

# Digital Signature

Yu Zhang

Harbin Institute of Technology

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- 1** Definitions of Digital Signatures
- 2** RSA Signatures
- 3** Digital Signature from the Discrete-Log Problem
- 4** One-Time Signature Scheme
- 5** Certificates and Public-Key Infrastructures

## **1** Definitions of Digital Signatures

## 2 RSA Signatures

## 3 Digital Signature from the Discrete-Log Problem

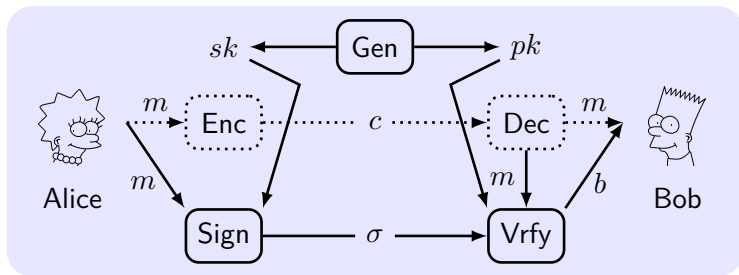
## 4 One-Time Signature Scheme

## 5 Certificates and Public-Key Infrastructures

# Digital Signatures – An Overview

- **Digital signature scheme** is a mathematical scheme for demonstrating the authenticity/integrity of a digital message
- allow a **signer**  $S$  to “**sign**” a message with its own  $sk$ , anyone who knows  $S$ 's  $pk$  can **verify** the authenticity/integrity
- (Comparing to MAC) digital signature is:
  - publicly verifiable
  - transferable
  - non-repudiation
  - but slow
- Q: What are the differences between digital signatures and handwritten signatures?
- Digital signature is NOT the “inverse” of public-key encryption

# The Syntax of Digital Signature Scheme



- signature  $\sigma$ , a bit  $b$  means valid if  $b = 1$ ; invalid if  $b = 0$ .
- **Key-generation** algorithm  $(pk, sk) \leftarrow \text{Gen}(1^n), |pk|, |sk| \geq n$ .
- **Signing** algorithm  $\sigma \leftarrow \text{Sign}_{sk}(m)$ .
- **Verification** algorithm  $b := \text{Vrfy}_{pk}(m, \sigma)$ .
- **Basic correctness requirement:**  $\text{Vrfy}_{pk}(m, \text{Sign}_{sk}(m)) = 1$ .

# Defining Signature Security

The signature experiment  $\text{Sigforge}_{\mathcal{A}, \Pi}(n)$ :

- 1  $(pk, sk) \leftarrow \text{Gen}(1^n)$ .
- 2  $\mathcal{A}$  is given input  $1^n$  and oracle access to  $\text{Sign}_{sk}(\cdot)$ , and outputs  $(m, \sigma)$ .  $\mathcal{Q}$  is the set of queries to its oracle.
- 3  $\text{Sigforge}_{\mathcal{A}, \Pi}(n) = 1 \iff \text{Vrfy}_{pk}(m, \sigma) = 1 \wedge m \notin \mathcal{Q}$ .

## Definition 1

A signature scheme  $\Pi$  is **existentially unforgeable under an adaptive CMA** if  $\forall$  PPT  $\mathcal{A}$ ,  $\exists$   $\text{negl}$  such that:

$$\Pr[\text{Sigforge}_{\mathcal{A}, \Pi}(n) = 1] \leq \text{negl}(n).$$

**Q: What's the difference on the ability of adversary between MAC and digital signature? What if an adversary is not limited to PPT?**

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# Insecurity of “Textbook RSA”

## Construction 2

- Gen: on input  $1^n$  run  $\text{GenRSA}(1^n)$  to obtain  $N, e, d$ .  
 $pk = \langle N, e \rangle$  and  $sk = \langle N, d \rangle$ .
- Sign: on input  $sk$  and  $m \in \mathbb{Z}_N^*$ ,  $\sigma := [m^d \bmod N]$ .
- Vrfy: on input  $pk$  and  $m \in \mathbb{Z}_N^*$ ,  $m \stackrel{?}{=} [\sigma^e \bmod N]$ .
- **A no-message attack:** choose an arbitrary  $\sigma \in \mathbb{Z}_N^*$  and compute  $m := [\sigma^e \bmod N]$ . Output the forgery  $(m, \sigma)$ .

$$pk = \langle 15, 3 \rangle, \sigma = 2, m = ? \quad m^d = ?$$

- **Forging a signature on an arbitrary message:**  
To forge a signature on  $m$ , choose a random  $m_1$ , set  $m_2 := [m/m_1 \bmod N]$ , obtain signatures  $\sigma_1, \sigma_2$  on  $m_1, m_2$ .  
Q:  $\sigma := [\text{____} \bmod N]$  is a valid signature on  $m$ .



# Hashed RSA

- Gen: a hash function  $H : \{0, 1\}^* \rightarrow \mathbb{Z}_N^*$  is part of public key.
- Sign:  $\sigma := [H(m)^d \bmod N]$ .
- Vrfy:  $\sigma^e \stackrel{?}{=} H(m) \bmod N$ .

If  $H$  is not efficiently invertible, then the no-message attack and forging a signature on an arbitrary message is difficult.

## Insecurity

There is NO known function  $H$  for which hashed RSA signatures are secure.

**RSA-FDH Signature Scheme:** Random Oracle as a **Full Domain Hash (FDH)** whose image size = the RSA modulus  $N - 1$ .

# The “Hash-and-Sign” Paradigm

## Construction 3

$\Pi = (\text{Gen}_S, \text{Sign}, \text{Vrfy})$ ,  $\Pi_H = (\text{Gen}_H, H)$ . A signature scheme  $\Pi'$ :

- $\text{Gen}'$ : on input  $1^n$  run  $\text{Gen}_S(1^n)$  to obtain  $(pk, sk)$ , and run  $\text{Gen}_H(1^n)$  to obtain  $s$ . The public key is  $pk' = \langle pk, s \rangle$  and the private key is  $sk' = \langle sk, s \rangle$ .
- $\text{Sign}'$ : on input  $sk'$  and  $m \in \{0, 1\}^*$ ,  $\sigma \leftarrow \text{Sign}_{sk}(H^s(m))$ .
- $\text{Vrfy}'$ : on input  $pk'$ ,  $m \in \{0, 1\}^*$  and  $\sigma$ , output  $1 \iff \text{Vrfy}_{pk}(H^s(m), \sigma) = 1$ .

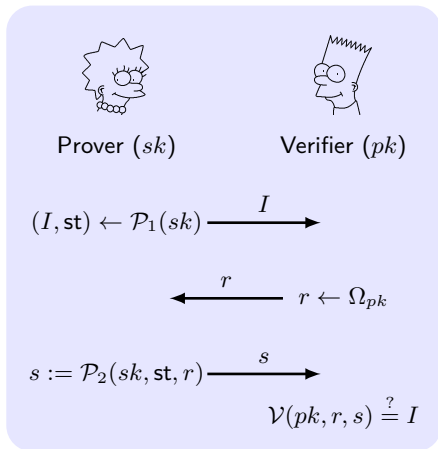
## Theorem 4

*If  $\Pi$  is existentially unforgeable under an adaptive CMA and  $\Pi_H$  is collision resistant, then Construction is existentially unforgeable under an adaptive CMA.*

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# Identification Schemes

An identification scheme  $\Pi = (\text{Gen}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{V})$  is a 3-round protocol between the prover and the verifier. The attacker can do eavesdropping and has an access to an oracle  $\text{Trans}_{sk}$  to learn  $(I, r, s)$  by executing the protocol as a verifier.



# Identification Schemes: Definition

The identification experiment  $\text{Ident}_{\mathcal{A}, \Pi}(n)$ :

- 1  $(pk, sk) \leftarrow \text{Gen}(1^n)$ .
- 2  $\mathcal{A}$  is given input  $1^n$  and oracle access to  $\text{Trans}_{sk}(\cdot)$ , and outputs a message  $I$ .
- 3 A uniform challenge  $r$  is chosen and given to  $\mathcal{A}$ , and  $\mathcal{A}$  outputs  $s$ . ( $\mathcal{A}$  may continue to query the oracle.)
- 4  $\text{Ident}_{\mathcal{A}, \Pi}(n) = 1 \iff \mathcal{V}(pk, r, s) \stackrel{?}{=} I$ .

## Definition 5

An identification scheme  $\Pi = (\text{Gen}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{V})$  is **secure** if  $\forall$  PPT  $\mathcal{A}$ ,  $\exists$   $\text{negl}$  such that:

$$\Pr[\text{Ident}_{\mathcal{A}, \Pi}(n) = 1] \leq \text{negl}(n).$$

# The Schnorr Identification Scheme



Prover ( $x$ )



Verifier ( $\mathbb{G}, q, g, y$ )

$$k \leftarrow \mathbb{Z}_q; I := g^k \xrightarrow{I}$$

$$\xleftarrow{r} r \leftarrow \mathbb{Z}_q$$

$$s := [rx + k \bmod q] \xrightarrow{s} g^s \cdot y^{-r} \stackrel{?}{=} I$$

## Theorem 6

*If the discrete-log problem is hard, then the Schnorr identification scheme is secure.*

# Proof of the Schnorr Identification Scheme

**Idea:** If the attacker can let  $g^s \cdot y^{-r} = I$ , then the attacker can compute  $x$ .

## Proof.

Reduce  $\mathcal{A}'$  inverting  $y$  to  $\mathcal{A}$  attacking the Schnorr scheme:

- 1  $\mathcal{A}'$  as a verifier, answering all queries, runs  $\mathcal{A}$  as a prover.
- 2 When  $\mathcal{A}$  outputs  $I$ ,  $\mathcal{A}'$  choose  $r_1 \in \mathbb{Z}_q$  and give it to  $\mathcal{A}$ , who responds with  $s_1$ .
- 3 Run  $\mathcal{A}$  a second time, send  $r_2 \in \mathbb{Z}_q$  to  $\mathcal{A}$  who responds with  $s_2$ .
- 4 If  $g^{s_1} \cdot h^{-r_1} = I$  and  $g^{s_2} \cdot h^{-r_2} = I$  and  $r_1 \neq r_2$  then output  $x = [(s_1 - s_2) \cdot (r_1 - r_2)^{-1} \bmod q]$ . Else, output nothing.



# The Fiat-Shamir Transform

The Fiat-Shamir transform constructs a (non-interactive) signature scheme by letting the signer run the protocol by itself.

## Construction 7

Let  $\Pi = (\text{Gen}_{\text{id}}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{V})$  be an identification scheme.

- **Gen:**  $(pk, sk) \leftarrow \text{Gen}_{\text{id}}$ . A function  $H : \{0, 1\}^* \rightarrow \Omega_{pk}$  (a set of challenges).
- **Sign:** On input  $sk$  and  $m \in \{0, 1\}^*$ , do
  - 1 Compute  $(I, \text{st}) \leftarrow \mathcal{P}_1(sk)$
  - 2 Compute  $r := H(I, m)$
  - 3 Compute  $s := \mathcal{P}_2(sk, \text{st}, r)$

Output the signature  $r, s$ .

- **Vrfy:**  $I := \mathcal{V}(pk, r, s)$ . Output  $1 \iff H(I, m) \stackrel{?}{=} r$ .

## Theorem 8

If  $\Pi$  is a secure identification scheme and  $H$  is a random oracle, then the Fiat-Shamir transform results a secure signature scheme.



# The Schnorr Signature Scheme

## Construction 9

- Gen:  $(\mathcal{G}, q, g) \leftarrow \mathcal{G}(1^n)$ . Choose  $x \in \mathbb{Z}_q$  and set  $y := g^x$ . The private key is  $x$  and the public key is  $(\mathcal{G}, q, g, y)$ . A function  $H : \{0, 1\}^* \rightarrow \mathbb{Z}_q$ .
- Sign: On input  $x$  and  $m \in \{0, 1\}^*$ , do
  - 1 Compute  $I := g^k$ , where a uniform  $k \in \mathbb{Z}_q$
  - 2 Compute  $r := H(I, m)$
  - 3 Compute  $s := [rx + k \bmod q]$Output the signature  $(r, s)$ .
- Vrfy: Compute  $I := g^s \cdot y^{-r}$  and output  $1 \iff H(I, m) \stackrel{?}{=} r$ .

DSS (Digital Signature Standard) uses Digital Signature Algorithm (DSA, a variant of ElGamal signature scheme). [FIPS 186]

## Construction 10

- $\mathcal{G}$  outputs  $(p, q, g)$ : (1)  $p$  and  $q$  are primes with  $\|q\| = n$ ;  
(2)  $q|(p-1)$  but  $q^2 \nmid (p-1)$ ;  
(3)  $g$  is a generator of the subgroup of  $\mathbb{Z}_p^*$  of order  $q$ .
- Gen:  $(p, q, g) \leftarrow \mathcal{G}$ . hash function  $H : \{0, 1\}^* \rightarrow \mathbb{Z}_q$ .  
 $x \leftarrow \mathbb{Z}_q$  and  $y := [g^x \bmod p]$ .  
 $pk = \langle H, p, q, g, y \rangle$ .  $sk = \langle H, p, q, g, x \rangle$ .
  - Sign:  $k \leftarrow \mathbb{Z}_q^*$  and  $r := [[g^k \bmod p] \bmod q]$ ,  
 $s := [(H(m) + xr) \cdot k^{-1} \bmod q]$ . Output  $(r, s)$ .
  - Vrfy:  $u_1 := [H(m) \cdot s^{-1} \bmod q]$ ,  $u_2 := [r \cdot s^{-1} \bmod q]$ .  
Output 1  $\iff r \stackrel{?}{=} [[g^{u_1} y^{u_2} \bmod p] \bmod q]$ .

# Correctness and Security of DSS/DSA

$r = [[g^k \bmod p] \bmod q]$  and  $s = [(\hat{m} + xr) \cdot k^{-1} \bmod q]$ ,  $\hat{m} = H(m)$ .

$$\begin{aligned} g^{\hat{m}s^{-1}} y^{rs^{-1}} &= g^{\hat{m} \cdot (\hat{m} + xr)^{-1} k} g^{xr \cdot (\hat{m} + xr)^{-1} k} \pmod{p} \\ &= g^{(\hat{m} + xr) \cdot (\hat{m} + xr)^{-1} k} \pmod{p} \\ &= g^k \pmod{p}. \end{aligned}$$

$$[[g^k \bmod p] \bmod q] = r.$$

Security of DSS relies on the hardness of discrete log problem.  
The entropy, secrecy and uniqueness of  $k$  is critical.

## Insecurity

No proof of security for DSS based on discrete log assumption.

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# One-Time Signature (OTS)

**One-Time Signature (OTS):** Under a weaker attack scenario, sign only one message with one secret.

The OTS experiment  $\text{Sigforge}_{\mathcal{A}, \Pi}^{1\text{-time}}(n)$ :

- 1  $(pk, sk) \leftarrow \text{Gen}(1^n)$ .
- 2  $\mathcal{A}$  is given input  $1^n$  and a **single query**  $m'$  to  $\text{Sign}_{sk}(\cdot)$ , and outputs  $(m, \sigma)$ ,  $m \neq m'$ .
- 3  $\text{Sigforge}_{\mathcal{A}, \Pi}^{1\text{-time}}(n) = 1 \iff \text{Vrfy}_{pk}(m, \sigma) = 1$ .

## Definition 11

A signature scheme  $\Pi$  is **existentially unforgeable under a single-message attack** if  $\forall$  PPT  $\mathcal{A}$ ,  $\exists$   $\text{negl}$  such that:

$$\Pr[\text{Sigforge}_{\mathcal{A}, \Pi}^{1\text{-time}}(n) = 1] \leq \text{negl}(n).$$

**Idea:** OTS from OWF; one mapping per bit.

## Construction 12

*f is a one-way function.*

■ **Gen:** on input  $1^n$ , for  $i \in \{1, \dots, \ell\}$ :

1 choose random  $x_{i,0}, x_{i,1} \leftarrow \{0, 1\}^n$ .

2 compute  $y_{i,0} := f(x_{i,0})$  and  $y_{i,1} := f(x_{i,1})$ .

$$pk = \begin{pmatrix} y_{1,0} & y_{2,0} & \cdots & y_{\ell,0} \\ y_{1,1} & y_{2,1} & \cdots & y_{\ell,1} \end{pmatrix} \quad sk = \begin{pmatrix} x_{1,0} & x_{2,0} & \cdots & x_{\ell,0} \\ x_{1,1} & x_{2,1} & \cdots & x_{\ell,1} \end{pmatrix}.$$

■ **Sign:**  $m = m_1 \cdots m_\ell$ , output  $\sigma = (x_{1,m_1}, \dots, x_{\ell,m_\ell})$ .

■ **Vrfy:**  $\sigma = (x_1, \dots, x_\ell)$ , output  $1 \iff f(x_i) = y_{i,m_i}$ , for all  $i$ .

## Theorem 13

*If f is OWF,  $\Pi$  is OTS for messages of length polynomial  $\ell$ .*

# Example of Lamport's OTS

**Signing**  $m = 011$

$$sk = \begin{pmatrix} x_{1,0} & x_{2,0} & x_{3,0} \\ x_{1,1} & x_{2,1} & x_{3,1} \end{pmatrix} \Rightarrow \sigma = \underline{\hspace{2cm}}$$

$\sigma = (x_1, x_2, x_3)$ :

$$pk = \begin{pmatrix} y_{1,0} & y_{2,0} & y_{3,0} \\ y_{1,1} & y_{2,1} & y_{3,1} \end{pmatrix} \Rightarrow \begin{array}{l} f(x_1) \stackrel{?}{=} \underline{\hspace{2cm}} \\ f(x_2) \stackrel{?}{=} \underline{\hspace{2cm}} \\ f(x_3) \stackrel{?}{=} \underline{\hspace{2cm}} \end{array}$$

# Proof of Lamport's OTS Security

**Idea:** If  $m \neq m'$ , then  $\exists i^*, m_{i^*} = b^* \neq m'_{i^*}$ . So to forge a signature on  $m$  can invert a single  $y_{i^*, b^*}$  at least.

## Proof.

Reduce  $\mathcal{I}$  inverting  $y$  to  $\mathcal{A}$  attacking  $\Pi$ :

- 1 Construct  $pk$ : Choose  $i^* \leftarrow \{1, \dots, \ell\}$  and  $b^* \leftarrow \{0, 1\}$ , set  $y_{i^*, b^*} := y$ . For  $i \neq i^*$ ,  $y_{i, b} := f(x_{i, b})$ .
- 2  $\mathcal{A}$  queries  $m'$ : If  $m'_{i^*} = b^*$ , stop. Otherwise, return  $\sigma = (x_{1, m'_1}, \dots, x_{\ell, m'_\ell})$ .
- 3 When  $\mathcal{A}$  outputs  $(m, \sigma)$ ,  $\sigma = (x_1, \dots, x_\ell)$ , if  $\mathcal{A}$  output a forgery at  $(i^*, b^*)$ :  $\text{Vrfy}_{pk}(m, \sigma) = 1$  and  $m_{i^*} = b^* \neq m'_{i^*}$ , then output  $x_{i^*, b^*}$ .

$$\Pr[\mathcal{I} \text{ succeeds}] \geq \frac{1}{2\ell} \Pr[\mathcal{A} \text{ succeeds}]$$





# Stateful Signature Scheme

**Idea:** OTS by signing with “**new**” key derived from “**old**” state.

## Definition 14 (Stateful signature scheme)

- **Key-generation** algorithm  $(pk, sk, s_0) \leftarrow \text{Gen}(1^n)$ .  $s_0$  is initial state.
- **Signing** algorithm  $(\sigma, s_i) \leftarrow \text{Sign}_{sk, s_{i-1}}(m)$ .
- **Verification** algorithm  $b := \text{Vrfy}_{pk}(m, \sigma)$ .

### A simple stateful signature scheme for OTS:

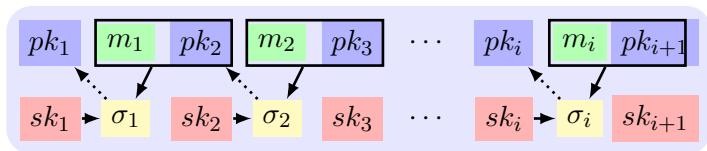
Generate  $(pk_i, sk_i)$  independently, set  $pk := (pk_1, \dots, pk_\ell)$  and  $sk := (sk_1, \dots, sk_\ell)$ .

Start from the state 1, sign the  $s$ -th message with  $sk_s$ , verify with  $pk_s$ , and update the state to  $s + 1$ .

**Weakness:** the upper bound  $\ell$  must be fixed in advance.

# “Chain-Based” Signatures

**Idea:** generate keys “on-the-fly” and sign the key chain.



Use a single public key  $pk_1$ , sign each  $m_i$  and  $pk_{i+1}$  with  $sk_i$ :

$$\sigma_i \leftarrow \text{Sign}_{sk_i}(m_i || pk_{i+1}),$$

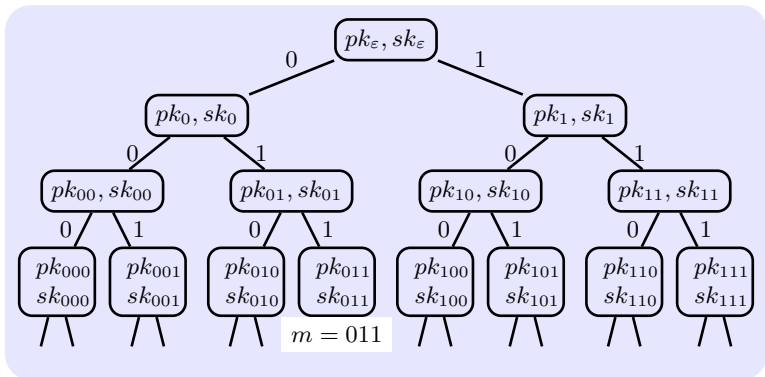
output  $\langle pk_{i+1}, \sigma_i \rangle$ , and verify  $\sigma_i$  with  $pk_i$ .

The signature is  $(pk_{i+1}, \sigma_i, \{m_j, pk_{j+1}, \sigma_j\}_{j=1}^{i-1})$ .

**Weakness:** stateful, not efficient, revealing all previous messages.

# “Tree-Based” Signatures

**Idea:** generate a chain of keys for each message and sign the key chain.



- root is  $\epsilon$  (empty string), leaf is a message  $m$ , and internal nodes  $(pk_w, sk_w)$ , where  $w$  is the prefix of  $m$ .
- each node  $pk_w$  “certifies” its children  $pk_{w0}||pk_{w1}$  or  $w$ .

# A Stateless Solution

**Idea:** use deterministic randomness to emulate the state of tree.

Use PRF  $F$  and two keys  $k, k'$  (secrets) to generate  $pk_w, sk_w$ :

- 1 compute  $r_w := F_k(w)$ .
- 2 compute  $(pk_w, sk_w) := \text{Gen}(1^n; r_w)$ , using  $r_w$  as random coins.

$k'$  is used to generate  $r'_w$  that is used to compute  $\sigma_w$ .

## Lemma 15

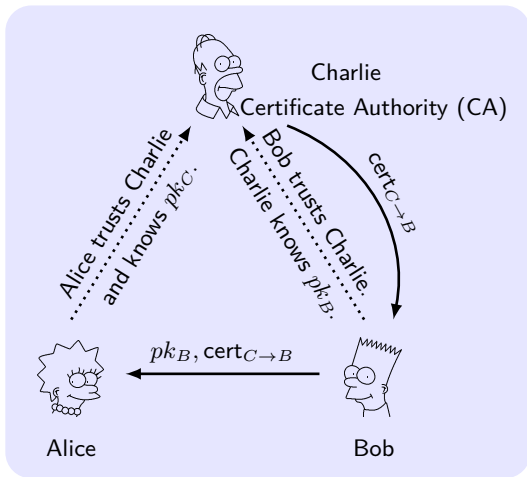
*If OWF exist, then  $\exists$  OTS (for messages of arbitrary length).*

## Theorem 16

*If OWF exists, then  $\exists$  (stateless) secure signature scheme.*

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# Certificates



**Certificates**  $cert_{C \rightarrow B} \stackrel{\text{def}}{=} \text{Sign}_{sk_C}(\text{'Bob's key is } pk_B\text{'})$ .

# Public-Key Infrastructure (PKI)

- **A single CA:** is trusted by everybody.
  - Strength: simple
  - Weakness: single-point-of-failure
- **Multiple CAs:** are trusted by everybody.
  - Strength: robust
  - Weakness: cannikin law
- **Delegation and certificate chains:** The trust is transitive.
  - Strength: ease the burden on the root CA.
  - Weakness: difficult for management, cannikin law.
- **“Web of trust”:** No central points of trust, e.g., PGP.
  - Strength: robust, work at “grass-roots” level.
  - Weakness: difficult to manage/give a guarantee on trust.

# Invalidating Certificates

- **Expiration:** include an *expiry date* in the certificate.

$$\text{cert}_{C \rightarrow B} \stackrel{\text{def}}{=} \text{Sign}_{sk_C}(\text{'bob's key is } pk_B', \text{ date}).$$

- **Revocation:** explicitly revoke the certificate.

$$\text{cert}_{C \rightarrow B} \stackrel{\text{def}}{=} \text{Sign}_{sk_C}(\text{'bob's key is } pk_B', \text{ ###}).$$

“###” represents the serial number of this certificate.

**Cumulated Revocation:** CA generates *certificate revocation list* (CRL) containing the serial numbers of all revoked certificates, signs CRL with the current date.



- Textbook RSA, Hashed RSA, Hash-and-Sign
- Identification, Fiat-Shamir Transform, Schnorr Signature, DSS/DSA
- Lamport's OTS, Stateful/Chain-based/Tree-based/Stateless Signature
- Certificates, PKI, CA, Web-of-trust, Invalidation