Rules of Inference

Section 1.6

An argument is a sequence of statements called premises followed by a statement called the conclusion.

An argument is valid when and only when it is imposible for the conclusion to be false when all the premises are true.

Revisiting the Socrates Example

We have the two premises:

- "All men are mortal."
- "Socrates is a man."

And the conclusion:

- "Socrates is mortal."

How do we get the conclusion from the premises?

The Argument

What we see

We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

What we see

$$\begin{array}{l} \forall x\left(Man(x) \rightarrow Mortal(x)\right) \\ \underline{\quad Man(Socrates) \quad} \\ \therefore \quad Mortal(Socrates) \end{array}$$

later in this section

We will see that this is a valid argument.

Arguments in Propositional Logic

A *argument* in propositional logic is a sequence of propositions. All but the final proposition are called *premises*. The last statement is the *conclusion*.

The argument is valid if the premises imply the conclusion. An *argument form* is an argument that is valid no matter what propositions are substituted into its propositional variables.

This means that

If the premises are $p_1, p_2, \ldots, p_n$ and the conclusion is q then

$(p_1 \land p_2 \land \ldots \land p_n) \rightarrow q$ is a tautology.

Inference rules are all simple argument forms that will be used to construct more complex argument forms.

Rules of Inference for Propositional Logic: Modus Ponens

mood that affirms

$$
\begin{array}{l}
p \rightarrow q \\
\underline{\quad p \quad} \\
\therefore q
\end{array}
$$

Corresponding Tautology:

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

Example:

Let *p* be "It is snowing."

Let *q* be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"It is snowing."

"Therefore , I will study discrete math."

Modus Tollens

mood that denies

$$p \to q$$
$$\neg q$$
$$\therefore \neg p$$

Corresponding Tautology:

$$(\neg q \wedge (p \to q)) \to \neg p$$

Example:

Let *p* be "it is snowing."

Let *q* be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"I will not study discrete math."

"Therefore , it is not snowing."

Hypothetical Syllogism

$$\begin{array}{r} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology:

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Example:

Let p be "it snows."

Let q be "I will study discrete math."

Let r be "I will get an A."

"If it snows, then I will study discrete math."

"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism

$$\begin{array}{r} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$$(\neg p \wedge (p \vee q)) \rightarrow q$$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will study English literature."

"I will study discrete math or I will study English literature."

"I will not study discrete math."

"Therefore , I will study English literature."

Addition

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:

$$p \rightarrow (p \vee q)$$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit
Las Vegas."

Simplification

$$\frac{p \wedge q}{\therefore p}$$

Corresponding Tautology:

$$(p \wedge q) \rightarrow p$$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

$$\begin{array}{r} p \\ \underline{\quad q \quad} \\ \therefore p \wedge q \end{array}$$

Corresponding Tautology:

$$((p) \wedge (q)) \rightarrow (p \wedge q)$$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will study English literature."

"I will study discrete math."

"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Resolution

$$\begin{array}{l} \neg p \vee r \\ \underline{p \vee q} \\ \therefore q \vee r \end{array}$$

Corresponding Tautology:

$$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$$

Example:

Let *p* be "I will study discrete math."

Let *r* be "I will study English literature."

Let q be "I will study databases."

"I will not study discrete math or I will study English literature."

"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will study English literature."

Using the Rules of Inference to Build Valid Arguments

Rules of inference may be ussed as an aid in showing that the premises imply the conclusion.

A *valid argument* is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.

the

A valid argument takes the following form:

$$
\begin{array}{l}
S_1 \\
S_2 \\
\cdot \\
\cdot \\
\cdot \\
S_n \\
\therefore C
\end{array}
$$

Valid Arguments [2]

Example 1: From the single proposition

$$p \wedge (p \rightarrow q)$$

Show that q is a conclusion.

Solution:

Step	**Reason**
1. $p \wedge (p \rightarrow q)$	Premise
2. p	Simplification using (1)
3. $p \rightarrow q$	Simplification using (1)
4. q	Modus Ponens using (2) and (3)

Valid Arguments 3

Example 2:

With these hypotheses:

"It is not sunny this afternoon and it is colder than yesterday."

"We will go swimming only if it is sunny."

"If we do not go swimming, then we will take a canoe trip."

"If we take a canoe trip, then we will be home by sunset."

Using the inference rules, construct a valid argument for the conclusion:

"We will be home by sunset."

Solution:

1. Choose propositional variables:

p : "It is sunny this afternoon." *r* : "We will go swimming." *t : "*We will be home by sunset."

q : "It is colder than yesterday." *s : "*We will take a canoe trip."

2. Translation into propositional logic:

$$\text{Hypotheses: } \neg p \wedge q,\ r \to p,\ \neg r \to s,\ s \to t$$
$$\text{Conclusion: } t$$

Continued on next slide →

Valid Arguments 4

3. Construct the Valid Argument

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Handling Quantified Statements

Valid arguments for quantified statements are a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference which include:

- Rules of Inference for Propositional Logic
- Rules of Inference for Quantified Statements

The rules of inference for quantified statements are introduced in the next several slides.

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all dogs and Fido is a dog.

"All dogs are cuddly."

"Therefore, Fido is cuddly."

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

"There is someone who got an A in the course."

"Let's call her *a* and say that *a* got an A"

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

"Michelle got an A in the class."

"Therefore, someone got an A in the class."

Using Rules of Inference [1]

Example 1: Using the rules of inference, construct a valid argument to show that

"John Smith has two legs"

is a consequence of the premises:

"Every man has two legs." "John Smith is a man."

Solution: Let $M(x)$ denote "x is a man" and $L(x)$ "x has two legs" and let John Smith be a member of the domain.

Valid Argument:

Step	**Reason**
1. $\forall x\left(M(x) \rightarrow L(x)\right)$	Premise
2. $M(J) \rightarrow \mathrm{L}(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Using Rules of Inference 2

Example 2: Use the rules of inference to construct a valid argument showing that the conclusion

"Someone who passed the first exam has not read the book."

follows from the premises

"A student in this class has not read the book."

"Everyone in this class passed the first exam."

Solution: Let $C(x)$ denote "x is in this class," $B(x)$ denote " x has read the book," and $P(x)$ denote "x passed the first exam."

First we translate the

premises and conclusion

into symbolic form.

$$\begin{array}{l} \exists x\big(C(x) \land \neg B(x)\big) \\ \forall x\big(C(x) \rightarrow P(x)\big) \\ \hline \therefore \exists x\big(P(x) \land \neg B(x)\big) \end{array}$$

Continued on next slide →

Using Rules of Inference [3]

Valid Argument:

Step	**Reason**
1. $\exists x\left(C(x) \wedge \neg B(x)\right)$	Premise
2. $C(a) \wedge \neg \mathrm{B}(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x\left(C(x) \rightarrow P(x)\right)$	Premise
5. $C(a) \rightarrow P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conj from (6) and (7)
9. $\exists x\left(P(x) \wedge \neg B(x)\right)$	EG from (8)

Returning to the Socrates Example

$$
\begin{array}{l}
\forall x\left(Man\left(x\right)\rightarrow Mortal\left(x\right)\right) \\
\qquad Man\left(Socrates\right) \\
\hline
\therefore \quad Mortal\left(Socrates\right)
\end{array}
$$

Solution for Socrates Example

Valid Argument

Step	**Reason**
1. $\forall x\left(Man\left(x\right)\rightarrow Mortal\left(x\right)\right)$	Premise
2. $Mam\left(Socrates\right)\rightarrow Mortal\left(Socrates\right)$	UI from $\left(1\right)$
3. $Mam\left(Socrates\right)$	Premise
4. $Mortal\left(Socrates\right)$	MP from $\left(2\right)$ and $\left(3\right)$

Universal Modus Ponens

Universal Modus Ponens combines universal instantiation and modus ponens into one rule.

$$\begin{array}{l} \forall x\left(P(x) \rightarrow Q(x)\right) \\ P(a), \text{ where } a \text{ is a particular} \\ \underline{\text{element in the domain}} \\ \qquad \therefore Q(a) \end{array}$$

This rule could be used in the Socrates example.