Predicate Logic

Shirley Chu

De La Salle University

updated February 11, 2020

Introduction

Predicate Logic

- a more powerful type of logic
- allow us to reason and explore object relationships
- Concepts:
  - Predicates
  - Quantifiers

Introduction

Propositions or not?

- $x + y = 25$
- Let $x = 8, xy = 1$
- x is my favorite subject.

Predicates

Given the statement,

x is greater than 5.

Variable the subject of the statement.

Predicates refers to the property that the subject of the statement can have.

- Let P be the predicate *is greater than 5*.
- The statement can be represented as: $P(x)$

Propositional Functions

Let $P(x)$ be x *is greater than 5*.

- Is $P(x)$ a proposition? **No!**
- $P(x)$ is a proposition once x **has a value**.
  Example:
  - $P(4) = F$
  - $P(10) = T$

$P(x_1, x_2, \cdots, x_n)$ is value of the **Propositional Function** P at the n-tuple $(x_1, x_2, \cdots, x_n)$.

P is also called *n-ary predicate* or *n-place predicate*.

Convention: Uppercase letters for propositional functions, and lowercase letters for propositions.

Remember!

- A propositional function whose variables do not have specific values is **not** a proposition.
- There are two ways wherein we can convert propositional functions into propositions.
  1. by value assignment.
     Example: $P(5)$, $P(-1)$
  2. through *quantification*

Definition

Quantification expresses the extent to which the predicate is true over a range of elements.

The use of English words *some*, *all*, *none* are used in quantification.

We study two types of quantification:

1. Universal Quantification
2. Existential Quantification

Quantification

Definition

Universal Quantification expresses that the predicate is true for all elements in a given domain.

English words used to express such are *all*, *every*, *each*.

The universal quantification of $P(x)$ is the statement:

- In English: $P(x)$ *is true for all values of* x *in the domain.*
- Notation: $\forall x P(x)$

$\forall x P(x)$

$\forall x P(x)$

Read as *For all x, P(x).*

Meaning $P(x)$ is true for every element of x in the domain.

$\equiv$ $P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \cdots \wedge P(x_n)$, where $x_1, x_2, \cdots, x_n$ are all the elements in the domain.

Note:

- $\forall$ is called the *Universal Quantifier*.
- The *domain* of the statement is also called the *universe of discourse*. It specifies all the possible values of the variable.
- The meaning of the statement changes when the domain changes.

Example

Let $P(x)$ be $x > 5$. Determine the truth value of $\forall x P(x)$ when the domain for x is:

1. $\{8, 10, 12, 14, 16\}$
   $P(8) \wedge P(10) \wedge P(12) \wedge P(14) \wedge P(16) \equiv T$
2. $\{5, 6, 7, 8, 9, 10\}$
   $P(5) \wedge P(6) \wedge P(7) \wedge P(8) \wedge P(9) \wedge P(10) \equiv F$
   $P(5)$ is the *counterexample*.
3. the set of all positive integers.
4. the set of all integers.

Definition

An element for which $P(x)$ is false is called a *counterexample*.

Quantification

Definition

Existential Quantification expresses that the predicate is true for at least one of the elements in a given domain.

English words used to express such are *some*, *at least one*, *there exists*.

The existential quantification of $P(x)$ is the statement:

- In English: $P(x)$ *is true for at least one value of* x *in the domain.*
- Notation: $\exists x P(x)$

$\exists x P(x)$

$\exists x P(x)$

Read as	*There exists x, $P(x)$.*
Meaning	$P(x)$ is true for at least one element of x in the domain.
$\equiv$	$P(x_1) \vee P(x_2) \vee P(x_3) \vee \cdots \vee P(x_n)$, where $x_1, x_2, \cdots, x_n$ are all the elements in the domain.

Note:

- $\exists$ is called the *Existential Quantifier*.
- The *domain* of the statement is also called the *universe of discourse*. It specifies all the possible values of the variable.
- The meaning of the statement changes when the domain changes.

Example

Let $P(x)$ be $x > 5$. Determine the truth value of $\exists x P(x)$ when the domain for x is:

1. $\{8, 10, 12, 14, 16\}$
   $P(8) \vee P(10) \vee P(12) \vee P(14) \vee P(16) \equiv T$
2. $\{5, 6, 7, 8, 9, 10\}$
   $P(5) \vee P(6) \vee P(7) \vee P(8) \vee P(9) \vee P(10) \equiv T$
3. the set of all positive integers.
4. the set of all negative integers.

Note that the statement $\exists x P(x)$ is false if and only if **no** element x in the domain makes $P(x)$ true.

General Note

- Generally, an implicit assumption is made that *No universe of discourse is empty.* However, if the domain is empty:

  $\forall x P(x) \equiv T$, since no element x in the domain where $P(x)$ can be false.

  $\exists x P(x) \equiv F$, since no element x in the domain where $P(x)$ can be true.

- Avoid using the word *any*.

General Note

- A propositional function becomes a proposition when all its variables are *bound*, or *given specific values*.

Definition

When a variable in a propositional function quantified, this variable is *bound*. If not, the variable is *free*.

Example:

$\forall x Q(x)$ is a proposition. The variable x is bound by the universal quantifier.

$\exists x R(x, y)$ is not a proposition. Though the variable x is bound by the existential quantifier, the other variable y is free.

$\forall x \exists y Q(x, y)$ is a proposition. Both variables x and y are bound.

Precedence and Equivalences

- The quantifiers $\forall$ and $\exists$ have higher precedence over the logical operators from propositional calculus.
  Note that $\forall x P(x) \land Q(x) \not\equiv \forall x (P(x) \land Q(x))$.
- Logical Equivalences
  - You can distribute **universal quantifier** over a **conjunction**.
    $\forall x (P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$
  - You can distribute **existential quantifier** over a **disjunction**.
    $\exists x (P(x) \lor Q(x)) \equiv \exists x P(x) \lor \exists x Q(x)$
  - De Morgan's Laws for Quantifiers
    $\neg \forall x P(x) \equiv \exists x \neg P(x)$
    $\neg \exists x P(x) \equiv \forall x \neg P(x)$

Negation Exercises

Simplify the following by removing the negation using De Morgan's Laws.

1. $\neg \exists x (P(x) \wedge Q(x))$
2. $\neg \forall x (\neg Q(x) \rightarrow P(x))$

Translating English Sentences to Logical Expressions

Steps:

1. Identify the predicate.
2. Identify the universe of discourse.

   Note: You may have to include additional predicates, depending on the domain you've specified.
3. Translate.

Translation 1

Example 1: **All students studied for the exam.**

Translation: $\forall x S(x)$

Example 2: **Some students studied for the exam.**

Translation: $\exists x S(x)$

Example 3: **Sarah studied for the exam.**

Translation: $S(Sarah)$

Let:

$S(x)$ x studied for the exam.

The universe of discourse for x is the set of students.

During translation:

1. If the subject is a specific value (i.e. not quantified), no need to use quantifiers! Use the value as the parameter and not the variable.

Translation 2

Let:

$S(x)$ x studied for the exam.
$C(x)$ x is a CS student.

The universe of discourse for x is the set of students.

Example 4: **All CS students studied for the exam.**

Translation: $\forall x(C(x) \rightarrow S(x))$

Note: When the domain specified is different from the subject of the statement,

- update your list of predicates to include the predicate for the subject.
- during translation:
  1. If universal quantifier is used in the sentence, use **implication** $\rightarrow$, where the subject is the premise and the rest of the sentence is the conclusion.
  2. Do not forget to put parenthesis whenever necessary.

Translation 3

Let:

$S(x)$ x studied for the exam.

$C(x)$ x is a CS student.

The universe of discourse for x is the set of students.

Example 5: **Some CS students studied for the exam.**

Translation: $\exists x(C(x) \wedge S(x))$

During translation:

1. If existential quantifier is used in the sentence, use **conjunction** $\wedge$, where the subject is the an operand in the expression *and-ed* with the rest of the predicates in the sentence.
2. Do not forget to put parenthesis whenever necessary.

Exercise 1

Evaluate the following. Determine the truth value of each of the statements below.

1. Let $C(x)$ denote the statement "*The word x contains the letter a.*"
    1. $C(mango)$
    2. $C(lemon)$
    3. $C(pomegranate)$

    - *What is the domain of x?*
2. Let $P(x)$ denote the statement "$x^2 > 0$", where the domain of x is $\mathbb{Z}$.
    1. $P(4)$
    2. $P(-5)$
    3. $\forall a P(a)$
    4. $\neg \exists y P(y)$

Exercise 2

Simplify the following expressions by removing the negation beside the quantifiers.

1. $\neg\forall x(P(x) \rightarrow (S(x) \wedge Q(x)))$
2. $\neg\exists x\neg(P(x) \vee Q(x) \vee \neg(Q(x) \wedge \neg P(x)))$

Exercise 3

Translate the following English statements to logical expressions.

1. Every student in this class has passed CCSALGE.
2. Some freshmen students are taking COMPRO2.
3. No one answered the exercises.
4. Not everyone answered the exercises.

Just a Note

Definition

The area of logic that deals with predicates and quantifiers is called *Predicate Calculus*.

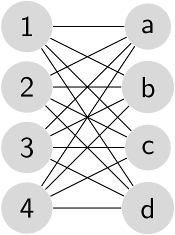
Nested Quantifiers

- Two quantifiers are nested if one is within the scope of the other.
- Examples:
  - $\forall x \exists y (x + y = 0)$
  - $\exists x \forall y P(x, y)$

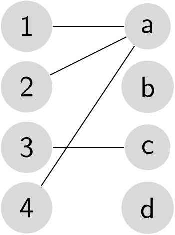
Nested Quantifiers

Let the domain for x be $\{1, 2, 3, 4\}$ and the domain for y be $\{a, b, c, d\}$.

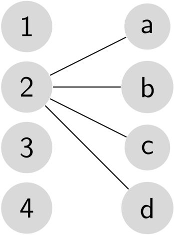
$\forall x \forall y P(x, y)$



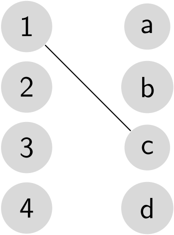
$\forall x \exists y P(x, y)$



$\exists x \forall y P(x, y)$



$\exists x \exists y P(x, y)$



Exercise 4

Given that

$C(x, y)$ x has been a contestant in quiz show y.

Translate the following sentences to expressions where the domain of x in $C(x, y)$ is the set of all students at your school and the domain of y in $C(x, y)$ is the set of all quiz shows on tv.

1. There is a student at your school who has been a contestant on a television quiz show.
2. No student at your school who has ever been a contestant on a television quiz show.
3. There is a student at your school who has been a contestant on *Jeopardy* and on *Wheel of Fortune*.
4. Every television quiz show has a student from your school as a contestant.

Exercise 5

Determine the truth value of each expression, given that the domain of all the variables is the set of real numbers.

1. $\forall x \exists y (x^2 = y)$
2. $\forall x \exists y (x = y^2)$
3. $\forall y \exists x (x = y^2)$
4. $\exists m \forall n (mn = 0)$
5. $\exists a \exists b (a + b \neq b + a)$
6. $\forall x \exists y (x + y = 1)$

References

[Rosen, 2007] Kenneth Rosen.
Discrete Mathematics and Its Applications 7^{th} edition, 2007