LOGIC-BASED MODELS

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Logic-Based Models

- **Goal:** Model **real world entities and relationships** into something that the computer can process.

Represent knowledge about the world

Reason with that knowledge

- **Applications:** theorem proving, reasoning, expert systems

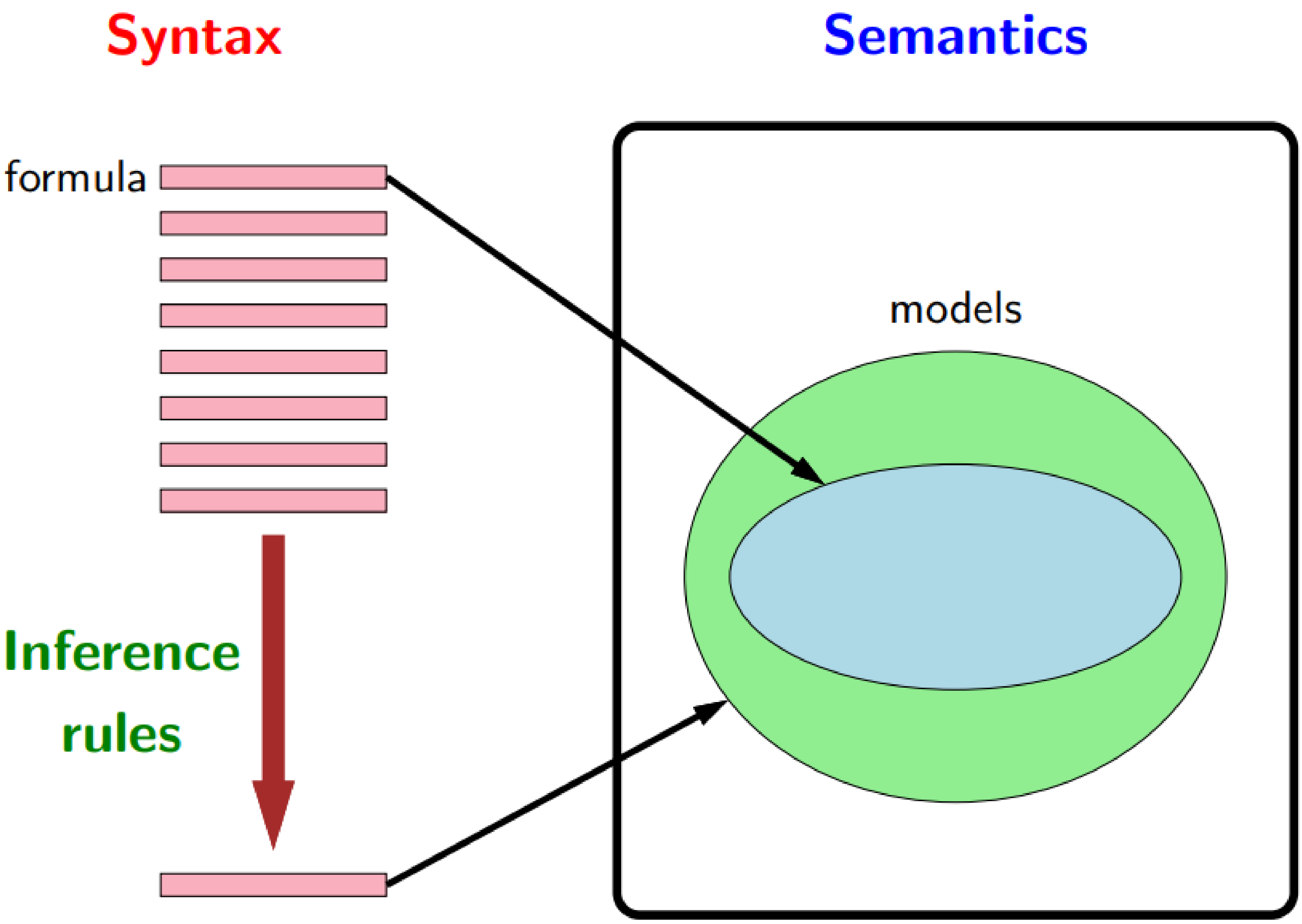
Language

- We need to represent general knowledge in the computer.
- However, computers have difficulties handling the ambiguities of informal languages.

- **Natural Language (informal)**
  - *A number that is divisible by 2 and generates a remainder of 0 is called an even number.*
- **Programming Language (formal)**
  - **def even(x): return x % 2 == 0**
- **Logical Language (formal)**
  - $\forall x \; \boldsymbol{Even}(\boldsymbol{x}) \rightarrow \boldsymbol{Divides}(\boldsymbol{x}, \mathbf{2})$

Ingredients of Logic Language

- **Syntax:** defines what makes a valid sentence / formula
- **Semantics:** maps a formula to a set of configurations that represent some meaning in the world
- **Inference Rules:** given a formula f, what other formulas are guaranteed to follow?


Syntax
Semantics
formula
models
Inference
rules

Modeling: Propositional Logic

- A logical language based around **propositions**.
- **Proposition** is a something that is either **TRUE** or **FALSE**
  - (but not both)

- A **propositional symbol** $(A, B, C, \dots)$ represents a single proposition.

- **Propositional calculus** is a method of manipulating formulas in propositional logic language.

Propositional Logic: Syntax

- A propositional logic formula can be a propositional symbol: A, B, C
- Logical connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- If f and g are formulas, so are the following:
  - **Negation:** $\neg f$
  - **Conjunction:** $f \wedge g$
  - **Disjunction:** $f \vee g$
  - **Implication:** $f \rightarrow g$
  - **Biconditional:** $f \leftrightarrow g$

Propositional Logic: Semantics

f	g	$\neg f$	$f \wedge g$	$f \vee g$	$f \rightarrow g$	$f \leftrightarrow g$
0	0	1	0	0	1	1
0	1	1	0	1	1	0
1	0	0	0	1	0	0
1	1	0	1	1	1	1

- **Key idea:** A formula is a compact representation of a set of configurations.

Truth Tables

Negation

p	$\sim p$
F	T
T	F

Conjunction

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Disjunction

P	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

Bi-conditional

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Precedence and Associativity

- **Precedence** is to determine the order in which different connectives in a compound proposition are evaluated.
- **Associativity** is to determine the order in which connectives in the same precedence in a compound proposition are evaluated.

Connectives	Associativity
$\neg$	
$\land$, $\lor$	Left to right
$\rightarrow$, $\leftrightarrow$	Right to left

precedence

Example

- Evaluate the following expression if P is true, Q is false, and R is true.

$$\neg Q \rightarrow Q \wedge P \rightarrow R$$

Example

- Evaluate the following expression if P is true, Q is false, and R is true.

$$\neg Q \rightarrow Q \land P \rightarrow R$$

Example

- Evaluate the following expression if P is true, Q is false, and R is true.

$$\neg Q \rightarrow Q \wedge P \rightarrow R$$

$$1 \rightarrow Q \wedge P \rightarrow R$$

Example

- Evaluate the following expression if P is true, Q is false, and R is true.

$$\neg Q \rightarrow Q \wedge P \rightarrow R$$

$$1 \rightarrow Q \wedge P \rightarrow R$$

Example

- Evaluate the following expression if P is true, Q is false, and R is true.

$$\neg Q \rightarrow Q \wedge P \rightarrow R$$

$$1 \rightarrow Q \wedge P \rightarrow R$$

$$1 \rightarrow 0 \rightarrow R$$

Example

- Evaluate the following expression if P is true, Q is false, and R is true.

$$\neg Q \to Q \wedge P \to R$$

$$1 \to Q \wedge P \to R$$

$$1 \to 0 \to R$$

Example

- Evaluate the following expression if P is true, Q is false, and R is true.

$$\neg Q \rightarrow Q \land P \rightarrow R$$

$$1 \rightarrow Q \land P \rightarrow R$$

$$1 \rightarrow 0 \rightarrow R$$

$$1 \rightarrow 1$$

Example

- Evaluate the following expression if P is true, Q is false, and R is true.

$$\neg Q \rightarrow Q \land P \rightarrow R$$

$$1 \rightarrow Q \land P \rightarrow R$$

$$1 \rightarrow 0 \rightarrow R$$

$$1 \rightarrow 1$$

Example

- Evaluate the following expression if P is true, Q is false, and R is true.

$$\neg Q \to Q \land P \to R$$

$$1 \to Q \land P \to R$$

$$1 \to 0 \to R$$

$$1 \to 1$$

$$1$$

Evaluation using Truth Table

- Methods to construct truth value of compound statement:
  - Step 1: Resolve ambiguity of propositions (use the precedence and associativity rules)
  - Step 2: Extract simple proposition and place them on the LHS of the table.
  - Step 3: Enumerate all possible truth assignments to the simple propositions.
  - Step 4: Fill the table from “small expressions” to “big expressions”
- Example:

$\sim p \vee \sim q \rightarrow p \wedge r$

$= (\sim p) \vee (\sim q) \rightarrow p \wedge r$

$= ((\sim p) \vee (\sim q)) \rightarrow (p \wedge r)$

$= (((\sim p) \vee (\sim q)) \rightarrow (p \wedge r))$

Note:
There are 3 propositions (p,q,r) giving 2^3 possible truth value i.e. 8.

p	q	r	~p	~q	~p V ~q	p Λ r	~p V ~q → p Λ r
F	F	F	T	T	T	F	**F**
F	F	T	T	T	T	F	**F**
F	T	F	T	F	T	F	**F**
F	T	T	T	F	T	F	**F**
T	F	F	F	T	T	F	**F**
T	F	T	F	T	T	T	**T**
T	T	F	F	F	F	T	**T**
T	T	T	F	F	F	T	**T**

Derivation

- Derivation is used to find out equivalence of a proposition equation (proofing).
- Some definitions:
  - Tautology: A compound proposition is a tautology if it is true for every possible assignment to its simple proposition.
  - Contradiction: A compound proposition is a contradiction if it is false for every possible assignment to its simple proposition.

p	q	$p \land q$	$p \lor q$	$p \land q \to p \lor q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Tautology

p	q	¬p	¬q	p ∨ q	(¬p) ∧ (¬q)	(p ∨ q) ∧ ((¬p) ∧ (¬q))
F	F	T	T	F	T	F
F	T	T	F	T	F	F
T	F	F	T	T	F	F
T	T	F	F	T	F	F

Contradiction

Logical Equivalence Laws

- **Commutative Laws**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative Laws**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive Laws**
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- **De Morgan's Laws**
  - $\neg (p \vee q) \equiv \neg p \wedge \neg q$
  - $\neg (p \wedge q) \equiv \neg p \vee \neg q$
- **Contrapositive Laws**
  - $p \to q \equiv \neg q \to \neg p$

- **Negative Law**
  - $p \vee \neg p \equiv t$
  - $p \wedge \neg p \equiv c$

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Contingency Tautology Contradiction

- **Double Negative Laws**
  - $\neg(\neg p) \equiv p$
- **Idempotent Laws**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Universal Bound Laws**
  - $p \vee t \equiv t$
  - $p \wedge c \equiv c$
- **Identity Laws**
  - $p \wedge t \equiv p$
  - $p \vee c \equiv p$

Example

Use the logical equivalence laws to verify the following logical equation:

$\sim(\sim p \land q) \land (p \lor q) \equiv p$

Answer:

$\sim(\sim p \land q) \land (p \lor q)$

$\equiv (\sim(\sim p) \lor \sim q) \land (p \lor q)$ by De Morgan’s Law

$\equiv (p \lor \sim q) \land (p \lor q)$ by Double Negative Law

$\equiv p \lor (\sim q \land q)$ by Distributive Law

$\equiv p \lor (q \land \sim q)$ by Commutative Law

$\equiv p \lor c$ by Negative Law

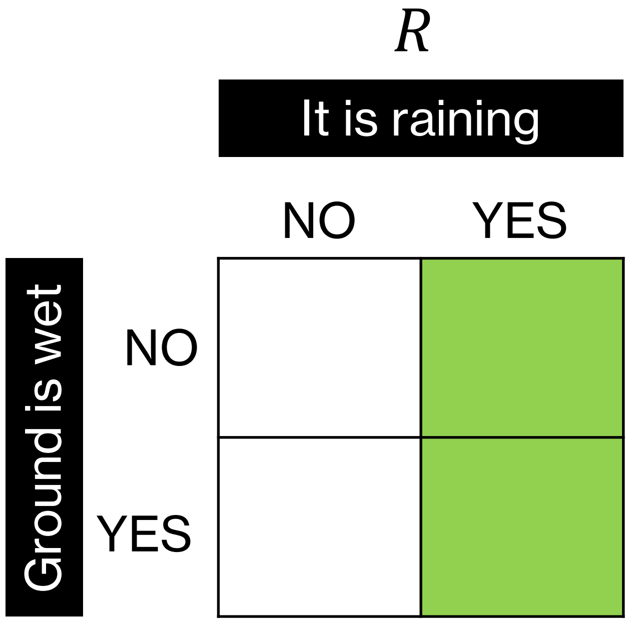
$\equiv p$ by Identity Law

Propositional Logic: Semantics

- **Example:** R – it is raining W – the ground is wet

Propositional Logic: Semantics

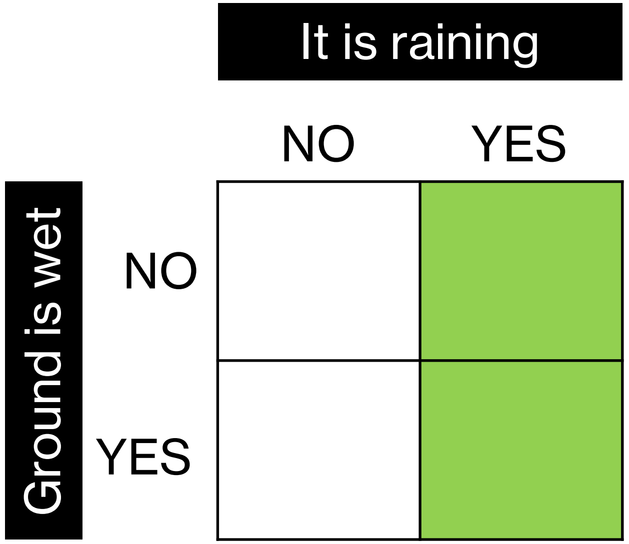
- **Example:** R – it is raining W – the ground is wet



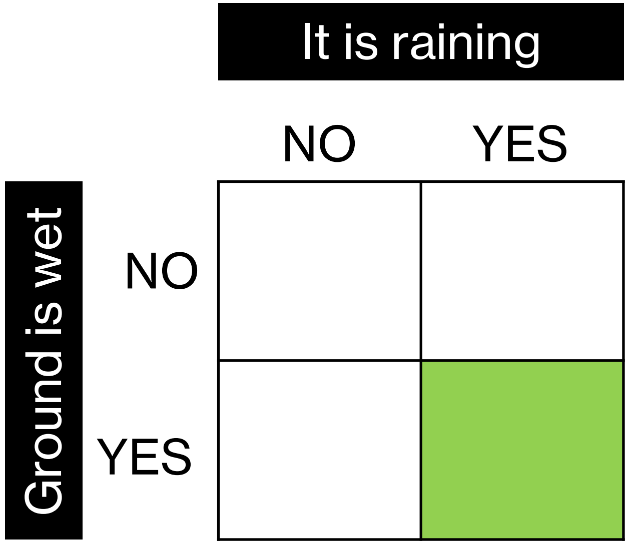
Propositional Logic: Semantics

- **Example:** R – it is raining W – the ground is wet

R



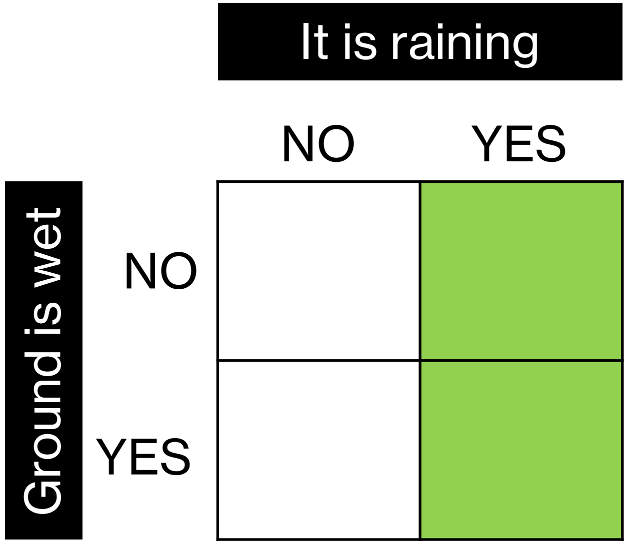
$R \land W$



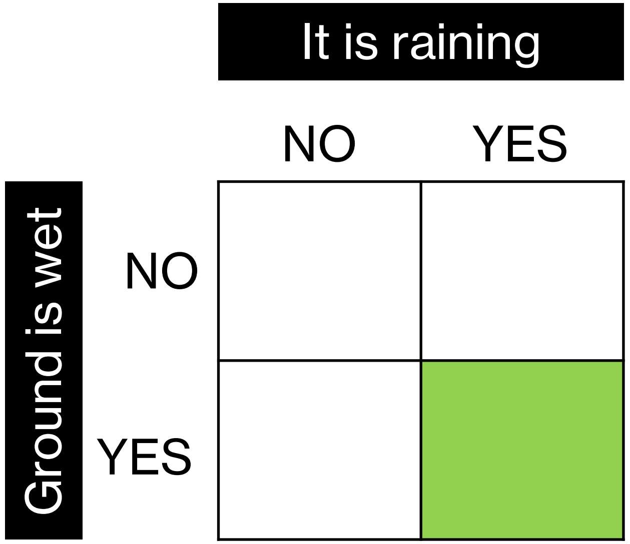
Propositional Logic: Semantics

- **Example:** R – it is raining W – the ground is wet

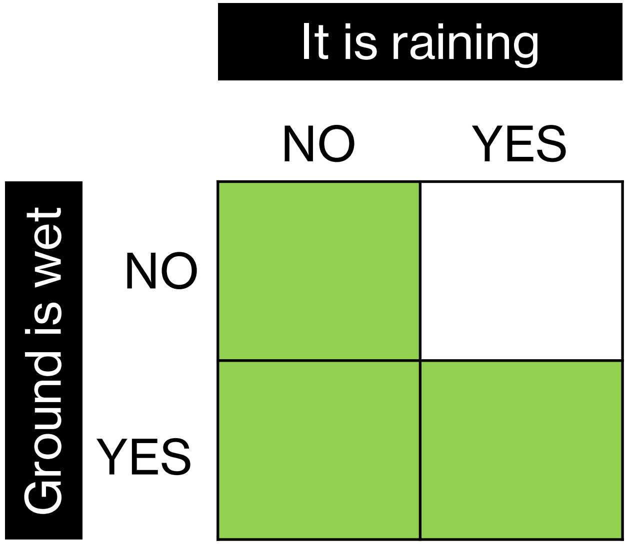
R



$R \wedge W$

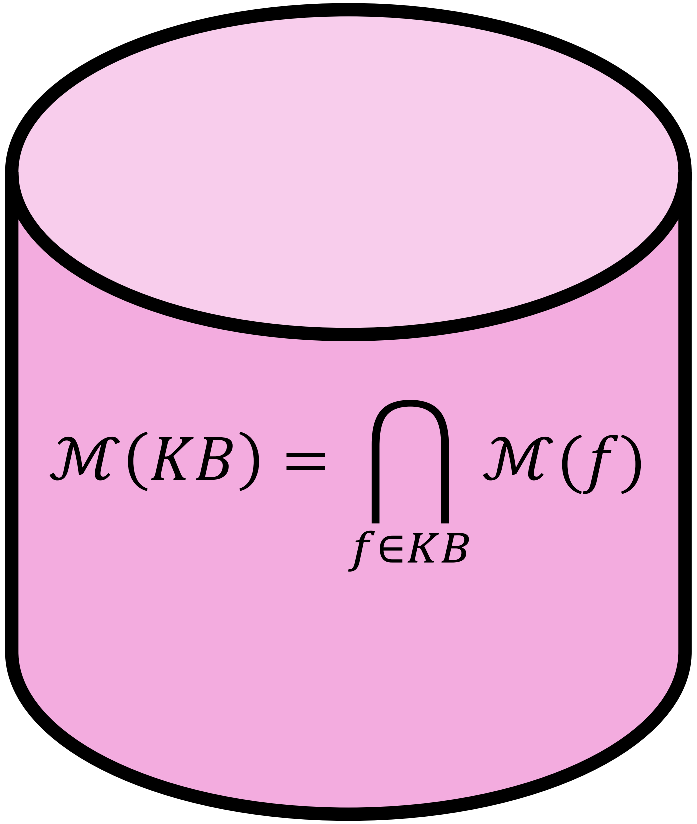


$R \rightarrow W$

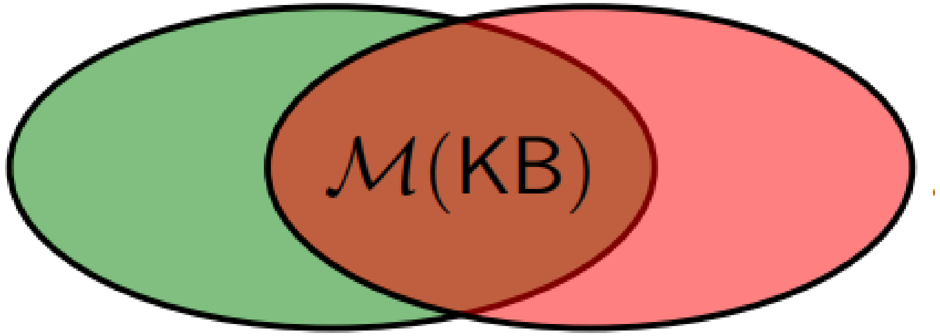


Knowledge Base

- **List of formulas** that represent truths about the world



$$\mathcal{M}(KB) = \bigcap_{f \in KB} \mathcal{M}(f)$$



Propositional Logic: Semantics

- **Example:** R – it is raining W – the ground is wet

KB

$$R$$
$$R \rightarrow W$$

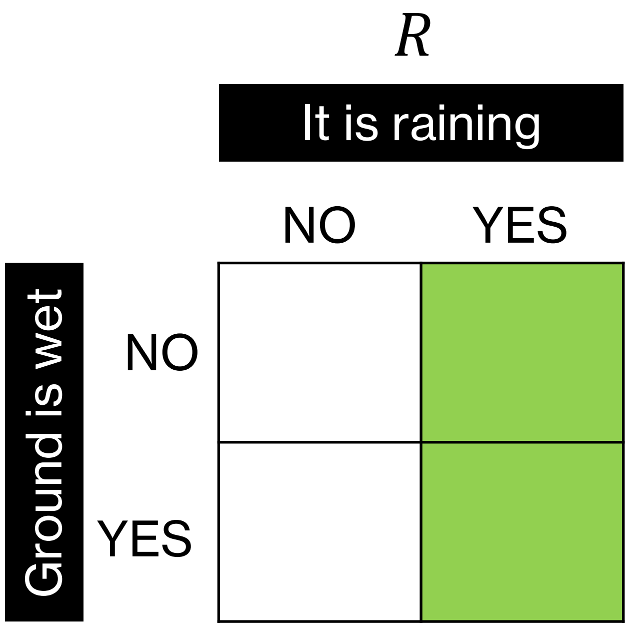
Propositional Logic: Semantics

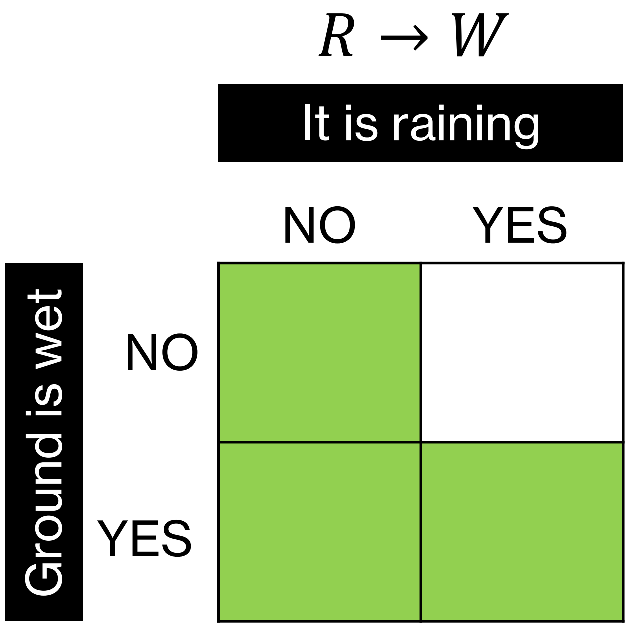
- **Example:** R – it is raining W – the ground is wet

KB

R

$R \to W$





Propositional Logic: Semantics

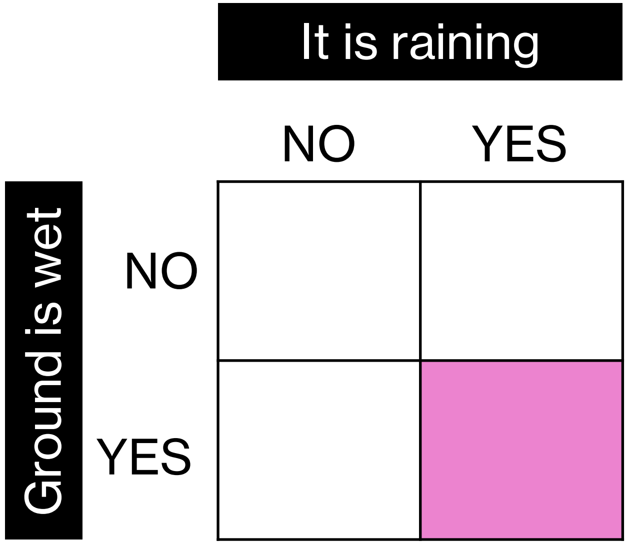
- **Example:** R – it is raining W – the ground is wet

KB

R

$R \to W$

Get the intersection:



Adding to the Knowledge Base

- When we add a new formula to the knowledge base, three things could happen:
  - **Entailment**
  - **Contradiction**
  - **Contingency**

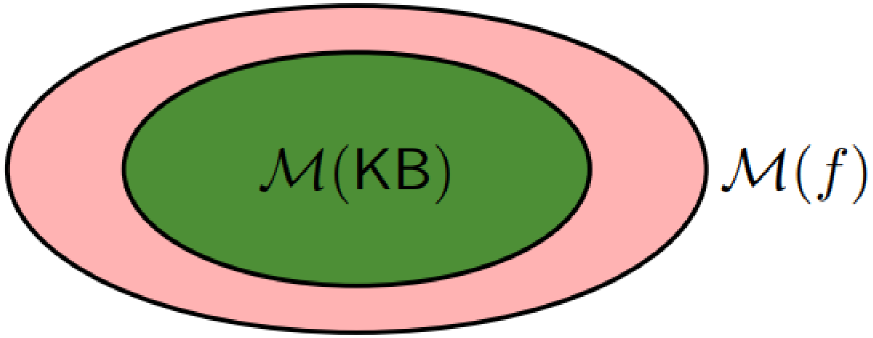
Entailment

- The new formula **did not add anything new** to the knowledge.
- KB: *"I already knew that"*

KB **entails** f (written $KB \vDash f$)

if and only if

$$\mathcal{M}(KB) \subseteq \mathcal{M}(f)$$



Entailment

- Example (2 variables):

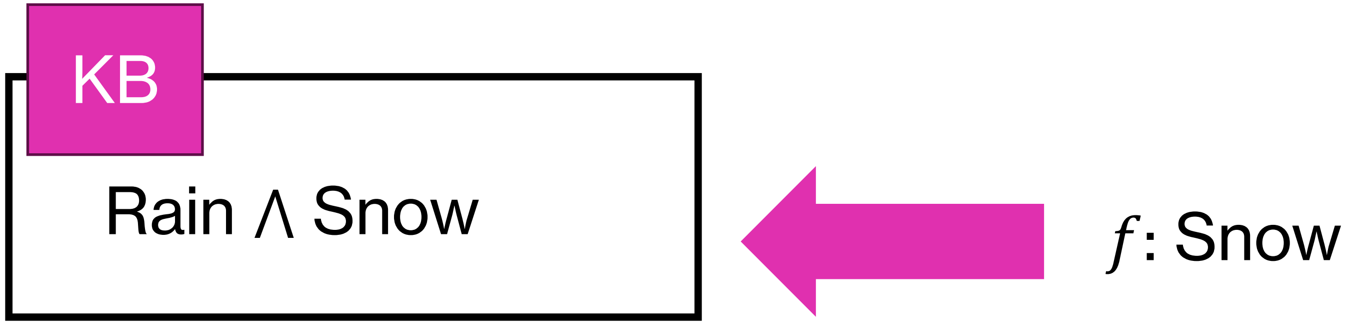
KB

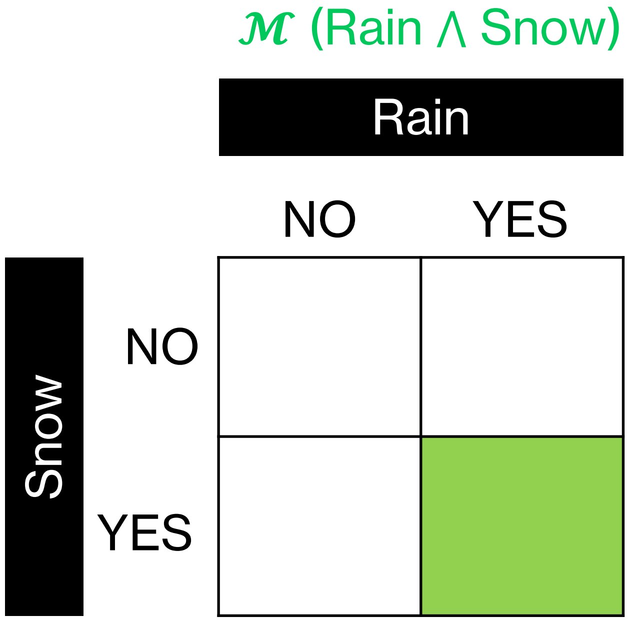
Rain ∧ Snow

f: Snow

Entailment

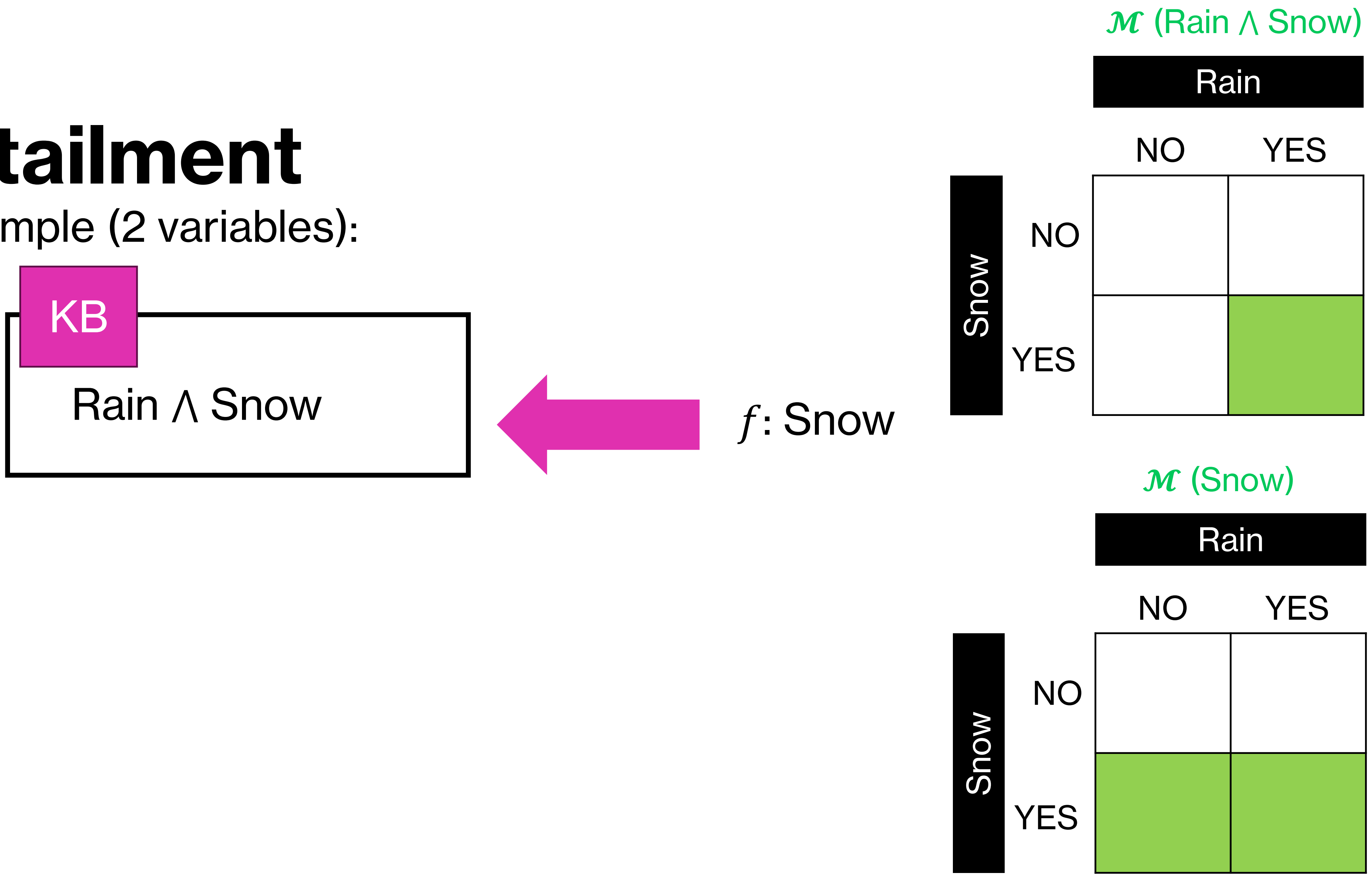
- Example (2 variables):





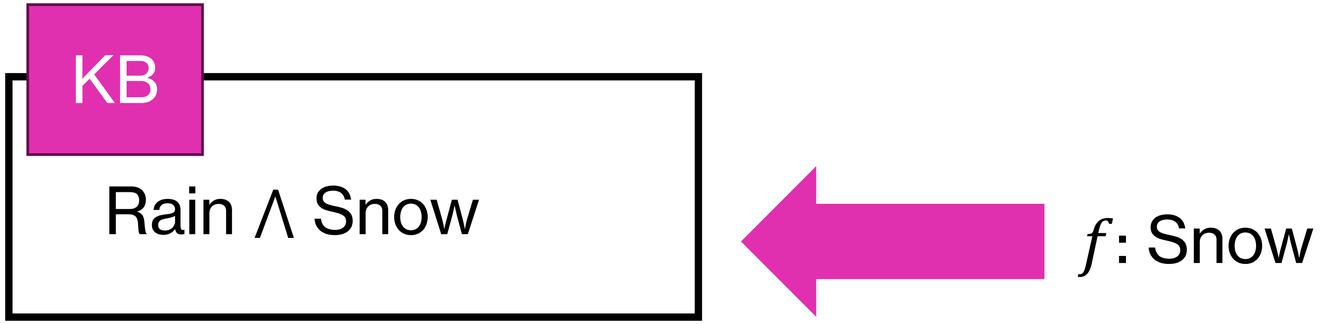
Entailment

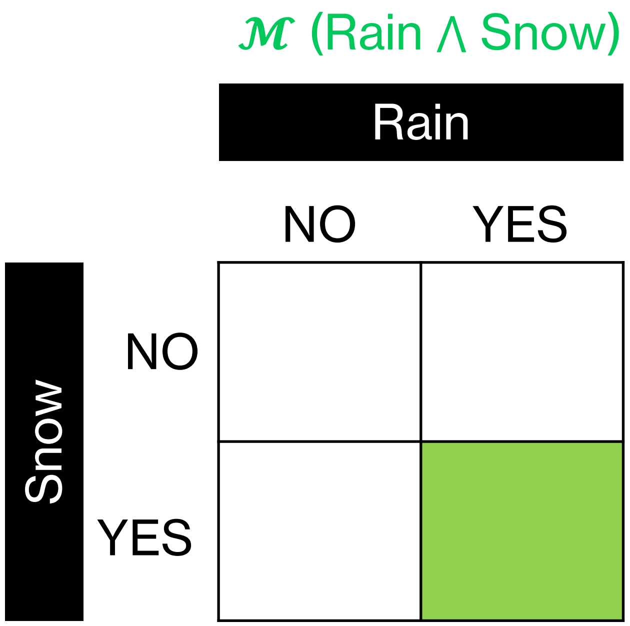
- Example (2 variables):

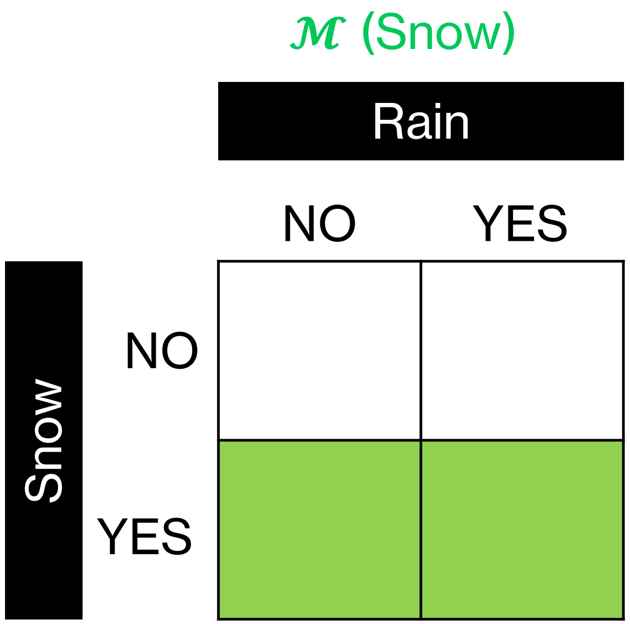


Entailment

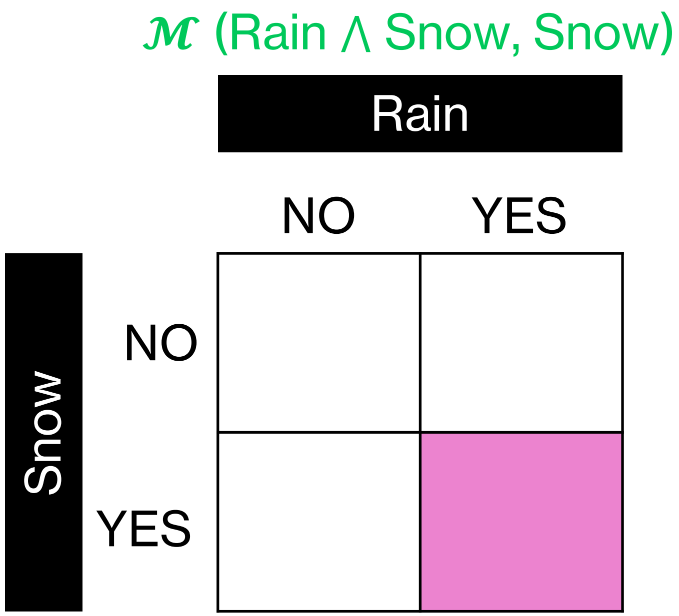
- Example (2 variables):







- **KB entails *f***



Entailment

- Example (3 variables):

KB

Rain ∧ Snow

Rain → Traffic

f: Traffic

Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Entailment

- Example (3 variables):

KB

$\boxed{\text{Rain} \land \text{Snow}}$

Rain → Traffic

f: Traffic

Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Entailment

- Example (3 variables):

KB

$\text{Rain} \land \text{Snow}$

$\text{Rain} \rightarrow \text{Traffic}$

f: Traffic

Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Reference: Implication

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Entailment

- Example (3 variables):

KB

Rain ∧ Snow

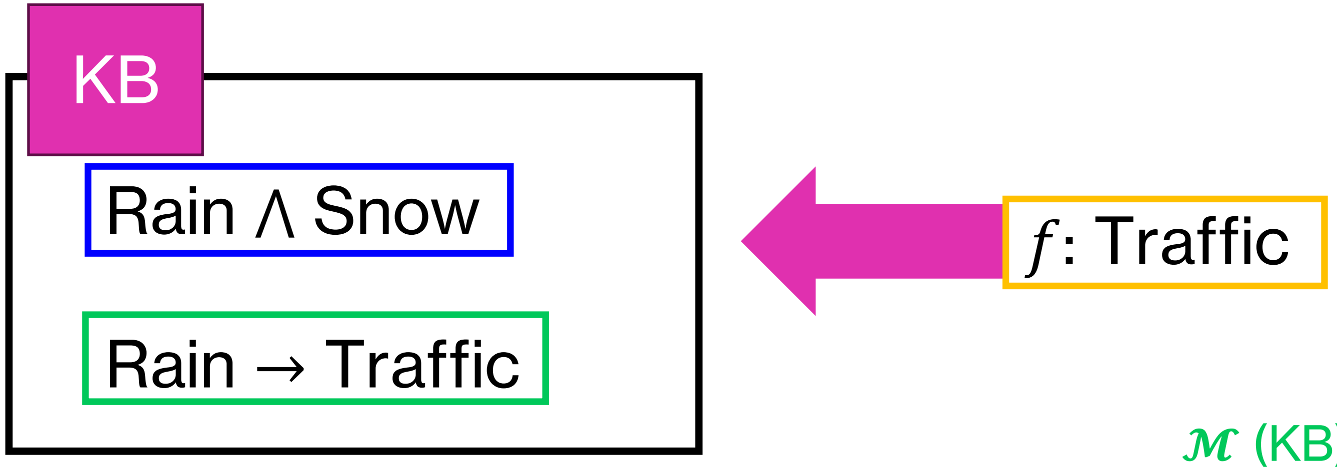
Rain → Traffic

f: Traffic

Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Entailment

- Example (3 variables):



Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

$\mathcal{M}$ (KB)

- **KB entails f**

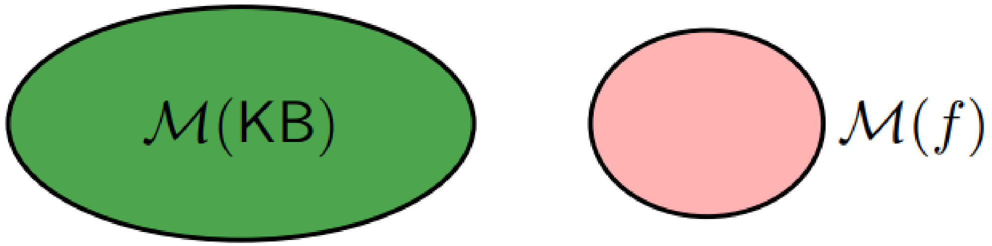
Contradiction

- The new formula **is incompatible** with the knowledge.
- KB: *"That's impossible – you can't add that"*

KB **contradicts** f

if and only if

$$\mathcal{M}(KB) \cap \mathcal{M}(f) = \phi$$



Contradiction

- Example (2 variables):

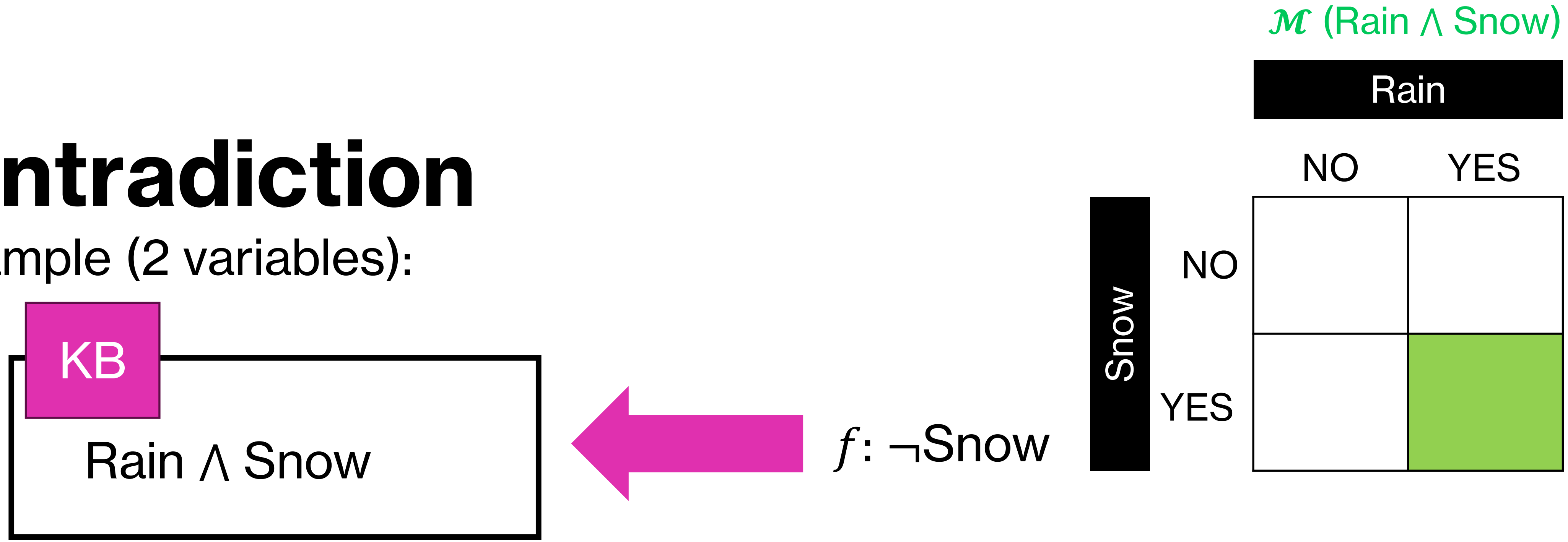
KB

Rain $\land$ Snow

f: $\neg$Snow

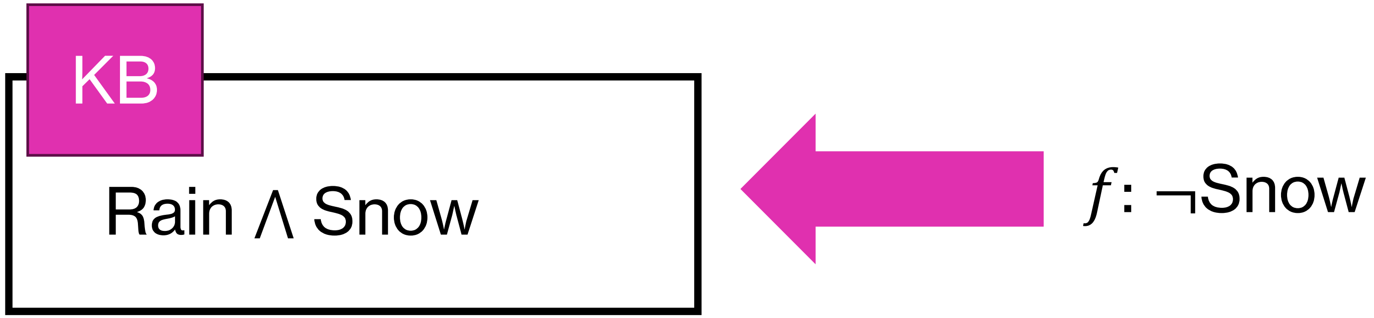
Contradiction

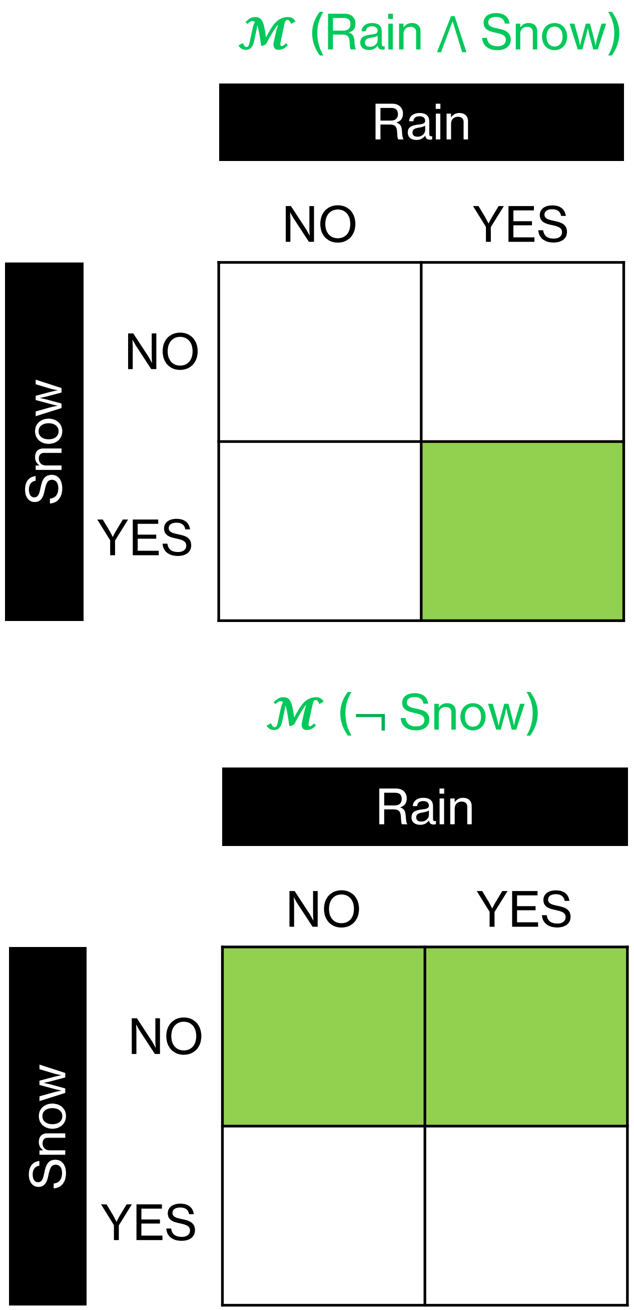
- Example (2 variables):



Contradiction

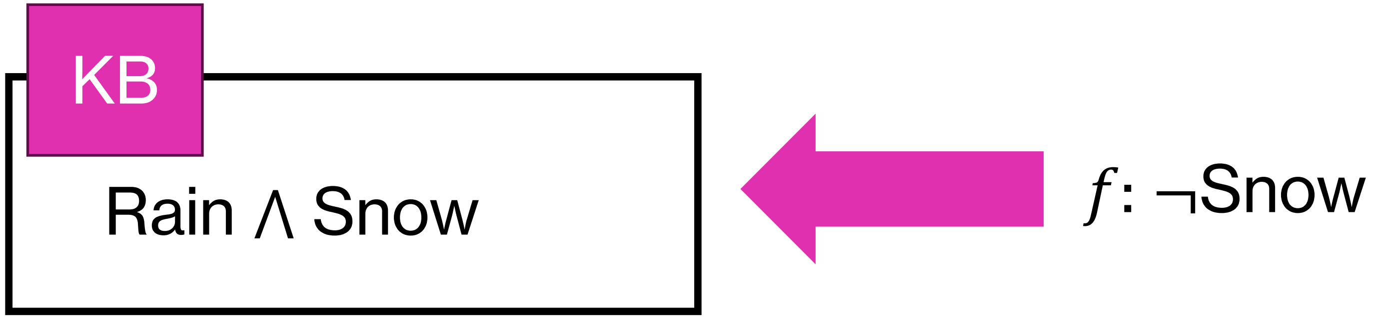
- Example (2 variables):

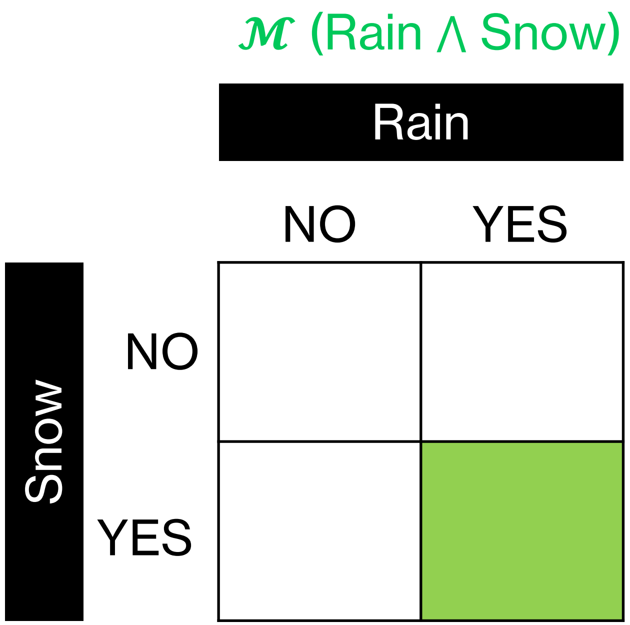


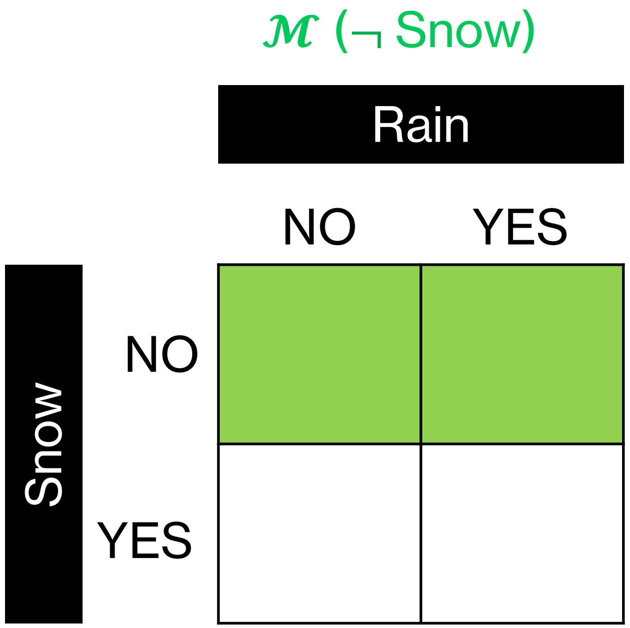


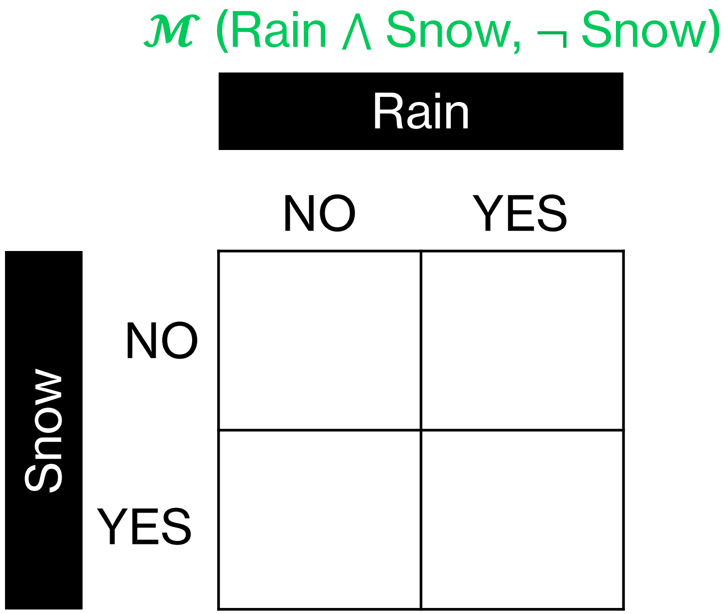
Contradiction

- Example (2 variables):









- **KB contradicts *f***

Contradiction

- Example (3 variables):

KB

Rain ∧ Snow

Rain → Traffic

f: ¬Traffic

Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Contradiction

- Example (3 variables):

KB

$\boxed{\text{Rain} \land \text{Snow}}$

Rain → Traffic

← f: ¬Traffic

Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
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1	0	1
1	1	0
1	1	1

Contradiction

- Example (3 variables):

KB

Rain ∧ Snow

Rain → Traffic

f: ¬Traffic

Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Reference: Implication

p	q	p ⇒ q
0	0	1
0	1	1
1	0	0
1	1	1

Contradiction

- Example (3 variables):

KB

Rain ∧ Snow

Rain → Traffic

f: ¬Traffic

No intersection!

Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

- **KB contradicts f**

Contradiction

- Try this:

KB

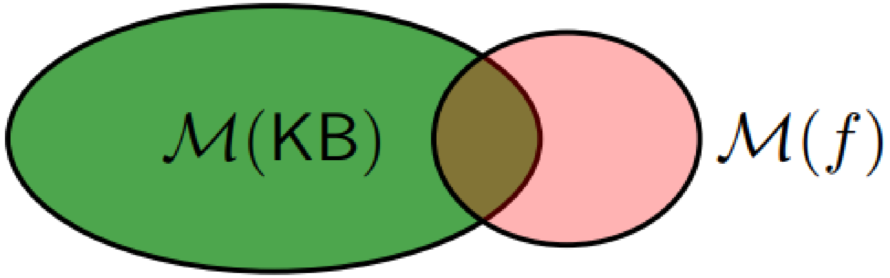
Rain ∨ Snow
Rain → Traffic

f: ¬Rain ∧ ¬Snow

Contingency

- The new formula **added something non-trivial** to the knowledge.
- KB: *"I learned something new"*

Contingency happens if and only if
$$\phi \subset \mathcal{M}(KB) \cap \mathcal{M}(f) \subset \mathcal{M}(KB)$$



Contingency

- Example (2 variables):

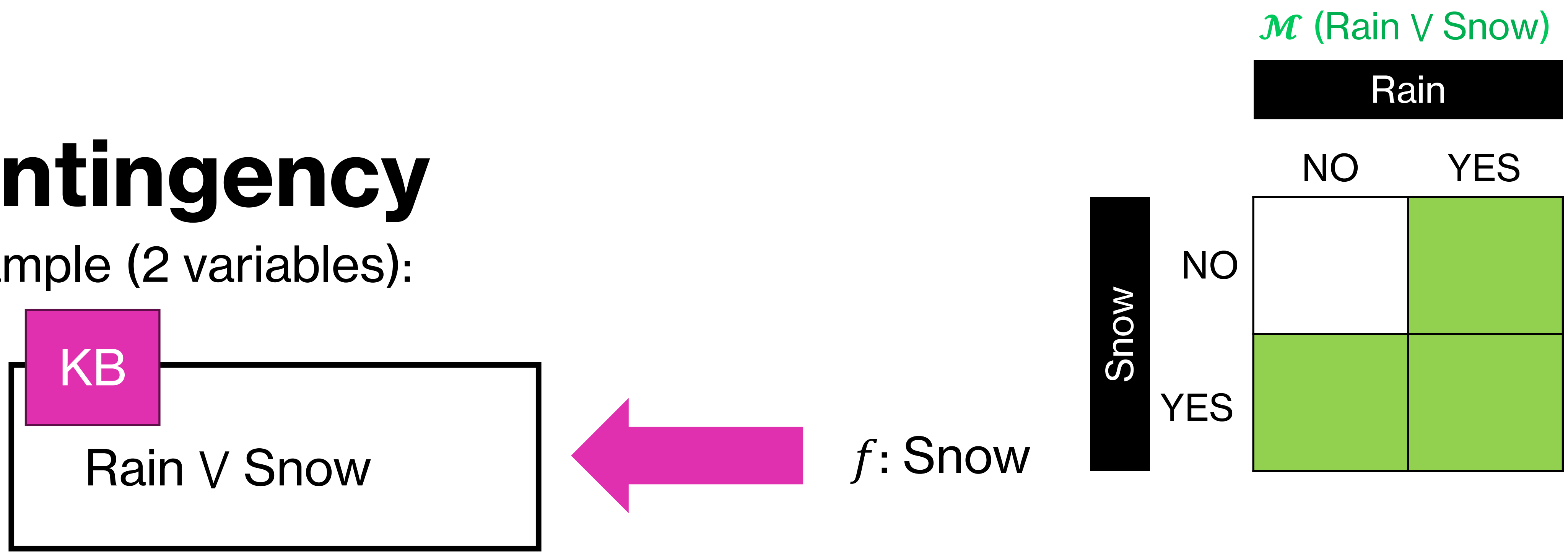
KB

Rain ∨ Snow

f: Snow

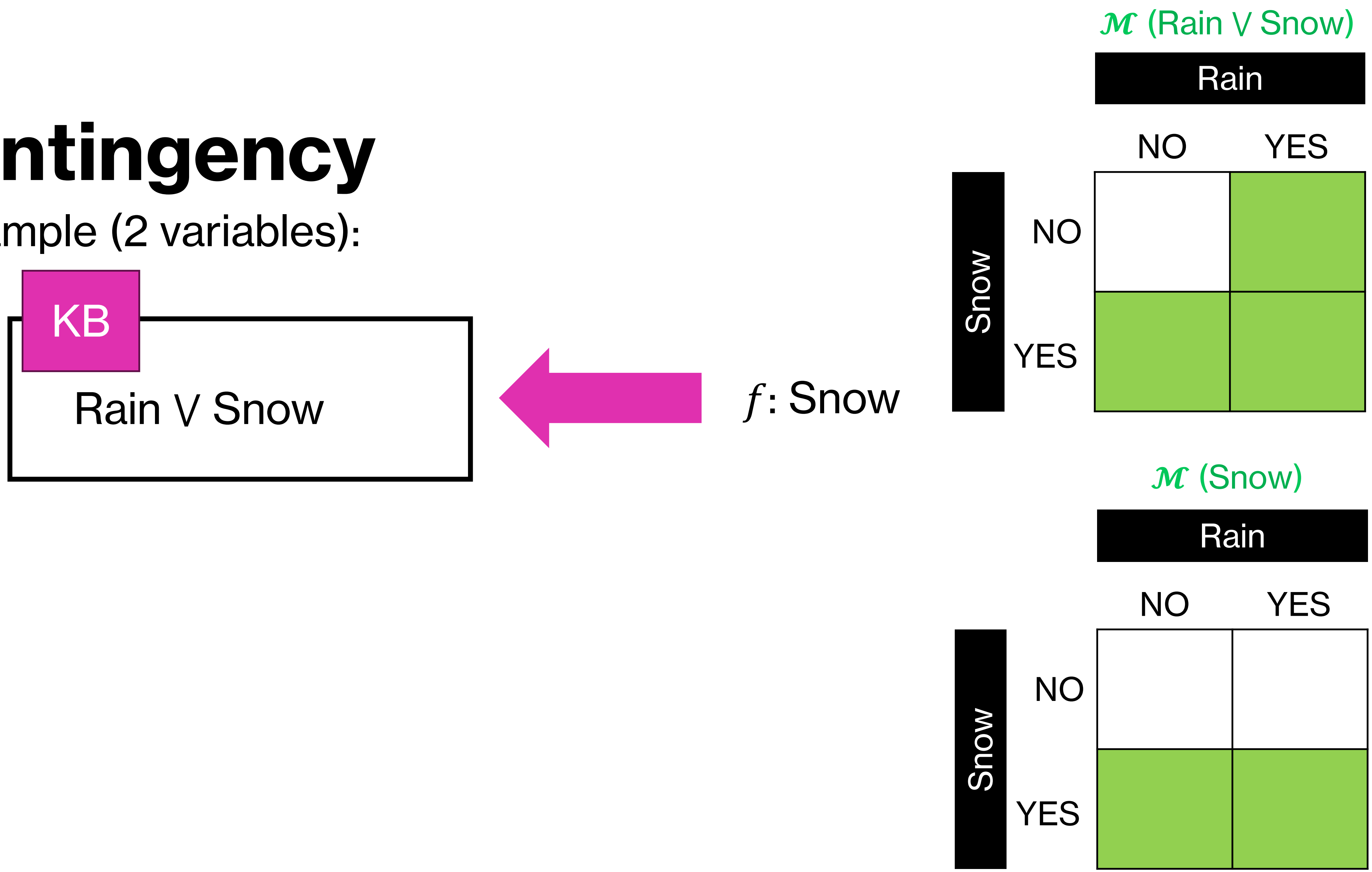
Contingency

- Example (2 variables):



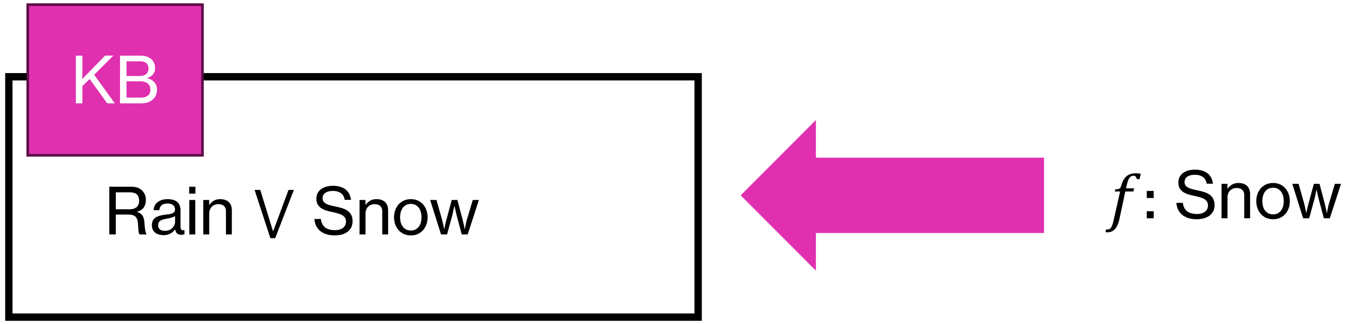
Contingency

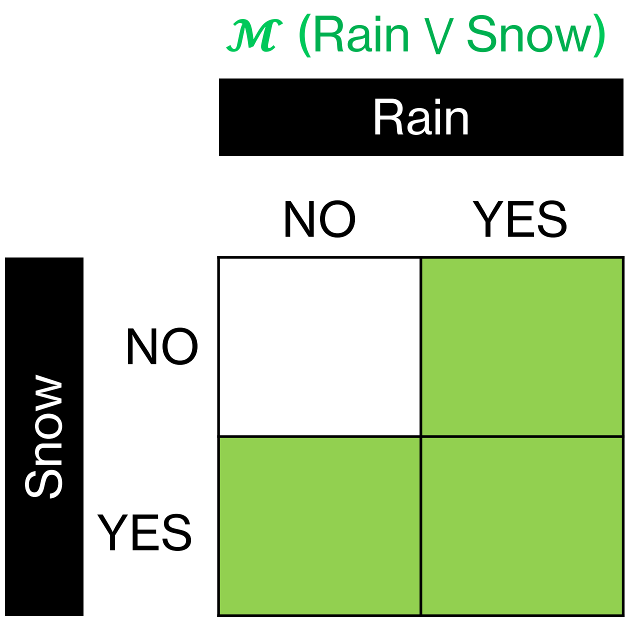
- Example (2 variables):

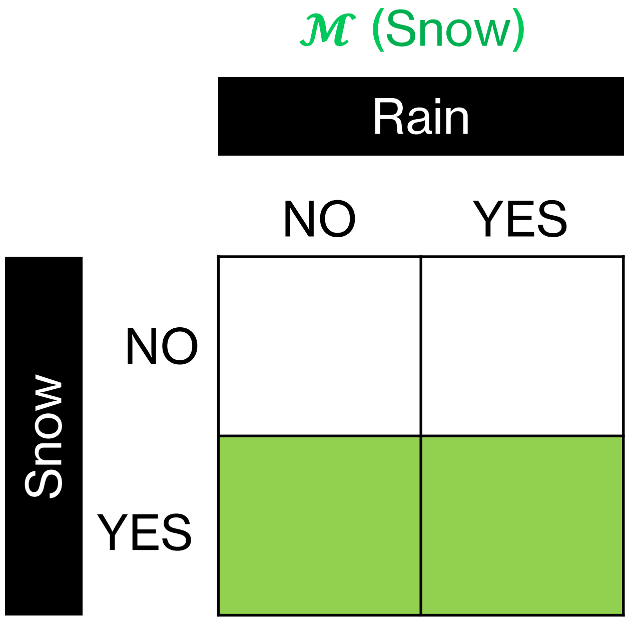


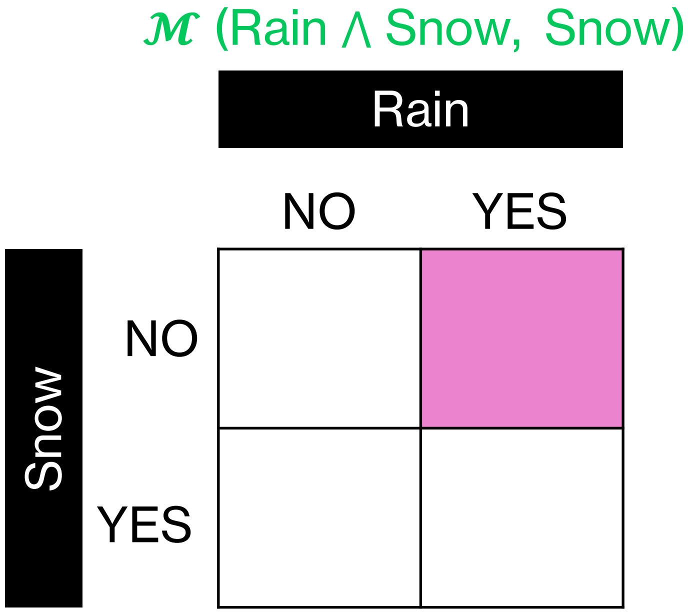
Contingency

- Example (2 variables):









- ***f* is added to KB**

Contingency

- Example (3 variables):

KB

Rain ∨ Snow

Rain → Traffic

f: Traffic

Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Contingency

- Example (3 variables):

KB

$\boxed{\text{Rain} \lor \text{Snow}}$

Rain → Traffic

f: Traffic

Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Contingency

- Example (3 variables):

KB

$\boxed{\text{Rain} \lor \text{Snow}}$

$\boxed{\text{Rain} \rightarrow \text{Traffic}}$

f: Traffic

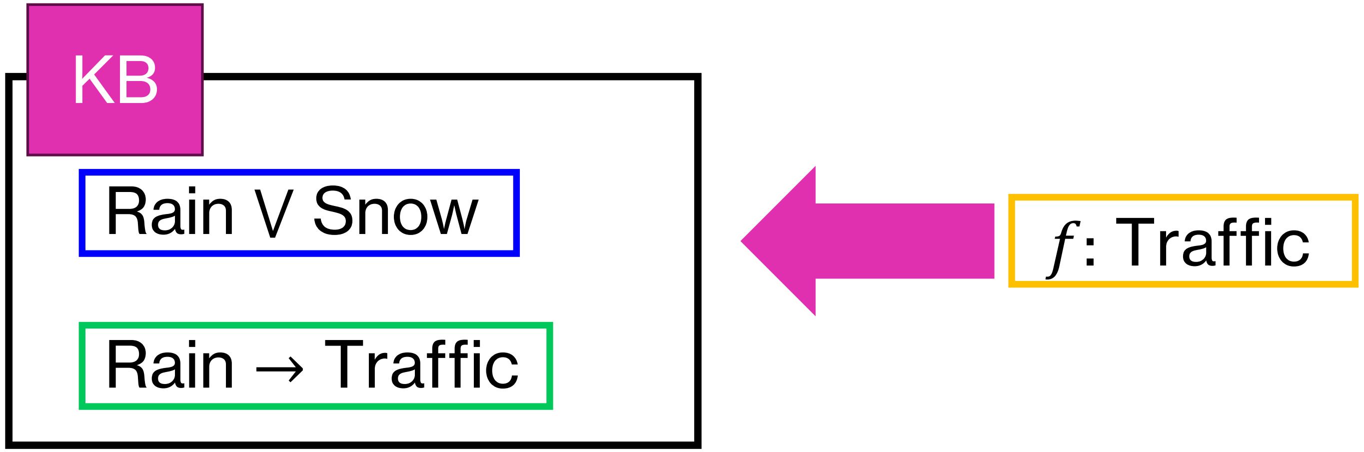
Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Reference: Implication

p	q	p ⇒ q
0	0	1
0	1	1
1	0	0
1	1	1

Contingency

- Example (3 variables):



Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Contingency

- Example (3 variables):

KB

Rain ∨ Snow

Rain → Traffic

f: Traffic

Possibly new knowledge! →

Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

- **f is added to KB**
- **Contingency happens (non-trivial knowledge was added to KB)**

Contingency

- Try this:

KB

Rain ∨ Snow
Rain → Traffic

f: ¬Traffic

Knowledge Base Queries

	Tell (add something to the KB)	Ask (query if a formula is true)
Entailment	“I already know that”	“Yes”
Contradiction	“That’s impossible”	“No”
Contingency	“I learned something new”	“I’m not sure”

Satisfiability

- A knowledge base is **satisfiable** if $\mathcal{M}(KB) \neq \emptyset$

Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

KB

- Rain ∧ Snow
- Rain → Traffic
- ¬Traffic

Not Satisfiable!

KB

- Rain ∨ Snow
- Rain → Traffic
- ¬Traffic

Satisfiable!

Rain	Snow	Traffic
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

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  - Norshuhani Zamin, PhD