UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Solution 2
Spring 2017

Issued: Thursday, January 26, 2017

Self-graded Scores Due: 5pm, Monday, February 6, 2017
Submit your self-graded scores via the google form:
https://goo.gl/forms/w8vvs2erGiSaOJdW2. Make sure that you use your **SORTABLE NAME** on bCourses.

Problem 1. (a) $Y \in \{0, 1, 2\}$:

$$\begin{aligned} Pr(Y = 0) &= Pr(X \in \{0, 3, 6, 9\}) = 4/10 \\ Pr(Y = 1) &= Pr(X \in \{1, 4, 7\}) = 3/10 \\ Pr(Y = 2) &= Pr(X \in \{2, 5, 8\}) = 3/10 \end{aligned}$$

$$\implies p_Y(y) = \begin{cases} 4/10 & \text{if } y = 0, \\ 3/10 & \text{if } y \in \{1, 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

(b) $Z \in \{0, 1, 2, 5\}$:

$$\begin{aligned} Pr(Z = 0) &= Pr(X \in \{0, 4\}) = 1/5 \\ Pr(Z = 1) &= Pr(X \in \{1, 3\}) = 1/5 \\ Pr(Z = 2) &= Pr(X \in \{2\}) = 1/10 \\ Pr(Z = 5) &= Pr(X \in \{5, 6, 7, 8, 9\}) = 1/2 \end{aligned}$$

$$\implies p_Z(z) = \begin{cases} 1/5 & \text{if } z \in \{0, 1\}, \\ 1/10 & \text{if } z = 2, \\ 1/2 & \text{if } z = 5, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 2. Let X_i be the indicator that couple i remains. Thus, the number of couples remaining at the end of two hours is $X = \sum_{i=1}^{n} X_i$ and $E[X] = \sum_{i=1}^{n} E[X_i]$. Now, consider couple i. The probability that couple i remains, call it p_c is $\binom{2N-2}{N}/\binom{2N}{N} = \frac{N-1}{2(2N-1)}$. Thus, we have that $E[X_i] = p_c = \frac{N-1}{2(2N-1)}$. Thus:

$$\begin{aligned} E[X] &= N p_c \\ &= \frac{N(N-1)}{2(2N-1)} \end{aligned}$$

Problem 3. Let T be the event that for every city in $N_2(A_1)$ (except A_1 itself), there is a unique flight route with at most 2 flights from A_1 to that city. Then we have

$$P(T) = \sum_{k=0}^{m} P(T|\deg(A_1) = k)P(\deg(A_1) = k).$$

Here, by $\deg(A_1)$, we mean the number of cities that are connected to A_1 by one flight. Given A_1 is connected to k cities $\{B_{i_1}, \dots, B_{i_k}\}$, the event T is equivalent to the event that each city in $\{A_2, \cdots, A_n\}$ is connected to at most one city in the set $\{B_{i_1}, \dots, B_{i_k}\}$. Therefore, we have

$$P(T) = \sum_{k=0}^{m} \left[\binom{m}{k} p^k (1-p)^{m-k} [(1-p)^k + kp(1-p)^{k-1}]^{n-1} \right].$$

Now, let N be the event that there is at least 1 city in country A in $N_2(A_1)$. We are looking for $P(T \cap N) = P(T) - P(T \cap N^c)$ by the law of total probability. Since we have already found $P(T)$, we must now find $P(T \cap N^c) = P(T|N^c)P(N^c) = P(N^c)$ where the last equality follows since N^c is the event that there is no country in A other than A_1 that is also in $N_2(A_1)$, thus, the probability that there is a unique path of at most two flights to all cities in $N_2(A_1)$ from A_1 must be 1 since there are only paths of length 1 in the neighborhood. Also we have:

$$\begin{aligned} P(N^c) &= \sum_{k=0}^{m} P(N^c|\deg(A_1) = k)P(\deg(A_1) = k) \\ &= \sum_{k=0}^{m} \left[\binom{m}{k} p^k (1-p)^{m-k} [(1-p)^k]^{n-1} \right] \end{aligned}$$

The result follows.

Problem 4. Define random variables T_i, $1 \le i \le n-1$ as follows:

$$T_i = \begin{cases} 1 & \text{if } X_i \neq X_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can find that $\Pr(T_i = 1) = 2p(1-p)$, then $\mathbb{E}[T_i] = 2p(1-p)$. Let N be the number of runs of X^n. We have $N = 1 + \sum_{i=1}^{n-1} T_i$. Then $\mathbb{E}[N] = 1 + 2(n-1)p(1-p)$.

Problem 5. (a) First note that the left degree distribution is $\text{Bin}(M, p)$. We approximate it by $\text{Poi}(Mp)$ for ease of calculations since M is large and Mp is fixed. Then for a particular edge, the degree distribution of a right node which is connected to this edge is

$$\Pr(X_r = i) = \binom{K-1}{i-1} p^{i-1} (1-p)^{K-i} \quad \text{for } i = 1, 2, \dots, K-1$$

Since K and M get large and Kp and Mp are fixed we use Poisson approximation:

$$\Pr(X_r = i) = \frac{(Kp)^{i-1} e^{-Kp}}{(i-1)!}.$$

Let $\rho_1 = \Pr(X_r = 1) = e^{-Kp}$. Then, for a left node of degree d the probability that it is not connected to any singleton is $(1-\rho_1)^d$. Thus the probability that a left node is not connected to a singleton is

$$\begin{aligned}
&\sum_d \Pr(\text{left node connected to no singleton}|d)\Pr(d) \\
&\quad = \sum_d (1-\rho_1)^d \frac{(Mp)^d e^{-Mp}}{d!} \\
&\quad = \sum_d \frac{[(1-\rho_1)Mp]^d}{d!} e^{-Mp(1-\rho_1)} e^{-Mp\rho_1} \\
&\quad = e^{-Mp\rho_1}.
\end{aligned}$$

Then the expected number of left nodes connected to singletons is

$$K(1-e^{-Mp\rho_1}) \triangleq Kq_s.$$

(b) The probability of a right node being a doubleton is $\binom{K}{2}p^2(1-p)^{K-2} \simeq \frac{e^{-Kp}(Kp)^2}{2!}$. Thus, the expected number of doubletons is approximately

$$M\frac{e^{-Kp}(Kp)^2}{2!}.$$

(c) We claim that this probability approaches 1. To see this, consider a doubleton that is connected to left nodes (l_1, l_2). The probability that another randomly chosen doubleton is also connected to (l_1, l_2) is $1/\binom{K}{2}$. Since the number of doubletons grow linearly in M and K (part (b)), this probability goes to 0.

Problem 6. (a) Let D be the event that a left node is in a doubleton, and S be the event that a left node is in a singleton. Let ρ_2 be the probability that a randomly selected edge is connected to a doubleton. First note that by part (a) of the previous problem,

$$\Pr(\bar{S}) = 1 - q_s = e^{-Mp\rho_1}.$$

Second, similarly we have

$$\Pr(\bar{D}) = \sum_d (1-\rho_2)^d \frac{(Mp)^d e^{-Mp}}{d!} = e^{-Mp\rho_2}.$$

Third, similarly we have

$$\Pr(\bar{D}\cap\bar{S}) = \sum_d (1-\rho_1-\rho_2)^d \frac{(Mp)^d e^{-Mp}}{d!} = e^{-Mp(\rho_1+\rho_2)}.$$

We compute the following condition probabilities:

$$\begin{aligned} p_1 \triangleq \Pr(D|S) &= \frac{\Pr(D \cap S)}{\Pr(S)} \\ &= \frac{1 - \Pr(\bar{S}) - \Pr(\bar{D}) + \Pr(\bar{S} \cap \bar{D})}{1 - \Pr(\bar{S})} \\ p_2 \triangleq \Pr(D|\bar{S}) &= 1 - \Pr(\bar{D}|\bar{S}) \\ &= 1 - \frac{\Pr(\bar{S} \cap \bar{D})}{\Pr(\bar{S})}. \end{aligned}$$

Now we use Bayes' rule to find that

$$\Pr(S|D) = \frac{\Pr(D|S)\Pr(S)}{\Pr(D|S)\Pr(S) + \Pr(D|\bar{S})\Pr(\bar{S})} = \frac{p_1 q_s}{p_1 q_s + p_2(1 - q_s)}.$$

Thus,

$$M_s = M \frac{e^{-Kp}(Kp)^2}{2!} (\Pr(S|D))^2.$$

(b) Since G_1 is constructed randomly, any of the M_s doubletons are equally likely to be any edges in the $\binom{K_s}{2}$ possible edges of G_2. Now since K_s and M_s get large, it is equivalent to say that each of the $\binom{K_s}{2}$ possible edges of G_2 is connected with probability $q = \frac{M_s}{\binom{K_s}{2}}$, which constructs an Erdos-Renyi graph.

(c) We have already done all the necessary calculations. We need that

$$K_s \frac{M_s}{\binom{K_s}{2}} > 1.$$

One can then find p such that the condition is satisfied numerically.