Statistical Learning
Homework 1

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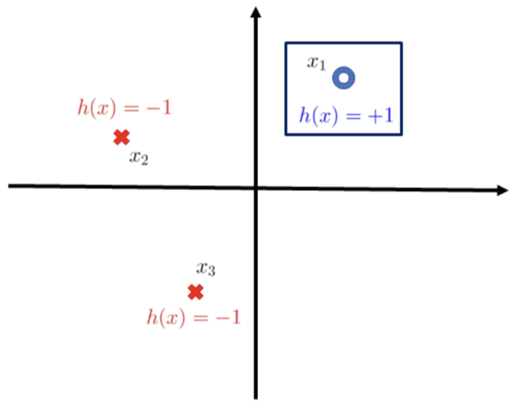

Figure 1: $\mathcal{H}$ - Positive rectangles, $N = 3$.

2. Consider the learning model of positive rectangles i.e. $\mathcal{H}$ contains rectangles that are aligned horizontally or vertically and are positive in the inside and negative elsewhere as it is shown in Figure 1. Determine:

   (a) The breaking point k and the VC dimension, d_{VC}. Note: Demonstrate your results, showing a set of points that cannot be shattered by $\mathcal{H}$.

   (b) A bound for the growth function:

$$m_{\mathcal{H}}(N) \le \sum_{i=0}^{d_{VC}} \binom{N}{i}$$

Solution:
(a). As its a convex set, we may have

$$m_{\mathcal{H}}(3) \le 2^3 = 8 < 2^4$$

As a result, we are supposed to have $k = 4$, and $d_{VC} = 3$.
However, since it is a rectangular that aligned with axises. Let's consider more.
For the case of three points, we can always shatter the label with only one point by the rectangular, and thus successfully classifying the points.
For the case that 4 points cannot be shattered by $\mathcal{H}$, simply consider 4 points in a line, with each conjecture points not the same type.
In conclusion we have breaking point $k = 4$, and VC dimension $d_{VC} = 3$

(b). Since $d_{VC} = 3$, we have

$$m_{\mathcal{H}}(N) \le \sum_{i=0}^{3} \binom{N}{i} = 1 + N + \frac{N(N-1)}{2} + \frac{N(N-1)(N-2)}{6}$$

To explain:
when $N = 0$, upper bound is 1, meaning at most one dichotomies;
when $N = 1$, upper bound is 2, meaning at most two dichotomies;
when $N = 2$, upper bound is 4, meaning at most four dichotomies;
when $N = 3$, upper bound is 8, meaning at most eight dichotomies.

3. Remember the inequality for multiple hypotheses:

$$\Pr\left[|E_{\text{in}}(g) - E_{\text{out}}(g)| \geq \epsilon\right] \leq 2Me^{-2N\epsilon}.$$

If we replace M by $m_{\mathcal{H}}(N)$ which can be bounded by a polynomial, the generation error will go to zero as $N \to \infty$ which implies learning is feasible. To prove this, assume $m_{\mathcal{H}}(N)$ can be bounded by the polynomial N^{k-1} and compute the following simplified limit for $\epsilon > 0$ and k being a finite positive integer (i.e. $k \in \mathbb{Z}^+, 0 < k < \infty$):

$$\lim_{N\to\infty} N^{k-1}e^{-N}.$$

Solution:
To prove

$$\lim_{N\to\infty} N^{k-1}e^{-N} = \lim_{N\to\infty} \frac{N^{k-1}}{e^N}$$

simply use L'Hopital's rule as N is differentiable when approaching infinity and,

$$\lim_{N\to\infty} N^{k-1} \to \infty$$

,

$$\lim_{N\to\infty} e^N \to \infty$$

$$\lim_{N\to\infty} \frac{N^{k-1}}{e^N} = \lim_{N\to\infty} \frac{\frac{dN^{k-1}}{dN}}{\frac{de^N}{dN}} = \lim_{N\to\infty} \frac{(k-1)N^{k-2}}{e^N}$$

After applying the L'Hopital's rule for $k-1$ times, we have

$$\lim_{N\to\infty} \frac{N^{k-1}}{e^N} = \lim_{N\to\infty} \frac{(k-1)!}{e^N} = 0$$

Thus, the bound has been proved.