UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 6
Spring 2017

Issued: Thursday, March 2, 2017 **Due:** 8:00am Thursday, March 9, 2017

Problem 1. Consider the Markov chain with state X_n, $n \geq 0$, shown in Figure 1, where $\alpha, \beta \in (0, 1)$.

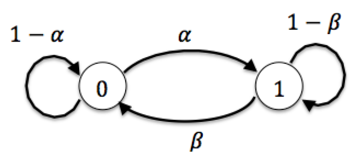

Figure 1: Markov chain for Problem 1

(a) Find the probability transition matrix P and the invariant distribution π of the Markov chain.

(b) Find two real numbers λ_1 and λ_2 such that there exists two non-zero vectors u_1 and u_2 such that $Pu_i = \lambda_i u_i$ for $i = 1, 2$. Further, show that P can be written as $P = U \Lambda U^{-1}$, where U and Λ are 2×2 matrices and Λ is a diagonal matrix.

Hint: This is called the eigendecomposition of a matrix.

(c) Find P^n in terms of U and Λ.

(d) Assume that $X_0 = 0$. Use the result in part (c) to compute the PMF of X_n for all $n \geq 0$. Verify that it converges to the invariant distribution.

Problem 2. For each of the following case, explain whether $\{Y_n\}_{n\geq 0}$ satisfies the Markov property.

(a) $Y_0, X_1, X_2, \ldots$ are mutually independent discrete random variables and

$$Y_{n+1} = (Y_n + X_{n+1})^{(n+1)} \quad \text{for } n \geq 0.$$

(b) $Y_0 = 1$ and $Y_n = U_1 U_2 U_3 \ldots U_n$, where $U_1, U_2, \ldots$ are independent random variables, each uniformly distributed on the interval $[0, 1]$.

(c) Let $\{X_n\}_{n\geq 0}$ be a Markov chain with two states, -1 and 1, and the transition probabilities $P(-1,1) = P(1,-1) = a$ for $a \in (0,1)$. Define

$$Y_n = X_0 + X_1 + \cdots + X_n.$$

Problem 3. (a) Find the steady-state probabilities $\pi_0, \ldots, \pi_{k-1}$ for the Markov chain in Figure 2. Express your answer in terms of the ratio $\rho = p/q$, where $q = 1-p$. Pay particular attention to the special case $\rho = 1$.

(b) Find the limit of π_0 as k approaches infinity; give separate answers for $\rho < 1$, $\rho = 1$, and $\rho > 1$. Find limiting values of π_{k-1} for the same cases.

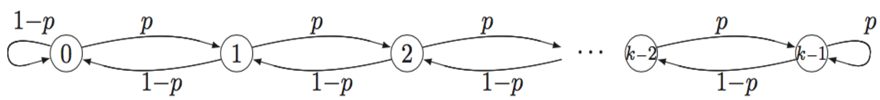

Figure 2: Markov chain for Problem 3

Problem 4. Consider the Markov chain with 6 states given in Figure 3. Suppose that $P(X_0 = 1) = 1$, and given $X_k = i$, the next state X_{k+1} is one of the two neighbors of i, selected with probability 0.5 each.

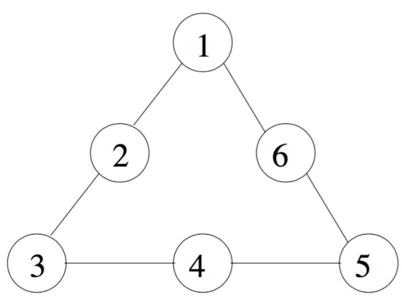

Figure 3: Markov chain for Problem 4

(a) Let τ_C be the first time $k \geq 1$ such that both states 3 and 5 have been visited, in either order, by time k. Find $E[\tau_C]$.

(b) Let τ_R denote the first time $k \geq \tau_C$ such that $X_k = 1$. That is, τ_R is the first time the process returns to vertex 1 of the triangle after visiting both of the other vertices. Find $E[\tau_R]$.

Problem 5. You have a database of an infinite number of movies. Each movie has a rating that is uniformly distributed in $[0, 5]$ and you want to find two movies such that the sum of their ratings is greater than 7.5. Assume that you choose movies from the database one by one and keep the movie with the highest rating. You stop when you find that the sum of the ratings of the last movie you have chosen and the movie with the highest rating among all the previous movies is greater than 7.5. What is the expected number of movies you will have to choose?