A game theoretic model of the behavioural gaming that takes place at the EMS - ED interface

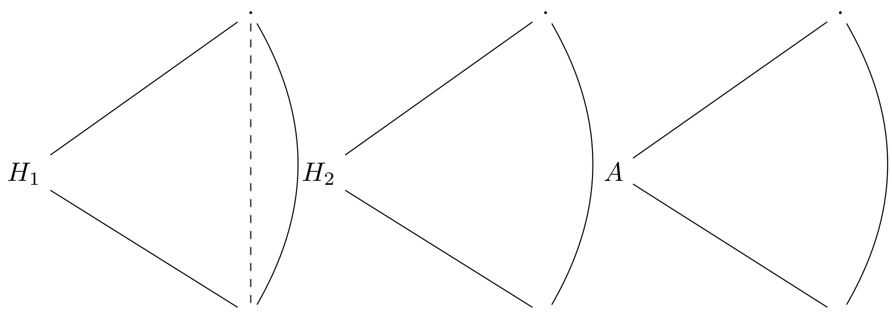

Figure 1: Ambulance Decision Problem

States:

1. A = Ambulance
2. H_i = Hospital i

Notation:

- Λ = total number of patients that need to be hospitalised
- p_i = proportion of patients going to Hospital i ($p_i\Lambda$ = number of patients going to hospital i)
- d_i = distance from Hospital i
- $\hat{c}_i$ = capacity of hospital i
- $W(c, \lambda\, \mu)$ = waiting time in the system function
- μ_i = service rate of hospital i
- λ_i^o = arrival rate of other patients to the hospital (not by ambulance)
- $C_i(p_i) = d_i + W(c = \hat{c}_i,\ \lambda = p_i\Lambda + \lambda_i^o,\ \mu = \mu_i)$

1 Game Theory component:

Players:

- Ambulance
- Hospital A
- Hospital B

Strategies of players:

- Hospital i:
  1. Close doors at $\hat{c}_i = 1$
  2. Close doors at $\hat{c}_i = 2$
  3. ...
  4. Close doors at $\hat{c}_i = C_i$
- Ambulance:
  1. Choose $p_1 \in [0, 1]$

Cost Functions: Waiting times + the distance to each hospital.

2 Formulas

$$\hat{c}_i \in \{1, 2, \dots, C_i\}$$

$$\rho_i = \frac{p_i \Lambda + \lambda_i^o}{\hat{c}_i \mu_i}$$

$$(W_q)_i = \frac{1}{\hat{c}_i \mu_i} \frac{(\hat{c}_i \rho_i)^{\hat{c}_i}}{\hat{c}_i!(1-\rho_i)^2} (P_0)_i$$

$$(P_0)_i = \frac{1}{\sum_{n=0}^{\hat{c}_i - 1} \left[\frac{(\hat{c}_i \rho_i)^n}{n!} \right] + \frac{(\hat{c}_i \rho_i)^{\hat{c}_i}}{\hat{c}_i!(1-\rho_i)}}$$

3 Quick Methodology

- Fix the parameters Λ, λ_i^o, μ_i and C_i.
- $\forall\ \hat{c_i} \in \{1, 2, \ldots, C_A\}$ and $\forall\ \hat{c_j} \in \{1, 2, \ldots, C_B\}$
- Calculate p_A and $p_B = 1 - p_A$ s.t. $(W_q)_A = (W_q)_B$.
- Calculate the probability $P((W_q)_i \leq 4$ hours
- Fill matrix A with $U^A_{\hat{c_i},\hat{c_j}} = 1 - |0.95 - P((W_q)_A \leq 4)|$ and
- fill matrix B with $U^B_{\hat{c_i},\hat{c_j}} = 1 - |0.95 - P((W_q)_B \leq 4)|$

$$A = \begin{array}{|c|c|c|c|} \hline U^A_{1,1} & U^A_{1,2} & \cdots & U^A_{1,C_B} \\ \hline U^A_{2,1} & U^A_{2,2} & \cdots & U^A_{2,C_B} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline U^A_{C_A,1} & U^A_{C_A,2} & \cdots & U^A_{C_A,C_B} \\ \hline \end{array}$$

$$B = \begin{array}{|c|c|c|c|} \hline U^B_{1,1} & U^B_{1,2} & \cdots & U^B_{1,C_B} \\ \hline U^B_{2,1} & U^B_{2,2} & \cdots & U^B_{2,C_B} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline U^B_{C_A,1} & U^B_{C_A,2} & \cdots & U^B_{C_A,C_B} \\ \hline \end{array}$$

- Ambulance decides the proportion of people to distribute to each hospital based on optimal patient distribution.

4 Distribution of waiting time

$$P(W_q > T) = \frac{(\frac{\lambda}{\mu})^c P_0}{c!(1 - \frac{\lambda}{c\mu})}(e^{-(c\mu-\lambda)T}) \tag{1}$$

5 Proper Methodology

The problem is formulated as a normal form game where the players are the two hospitals. Each hospital is given C_A and C_B number of strategies where C_A and C_B are the total capacities of the hospitals. In other words, depending on the capacity of each hospital, they may choose to stop receiving patients from arriving ambulances whenever they reach a certain capacity threshold. The goal of this problem is to satisfy the ED regulations which state that 95% of the patients should see a specialist within 4 hours of their arrival to the hospital. The mean of the random variable W_q is the average waiting time in the queue for hospital i.

$$W_q(\lambda_i, \mu_i, \hat{c_i}) = \frac{1}{\hat{c_i}\mu_i} \frac{(\hat{c_i}\rho_i)^{\hat{c_i}}}{\hat{c_i}!(1-\rho_i)^2} P_0, \quad i \in \{A, B\} \tag{2}$$

Thus, the utilities of the two players should be the proportion of people that fall within the 4 hours target. This is also equivalent to the probability of the waiting time of an individual to be less than or equal to 4 hours.

$$P(W_q(\lambda_i, \mu_i, \hat{c_i}) \leq 4), \quad i \in \{A, B\} \tag{3}$$

Therefore, a sensible goal for each player should be to minimise that probability, but the actual target of the hospitals is to satisfy 95% of those patients within the 4-hour time limit. Therefore, the goal should be to get that probability as close to 0.95 as possible. Thus each player should aim to minimise:

$$|0.95 - P(W_q(\lambda_i, \mu_i, \hat{c_i}) \leq 4)|, \quad i \in \{A, B\} \tag{4}$$

The classic formulation of a normal form game looks into the maximisation of each player's payoff. Consequently the utilities can be altered such that the goal of each player is to maximise:

$$U^A_{\hat{c_A}, \hat{c_B}} = 1 - |0.95 - P(W_q(\lambda_A, \mu_A, \hat{c_A}) \leq 4)| \tag{5}$$
$$U^B_{\hat{c_A}, \hat{c_B}} = 1 - |0.95 - P(W_q(\lambda_B, \mu_B, \hat{c_B}) \leq 4)| \tag{6}$$

Finally, the problem can be expressed as a normal form game with two players where each player/hospital has C_A and C_B strategies respectively. The two $C_A \times C_B$ payoff matrices for the utilities of the two hospitals can be defined as:

$$\text{A} = \begin{array}{|c|c|c|c|} \hline U^A_{1,1} & U^A_{1,2} & \dots & U^A_{1,C_2} \\ \hline U^A_{2,1} & U^A_{2,2} & \dots & U^A_{2,C_2} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline U^A_{C_1,1} & U^A_{C_1,2} & \dots & U^A_{C_1,C_2} \\ \hline \end{array} \qquad \text{B} = \begin{array}{|c|c|c|c|} \hline U^B_{1,1} & U^B_{1,2} & \dots & U^B_{1,C_2} \\ \hline U^B_{2,1} & U^B_{2,2} & \dots & U^B_{2,C_2} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline U^B_{C_1,1} & U^B_{C_1,2} & \dots & U^B_{C_1,C_2} \\ \hline \end{array}$$

Once the certain strategies of the game have been selected the ambulance service can decide what would be the optimal way to distribute patients. However, the way the ambulance service distributes patients can affect the utilities of the game. So how would one solve this kind of problem?

5.1 Solution

As mentioned before the problem requires the construction of two queuing models that will be needed for the formulation of the normal form game. Based on those utilities the ambulance service will then decide the percentage of patients that will distribute to each hospital.

First and foremost, the model consists of several parameters that are unknown and are assumed to be fixed. The model will be run multiple times for various values of these parameters.

Λ	Number of patients that need to be distributed
λ_i^o	Arrival rate of other patients that enter hospital i
μ_i	Service rate of hospital i
C_i	Total capacity of hospital i

Table 1: Fixed Parameters

Having established the fixed parameters of the model, the hospitals' utilities need to be calculated. In order to do so a backwards induction approach will be used. The EMS aims to distribute the patients such that the mean waiting time of patients is minimal. This can be further interpreted as when the mean waiting time of hospital A equals the mean waiting time of hospital B. Thus, the minimal mean waiting time can be found for the values of p_A and p_B that solve the following equation:

$$W_q(\lambda_A, \mu_A, \hat{c_A}) = W_q(\lambda_B, \mu_B, \hat{c_B}) \tag{7}$$

Equation (7) needs to be solved for all values of $c_i \in \{1, 2, \dots C_A\}$ and $c_j \in \{1, 2, \dots C_B\}$. Then, for every c_i and c_j the utility equation (5) has to be calculated for both hospitals. In order to solve it though, one must first estimate the probability $P[(W_q)_{\{A,B\}}] \leq 4]$. That is the probability that the waiting time in the queue for one of the hospitals is less than 4 hours. For a multi-server system, the distribution of the waiting time can be given by equation 8. The above expression returns the probability that the waiting time in the queue is less than some time T.

$$P(W_q > T) = \frac{(\frac{\lambda}{\mu})^c P_0}{c!(1 - \frac{\lambda}{c\mu})} (e^{-(c\mu - \lambda)T}) \tag{8}$$

Consequently when incorporating equation (8) into (5) a newer utility equation can be acquired:

$$U_{\hat{c_i}, \hat{c_j}}^{\{A,B\}} = 1 - \left| \left[\frac{(\frac{\lambda}{\mu})^c P_0}{c!(1 - \frac{\lambda}{c\mu})} \left(e^{-(c\mu - \lambda)T} \right) \right] - 0.05 \right| \tag{9}$$

$$A = \begin{array}{|c|c|c|c|} \hline U_{1,1}^A & U_{1,2}^A & \dots & U_{1,C_2}^A \\ \hline U_{2,1}^A & U_{2,2}^A & \dots & U_{2,C_2}^A \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline U_{C_1,1}^A & U_{C_1,2}^A & \dots & U_{C_1,C_2}^A \\ \hline \end{array} \qquad B = \begin{array}{|c|c|c|c|} \hline U_{1,1}^B & U_{1,2}^B & \dots & U_{1,C_2}^B \\ \hline U_{2,1}^B & U_{2,2}^B & \dots & U_{2,C_2}^B \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline U_{C_1,1}^B & U_{C_1,2}^B & \dots & U_{C_1,C_2}^B \\ \hline \end{array}$$

6 Markov Chain Representation of Hospital

The following Markov chain represents the transition between states of a hospital while capturing the EMS interaction with it. The hospital accepts both ambulance and other patients normally until a certain threshold T is reached. When it is reached all ambulances that arrive will be marked as *"parked outside"* until the number of people in the system is reduced below T. Alternatively, if the patients in the system keep rising, they may reach the total capacity C of the hospital and the other patients will be waiting for until a service becomes free. The states of the Markov chain are denoted by (u, v) where:

- u = number of ambulances parked outside of the hospital
- v = number of patients in the hospital

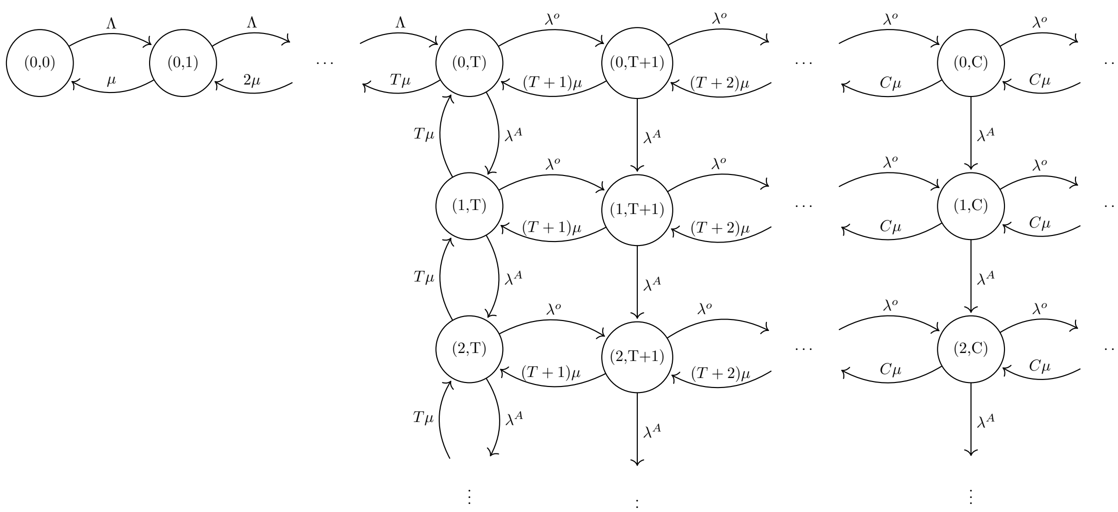

Figure 2: Markov chains

6.1 Markov-chain state mapping function

The transition matrix of the Markov-chain representation described above can be denoted by a state mapping function. The state space of this function is defined as:

$$
\begin{aligned}
S(T) =& S_1(T) \cup S_2(T) \text{ where:} \\
S_1(T) =& \left\{(0, v) \in \mathbb{N}_0^2 \; \| v < T\right\} \\
S_2(T) =& \{(u, v) \in \mathbb{N}_0^2 \; \| v \geq T\}
\end{aligned}
\tag{10}
$$

Therefore, the entries of the transition matrix Q, can be given by $q_{i,j} = q_{(u_i,v_i),(u_j,v_j)}$ which is the transition rate from state $i = (u_i, v_i)$ to state $j = (u_j, v_j)$ for all $(u_i, v_i), (u_j, v_j) \in S$.

$$q_{i,j} = \begin{cases} \Lambda, & \textbf{if } (u_i, v_i) - (u_j, v_j) = (0, -1) \textbf{ and } v_i < \text{t} \\ \lambda^o, & \textbf{if } (u_i, v_i) - (u_j, v_j) = (0, -1) \textbf{ and } v_i \geq \text{t} \\ \lambda^a, & \textbf{if } (u_i, v_i) - (u_j, v_j) = (-1, 0) \\ v_i \mu, & \textbf{if } \{(u_i, v_i) - (u_j, v_j) = (0, 1) \textbf{ and } v_i \leq C\} \\ C\mu, & \textbf{if } \{(u_i, v_i) - (u_j, v_j) = (0, 1) \textbf{ and } v_i > C\} \\ T\mu & \quad \{(u_i, v_i) - (u_j, v_j) = (1, 0) \textbf{ and } v_i = \text{t}\} \\ 0, & \textbf{otherwise} \end{cases} \tag{11}$$

In order to acquire an exact solution of the problem a slight adjustment needs to be considered. The problem defined above assumes no upper boundary to the number of people that can wait for service or the number of ambulances that can be parked outside. Therefore, a different state space $\tilde{S}$ needs to be constructed where $\tilde{S} \subseteq S$ and there is a maximum allowed number of people N that can be in the system and a maximum allowed number of ambulances M parked outside:

$$\tilde{S} = \{(u, v) \in S \; | u \leq M, v \leq N \; \} \tag{12}$$

6.2 Steady State

Having calculated the transition matrix Q for a given set of parameters the probability vector π needs to be considered. The vector π is commonly used to study such stochastic systems and it's main purpose is to keep track of the probability of being at any given state. Thus, by the definition of the steady state vector π the relationship between it and Q is given by:

$$\frac{d\pi}{dt} = \pi Q$$

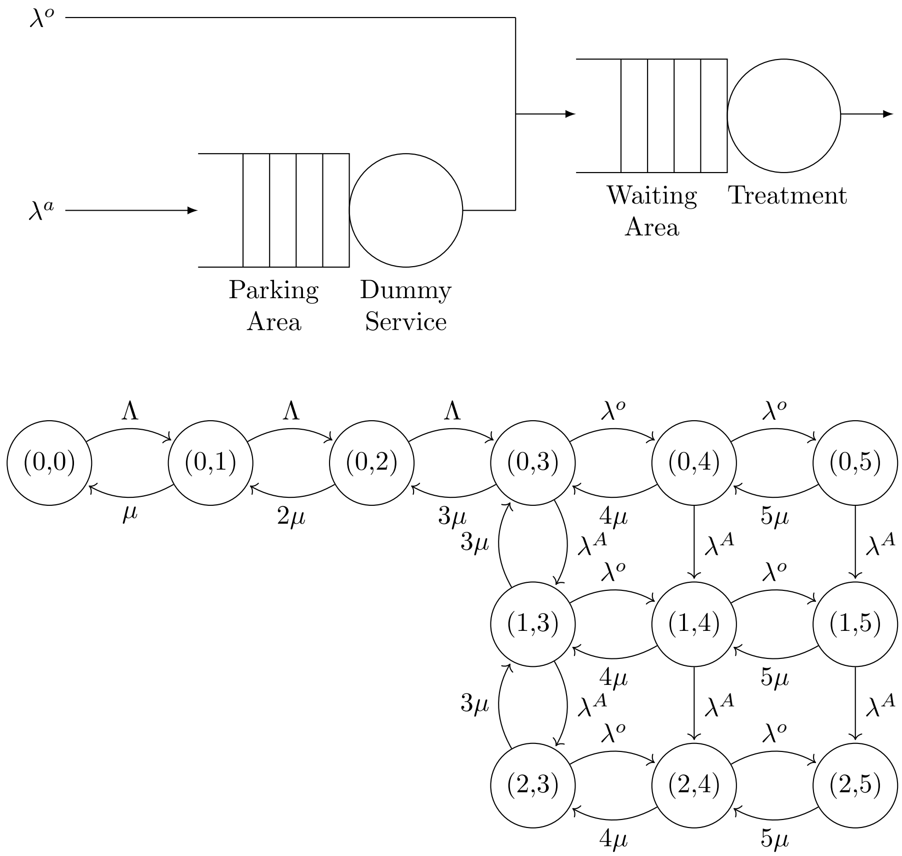

Figure 3: Markov chains