Sums and Sequences

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Sequences

- a discrete stucture to represent an *ordered list*.
  - finite sequence
  - infinite sequence

Definition

A *sequence* is a function from a subset of the set of integers.

The notation $\{a_n\}$ is used to describe a sequence, and

a_n is the n^{th} *term* in the sequence.

Example

- $\{a_n\}$, where $a_n = \dfrac{1}{n}$. Give the first 5 terms in the sequence starting with a_1.
- $\{b_n\}$, where $b_n = (-1)^n$. Give the first 8 terms in the sequence where n starts at 0.
- $\{c_n\}$, where $c_n = 1 + 3n$. Give the first 8 terms in the sequence where n starts at 0.

Geometric Progression

Definition

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \cdots, ar^n, \cdots$$

where *initial term* a, and *common ratio* r are both real numbers.

Examples are $\{b_n\}$ and $\{d_n\}$, where $d_n = 2 \cdot 5^n$

Arithmetic Progression

Definition

A *artithmetic progression* is a sequence of the form

$$a, a+d, a+2d, \cdots, a+nd, \cdots$$

where *initial term* a, and *common difference* d are both real numbers.

Examples are $\{c_n\}$ and $\{t_n\}$, where $t_n = 4 - 5n$

Exercise 1

Determine the next 3 terms and the formula for each of the following sequences.

1. $1, 3, 5, 7, \cdots$
2. $2, 4, 8, 16, \cdots$
3. $2, -4, 8, -16, \cdots$
4. $2, 3, 7, 25, 121, \cdots$
5. $1, 3, 9, 27, \cdots$

Recurrence Relations

Definition

A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence.

A sequence that satisfies the recurrence relation is called a *solution*.

Example: $a_n = a_{n-1} + x$, $a_n = 3a_{n-1} + 2$

Exercise 2

Determine the recurrence relation for each of the following sequences.

1. $1, 3, 5, 7, \cdots$
2. $2, 4, 8, 16, \cdots$
3. $0, 1, 1, 2, 3, 5, 8, \cdots$
4. $1, 1, 2, 6, 24, \cdots$
5. $1, 3, 9, 27, \cdots$

Exercise 3

Determine the next term and the formula for a_n for each of the following sequences.

1. $5, 11, 29, 83, \cdots$
2. $1, 9, 17, 25, 33, \cdots$
3. $-1, 1, 9, 45, 237, \cdots$
4. $2, 10, 24, 44, \cdots$

Summation Notation

The sum of the terms $a_m, a_{m+1}, a_{m+2}, \cdots, a_n$ may be written as

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

may also be written as:

$$\textstyle\sum_{j=m}^{n} a_j \qquad \displaystyle\sum_{j=m}^{n} a_j \qquad \sum_{m \le j \le n} a_j$$

Summation Notation

$$\sum_{j=m}^{n} a_j$$

- Greek letter for uppercase S, summation symbol
- summand, addend
- index of the summation
- lower limit
- upper limit

Read as: *The summation of* a_j *where* j *is from* m *to* n.

Examples

1. $$\sum_{j=1}^{50} j = 1 + 2 + 3 + \cdots + 50$$

2. $$\sum_{j=1}^{50} (j+1) = 2 + 3 + 4 + \cdots + 51$$

3. $$\sum_{j=1}^{50} 2j = 2 + 4 + 6 + \cdots + 100$$

4. $$\sum_{j=1}^{50} j^2 = 1 + 4 + 9 + \cdots + 2500$$

Summations and their Closed Form

$$\sum_{j=m}^{n} 1 = n - m + 1$$

$$\sum_{j=m}^{n} c = c(n - m + 1)$$

(c is any constant)

$$\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1}$$

($r \neq 1$)

$$\sum_{j=0}^{n} j = \frac{n(n+1)}{2}$$

$$\sum_{j=0}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{j=0}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

Summations and their Closed Form - Simplified

$$\sum_{j=m}^{n} 1 = n - m + 1$$

$$\sum_{j=0}^{n} r^j = \frac{r^{n+1} - 1}{r - 1}$$

$(r \neq 1)$

$$\sum_{j=0}^{n} j = \frac{n(n+1)}{2}$$

$$\sum_{j=0}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{j=0}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

Exercise 1

1. $\displaystyle\sum_{j=0}^{8} 4$
2. $\displaystyle\sum_{j=0}^{n} n$
3. $\displaystyle\sum_{j=0}^{n} n^3$
4. $\displaystyle\sum_{j=0}^{n} 2j$
5. $\displaystyle\sum_{j=0}^{n} 2j^2$

Exercise 2

1. $\sum_{j=0}^{n+5} j^3$
2. $\sum_{j=1}^{2n} 3j$
3. $\sum_{j=0}^{n^2} (j^2 + 3j) \quad = \sum_{j=0}^{n^2} j^2 + \sum_{j=0}^{n^2} 3j$
4. $\sum_{j=0}^{n} (k+3)^2 \quad = (k+3)^2 \sum_{j=0}^{n} 1$

Exercise 3

1. $\sum_{j=0}^{n} 3k \qquad = 3k \sum_{j=0}^{n} 1$

2. $\sum_{j=4}^{10} j \qquad = \sum_{j=0}^{10} j - \sum_{j=0}^{3} j$

$$\sum_{j=0}^{10} j = 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

$$-\left(\sum_{j=0}^{3} j = 0 + 1 + 2 + 3 \right)$$

$$\sum_{j=4}^{10} j = 4 + 5 + 6 + 7 + 8 + 9 + 10$$

Double Summation

To evaluate double (or multiple) summations, expand the inner (rightmost) summation first. Continue evaluating until all summations have been evaluated.

$$
\begin{aligned}
\sum_{j=1}^{n}\sum_{k=1}^{n} jk \quad = \sum_{j=1}^{n} j \sum_{k=1}^{n} k \quad &= \sum_{j=1}^{n} j \frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{2} \sum_{j=1}^{n} j \\
&= \frac{n(n+1)}{2} \cdot \frac{n(n+1)}{2} \\
&= \frac{n^2(n+1)^2}{4} \\
&= \frac{n^2(n^2+2n+1)}{4} \\
&= \frac{n^4+2n^3+n^2}{4}
\end{aligned}
$$

Exercise 4

Find the next term and a_n where $n \geq 1$ of each sequence below.

1. $3, 5, 13, 49, 241, \ldots$
2. $9, 24, 49, 84, 129, \ldots$
3. $2, 5, 10, 17, 26, \ldots$

Exercise 5

Evaluate the following

1. $$\sum_{j=0}^{8} 2j$$
2. $$\sum_{k=2}^{n} (2k+n)^2$$
3. $$\sum_{j=1}^{4} \sum_{k=0}^{3n} (k+j)$$
4. $$\sum_{j=0}^{8} 3 \cdot 2^j$$

Practice Makes Perfect

Image courtesy of `http://fridayreflections.typepad.com/weblog/2008/08/practice-makes.html`

References

[Rosen, 2007] Kenneth Rosen.
Discrete Mathematics and Its Applications 7^{th} edition, 2007