A game theoretic model of the behavioural gaming that takes place at the EMS - ED interface

Michalis Panayides

Contents

1 Introduction

2 Overview of game theoretic model

The problem studied is a 3-player normal form game. The players are:

- the decision makers of two queueing systems;
- a service that distributes individuals to these two queueing systems.

This is a standard Normal form game [7], in that each player in this game has their own objectives which they aim to optimise. More specifically, the queueing systems' objective is captured by an upper bound of the time that a fixed proportion of individuals spend in the system, while the distributor aims to minimise the time that its individuals are blocked.

The queueing systems are designed in such a way where they can accept two types of individuals. These are the individuals that the distributor allocates to them and other individuals from other sources. Each queueing system may then choose to block the individuals that arrive from the distributor when the system reaches a certain capacity. The strategy sets for each queueing system is the set $\{T \in \mathbb{N} \mid 1 \leq T \leq N\}$ where $N \in \{N_A, N_B\}$ are the total capacities of the two queueing systems. We denote the chosen actions from the strategy set as T_A, T_B and call these *thresholds*.

Both queueing systems follow a queueing model that has two waiting spaces for individuals. The first waiting zone is where the individuals queue right before receiving their service and has a capacity of $N - C$, where N is the total capacity of the waiting space and C is the number of servers. The second waiting zone is where the individuals, that are sent from the distributor, remain until they are allowed to enter the first waiting zone. The second waiting zone has a capacity of M and no servers.

This is shown diagrammatically in Figure 1.

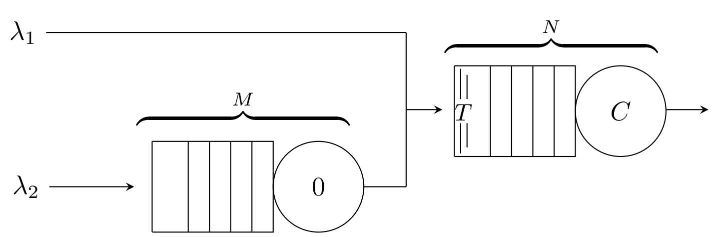

Figure 1: A diagrammatic representation of the queueing model. The threshold T only applies to arrivals from the first buffer. If the second buffer is at that threshold only individuals of the first type are accepted (at a rate λ_1) and individuals of the second type (arriving at a rate λ_2) are held blocked in the first buffer.

Note here that both types of individuals can become lost to the system. Individual allocated from the distributor become lost to the system whenever an arrival occurs and the second waiting zone is at full capacity (M individuals already waiting). Similarly, other individuals get lost whenever they arrive at the first waiting zone and it is at full capacity ($N - C$ individuals already waiting).

Following this queuing model, the two queueing systems' choice of strategy will then rely solely on satisfying their own objective, which is to make sure that the waiting time in the first waiting zone of a proportion of individuals will be below the predefined target time.

$$P(W < R) \geq \hat{P} \tag{1}$$

where W is the mean waiting time of all individuals, R is the time target and $\hat{P}$ is the percentage of individuals need to be within that target. There are numerous objective functions that can be used to capture this behaviour. For example one approach is to use the threshold that maximises the probability that the mean waiting is more than the target time, and completely ignore the percentage goal.

$$\arg\max_{T_i} \quad P(W_i < R) \tag{2}$$

A more sophisticated objective function would be to get the proportion of individuals as close to the percentage aim. In other words, to find the threshold that minimises the difference between the probability and the percentage goal (or maximise its negation).

$$\arg\max_{T_i} \quad -\left(\hat{P} - P(W_i < R)\right)^2 \tag{3}$$

The third player, the distributor has their own choices to make and their own goals to satisfy. The strategy set of the third player is the proportion $0 \leq p \leq 1$ of individuals it sends to the first queueing system (the proportion $1 - q$ is sent to the second queueing system). In addition, the distributor aims to minimise any potential blockages that may occur, given the pair of thresholds chosen by the two queueing systems. Thus, its objective is to minimise the blocked time of the individuals that they send to the two queueing systems. Apart from the time being blocked, an additional aspect that may affect the decision of the distributor is the proportion of lost individuals. Equation 4 can be used to capture a mixture between the two objectives.

$$\alpha P(L_A) + (1 - \alpha) B_A = \alpha P(L_B) + (1 - \alpha) B_B \tag{4}$$

Here, α represents the "importance" of each objective, where high α indicates a higher weight on the proportion of lost individuals and smaller α a higher weight on the time blocked.

Using equations 3 and 4 gives an imperfect information extensive form game. An imperfect information game is defined as an extensive form game where some of the information about the game state is hidden for at least one of the players [2]. In this study the state of the problem that is hidden is the threshold that each of the first two players chooses to play. In other words, each queueing system chooses to play a strategy without the knowing the other system's strategy. The distributor then, fully aware of the chosen threshold strategies, distributes individuals among the two systems in order to minimise the time that its individuals will be blocked. Figure 2 illustrates this.

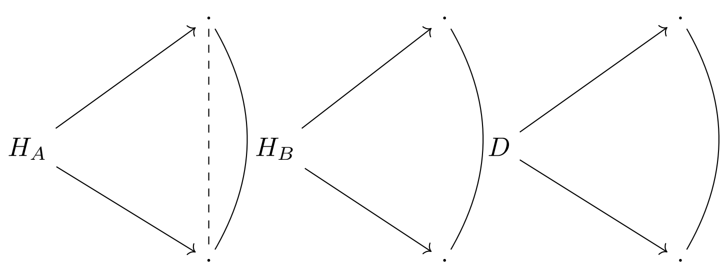

Figure 2: Imperfect information Extensive Form Game between the distributor and the 2 queueing systems

The first queueing system H_A decides on a threshold, then the second system H_B chooses its own threshold, without knowing the strategy of H_A, and finally the distributor makes its choice. Note here that the dotted line represents the fact that H_B is unaware of the state of the game when making its own decisions. The game can thus be partitioned into a normal form game between the two queueing systems and finding the distributor's best choice.

In order to define the normal form game the two payoff matrices of the players are required. From equation 3 the utilities of the players can be formulated as:

$$U^i_{T_1,T_2} = \quad -\left(\hat{P} - P(W_i < R)\right)^2 \tag{5}$$

Consequently, the payoff matrices of the game can be populated by these utilities:

$$A = \begin{pmatrix} U^A_{1,1} & U^A_{1,2} & \dots & U^A_{1,N_B} \\ U^A_{2,1} & U^A_{2,2} & \dots & U^A_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ U^A_{N_A,1} & U^A_{N_A,2} & \dots & U^A_{N_A,N_B} \end{pmatrix}, B = \begin{pmatrix} U^B_{1,1} & U^B_{1,2} & \dots & U^B_{1,N_B} \\ U^B_{2,1} & U^B_{2,2} & \dots & U^B_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ U^B_{N_A,1} & U^B_{N_A,2} & \dots & U^B_{N_A,N_B} \end{pmatrix} \tag{6}$$

Based on the choice of strategy of these two players the distributor will then make their own choice of the proportion of individuals to send to each system.

3 A queueing model with 2 consecutive buffer centres

In this section, a more in-depth explanation of the queueing model shown in figure 1 will be given. This is a queuing model that consists of two waiting spaces, one for each type of individual.

The model consists of two types of individuals; type 1 and type 2. Type 1 individuals arrive instantly at waiting zone 1 and proceed to wait to receive their service. Type 2 individuals arrive at waiting zone 2 and wait there until they are allowed to move to waiting zone 1. They are allowed to proceed only when the number of individuals in waiting zone 1 **and** in service is less than a pre-determined threshold T. When the number of individuals is equal to or exceeds this threshold, all second type individuals that arrive will remain *"blocked"* in waiting zone 1 until the number of people in the system is reduced below T. This is shown diagrammatically in figure 1. The parameters of the described queueing model are:

- λ_i: The arrival rate of individuals of type $i \in \{1, 2\}$
- μ: The service rate for individuals receiving service
- C: The number of servers
- T: The threshold at which individuals of the second type are blocked

Under the assumption that all rates (arrival and service) are Markovian the queuing system corresponds to a Markov chain [4]. The states of the Markov chain are denoted by (u, v) where:

- u is the number of individuals blocked
- v is the number of individuals either in waiting zone 1 or in the service centre

We denote the state space of the Markov chain as $S = S(T)$ which can be written as the disjoint union (7).

$$\begin{aligned} S(T) =& S_1(T) \cup S_2(T) \text{ where:} \\ S_1(T) =& \left\{(0, v) \in \mathbb{N}_0^2 \mid v < T\right\} \\ S_2(T) =& \{(u, v) \in \mathbb{N}_0^2 \mid v \geq T\} \end{aligned} \tag{7}$$

The transition matrix Q of the Markov chain consists of the transition rates between the numerous states of the model. Every entry $Q_{ij} = Q_{(u_i,v_i),(u_j,v_j)}$ represents the transition rate from state $i = (u_i, v_i)$ to state $j = (u_j, v_j)$ for all $(u_i, v_i), (u_j, v_j) \in S$. The entries of Q can be calculated using the state-mapping function described in (8):

$$
Q_{ij} = \begin{cases}
\Lambda, & \textbf{if } (u_i, v_i) - (u_j, v_j) = (0, -1) \textbf{ and } v_i < \mathrm{t} \\
\lambda_1, & \textbf{if } (u_i, v_i) - (u_j, v_j) = (0, -1) \textbf{ and } v_i \geq \mathrm{t} \\
\lambda_2, & \textbf{if } (u_i, v_i) - (u_j, v_j) = (-1, 0) \\
v_i \mu, & \textbf{if } (u_i, v_i) - (u_j, v_j) = (0, 1) \textbf{ and } v_i \leq C \textbf{ or} \\
& \quad (u_i, v_i) - (u_j, v_j) = (1, 0) \textbf{ and } v_i = T \leq C \\
C\mu, & \textbf{if } (u_i, v_i) - (u_j, v_j) = (0, 1) \textbf{ and } v_i > C \textbf{ or} \\
& \quad (u_i, v_i) - (u_j, v_j) = (1, 0) \textbf{ and } v_i = T > C \\
-\sum_{j=1}^{|Q|} Q_{ij} & \textbf{if } i = j \\
0, & \textbf{otherwise}
\end{cases} \tag{8}
$$

Note that Λ here denotes the overall arrival rate in the model by both types of individuals (i.e. $\Lambda = \lambda_1 + \lambda_2$). A visualisation of how the transition rates relate to the states of the model can be seen in the general Markov chain model shown in figure 3.

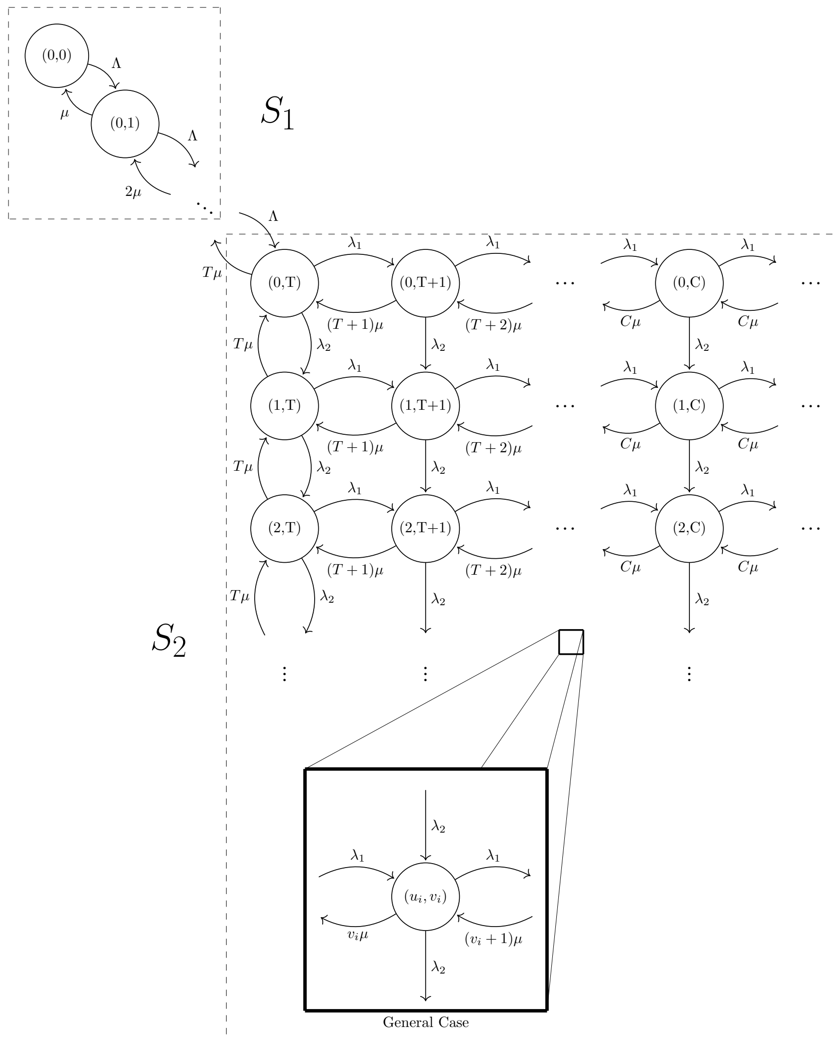

Figure 3: General case of the Markov chain model

In order to consider this model numerically an adjustment needs to be made. The problem defined above assumes no upper boundary to the number of individuals that can wait for service or for the ones that are blocked in the buffer centre. Therefore, a different state space $\tilde{S}$ is constructed where $\tilde{S} \subseteq S$ and there is a maximum allowed number of individuals N that can be in the system and a maximum allowed number of individuals M that can be blocked in the buffer centre:

$$\tilde{S} = \{(u, v) \in S \,|u \leq M, v \leq N\} \tag{9}$$

3.1 Performance Measures

The transition matrix Q defined in (8) can be used to get the probability vector π. The vector π is commonly used to study stochastic systems and it's main purpose is to keep track of the probability of being at any given state of the system. π_i is the steady state probability of being in state $(u_i, v_i) \in \tilde{S}$ which is the i^{th} state of $\tilde{S}$ for some ordering of $\tilde{S}$. The term *steady state* refers to the instance of the vector π where the probabilities of being at any state become stable over time. Thus, by considering the steady state vector π the relationship between it and Q is given by:

$$\frac{d\pi}{dt} = \pi Q = 0$$

Using vector π there are numerous performance measures of the model that can be calculated. The following equations utilise π to get performance measures on the average number of people at certain sets of state.

- Average number of people in the system:

$$L = \sum_{i=1}^{|\pi|} \pi_i (u_i + v_i)$$

- Average number of people in the service centre:

$$L_H = \sum_{i=1}^{|\pi|} \pi_i v_i$$

- Average number of people in waiting zone 2:

$$L_A = \sum_{i=1}^{|\pi|} \pi_i u_i$$

Consequently, there are some additional performance measures of interest that are not as straightforward to calculate. Such performance measures are the mean waiting time in the system (for both type 1 and type 2 individuals), the mean time blocked in waiting zone 2 (only valid for type 2 individuals) and the proportion of individuals that wait in waiting zone 1 within a predefined time target.

3.1.1 Waiting time

Waiting time is the amount of time that individuals from either type wait in waiting zone 1 so that they can receive their service. For a given set of parameters there are three different performance measures around the mean waiting time that can be calculated; the mean waiting time of type 1 individuals, the mean waiting time of type 2 individuals and the overall mean waiting time.

Since some of the individuals can be lost to the model, a new set of states needs to be defined; the set of *accepting states*. That is the set of states that the model is able to accept a certain type of individual. The set of accepting states for type 1 individuals is defined as:

$$S_A^{(1)} = \{(u, v) \in S \mid v < N\} \tag{10}$$

In essence, for type 1 individuals, this is the set of states that are not on the last column of states in the Markov chain. Equivalently, the set of accepting states for type 2 individuals is defined as:

$$S_A^{(2)} = \begin{cases} \{(u, v) \in S \mid u < M\}, & \textbf{if } T \leq N \\ \{(u, v) \in S \mid v < N\}, & \textbf{otherwise} \end{cases} \tag{11}$$

Note here that if the threshold is less than or equal the total capacity of the system the set includes all states that are not on the last column of the Markov chain. Otherwise, the set of accepting state is identical to (10). Thus, the expressions for the waiting times for type 1 and type 2 individuals are given by:

$$W^{(1)} = \frac{\sum_{\substack{(u,v) \in S_A^{(1)} \\ v \geq C}} \frac{1}{C\mu} \times (v - C + 1) \times \pi(u, v)}{\sum_{(u,v) \in S_A^{(1)}} \pi(u, v)} \tag{12}$$

$$W^{(2)} = \frac{\sum_{\substack{(u,v) \in S_A^{(2)} \\ min(v,T) \geq C}} \frac{1}{C\mu} \times (\min(v + 1, T) - C) \times \pi(u, v)}{\sum_{(u,v) \in S_A^{(2)}} \pi(u, v)} \tag{13}$$

Consequently, the overall waiting time can be estimated by a linear combination of W_1 and W_2. Thus, the overall waiting time can calculated by the following equation where c_1 and c_2 are the coefficients of the terms:

$$W = c_1 W^{(1)} + c_2 W^{(2)} \tag{14}$$

The two coefficients represent the proportion of individuals of each type that did not get lost and traversed through the model. Thus, one should account for the probability that an individual is lost to the system. This probability can be easily calculated by using the two sets of accepting states $S_A^{(2)}$ and $S_A^{(1)}$ defined in equations (10) and (11). Using these equations the probability, for either individual type, that an individual is not lost in the system is given by:

$$P(L'_1) = \sum_{(u,v) \in S_A^{(1)}} \pi(u, v) \qquad\qquad P(L'_2) = \sum_{(u,v) \in S_A^{(2)}} \pi(u, v)$$

Thus, by using these values as the coefficient of equation (14) the resultant equation can be used to get the overall waiting time.

$$W = \frac{\lambda_1 P(L_1')}{\lambda_2 P(L_2') + \lambda_1 P(L_1')} W^{(1)} + \frac{\lambda_2 P(L_2')}{\lambda_2 P(L_2') + \lambda_1 P(L_1')} W^{(2)} \tag{15}$$

3.1.2 Blocking time

Unlike the waiting time, the blocking time is only calculated for individuals of the second type. That is because individuals of the first type cannot be blocked. Thus, one only needs to consider the pathway of type 2 individuals to get the mean blocking time of the system. The set of states where individuals can be blocked is defined as:

$$S_b = \{(u, v) \in S \mid u > 0\} \tag{16}$$

In order to not consider individuals that will be lost to the system, the set of accepting states needs to be taken into consideration. As defined in section 3.1.1, the set of accepting states is given by (11):

$$S_A^{(2)} = \begin{cases} \{(u, v) \in S \mid u < M\} & \textbf{if } T \leq N \\ \{(u, v) \in S \mid v < N\} & \textbf{otherwise} \end{cases}$$

The mean sojourn time for each state is given by the inverse of the out-flow of that state. However, whenever a type 2 individual arrives at the system, no subsequent arrival of another type 2 individual can affect its pathway or total time in the system. Therefore, looking at the mean time in the system from the perspective of an individual of the second type, all such type 2 arrivals need to be ignored. Note here that this is not the case for individuals of the first type. Whenever a type 2 individual is blocked and a type 1 individual arrives the type 2 individuals will remain blocked for some additional amount of time. Thus, the mean time that a type 2 individual spends at each state is given by:

$$c(u, v) = \begin{cases} \frac{1}{\min(v, C)\mu}, & \text{if } v = N \\ \frac{1}{\lambda_1 + \min(v, C)\mu}, & \text{otherwise} \end{cases} \tag{17}$$

In equation (17), both service completions and type 1 arrivals are considered. Thus, from a blocked individual's perspective whenever the system moves from one state (u, v) to another state it can either:

- be because of a service being completed: we will denote the probability of this happening by $p_s(u, v)$.
- be because of an arrival of an individual of type 1: denoting such probability by $p_o(u, v)$.

The probabilities are given by:

$$p_s(u, v) = \frac{\min(v, C)\mu}{\lambda_1 + \min(v, C)\mu}, \qquad p_o(u, v) = \frac{\lambda_1}{\lambda_1 + \min(v, C)\mu}$$

Having defined $c(u,v)$ and S_b a formula for the blocking time that is expected to occur at each state can be given by:

$$b(u,v) = \begin{cases} 0, & \textbf{if } (u,v) \notin S_b \\ c(u,v) + b(u-1,v), & \textbf{if } v = N = T \\ c(u,v) + b(u,v-1), & \textbf{if } v = N \neq T \\ c(u,v) + p_s(u,v)b(u-1,v) + p_o(u,v)b(u,v+1), & \textbf{if } u > 0 \textbf{ and } \\ & \quad v = T \\ c(u,v) + p_s(u,v)b(u,v-1) + p_o(u,v)b(u,v+1), & \textbf{otherwise} \end{cases} \tag{18}$$

A direct approach will be used to solve this equation here. By enumerating all equations of ((18)) for all states (u,v) that belong in S_b a system of linear equations arises where the unknown variables are all the $b(u,v)$ terms. Note here that these equations correspond to all blocking states as defined in (16). Equations that correspond to non-blocking states have a value of 0 as defined in (18) The general form of the equation in terms of C, T, N and M is given by:

$$\begin{aligned}
b(1,T) &= c(1,T) + p_o b(1,T+1) && (19)\\
b(1,T+1) &= c(1,T+1) + p_s b(1,T) + p_o b(1,T+1) && (20)\\
b(1,T+2) &= c(1,T+2) + p_s b(1,T+1) + p_o b(1,T+3) && (21)\\
&\vdots \\
b(1,N) &= c(1,N) + b(1,N-1) && (22)\\
b(2,T) &= c(2,T) + p_s b(1,T) + p_o b(2,T+1) && (23)\\
b(2,T+1) &= c(2,T+1) + p_s b(2,T) + p_o b(2,T+2) && (24)\\
&\vdots \\
b(M-1,N) &= c(M,N-1) + b(M,N-1) && (25)\\
b(M,T) &= c(T,N) + p_s b(T-1,N) + p_o b(T,N+1) && (26)\\
&\vdots \\
b(M,N) &= c(M,N) + b(M,N-1) && (27)
\end{aligned}$$

The equivalent matrix notation of the linear system of equations ((19)) -

((27)) is given by $Zx = y$, where:

$$Z = \begin{pmatrix} -1 & p_o & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ p_s & -1 & p_o & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & p_s & -1 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ p_s & 0 & 0 & \dots & 0 & 0 & -1 & p_o & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & p_s & -1 & p_o & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}, x = \begin{pmatrix} b(1,T) \\ b(1,T+1) \\ b(1,T+2) \\ \vdots \\ b(1,N) \\ b(2,T) \\ b(2,T+1) \\ \vdots \\ b(M,N) \end{pmatrix}, y = \begin{pmatrix} -c(1,T) \\ -c(1,T+1) \\ -c(1,T+2) \\ \vdots \\ -c(1,N) \\ -c(2,T) \\ -c(2,T+1) \\ \vdots \\ -c(M,N) \end{pmatrix} \tag{28}$$

Thus, having calculated the mean blocking time for all blocking states $b(u, v)$, it only remains to put them together in a formula. The resultant formula for the mean blocking time is given by:

$$B = \frac{\sum_{(u,v)\in S_A} \pi_{(u,v)} \; b(u,v)}{\sum_{(u,v)\in S_A} \pi_{(u,v)}} \tag{29}$$

To illustrate how the described formula works consider a Markov model where $C = 2, T = 2, N = 4, M = 2$ (figure 4). The equations that correspond to such a model are shown in ((30))-((35)) and their equivalent matrix notation form is shown in (36).

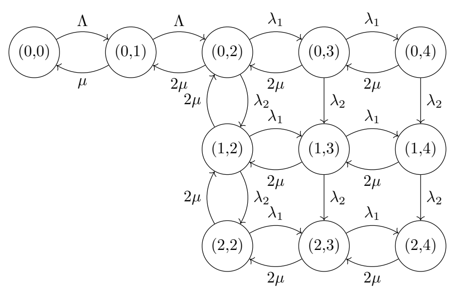

Figure 4: Example of Markov chain with $C = 2, T = 2, N = 4, M = 2$

$$b(1,2) = c(1,2) + p_o b(1,3) \tag{30}$$
$$b(1,3) = c(1,3) + p_s b(1,2) + p_o b(1,4) \tag{31}$$
$$b(1,4) = c(1,4) + b(1,3) \tag{32}$$
$$b(2,2) = c(2,2) + p_s b(1,2) + p_o b(2,3) \tag{33}$$
$$b(2,3) = c(2,3) + p_s b(2,2) + p_o b(1,4) \tag{34}$$
$$b(2,4) = c(2,4) + b(2,3) \tag{35}$$

$$Z = \begin{pmatrix} -1 & p_o & 0 & 0 & 0 & 0 \\ p_s & -1 & p_o & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ p_s & 0 & 0 & -1 & p_o & 0 \\ 0 & 0 & 0 & p_s & -1 & p_o \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}, x = \begin{pmatrix} b(1,2) \\ b(1,3) \\ b(1,4) \\ b(2,2) \\ b(2,3) \\ b(2,4) \end{pmatrix}, y = \begin{pmatrix} -c(1,2) \\ -c(1,3) \\ -c(1,4) \\ -c(2,2) \\ -c(2,3) \\ -c(2,4) \end{pmatrix} \tag{36}$$

3.1.3 Proportion of individuals within target

Another performance measure that needs to be taken into consideration is the proportion of individuals whose waiting and service times lie within a specified time target. In order to consider such measure though one would need to obtain the distribution of time in the system for all individuals. The complexity of such task lies on the fact that different individuals arrive at different states of the Markov model. Consider the case when an arrival occurs when the model is at a specific state.

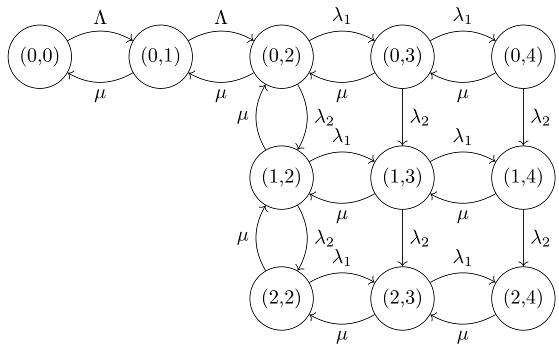

Figure 5: Example Markov model $C = 1, T = 2, N = 4, M = 2$

Time distribution at specific state (1 server): Consider the Markov model of figure 5 with one server and a threshold of two individuals. Assume that an individual of the first type arrives when the model is at state $(0, 3)$, thus forcing the model to move to state $(0, 4)$. The distribution of the time needed for the specified individual to exit the system from state $(0, 4)$ is given by the sum of exponentially distributed random variables with the same parameter μ. The sum of such random variables forms an Erlang distribution which is defined by the number of random variables that are added and their exponential parameter. Note here that these random variables represent the individual's pathway from the perspective of the individual. Thus, X_i represents the time that it takes to move from the i^{th} position of the queue to the $(i-1)^{\text{th}}$ position (i.e. for someone in front of them to finish their service) and X_0 is the time it takes to move from having a service to exiting the system.

$$
\begin{aligned}
(0,4) \Rightarrow &\quad X_3 \sim Exp(\mu) \\
(0,3) \Rightarrow &\quad X_2 \sim Exp(\mu) \\
(0,2) \Rightarrow &\quad X_1 \sim Exp(\mu) \\
(0,1) \Rightarrow &\quad X_0 \sim Exp(\mu) \\
S = X_3 + X_2 + &X_1 + X_0 = Erlang(4, \mu)
\end{aligned} \tag{37}
$$

Thus, the waiting and service time of an individual in the model of figure 5 can be captured by an erlang distributed random variable. The general CDF of the erlang distribution $Erlang(k, \mu)$ is given by:

$$P(S < t) = 1 - \sum_{i=0}^{k-1} \frac{1}{i!} e^{-\mu t} (\mu t)^i \tag{38}$$

Unfortunately, the erlang distribution can only be used for the sum of identically distributed random variables from the exponential distribution. Therefore, this approach cannot be used when one of the random variables has a different parameter than the others. In fact the only case where it can be used is only when the number of servers are $C = 1$, or when an individual arrives and goes straight to service (i.e. when there is no other individual waiting and there is an empty server).

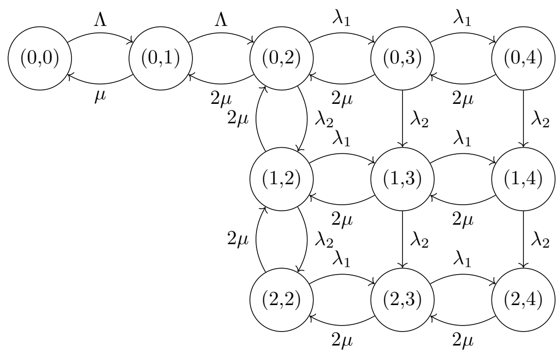

Figure 6: Example Markov model $C = 2, T = 2, N = 4, M = 2$

Time distribution at a state (multiple servers): Figure 6 represents the same Markov model as figure 5 with the only exception that there are 2 servers here. By applying the same logic, assuming that an individual arrives at state $(0, 4)$, the sum of the following random variables arises.

$$\begin{aligned}
(0,4) \Rightarrow &\quad X_2 \sim Exp(2\mu) \\
(0,3) \Rightarrow &\quad X_1 \sim Exp(2\mu) \\
(0,2) \Rightarrow &\quad X_0 \sim Exp(\mu)
\end{aligned} \tag{39}$$

Since these exponentially distributed random variables do not share the same parameter, an erlang distribution cannot be used. In fact, the problem can now be viewed either as the sum of exponentially distributed random variables with different parameters or as the sum of erlang distributed random variables. The sum of erlang distributed random variables is said to follow the hypoexponential distribution. The hypoexponential distribution is defined with two vectors of size equal to the number of Erlang random variables [1], [5]. The vector $\vec{r}$ contains all the k-values of the erlang distributions and $\vec{\lambda}$ is a vector of the distinct parameters as illustrated in equation (40).

$$
\left.\begin{array}{c} Erlang(k_1,\lambda_1) \\ Erlang(k_2,\lambda_2) \\ \vdots \\ Erlang(k_n,\lambda_n) \end{array}\right\} Hypo(\underbrace{(k_1,k_2,\dots k_n)}_{\vec{k}},\underbrace{(\lambda_1,\lambda_2,\dots \lambda_n)}_{\vec{\lambda}}) \tag{40}
$$

Equivalently, for this particular example:

$$
\left.\begin{array}{l} \left.\begin{array}{l} X_2 \sim Exp(2\mu) \\ X_1 \sim Exp(2\mu) \end{array}\right\} X_1+X_2=S_1\sim Erlang(2,2\mu) \\ X_0\sim Exp(\mu) \Rightarrow \qquad X_0=S_2\sim Erlang(1,\mu) \end{array}\right\} S_1+S_2=H\sim Hypo((2,1),(2\mu,\mu)) \tag{41}
$$

Therefore, the CDF of this distribution can be used to get the probability of the time in spent in the system being less than a given target. The general CDF of the hypoexponential distribution $Hypo(\vec{r},\vec{\lambda})$, is given by the following expression [3]:

$$
\begin{aligned}
& P(H<t) = 1 - \left(\prod_{j=1}^{|\vec{r}|} \lambda_j^{r_j}\right) \sum_{k=1}^{|\vec{r}|} \sum_{l=1}^{r_k} \frac{\Psi_{k,l}(-\lambda_k) t^{r_k-l} e^{-\lambda_k t}}{(r_k-l)!(l-1)!} \\
& \textbf{where} \qquad \Psi_{k,l}(t) = -\frac{\partial^{l-1}}{\partial t^{l-1}} \left(\prod_{j=0,j\neq k}^{|\vec{r}|} (\lambda_j+t)^{-r_j}\right) \\
& \textbf{and} \qquad \lambda_0 = 0, r_0 = 1
\end{aligned} \tag{42}
$$

The computation of the derivative makes equation (42) computationally expensive. In [6] an alternative linear version of that CDF is explored via matrix analysis, and is given by the following formula:

$$
\begin{aligned}
F(x) =& 1 - \sum_{k=1}^{n}\sum_{l=0}^{k-1}(-1)^{k-1}\binom{n}{k}\binom{k-1}{l}\sum_{j=1}^{n}\sum_{s=1}^{j-1}e^{-x\lambda_s}\prod_{l=1}^{s-1}\left(\frac{\lambda_l}{\lambda_l-\lambda_s}\right)^{k_s} \\
& \times \sum_{s<a_1<\dots<a_{l-1}<j}\left(\frac{\lambda_s}{\lambda_s-\lambda_{a_1}}\right)^{k_s}\prod_{m=s+1}^{a_1-1}\left(\frac{\lambda_m}{\lambda_m-\lambda_{a_1}}\right)^{k_m} \\
& \times \prod_{n=a_1}^{a_2-1}\left(\frac{\lambda_n}{\lambda_n-\lambda_{a_2}}\right)^{k_n}\cdots\prod_{r=a_l-1}^{j-1}\left(\frac{\lambda_r}{\lambda_r-\lambda_{a_j}}\right)^{k_r}\sum_{q=0}^{k_s-1}\frac{((\lambda_s-\lambda_{a_1})x)^q}{q!}, \\
& \text{for } x \geq 0
\end{aligned} \tag{43}
$$

Specific CDF of hypoexponential distribution Equations (42) and (43) refers to the general CDF of the hypoexponential distribution where the size of the vector parameters can be of any size [3]. In the Markov chain models described in figures 5 and 6 the parameter vectors of the hypoexponential distribution are of size two, and in fact, for any possible version of the investigated Markov chain model the vectors can only be of size two. This is true since for any dimensions of this Markov chain model there will always be at most two distinct exponential parameters; the parameter for finishing a service (μ) and the parameter for moving forward in the queue ($C\mu$). For the special case of $C = 1$ the hypoexponential distribution will not be used as this is equivalent to an erlang distribution. Therefore, by fixing the sizes of $\vec{r}$ and $\vec{\lambda}$ to 2, the following specific expression for the CDF of the hypoexponential distribution arises, where the derivative is removed:

$$\begin{aligned} &P(H < t) = 1 - \left(\prod_{j=1}^{|\vec{r}|} \lambda_j^{r_j} \right) \sum_{k=1}^{|\vec{r}|} \sum_{l=1}^{r_k} \frac{\Psi_{k,l}(-\lambda_k) t^{r_k - l} e^{-\lambda_k t}}{(r_k - l)!(l-1)!} \\ &\textbf{where} \qquad \Psi_{k,l}(t) = \begin{cases} \frac{(-1)^l (l-1)!}{\lambda_2} \left[\frac{1}{t^l} - \frac{1}{(t+\lambda_2)^l} \right], & k = 1 \\ -\frac{1}{t(t+\lambda_1)^{r_1}}, & k = 2 \end{cases} \\ &\textbf{and} \qquad \lambda_0 = 0, r_0 = 1 \end{aligned} \tag{44}$$

Note here that the only difference between equations (42) and (44) is the Ψ function. The next part proves that the expression for Ψ can be simplified for the cases of $k = 1, 2$. Equation (45) shows the expression to be proved.

$$\Psi_{(k,l)}(t) = -\frac{\partial^{l-1}}{\partial t^{l-1}} \left(\prod_{j=0, j \neq k}^{|\vec{r}|} (\lambda_j + t)^{-r_j} \right) = \begin{cases} \frac{(-1)^l (l-1)!}{\lambda_2} \left[\frac{1}{t^l} - \frac{1}{(t+\lambda_2)^l} \right], & k = 1 \\ -\frac{1}{t(t+\lambda_1)^{r_1}}, & k = 2 \end{cases} \tag{45}$$

Proof of equation (45) This section aims to show that there exists a simplified version of equation (42) that is specific to the proposed Markov model. Function Ψ is defined using the parameter t and the variables k and l. Given the Markov model, the range of values that k and l can take can be bounded. First of all, from the range of the double summation in equation (42), it can be seen that $k = 1, 2, \dots, |\ \vec{r}\ |$. Now, $|\ \vec{r}\ |$ represents the size of the parameter vectors that, for the Markov model, will always be 2. That is because, for all the exponentially distributed random variables that are added together to form the new distribution, there only two distinct parameters, thus forming two erlang distributions. Therefore:

$$k = 1, 2$$

By observing equation (42) once more, the range of values that l takes are $l = 1, 2, \dots, r_k$, where r_1 is subject to the individual's position in the queue and

$r_2 = 1$. In essence, the hypoexponential distribution will be used with these bounds:

$$\begin{aligned} k = 1 \quad &\Rightarrow \quad l = 1, 2, \ldots, r_1 \\ k = 2 \quad &\Rightarrow \quad l = 1 \end{aligned} \tag{46}$$

Thus the left hand side of equation (45) needs only to be defined for these bounds. The specific hypoexponential distribution investigated here is of the form $Hypo((r_1, 1)(\lambda_1, \lambda_2))$. Note the initial conditions $\lambda_0 = 0, r_0 = 1$ defined in equation (42) also hold here. Thus the proof is split into two parts, for $k = 1$ and $k = 2$.

- $k = 2, l = 1$

$$\begin{aligned} LHS &= -\frac{\partial^{1-1}}{\partial t^{1-1}} \left(\prod_{j=0, j\neq 2}^{2} (\lambda_j + t)^{-r_j} \right) \\ &= -\left((\lambda_0 + t)^{-r_0} \times (\lambda_1 + t)^{-r_1} \right) \\ &= -\left(t^{-1} \times (\lambda_1 + t)^{-r_1} \right) \\ &= -\frac{1}{t(t + \lambda_1)^{r_1}} \end{aligned}$$

$\square$

- $k = 1, l = 1, \ldots, r_1$

$$\begin{aligned} LHS &= -\frac{\partial^{l-1}}{\partial t^{l-1}} \left(\prod_{j=0, j\neq 1}^{2} (\lambda_j + t)^{-r_j} \right) \\ &= -\frac{\partial^{l-1}}{\partial t^{l-1}} \left((\lambda_o + t)^{-r_0} \times (\lambda_2 + t)^{-r_2} \right) \\ &= -\frac{\partial^{l-1}}{\partial t^{l-1}} \left(\frac{1}{t(t + \lambda_2)} \right) \end{aligned}$$

In essence, it only remains to show that:

$$-\frac{\partial^{l-1}}{\partial t^{l-1}} \left(\frac{1}{t(t + \lambda_2)} \right) = \frac{(-1)^l (l-1)!}{\lambda_2} \left[\frac{1}{t^l} - \frac{1}{(t + \lambda_2)^l} \right]$$

Proof by Induction:

1. Base case ($l = 1$):

$$
\begin{aligned}
LHS &= -\frac{\partial^{1-1}}{\partial t^{1-1}} \left(\frac{1}{t(t+\lambda_2)} \right) = -\frac{1}{t(t+\lambda_2)} \\
RHS &= \frac{(-1)^1(1-1)!}{\lambda_2} \left[\frac{1}{t^1} - \frac{1}{(t+\lambda_2)^1} \right] \\
&= -\frac{t+\lambda_2-t}{\lambda_2 t(t+\lambda_2)} \\
&= -\frac{1}{t(t+\lambda_2)} \\
LHS &= RHS
\end{aligned}
$$

2. Assume true for $l = x$:

$$
-\frac{\partial^{x-1}}{\partial t^{x-1}} \left(\frac{1}{t(t+\lambda_2)} \right) = \frac{(-1)^x(x-1)!}{\lambda_2} \left[\frac{1}{t^x} - \frac{1}{(t+\lambda_2)^x} \right]
$$

3. Prove true for $l = x + 1$. Need to show that:

$$
\frac{\partial^{x}}{\partial t^{x}} \left(\frac{-1}{t(t+\lambda_2)} \right) = \frac{(-1)^{x+1}(x)!}{\lambda_2} \left[\frac{1}{t^{x+1}} - \frac{1}{(t+\lambda_2)^{x+1}} \right]
$$

$$
\begin{aligned}
LHS &= \frac{\partial}{\partial t} \left[\frac{\partial^{x-1}}{\partial t^{x-1}} \left(\frac{-1}{t(t+\lambda_2)} \right) \right] \\
&= \frac{\partial}{\partial t} \left[\frac{(-1)^x(x-1)!}{\lambda_2} \left(\frac{1}{t^x} - \frac{1}{(t+\lambda_2)^x} \right) \right] \\
&= \frac{(-1)^x(x-1)!}{\lambda_2} \left(\frac{(-x)}{t^{x+1}} - \frac{(-x)}{(t+\lambda_2)^x} \right) \\
&= \frac{(-1)^x(x-1)!(-x)}{\lambda_2} \left(\frac{1}{t^{x+1}} - \frac{1}{(t+\lambda_2)^x} \right) \\
&= \frac{(-1)^{x+1}(x)!}{\lambda_2} \left(\frac{1}{t^{x+1}} - \frac{1}{(t+\lambda_2)^x} \right) \\
&= RHS
\end{aligned}
$$

□

Proportion within target for both types of individuals Given the two CDFs of the Erlang and Hypoexponential distributions a new function has to be defined to decide which one to use among the two. Based on the state of the model, there can be three scenarios when an individual arrives.

1. There is a free server and the individual does not have to wait

$$
X_{(u,v)} \sim Erlang(1, \mu)
$$

2. The individual arrives at a queue at the n^{th} position and the model has $C > 1$ servers
$$X_{(u,v)} \sim Hypo((n,1),(C\mu,\mu))$$

3. The individual arrives at a queue at the n^{th} position and the model has $C = 1$ servers
$$X_{(u,v)} \sim Erlang(n+1,\mu)$$

Note here that for the first case $Erlang(1,\mu)$ is equivalent to $Exp(\mu)$. Consider $X^{(1)}_{(u,v)}$ to be the distribution of type 1 individuals and $X^{(2)}_{(u,v)}$ the distribution of type 2 individuals, when arriving at state (u,v) of the model.

$$X^{(1)}_{(u,v)} \sim \begin{cases} \textbf{Erlang}(v,\mu), & \textbf{if } C=1 \textbf{ and } v>1 \\ \textbf{Hypo}\left([v-C,1],[C\mu,\mu]\right), & \textbf{if } C>1 \textbf{ and } v>C \\ \textbf{Erlang}(1,\mu), & \textbf{if } v \leq C \end{cases} \tag{47}$$

$$X^{(2)}_{(u,v)} \sim \begin{cases} \textbf{Erlang}(\min(v,T),\mu), & \textbf{if } C=1 \textbf{ and } v,T>1 \\ \textbf{Hypo}\left([\min(v,T)-C,1],[C\mu,\mu]\right), & \textbf{if } C>1 \textbf{ and } v,T>C \\ \textbf{Erlang}(1,\mu), & \textbf{if } v \leq C \textbf{ or } T \leq C \end{cases} \tag{48}$$

Thus, the CDF of the random variables $X^{(1)}_{(u,v)}$ and $X^{(2)}_{(u,v)}$ can be calculated using equations (38) and (44):

$$P(X^{(1)}_{(u,v)} < t) = \begin{cases} 1 - \sum_{i=0}^{v-1} \frac{1}{i!} e^{-\mu t} (\mu t)^i, & \textbf{if } C=1 \\ & \textbf{and } v>1 \\ \\ 1 - (\mu C)^{v-C} \mu \sum_{k=1}^{|\vec{r}|} \sum_{l=1}^{r_k} \frac{\Psi_{k,l}(-\lambda_k) t^{r_k-l} e^{-\lambda_k t}}{(r_k-l)!(l-1)!}, & \textbf{if } C>1 \\ \textbf{where } \vec{r} = (v-C,1) \textbf{ and } \vec{\lambda} = (C\mu,\mu) & \textbf{and } v>C \\ \\ 1 - e^{-\mu t}, & \textbf{if } v \leq C \end{cases} \tag{49}$$

$$
P(X^{(2)}_{(u,v)} < t) = \begin{cases} 1 - \sum_{i=0}^{\min(v,T)-1} \frac{1}{i!} e^{-\mu t} (\mu t)^i, & \textbf{if } C = 1 \\ & \textbf{and } v, T > 1 \\ \\ 1 - (\mu C)^{\min(v,T)-C} \mu & \textbf{if } C > 1 \\ \quad \times \sum_{k=1}^{|\vec{r}|} \sum_{l=1}^{r_k} \frac{\Psi_{k,l}(-\lambda_k) t^{r_k - l} e^{-\lambda_k t}}{(r_k - l)!(l-1)!}, & \textbf{and } v, T > C \\ \textbf{where } \vec{r} = (\min(v,T) - C, 1) \\ \qquad \vec{\lambda} = (C\mu, \mu) \\ \\ 1 - e^{-\mu t}, & \textbf{if } v \leq C \\ & \textbf{or } T \leq C \end{cases} \tag{50}
$$

In addition, the set of accepting states for type 1 $S_A^{(1)}$ and type 2 $S_A^{(2)}$ individuals defined in (10) and (11) are also needed here. Note here that, S denotes the set of all states of the Markov chain model.

$$
S_A^{(1)} = \{(u,v) \in S \mid v < N\}
$$

$$
S_A^{(2)} = \begin{cases} \{(u,v) \in S \mid u < M\}, & \textbf{if } T \leq N \\ \{(u,v) \in S \mid v < N\}, & \textbf{otherwise} \end{cases}
$$

The following formula uses the state probability vector π to get the weighted average of the probability below target of all states in the Markov model.

$$
P(X^{(1)} < t) = \frac{\sum_{(u,v) \in S_A^{(1)}} P(X^{(1)}_{u,v} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(1)}} \pi_{u,v}} \tag{51}
$$

$$
P(X^{(2)} < t) = \frac{\sum_{(u,v) \in S_A^{(2)}} P(X^{(2)}_{u,v} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(2)}} \pi_{u,v}} \tag{52}
$$

Overall proportion within target The overall proportion of individuals for both types of individuals is given by the equivalent formula of equations (14) and (15). The following formula uses the probability of lost individuals from both types to get the weighted sum of the two probabilities.

$$
P(L'_1) = \sum_{(u,v) \in S_A^{(1)}} \pi(u,v), \qquad P(L'_2) = \sum_{(u,v) \in S_A^{(2)}} \pi(u,v)
$$

$$
P(X < t) = \frac{\lambda_1 P(L'_1)}{\lambda_2 P(L'_2) + \lambda_1 P(L'_1)} P(X^{(1)} < t) \tag{53}
$$

$$
+ \frac{\lambda_2 P(L'_2)}{\lambda_2 P(L'_2) + \lambda_1 P(L'_1)} P(X^{(2)} < t) \tag{54}
$$

3.2 Example

4 Methodology

4.1 Backwards Induction

4.2 Nash Equilibrium

4.3 Learning Algorithms

5 EMS - ED application

5.1 Application

5.2 Data Analysis

6 Conclusion

References

[1] Mohamed Akkouchi. On the convolution of exponential distributions. *J. Chungcheong Math. Soc*, 21(4):501–510, 2008.

[2] Dietmar Berwanger and Laurent Doyen. On the power of imperfect information. *Leibniz International Proceedings in Informatics, LIPIcs*, 2:73–82, 2008.

[3] Stefano Favaro and Stephen G Walker. On the distribution of sums of independent exponential random variables via wilks' integral representation. *Acta applicandae mathematicae*, 109(3):1035–1042, 2010.

[4] John G Kemeny and J Laurie Snell. *Markov chains*, volume 6. Springer-Verlag, New York, 1976.

[5] Smaili Khaled, Therrar Kadri, and Seifedine Kadry. Hypoexponential distribution with different parameters. *Journal of Applied Mathematics*, 4:624–631, 04 2013.

[6] Benjamin Legros and Oualid Jouini. A linear algebraic approach for the computation of sums of erlang random variables. *Applied Mathematical Modelling*, 39(16):4971–4977, 2015.

[7] Michael Maschler, Eilon Solan, and Eilon Zamir. *Game Theory.* Cambridge University Press, 2013.