Multivariate Gaussians

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Proof: $\mathrm{cov}((AZ)_i, (AZ)_j) = \mathrm{cov}(\sum_{k=1}^n A_{i,k}Z_k, \sum_{l=1}^n A_{j,l}Z_l)$

$= \sum_{k=1}^n A_{i,k}A_{j,k} = (AA^\top)_{i,j}$.

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$\Pr(X - Y = 0) = 1/2$.

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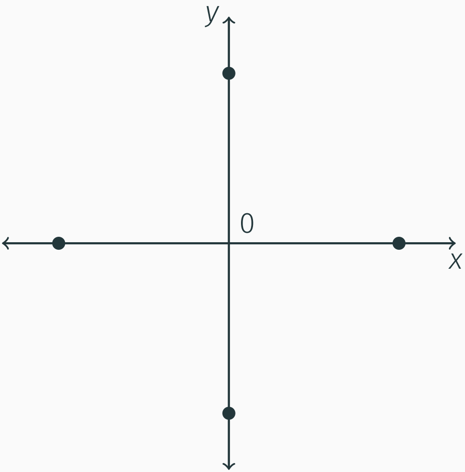

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Joint density factors into product of marginal densities.

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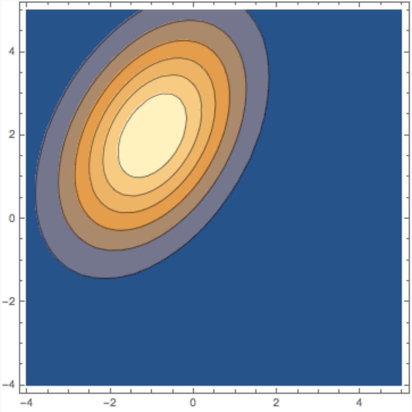
$\implies$ $Y - L[Y \mid X]$ is orthogonal to any $\phi(X)$, so $L[Y \mid X] = E[Y \mid X]$.

$$X \sim \mathcal{N}\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\right).$$



Eigenvectors of $\boldsymbol{\Sigma}$:

$$\lambda_1 = 3.62, \qquad v_1 = \begin{bmatrix} 0.526 \\ 0.851 \end{bmatrix}$$

$$\lambda_2 = 1.38, \qquad v_2 = \begin{bmatrix} -0.851 \\ 0.525 \end{bmatrix}$$