UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

**Problem Set 7**
Spring 2017

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**Self-Graded Scores Due:** Monday, March 20, 5:00 PM
Submit your self-graded scores via the Google form:
https://goo.gl/forms/GrwogQuL6lrDrSOJ3.
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1. **Inventory Management**

   Consider a Markov chain $(X_n)$, where $X_n$ represents the quantity of an item in stock at time $n$. We will assume that the changes in stock are modeled by a simple random walk, that is,

   $$\Pr(X_{n+1} = i + 1 \mid X_n = i) = \Pr(X_{n+1} = i - 1 \mid X_n = i) = \frac{1}{2}.$$

   A $(s, S)$ policy (for $S > s$) is given as follows:

   - If the stock ever drops to 0, then buy enough items until we have replenished our stock to $s$.
   - If the stock ever reaches $S$, then sell enough items until our stock is back down to $s$.

   In other words, $s$ is the *baseline*, and we return the baseline as soon as the stock drops to 0 or increases to $S$.

   (a) Let $V_k$ denote the number of visits to $k$ starting from $i$, before we reach 0 or $S$. Calculate $E_i[V_k] = E[V_k \mid X_0 = i]$.

   (b) $E_s[V] = \sum_{k=1}^{S-1} kE_s[V_k]$ represents the expected total quantity of the item that we will have in stock, starting from the baseline $s$, until we reach 0 or $S$. Calculate $E_s[V]$.

   (c) A *cycle* starts at the baseline and ends when we reach 0 or $S$ (and then the next cycle starts). Let $T_i$ denote the length of the $i$th cycle. Associated with each cycle is a *transaction cost* $c_t$, which represents the need to buy or sell items to meet our policy. Also, there is a *holding cost* $c_h$ which is assumed to be proportional to $V_k$: $c_h = hV_k$. Therefore, the long-run average cost incurred by the policy is

   $$\text{LRAC} = \frac{c_t + hE_s[V_k]}{E[T_i]}.$$

   (The LRAC is an abbreviation for long-run average cost.) Find the optimal policy $(s^*, S^*)$ which minimizes the LRAC.

**Solution:**

(a) The first-step equations give

$$E_i[V_k] = \frac{1}{2}E_{i-1}[V_k] + \frac{1}{2}E_{i+1}[V_k], \quad i \neq k,$$
$$E_k[V_k] = 1 + \frac{1}{2}E_{k-1}[V_k] + \frac{1}{2}E_{k+1}[V_k].$$

We claim that

$$E_i[V_k] = \begin{cases} \dfrac{2i(S-k)}{S}, & i \leq k, \\ 2\left(\dfrac{i(S-k)}{S} - (i-k)\right), & i \geq k. \end{cases}$$

Indeed, for $i < k$,

$$E_i[V_k] = \frac{(i-1)(S-k)}{S} + \frac{(i+1)(S-k)}{S} = \frac{2i(S-k)}{S},$$

and for $i = k$,

$$E_k[V_k] = 1 + \frac{(k-1)(S-k)}{S} + \frac{(k+1)(S-k)}{S} - 1 = \frac{2k(S-k)}{S},$$

and for $i > k$,

$$\begin{aligned} E_k[V_k] &= \frac{(i-1)(S-k)}{S} - (i-k-1) + \frac{(i+1)(S-k)}{S} - (i-k+1) \\ &= 2\left(\frac{i(S-k)}{S} - (i-k)\right). \end{aligned}$$

(b) One has

$$\begin{aligned} E_s[V] &= \frac{2s}{S}\sum_{k=1}^{S-1} k(S-k) - 2\sum_{k=1}^{s-1} k(s-k) \\ &= \frac{2s}{S}\frac{(S-1)S(S+1)}{6} - 2\cdot\frac{(s-1)s(s+1)}{6} = \frac{s}{3}(S^2-1-s^2+1) \\ &= \frac{s(S^2-s^2)}{3}. \end{aligned}$$

To evaluate the summation, we note that

$$\begin{aligned} \sum_{k=1}^{n-1} k(n-k) &= n\sum_{k=1}^{n-1} k - \sum_{k=1}^{n-1} k^2 = \frac{n(n-1)n}{2} - \frac{(n-1)n(2n-1)}{6} \\ &= \frac{n}{6}(3n^2 - 3n - 2n^2 + 3n - 1) = \frac{(n-1)n(n+1)}{6}. \end{aligned}$$

(c) First we note that $E[T_i] = s(S-s)$ (this is the result of the gambler's ruin problem). The function we must minimize is

$$\text{LRAC} = \frac{c_t + hs(S^2-s^2)/3}{s(S-s)},$$

w.r.t. $s$ and $S$. It will be convenient to instead write the problem in terms of $r = s/S$, the ratio of the two quantities.

$$\text{LRAC} = \frac{c_t + hS^3 r(1-r^2)/3}{S^2 r(1-r)}.$$

Now, we differentiate:

$$\begin{aligned}
\frac{\partial \text{LRAC}}{\partial r} &= -\frac{c_t(1-2r)}{S^2r^2(1-r)^2} + \frac{1}{3}hS, \\
\frac{\partial \text{LRAC}}{\partial S} &= -\frac{2c_t}{S^3 r(1-r)} + \frac{h(1+r)}{3}.
\end{aligned}$$

The two equations give us the two conditions

$$\begin{aligned}
3c_t(1-2r) &= hS^3r^2(1-r)^2, \\
6c_t &= hS^3 r(1-r^2).
\end{aligned}$$

Dividing the first condition by the second condition, we have

$$\frac{1-2r}{2} = \frac{r(1-r)}{1+r},$$

which is readily solved to give $r = 1/3$. Now, we add $r$ times the second condition to the first condition to obtain

$$3c_t = 2hS^3r^2 - 2hS^3r^3 = 2hS^3r^2(1-r).$$

Using $r^2(1-r) = 2/27$ and solving for $S$, one has

$$S = 3\sqrt[3]{\frac{3c_t}{4h}}.$$

Hence, the optimal pair is

$$(s^*, S^*) = \left(\sqrt[3]{\frac{3c_t}{4h}}, 3\sqrt[3]{\frac{3c_t}{4h}}\right).$$

2. **Poisson Process Warm-Up**

Consider a Poisson process $\{N_t, t \geq 0\}$ with rate $\lambda = 1$. Let random variable $S_i$ denote the time of the $i$th arrival.

(a) Given $S_3 = s$, find the joint distribution of $S_1$ and $S_2$.

(b) Find $E[S_2 \mid S_3 = s]$.

(c) Find $E[S_3 \mid N_1 = 2]$.

(d) Give an interpretation, in terms of a Poisson process with rate $\lambda$, of the following fact:

*If $N$ is a geometric random variable with parameter $p$, and $X_i$ are IID exponential random variables with parameter $\lambda$, then $X_1 + \cdots + X_N$ has the exponential distribution with parameter $\lambda p$.*

**Solution:**

(a) We know the distribution of sum of IID exponential random variables is Erlang. So, since the inter-arrival times of Poisson process are exponentially distributed we have

$$f_{S_i}(s) = \frac{s^{i-1}e^{-s}}{(i-1)!}1\{s \ge 0\}.$$

$$\begin{aligned} f_{S_1,S_2|S_3}(s_1, s_2 \mid S_3 = s) &= \frac{f_{S_1,S_2,S_3}(s_1, s_2, s)}{f_{S_3}(s)} \\ &= \frac{e^{-s_1}e^{-(s_2-s_1)}e^{-(s-s_2)}}{s^2e^{-s}/2!}1\{0 \le s_1 \le s_2 \le s\} \\ &= \frac{2}{s^2}1\{0 \le s_1 \le s_2 \le s\}. \end{aligned}$$

Thus, $S_1$ and $S_2$ are uniformly distributed on the feasible region $\{0 \le s_1 \le s_2 \le s\}$.

(b) By part (a), $S_2$ is the maximum of two uniform random variables between 0 and $s$. Thus, if $0 \le x \le s$,

$$F_{S_2|S_3=s}(x) = \Pr(S_2 \le x \mid S_3 = s) = \left(\frac{x}{s}\right)^2$$

and

$$f_{S_2|S_3=s}(x) = \frac{2x}{s^2}1\{0 \le x \le s\}.$$

Finally,

$$E[S_2 \mid S_3 = s] = \int_0^s \frac{2x^2}{s^2}\,\mathrm{d}x = \frac{2s}{3}.$$

(c) By the memoryless property, $E[S_3 \mid N_1 = 2] = 1 + E[S_1] = 2$.

(d) Consider a Poisson process with rate $\lambda$ and split the process by keeping each arrival with probability $p$, independently of the other arrivals. In the original process, the inter-arrival times are IID exponential random variables with parameter $\lambda$, and $X_1 + \cdots + X_N$ represents the amount of time until the first arrival we keep. By Poisson splitting, we know that the split process is a Poisson process with rate $\lambda$, and so the time until the first arrival is an exponential random variable with parameter $\lambda p$.

## 3. Bus Arrivals at Cory Hall

Starting at time 0, the F line makes stops at Cory Hall according to a Poisson process of rate $\lambda$. Students arrive at the stop according to an independent Poisson process of rate $\mu$. Every time the bus arrives, all students waiting get on.

(a) Given that the interarrival time between bus $i-1$ and bus $i$ is $x$, find the distribution for the number of students entering the $i$th bus.

(b) Given that a bus arrived at 9:30 AM, find the distribution for the number of students that will get on the next bus.

(c) Find the distribution of the number of students getting on the next bus to arrive after 11:00 AM. (You can assume that time 0 was infinitely far in the past.)

**Solution:**

(a) We note that the student arrival process is independent of the bus arrival process and thus, the number of arrivals to the student arrival process in the interval of size $x$ is a Poisson random variable with parameter $\mu x$.

(b) We consider a related problem, where we would like to find the distribution of the number of students entering the $i$th bus. In fact, as we will see, the number of students entering the $i$th bus is independent of $i$, and thus, we may consider 9:30 AM to be the arrival time of the $(i-1)$th bus for any $i > 1$, and the result will be the same. We now find the distribution of the number of students entering the $i$th bus.

We note that the student arrival process and the bus arrival process are independent Poisson processes, and we can thus consider the merged Poisson process with parameter $\lambda + \mu$. As we saw in discussion, each arrival for the combined process is a bus with probability $\lambda/(\lambda+\mu)$ and likewise each arrival for the combined process is a student with probability $\mu/(\lambda+\mu)$. The sequence of bus/student choices is an IID sequence, so starting immediately after bus $i-1$, the number of students before a bus arrival is a geometric random variable with parameter $\lambda/(\lambda+\mu)$. Thus, if we let $N_i$ give the number of students entering the $i$th bus, we see that:

$$\Pr(N_i = k) = \left(\frac{\mu}{\lambda+\mu}\right)^k \cdot \frac{\lambda}{\lambda+\mu}.$$

Note that this is independent of $i$, so this also gives the number of students that will get on the next bus given that there was an arrival at 9:30 AM. What this problem is essentially telling us is that given the process started infinitely far in the past, if we pick some random time $t$, then the number of students arriving after $t$, but before the next bus has the same distribution as $N_i$.

(c) This is a slight variation on random incidence. We note that in part (b), we found the number of future student arrivals before the next bus. What we are looking for is the sum of the number of students waiting at 11:00 AM and the number of future student arrivals before the next bus. We see that by definition of the Poisson process, these are IID, so we may convolve their PMFs. Now, we find the PMF of the number of students waiting, $W$.

Note that since time 0 was infinitely far in the past, we consider $\{Z_i, -\infty < i < \infty\}$, the doubly infinite IID sequence of bus/student choices where $Z_i = 0$ if it the $i$th combined arrival is a bus, and $Z_i = 1$ if it was a student. We may index this sequence so that $-1$ is the index of the most recent combined arrival prior to 11:00 AM. We then see that if

$Z_{-1} = 0$, then no customers are waiting at 11:00 AM. Also, if $Z_{-n} = 0$ and $Z_{-m} = 1$ for $1 \leq m < n$, then $n$ customers are waiting. Since the $Z_i$ are IID, the distribution is also geometric with parameter $\lambda/(\lambda+\mu)$. In other words:

$$\Pr(W = n) = \left(\frac{\mu}{\lambda+\mu}\right)^n \cdot \frac{\lambda}{\lambda+\mu}.$$

Returning to the original problem, we now want:

$$\begin{aligned}
\Pr(W + N_i = n) &= \sum_{m=0}^{n} \left(\frac{\mu}{\lambda+\mu}\right)^m \cdot \frac{\lambda}{\lambda+\mu} \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{n-m} \cdot \frac{\lambda}{\lambda+\mu} \\
&= (n+1)\left(\frac{\mu}{\lambda+\mu}\right)^n \left(\frac{\lambda}{\lambda+\mu}\right)^2.
\end{aligned}$$

4. **Sum-Quota Sampling**

Consider the problem of estimating the mean inter-arrival time of a Poisson process. In what follows, recall that $N_t$ denotes the number of arrivals by time $t$.

*Sum-quota sampling* is a form of sampling in which the number of samples is not fixed in advance; instead, we wait until a fixed *time* $t$, and take the average of the interarrival times seen so far. If we let $X_i$ denote the $i$th inter-arrival time, then

$$\bar{X} = \frac{X_1 + \cdots + X_{N_t}}{N_t}.$$

Of course, the above quantity is not defined when $N_t = 0$, so instead we must condition on the event $\{N_t > 0\}$. Compute $E[\bar{X} \mid N_t > 0]$, assuming that $N_t$ is a Poisson process of rate $\lambda$.

**Solution:**

We proceed by conditioning on the values of $N_t$. Note that

$$\Pr(N_t = n \mid N_t > 0) = \frac{e^{-\lambda t}(\lambda t)^n}{n!(1-e^{-\lambda t})}.$$

Now, we use the law of total probability:

$$E[\bar{X} \mid N_t > 0] = \sum_{n=1}^{\infty} E[N_t^{-1}(X_1 + \cdots + X_{N_t}) \mid N_t = n] \Pr(N_t = n \mid N_t > 0)$$

Conditioned on $\{N_t = n\}$, the sum $X_1 + \cdots + X_{N_t}$ is the maximum of $n$ uniform $[0, t)$ random variables:

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{n} \frac{tn}{n+1} \frac{e^{-\lambda t}(\lambda t)^n}{n!(1-e^{-\lambda t})} = \frac{e^{-\lambda t}}{\lambda(1-e^{-\lambda t})} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n+1}}{(n+1)!} \\
&= \frac{e^{-\lambda t}}{\lambda(1-e^{-\lambda t})}(e^{\lambda t} - 1 - \lambda t) = \frac{1}{\lambda}\left(1 - \frac{\lambda t e^{-\lambda t}}{1-e^{-\lambda t}}\right).
\end{aligned}$$

The expectation $E[\bar{X} \mid N_t > 0]$ does not quite equal $1/\lambda$, which is what we want to estimate.

5. **Taxi Queue**

Empty taxis pass by a street corner at a Poisson rate of two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner at a Poisson rate of one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. John arrives at the street corner at a given time. Find his expected waiting time, given that he joins the queue. Assume that the process is in steady state.

**Solution:**

Consider a continuous time Markov chain with states $X \in \{0, 1, 2, 3, 4\}$ which denotes the number of people waiting. For $n = 0, 1, 2, 3$, the transitions from $n$ to $n + 1$ have rate 1, and the transitions from $n + 1$ to $n$ have rate 2. The balance equations are then

$$\pi(n) = \frac{1}{2}\pi(n-1), \qquad n = 1, 2, 3, 4.$$

Using the above equations and $\sum_{i=0}^{4} \pi(i) = 1$ we find that $\pi(i) = 2^i \pi(0)$ and $\pi(0) = 16/31$. Since the expected waiting time for a new taxi is 0.5, the expected waiting time of John given that he joins the queue can be computed as follows.

$$E[T] = \pi(1) \times 0.5 + \pi(2) \times 1 + \pi(3) \times 1.5 + \pi(4) \times 2 = \frac{13}{15}.$$

6. **Poisson Queues**

A continuous-time queue has Poisson arrivals with rate $\lambda$, and it is equipped with infinitely many servers. The servers can work in parallel on multiple customers, but they are non-cooperative in the sense that a single customer can only be served by one server. Thus, when there are $k$ customers in the queue, $k$ servers are active. Suppose that the service time of each customer is exponentially distributed with rate $\mu$ and they are i.i.d.

(a) Argue that the queue-length is a Markov chain. Draw the transition diagram of the Markov chain.

(b) Prove that for all finite values of $\lambda$ and $\mu$ the Markov chain is positive-recurrent and find the invariant distribution.

**Solution:**

(a) The queue length is a MC as customer arrivals are independent of the current number of customers in the queue. Also, the departures only depend on the current number of customers being served. Next, even when $k$ customers are being served, the completion of their service is independent of one another. Finally, even if one of the $k$ customers has been completely served, the other customer has the same service time distribution as before as the exponential distribution is memoryless.

The only non-zero transition rates are

$$\begin{aligned} Q(k, k+1) &= \lambda, \qquad k \geq 0, \\ Q(k, k-1) &= k\mu, \qquad k \geq 1. \end{aligned}$$

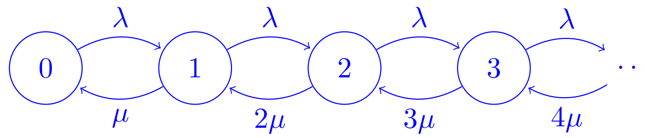


Figure 1: Markov chain of a memoryless queue with infinitely many servers.

(b) By flow conservation equations,

$$\pi(k)Q(k, k+1) = \pi(k+1)Q(k+1, k), \qquad k \geq 0.$$

Thus,

$$\pi(k) = \pi(0)\left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}.$$

Let $\rho = \lambda/\mu$. Then, $\pi(0)\sum_{k=0}^{\infty} \rho^k/k! = 1$. Thus, $\pi(0) = e^{-\rho}$ and the MC is positive-recurrent for all finite $\lambda$ and $\mu$.