Statistical Learning
Homework 1

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Q1.

Run a computer simulation for flipping 1,000 fair coins. Flip each coin independently 10 times. Let's focus on 3 coins as follows: c_1 is the first coin flipped; c_{rand} is a coin you choose at random; c_{min} is the coin that had the minimum frequency of heads (pick the earlier one in case of a tie). Let ν_1, ν_{rand} and ν_{min} be the fraction of heads you obtain for the respective three coins. For a coin, let μ be its probability of heads.

In [6]:
```
import numpy as np
import matplotlib.pyplot as plt
import  pandas as pd
```

In [7]:
```
# run simulation here, for 1000 fair coin flips, 10 times, and store the res
n = 1000
num_trials = 10
results = np.zeros((num_trials, n))
for i in range(num_trials):
    flips = np.random.randint(2, size=n)
    # save each trial as a row in the results array
    results[i] = flips
results.shape
```

Out[7]: (10, 1000)

In [8]:
```
randam_int = np.random.randint(1, n)
randam_int
```

Out[8]: 560

In [9]:
```
# suppose 0 is tails and 1 is heads
# select from results array
# c_1, the first coin flip of each trial
c_1 = results[:, 0]
# c_rand, the random coin flip of each trial
c_rand = results[:, randam_int]
# c_min, the coin with minimum frequency of heads
c_min = results[:, np.argmin(np.sum(results, axis=0))]
```

(a).

What is μ for the three coins selected?

In [10]:
```
# print the number of heads for each coin
print("Number of heads for c_1: ", np.sum(c_1))
print("Number of heads for c_rand: ", np.sum(c_rand))
print("Number of heads for c_min: ", np.sum(c_min))
```

```
Number of heads for c_1:  1.0
Number of heads for c_rand:  4.0
Number of heads for c_min:  0.0
```

If it is asking for the simple probability of the coin itself, the μs of the three coins should all be 0.5.

However, if it is asking for a more complicated case for these specific three coins, we shall need to use Bayesian estimation with bernoulli distribution as prior, and the simulation samples as the data, and then calculate the μ using the posterior distribution, which is a Beta distribution in this case. And here's an result showing the estiamation by Bayesian estimation. As $Beta(\text{number of heads} + 1, \text{number of trails} + 1)$, be the posterior, thus the expected value being $\mu = \frac{\text{number of heads}+1}{\text{number of heads}+1+\text{number of trails}+1}$. Then the expected probability of heads are $\mu_{c_1} = \frac{2}{13}$, $\mu_{c_{rand}} = \frac{5}{16}$, $\mu_{c_{min}} = \frac{1}{12}$

But I think this question does not go that far. Let's give the answer of the question a simple 0.5 for all the three coins.

(b)

Repeat this entire experiment a large number of times (e.g., 100,000 runs of the entire experiment) to get several instances of ν_1,ν_{rand} and ν_{min} and plot the histograms of the distributions of ν_1,ν_{rand} and ν_{min}. Notice that the coins that end up being c_{rand} and c_{min} may differ from one run to another

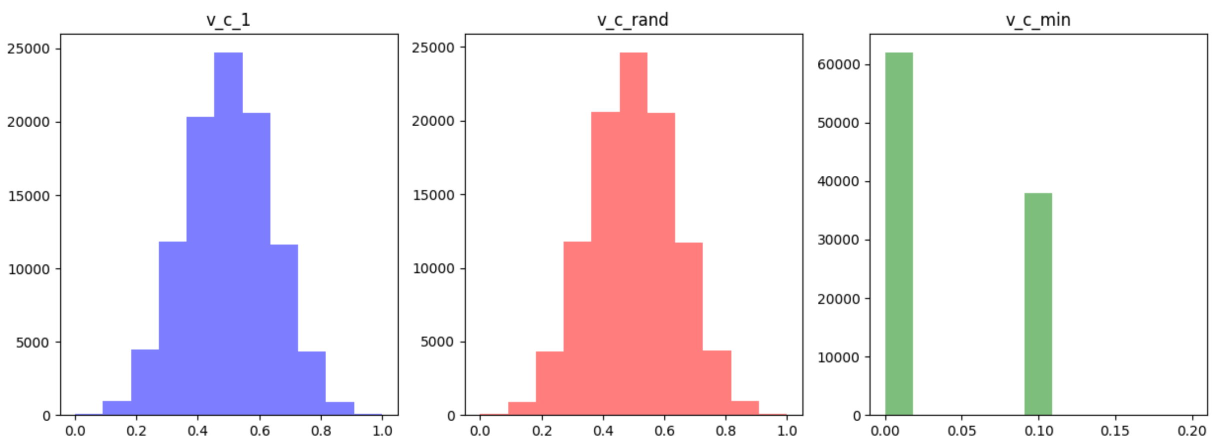
In [13]:
```
# repeat the experiment for 100000 rounds for each complete experiment
# run simulation here, for 1000 fair coin flips, 10 times, and store the res
rounds = 100000
df = pd.DataFrame(columns=['v_c_1', 'v_c_rand', 'v_c_min'])
for i in range(rounds):
    n = 1000
    num_trials = 10
    results = np.zeros((num_trials, n))
    for i in range(num_trials):
        flips = np.random.randint(2, size=n)
        # save each trial as a row in the results array
        results[i] = flips
    # suppose 0 is tails and 1 is heads
    # select from results array
    # c_1, the first coin flip of each trial
    c_1 = results[:, 0]
    # c_rand, the random coin flip of each trial
    c_rand = results[:, randam_int]
    # c_min, the coin with minimum frequency of heads
    c_min = results[:, np.argmin(np.sum(results, axis=0))]
    # calcualte the v_c_1, v_c_rand, v_c_min
    v_c_1 = np.mean(c_1)
    v_c_rand = np.mean(c_rand)
    v_c_min = np.mean(c_min)
    # save the results as a row in the dataframe
```

```
    df = pd.concat([df, pd.DataFrame({'v_c_1': [v_c_1], 'v_c_rand': [v_c_rar
df.shape
```

```
/var/folders/d8/t7ct8drs7yx5fdqcplhp49k00000gn/T/ipykernel_39494/3057762468.
py:26: FutureWarning: The behavior of DataFrame concatenation with empty or
all-NA entries is deprecated. In a future version, this will no longer exclu
de empty or all-NA columns when determining the result dtypes. To retain the
old behavior, exclude the relevant entries before the concat operation.
  df = pd.concat([df, pd.DataFrame({'v_c_1': [v_c_1], 'v_c_rand': [v_c_ran
d], 'v_c_min': [v_c_min]})], ignore_index=True)
```

Out[13]: (100000, 3)

In [15]:
```
# plot the histogram of v_c_1, v_c_rand, v_c_min in the same plot with diffe
# subplot for 1*3 subplots
fig, ax = plt.subplots(1, 3, figsize=(15, 5))
# plot the histogram of v_c_1
ax[0].hist(df['v_c_1'], bins=11, color='blue', alpha=0.5)
ax[0].set_title('v_c_1')
# plot the histogram of v_c_rand
ax[1].hist(df['v_c_rand'], bins=11, color='red', alpha=0.5)
ax[1].set_title('v_c_rand')
# plot the histogram of v_c_min
ax[2].hist(df['v_c_min'], bins=11, color='green', alpha=0.5)
ax[2].set_title('v_c_min')
plt.show()
```



(c)

Using (b), plot estimates for $P[|\nu - \mu| > \varepsilon]$ as a function of ϵ, together with the Hoeffding bound $2e^{-2\varepsilon^2 N}$ (on the same graph).

In [56]:
```
df['v_c_1'].mean(), df['v_c_rand'].mean(), df['v_c_min'].mean()
```

Out[56]:
```
(np.float64(0.49937600000000004),
 np.float64(0.5003179999999999),
 np.float64(0.037982999999999996))
```

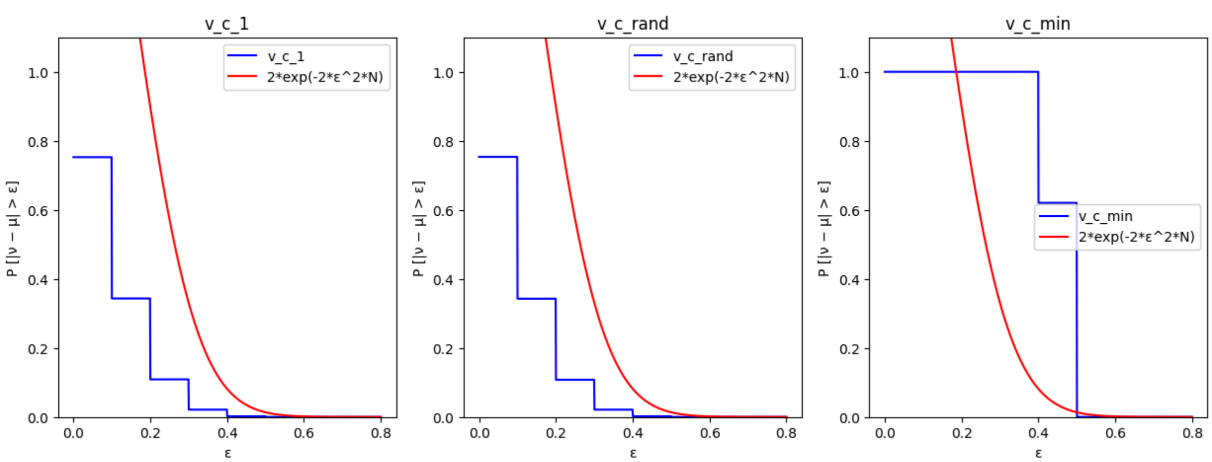
In [54]:
```
def subplot(j, title, mu=0.5, interval=1000, max_epsilon=0.5, df=df):
    N = 10
    bias = np.abs(df[title] - mu)
```

```
    # plot the line plot of bias_v_c_1
    ax[j].plot(np.linspace(0, max_epsilon, interval), [np.mean(bias > i) for
    ax[j].set_title('v_c_1')
    # set the x-axis label
    ax[j].set_xlabel('ε')
    # set the y-axis label
    ax[j].set_ylabel('P [|ν − μ| > ε]')
    # also plot the line of 2*exp(-2*ε^2*n) with n=N
    ax[j].plot(
        np.linspace(0, max_epsilon, interval),
        [2*np.exp(-2*i**2*N) for i in np.linspace(0, max_epsilon, interval)]
        color='red')
    # set the title of the plot
    ax[j].set_title(title)
    # set the x-axis label
    ax[j].set_xlabel('ε')
    # set the y-axis label
    ax[j].set_ylabel('P [|ν − μ| > ε]')
    # set ylim to be 0 to 1
    ax[j].set_ylim(0, 1.1)
    # set the legend
    ax[j].legend([title, '2*exp(-2*ε^2*N)'])
```

In [48]:
```
2*np.exp(-2*0.1**2*10)
```

Out[48]:
```
np.float64(1.6374615061559636)
```

In [55]:
```
# suppose mu be 0.5 for all coins
# calculate the bias of v_c_1, v_c_rand, v_c_min in absolute value
mu = 0.5
# line plot with x-axis being \epsilon
# # and y-axis being probability that the bias is greater than \epsilon
# subplot for 1*3 subplots,
fig, ax = plt.subplots(1, 3, figsize=(15, 5))
subplot(0, 'v_c_1', max_epsilon=0.8, df=df)
subplot(1, 'v_c_rand', max_epsilon=0.8, df=df)
subplot(2, 'v_c_min', max_epsilon=0.8, df=df)
```



(d)

Which coins obey the Hoeffding bound, and which ones do not? Ex- plain why.

Both the first coin and the randomly picked coins obbey the Hoeffding bound as their simulations can all be considered independent variable.

However, for the one always with smallest number of heads, it does not hold the Hoeffding bound, as it cannot be considered independent, because it is the order statistics that would have dependency.

Check q2&3 next pages

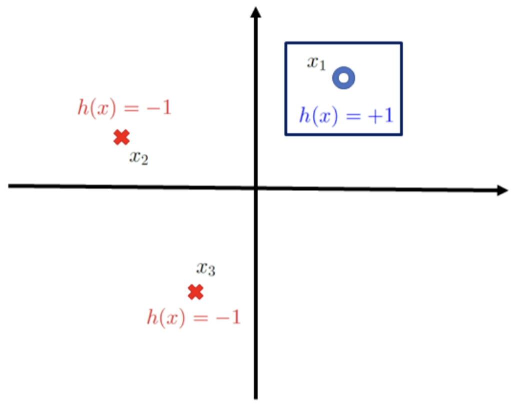


Figure 1: $\mathcal{H}$ - Positive rectangles, $N = 3$.

2. Consider the learning model of positive rectangles i.e. $\mathcal{H}$ contains rectangles that are aligned horizontally or vertically and are positive in the inside and negative elsewhere as it is shown in Figure 1. Determine:

   (a) The breaking point k and the VC dimension, d_{VC}. Note: Demonstrate your results, showing a set of points that cannot be shattered by $\mathcal{H}$.

   (b) A bound for the growth function:

$$m_{\mathcal{H}}(N) \leq \sum_{i=0}^{d_{VC}} \binom{N}{i}$$

Solution:
(a). As its a convex set, we may have

$$m_{\mathcal{H}}(3) \leq 2^3 = 8 < 2^4$$

As a result, we are supposed to have $k = 4$, and $d_{VC} = 3$.
However, since it is a rectangular that aligned with axises. Let's consider more. For the case of three points, we can always shatter the label with only one point by the rectangular, and thus successfully classifying the points.
For the case that 4 points cannot be shattered by $\mathcal{H}$, simply consider 4 points in a line, with each conjecture points not the same type.
In conclusion we have breaking point $k = 4$, and VC dimension $d_{VC} = 3$

(b). Since $d_{VC} = 3$, we have

$$m_{\mathcal{H}}(N) \leq \sum_{i=0}^{3} \binom{N}{i} = 1 + N + \frac{N(N-1)}{2} + \frac{N(N-1)(N-2)}{6}$$

To explain:
when $N = 0$, upper bound is 1, meaning at most one dichotomies;
when $N = 1$, upper bound is 2, meaning at most two dichotomies;
when $N = 2$, upper bound is 4, meaning at most four dichotomies;
when $N = 3$, upper bound is 8, meaning at most eight dichotomies.

3. Remember the inequality for multiple hypotheses:

$$\Pr\left[|E_{\text{in}}(g) - E_{\text{out}}(g)| \geq \epsilon\right] \leq 2Me^{-2N\epsilon}.$$

If we replace M by $m_{\mathcal{H}}(N)$ which can be bounded by a polynomial, the generation error will go to zero as $N \to \infty$ which implies learning is feasible. To prove this, assume $m_{\mathcal{H}}(N)$ can be bounded by the polynomial N^{k-1} and compute the following simplified limit for $\epsilon > 0$ and k being a finite positive integer (i.e. $k \in \mathbb{Z}^+, 0 < k < \infty$):

$$\lim_{N\to\infty} N^{k-1}e^{-N}.$$

Solution:
To prove

$$\lim_{N\to\infty} N^{k-1}e^{-N} = \lim_{N\to\infty} \frac{N^{k-1}}{e^N}$$

simply use L'Hopital's rule as N is differentiable when approaching infinity and,

$$\lim_{N\to\infty} N^{k-1} \to \infty$$

,

$$\lim_{N\to\infty} e^N \to \infty$$

$$\lim_{N\to\infty} \frac{N^{k-1}}{e^N} = \lim_{N\to\infty} \frac{\frac{dN^{k-1}}{dN}}{\frac{de^N}{dN}} = \lim_{N\to\infty} \frac{(k-1)N^{k-2}}{e^N}$$

After applying the L'Hopital's rule for $k-1$ times, we have

$$\lim_{N\to\infty} \frac{N^{k-1}}{e^N} = \lim_{N\to\infty} \frac{(k-1)!}{e^N} = 0$$

Thus, the bound has been proved.