Set Theory

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Sets

Definition

A *set* is an unordered collection of objects.

These objects are called *elements* or *members*.

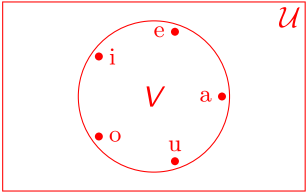
$a \in A$ denotes that a is an element of set A.
$a \notin A$ denotes that a is not an element of set A.

Convention: Uppercase letters are used to name sets and lowercase letters are used to indicate elements of the set.

Describing a Set

There are several ways to describe a set.

- using the **roster method**;
  $V = \{a, e, i, o, u\}$
  $A = \{1, 2, 3, \ldots\}$
  $B = \{2, 4, 6, \ldots, 20\}$
- using a **set builder notation**;
  $V = \{x | x \text{ is a vowel}\}$
  $A = \{x | x \in \mathbb{Z}^+\}$
  $B = \{x \in \mathbb{Z}^+ | x \mod 2 = 0 \wedge x \leq 20\}$
- using a **Venn diagram**



- universe of discourse
- name of the set
- elements of the set

Important Sets

$\mathbb{N}$ the set of natural numbers.
$= \{0, 1, 2, 3, \ldots\}$[1]

$\mathbb{Z}$ the set of integers.
$= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Z}^+$ the set of positive integers.
$= \{1, 2, 3, \ldots\}$

$\mathbb{Z}^-$ the set of negative integers.
$= \{\ldots, -3, -2, -1\}$

$\mathbb{Q}$ the set of rational numbers.
$= \{\frac{p}{q} | p \in \mathbb{Z} \land q \in \mathbb{Z} \land q \neq 0\}$

$\mathbb{C}$ the set of complex numbers.

[1] There is no general agreement whether 0 is included in the set or not. (http://mathworld.wolfram.com/NaturalNumber.html)

Equal Sets

Definition

Two set are *equal* if and only if they have the same elements.

If A and B are sets and are equal, we write $A = B$.

$$\forall x (x \in A \leftrightarrow x \in B)$$

Which among the sets are equal?

$A = \{1, 2, 3\}$ $B = \{1, 3, 2, 4\}$ $C = \{1, 3, 2\}$ $D = \{3, 1, 2, 3, 1, 3\}$

Remember:

- The order on how the elements are listed in the set does not matter.
- The number of times an element is listed in the set does not matter.

Sets

Definition

A set that contains no elements is called an *empty set* or a *null set*.

$\{\}$ and $\varnothing$ are used to denote an empty set.

Definition

A set that contains exactly one element is called a *singleton set*.

Note that $\varnothing \neq \{\varnothing\}$.

Size of a Set

Definition

The number of distinct elements in a set S is called the *cardinality of S* or $|S|$.

Determine the sizes of the following sets.

1. $\{a, e, i, o, u\}$
2. $\{1, 2, 3, 2, 1, 2, 3\}$
3. $\{x | x \in \mathbb{Z}^+ \wedge x \mod 2 = 1 \wedge x < 20\}$
4. $\{\}$
5. $\{\varnothing\}$
6. $\{a, a, b, a, b, c\}$

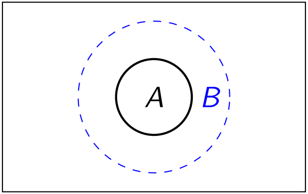
Subset

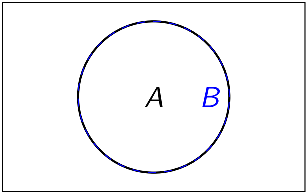
Definition

Set A is a *subset of* set B if and only if every element of A is also in B.

The notation $A \subseteq B$ is used to indicate that A is a subset of B.

$$\forall x(x \in A \rightarrow x \in B)$$





Note that $\varnothing \subseteq$ all sets and a set is a subset of itself (i.e. $S \subseteq S$).

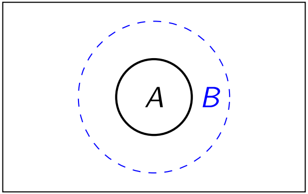
Subset vs Proper Subset

Definition

When set A is a subset of set B, but A and B are not equal sets, we say that set A is a *proper subset* of set B.

Notation: $A \subset B$

$$\forall x (x \in A \rightarrow x \in B) \wedge \exists x (x \in B \wedge x \notin A)$$



Exercise

True or False.

1. $\varnothing \subset \{\varnothing\}$
2. $\varnothing \subseteq \{\varnothing\}$
3. $\{2\} \subseteq \{2\}$
4. $\{2\} \subseteq \{2, \{2\}\}$
5. $\varnothing \subseteq \{2\}$

Power Sets

Definition

Given a set S, the *power set of S* is the set of all subsets of S.

The power set of S is denoted by $\mathcal{P}(S)$.

Let $A = \{1, 2, 3\}$.

$$\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Let $B = \{a\}$.

$$\mathcal{P}(B) = \{\varnothing, \{a\}\}$$

$$\mathcal{P}(\varnothing) = \{\varnothing\}$$

Note that $|\mathcal{P}(S)| = 2^{|S|}$.

Exercise

Answer the following.

1. $\mathcal{P}(\{2\})$
2. $A = \{a, b\}$, $\mathcal{P}(A)$
3. $B = \{\varnothing\}$, $\mathcal{P}(B)$
4. $C = \{\varnothing, \{\varnothing\}\}$, $\mathcal{P}(C)$

Cartesian Product

Definition

Let A and B be sets. The *cartesian product of A and B*, denoted by $A \times B$, is the set of all ordered pairs (a, b), such that $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$.

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

$$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

Note

- $A \times B \neq B \times A$, unless $A = \varnothing$ or $B = \varnothing$ or $A = B$.
- $|A \times B| = |A||B|$
- $(A \times B) \times C \neq A \times B \times C$
- A^2 denotes the cartesian product of A with itself, thus $A^3 = A \times A \times A$ and so on.

Exercise

Assume that

$$\begin{aligned} A &= \{1, 2, 3\} \\ B &= \{x | x \in \mathbb{Z}^- \wedge x \mod 5 = 0 \wedge x > -10\} \\ C &= \{y | y \text{ is a vowel in the word } \textit{disctru}\} \end{aligned}$$

Answer the following:

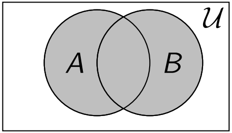
1. $A \times C$
2. $B \times A$
3. $A \times B \times C$
4. C^2
5. $B \times B \times B$

Set Operations

Assume A and B are sets.

The Union of the sets A and B

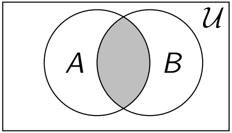
$$A \cup B = \{x | x \in A \vee x \in B\}$$



The Intersection of the sets A and B

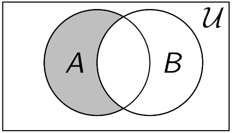
$$A \cap B = \{x | x \in A \wedge x \in B\}$$

If $A \cap B = \varnothing$, A and B are disjoint sets.



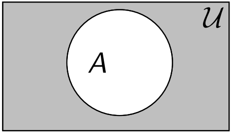
The Difference of the sets A and B

$$A - B = \{x | x \in A \wedge x \notin B\}$$



The Complement of the set A

$$\overline{A} = \{x \in \mathcal{U} | x \notin A\}$$



Set Identities

Identity Laws

$$A \cap \mathcal{U} = A$$

$$A \cup \varnothing = A$$

Domination Laws

$$A \cap \varnothing = \varnothing$$

$$A \cup \mathcal{U} = \mathcal{U}$$

Idempotent Laws

$$A \cap A = A$$

$$A \cup A = A$$

Complement Laws

$$A \cap \overline{A} = \varnothing$$

$$A \cup \overline{A} = \mathcal{U}$$

Complementation Laws

$$\overline{\overline{A}} = A$$

De Morgan's Laws

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Set Identities

Commutative Laws

$$A \cap B = B \cap A$$

$$A \cup B = B \cup A$$

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Associative Laws

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cup (B \cup C) = (A \cup B) \cup C$$

Absorption Laws

$$A \cap (A \cup C) = A$$

$$A \cup (A \cap C) = A$$

Exercise

Derive the following:

- De Morgan's Laws
- $A - B = A \cap \overline{B}$

Exercise 1

1. List the members of the set $\{x \in \mathbb{R} | x^2 = 1\}$
2. List the members of the set $\{x | x^2 = 2 \wedge x \in \mathbb{Z}\}$
3. $|\{x | x^2 < 100 \wedge x \in \mathbb{Z}\}|$
4. Describe $\{0, 3, 6, 9, 12\}$ using the set builder notation.

Let $D = \{2, 4, 6\}$, $L = \{2, 6\}$, $S = \{2, 4, 2, 6, 4, 2\}$ and $U = \{4, 6, 4, 6, 4\}$.

5. Determine which of these sets are subsets of which other sets.
6. Determine which of these sets are proper subsets of which other sets.
7. Determine which of these sets are equal sets.
8. $\mathcal{P}(U)$
9. $|S|$
10. $L \times D$

Exercise 2

Determine whether 2 is an element of the sets below.

1. $\{x \in \mathbb{R} | x \text{ is an integer greater than } 1\}$
2. $\{x \in \mathbb{R} | x \text{ is the square of an integer}\}$
3. $\{2, \{2\}\}$
4. $\{\{2\}, \{\{2\}\}\}$
5. $\{\{2\}, \{2, \{2\}\}\}$
6. $\{\{\{2\}\}\}$

Exercise 3

True or False.

1. $0 \in \varnothing$
2. $\{0\} \in \varnothing$
3. $\varnothing \in \{0\}$
4. $\{0\} \subset \varnothing$
5. $\varnothing \subset \{0\}$
6. $\{0\} \in \{0\}$
7. $\{0\} \subset \{0\}$
8. $\{0\} \subseteq \{0\}$
9. $\{\varnothing\} \subseteq \{\varnothing\}$
10. $\{\{\varnothing\}\} \subset \{\varnothing, \{\varnothing\}\}$

Exercise 4

Let $A = \{x | x \in \mathbb{Z}^+ \land x < 6\}$, $B = \{0, 3, 6\}$. Answer the following:

1. $A \cup B$
2. $A - B$
3. $A \cap B$
4. $B - A$
5. $\overline{A}$

References

[Rosen, 2007] Kenneth Rosen.
Discrete Mathematics and Its Applications 7^{th} edition, 2007