UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Solutions 4
Spring 2017

Issued: Thursday, February 16, 2017

Self-graded Scores Due: 5pm, Monday, February 27, 2017
Submit your self-graded scores via the google form:
https://goo.gl/forms/o10VTMk9Ks29WyWs1. Make sure that you use your **SORTABLE NAME** on bCourses.

Problem 1. See the solution for Midterm 01.

Problem 2. Consider a population of N individuals. At the end of each year, each individual, independently of others, leaves behind ξ offspring. Assume $E[\xi] = \mu$ and $\mathrm{Var}(\xi) = \sigma^2$. Let X_n denote the size of the population at the end of the n^{th} year. Compute $E[X_n]$ and $\mathrm{Var}(X_n)$.

Solution: Conditioning on X_{n-1}, we have

$$E[X_n|X_{n-1}] = E[\xi_1 + \xi_2 + \cdots + \xi_{X_{n-1}}|X_{n-1}] = \mu X_{n-1},$$

and

$$E[X_n] = E[E[X_n|X_{n-1}]] = E[\mu X_{n-1}] = \mu E[X_{n-1}].$$

By recursion, we find $E[X_n] = \mu^n N$.

The conditional variance is

$$\mathrm{Var}(X_n|X_{n-1}) = \mathrm{Var}(\xi_1 + \xi_2 + \cdots + \xi_{X_{n-1}}) = X_{n-1}\mathrm{Var}(\xi) = \sigma^2 X_{n-1}.$$

Then, by the law of total variance, we have

$$\begin{aligned}\mathrm{Var}(X_n) &= E[\sigma^2 X_{n-1}] + \mathrm{Var}(\mu X_{n-1}) \\ &= \sigma^2 \mu^{n-1} N + \mu^2 \mathrm{Var}(X_{n-1})\end{aligned}$$

First, suppose that $\mu = 1$. Then, the recurrence simplifies to

$$\mathrm{Var}(X_n) = \mathrm{Var}(X_{n-1}) + \sigma^2 N,$$

which means that the variance increases linearly:

$$\mathrm{Var}(X_n) = \sigma^2 N n.$$

(Here, we are assuming that the initial population is fixed, so $\mathrm{Var}(X_0) = 0$.)

Now, assume that $\mu \neq 1$. Then,

$$\begin{aligned}
\mathrm{Var}(X_n) &= \sigma^2\mu^{n-1}N + \mu^2\mathrm{Var}(X_{n-1}) \\
&= \sigma^2\mu^{n-1}N + \mu^2(\sigma^2\mu^{n-2}N + \mathrm{Var}(X_{n-2}))(\mu^2+\sigma^2) \\
&\vdots \\
&= \sigma^2\mu^{n-1}N\sum_{i=0}^{n-1}\mu^i = \sigma^2\mu^{n-1}N\frac{1-\mu^n}{1-\mu}.
\end{aligned}$$

Problem 3. Assume that you have a random variable U which is uniformly distributed on $(0,1)$. Using U, you want to simulate an exponential random variable T with rate λ. Find a strictly increasing function $h:(0,1)\to(0,\infty)$ such that $T \sim h(U)$.

Solution: T is an exponential random variable with rate λ:

$$f_T(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

For $x \in (0,\infty)$:

$$P(T \leq x) = \int_0^x \lambda e^{-\lambda x}dx = 1 - e^{-\lambda x}.$$

We want $h(U)$ and T to have the same distribution:

$$\begin{aligned}
P(h(U) \leq x) &= P(T \leq x) \\
&= 1 - e^{-\lambda x} \quad \forall x \geq 0.
\end{aligned}$$

Since h is strictly increasing, its inverse is also a strictly increasing function. Then,

$$\begin{aligned}
P(h(U) \leq x) &= P(h^{-1}(h(U)) \leq h^{-1}(x)) \\
&= P(U \leq h^{-1}(x)) \\
&= h^{-1}(x).
\end{aligned}$$

As a result, we need

$$h^{-1}(x) = 1 - e^{-\lambda x} \iff h(x) = -\frac{1}{\lambda}\log(1-x).$$

Problem 4. A bin contains balls numbered $1, 2, \ldots, n$. You select m balls at random without replacement and order them: $X_{(1)} < X_{(2)} < \cdots < X_{(m)}$. For this question, you may assume $a, b \in \mathbb{N}$.

(a) Find $P(X_{(1)} = a)$ for $1 \leq a \leq n-m+1$.

(b) Find $P(X_{(2)} = b)$ for $2 \leq b \leq n-m+2$.

(c) Assume $m = 10$, $n = 100$.

(i) Find $P(X_{(1)} = 3, X_{(2)} = 7, X_{(3)} = 15 | X_{(5)} = 20)$.

(ii) Find $P(X_{(1)} = 8, X_{(2)} = 11, X_{(3)} = 15 | X_{(5)} = 20)$.

Solution:

(a)

$$\begin{aligned} P(X_{(1)} = a) &= \frac{\#\text{ orderings } a+1 \leq X_{(2)} < X_{(3)} < \cdots < X_{(m)} \leq n}{\#\text{ orderings } 1 \leq X_{(1)} < X_{(2)} < \cdots < X_{(m)} \leq n} \\ &= \frac{\binom{n-a}{m-1}}{\binom{n}{m}} \end{aligned}$$

(b)

$$\begin{aligned} P(X_{(2)} = b) &= \frac{\#\text{ orderings } 1 \leq X_{(1)} < b < X_{(3)} < \cdots < X_{(m)} \leq n}{\#\text{ orderings } 1 \leq X_{(1)} < X_{(2)} < \cdots < X_{(m)} \leq n} \\ &= \frac{\binom{b-1}{1}\binom{n-b}{m-2}}{\binom{n}{m}} \end{aligned}$$

(c)

$$\begin{aligned} P(X_{(1)} = a, X_{(2)} = b, X_{(3)} = c | X_{(5)} = d) &= \frac{P(X_{(1)} = a, X_{(2)} = b, X_{(3)} = c, X_{(5)} = d)}{P(X_{(5)} = d)} \\ &= \frac{\binom{d-c-1}{1}\binom{n-d}{m-5}}{\binom{d-1}{4}\binom{n-d}{m-5}} \\ &= \frac{d-c-1}{\binom{d-1}{4}} \end{aligned}$$

(i) $P(X_{(1)} = 3, X_{(2)} = 7, X_{(3)} = 15 | X_{(5)} = 20) = 4\binom{19}{4}^{-1}$

(ii) $P(X_{(1)} = 8, X_{(2)} = 11, X_{(3)} = 15 | X_{(5)} = 20) = 4\binom{19}{4}^{-1}$

Problem 5. Consider a random variable Z with transform:

$$M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8}$$

(a) Find the numerical value for the parameter a.

(b) Find $P(Z \geq 0.5)$.

(c) Find $E[Z]$ by using the probability distribution of Z.

(d) Find $E[Z]$ by using the transform of Z and without explicitly using the probability distribution of Z.

(e) Find Var(Z) by using the probability distribution of Z.

(f) Find Var(Z) by using the transform of Z and without explicity using the probability distribution of Z.

Solution:

(a) By definition, we know that $M_Z(s) = E[e^{sZ}]$. Thus, we know the following must be true:

$$M_Z(0) = E[e^{0Z}] = 1 = \frac{a}{8}$$

It follows that $a = 8$.

(b) We should find the pdf of Z. Expanding the transform, we write $M_Z(s) = \frac{A}{s-4} + \frac{B}{s-2}$. We may solve for A, B to see that $A = -2$ and $B = -1$. Thus:

$$M_Z(s) = \frac{1}{2}\left(\frac{4}{4-s} + \frac{2}{2-s}\right)$$

It follows that $f_Z(z) = \frac{1}{2}(4e^{-4z} + 2e^{-2z})1\{z \geq 0\}$ We may thus integrate to see that:

$$\int_{0.5}^{\infty} f_Z(z)dz = \frac{1}{2}(e^{-2} + e^{-1})$$

(c) From the expectation of exponential random variables, we see that:

$$E[Z] = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8}$$

(d) $E[Z] = \frac{d}{ds}M_Z(s)\Big|_{s=0} = \frac{2}{(4-s)^2} + \frac{1}{(2-s)^2}\Big|_{s=0} = \frac{3}{8}$

(e) We see that: $E[Z^2] = \int_0^\infty z^2 f_Z(z)dz = \frac{5}{16}$ and thus $\mathrm{Var}(Z) = E[Z^2] - E[Z]^2 = \frac{11}{64}$.

(f) Note that $E[Z^2] = \frac{d^2}{ds^2}M_Z(2)\Big|_{s=0} = \frac{4}{(4-s)^3} + \frac{2}{(2-s)^3}\Big|_{s=0} = \frac{5}{16}$. Thus, $\mathrm{var}(Z) = \frac{11}{64}$.

Problem 6. Suppose $E[X] = 0, \mathrm{Var}(X) = \sigma^2 < \infty$ and $\alpha > 0$. Prove that

$$P(X \geq \alpha) \leq \frac{\sigma^2}{\alpha^2 + \sigma^2}.$$

Note that

$$\begin{aligned} P(X \geq \alpha) &= P(X + \sigma^2/\alpha \geq \alpha + \sigma^2/\alpha) \\ &\leq P(|X + \sigma^2/\alpha| \geq \alpha + \sigma^2/\alpha) \\ &= P((X + \sigma^2/\alpha)^2 \geq (\alpha + \sigma^2/\alpha)^2). \end{aligned}$$

Now using Markov inequality,

$$\begin{aligned}
P(X \geq \alpha) &\leq \frac{E[(X + \sigma^2/\alpha)^2]}{(\alpha + \sigma^2/\alpha)^2} \\
&= \frac{E[X^2] + 2E[X]\sigma^2/\alpha + \sigma^4/\alpha^2}{(\alpha + \sigma^2/\alpha)^2} \\
&= \frac{\sigma^2 + \sigma^4/\alpha^2}{(\alpha + \sigma^2/\alpha)^2} \\
&= \frac{\sigma^2}{\alpha^2 + \sigma^2}.
\end{aligned}$$

Remark: How does one obtain the above solution? A good approach would be to observe that the desired bound has a number of squared terms, and thus conjecture the use of Markov's Inequality with a quadratic function $\phi(x) = (x + c)^2$. One can then optimize over values of c to obtain $c = \sigma^2/\alpha$.