Functions

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Introduction

Definition

Let A and B be sets. f is a *function* from A to B when exactly one element of B to each element of A.

$f(a) = b$ $\qquad\qquad$ $f : A \to B$

a.k.a. *mappings* or *transformations*

Example: $f(x) = x + 1$

Introduction

Definition

Given $f : A \to B$, i.e. $f(a) = b$

- A is the domain of f
- B is the codomain of f
- b is the image of a
- a is the preimage of b

Note that the *range of f* is the set of all images of f.

The range of f is also known as *image of f*.

Example

Determine if each is a function or not. If so, determine the domain, codomain and range of each function.

Example 1: $f : \{a, b, c, d, e\} \to \mathbb{Z}$ and $f(a) = 1$, $f(b) = 2$, $f(c) = -1$, $f(d) = 5$, $f(d) = 1$ and $f(e) = 1$.

not a function

Example 2: $f : \{a, b, c, d, e\} \to \mathbb{Z}$ and $f(a) = 1$, $f(b) = 2$, $f(c) = -1$, $f(d) = 5$ and $f(e) = 1$.

a function

- domain of f: $\{a, b, c, d, e\}$
- codomain of f: $\mathbb{Z}$
- range of f: $\{1, 2, -1, 5\}$

Example 3: f : set of students $\to \mathbb{Z}$ and $f(\text{Abby}) = 82$, $f(\text{Bren}) = 62$, $f(\text{Carla}) = 71$, $f(\text{Desiree}) = 95$ and $f(\text{Eddie}) = 50$.

a function

- domain of f: $\{\text{Abby}, \text{Bren}, \text{Carla}, \text{Desiree}, \text{Eddie}\}$
- codomain of f: $\mathbb{Z}$
- range of f: $\{82, 62, 71, 95, 50\}$

Operations on Functions

Let f_1 and f_2 be functions from A to $\mathbb{R}$. Functions $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$.
(let $x \in A$)

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- $(f_1 f_2)(x) = f_1(x) f_2(x)$

Example: Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ where $f(x) = x + 5$ and $g(x) = x - x^2$.

- $(f + g)(x) = 2x + 5 - x^2$
- $(f \cdot g)(x) = 5x - 4x^2 - x^3$

One-to-One Functions

Definition

A function f from A to B is *one-to-one* or an *injection*, if and only if $f(a) = f(b)$ for all $a = b$ in the domain of f.

A function is said to be *injective* if it is one-to-one.

Example 1: $f : \{a, b, c, d\} \to \{1, 2, 3, 4, 5\}$ and $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, $f(d) = 3$

one-to-one

Example 2: $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2$

not one-to-one, *since* $f(1) = f(-1) = 1$ *and* $1 \neq -1$.

Onto Functions

Definition

A function f from A to B is *onto* or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

A function is said to be *surjective* if it is onto.

Example 1: $f : \{a, b, c, d\} \to \{1, 2, 3, 4, 5\}$ and $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, $f(d) = 3$

not onto, *since there is not x such that $f(x) = 2$.*

Example 2: $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2$

not onto, *since there is no x such that $f(x) < 0$.*

Example 3: $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x + 1$

onto

One-to-One Correspondence

Definition

A function f is a *one-to-one correspondence* or a *bijection*, if it is **both** one-to-one and onto.

A function is said to be *bijective* if it is a one-to-one correspondence.

Example 1: $f : \{a, b, c, d\} \to \{1, 2, 3, 4, 5\}$ and $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, $f(d) = 3$

not a one-to-one correspondence, *since it is not onto.*

Example 2: $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x + 1$

one-to-one correspondence

Summary

Supposed $f : A \to B$ and $x \in A$ and $y \in B$

- f is **injective**: If $f(x) = f(y)$ then $x = y$
- f is **surjective**: $\forall y \exists x (f(x) = y)$

Thus,

- f is **not injective**: Found $f(x) = f(y)$ where $x \neq y$.
- f is **not surjective**: Found y such that $\forall x\ (f(x) \neq y)$.

Exercise 1

- Determine if f is a function from $\mathbb{R}$ to $\mathbb{R}$?
  1. $f(x) = \frac{1}{x}$
  2. $f(x) = \sqrt{x}$
- Determine if f is a function from $\mathbb{Z}$ to $\mathbb{R}$?
  1. $f(n) = \pm n$
  2. $f(n) = \sqrt{n^2 + 1}$
- Find these values.
  1. $\lfloor 1.1 \rfloor$
  2. $\lfloor -0.1 \rfloor$
  3. $\lceil -\frac{3}{4} \rceil$
  4. $\left\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \right\rfloor$

Exercise 2

- Determine whether function f from $\mathbb{Z}$ to $\mathbb{Z}$ is one-to-one, onto or one-to-one correspondence.
  1. $f(x) = -3x + 4$
  2. $f(x) = -3x^2 + 7$
  3. $f(x) = x^5 + 1$
  4. $f(x) = \dfrac{x+1}{x+2}$
- Determine whether function f from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$ is one-to-one, onto or one-to-one correspondence.
  1. $f(m, n) = m + n$
  2. $f(m, n) = m^2 + n^2$
  3. $f(m, n) = m$
  4. $f(m, n) = |n|$

References

[Rosen, 2007] Kenneth Rosen.
Discrete Mathematics and Its Applications 7^{th} edition, 2007