

# Theoretical Constructions of Pseudorandom Objects

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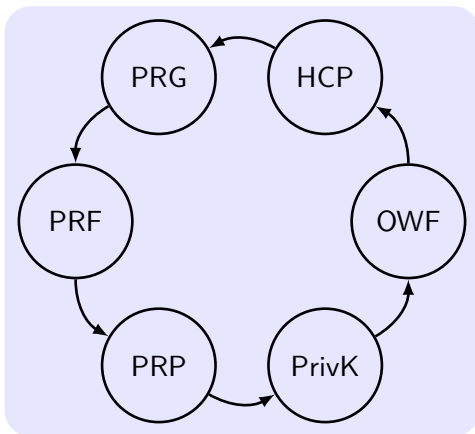
Cryptography, Autumn, 2020

**1** One-Way Functions

**2** From OWF to PRP

## **1** One-Way Functions

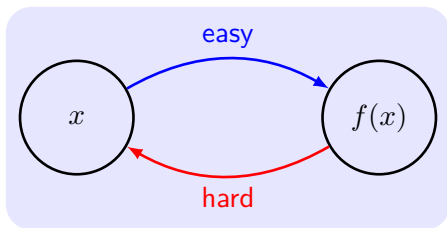
## **2** From OWF to PRP



## One of contributions of modern cryptography

The existence of one-way functions is equivalent to the existence of all (non-trivial) private-key cryptography.

# One-Way Functions (OWF)



The inverting experiment  $\text{Invert}_{\mathcal{A},f}(n)$ :

- 1 Choose input  $x \leftarrow \{0, 1\}^n$ . Compute  $y := f(x)$ .
- 2  $\mathcal{A}$  is given  $1^n$  and  $y$  as input, and outputs  $x'$ .
- 3  $\text{Invert}_{\mathcal{A},f}(n) = 1$  if  $f(x') = y$ , otherwise 0.

# Definitions of OWF/OWP [Yao]

For polynomial-time algorithm  $M_f$  and  $\mathcal{A}$ .

## Definition 1

A function  $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$  is **one-way** if:

- 1 (Easy to compute):  $\exists M_f: \forall x, M_f(x) = f(x)$ .
- 2 (Hard to invert):  $\forall \mathcal{A}, \exists \text{negl}$  such that

$$\Pr[\text{Invert}_{\mathcal{A},f}(n) = 1] \leq \text{negl}(n).$$

or

$$\Pr_{x \leftarrow \{0,1\}^n}[\mathcal{A}(f(x)) \in f^{-1}(f(x))] \leq \text{negl}(n).$$

## Definition 2

Let  $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$  be length-preserving, and  $f_n$  be the restriction of  $f$  to the domain  $\{0, 1\}^n$ . A OWP  $f$  is a **one-way permutation** if  $\forall n$ ,  $f_n$  is a bijection.

# Candidate One-Way Function

- **Multiplication and factoring:**

$f_{\text{mult}}(x, y) = (xy, \|x\|, \|y\|)$ ,  $x$  and  $y$  are equal-length primes.

- **Modular squaring and square roots:**

$f_{\text{square}}(x) = x^2 \bmod N$ .

- **Discrete exponential and logarithm:**

$f_{g,p}(x) = g^x \bmod p$ .

- **Subset sum problem:**

$f(x_1, \dots, x_n, J) = (x_1, \dots, x_n, \sum_{j \in J} x_j)$ .

- **Cryptographically secure hash functions:**

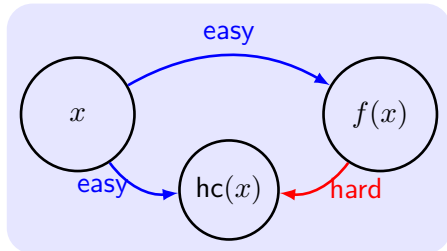
Practical solutions for one-way computation.

$f : \{0, 1\}^{128} \rightarrow \{0, 1\}^{128}$  is a OWF. Is  $f'$  OWF?

- $f'(x) = f(x) \| x$
- $f'(x \| x') = f(x) \| x'$
- $f'(x) = f(x) \oplus f(x)$
- $f'(x) = \begin{cases} f(x) & \text{if } x[0, 1, 2, 3] \neq 1010 \\ x & \text{otherwise} \end{cases}$
- $f'(x) = \begin{cases} f(x) & \text{if } x \neq 1010 \| 0^{124} \\ x & \text{otherwise} \end{cases}$
- more examples in homework



# Hard-Core Predicates (HCP) [Blum-Micali]



## Definition 3

A function  $hc : \{0, 1\}^* \rightarrow \{0, 1\}$  is a **hard-core predicate of a function**  $f$  if (1)  $hc$  can be computed in polynomial time, and (2)  $\forall$  PPT  $\mathcal{A}$ ,  $\exists$   $\text{negl}$  such that

$$\Pr_{x \leftarrow \{0,1\}^n} [\mathcal{A}(f(x)) = hc(x)] \leq \frac{1}{2} + \text{negl}(n).$$

# A HCP for Any OWF

## Theorem 4

*$f$  is OWF. Then  $\exists$  an OWF  $g$  along with an HCP  $gl$  for  $g$ . If  $f$  is a permutation then so is  $g$ .*

Q: is  $gl(x) = \bigoplus_{i=1}^n x_i$  the HCP of any OWF?

## Proof.

$g(x, r) \stackrel{\text{def}}{=} (f(x), r)$ , for  $|x| = |r|$ , and define

$$gl(x, r) \stackrel{\text{def}}{=} \bigoplus_{i=1}^n x_i \cdot r_i.$$

$r$  is generated uniformly at random. [Goldreich and Levin]



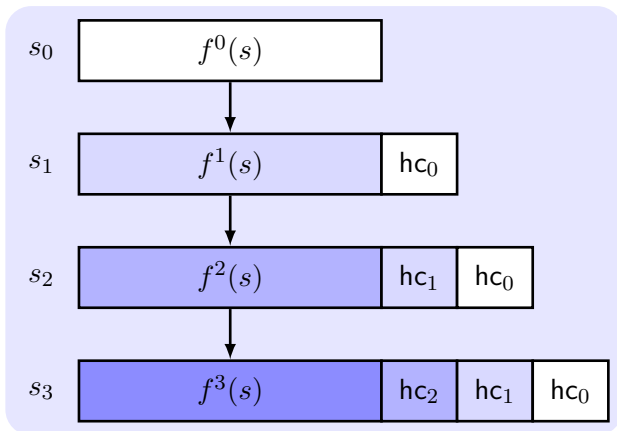
## 1 One-Way Functions

## 2 From OWF to PRP

# PRG from OWP: Blum-Micali Generator

## Theorem 5

*$f$  is an OWP and  $hc$  is an HCP of  $f$ . Then  $G(s) \stackrel{\text{def}}{=} (f(s), hc(s))$  constitutes a PRG with expansion factor  $\ell(n) = n + 1$ , then  $\forall$  polynomial  $p(n) > n$ ,  $\exists$  a PRG with expansion factor  $\ell(n) = p(n)$ .*

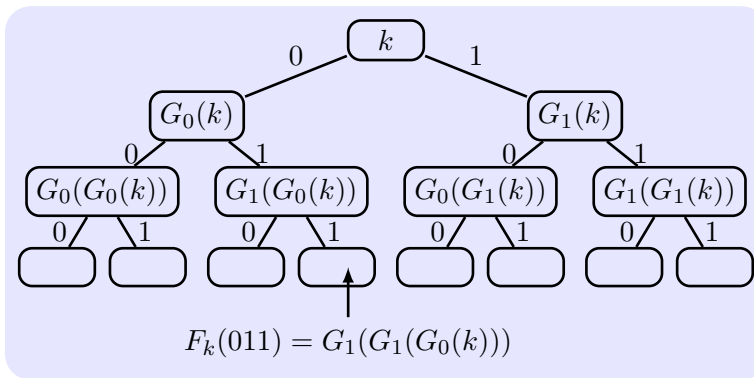


# PRF from PRG [Goldreich, Goldwasser, Micali]

## Theorem 6

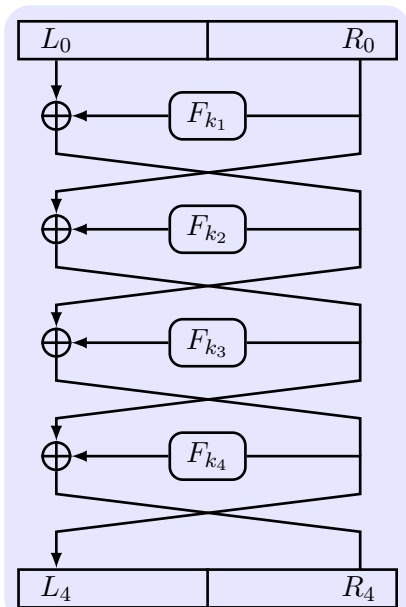
If  $\exists$  a PRG with expansion factor  $\ell(n) = 2n$ , then  $\exists$  a PRF.

$$G(k) = G_0(k) \| G_1(k)$$



$$F_k(x_1 x_2 \cdots x_n) = G_{x_n}(\cdots (G_{x_2}(G_{x_1}(k))) \cdots), G(s) = (G_0(s), G_1(s)).$$

# PRP from PRF [Lucy, Rackoff]



$F^{(r)}$  is an  $r$ -round Feistel network with the mangler function  $F$ .

## Theorem 7

*If  $F$  is a length-preserving PRF, then  $F^{(3)}$  is a PRP that maps  $2n$ -bit strings to  $2n$ -bit strings (and uses a key of length  $3n$ ).*

## Theorem 8

*If  $F$  is a length-preserving PRF, then  $F^{(4)}$  is a strong PRP that maps  $2n$ -bit strings to  $2n$ -bit strings (and uses a key of length  $4n$ ).*

# Necessary Assumptions

## Theorem 9

*Assume that  $\exists$  OWP. Then  $\exists$  PRG, PRF, strong PRP, and CCA-secure private-key encryption schemes.*

## Proposition 10

*If  $\exists$  a private-key encryption scheme that has indistinguishable encryptions in the presence of an eavesdropper, then  $\exists$  an OWF.*

## Proof.

$$f(k, m, r) \stackrel{\text{def}}{=} (\text{Enc}_k(m, r), m).$$



- OWF implies secure private-key encryption scheme
- Secure private-key encryption scheme implies OWF