A game theoretic model of the behavioural gaming that takes place at the EMS - ED interface

Michalis Panayides

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2 Overview of game theoretic model

The problem studied is a 3-player normal form game. The players are:

- the decision makers of two queueing systems;
- a service that distributes individuals to these two queueing systems.

This is a standard Normal form game [1], in that each player in this game has their own objectives which they aim to optimise. More specifically, the queueing systems' objective is captured by an upper bound of the time that a fixed proportion of individuals spend in the system, while the distributor aims to minimise the time that its individuals are blocked.

The queueing systems are designed in such a way where they can accept two types of individuals. These are the individuals that the distributor allocates to them and other individuals from other sources. Each queueing system may then choose to block the individuals that arrive from the distributor when the system reaches a certain capacity. The strategy sets for each queueing system is the set $\{T \in \mathbb{N} \mid 1 \leq T \leq N\}$ where $N \in \{N_A, N_B\}$ are the total capacities of the two queueing systems. We denote the chosen actions from the strategy set as T_A, T_B and call these *thresholds*.

Both queueing systems follow a queueing model that has two waiting spaces for individuals. The first waiting zone is where the individuals queue right before receiving their service and has a capacity of $N-C$, where N is the total capacity of the waiting space and C is the number of servers. The second waiting zone is where the individuals, that are sent from the distributor, remain until they are allowed to enter the first waiting zone. The second waiting zone has a capacity of M and no servers.

This is shown diagrammatically in Figure 1.

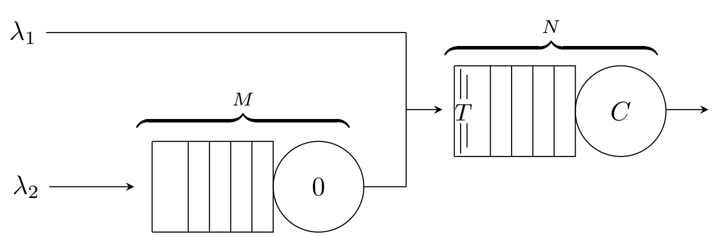

Figure 1: A diagrammatic representation of the queueing model. The threshold T only applies to arrivals from the first buffer. If the second buffer is at that threshold only individuals of the first type are accepted (at a rate λ_1) and individuals of the second type (arriving at a rate λ_2) are held blocked in the first buffer.

Note here that both types of individuals can become lost to the system. Individual allocated from the distributor become lost to the system whenever an arrival occurs and the second waiting zone is at full capacity (M individuals already waiting). Similarly, other individuals get lost whenever they arrive at the first waiting zone and it is at full capacity ($N - C$ individuals already waiting).

Following this queuing model, the two queueing systems' choice of strategy will then rely solely on satisfying their own objective, which is to make sure that the waiting time in the first waiting zone of a proportion of individuals will be below the predefined target time.

$$P(W < R) \geq \hat{P} \tag{1}$$

where W is the mean waiting time of all individuals, R is the time target and $\hat{P}$ is the percentage of individuals need to be within that target. There are numerous objective functions that can be used to capture this behaviour. For example one approach is to use the threshold that maximises the probability that the mean waiting is more than the target time, and completely ignore the percentage goal.

$$\arg\max_{T_i} \quad P(W_i < R) \tag{2}$$

A more sophisticated objective function would be to get the proportion of individuals as close to the percentage aim. In other words, to find the threshold that minimises the difference between the probability and the percentage goal (or maximise its negation).

$$\arg\max_{T_i} \quad -\left(\hat{P} - P(W_i < R)\right)^2 \tag{3}$$

The third player, the distributor has their own choices to make and their own goals to satisfy. The strategy set of the third player is the proportion $0 \leq p \leq 1$ of individuals it sends to the first queueing system (the proportion $1 - q$ is sent to the second queueing system). In addition, the distributor aims to minimise any potential blockages that may occur, given the pair of thresholds chosen by the two queueing systems. Thus, its objective is to minimise the blocked time of the individuals that they send to the two queueing systems. Apart from the time being blocked, an additional aspect that may affect the decision of the distributor is the proportion of lost individuals. Equation 4 can be used to capture a mixture between the two objectives.

$$\alpha P(L_A) + (1 - \alpha)B_A = \alpha P(L_B) + (1 - \alpha)B_B \tag{4}$$

Here, α represents the "importance" of each objective, where high α indicates a higher weight on the proportion of lost individuals and smaller α a higher weight on the time blocked.

Using equations 3 and 4 gives an imperfect information extensive form game. An imperfect information game is defined as an extensive form game where some of the information about the game state is hidden for at least one of the players [2]. In this study the state of the problem that is hidden is the threshold that each of the first two players chooses to play. In other words, each queueing system chooses to play a strategy without the knowing the other system's strategy. The distributor then, fully aware of the chosen threshold strategies, distributes individuals among the two systems in order to minimise the time that its individuals will be blocked. Figure 2 illustrates this.

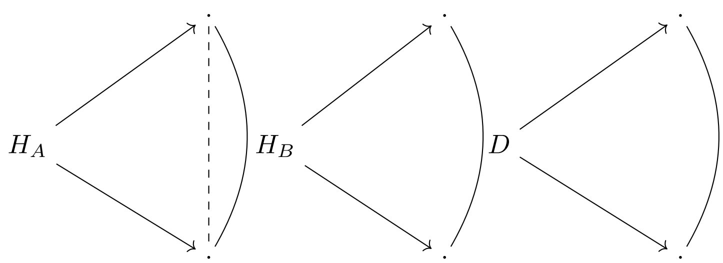

Figure 2: Imperfect information Extensive Form Game between the distributor and the 2 queueing systems

The first queueing system H_A decides on a threshold, then the second system H_B chooses its own threshold, without knowing the strategy of H_A, and finally the distributor makes its choice. Note here that the dotted line represents the fact that H_B is unaware of the state of the game when making its own decisions. The game can thus be partitioned into a normal form game between the two queueing systems and finding the distributor's best choice.

In order to define the normal form game the two payoff matrices of the players are required. From equation 3 the utilities of the players can be formulated as:

$$U^i_{T_1,T_2} = \quad -\left(\hat{P} - P(W_i < R)\right)^2 \tag{5}$$

Consequently, the payoff matrices of the game can be populated by these utilities:

$$A = \begin{pmatrix} U^A_{1,1} & U^A_{1,2} & \dots & U^A_{1,N_B} \\ U^A_{2,1} & U^A_{2,2} & \dots & U^A_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ U^A_{N_A,1} & U^A_{N_A,2} & \dots & U^A_{N_A,N_B} \end{pmatrix}, B = \begin{pmatrix} U^B_{1,1} & U^B_{1,2} & \dots & U^B_{1,N_B} \\ U^B_{2,1} & U^B_{2,2} & \dots & U^B_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ U^B_{N_A,1} & U^B_{N_A,2} & \dots & U^B_{N_A,N_B} \end{pmatrix} \tag{6}$$

Based on the choice of strategy of these two players the distributor will then make their own choice of the proportion of individuals to send to each system.

3 A queueing model with 2 consecutive buffer centres

In this section, a more in-depth explanation of the queueing model shown in figure 1 will be given. This is a queuing model that consists of two waiting spaces, one for each type of patient.

The model consists of two types of individuals; class 1 and class 2. Class 1 individuals arrive instantly at waiting zone 1 and proceed to wait to receive their service. Class 2 individuals arrive at waiting zone 2 and wait there until they are allowed to move to waiting zone 1. They are allowed to proceed only when the number of individuals in waiting zone 1 **and** in service is less than a predetermined threshold T. When the number of individuals is equal to or exceeds this threshold, all second type individuals that arrive will remain *"blocked"* in waiting zone 1 until the number of people in the system is reduced below T. This is shown diagrammatically in figure 1. The parameters of the described queueing model are:

- λ_i: The arrival rate of individuals of type $i \in \{1, 2\}$
- μ: The service rate for individuals receiving service
- C: The number of servers
- T: The threshold at which individuals of the second type are blocked

Under the assumption that all rates (arrival and service) are Markovian the queuing system corresponds to a Markov chain [3]. The states of the Markov chain are denoted by (u, v) where:

- u is the number of individuals blocked
- v is the number of individuals either in waiting zone 1 or in the service centre

We denote the state space of the Markov chain as $S = S(T)$ which can be written as the disjoint union 7.

$$\begin{aligned} S(T) =& S_1(T) \cup S_2(T) \text{ where:} \\ S_1(T) =& \left\{(0, v) \in \mathbb{N}_0^2 \mid v < T\right\} \\ S_2(T) =& \{(u, v) \in \mathbb{N}_0^2 \mid v \geq T\} \end{aligned} \tag{7}$$

The transition matrix Q of the Markov chain consists of the transition rates between the numerous states of the model. Every entry $Q_{ij} = Q_{(u_i,v_i),(u_j,v_j)}$ represents the transition rate from state $i = (u_i, v_i)$ to state $j = (u_j, v_j)$ for all $(u_i, v_i), (u_j, v_j) \in S$. The entries of Q can be calculated using the state-mapping function described in 8:

$$
Q_{ij} = \begin{cases}
\Lambda, & \textbf{if } (u_i, v_i) - (u_j, v_j) = (0, -1) \textbf{ and } v_i < \mathrm{t} \\
\lambda_1, & \textbf{if } (u_i, v_i) - (u_j, v_j) = (0, -1) \textbf{ and } v_i \geq \mathrm{t} \\
\lambda_2, & \textbf{if } (u_i, v_i) - (u_j, v_j) = (-1, 0) \\
v_i \mu, & \textbf{if } (u_i, v_i) - (u_j, v_j) = (0, 1) \textbf{ and } v_i \leq C \textbf{ or} \\
 & \quad (u_i, v_i) - (u_j, v_j) = (1, 0) \textbf{ and } v_i = T \leq C \\
C\mu, & \textbf{if } (u_i, v_i) - (u_j, v_j) = (0, 1) \textbf{ and } v_i > C \textbf{ or} \\
 & \quad (u_i, v_i) - (u_j, v_j) = (1, 0) \textbf{ and } v_i = T > C \\
-\sum_{j=1}^{|Q|} Q_{ij} & \textbf{if } i = j \\
0, & \textbf{otherwise}
\end{cases} \tag{8}
$$

Note that Λ here denotes the overall arrival rate in the model by both classes of individuals (i.e. $\Lambda = \lambda_1 + \lambda_2$). A visualisation of how the transition rates relate to the states of the model can be seen in the general Markov chain model shown in figure 3.

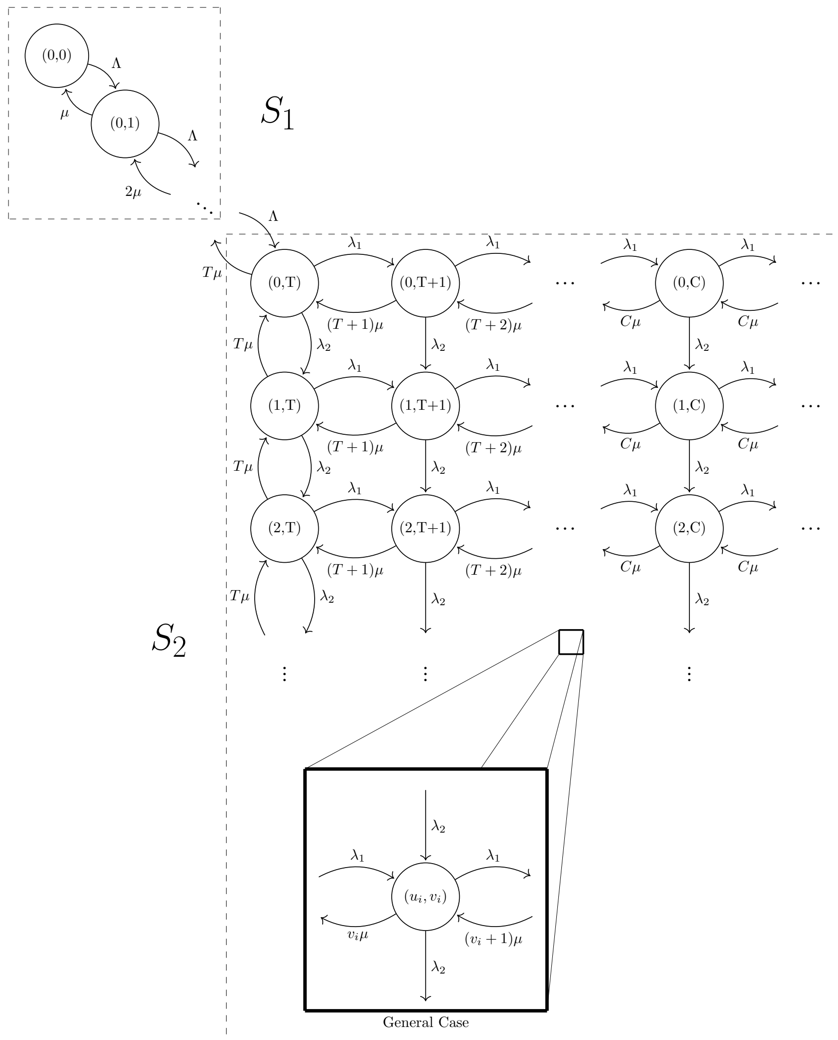

Figure 3: General case of the Markov chain model

In order to acquire an exact solution of the problem a slight adjustment needs to be considered. The problem defined above assumes no upper boundary to the number of individuals that can wait for service or for the ones that are blocked in the buffer centre. Therefore, a different state space $\tilde{S}$ needs to be constructed where $\tilde{S} \subseteq S$ and there is a maximum allowed number of people N that can be in the system and a maximum allowed number of people M that can be blocked in the buffer centre:

$$\tilde{S} = \{(u, v) \in S \, | u \leq M, v \leq N\} \tag{9}$$

3.1 Performance Measures

The transition matrix Q defined in 8 can be used to get the probability vector π. The vector π is commonly used to study stochastic systems and it's main purpose is to keep track of the probability of being at any given state of the system. The term *steady state* refers to the instance of the vector π where the probabilities of being at any state become stable over time. Thus, by considering the steady state vector π the relationship between it and Q is given by:

$$\frac{d\pi}{dt} = \pi Q = 0$$

Using vector π there are numerous performance measures of the model that can be calculated. The following equations utilise π to get performance measures on the average number of people at certain sets of state.

- Average number of people in the system:

$$L = \sum_{i=1}^{|\pi|} \pi_i (u_i + v_i)$$

- Average number of people in the service centre:

$$L_H = \sum_{i=1}^{|\pi|} \pi_i v_i$$

- Average number of people in waiting zone 2:

$$L_A = \sum_{i=1}^{|\pi|} \pi_i u_i$$

Consequently, there are some additional performance measures of interest that are not as straightforward to calculate. Such performance measures are the mean waiting time in the system (for both class 1 and class 2 individuals), the mean time blocked in waiting zone 2 (only valid for class 2 individuals) and the proportion of individuals that wait in waiting zone 1 within a predefined time target.

3.1.1 Waiting time

Waiting time is the amount of time that individuals from either class wait in waiting zone 1 so that they can receive their service. For a given set of parameters there are three different performance measures around the mean waiting time that can be calculated; the mean waiting time of class 1 individuals, the mean waiting time of class 2 individuals and the overall mean waiting time.

Since some of the individuals can be lost to the model, a new set of states needs to be defined; the set of *accepting states*. That is the set of states that the model is able to accept a certain type of individual. The set of accepting states for class 1 individuals is defined as:

$$S_A^{(1)} = \{(u,v) \in S \mid v < N\} \tag{10}$$

In essence, for class 1 individuals, this is the set of states that are not on the last column of states in the Markov chain. Equivalently, the set of accepting states for class 2 individuals is defined as:

$$S_A^{(2)} = \begin{cases} \{(u,v) \in S \mid u < M\}, & \textbf{if } T \leq N \\ \{(u,v) \in S \mid v < N\}, & \textbf{otherwise} \end{cases} \tag{11}$$

Note here that if the threshold is less than or equal the total capacity of the system the set includes all states that are not on the last column of the Markov chain. Otherwise, the set of accepting state is identical to 10. Thus, the expressions for the waiting times for class 1 and class 2 individuals are given by:

$$W^{(1)} = \frac{\sum_{\substack{(u,v)\in S_A^{(1)} \\ v \geq C}} \frac{1}{C\mu} \times (v - C + 1) \times \pi(u,v)}{\sum_{(u,v)\in S_A^{(1)}} \pi(u,v)} \tag{12}$$

$$W^{(2)} = \frac{\sum_{\substack{(u,v)\in S_A^{(2)} \\ min(v,T) \geq C}} \frac{1}{C\mu} \times (\min(v+1, T) - C) \times \pi(u,v)}{\sum_{(u,v)\in S_A^{(2)}} \pi(u,v)} \tag{13}$$

Consequently, the overall waiting time can be estimated by a linear combination of W_1 and W_2. Thus, the overall waiting time can calculated by the following equation where c_1 and c_2 are the coefficients of the terms:

$$W = c_1 W^{(1)} + c_2 W^{(2)} \tag{14}$$

The two coefficients represent the proportion of individuals of each type that did not get lost and traversed through the model. Thus, one should account for the probability that an individual is lost to the system. This probability can be easily calculated by using the two sets of accepting states $S_A^{(2)}$ and $S_A^{(1)}$ defined in equations 10 and 11. Using these equations the probability, for either class type, that an individual is not lost in the system is given by:

$$P(L'_1) = \sum_{(u,v)\in S_A^{(1)}} \pi(u,v) \qquad\qquad P(L'_2) = \sum_{(u,v)\in S_A^{(2)}} \pi(u,v)$$

Thus, by using these values as the coefficient of equation 14 the resultant equation can be used to get the overall waiting time.

$$W = \frac{\lambda_1 P(L'_1)}{\lambda_2 P(L'_2) + \lambda_1 P(L'_1)} W^{(1)} + \frac{\lambda_2 P(L'_2)}{\lambda_2 P(L'_2) + \lambda_1 P(L'_1)} W^{(2)} \tag{15}$$

3.2 Example

4 Methodology

4.1 Backwards Induction

4.2 Nash Equilibrium

4.3 Learning Algorithms

5 EMS - ED application

5.1 Application

5.2 Data Analysis

6 Conclusion

References

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[3] J. G. Kemeny and J. L. Snell, *Markov chains*, vol. 6. Springer-Verlag, New York, 1976.