

A 3-player game theoretic model of a choice  
between two queueing systems with strategic  
managerial decision making

**Michalis Panayides**

**EURO 2021 Athens**

# About me



About me



THIS.

## Queues - M/M/1

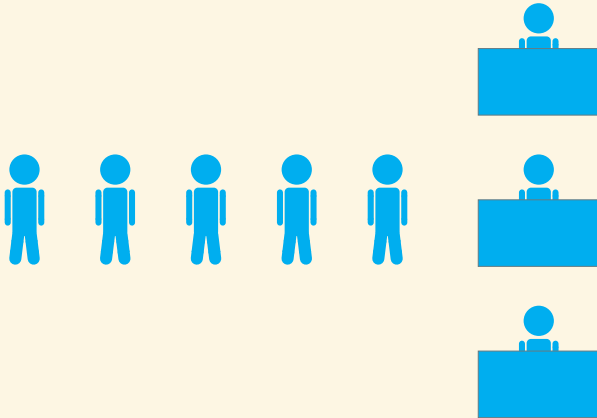


# Queues - M/M/1

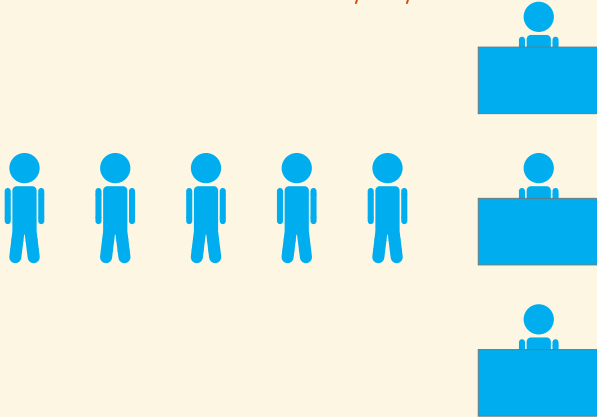


- ▶ Shone R, Knight VA, Williams JE. Comparisons between observable and unobservable M/M/1 queues with respect to optimal customer behaviour
- ▶ Kerner Y, Shmuel-Bittner O. Strategic behaviour and optimization in a hybrid M/M/1 queue with retrials.
- ▶ Gai Y, Liu H, Krishnamachari B. A packet dropping mechanism for efficient operation of M/M/1 queues with selfish users.

## Queues - M/M/3



## Queues - M/M/3



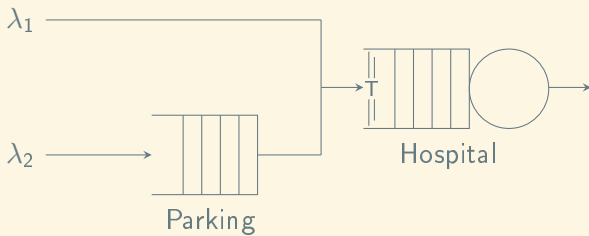
- ▶ Knight V, Harper P. The Impact of Choice on Public Services.
- ▶ Haviv M, Roughgarden T. The price of anarchy in an exponential multi-server
- ▶ Shone R, Knight V, Harper P. A conservative index heuristic for routing problems with multiple heterogeneous service facilities

## Queues - Custom network of queues

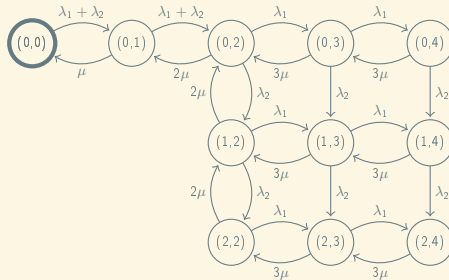
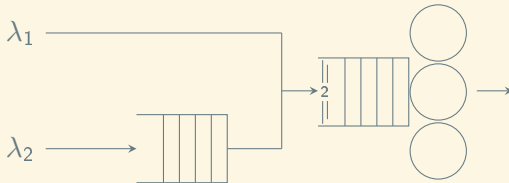




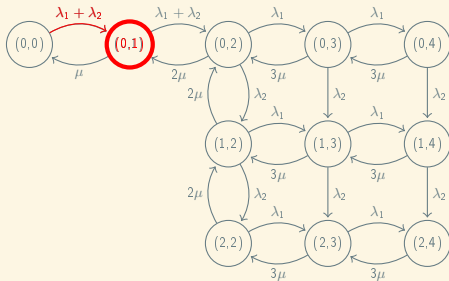
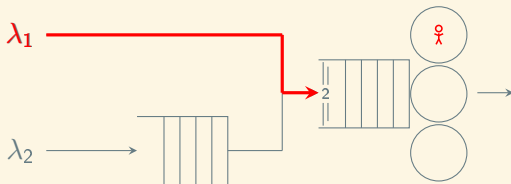
## Queueing network structure



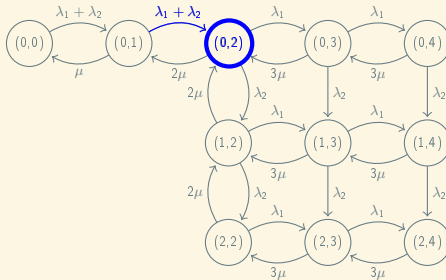
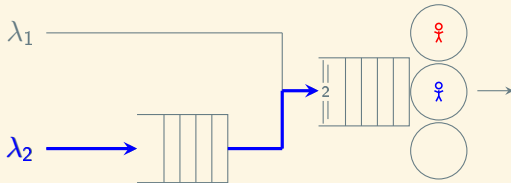
# Markov Chain - Custom network



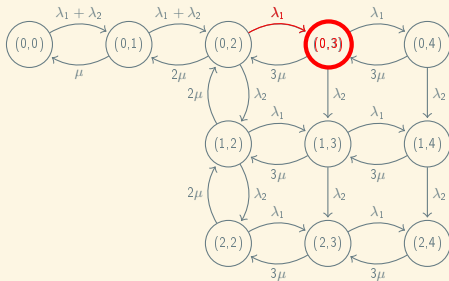
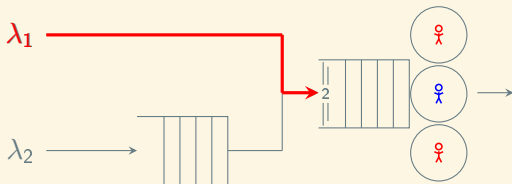
## Markov Chain - Custom network



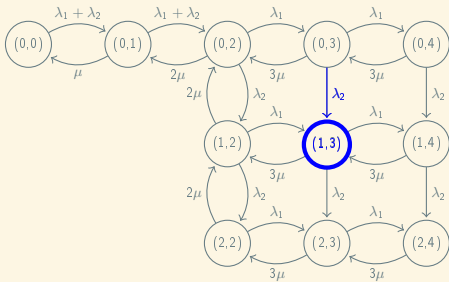
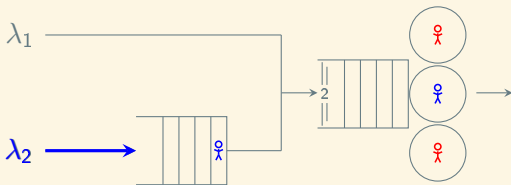
# Markov Chain - Custom network



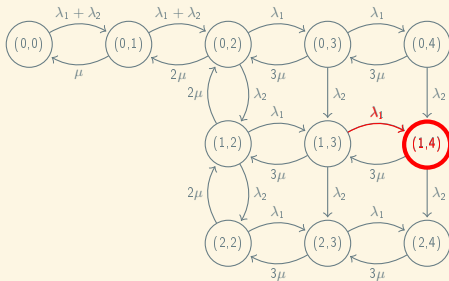
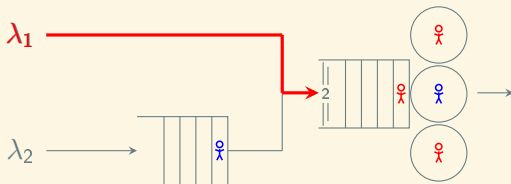
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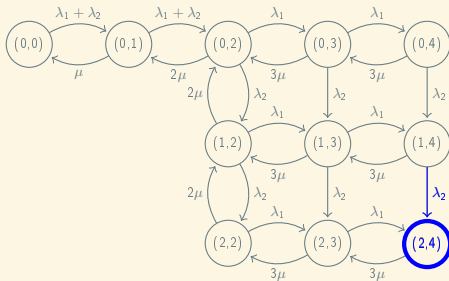
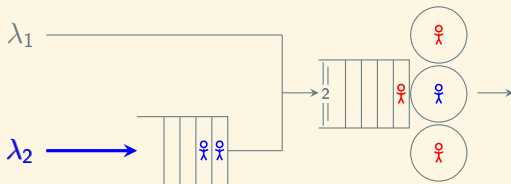
## Markov Chain - Custom network



# Markov Chain - Custom network

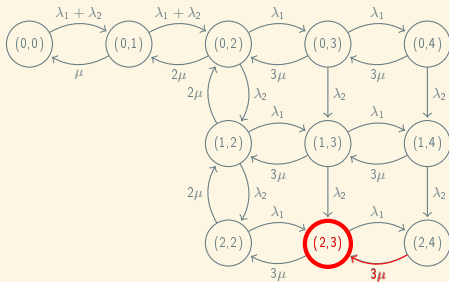
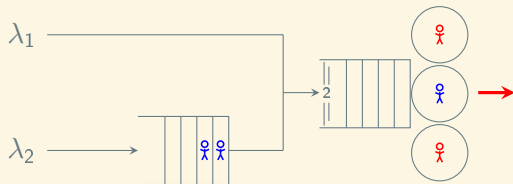


# Markov Chain - Custom network

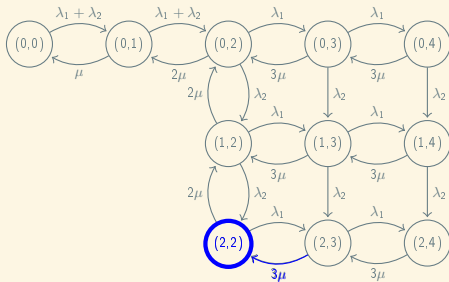
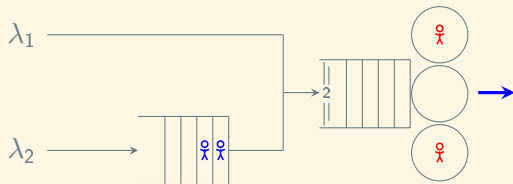




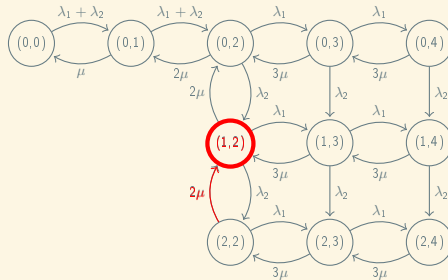
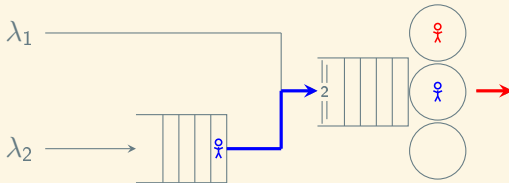
# Markov Chain - Custom network



# Markov Chain - Custom network

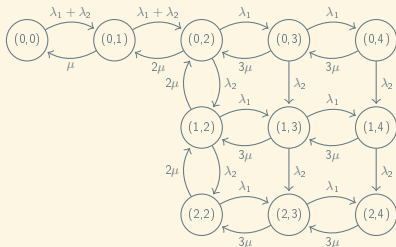


# Markov Chain - Custom network



# Steady state probabilities - Custom network

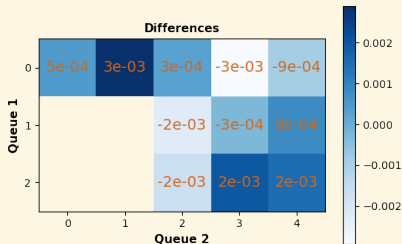
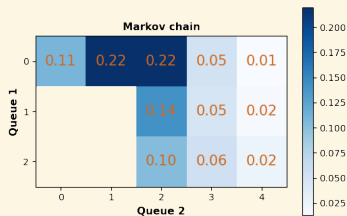
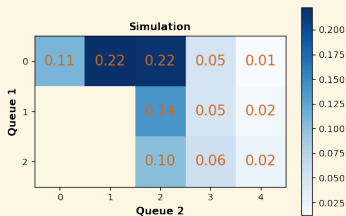
$$Q = \begin{matrix} & \begin{matrix} (0,0) & (0,1) & (0,2) & \dots & (2,3) & (2,4) \end{matrix} \\ \begin{matrix} (0,0) \\ (0,1) \\ (0,2) \\ \vdots \\ (2,3) \\ (2,4) \end{matrix} & \begin{pmatrix} -\lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 & 0 & \dots & 0 & 0 \\ \mu & -\mu - \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 & \dots & 0 & 0 \\ 0 & 2\mu & -2\mu - \lambda_1 - \lambda_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda_1 - 3\mu & \lambda_1 \\ 0 & 0 & 0 & \dots & 3\mu & -3\mu \end{pmatrix} \end{matrix}$$



$$\frac{d\pi}{dt} = \pi Q = 0, \quad \sum \pi_{(u,v)} = 1$$

$$\pi = \begin{bmatrix} \pi(0,0) \\ \pi(0,1) \\ \pi(0,2) \\ \vdots \\ \pi(2,3) \\ \pi(2,4) \end{bmatrix}$$

# Steady state probabilities - Comparison



# Game - Definition

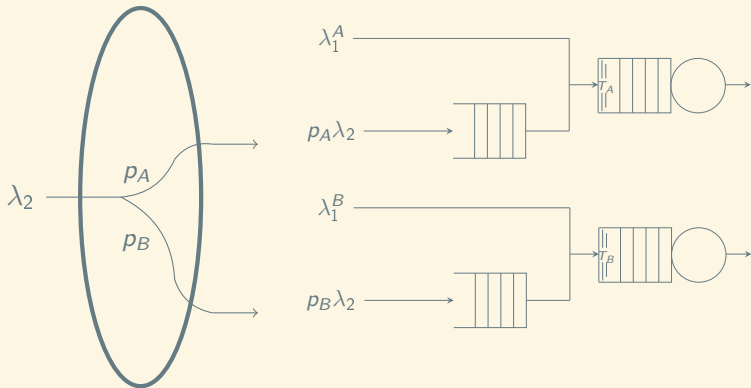


# Game - Definition



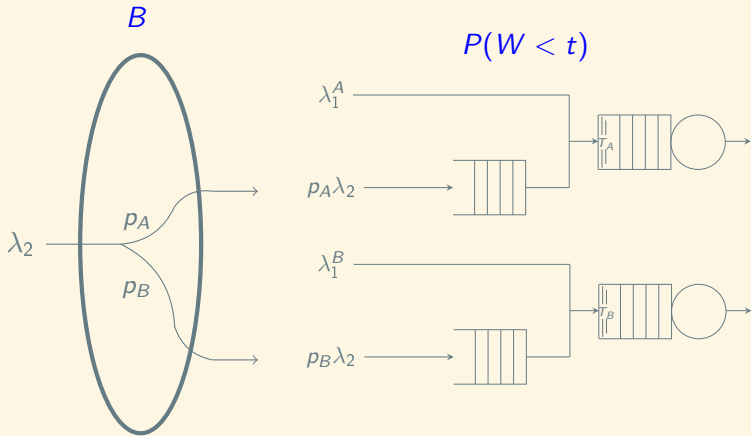
- ▶ Misra S, Sarkar S. Priority-based time-slot allocation in wireless body area networks during medical emergency situations: An evolutionary game-theoretic perspective
- ▶ Song J, Wen J. A non-cooperative game with incomplete information to improve patient hospital choice

## Game - Players and objectives





## Game - Players and objectives



## Performance Measures - Blocking time

$$B = \frac{\sum_{(u,v) \in S_A^{(2)}} \pi(u,v) b(u,v)}{\sum_{(u,v) \in S_A^{(2)}} \pi(u,v)}$$

## Performance Measures - Blocking time

$$B = \frac{\sum_{(u,v) \in S_A^{(2)}} \pi(u,v) b(u,v)}{\sum_{(u,v) \in S_A^{(2)}} \pi(u,v)}$$

# Performance Measures - Blocking time

$$B = \frac{\sum_{(u,v) \in S_A^{(2)}} \pi(u,v) b(u,v)}{\sum_{(u,v) \in S_A^{(2)}} \pi(u,v)}$$

$$b(u, v) = \begin{cases} 0, & \text{if } (u, v) \notin S_b \\ c(u, v) + b(u-1, v), & \text{if } v = N = T \\ c(u, v) + b(u, v-1), & \text{if } v = N \neq T \\ c(u, v) + p_s(u, v)b(u-1, v) + p_a(u, v)b(u, v+1), & \text{if } u > 0 \text{ and } v = T \\ c(u, v) + p_s(u, v)b(u, v-1) + p_a(u, v)b(u, v+1), & \text{otherwise} \end{cases}$$

$$S_b = \{(u, v) \in S \mid u > 0\}$$

$$c(u, v) = \begin{cases} \frac{1}{\min(v, C)\mu}, & \text{if } v = N \\ \frac{1}{\lambda_1 + \min(v, C)\mu}, & \text{otherwise} \end{cases}$$

$$p_s(u, v) = \frac{\min(v, C)\mu}{\lambda_1 + \min(v, C)\mu}, \quad p_a(u, v) = \frac{\lambda_1}{\lambda_1 + \min(v, C)\mu}$$

## Performance Measures - Proportion within time

$$P(W < t) = \frac{\lambda_1 P_{L'_1}}{\lambda_2 P_{L'_2} + \lambda_1 P_{L'_1}} P(W^{(1)} < t) + \frac{\lambda_2 P_{L'_2}}{\lambda_2 P_{L'_2} + \lambda_1 P_{L'_1}} P(W^{(2)} < t)$$

$$P(W^{(1)} < t) = \frac{\sum_{(u,v) \in S_A^{(1)}} P(W_{(u,v)}^{(1)} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(1)}} \pi_{u,v}}$$

$$P(W^{(2)} < t) = \frac{\sum_{(u,v) \in S_A^{(2)}} P(W_{(u,v)}^{(2)} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(2)}} \pi_{u,v}}$$

## Performance Measures - Proportion within time

$$P(W^{(i)} < t) = \frac{\sum_{(u,v) \in S_A^{(i)}} P(W_{u,v}^{(i)} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(i)}} \pi_{u,v}}, \quad \text{for } i = \{1, 2\}$$

# Performance Measures - Proportion within time

$$P(W^{(i)} < t) = \frac{\sum_{(u,v) \in S_A^{(i)}} P(W_{u,v}^{(i)} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(i)}} \pi_{u,v}}, \quad \text{for } i = \{1, 2\}$$

$$W_{(u,v)}^{(1)} \sim \begin{cases} \text{Erlang}(v, \mu), & \text{if } C = 1 \text{ and } v > 1 \\ \text{Hypo}([v - C, 1], [C\mu, \mu]), & \text{if } C > 1 \text{ and } v > C \\ \text{Erlang}(1, \mu), & \text{if } v \leq C \end{cases}$$

$$W_{(u,v)}^{(2)} \sim \begin{cases} \text{Erlang}(\min(v, T), \mu), & \text{if } C = 1 \text{ and } v, T > 1 \\ \text{Hypo}([\min(v, T) - C, 1], [C\mu, \mu]), & \text{if } C > 1 \text{ and } v, T > C \\ \text{Erlang}(1, \mu), & \text{if } v \leq C \text{ or } T \leq C \end{cases}$$

# Performance Measures - Proportion within time

$$P(W^{(i)} < t) = \frac{\sum_{(u,v) \in S_A^{(i)}} P(W_{u,v}^{(i)} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(i)}} \pi_{u,v}}, \quad \text{for } i = \{1, 2\}$$

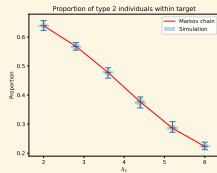
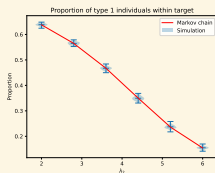
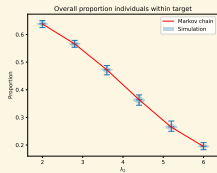
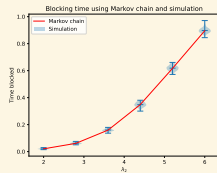
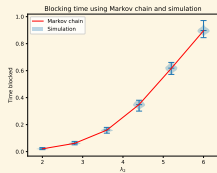
$$P(W_{(u,v)}^{(1)} < t) = \begin{cases} 1 - \sum_{i=0}^{v-1} \frac{1}{i!} e^{-\mu t} (\mu t)^i, & \text{if } C = 1 \text{ and } v > 1 \\ 1 - (\mu C)^{v-C} \mu \sum_{k=1}^{\lceil \vec{r} \rceil} \sum_{l=1}^{r_k} \frac{\Psi_{k,l}(-\lambda_k) t^{r_k-l} e^{-\lambda_k t}}{(r_k-l)!(l-1)!}, & \text{if } C > 1 \text{ and } v > C \\ \quad \text{where } \vec{r} = (v-C, 1) \text{ and } \vec{\lambda} = (C\mu, \mu) \\ 1 - e^{-\mu t}, & \text{if } v \leq C \end{cases}$$

$$P(W_{(u,v)}^{(2)} < t) = \begin{cases} 1 - \sum_{i=0}^{\min(v,T)-1} \frac{1}{i!} e^{-\mu t} (\mu t)^i, & \text{if } C = 1 \text{ and } v, T > 1 \\ 1 - (\mu C)^{\min(v,T)-C} \sum_{k=1}^{\lceil \vec{r} \rceil} \sum_{l=1}^{r_k} \frac{\Psi_{k,l}(-\lambda_k) t^{r_k-l} e^{-\lambda_k t}}{(r_k-l)!(l-1)!}, & \text{if } C > 1 \text{ and } v, T > C \\ \quad \text{where } \vec{r} = (\min(v, T) - C, 1) \text{ and } \vec{\lambda} = (C\mu, \mu) \\ 1 - e^{-\mu t}, & \text{if } v \leq C \text{ or } T \leq C \end{cases}$$

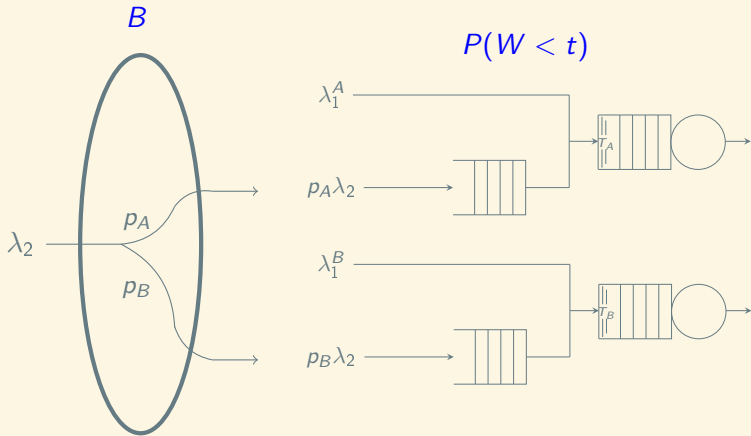


# Comparisons

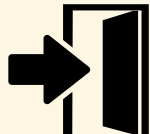
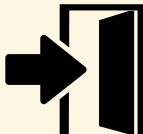
N/A



## Game - Players and objectives



## Game - Strategies



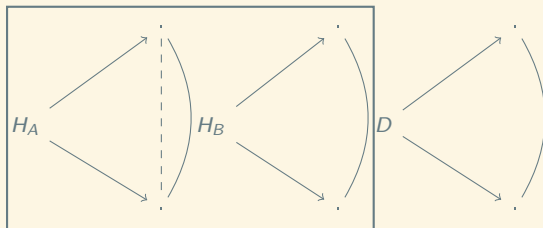
$$p_A, p_B \in [0, 1]$$

$$T_A \in [1, N_A]$$

$$T_B \in [1, N_B]$$

$$p_A + p_B = 1$$

# Game - Formulation

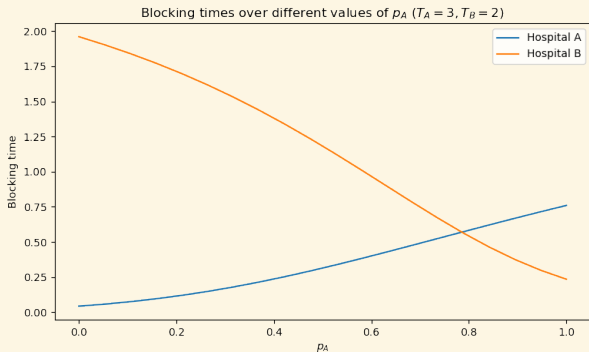


## Ambulance's Decision

$$R = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,N_B} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N_A,1} & p_{N_A,2} & \cdots & p_{N_A,N_B} \end{pmatrix}$$

# Ambulance's Decision

$$\begin{matrix} T_A = 3 \\ T_B = 2 \end{matrix} \Rightarrow \begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & x & - & - \\ - & - & - & - \end{pmatrix} \Rightarrow B_A(T = 3) = B_B(T = 2)$$



# Hospitals' Decision

$$A = \begin{pmatrix} U_{1,1}^A & U_{1,2}^A & \cdots & U_{1,N_B}^A \\ U_{2,1}^A & U_{2,2}^A & \cdots & U_{2,N_B}^A \\ \vdots & \vdots & \ddots & \vdots \\ U_{N_A,1}^A & U_{N_A,2}^A & \cdots & U_{N_A,N_B}^A \end{pmatrix}, \quad B = \begin{pmatrix} U_{1,1}^B & U_{1,2}^B & \cdots & U_{1,N_B}^B \\ U_{2,1}^B & U_{2,2}^B & \cdots & U_{2,N_B}^B \\ \vdots & \vdots & \ddots & \vdots \\ U_{N_A,1}^B & U_{N_A,2}^B & \cdots & U_{N_A,N_B}^B \end{pmatrix}$$

$$U_{T_A, T_B}^{(i)} = 1 - \left[ (P(X^{(i)} < t) - 0.95)^2 \right]$$

# Nash Equilibrium

$$R = \begin{pmatrix} 0.5 & 0.1 & 0 & 0 \\ 0.9 & 0.5 & 0.2 & 0 \\ 1 & 0.8 & 0.5 & 0.3 \\ 1 & 1 & 0.7 & 0.5 \end{pmatrix}$$

$$A = \begin{pmatrix} 0.99998394 & 0.99998394 & 0.99998394 & 0.99998394 \\ 0.99998955 & 0.99998848 & 0.99998649 & 0.9999845 \\ 0.99999952 & 0.9999987 & 0.99999596 & 0.99999199 \\ 0.99994372 & 0.99995113 & 0.99998603 & 0.99999911 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.99998394 & 0.99998955 & 0.99999952 & 0.99994372 \\ 0.99998394 & 0.99998848 & 0.9999987 & 0.99995113 \\ 0.99998394 & 0.99998649 & 0.99999596 & 0.99998603 \\ 0.99998394 & 0.9999845 & 0.99999199 & 0.99999911 \end{pmatrix}$$

**Nash Equilibria:**  $\frac{A}{(0, 0, 0.4, 0.6)}$   $\frac{B}{(0, 0, 0.4, 0.6)}$



# Asymmetric Replicator Dynamics

$$\frac{dx}{dt}_i = x_i((f_x)_i - \phi_x), \quad \text{for all } i$$

$$\frac{dy}{dt}_i = y_i((f_y)_i - \phi_y), \quad \text{for all } i$$

- ▶ Fudenberg, Drew, et al. The theory of learning in games. Vol. 2. MIT press, 1998.
- ▶ Elvio, Accinelli and Carrera, Edgar. 2011. Evolutionarily Stable Strategies and Replicator Dynamics in Asymmetric Two-Population Games. 10.1007/978-3-642-11456-4\_3.

## Inefficiency measure

$$PoA = \frac{\max_{s \in E} Cost(s)}{\min_{s \in S} Cost(S)}$$

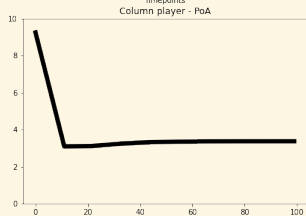
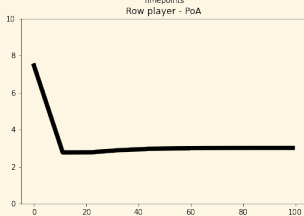
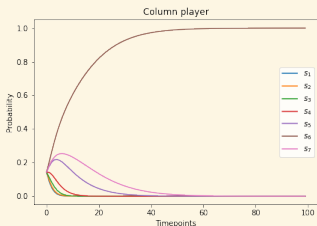
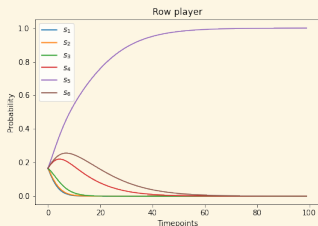
## Inefficiency measure

$$PoA = \frac{\max_{s \in E} Cost(s)}{\min_{s \in S} Cost(S)}$$

$$PoA_A(s_r) = \frac{Cost(s_r)}{\min_{s \in S} Cost(S)},$$

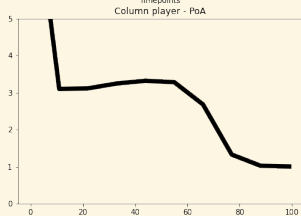
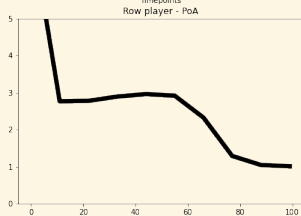
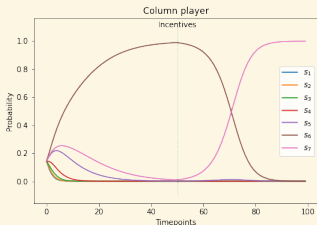
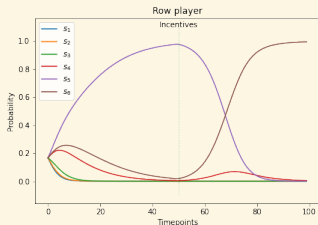
$$PoA_B(s_c) = \frac{Cost(s_c)}{\min_{s \in S} Cost(S)}$$

# Learning algorithms - Asymmetric replicator dynamics



“Inefficiencies can be learned  
and emerge naturally”

# Learning algorithms - Asymmetric replicator dynamics



“Targeted incentivisation of behaviours can help escape learned inefficiencies”

Thank you!

“Inefficiencies can be learned and emerge naturally”

“Targeted incentivisation of behaviours can help escape learned inefficiencies”

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🐦 @Michalis\_Pan

🐙 @11michalis11

<https://github.com/11michalis11/AmbulanceDecisionGame>