

Elementary Model Theory in Rocq

Current Progress Report

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1 Introduction

In math, we study different types of structures and how they interact with others

For example: group theory is the study of Groups, and field theory is the study of Fields

Model theory is the study of mathematical structures as a whole

A model is a representation of a mathematical theory

A mathematical theory is a collection of axioms - formal statements in a language

Let's consider groups as a simple example

A “group” is a set, G , along with a multiplication function $\odot : G \times G \rightarrow G$ that obeys the following axioms:

- $\forall a, b, c \in G, a \odot (b \odot c) = (a \odot b) \odot c$
- $\exists e$ such that $\forall a \in G, e \odot a = a \odot e = a$
- $\forall a \in G, \exists b$ such that $a \odot b = b \odot a = e$

One model of a group would be \mathbb{Z} - the integers

One can check it obeys all the axioms

Model theory is about arbitrary collections of functions, axioms and their models

Questions we want to answer are “what can we prove using these axioms?” and “is a given theory consistent?”

- Implement Elementary Model Theory in Rocq
 - Gödel's First Incompleteness Theorem
 - Łoś's Theorem (ultraproducts)
 - Löwenheim–Skolem Theorem
 - Concerning completeness/compactness
 - Stability

Proofs requires non-trivial definitions, theorems and lemmas.

We have the basics of Peano arithmetic in the Rocq standard library

Additionally, there was a crowdsourced effort that ended ~3 years ago which was able to prove Godel's first incompleteness theorem: <https://github.com/rocq-community/goedel/>

From what he have seen, it is very difficult to find Model Theory libraries and or implementations in Rocq.

- Why?

2 Our Progress

Definitions so Far:

- First-order language: \mathcal{L}
 - Logical symbols of \mathcal{L} : $\text{Var}, \doteq, \wedge, \vee, \neg, \rightarrow, \leftrightarrow, \forall, \exists$
 - Signature of \mathcal{L} : $\text{Const}(\mathcal{L}), \text{Rel}(\mathcal{L}), \text{Fun}(\mathcal{L})$
- Terms of \mathcal{L} or \mathcal{L} -terms
- Formulas of \mathcal{L} or \mathcal{L} -formulas

Sections:

- Languages
- Semantics

- Translating definitions into Rocq:
 - Enforcing properties?
- Weak definitions and or induction schema
- Proving simple things are hard
- Proving in Rocq requires deep knowledge of proof.
 - Learning Model Theory
- Getting stuck

3 What's Next

Fallbacks:

- Focus on most important results:
 - Gödel's Results
 - Completeness/Compactness

- Rocq's "Limitations"
- No Sandboxing
- Acknowledging Previous Efforts

