Relations and Their Properties

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February 19, 2020

Introduction

Defintion

Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

a R b denotes that $(a, b) \in R$

Relations on a Set

Definition

A *relation on the set A* is a relation from A to A.

Let $A = \{1, 2, 3, 4\}$

Example 1: $R = \{(a, b) | a \text{ divides } b\}$. What are the elements of R?

a divides b means b is divisible by a.

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$$

Example 2: $S = \{(a, b) | a + b \leq 3\}$

$$S = \{(1,1),(1,2),(2,1)\}$$

Relations on a Set

How many relations are there on a set with n elements?

- A relation on set A is formed by $A \times A$.
  $|A \times A| = n^2$
- A relation on set A is a subset of $A \times A$.
  $|P(A \times A)| = 2^{n^2}$

$\boxed{\text{Thus, there are } 2^{n^2} \text{ relations on a set with } n \text{ elements.}}$

Reflexive

Definition

A relation R on a set A is *reflexive*, if $(a, a) \in R$ for every element $a \in A$.

$\forall a \, ((a, a) \in R)$

Let $A = \{a, b, c\}$, and R are relations on set A.
Example 1: $R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (c, c)\}$

reflexive

Example 2: $R = \{(a, a), (a, c), (b, b), (c, a)\}$

not reflexive, *since* $(c, c) \notin R$

Symmetric

Definition

A relation R on a set A is *symmetric*, if $(b, a) \in R$ whenever $(a, b) \in R$.

$\forall a \forall b((b, a) \in R \rightarrow (a, b) \in R)$

Let $A = \{a, b, c\}$, and R are relations on set A.
Example 1: $R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (c, c)\}$

not symmetric, *since* $(a, c) \in R$, *but* $(c, a) \notin R$

Example 2: $R = \{(a, a), (a, c), (b, b), (c, a)\}$

symmetric

Antisymmetric

Definition

A relation R on a set A is *antisymmetric*, when for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

$\forall a \forall b(((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b))$

Let $A = \{a, b, c\}$, and R are relations on set A.
Example 1: $R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (c, c)\}$

not antisymmetric, *since $(a, b) \in R$ and $(b, a) \in R$, but $a \neq b$.*

Example 2: $R = \{(a, a), (a, c), (b, b), (b, c)\}$

antisymmetric

Transitive

Definition

A relation R on a set A is *transitive*, if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$, for every element $a, b, c \in A$.

$\forall a \forall b \forall c(((a, b) \in R \land (b, c) \in R) \to (a, c) \in R)$

Let $A = \{a, b, c\}$, and R are relations on set A.
Example 1: $R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, c)\}$

transitive

Example 2: $R = \{(a, a), (a, c), (b, b), (b, c)\}$

transitive

Properties of a Relation

Let R be a relation on set A, and $a, b, c \in A$

Reflexive $\forall a((a, a) \in R)$

Check if all elements of the set A has a reflexive pair (a, a) in the relation.

Symmetric $\forall a \forall b((b, a) \in R \rightarrow (a, b) \in R)$

Check the elements of R. For all pairs (a, b) in R, you must find (b, a) also in R.

Antisymmetric $\forall a \forall b(((a, b) \in R \land (b, a) \in R) \rightarrow (a = b))$

Check the elements of R. You must not find both (a, b) and (b, a) pairs in R where $(a \neq b)$.

Transitive $\forall a \forall b \forall c(((a, b) \in R \land (b, c) \in R) \rightarrow (a, c) \in R)$

Check the elements of R. For every (a, b) and (b, c) pair you find in R, you must also find (a, c) in R.

Exercise 1

Let $A = \{1, 2, 3, 4\}$

Given the following relations on set A, which are reflexive, symmetric, antisymmetric and transitive?

- $R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$
- $R_2 = \{(1,1),(1,2),(2,1)\}$
- $R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$
- $R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$
- $R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$
- $R_6 = \{(3,4)\}$
- $R_7 = \{(1,1),(2,2),(3,3),(4,4)\}$
- $R_8 = \{(2,2),(4,4)\}$

Exercise 2

Determine whether the relations on the set of integers below are reflexive, symmetric, antisymmetric and/or transitive. For each property not satisfied, give a counter example.

1. $R = \{(a, b) | a = b \vee a = -b\}$
2. $R = \{(a, b) | a < b\}$
3. $R = \{(a, b) | a \textit{ divides } b\}$

References

[Rosen, 2007] Kenneth Rosen.
Discrete Mathematics and Its Applications 7^{th} edition, 2007