UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Problem Set 3
Spring 2017

Self-Graded Scores Due: 5 pm, Monday, February 13, 2017
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1. **Triangle Density**

   Let (X, Y) be uniformly distributed over the triangle with vertices $(0, 0)$, $(1, 0)$, and $(2, 1)$. Find the following:

   (a) $f_{X,Y}(x, y)$.

   (b) $f_X(x)$.

   (c) $E[Y \mid X = x]$.

   Solution:

   (a) Since the density is uniform, we need only calculate the total area of the triangle, which is $(1 \cdot 1)/2 = 1/2$. Therefore, the density is

   $$f_{X,Y}(x, y) = 2, \qquad (x, y) \in T,$$

   where T is the triangle.

   (b) We integrate out y:

   $$\begin{aligned} f_X(x) &= 1\{0 \le x \le 1\} \int_0^{x/2} 2 \, \mathrm{d}y + 1\{1 \le x \le 2\} \int_{x-1}^{x/2} 2 \, \mathrm{d}y \\ &= x \cdot 1\{0 \le x \le 1\} + (2 - x) \cdot 1\{1 \le x \le 2\}. \end{aligned}$$

   The density can equivalently be written as

   $$f_X(x) = (1 - |x - 1|)^+,$$

   where $x^+ = \max(0, x)$.

   (c) If $0 \le x \le 1$, then $Y \mid X = x$ is uniform on a slice from 0 to $x/2$, so $E[Y \mid X = x] = x/4$. Similarly, if $1 \le x \le 2$, then $Y \mid X = x$ is uniform on a slice from $x - 1$ to $x/2$, so $E[Y \mid X = x] = (3x - 2)/4$.

2. **Graphical Density**

   Figure 1 shows the joint density $f_{X,Y}$ of the random variables X and Y.

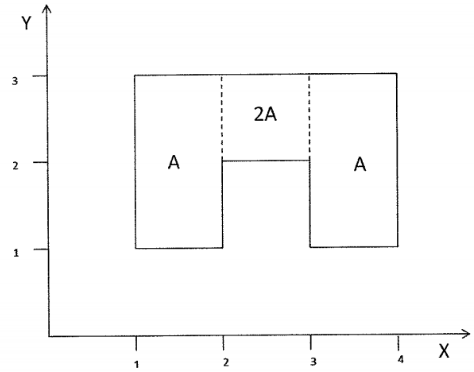

Figure 1: Joint density of X and Y.

(a) Find A and sketch f_X, f_Y, and $f_{X|X+Y\leq 3}$.

(b) Find $E[X \mid Y = y]$ for $1 \leq y \leq 3$ and $E[Y \mid X = x]$ for $1 \leq x \leq 4$.

(c) Find $\mathrm{cov}(X, Y)$.

Solution:

(a) The integration over the total shown area should be 1 so $2A+2A+2A = 1$ so $A = 1/6$. We find the densities as follows. X is clearly uniform in intervals $(1, 2)$, $(2, 3)$, and $(3, 4)$. The probability of X being in any of these intervals is $2A = 1/3$ so $f_X(x) = (1/3)\cdot 1\{1 \leq x \leq 4\}$. Y is uniform in intervals $(1, 2)$ and $(2, 3)$. The probability of the first interval is $1/3$ and the probability of being in second one is $2/3$. So

$$f_Y(y) = \frac{1}{3} \cdot 1\{1 \leq y \leq 2\} + \frac{2}{3} \cdot 1\{2 < y \leq 3\}.$$

Finally, given that $X + Y \leq 3$, (X, Y) is chosen randomly in the triangle constructed by $(1, 1), (1, 2), (2, 1)$. Thus,

$$f_{X|X+Y\leq 3}(x) = \int_1^{3-x} 2\,\mathrm{d}y = 2(2 - x) \cdot \{1 \leq x \leq 2\}.$$

Sketching the densities is then straightforward.

(b) Given any value of y, X has a symmetric distribution with respect to the line $x = 2.5$. Thus, $E[X \mid Y = y] = 2.5$ for all y, $1 \leq y \leq 3$. To calculate $E[Y \mid X = x]$, we consider two cases:

(a) $2 \leq x \leq 3$, then $E[Y \mid X = x] = 2.5$,

(b) $1 \leq x < 2$ or $3 < x \leq 4$, then $E[Y \mid X = x] = 2$.

(c) Since $E[X \mid Y = y] = E[X]$ we have

$$E[XY] = \int_y E[XY \mid Y = y]f_Y(y)\,\mathrm{d}y = \int_y yf_Y(y)E[X]\,\mathrm{d}y = E[X]E[Y].$$

So the covariance is 0.

3. **Office Hours**

In an EE126 office hour, students bring either a difficult-to-answer question with probability $p = 0.2$ or an easy-to-answer question with probability $1-p = 0.8$. A GSI takes a random amount of time to answer a question, with this time duration being exponentially distributed with rate $\mu_D = 1$ (question per minute)-where D denotes difficult- if the problem is difficult, and $\mu_E = 2$ (questions per minute)-where E denotes easy-if the problem is easy.

(a.) You visit office hours and find a GSI answering the question of another student. Conditioned on the fact that the GSI has been busy with the other students question for T minutes, let q be the conditional probability that the problem is difficult. Find the value of q .

(b.) Conditioned on the information above, find the expected amount of time you have to wait from the time you arrive until the other students question is answered.

(c.) Now suppose two GSI's share a room and the professor is holding office hours in a different room. Both GSI's in the shared room are busy helping a student, and each has been answering questions for T minutes (there are no other students in the room). The amount of time the professor takes to answer a question is exponentially distributed with rate $\lambda = 6$ regardless of the difficulty. Supposing that the professor's room has two students (one of whom is being helped), in which room should you ask your question?

Solution:

(a.) Let X be the random amount of time to answer a question and Z the indicator that the problem being answered is difficult. We have:

$$
\begin{aligned}
P(X > t | Z = 0) &= e^{-\mu_E t} \\
P(X > t | Z = 1) &= e^{-\mu_D t}
\end{aligned}
$$

for $t \geq 0$. Thus, we have:

$$
P(X > t) = pe^{-\mu_D t} + (1-p)e^{-\mu_E t} = 0.2e^{-t} + 0.8e^{-2t}.
$$

We are interested in $q = P(Z = 1 | X > T)$. Using Bayes' Rule, we have:

$$
\begin{aligned}
q &= P(Z = 1 | X > T) \\
&= \frac{P(Z = 1, X > T)}{P(X > T)} \\
&= \frac{pe^{-\mu_D T}}{pe^{-\mu_D T} + (1-p)e^{-\mu_E T}} \\
&= \frac{1}{1 + 4e^{-T}}
\end{aligned}
$$

(b.) We are interested in $E[X-T|X>T]$. Thus, we have:

$$\begin{aligned}E[X-T|X>T] &= E[X-T|X>T,Z=0]P(Z=0|X>T)+E[X-T|X>T,Z=1]P(Z=\\ &= (1-q)\frac{1}{\mu_E}+q\frac{1}{\mu_D}\\ &= \frac{1+q}{2}\end{aligned}$$

(c.) Let X_1 and X_2 be the amount of time that the two GSIs still need to take to answer their questions. The amount time to wait for the GSIs is $\min X_1, X_2$. Let X_3 be the amount of time that the lecturer need to take to finish the two students questions. Let a be the probability that a particular GSI is answering a difficult question, i.e.,

$$a=\frac{0.2e^{-T}}{0.2e^{-T}+0.8e^{-2T}}=\frac{1}{1+4Y}$$

where $Y=e^{-T}$. Thus,

$$E[\min X1, X2]=\frac{a^2}{2\mu_D}+\frac{2a(1-a)}{\mu_D+\mu_E}+\frac{(1-a)^2}{2\mu_E}=\frac{6a^2+8a(1-a)+3(1-a)^2}{12}$$
$$E[X_3]=\frac{2}{\mu'}=\frac{1}{3}$$

We equate the two equations to see:

$$6a^2+8a(1-a)+3(1-a)^2=4$$

Solving gives $a=\sqrt{2}-1$ and $Y=\frac{\sqrt{2}}{4}$. Therefore, if $T>\ln 2\sqrt{2}$, you should choose the GSI room, and otherwise choose the lecturer's room.

4. **Drawing Batteries I**

You have an endless box of used batteries. Assume that the number of hours remaining in a battery is i.i.d., uniformly distributed on $[0,1]$.

(a.) Suppose you draw n batteries. Suppose that the ith battery you draw has X_i hours remaining. What is $P(X_1 \leq X_2 \leq \cdots \leq X_n)$?

Now, you draw batteries until you have enough batteries to last one hour. Let N be the number of batteries you draw.

(b.) What is $P(X_1+X_2 \leq 1)$? What about $P(X_1+X_2+X_3 \leq 1)$?

(c.) Find the distribution and expectation of N.

Hint: Try to relate $P(X_1+X_2+\cdots+X_N \leq 1)$ to the quantity you found in part a.

Solution:

(a.) Note that each ordering of the random variables is equally likely and there are $n!$ such orderings. Thus, we have:

$$P(X_1 \leq X_2 \leq X_3 \leq \cdots \leq X_n)=\frac{1}{n!}$$

(b.) We first find $P(X_1 + X_2 \leq 1)$. Consider the (X_1, X_2) plane and draw the square with endpoints $(0,0),(0,1),(1,1),(1,0)$. The tuple (X_1, X_2) is uniform on the area of this square. Now, if $X_1 + X_2 \leq 1$, then we draw the diagonal from $(1,0)$ to $(0,1)$. If the point lies within the boundary of the square and below this diagonal, $X_1 + X_2 \leq 1$. Thus, $P(X_1 + X_2 \leq 1) = \frac{1}{2}$. We now tackle $P(X_1 + X_2 + X_3 \leq 1)$. We do the same, drawing the (X_1, X_2, X_3) axes and considering the unit cube in the first orthant. Now, we know (X_1, X_2, X_3) is uniform on the cube. The region we are interested is the triangular pyramid with vertices $(0,0,0),(0,0,1),(0,1,0),(1,0,0)$. Thus, $P(X_1+X_2+X_3 \leq 1) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

(c.) **Solution 1:** One might see from part b. that there is a pattern and conjecture that $P(X_1 + X_2 + \cdots + X_n \leq 1) = \frac{1}{n!}$ which is the result of part a. We now prove this conjecture. Consider the random variables $Y_1, Y_2, \ldots, Y_n$ where:

$$Y_k := \sum_{i=1}^{k} X_i - \lfloor \sum_{i=1}^{k} X_i \rfloor$$

Note that $Y_k \sim U[0,1]$. To see this, consider taking the unit interval $[0,1]$ and bending it into a circle so that 0 and 1 coincide. Now, Y_k is the decimal part of $\sum_{i=1}^{k} X_i$. Since each of the X_i is iid uniform on the unit interval, the $k+1$th partial sum is equally likely to lie anywhere on the unit circle, implying that it is $U[0,1]$. Now, we note that if $X_1 + \cdots + X_n \leq 1$, this implies that $Y_1 \leq Y_2 \leq \cdots \leq Y_n$. Additionally, if $Y_1 \leq Y_2 \leq \cdots \leq Y_n$, it follows that $X_1 + \cdots + X_n \leq 1$. Thus, we have a bijection and:

$$P(X_1 + \cdots + X_n \leq 1) = P(Y_1 \leq Y_2 \leq \cdots \leq Y_n) = \frac{1}{n!}$$

Thus,

$$\begin{aligned} P(N = n) &= P(X_1 + X_2 + \cdots + X_{n-1} \leq 1) - P(X_1 + X_2 + \cdots + X_n \leq 1) \\ &= \frac{1}{(n-1)!} - \frac{1}{n!} \\ &= \frac{n-1}{n!} \end{aligned}$$

Thus, we have:

$$E[N] = \sum_{n=2}^{\infty} n \cdot \frac{n-1}{n!} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = e.$$

Where the last equality follows from the Taylor expansion of e^1.

Solution 2: Let the battery lifetimes be denoted X_i and let f_n denote the density of $X_1 + \cdots + X_n$. In general, f_n is rather complicated, but we only care about values of x in the interval $[0,1]$. Therefore, let $0 < x < 1$, and we have

$$f_n(x) = \int_0^x f_{n-1}(s) f_1(x-s)\, \mathrm{d}s = \int_0^x f_{n-1}(s)\, \mathrm{d}s.$$

We have $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2/2$, and in general,

$$f_n(x) = \frac{x^{n-1}}{(n-1)!}, \qquad 0 < x < 1.$$

The distribution of N is given by

$$\begin{aligned}
\Pr(N = n) &= \Pr(X_1 + \cdots + X_{n-1} < 1 < X_1 + \cdots + X_n) \\
&= \Pr(X_1 + \cdots + X_{n-1} < 1) - \Pr(X_1 + \cdots + X_n < 1) \\
&= \int_0^1 f_{n-1}(x)\,\mathrm{d}x - \int_0^1 f_n(x)\,\mathrm{d}x \\
&= \int_0^1 \frac{x^{n-2}}{(n-2)!}\,\mathrm{d}x - \int_0^1 \frac{x^{n-1}}{(n-1)!}\,\mathrm{d}x = \frac{1}{(n-1)!} - \frac{1}{n!} \\
&= \frac{n-1}{n!}.
\end{aligned}$$

The expectation is computed to be

$$E[N] = \sum_{n=2}^{\infty} n \cdot \frac{n-1}{n!} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = e.$$

5. **Finite Population Correction**

Consider a model of sampling in which we randomly draw a sample of n people, without replacement, from a population of N members. We are interested in, say, the height of the population. Assume that an individual's height is distributed with mean μ and variance σ^2. Let X_i denote the height of the ith individual in our sample, $i = 1, \ldots, n$. Let x_k, $k = 1, \ldots, m$ denote the possible values for X_i. (Since the population is finite, there are only finitely many possible values for X_i.)

(a) Calculate $E[X_j]$.

(b) Prove that for random variables $X_1, \ldots, X_N$,

$$\operatorname{var}(X_1 + \cdots + X_N) = \sum_{i=1}^{N} \operatorname{var}(X_i) + \sum_{i \neq j} \operatorname{cov}(X_i, X_j),$$

where the second summation ranges over all $(i, j) \in \{1, \ldots, N\}^2$ such that $i \neq j$. Do not assume that the random variables are independent.

(c) Calculate $\operatorname{cov}(X_i, X_j)$ for $i \neq j$.

(d) Using the result you just calculated, calculate $\operatorname{var}(\bar{X})$, where

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

is the sample mean. What is $\operatorname{var}(\bar{X})/s^2$, where $s^2 = \sigma^2/n$ is the variance when sampling with replacement? (This is known as the **finite population correction**. Let $N \to \infty$ to see why an "infinite population" does not suffer from the same problem.)

Solution:

(a) The probability of choosing an individual with height x_k is n_k/N, if there are n_k total individuals with height x_k. Then, we have

$$\sum_{i=1}^{N} x_i \frac{n_i}{N} = \sum_{i=1}^{N} x_i \Pr(X_j = x_i) = \mu.$$

(b) One can simply use bilinearity of covariance.

$$\begin{aligned}
\operatorname{var}(X_1 + \cdots + X_N) &= \operatorname{cov}(X_1 + \cdots + X_N, X_1 + \cdots + X_N) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \operatorname{cov}(X_i, X_j) \\
&= \sum_{i=1}^{N} \operatorname{var}(X_i) + \sum_{i \neq j} \operatorname{cov}(X_i, X_j)
\end{aligned}$$

(c) Suppose that there are n_k individuals in the population with height x_k. Observe that

$$\begin{aligned}
\mu &= \sum_{i=1}^{N} x_i \frac{n_i}{N}, \\
\sigma^2 + \mu^2 &= \sum_{i=1}^{N} x_i^2 \frac{n_i}{N}.
\end{aligned}$$

We let $X_{i,k}$ denote the indicator that $X_i = x_k$. Hence,

$$E[X_i X_j] = \sum_{k=1}^{m} \sum_{\ell=1}^{m} x_k x_\ell E[X_{i,k} X_{j,\ell}].$$

The expectation of this indicator is given by

$$E[X_{i,k} X_{j,\ell}] = \begin{cases} \dfrac{n_k}{N} \cdot \dfrac{n_k - 1}{N - 1}, & k = \ell, \\[2ex] \dfrac{n_k}{N} \cdot \dfrac{n_\ell}{N - 1}, & k \neq \ell. \end{cases}$$

Now, we can write

$$E[X_i X_j] = \sum_{k=1}^{m} x_k^2 \frac{n_k(n_k - 1)}{N(N-1)} + \sum_{k \neq \ell} x_k x_\ell \frac{n_k n_\ell}{N(N-1)}.$$

We rewrite the last sum as

$$\sum_{k \neq \ell} x_k x_\ell \frac{n_k n_\ell}{N(N-1)} = \frac{1}{N(N-1)} \left(\sum_{k=1}^{m} x_k n_k \right)^2 - \frac{1}{N(N-1)} \sum_{k=1}^{m} x_k^2 n_k^2.$$

Hence, one has

$$\begin{aligned} E[X_iX_j] &= \frac{1}{N(N-1)}\left(\sum_{k=1}^{m} x_k n_k\right)^2 - \frac{1}{N(N-1)}\sum_{k=1}^{m} x_k^2 n_k \\ &= \frac{N}{N-1}\left(\sum_{k=1}^{m} x_k \frac{n_k}{N}\right)^2 - \frac{1}{N-1}\sum_{k=1}^{m} x_k^2 \frac{n_k}{N} \\ &= \frac{N\mu^2}{N-1} - \frac{\mu^2+\sigma^2}{N-1} = \mu^2 - \frac{\sigma^2}{N-1}. \end{aligned}$$

Now, we subtract $E[X_i]E[X_j] = \mu^2$ to obtain

$$\operatorname{cov}(X_i, X_j) = -\frac{\sigma^2}{N-1}.$$

(d) The hard work is already done.

$$\begin{aligned} \operatorname{var}(\bar{X}) &= \frac{1}{n^2}\left(\sum_{i=1}^{n} \operatorname{var}(X_i) + \sum_{i\neq j} \operatorname{cov}(X_i, X_j)\right) \\ &= \frac{1}{n^2}\left(n\sigma^2 - n(n-1)\frac{\sigma^2}{N-1}\right) \\ &= \frac{\sigma^2}{n}\left(1 - \frac{n-1}{N-1}\right), \end{aligned}$$

where the last term is the finite population correction. Letting $N \to \infty$, we see that the correction goes to 1.

6. **Auction Theory**

This problem explores auction theory and is meant to be done at the same time as the lab.

In auction theory, n bidders have **valuations** which represent how much they value an item; we will make the simplifying assumption that the valuations are i.i.d. with density $f(x)$. In the first-price auction, the bidder who makes the highest bid wins the item and pays his/her bid. In the second-price auction, the bidder who makes the highest bid wins the auction, and pays an amount equal to the *second-highest* bid. A strategy for the auction is a **bidding function** $\beta(x)$, where x is the bidder's valuation. The bidding function determines how much to bid as a function of the bidder's valuation, and the goal is to find a bidding function $\beta(\cdot)$ which maximizes your expected utility (0 if you do not win, and your valuation minus the amount of money you bid if you do win).

(a) For the first-price auction, consider the following scenario: each person draws his/her valuation uniformly from the interval $(0, 1)$ (so $f(x) = 1$ for $x \in (0, 1)$). Suppose that the other bidders bid their own valuations (they use $\beta(x) = x$, the identity bidding function). Consider the case where there is only one other bidder. The Donald insists that you should make a 'yuge' bid and always bid $\beta(x) = 1$. Your friend Bernie tells The Donald that it would be better to bid $\beta(x) = \frac{x}{2}$. Who is correct?

(b) Consider the same situation as the previous part, but now assume that there are n other bidders. The Donald again suggests that $\beta(x) = 1$ is a great bid, a fantastic bid, the best bid. Your friend Bernie suggests $\beta(x) = \frac{n}{n+1}x$. Who is correct this time?

(c) Consider a second-price auction where the bidders' valuations are i.i.d. with the exponential density (with parameter λ). Again, they use the identity bidding function, $\beta(x) = x$. What is the distribution of the price P at which the item sells?

Solution:

(a) Suppose that your valuation is x, and you choose to bid b. The probability that you win the auction is the probability that the other bidder has a valuation which is less than b, which occurs with probability b. Therefore, the expected utility is the probability that you win the auction, multiplied by $x - b$, which gives $b(x - b)$. The optimal bid b is therefore

$$\beta(x) = \frac{1}{2}x.$$

(b) Now, the probability that you win is the probability that all n other bidders have a valuation less than b, which is b^n. The expected utility is $b^n(x - b)$, and optimizing over b gives $nb^{n-1}(x - b) - b^n = 0$, or

$$\beta(x) = \frac{n}{n+1}x.$$

(c) We let $X^{(2)}$ be the second-largest bid. We can specify the distribution simply by specifying the CDF, so we aim to find $\Pr(X^{(2)} < x)$. There are two disjoint events in which $X^{(2)} < x$. The first case is when each of the $X_i < x$ and the second is when exactly $n-1$ of the $X_i < x$ and one is greater. Concretely, we may let A be the event that each of the $X_i < x$ and B be the event that $n - 1$ of the $X_i < x$. We thus have:

$$\begin{aligned}\Pr(X^{(2)} < x) &= \Pr(A \cup B) = \Pr(A) + \Pr(B) \\ &= (1 - e^{-\lambda x})^n + (1 - e^{-\lambda x})^{n-1} \cdot n \cdot e^{-\lambda x}\end{aligned}$$

We may take the derivative to compute the density:

$$f_{X^{(2)}} = \lambda \cdot n \cdot (n - 1) \cdot (1 - e^{-\lambda x})^{n-2} \cdot e^{-2\lambda x}$$

7. **Auctions: Bayesian Nash Equilibrium (Optional: This problem is of a more theoretical nature and will not be tested on the exam)**

A **Bayesian Nash equilibrium** is a strategy for each player, such that no player has an incentive to change strategies. In other words, no player can improve his/her expected utility by changing his/her strategy. *The contents of this question will not be tested, but this question is provided as a way for you to further explore auction theory if you are interested.*

In this question, we will derive the Bayesian Nash equilibrium for the first-price auction, under the assumption that the valuations are i.i.d. with common density function $f(x)$. By symmetry, in the Bayesian Nash equilibrium, each bidder should use the same bid function $\beta(\cdot)$. We further assume that $\beta(\cdot)$ is differentiable and strictly increasing.

(a) Suppose that your valuation is x. Let X_i denote the valuation of player i, $i = 1, \ldots, n-1$ (your own valuation is known to you as a fixed real number, whereas the valuations of other players are modeled as random variables whose value is unknown). What is your expected utility when you bid b, assuming that the other $n-1$ bidders bid according to $\beta(\cdot)$? Write your answers in terms of the CDF $F(x) := \int_{-\infty}^{x} f(x)\,\mathrm{d}x$.

(b) Differentiate the expression you obtained with respect to b. *Hint*: You may need to use the Inverse Function Theorem, which states that the derivative of an inverse function is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

(c) Now, suppose that you bid according to $\beta(\cdot)$ as well, i.e. $b = \beta(x)$. Under this assumption, set the result from the previous part to 0 and solve for $\beta(x)$.

(d) For the second-price auction, suppose that the other $n-1$ bidders bid their own valuations, i.e. they use the identity bidding function $\beta(x) = x$. Prove that it is optimal for you to bid your own valuation. The strategy of bidding your own valuation in the second-price auction is thus a Bayesian Nash equilibrium.

Solution:

(a) The expected utility U is the probability that you win the auction, multiplied by the payoff $x - b$.

$$\begin{aligned} U &= \Pr(\beta(X_1) < b, \ldots, \beta(X_{n-1}) < b)(x - b) \\ &= \Pr(X_1 < \beta^{-1}(b), \ldots, X_{n-1} < \beta^{-1}(b))(x - b) \\ &= F^{n-1}(\beta^{-1}(b))(x - b) \end{aligned}$$

(b) With a little calculus, we obtain

$$\frac{\mathrm{d}U}{\mathrm{d}b} = (n-1)F^{n-2}(\beta^{-1}(b))f(\beta^{-1}(b))\frac{1}{\beta'(\beta^{-1}(b))}(x - b) - F^{n-1}(\beta^{-1}(b)).$$

(c) Setting $b = \beta(x)$, the expression simplifies to

$$\frac{\mathrm{d}U}{\mathrm{d}b} = (n-1)F^{n-2}(x)f(x)\frac{1}{\beta'(x)}(x - \beta(x)) - F^{n-1}(x).$$

Writing

$$(n-1)F^{n-2}(x)f(x) = \frac{\mathrm{d}}{\mathrm{d}x}F^{n-1}(x),$$

we have

$$0 = \left(\frac{\mathrm{d}}{\mathrm{d}x}F^{n-1}(x)\right)x - \left(\frac{\mathrm{d}}{\mathrm{d}x}F^{n-1}(x)\right)\beta(x) - F^{n-1}(x)\frac{\mathrm{d}}{\mathrm{d}x}\beta(x).$$

By the Product Rule, the last two terms are

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F^{n-1}(x)\beta(x)\right).$$

Putting it together, we can solve the differential equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(F^{n-1}(x)\beta(x)\right) &= x\left(\frac{\mathrm{d}}{\mathrm{d}x}F^{n-1}(x)\right),\\ F^{n-1}(x)\beta(x) &= \int_{-\infty}^{x} x\left(\frac{\mathrm{d}}{\mathrm{d}x}F^{n-1}(s)\right)\mathrm{d}s.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}F^{n-1}(x)\beta(x) &= sF^{n-1}(s)\Big|_{-\infty}^{x} - \int_{-\infty}^{x} F^{n-1}(s)\,\mathrm{d}s,\\ \beta(x) &= x - \frac{1}{F^{n-1}(x)}\int_{-\infty}^{x} F^{n-1}(s)\,\mathrm{d}s.\end{aligned}$$

In the last line, we have used the fact that a distribution function satisfies $\lim_{x\to-\infty} F(x) = 0$.

(d) Fix the bids of the other $n-1$ bidders and observe that your bid does not influence how much you pay (your bid only determines whether you win or not). Let b_{-1} denote the highest bid among the other $n-1$ bidders and x denote your valuation. Bidding your valuation ensures that you win exactly when $x - b_{-1} > 0$ and you lose exactly when $x - b_{-1} < 0$.

Suppose $x - b_{-1} > 0$. Any bid which is greater than b_{-1} ensures that you win the auction, which is good in this case because you value the object more than what you pay. Therefore, any bid greater than b_{-1} is optimal, so bidding x is optimal.

Suppose $x - b_{-1} < 0$. Any bid which is less than b_{-1} ensures that you lose the auction, which is good in this case because you value the object less than what you would pay. Therefore, any bid less than b_{-1} is optimal, so bidding x is optimal.

Hence, bidding x is always optimal.