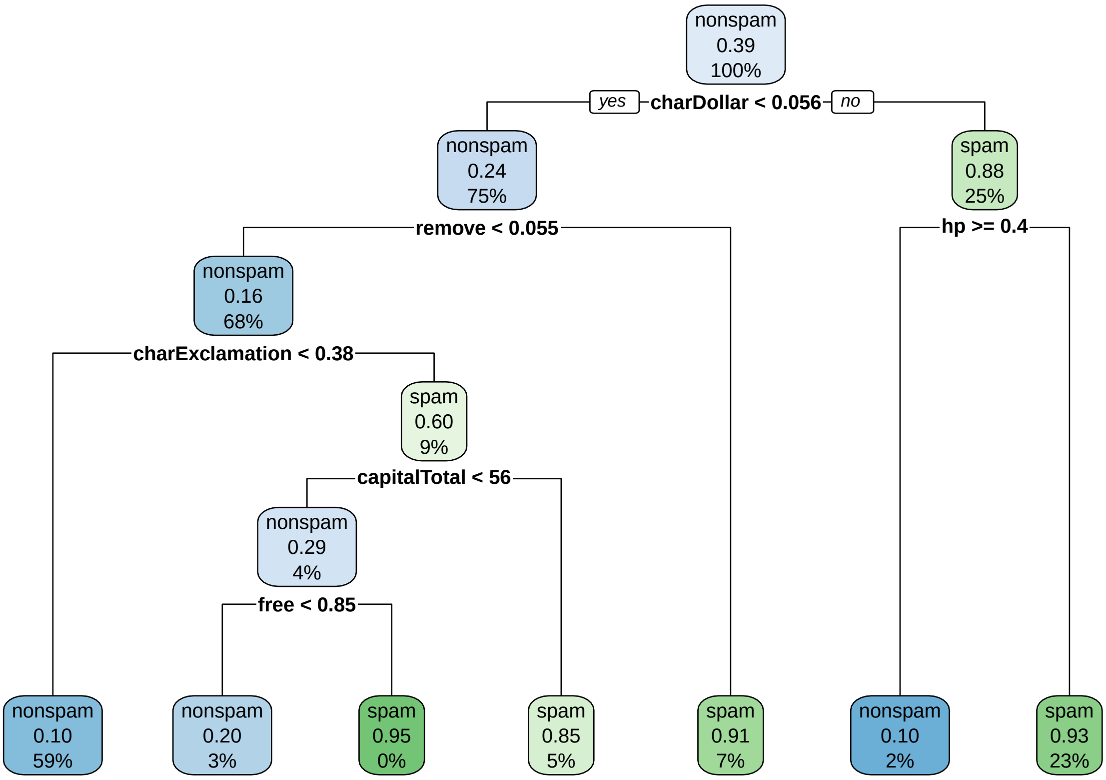
Solution 1:

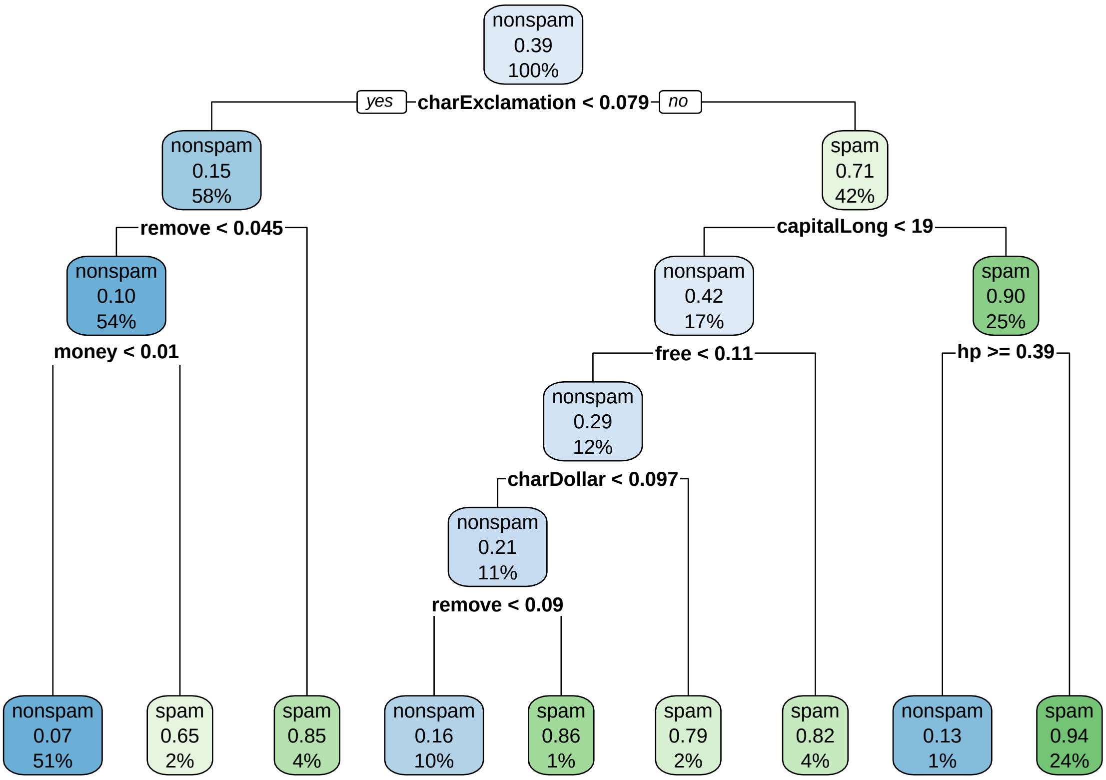
a) The spam data is a binary classification task where the aim is to classify an email as spam or no-spam.

```
library(mlr)
spam.task

## Supervised task: spam-example
## Type: classif
## Target: type
## Observations: 4601
## Features:
##    numerics     factors     ordered functionals
##          57           0           0           0
## Missings: FALSE
## Has weights: FALSE
## Has blocking: FALSE
## Has coordinates: FALSE
## Classes: 2
## nonspam    spam
##    2788    1813
## Positive class: nonspam
```

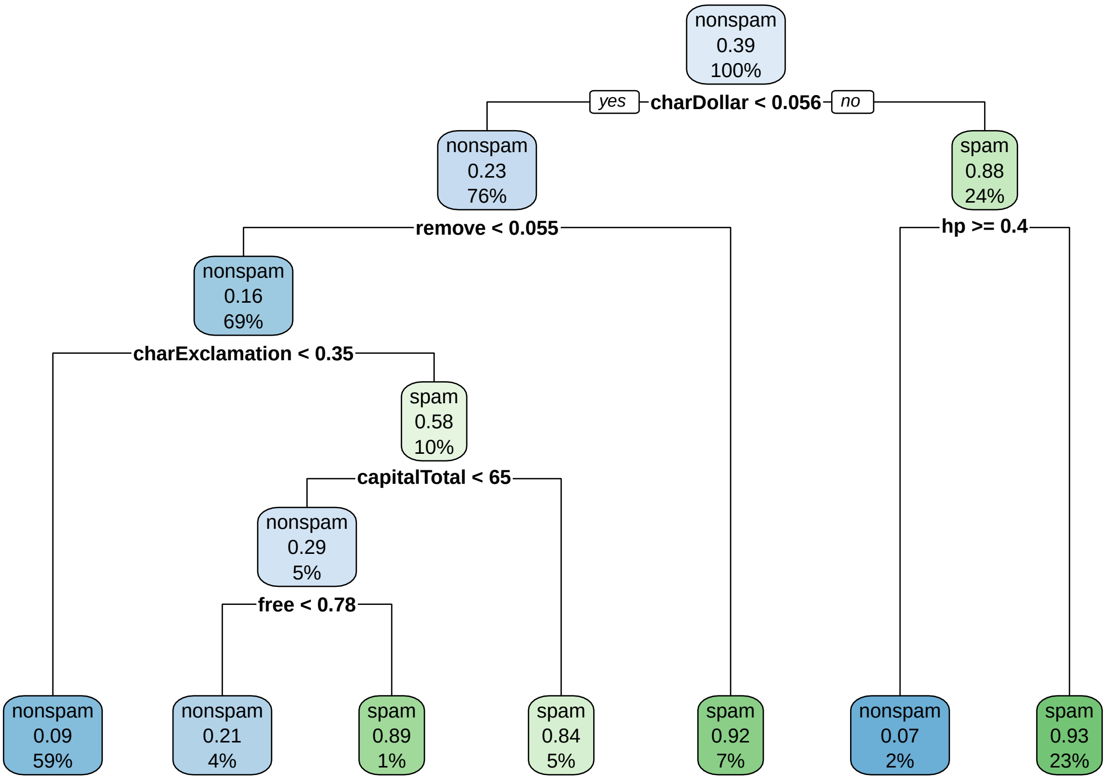
b)
```
library(rpart.plot)
## Loading required package:  rpart

lrn = makeLearner("classif.rpart")
model = train(lrn, spam.task)
mod = getLearnerModel(model)
rpart.plot(mod)

## Warning:  Cannot retrieve the data used to build the model (so cannot determine
roundint and is.binary for the variables).
## To silence this warning:
##    Call rpart.plot with roundint=FALSE,
##    or rebuild the rpart model with model=TRUE.
```




```
set.seed(42)
subset1 = sample.int(getTaskSize(spam.task), size = 0.8 * getTaskSize(spam.task))
subset2 = sample.int(getTaskSize(spam.task), size = 0.8 * getTaskSize(spam.task))

model = train(lrn, spam.task, subset = subset1)
mod = getLearnerModel(model)
rpart.plot(mod)

## Warning:  Cannot retrieve the data used to build the model (so cannot determine
roundint and is.binary for the variables).
## To silence this warning:
##     Call rpart.plot with roundint=FALSE,
##     or rebuild the rpart model with model=TRUE.
```

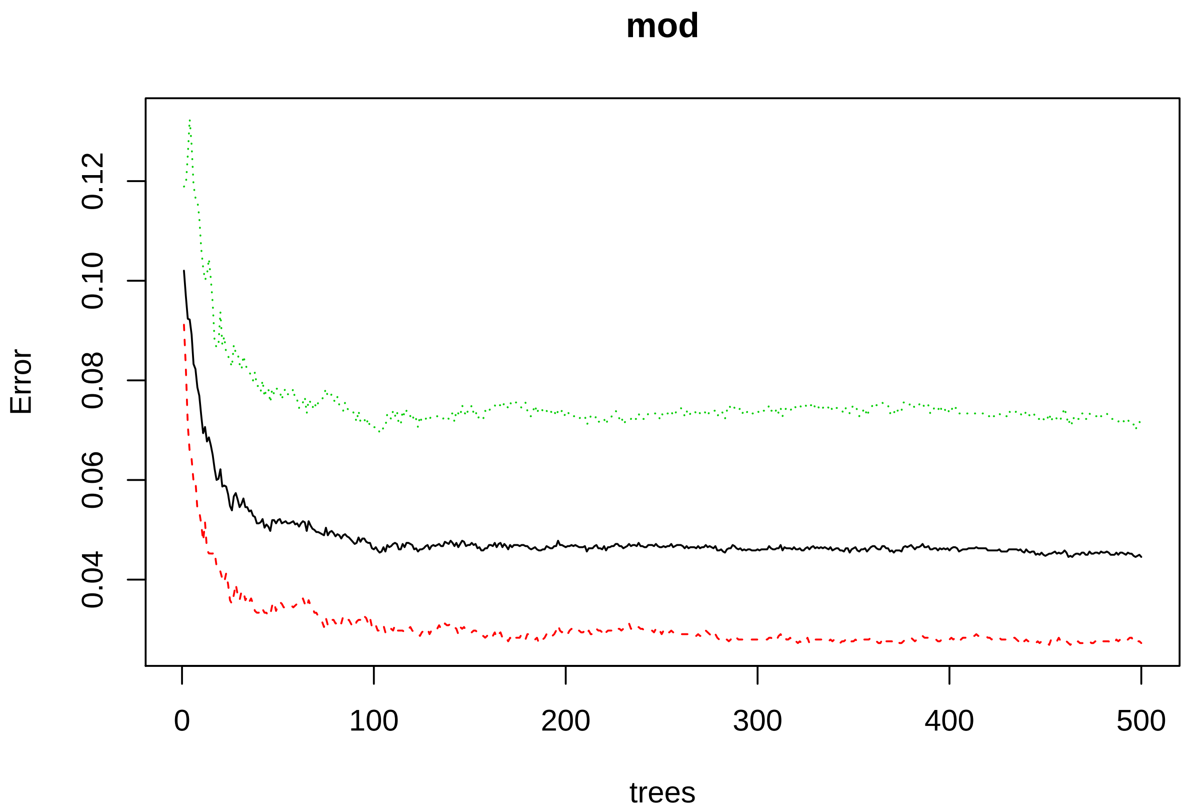



```
model = train(lrn, spam.task, subset = subset2)
mod = getLearnerModel(model)
rpart.plot(mod)

## Warning:  Cannot retrieve the data used to build the model (so cannot determine
roundint and is.binary for the variables).
## To silence this warning:
##     Call rpart.plot with roundint=FALSE,
##     or rebuild the rpart model with model=TRUE.
```

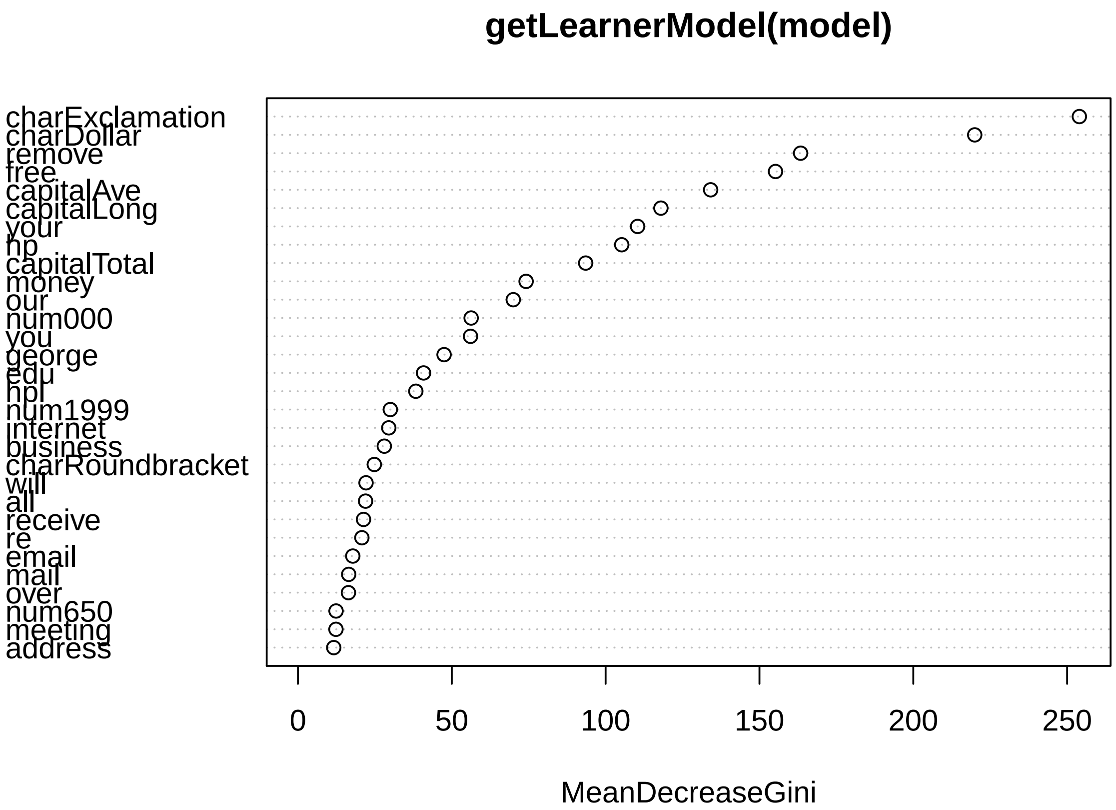


Observation: Trees with different sample find different split points and variables, leading to different trees!

c)
```
lrn = makeLearner("classif.randomForest")
model = train(lrn, spam.task)
mod = getLearnerModel(model)
mod

##
## Call:
##  randomForest(formula = f, data = data, classwt = classwt, cutoff = cutoff)
##                Type of random forest: classification
##                      Number of trees: 500
## No. of variables tried at each split: 7
##
##         OOB estimate of  error rate: 4.46%
## Confusion matrix:
##         nonspam spam class.error
## nonspam    2712   76     0.02726
## spam        129 1684     0.07115

plot(mod)
```



d)
```
imp = getFeatureImportance(model)
sort(imp$res, decreasing = TRUE)

##   charExclamation charDollar remove  free capitalAve capitalLong  your
## 1             254        220  163.4 155.2      134.1         118 110.4
##      hp capitalTotal money   our num000   you george   edu   hpl num1999
## 1 105.2        93.52 74.15 69.98   56.3 56.09   47.5 40.82 38.31   30.03
##   internet business charRoundbracket  will   all receive    re email  mail
## 1    29.51    28.06            24.82 22.13 21.98   21.31 20.77 17.82 16.49
##    over num650 meeting address charSemicolon order credit make  labs
## 1 16.41  12.38   12.36   11.63         10.85 9.045  8.255 8.01 7.917
##   charHash num85 people technology    pm  data charSquarebracket font
## 1    7.689 7.663  7.415      6.834 6.589 5.838             5.443 5.09
##   project report   lab telnet original addresses conference direct    cs
## 1   4.693  4.506 4.175   3.96     3.63      2.94      2.724  2.449 2.025
##   num415 num3d num857  parts  table
## 1  1.689 1.686  1.625 0.9221 0.5594

# as alternative, the randomForest package provides a plotting function
randomForest::varImpPlot(getLearnerModel(model))
```



Solution 2:
See R code **randomForest_1_2.R**

Solution 3:

a) Proceed as follows:

(i) Split x in two groups using the following split points.
- (1), $(2, 7, 10, 20)$ (splitpoint 1.5)
- $(1, 2)$, $(7, 10, 20)$ (splitpoint 4.5)
- $(1, 2, 7)$, $(10, 20)$ (splitpoint 8.5)
- $(1, 2, 7, 10)$, (20) (splitpoint 15)

(ii) For each possible split point compute the sum of squares in both groups.

(iii) Use as split point the point that splits both groups best w.r.t. minimizing the sum of squares in both groups.

Here, we have only one split variable x. A split point t, leads to the following half-spaces:

$$\mathcal{N}_1(t) = \{(x, y) \in \mathcal{N} : x \leq t\} \text{ and } \mathcal{N}_2(t) = \{(x, y) \in \mathcal{N} : x > t\}.$$

Remember the minimization Problem (here only for one split variable x):

$$\min_t \left(\min_{c_1} \sum_{(x,y)\in\mathcal{N}_1} (y-c_1)^2 + \min_{c_2} \sum_{(x,y)\in\mathcal{N}_2} (y-c_2)^2 \right).$$

The inner minimization is solved through: $\hat{c}_1 = \bar{y}_1$ and $\hat{c}_2 = \bar{y}_2$

Which results in:

$$\min_t \left(\sum_{(x,y)\in\mathcal{N}_1} (y-\bar{y}_1)^2 + \sum_{(x,y)\in\mathcal{N}_2} (y-\bar{y}_2)^2 \right).$$

The sum of squares error of the parent is:

$$Impurity_{parent} = MSE_{parent} = \frac{1}{5}\sum_{i=1}^{5}(y_i - 4.7)^2 = 22.56$$

Calculate the risk for each split point:

$x \leq 1.5$

$$\begin{aligned} \mathcal{R}_{\text{emp}}(\mathcal{N}) &= \frac{1}{5}\text{MSE}_{left} + \frac{4}{5}\text{MSE}_{right} = \\ &= \frac{1}{5}\cdot\frac{1}{1}(1-1)^2 + \frac{4}{5}\cdot\frac{1}{4}((1-5.625)^2 + (0.5-5.625)^2 + (10-5.625)^2 + (11-5.625)^2) \\ &= 19.1375 \end{aligned}$$

$x \leq 4.5$ $\mathcal{R}_{\text{emp}}(\mathcal{N}) = 13.43$

$x \leq 8.5$ $\mathcal{R}_{\text{emp}}(\mathcal{N}) = 0.13$

$x \leq 15$ $\mathcal{R}_{\text{emp}}(\mathcal{N}) = 12.64$

Minimal empirical risk is obtained by choosing the split point 8.5.

Doing the same for the log-transformation gives:

$x \leq 0.3$ $\mathcal{R}_{\text{emp}}(\mathcal{N}) = 19.14$

$x \leq 1.3$ $\mathcal{R}_{\text{emp}}(\mathcal{N}) = 13.43$

$x \leq 2.1$ $\mathcal{R}_{\text{emp}}(\mathcal{N}) = 0.13$

$x \leq 2.6$ $\mathcal{R}_{\text{emp}}(\mathcal{N}) = 12.64$

Minimal empirical risk is obtained by choosing the split point 2.1.

b)
```r
x = c(1,2,7,10,20)
y = c(1,1,0.5,10,11)

calculateMSE = function (y) mean((y - mean(y))^2)
calculateTotalMSE = function (yleft, yright) {
  n_left = length(yleft)
  n_right = length(yright)

  mse_left = n_left / (n_left + n_right) * calculateMSE(yleft)
  mse_right = n_right / (n_left + n_right) * calculateMSE(yright)

  return(mse_left + mse_right)
}

split = function(x, y) {
```

```
  # try out all points as potential split points and ...
  split_points = 0.5 * diff(sort(x)) + x[-length(x)]
  node_mses = lapply(split_points, function(i) {
    y_left = y[x <= i]
    y_right = y[x > i]

    # ... compute SS in both groups
    mse_split = calculateTotalMSE(y_left, y_right)
    print(sprintf("Split at %.1f: empirical Risk = %.2f", i, mse_split))

    return(mse_split)
  })
  # select the split point yielding the maximum impurity reduction
  best = which.min(node_mses)
  split_points[best]
}

x

## [1]  1  2  7 10 20

split(x, y) # the 3rd observation is the best split point

## [1] "Split at 1.5: empirical Risk = 19.14"
## [1] "Split at 4.5: empirical Risk = 13.43"
## [1] "Split at 8.5: empirical Risk = 0.13"
## [1] "Split at 15.0: empirical Risk = 12.64"
## [1] 8.5

log(x)

## [1] 0.0000 0.6931 1.9459 2.3026 2.9957

split(log(x), y) # also here, the 3rd observation is the best split point

## [1] "Split at 0.3: empirical Risk = 19.14"
## [1] "Split at 1.3: empirical Risk = 13.43"
## [1] "Split at 2.1: empirical Risk = 0.13"
## [1] "Split at 2.6: empirical Risk = 12.64"
## [1] 2.124
```

Solution 4:

The fractions of the classes $k = 1, \dots, g$ in node $\mathcal{N}$ of a decision tree are $p(1|\mathcal{N}), \dots, p(g|\mathcal{N})$. Assume we replace the classification rule in node t

$$\hat{k}|\mathcal{N} = \arg\max_k p(k|\mathcal{N})$$

with a randomizing rule, in which we draw the classes in one node from their estimated probabilities.

Derive an estimator for the misclassification rate in node $\mathcal{N}$. What do you (hopefully) recognize?

Solution:

- For the feature space $\mathcal{Y} = \{1, \dots, g\}$ we assume the distribution P_Y. The distribution $P_{\hat{Y}}$ of our classifier $\hat{Y} = h(x)$ is defined as the individual class frequencies in a node $\mathcal{N}$ (estimated probability, that an object of class k is in node $\mathcal{N}$):
$$P(\hat{Y} = k \mid \mathcal{N}) = p(k|\mathcal{N})$$
- As estimate of the target distribution P_Y we use the distribution of the classifier $h(x)$ which we assume to be independent of $P_{\hat{Y}}$:
$$P_Y \overset{\text{ind.}}{\sim} P_{\hat{Y}}$$
- The individual error rate of wrongly predicting a true label k in node $\mathcal{N}$ ($\mathsf{err}_{k|\mathcal{N}}$) can be written as probability that $Y = k$ and $\hat{Y} \neq k$:
$$\begin{aligned} P(\mathsf{err}_{k|\mathcal{N}}) &= P(Y = k, \hat{Y} \neq k \mid \mathcal{N}) \\ &= P(Y = k \mid \mathcal{N}) P(\hat{Y} \neq k \mid \mathcal{N}) \\ &= p(k|\mathcal{N})(1 - p(k|\mathcal{N})) \end{aligned}$$
- The error rate is the combination of all individual error rates:
$$\mathsf{err}_{\mathcal{N}} = \bigcup_{k=1}^{g} \mathsf{err}_{k|\mathcal{N}}$$
- Finally, we are interested in the probability of the error rate $\mathsf{err}_{\mathcal{N}}$ as estimator for the missclassification rate:
$$\begin{aligned} P(\mathsf{err}_{\mathcal{N}}) &= P\left(\bigcup_{k=1}^{g} \mathsf{err}_{k|\mathcal{N}}\right) \\ &= \sum_{k=1}^{g} P\left(\mathsf{err}_{k|\mathcal{N}}\right) \\ &= \sum_{k=1}^{g} p(k|\mathcal{N})(1 - p(k|\mathcal{N})) \\ &= \sum_{k=1}^{g} p(k|\mathcal{N}) - \sum_{k=1}^{g} p(k|\mathcal{N})^2 \\ &= 1 - \sum_{k=1}^{g} p(k|\mathcal{N})^2 \end{aligned}$$

This is exactly the Gini-Index that CART uses for splitting the tree.

Solution 5:

$f(x) = \frac{1}{B} \sum_{b=1}^{B} f_b(x)$ is the bagging estimator based on B bootstrap samples. Then we can easily calculate:

$$\begin{aligned} \mathrm{Var}(f(x)) &= \frac{1}{B^2} \left(\sum_{b=1}^{B} \mathrm{Var}(f_b(x)) + \sum_{i \neq j}^{B} \mathrm{Cov}(f_i(x), f_j(x)) \right) \\ &= \frac{1}{B^2} \left(B\sigma^2 + (B^2 - B)\rho\sigma^2 \right) \\ &= \frac{1}{B}\sigma^2 + \rho\sigma^2 - \frac{1}{B}\rho\sigma^2 \\ &= \rho\sigma^2 + \frac{\sigma^2}{B}(1 - \rho) \end{aligned}$$

In the first line the rules for variance of a non-independent sum of random variables is used. All other steps are trivial.