3/2 lecture

recap of CLT

$$Z = \sum_{i=1}^{n} \frac{X_i}{\sqrt{n}}$$

$$M_Z(s) = \left[1 + \frac{s^2}{2n} + \frac{C_3}{n^{3/2}} + \frac{C_4}{n^2}\right]^n$$

$$\approx 1 + n\left(\frac{s^2}{2n} + \frac{C_3}{n^{3/2}} + \frac{C_4}{n^2} \cdots\right)$$

$$= \left[1 + \frac{s^2}{2n}(1+\varepsilon_n)\right]^n$$

$$\lim_{n\to\infty} \downarrow = \exp\left\{\lim_{n\to\infty} \left(\frac{s^2}{2} + \varepsilon_n'\right)\right\} \qquad \lim_{n\to\infty} \varepsilon_n = 0$$

$$= e^{\frac{s^2}{2}}$$

stochastic process

a stochastic process $X = \{X_t\}_{t \in T}$ is a collection of RVs.

the index t often used to represent time

X models the evolution of a sequence of RVs over time

ex) stock prices over days, # of students in OH over 15 min intervals, sequences of coin flips

I want to characterize the complete behavior of a discrete time random process $X = \{X_1, X_2, \ldots, X_n\}$

we need the joint pmf

$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$ $\forall$ too much to ask

Markov chains impose structure on the joint pmf to make study feasible

let $\mathcal{X}$ be a finite set of possible "states" (state space)

$$P(X_n = x_n \mid X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \dots X_0 = x_0)$$
$$= P(X_n = x_n \mid X_{n-1} = x_{n-1})$$
$$\forall n \;\; \& \; \forall x \in \mathcal{X}$$

note: $X_n \not\perp\!\!\!\perp X_{n-2}, X_{n-3}$

$X_n \perp\!\!\!\perp X_{n-i} \mid X_{n-1}$ called the Markov property (memoryless property)
$i > 1$

"given present, future independent of past"

$$P(X_0, X_1, \dots X_n) = P(X_0) P(X_1 \mid X_0) P(X_2 \mid X_1) \dots P(X_n \mid X_{n-1})$$

$P_{i,j} = P_r(X_n = j \mid X_{n-1} = i)$ time homogeneous MC

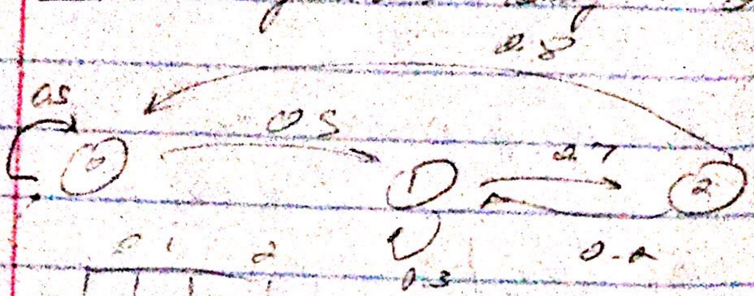
$\Downarrow$ $\quad \forall i, j \in \mathcal{X}$

state transition probabilities

e.g. $\mathcal{X} = \{1, 2, \dots m\}$

$$P = \begin{array}{c|c} & \text{to} \rightarrow \;\; 1 \;\; 2 \; \dots \; j \; \dots \; m \\ \text{from} \downarrow \begin{array}{c} 1 \\ 2 \\ \vdots \\ i \\ m \end{array} & \begin{bmatrix} P_{11} & P_{12} & & & \\ & & & & \\ & & P_{i,j} & & \\ & & & & P_{mm} \end{bmatrix} \end{array} = \{ \text{stochastic matrix, state transition matrix} \}$$

$\sum_{j \in \mathcal{X}} P_{i,j} = 1 \Rightarrow$ row sums of $P = 1$ for all rows ex "# of students coming to 'GH'"



State transition diagram

	0	1	2
0	0.5	0.5	0
1	0	0.3	0.7
2	0.8	0.2	0

$\mathcal{X} = \{0, 1, 2\}$

RP $\underline{X} = \{X_0, X_1, X_2 \ldots\}$

$\pi_0 = [P(X_0 = 0), \; P(X_0 = 1) \quad P(X_0 = 2)]$

State vector @ time 0 has 3 entries (# of entries = $|\mathcal{X}|$)

$\pi_1 = [\pi_1(0) \quad \pi_1(1) \quad \pi_1(2)]$

$P(X_1 = x_1) = \sum_{x_0=0}^{2} P(X_1 = x_1 \mid X_0 = x_0)\, P(X_0 = x_0)$

$\pi_1(x_1) = \sum_{x_0=0}^{2} P_{x_0, x_1}\, \pi_0(x_0)$

$\forall x_1 \in 0, 1, 2$

$\pi_1 = \pi_0 \cdot P \Rightarrow \pi_n = \pi_{n-1} P = \pi_{n-2} P^2 = \pi_0 P^n$

$\pi_1 = [\pi_0(0) \quad \pi_0(1) \quad \pi_0(2)] \begin{bmatrix} P_{00} & \cdots & P_{02} \\ & & \\ P_{20} & \cdots & P_{22} \end{bmatrix}$

@ time 0, state 0

$[x \; y \; z] = [x \; y \; z] \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.3 & 0.7 \\ 0.8 & 0.2 & 0 \end{bmatrix}$

$\Rightarrow x = \frac{56}{141}, \; y = \frac{50}{141}, \; z = \frac{35}{141}$

Some questions of interest:

(1) is there always an invariant dist ?

(2) is it unique ?

(3) is it true that the $\lim_{n\to\infty} \pi \to \pi_n$?

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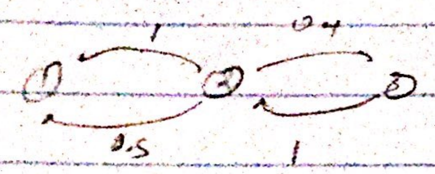

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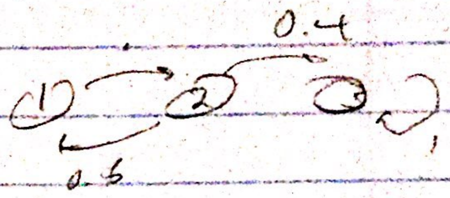

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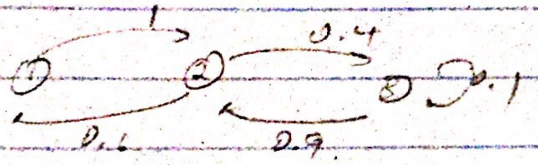

1. irreducible : can go from any state to any state

2. aperiodic vs periodic

$$d(i) = \gcd\{n \geq 1 \mid P^n(i,i) > 0\}$$

facts $d(i) = d \quad \forall i \in \mathcal{X}$

if $d=1$, called aperiodic MC

else, it is called a d-periodic MC

ex1 $d = 2$ $(d = \gcd(2, 4, 6, \ldots))$

ex2 $d = 1$ $(d = \gcd(2, 4, 5, \ldots))$

BIG theorem for FMC

(a) if MC is finite & irreducible, it has a unique invariant distribution π^∞, & $\pi^\infty(i)$

(b) if the MC is also aperiodic, then $\lim_{n\to\infty} \pi_n = \pi^\infty$

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recall. long term fraction of $X_n = i$ is

$$\lim_{n \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \{1\}_{X_n = i}$$

converge in the SLLN sense