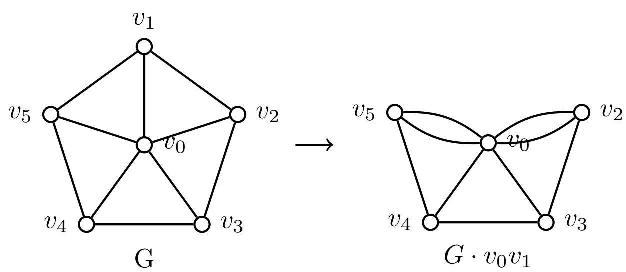
习题. 设在一个长为n的圈外再加一个新的顶点，并且新顶点与圈上每个顶点联结一条边， 所得到的图称为轮，新加的边称为轮的辐。在有n条辐的轮中，给出一个求生成树棵数的公式。

解. 我们利用c图的生成树棵数的一个递推公式推导出轮的生成树棵数的计算公式。 设G为一个图（允许有环和多重边），e为G的任意一条不为环的边，则G的生成树的棵数$\tau(G)$可以计算如下:

$$\tau(G) = \tau(G - e) + \tau(G \cdot e) \tag{1}$$

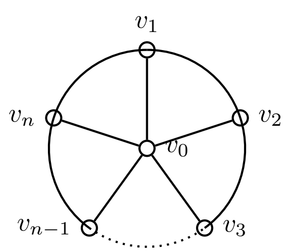
其中$G - e$表示从图G中去掉边e所得到的图。$G \cdot e$表示从图G中去掉边e并将e的两个端点视为同一个顶点所得到的图，这个过程称为从图G中收缩掉边e。

下图给出了从一个图G中收缩掉一条边的过程。



公式(1)可以推导如下：图G中不包含边e的生成树的棵数为$\tau(G - e)$， 包含边e的生成树的棵数为$\tau(G \cdot e)$，从而图G中所有生成树的棵数可以用公式 (1)计算。

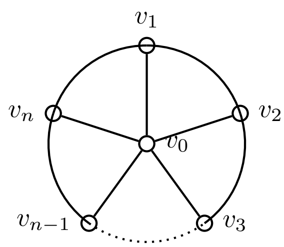
有n条辐的轮如下图所示:

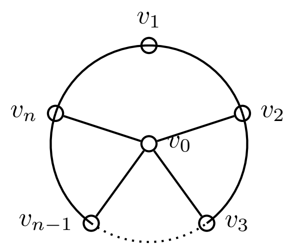


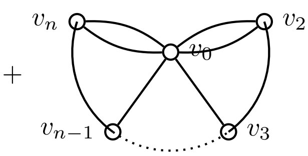
有n条辐的轮中生成树的棵数记为w_n，以下应用公式(1)推导出w_n的递推关系式如下:

$$\begin{cases} w_1 &= 1 \\ w_2 &= 5 \\ w_3 &= 16 \\ w_n &= 4w_{n-1} - 4w_{n-2} + w_{n-3} (n \geq 4) \end{cases} \tag{2}$$

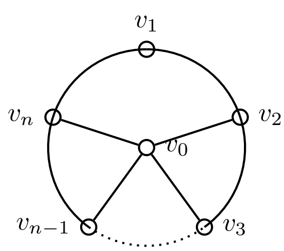
为了简洁，在这里，图中生成树的棵数用图本身表示。w_n的推导过程如下：

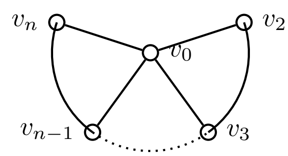
$w_n =$ 

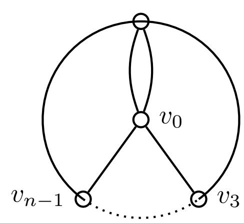
$=$ 

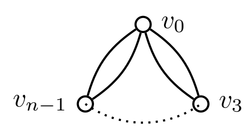
$+$ 

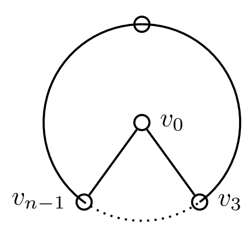
其中

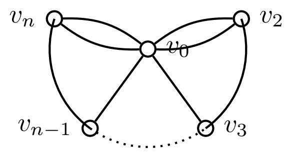


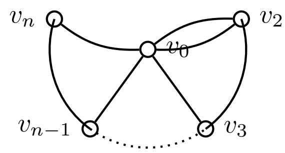
$= w_{n-1} +$ 

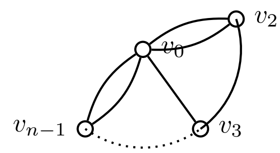
$= 2w_{n-1} -$ 

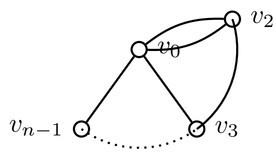
$= 2w_{n-1} - w_{n-2} -$ 

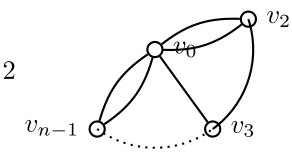
$= 2w_{n-1} - 2w_{n-2} +$ 

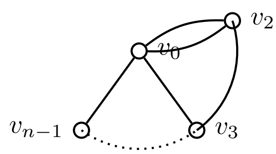


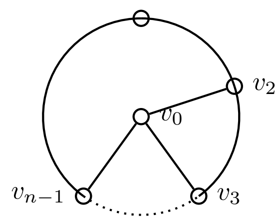
$=$ 

$+$ 

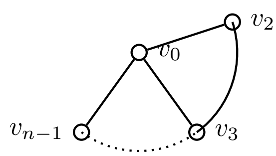
$=$ 

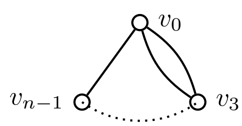
$+\,2$ 

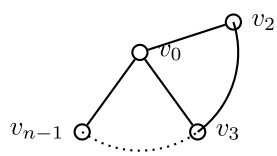
$=$ 

$+\,2(w_{n-1} -$ 

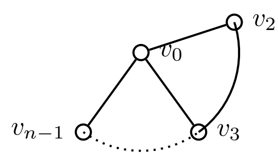
$)$

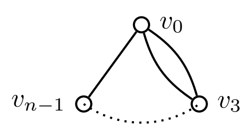
$=$ 

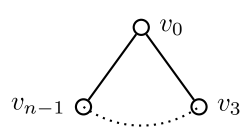
$+$ 

$+2w_{n-1} - 2($ 

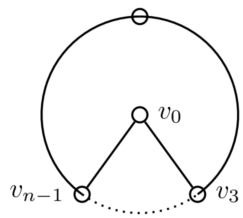
$+\, w_{n-2})$

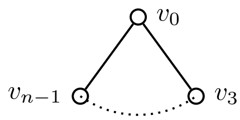
$=2w_{n-1} - 2w_{n-2} -$ 

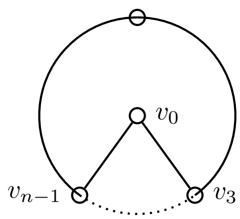
$+$ 

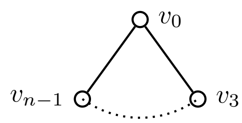
$=2w_{n-1} - 2w_{n-2} -$ 

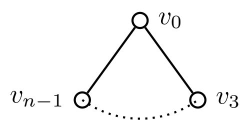
于是

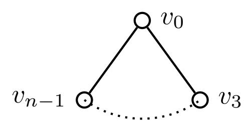
$w_n = (2w_{n-1} - 2w_{n-2} +$  $)$

$+(2w_{n-1} - 2w_{n-2} -$  $)$

$=4w_{n-1} - 4w_{n-2} +$ 

$-$ 

$=4w_{n-1} - 4w_{n-2} + (w_{n-3} +$ 

$) -$ 

$=4w_{n-1} - 4w_{n-2} + w_{n-3}$

w_n可以计算如下:

$$w_n = -2 + (\frac{3-\sqrt{5}}{2})^n + (\frac{3+\sqrt{5}}{2})^n \tag{3}$$

这是因为设方程

$$\lambda^3 - 4\lambda^2 + 4\lambda - 1 = 0 \tag{4}$$

的任意一个根为λ，则$w_n = \lambda^n$满足递推关系式:

$$w_n = 4w_{n-1} - 4w_{n-2} + w_{n-3}(n \geq 4) \tag{5}$$

方程(4)可以化为

$$(\lambda - 1)(\lambda - \frac{3-\sqrt{5}}{3})(\lambda - \frac{3+\sqrt{5}}{3}) = 0$$

其三个根为

$$\begin{cases} \lambda_1 & = 1 \\ \lambda_2 & = \frac{3-\sqrt{5}}{3} \\ \lambda_3 & = \frac{3+\sqrt{5}}{3} \end{cases}$$

则

$$w_n = C_1\lambda_1^n + C_2\lambda_2^n + C_3\lambda_3^n \tag{6}$$

满足递推关系式(5)。

将$w_1 = 1$，$w_2 = 5$，$w_3 = 16$代入(6)得:

$$\begin{cases} 1 & = C_1\lambda_1 + C_2\lambda_2 + C_3\lambda_3 \\ 5 & = C_1\lambda_1^2 + C_2\lambda_2^2 + C_3\lambda_3^2 \\ 16 & = C_1\lambda_1^3 + C_2\lambda_2^3 + C_3\lambda_3^3 \end{cases}$$

解得$C_1 = -2, C_2 = 1, C_3 = 1$，代入(6)便可得(3)。 □