Statistical Learning 2024 Fall

Homework 5

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1. 1. Bias-variance trade-off. Show that when the output is noisy, that is,

$$y(x) = f(x) + \epsilon,$$

with ϵ being independent random noise with zero mean and variance σ^2, the expected generalization error becomes:

$$\mathbb{E}_D[E_{\text{out}}(g^{(D)})] = \sigma^2 + \text{bias} + \text{var}$$

Solution
Given

$$E_{\text{out}}(g^{(D)}) = \mathbb{E}_x\{[g^{(D)}(x) - (f(x) + \epsilon)]^2\}$$
$$\mathbb{E}_D[E_{\text{out}}(g^{(D)})] = \mathbb{E}_D\{\mathbb{E}_x[g^{(D)}(x) - y(x)]^2\} = \mathbb{E}_x\{\mathbb{E}_D[g^{(D)}(x) - y(x)]^2\}$$

Check $\mathbb{E}_D\{[g^{(D)}(x) - y(x)]^2\}$, let's define $\bar{g}(x) = \mathbb{E}_D[g^{(D)}(x)]$.

arrow

$$\begin{aligned}
\mathbb{E}_D\{[g^{(D)}(x) - y(x)]^2\} &= \mathbb{E}_D\{[g^{(D)}(x) - \bar{g}(x) + \bar{g}(x) - y(x)]^2\} \\
&= \mathbb{E}_D\{[(g^{(D)}(x) - \bar{g}(x)) + (\bar{g}(x) - (f(x) + \epsilon))]^2\} \\
&= \mathbb{E}_D\{(g^{(D)}(x) - \bar{g}(x))^2 + (\bar{g}(x) - (f(x) + \epsilon))^2 \\
&\quad + 2 * (g^{(D)}(x) - \bar{g}(x))(\bar{g}(x) - (f(x) + \epsilon))\} \\
&= \mathbb{E}_D\{(g^{(D)}(x) - \bar{g}(x))^2 + (\bar{g}(x) - (f(x) + \epsilon))^2 \\
&\quad + 2 * (g^{(D)}(x) - \bar{g}(x))(\bar{g}(x) - (f(x) + \epsilon))\} \\
&= \mathbb{E}_D\{(g^{(D)}(x) - \bar{g}(x))^2\} + [(\bar{g}(x) - f(x)) + \epsilon]^2
\end{aligned}$$

arrow

$$\begin{aligned}
\mathbb{E}_D[E_{\text{out}}(g^{(D)})] &= \mathbb{E}_x\{\mathbb{E}_D[(g^{(D)}(x) - \bar{g}(x))^2] + [(\bar{g}(x) - f(x)) + \epsilon]^2\} \\
&= \mathbb{E}_x\{\mathbb{E}_D[(g^{(D)}(x) - \bar{g}(x))^2] + [(\bar{g}(x) - f(x))]^2 + 2\epsilon(\bar{g}(x) - f(x)) + \epsilon^2\} \\
&= \mathbb{E}_x[bias(x) + var(x) + \epsilon^2] \\
&= \sigma^2 + bias + var
\end{aligned}$$