Reguła de l'Hospitala

Zad 1. Oblicz granicę:

(a) $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{x}$; (b) $\lim\limits_{x\to 1^+}\frac{\ln x}{\sqrt{x^2-1}}$; (c) $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$; (d) $\lim\limits_{x\to 0}\frac{x-\sin x}{x^3}$; (e) $\lim\limits_{x\to\infty}\frac{\ln(\ln x)}{x}$;
(f) $\lim\limits_{x\to 0}\frac{e^{2x}-1}{\ln(1+2x)}$; (g) $\lim\limits_{x\to 1}\frac{1-x}{\ln x}$; (h) $\lim\limits_{x\to 1^-}(1-x)\ln(1-x)$; (i) $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$; (j) $\lim\limits_{x\to 0}\frac{x-\operatorname{tg}x}{x^2\operatorname{tg}x}$;
(k) $\lim\limits_{x\to 1}x^{\frac{1}{1-x}}$; (l) $\lim\limits_{x\to\infty}x^{\frac{1}{x}}$; (m) $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{x\sin x}\right)$; (n) $\lim\limits_{x\to 1}(1-x)\operatorname{tg}\left(\frac{\pi x}{2}\right)$; (o) $\lim\limits_{x\to 0^+}(\operatorname{tg}x)^{\operatorname{tg}x}$;
(p) $\lim\limits_{x\to 0^+}\left(\frac{1}{x}\right)^{\sin x}$; (q) $\lim\limits_{x\to 0^+}x^x$; (r) $\lim\limits_{x\to 0}\left(e^{2x}+x\right)^{\frac{1}{x}}$; (s) $\lim\limits_{x\to 1^-}\frac{\operatorname{tg}\frac{\pi}{2}x}{\ln(1-x)}$; (t) $\lim\limits_{x\to 0^+}\frac{\ln x}{1+2\ln(\sin x)}$;
(u) $\lim\limits_{x\to 1}\frac{x\ln x}{x-1}$; (v) $\lim\limits_{x\to\infty}\frac{x}{\ln x}$; (w) $\lim\limits_{x\to 0}\frac{e^{a\sqrt{x}}-1}{\sqrt{\sin bx}}$; (x) $\lim\limits_{x\to 2}\left(2-\frac{x}{2}\right)^{\operatorname{tg}\frac{\pi x}{4}}$; (y) $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\operatorname{ctg}^2 x\right)$;
(z) $\lim\limits_{x\to\infty}\frac{e^{\frac{1}{x^2}}-1}{2\operatorname{arc\,ctg}x^2-\pi}$;

Reguła de l'Hospitala - odpowiedzi

Zad 1.

(a) 2; (b) 0; (c) 2; (d) $\frac{1}{6}$; (e) 0; (f) ; (g) -1; (h) 0; (i) 0; (j) ; (k) e^{-1}; (l) 1;
(m) $-\frac{1}{6}$; (n) ; (o) ; (p) 1; (q) 1; (r) 0; (s) ; (t) ; (u) ; (v) ; (w) $\frac{a}{\sqrt{b}}$; (x) e^{π};
(y) $\frac{2}{3}$; (z) $-\frac{1}{2}$;

Różniczka funkcji

Zad 1. Wyznacz przybliżoną wartość wyrażenia:

(a) $\sqrt[3]{63}$; (b) $\operatorname{arc\,tg}(1,005)$; (c) $\sin(29^\circ)$; (d) $2^{2,9999}$;

Różniczka funkcji - odpowiedzi

Zad 1.

(a) $4-\frac{1}{48}\approx 3,9792$; (b) $\frac{\pi}{4}+\frac{0,005}{2}\approx 0,7879$; (c) $\frac{1}{2}-\frac{\pi\sqrt{3}}{360}\approx 0,4849$; (d) $8-0,0008\ln 2\approx 7,9994$;

Styczna i normalna

Zad 1. Napisać równanie stycznej i normalnej do wykresu zadanej funkcji we wskazanym punkcie:

(a) $f(x)=x^2$, $(2,f(2))$; (b) $f(x)=\arcsin\frac{x}{2}$, $(1,f(1))$; (c) $f(x)=\ln\left(x^2+e\right)$, $(0,f(0))$;
(d) $f(x)=e^{\operatorname{tg}x}$, $\left(\frac{\pi}{4},f\left(\frac{\pi}{4}\right)\right)$; (e) $f(x)=\sqrt{2x^2-2}$, $(3,f(3))$; (f) $f(x)=\frac{2x}{1+x^2}$, $\left(\sqrt{2},f\left(\sqrt{2}\right)\right)$;
(g) $f(x)=\operatorname{arc\,tg}x^2$, $(0,f(0))$; (h) $f(x)=x^x$, $(e,f(e))$; (i) $f(x)=\frac{x^2}{x+1}$, $(2,f(2))$;
(j) $f(x)=\frac{\ln x}{x}$, $(e,f(e))$; (k) $f(x)=\operatorname{arc\,tg}\frac{1-x}{1+x}$, $(1,f(1))$;

Zad 2. Znajdź kąt pod jakim przecinają się wykresy funkcji:

(a) $f(x)=x^2$, $g(x)=\sqrt[3]{x}$; (b) $f(x)=4-x$, $g(x)=4-\frac{x^2}{2}$; (c) $f(x)=\operatorname{tg}x$, $g(x)=\operatorname{ctg}x$, $x\in\left(0,\frac{\pi}{2}\right)$;
(d) $f(x)=\frac{1}{x}$, $g(x)=\sqrt{x}$;

Styczna i normalna - odpowiedzi

Zad 1.

(a) $y_s=4x-4$, $y_n=-\frac{1}{4}x+\frac{9}{2}$; (b) ; (c) ; (d) ; (e) ;
(f) $y_s=-\frac{2}{9}x+\frac{8}{9}\sqrt{2}$, $y_n=\frac{9}{2}x-\frac{23}{6}\sqrt{2}$; (g) ; (h) ; (i) ; (j) $y_s=\frac{1}{3}$, $x_n=e$; (k) ;

Zad 2.

(a) $\alpha=\frac{\pi}{2},(x=0)$; $\alpha=\frac{\pi}{4},(x=1)$; (b) ; (c) ; (d) $\alpha\approx -71.57^\circ$;

Przebieg zmienności funkcji

Zad 1. Wyznacz dziedzinę funkcji:

(a) $y = \ln\left(x - \frac{1}{x}\right)$; **(b)** $y = \ln(1 - \ln x)$;

Zad 2. Zbadaj granice funkcji na krańcach przedziału określoności:

(a) $y = \frac{x \ln x}{x-1}$; **(b)** $y = \frac{(x-1)^2}{\ln x}$;

Zad 3. Znaleźć asymptoty funkcji:

(a) $y = \frac{x^2-x-2}{2-x}$; **(b)** $y = x + 3\arctan x$; **(c)** $y = x^2 e^{-\frac{1}{x}}$; **(d)** $y = \frac{x^2-4}{x^2+x-6}$; **(e)** $y = x + \frac{\ln x}{x}$;
(f) $y = \frac{x^2}{(x+3)^2}$; **(g)** $y = x\sqrt[3]{\left(\frac{x+2}{x-1}\right)^2}$; **(h)** $y = 2x + 1 + \frac{1}{\sqrt{x}}$; **(i)** $y = \frac{3-x^2}{2-x}$; **(j)** $y = \frac{x^3+x^2-2x}{x^2-1}$;
(k) $y = e^{\frac{1}{x}} - x$; **(l)** $y = xe^{-\frac{1}{x}}$; **(m)** $y = 2x + \frac{2}{x-1}$;

Zad 4. Znaleźć przedziały monotoniczności i ekstrema funkcji:

(a) $y = 2x^3 - 3x^2$; **(b)** $y = x^2 \ln x$; **(c)** $y = x - \ln\left(1 + x^2\right)$; **(d)** $y = e^{-x^2}\left(1 + x^2\right)$;
(e) $y = \frac{\ln x}{\sqrt{x}}$; **(f)** $y = \ln x + \frac{1}{\ln x}$; **(g)** $y = e^{\frac{1}{x^2(x+1)}}$; **(h)** $y = 2x^3 - 6x^2 - 18x + 7$;
(i) $y = \frac{x}{\ln x}$; **(j)** $y = x^2 e^{-x}$; **(k)** $y = \frac{x}{e^x}$; **(l)** $y = x - 2\arctan x$;
(m) $y = \frac{2}{3}x - \sqrt[3]{(x-2)^2}$; **(n)** $y = \frac{1}{3}x - \sqrt[3]{x-2}$;

Zad 5. Znaleźć przedziały wypukłości i punkty przegięcia funkcji:

(a) $y = e^{\arctan x}$; **(b)** $y = 1 - \ln(x^2 - 4)$; **(c)** $y = \frac{x}{e^x}$; **(d)** $y = x^2 e^{-x}$;
(e) $y = \ln^3 x - 2\ln x$; **(f)** $y = x^3 - 3x^2 + 3x + 7$; **(g)** $y = \frac{x^3}{(1+x)^2}$; **(h)** $y = 3x^5 - 5x^4 + 3x - 2$;
(i) $y = x^4(12\ln x - 7)$; **(j)** $y = e^{\frac{1}{x}} - x$; **(k)** $y = x^3 e^{-x}$; **(l)** $y = x\sqrt[3]{\left(\frac{x+2}{x-1}\right)^2}$;
(m) $y = \frac{5}{9}x^2 - \sqrt[3]{(x-2)^5}$; **(n)** $y = \frac{2}{9}x^2 - \sqrt[3]{(x-2)^4}$; **(o)** $y = \frac{x^2}{2} + 5\ln(x+6)$;

Zad 6. Znaleźć wartość największą i najmniejsza funkcji we wskazanych przedziałach:

(a) $f(x) = 2x^3 - 3x^2 - 36x - 8, \quad x \in [-3, 6]$; **(b)** $f(x) = x - 2\sqrt{x}, \quad x \in [0, 5]$;
(c) $f(x) = 2\sin x + \sin 2x, \quad x \in [0, \frac{3}{2}\pi]$;

Zad 7. Zbadaj przebieg zmienności funkcji i naszkicuj jej wykres:

(a) $y = (x-2)^3(x+4)$; **(b)** $y = \frac{5}{6}x - \sqrt[6]{(x-2)^5}$; **(c)** $y = \frac{2}{3}x - \sqrt[3]{(x-2)^2}$; **(d)** $y = \frac{1}{\arcsin x}$;
(e) $y = x^2 e^{-\frac{1}{x}}$; **(f)** $y = x\sqrt[3]{\left(\frac{x+2}{x-1}\right)^2}$; **(g)** $y = \frac{3-x^2}{2-x}$; **(h)** $y = e^{\frac{1}{x}} - x$;
(i) $y = x - \ln\left(1 + x^2\right)$; **(j)** $y = x^2 \ln x$; **(k)** $y = e^{-x^2}\left(1 + x^2\right)$; **(l☠)** $y = \ln x + \frac{1}{\ln x}$;
(m) $y = \frac{x}{\ln x}$; **(n)** $y = x^2 e^{-x}$; **(o)** $y = \frac{x}{e^x}$; **(p)** $y = x - 2\arctan x$;
(q) $y = 1 - \ln\left|x^2 - 4\right|$; **(r)** $y = \ln^3 x - 2\ln x$; **(s)** $y = \frac{x^3}{(1+x)^2}$; **(t)** $y = x^3 e^{-x}$;
(u) $y = \frac{e^{-\frac{1}{x}}}{x}$; **(v)** $y = \frac{x-a}{\sqrt{x}}$ gdzie a jest parametrem; **(w)** $y = \arcsin\frac{x-2}{x}$;
(x) $y = \ln\left(e - \frac{1}{x}\right)$; **(y)** $y = \frac{2}{9}x^2 - \sqrt[3]{(x-2)^4}$; **(z)** $y = \frac{10}{9}x - \sqrt[3]{(x-2)^5}$;

Zad 8. Zbadaj przebieg zmienności funkcji i naszkicuj jej wykres:

(a) $y = \frac{4-x^2}{x^2-1}$; **(b)** $y = \frac{x^3}{x-1}$; **(c)** $y = x^x$; **(d)** $y = x\ln^2 x$; **(e)** $y = \frac{x\sqrt{x^2-9}}{x-3}$; **(f)** $y = \frac{2+x-x^2}{x-3}$; **(g)** $y = (x+3)e^{\frac{x+1}{x-1}}$

Przebieg zmienności funkcji - odpowiedzi

Zad 1.

(a) $x \in (-1,0) \cup (1,\infty)$; **(b)** $x \in (0,e)$;

Zad 2.

(a) $\lim\limits_{x\to 0^+} f(x) = 0$, $\lim\limits_{x\to 1} f(x) = 1$, $\lim\limits_{x\to\infty} f(x) = \infty$; **(b)** $\lim\limits_{x\to 0^+} f(x) = 0$, $\lim\limits_{x\to 1} f(x) = 1$, $\lim\limits_{x\to\infty} f(x) = \infty$;

Zad 3.

(a) $y = -x - 1$; **(b)** $y = x$; **(c)** $x = 0$; **(d)** $x = -3$, $y = 1$;
(e) $x = 0$, $y = x$; **(f)** $x = -3$, $y = 1$; **(g)** $x = 1$, $y = x + 2$; **(h)** ;
(i) $x = 2$, $y = x + 2$; **(j)** $x = -1$, $y = x + 1$; **(k)** $x = 0$, $y = -x + 1$; **(l)** $y = x - 1$;
(m) $x = 1$, $y = 2x$;

Zad 4.

(a) $f \nearrow: x \in (-\infty,0) \vee x \in (1,\infty)$, $f \searrow: x \in (0,1)$, $f_{\max} = (0,0)$, $f_{\min} = (1,-1)$;
(b) $f \searrow: x \in (0,\frac{1}{\sqrt{e}})$, $f \nearrow: x \in (\frac{1}{\sqrt{e}},\infty)$, $f_{\min} = \left(\frac{1}{\sqrt{e}}, -\frac{1}{2e}\right)$;
(c) $f \nearrow: x \in \mathbb{R}$, brak ekstremów;
(d) $f \nearrow: x \in (-\infty,0)$, $f \searrow: x \in (0,\infty)$, $f_{\max} = (0,1)$;
(e) $f \nearrow: x \in (0,e^2)$, $f \searrow: x \in (e^2,\infty)$, $f_{\max} = \left(e^2, \frac{2}{e}\right)$;
(f) $f \nearrow: x \in (0,\frac{1}{e}) \vee x \in (e,\infty)$, $f \searrow: x \in (\frac{1}{e},1) \vee x \in (1,e)$, $f_{\max} = (e^{-1},-2)$, $f_{\min} = (e,2)$;
(g) $f \nearrow: x \in (-\frac{2}{3},0)$, $f \searrow: x \in (-\infty,-1) \vee x \in (-1,-\frac{2}{3}) \vee x \in (0,\infty)$, $f_{\min} = \left(-\frac{2}{3}, e^{\frac{27}{4}}\right)$;
(h) $f \nearrow: x \in (-\infty,-1) \vee x \in (3,\infty)$, $f \searrow: x \in (-1,3)$, $f_{\min} = (3,-47)$, $f_{\max} = (-1,17)$;
(i) $f \nearrow: x \in (e,\infty)$, $f \searrow: x \in (0,1) \vee x \in (1,e)$, $f_{\min} = (e,e)$;
(j) $f \nearrow: x \in (0,2)$, $f \searrow: x \in (-\infty,0) \vee x \in (2,\infty)$, $f_{\min} = (0,0)$, $f_{\max} = \left(2, \frac{4}{e^2}\right)$;
(k) $f \nearrow: x \in (-\infty,1)$, $f \searrow: x \in (1,\infty)$, $f_{\max} = \left(1, \frac{1}{e}\right)$;
(l) $f \nearrow: x \in (-\infty,-1) \vee x \in (1,\infty)$, $f \searrow: x \in (-1,1)$, $f_{\min} = \left(1, 1 - \frac{\pi}{2}\right)$, $f_{\max} = \left(-1, \frac{\pi}{2} - 1\right)$;

Zad 5.

(a) $f\smile: x \in (-\infty,\frac{1}{2})$, $f\frown: x \in (\frac{1}{2},\infty)$, $P_p = (\frac{1}{2}, e^{\arctan\frac{1}{2}})$;
(b) $f\smile: x \in (-\infty,-2) \vee x \in (2,\infty)$, brak Pp;
(c) $f\smile: x \in (2,\infty)$, $f\frown: x \in (-\infty,2)$, $P_p = (2, \frac{2}{e^2})$;
(d) $f\smile: x \in (-\infty, 2-\sqrt{2}) \vee x \in (2+\sqrt{2},\infty)$, $f\frown: x \in (2-\sqrt{2}, 2+\sqrt{2})$,
$P_{p1} = (2-\sqrt{2}, \left(2-\sqrt{2}\right)^2 e^{-2+\sqrt{2}})$, $P_{p2} = (2+\sqrt{2}, \left(2+\sqrt{2}\right)^2 e^{-2-\sqrt{2}})$
(e) $f\smile: x \in (e^{1-\sqrt{\frac{5}{3}}}, e^{1+\sqrt{\frac{5}{3}}})$, $f\frown: x \in (0, e^{1-\sqrt{\frac{5}{3}}}) \vee x \in (e^{1+\sqrt{\frac{5}{3}}}, \infty)$,
$P_{p1} = (e^{1-\sqrt{\frac{5}{3}}}, 4 - \frac{8\sqrt{\frac{5}{3}}}{3})$; $P_{p2} = (e^{1+\sqrt{\frac{5}{3}}}, 4 + \frac{8\sqrt{\frac{5}{3}}}{3})$
(f) $f\smile: x \in (2,\infty)$, $f\frown: x \in (-\infty,2)$, $P_p(1,8)$;
(g) $f\smile: x \in (0,\infty)$, $f\frown: x \in (-\infty,-1) \vee x \in (-1,0)$, $P_p(0,0)$;
(h) $f\smile: x \in (1,\infty)$, $f\frown: x \in (-\infty,1)$, $P_p(1,-1)$;
(i) $f\smile: x \in (1,\infty)$, $f\frown: x \in (0,1)$, $P_p(1,-7)$;
(j) $f\smile: x \in (-\frac{1}{2},0) \vee x \in (0,\infty)$, $f\frown: x \in (-\infty,-\frac{1}{2})$, $P_p(-\frac{1}{2}, e^{-2} + \frac{1}{2})$;
(k) $f\smile: x \in (0, 3-\sqrt{3}) \vee x \in (3+\sqrt{3},\infty)$, $f\frown: x \in (-\infty,0) \vee x \in (3-\sqrt{3}, 3+\sqrt{3})$,
$P_{p1}(0,0), P_{p2}(3+\sqrt{3}, 6\left(9+5\sqrt{3}\right) e^{-3-\sqrt{3}}), P_{p3}(3-\sqrt{3}, -6\left(-9+5\sqrt{3}\right) e^{-3+\sqrt{3}})$;
(l) $f\smile: x \in (-4,-2) \vee x \in (-2,2) \vee x \in (2,\infty)$, $f\frown: x \in (-\infty,-3)$, $P_p(-4, -4\left(\frac{2}{5}\right)^{2/3})$;

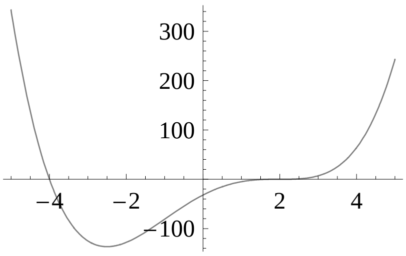
Zad 6.

(a) $f_{\min} = (3, -89)$, $f_{\max} = (6, 100)$; **(b)** $f_{\min} = (1, -1)$, $f_{\max} = (5, 5 - 2\sqrt{5})$;

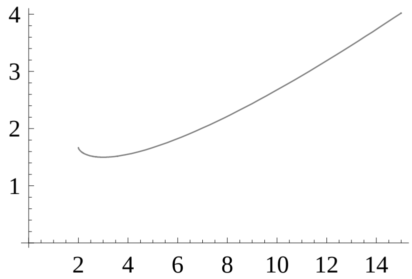
(c) $f_{\min} = \left(\frac{3}{2}\pi, -2\right)$, $f_{\max} = \left(\frac{\pi}{3}, \frac{3}{2}\sqrt{3}\right)$;

Zad 7.

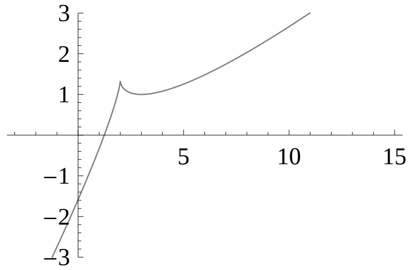
(a)



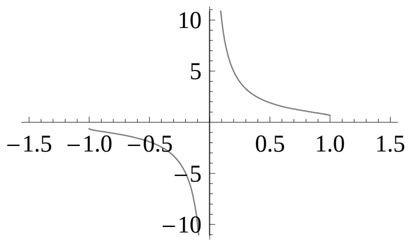
(b)



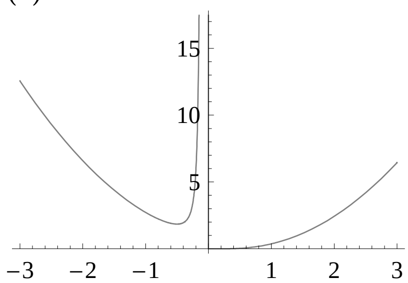
(c)



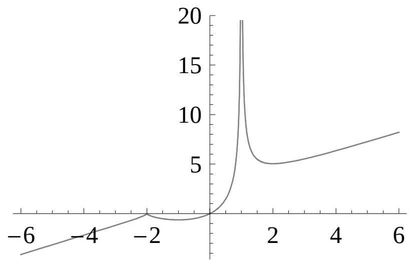
(d)



(e)



(f)



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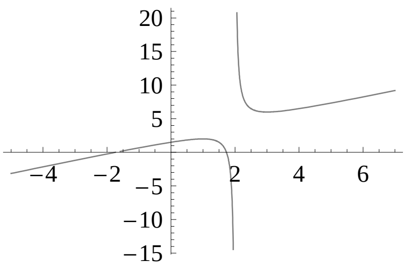

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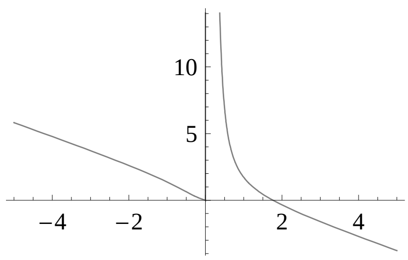
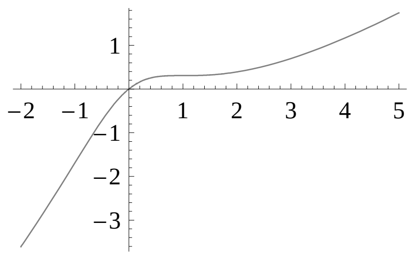


(i)



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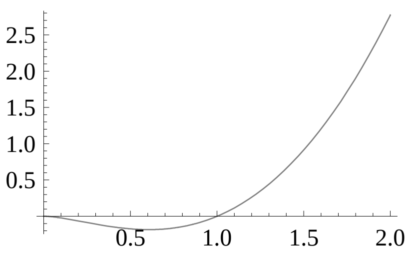

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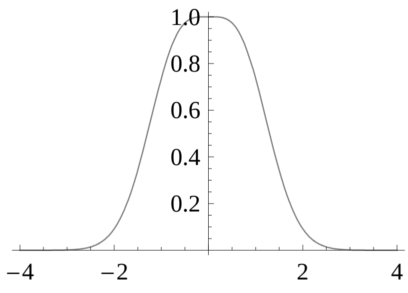

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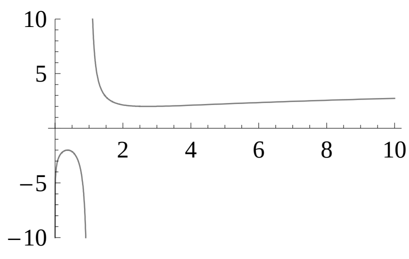

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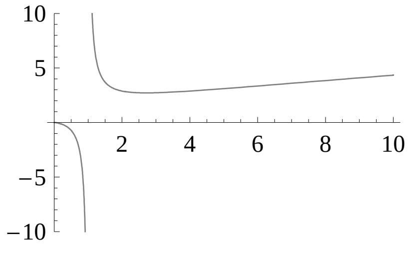

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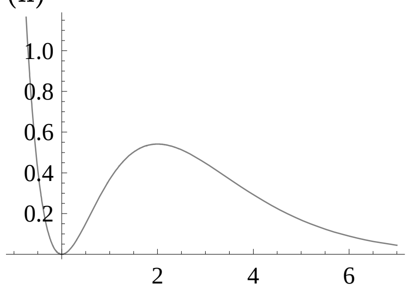

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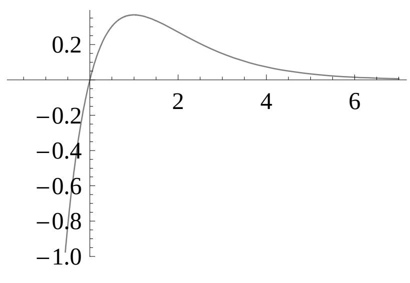
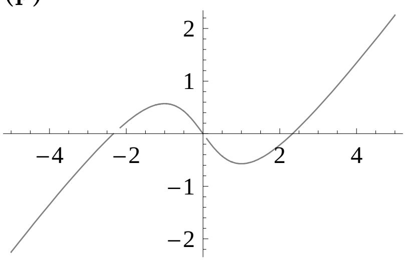
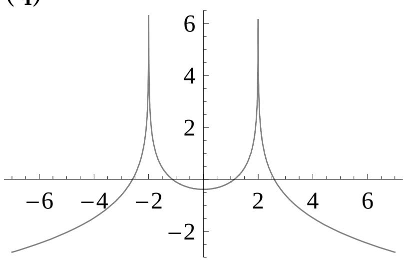


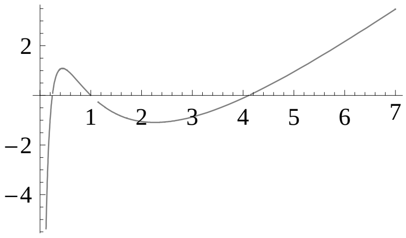
(p)


2
1
−4
−2
2
4
−1
−2

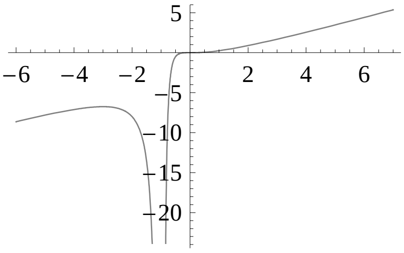
(q)


6
4
2
−6
−4
−2
2
4
6
−2

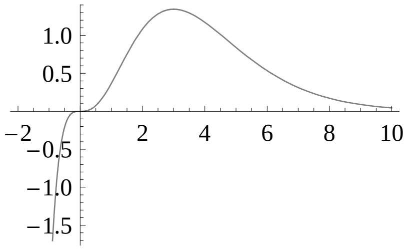
(r)


2
1
2
3
4
5
6
7
−2
−4

(s)


5
−6
−4
−2
2
4
6
−5
−10
−15
−20

(t)


1.0
0.5
−2
2
4
6
8
10
−0.5
−1.0
−1.5

(u)