Summations

Exercise 1. Evaluate the following summations.

1) $\sum_{j=0}^{8} 4 = 4(n-m+1) = 4(8-0+1) = 4(9) = 36$

constant c; upper limit n; lower limit m; $8-0$

2) $\sum_{j=0}^{n} n = n(n-0+1) = n(n+1) = n^2+n$

3) $\sum_{j=0}^{n} n^3 = n^3(n-0+1) = n^3(n+1) = n^4+n^3$

4) $\sum_{j=0}^{n} 2j = 2\sum_{j=0}^{n} j = 2\left(\boxed{\frac{n(n+1)}{2}}\right) = n(n+1) = n^2+n$

$2(0)+2(1)+2(2)+2(3)+\dots+2(n) = 2(0+1+2+3+\dots+n)$

$\frac{0}{2}+\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\dots+\frac{n}{2} = \frac{1}{2}(0+1+2+3+\dots+n)$

5) $\sum_{j=0}^{n} 2j^2 = \boxed{2}\sum_{j=0}^{n} j^2 = 2\left(\boxed{\frac{n(n+1)(2n+1)}{6}}\right) = \frac{(n^2+n)(2n+1)}{3} = \frac{2n^3+3n^2+n}{3}$

$$\sum_{j=m}^{n} 1 = n-m+1 \qquad \sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$$

exponent

$$\sum_{j=0}^{n} j = \frac{n(n+1)}{2} \qquad \sum_{j=0 \text{ or } 1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

base

$$\sum_{j=0 \text{ or } 1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

base

$\sum_{a=1}^{n} a^2,$ base or exponent?

$\therefore$ apply $\sum_{j=0 \text{ or } 1}^{n} j^2$

If summand/addend (formula) is multiplied by some constant, you can factor it out.

Exercise 2 Use the closed-form formulas to evaluate the following summations.

$$\sum_{j=0}^{n} c = c(n-m+1)$$

$$\sum_{j=0/1}^{n} j = \frac{n(n+1)}{2}$$

$$\sum_{j=0/1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{j=0/1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$$

1) $$\sum_{j=0}^{n+5} j^3 = \frac{(n+5)^2(n+5+1)^2}{4} = \frac{(n^2+10n+25)(n^2+12n+36)}{4}$$

$$= \boxed{\frac{n^4+22n^3+181n^2+660n+900}{4}}$$

2) $$\sum_{j=1}^{2n} 3j = 3\sum_{j=1}^{2n} j = 3\left(\frac{\cancel{2}n(2n+1)}{\cancel{2}}\right) = \boxed{6n^2+3n}$$

① Conform to formulas ② Apply formulas

3) $$\sum_{j=0}^{n^2} (j^2+3j) = \sum_{j=0}^{n^2} j^2 + 3\sum_{j=0}^{n^2} j = \frac{n^2(n^2+1)(2n^2+1)}{6} + 3\left(\frac{n^2(n^2+1)}{2}\right)$$

③ Simplify

$$= \frac{n^2(n^2+1)(2n^2+1)+9(n^2)(n^2+1)}{6}$$

$$= \frac{n^2(n^2+1)(2n^2+1+9)}{6} = \frac{2n^6+12n^4+10n^2}{6}$$

$$= \frac{\cancel{2}(n^6+6n^4+5n^2)}{3\cancel{6}}$$

$$= \boxed{\frac{n^6+6n^4+5n^2}{3}}$$ ④ Final Answer

4) $$\sum_{j=0}^{n} (k+3)^2 = (k+3)^2(n-0+1) = (k^2+6k+9)(n+1)$$

$$= \boxed{k^2n+6kn+9n+k^2+6k+9}$$

Exercise 3 Use the closed-form formulas to evaluate the following summations.

$\sum_{j=m}^{n} 1 = n-m+1$

$\sum_{j=0/1}^{n} j = \frac{n(n+1)}{2}$

$\sum_{j=0/1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{j=0/1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$

$\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$

1) $\sum_{j=0}^{n} 3k = 3k(n-0+1) = \boxed{3kn + 3k}$

2) $\sum_{j=4}^{10} j = 4+5+6+\ldots+10$

$= \sum_{j=0}^{10} j - \sum_{j=0}^{3} j$

$\sum_{j=0}^{10} j = 0+1+2+3+\boxed{4+5+6+7+8+9+10}$

$= \frac{10(10+1)}{2} - \frac{3(3+1)}{2}$

$\vdots$

3 lower bound - 1

$- \sum_{j=0}^{3} j = 0+1+2+3$

$4+5+6+7+8+9+10 = \boxed{49}$

Exercise 4 Find the next term and a_n where $n \geq 1$ of each sequence below.

1) 3, 5, 13, 49, 241, ...

1

2 8 36 192 → no common difference or ratio, so check growth

-1 2, 4, 12, 48, 240, ...

/2 1, 2, 6, 24, 120, ...

$\therefore n! * 2 + 1 = 2n! + 1$

n^2 → 1, 4, 9, 16, 25, 36, ...

$n!$ → 1, 2, 6, 24, 120, 720, ...

$\sum n$ → 1, 3, 6, 10, 15, 21, ...

→ growth match!

OR...

Goal	n	n!	*2		+1		
3	1	1	*2	= 2	+1	=	3
5	2	2	*2	= 4	+1	=	5
13	3	6	*2	= 12	+1	=	13
49	4	24	*2	= 48	+1	=	49
241	5	120	*2	= 240	+1	=	241
?	6	720	*2	= 1440	+1	=	1441

→ diffence fr. goal is high, more than double so try *2.

$\therefore \boxed{2n! + 1}$ is the closed-form formula.

$\boxed{1441}$ is the next term.

2) 9, 24, 49, 84, 129, . . .

15 25 35 45

10 10 10 → common diff. in differences

−4	5	20	45	80	125
/5	1	4	9	16	25

∴ $n^2 * 5 + 4$

$n^2 \rightarrow 1, 4, 9, 16, 25, \ldots$

$n! \rightarrow 1, 2, 6, 24, 120, \ldots$

$\Sigma n \rightarrow 1, 3, 6, 10, 15, 21, \ldots$

OR

Goal	n	$\boxed{n^2}$	$\boxed{*5}$	$\boxed{+4}$
9	1	1	5	9
24	2	4	20	24
49	3	9	45	49
84	4	16	80	84
129	5	25	125	129
?	6	36	180	184

Goal	n	$\sum_{j=1}^{n} 10j = \cancel{10}^{5}\left(\frac{n(n+1)}{\cancel{2}}\right) = 5n^2+5n$	$-(1+5(n-1))$	$5n^2+5n-1-5n+5 = \boxed{5n^2+4}$
9	1	10	−1	9
24	2	30	−6	24
49	3	60	−11	49
84	4	100	−16	84
129	5	150	−21	129
?	6	210	−26	184

5 5 5 5

$\boxed{5n^2 + 4}$ is the closed-form formula.

$\boxed{184}$ is the next term.

3) 2, 5, 10, 17, 26, ...

3 5 7 9

2 2 2

-1 1, 4, 9, 16, 25, ... n^2

$\therefore n^2 + 1$

Goal	n	n^2	$+ 1$
2	1	1	2
5	2	4	5
10	3	9	10
17	4	16	17
26	5	25	26
?	6	36	37

$\boxed{n^2 + 1}$ is the closed-form formula.

$\boxed{37}$ is the next term.

Exercise 4 Evaluate the following.

1) $\sum_{j=0}^{8} 2j \qquad 2\sum_{j=0}^{8} j = 2\left(\frac{8(8+1)}{2}\right) = 8(9) = \boxed{72}$

2) $\sum_{k=2}^{n} (2k+n)^2 = \sum_{k=2}^{n} (4k^2 + 4nk + n^2)$ ①

① Conform ② ③

$$= 4\sum_{k=2}^{n} k^2 + 4n\sum_{k=2}^{n} k + \sum_{k=2}^{n} n^2$$

$$= 4\left(\sum_{k=0}^{n} k^2 - \sum_{k=0}^{1} k^2\right) + 4n\left(\sum_{k=0}^{n} k - \sum_{k=0}^{1} k\right) + n^2(n-2+1)$$

④ Apply

$$= \frac{\cancel{4}^{2}\left(n(n+1)(2n+1) - 1(1+1)(2(1)+1)\right)}{\cancel{6}_{3}} + \frac{\cancel{4}^{2} n\left(n(n+1) - 1(1+1)\right)}{\cancel{2}_{1}} + n^2(n-1)$$

⑤ Simplify

$$= \frac{\left(\boxed{2}(n)(n+1)\boxed{(2n+1)} - 12\right) + \left(\boxed{6n}(n)(n+1) - 12n\right) + 3n^2(n-1)}{3}$$

$$= \frac{(n(n+1))(\boxed{4n+2} + \boxed{6n}) - 12 - 12n + 3n^3 - 3n^2}{3} = \boxed{\frac{13n^3 + 9n^2 - 10n - 12}{3}}$$

⑥ Final Answer

$$= \frac{(2(n)(n+1)(2n+1)-12) + (6n(n)(n+1)-12n) + 3\overset{(n)(n)}{n^2}(n-1)}{3}$$

$$= \frac{n(2(n+1)(2n+1) + 6n(n+1) - 12 + 3n(n-1))}{3} - 12$$

$$= \frac{n((n+1)(2(2n+1)+6n) - 12 + 3n(n-1))}{3} - 12$$

$$= \frac{n((n+1)(10n+2) - 12 + 3n^2 - 3n)}{3} - 12$$

$$= \frac{n(10n^2 + 12n + 2 - 12 + 3n^2 - 3n)}{3} - 12$$

$$= \frac{13n^3 + 9n^2 - 10n - 12}{3}$$

Alternative for number 2.

Exercise 4 Continuation . . .

3) $\sum_{j=1}^{4} \sum_{k=0}^{3n} (k + j) = \sum_{j=1}^{4} \left(\sum_{k=0}^{3n} k + \sum_{k=0}^{3n} j \right)$ ① Conform

$= \sum_{j=1}^{4} \left(\frac{3n(3n+1)}{2} + j(3n-0+1) \right)$ ② Apply

$= \sum_{j=1}^{4} \frac{3n(3n+1)}{2} + \sum_{j=1}^{4} j(3n+1)$

$= \sum_{j=1}^{4} \frac{3n(3n+1)}{2} + (3n+1)\sum_{j=1}^{4} j$ ③ Conform

$= \frac{3n(3n+1)}{\cancel{2}^{1}}(\cancel{4-1+1})^{2} + (3n+1)\left(\frac{\cancel{4}^{2}(4+1)}{\cancel{2}^{1}}\right)$ ④ Apply

$= 6n(3n+1) + (3n+1)10$ ⑤ Simplify

$= \boxed{18n^2 + 36n + 10}$ ⑥ Final Answer

4) $\sum_{j=0}^{8} 3 \cdot 2^j = \frac{3(2^{8+1}) - 3}{2-1}$

$a = 3$

$r = 2$

$= 3(2^9) - 3$

$= 3(512) - 3$

$= 1536 - 3$

$= \boxed{1533}$

$= 3\sum_{j=0}^{8} 2^j = 3\left(\frac{2^{8+1}-1}{2-1}\right)$

$= 3\left(\frac{2^9-1}{1}\right)$

$= 3(512-1)$

$= 3(511)$

$= 1533$

$$\sum_{j=1}^{n}\boxed{\sum_{k=1}^{n}}(j+k) = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} j + \sum_{k=1}^{n} k\right) = \sum_{j=1}^{n}\left(j(n-1+1) + \frac{n(n+1)}{2}\right) = n\sum_{j=1}^{n} j + \sum_{j=1}^{n}\frac{n(n+1)}{2}$$

$$= n\left(\frac{n(n+1)}{2}\right) + \frac{n(n+1)}{2}(n\cancel{-1+1})$$

$$= \cancel{2}\left(n\,\frac{n(n+1)}{\cancel{2}}\right) = \boxed{n^3+n^2}$$

$$\boxed{\sum_{j=1}^{n}\sum_{k=1}^{n}}(j+k) = \sum_{j=1}^{n}\sum_{k=1}^{n} j + \sum_{j=1}^{n}\sum_{k=1}^{n} k = \sum_{j=1}^{n} j(n\cancel{-1+1}) + \sum_{j=1}^{n}\frac{n(n+1)}{2}$$

$$= n\sum_{j=1}^{n} j + \frac{n(n+1)}{2}\sum_{j=1}^{n} 1 = n\left(\frac{n(n+1)}{2}\right) + \frac{n(n+1)}{2}(n\cancel{-1+1})$$

$$= \cancel{2}\left(n\left(\frac{n(n+1)}{\cancel{2}}\right)\right) = n(n^2+n) = \boxed{n^3+n^2}$$