + T_N + \tau$ and $T_i \leq \tau$ for $1 \leq i \leq N$. Then, using parts (a) and (b),

   $$\begin{aligned} E[T] &= \tau + \sum_{n=0}^{\infty} \Pr(N = n) E[T_1 + \cdots + T_N \mid N = n] \\ &= \tau + \sum_{n=0}^{\infty} \Pr(N = n) n E[T_i \mid T_i \leq \tau] \\ &= \tau + (e^{\lambda\tau} - 1) \times \frac{1/\lambda - (\tau + 1/\lambda)e^{-\lambda\tau}}{1 - e^{-\lambda\tau}}. \end{aligned}$$

2. **Two-Server System**

A company has two servers (the second server is a backup in case the first server fails, so only one server is ever used at a time). When a server is running, the time until it breaks down is exponentially distributed with rate μ. When a server is broken, it is taken to the repair shop. The repair shop can only fix one server at a time, and its repair time is exponentially distributed with rate λ. Find the long-run probability that no servers are operational.

Solution:

The idea is to model the number of operational servers as a continuous-time Markov chain on the state space $\{0, 1, 2\}$. By thinking about the infinitesimal transition probabilities (which are simply the rates of the exponential distributions), we have the following matrix:

$$\mathbf{Q} = \begin{bmatrix} 0 & \lambda & 0 \\ \mu & 0 & \lambda \\ 0 & \mu & 0 \end{bmatrix}.$$

Now, we write down the balance equations.

$$\begin{aligned} \lambda\pi(0) &= \mu\pi(1) \\ (\lambda + \mu)\pi(1) &= \lambda\pi(0) + \mu\pi(2) \\ \mu\pi(2) &= \lambda\pi(1) \\ 1 &= \pi(0) + \pi(1) + \pi(2) \end{aligned}$$

We eliminate $\pi(2)$ with $\pi(2) = (\lambda/\mu)\pi(1)$. Plugging this into the second and fourth equations, we have

$$\begin{aligned} \mu\pi(1) &= \lambda\pi(0), \\ 1 &= \pi(0) + \left(1 + \frac{\lambda}{\mu}\right)\pi(1). \end{aligned}$$

We next eliminate $\pi(1)$ with $\pi(1) = (\lambda/\mu)\pi(0)$. Plugging this into the second equation above, we have

$$\pi(0) = \frac{1}{1 + \lambda/\mu + (\lambda/\mu)^2}.$$

This is the long-run probability that we will be in state 0, i.e. there are no operational servers.

3. **$M/M/2$ Queue**

A queue has Poisson arrivals with rate λ. It has two servers that work in parallel. Where there are at least two customers in the queue, two are being served. When there is only one customer, only one server is active. The service times are i.i.d. exponential random variables with rate μ.

(a) Argue that the queue length is a Markov chain.

(b) Draw the state transition diagram.

(c) Find the minimum value of μ so that the queue is positive recurrent and solve the balance equations.

Solution:

(a) The queue length is a MC as customer arrivals are independent of the current number of customers in the queue. Also, the departures only depend on the current number of customers being served. Next, even when k ($k = 1, 2$) customers are being served, the completion of their service is independent of one another. Finally, when $k = 2$, even if one of the customers has been completely served, the other customer has the same service time distribution as before as the exponential distribution is memoryless.

(b) It is shown in the following figure.

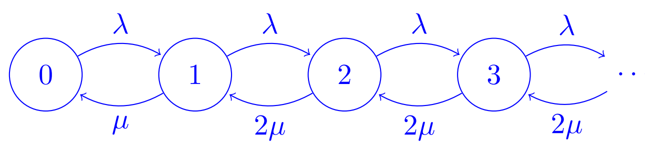

Figure 1: Markov chain for a queue with two servers.

(c) We write the flow conservation. Thus,

$$\pi(1) = \frac{\lambda}{\mu}\pi(0)$$
$$\pi(i+1) = \frac{\lambda}{2\mu}\pi(i), \qquad i \geq 1.$$

Thus,

$$\pi(i) = \left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{2\mu}\right)^{i-1}\pi(0).$$

Also, we know that

$$\sum_{i=0}^{\infty}\pi(i) = 1.$$

Thus,

$$\pi(0)\left(1 + \sum_{i=1}^{\infty}\left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{2\mu}\right)^{i-1}\right) = 1.$$

The series converges if $\lambda < 2\mu$. So the minimum value of μ for positive recurrence is $\mu > \lambda/2$. Then, solving the equation we have

$$\pi(0) = \frac{2\mu - \lambda}{2\mu + \lambda}.$$

Then,

$$\pi(i) = \frac{2\mu - \lambda}{2\mu + \lambda}\left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{2\mu}\right)^{i-1}, \qquad i \in \mathbb{Z}_+.$$