Inference Rules

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Introduction

Definition

An *argument* in propositional logic is a sequence of propositions. All, except the last, statements are called the *premises*. The final argument in the list is called *conclusion*.

Definition

An argument is *valid* if the truth of all premises implies that the conclusion is true.

Rules of Inference

Conjunction

$$\begin{array}{l} p \\ \underline{q} \\ \therefore p \land q \end{array}$$

Addition

$$\begin{array}{l} \underline{p} \\ \therefore p \lor q \end{array}$$

Simplification

$$\begin{array}{l} \underline{p \land q} \\ \therefore p \end{array}$$

Resolution

$$\begin{array}{l} p \lor q \\ \underline{\neg p \lor r} \\ \therefore q \lor r \end{array}$$

Rules of Inference

Hypothethical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Modus Ponens

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Disjunctive Syllogism

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

Modus Tollens

$$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$$

Example

It is not sunny this afternoon and it is colder than yesterday. We will go swimming only if it is sunny. If we do not go swimming then we will take a canoe trip. If we take a canoe trip, then we will be home by sunset. Therefore, we will be home by sunset.

Let

		Translation:
s	*It is sunny this afternoon.*	$\neg s \wedge c$
c	*It is colder than yesterday.*	$w \rightarrow s$
w	*We go swimming.*	$\neg w \rightarrow t$
t	*We will take a canoe trip.*	$t \rightarrow h$
h	*We will be home by sunset.*	Goal: h

Example

Goal: h

Statement	Reason
1 $\neg s \wedge c$	given
2 $w \rightarrow s$	given
3 $\neg w \rightarrow t$	given
4 $t \rightarrow h$	given
5 $\neg s$	simplification (1)
6 $\neg w$	modus tollens (5, 2)
7 $\neg w \rightarrow h$	hypothetical syllogism (3, 4)
8 h	modus ponens (6, 7)

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Exercise 1

1. If you send me an email message, then I will finish writing the program. If you do not send me an email message, then I will go to sleep early. If I go to sleep early, then I will wake up feeling refreshed. Therefore, If I do not finish writing the program, then I will wake up feeling refreshed.
2. Randy works hard. If Randy works hard, then he is a dull boy. If Randy is a dull boy, then he will not get the job. Therefore, Randy will not get the job.
3. If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on. If the sailing race is held, then the trophy will be awarded. The trophy was not awarded. Therefore, it rained.

Exercise 1

4. If Claghorn has wide support, then he'll be asked to run for the senate. If Claghorn yells *Eureka* in Iowa, he will not be asked to run for the senate. Claghorn yells *Eureka* in Iowa. Therefore, Claghorn does not have wide support.

5. $$p$$
$$p \to q$$
$$s \vee r$$
$$r \to \neg q$$
$$\therefore\ s \vee t$$

Rules of Inference for Quantified Statements

Let $c \in \mathcal{U}$

Universal Instantiation

$$\frac{\forall x\ \mathrm{P}(x)}{\therefore \mathrm{P}(c)}$$

Existential Instantiation

$$\frac{\exists x\ \mathrm{P}(x)}{\therefore \mathrm{P}(c) \text{ for some element } c}$$

Universal Generalization

$$\frac{\mathrm{P}(c) \text{ for an arbitrary } c}{\therefore \forall x\ \mathrm{P}(x)}$$

Existential Generalization

$$\frac{\mathrm{P}(c) \text{ for some element } c}{\therefore \exists x\ \mathrm{P}(x)}$$

Example 1

Everyone in this class has taken CCSALGE. Tomy is a student in this class. Therefore, Tomy has taken CCSALGE.

Let

the domain for x be all students, and:

$A(x)$ *x has taken CCSALGE.*

$C(x)$ *x is in this class.*

Translation:

$\forall x(C(x) \to A(x))$

$C(Tomy)$

Goal: $A(Tomy)$

Example 1

Goal: A(Tomy)

Statement	Reason
1 $\forall x(\mathrm{C}(x) \rightarrow \mathrm{A}(x))$	given
2 C(Tomy)	given
3 $\mathrm{C(Tomy)} \rightarrow \mathrm{A(Tomy)}$	universal instantiation (1, 2)
4 A(Tomy)	modus ponens (2, 3)

☑

Example 2

Not all students in this class read the book. Everyone in this class passed the exam. Therefore, someone who passed the exam did not read the book.

Let

the domain for x be all students, and

$R(x)$ *x read the book.*

$P(x)$ *x passed the exam.*

$C(x)$ *x is in this class.*

Translation:

$\neg\forall x(C(x) \rightarrow R(x))$

$\forall x(C(x) \rightarrow P(x))$

Goal:

$\exists x(P(x) \land \neg R(x))$

Exercise 2: **Goal:** $\exists x(\mathbf{P}(x) \wedge \neg \mathbf{R}(x))$

Statement	**Reason**
1 $\neg \forall x(C(x) \rightarrow R(x))$	given
2 $\forall x(C(x) \rightarrow P(x))$	given
3 $\exists x \neg (C(x) \rightarrow R(x))$	De Morgan's (1)
4 $\exists x \neg (\neg C(x) \vee R(x))$	Material Implication (3)
5 $\exists x(C(x) \wedge \neg R(x))$	De Morgan's (4)
6 $C(him) \wedge \neg R(him)$	Existential Instantiation (5, $him \in \mathcal{U}$)
7 $C(him) \rightarrow P(him)$	Universal Instantiation (2)
8 $C(him)$	Simplification (6)
9 $P(him)$	Modus Ponens (8, 7)
10 $\neg R(him)$	Simplification (6)
11 $P(him) \wedge \neg R(him)$	Conjunction (9, 10)
12 $\exists x(P(x) \wedge \neg R(x))$	Existential Generalization (11) ☑

Notes

- All given statements (hypothesis) are assumed to be true.
- New statements are derived from existing statements either by *Rules of Inference* or by *Logical Equivalence Rules*.
- The truth value of each statement in the list must be **true**.
- For quantified statements, whenever applicable, use existential instantiation first before universal instantiation.

Exercise 2

1. No man is an island. Manhattan is an island. Therefore, Manhattan is not a man.
2. Someone in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution.
3. Each of the 93 students in this class owns a personal computer. Everyone who owns a personal computer can use a word processing program. Therefore, Zeke, a student in this class, can use a word processing program.

References

[Rosen, 2007] Kenneth Rosen.
Discrete Mathematics and Its Applications 7^{th} edition, 2007