A 3-player game theoretic model of a choice between two queueing systems with strategic managerial decision making

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About me

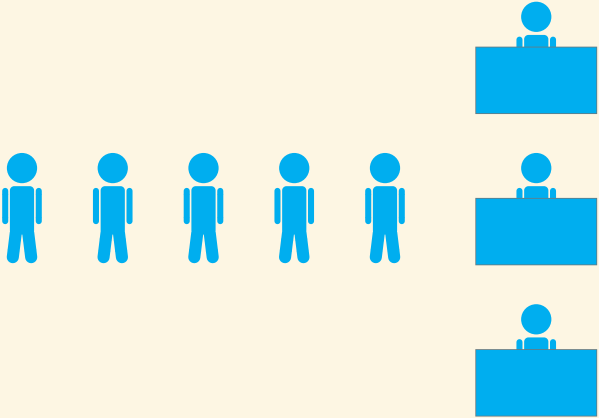
About me

Queues - Examples

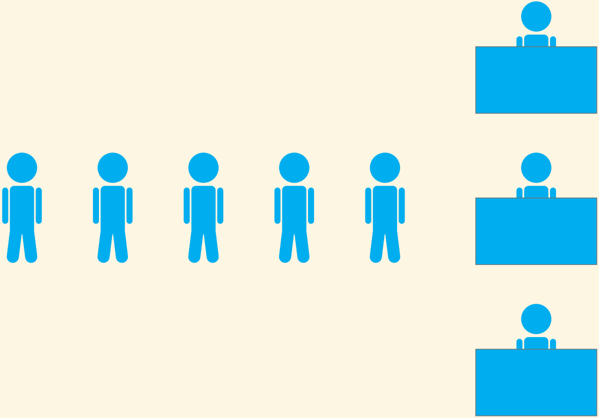
Queues - Examples

- Shone R, Knight VA, Williams JE. Comparisons between observable and unobservable M/M/1 queues with respect to optimal customer behaviour
- Kerner Y, Shmuel-Bittner O. Strategic behaviour and optimization in a hybrid M/M/1 queue with retrials.
- Gai Y, Liu H, Krishnamachari B. A packet dropping mechanism for efficient operation of M/M/1 queues with selfish users.

Queues - Examples



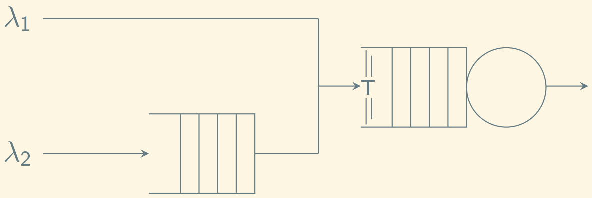
Queues - Examples



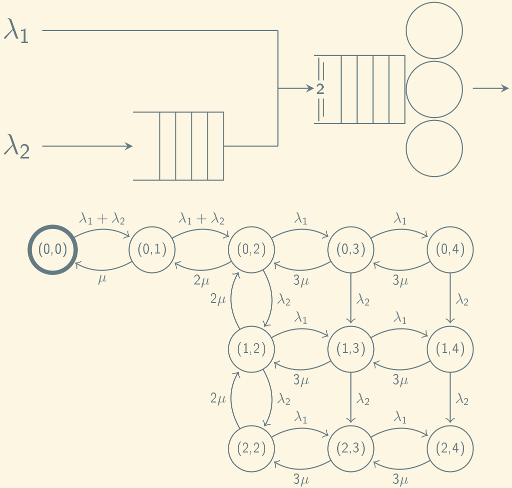
- Knight V, Harper P. The Impact of Choice on Public Services.
- Wang X, Song C, Zhuang J. Simulating a multi-stage screening network: A queueing theory and game theory application.

Queues - Examples

Queueing network structure



Markov Chain - Custom network



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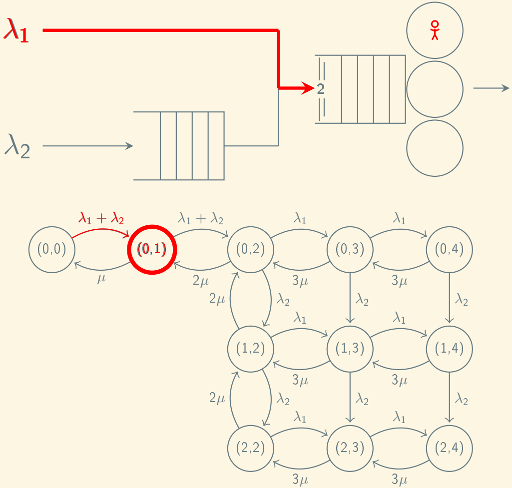

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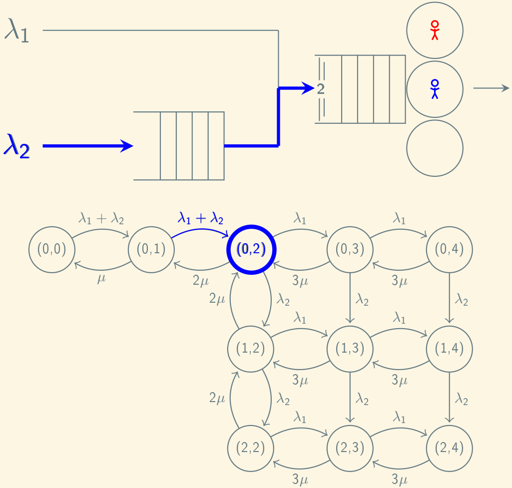

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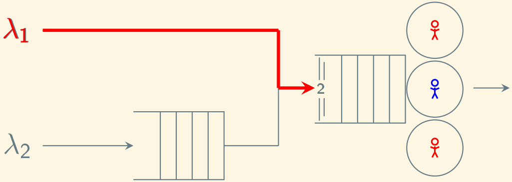

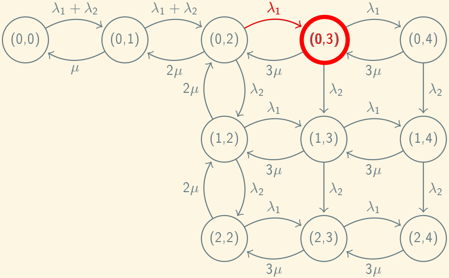

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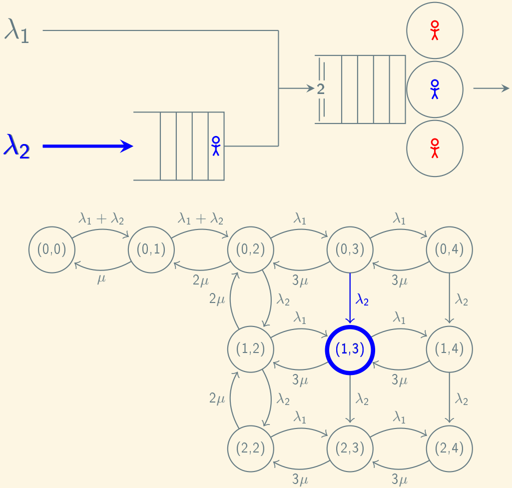
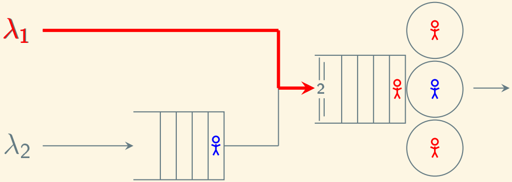
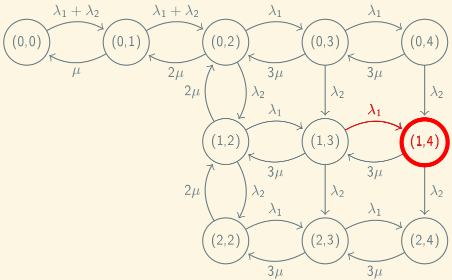


Markov Chain - Custom network





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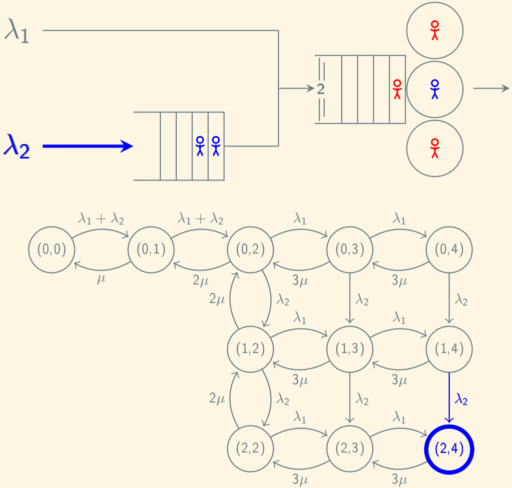

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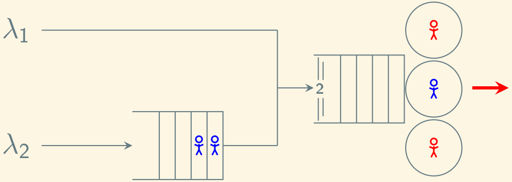
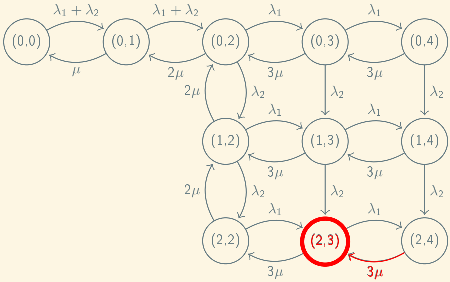
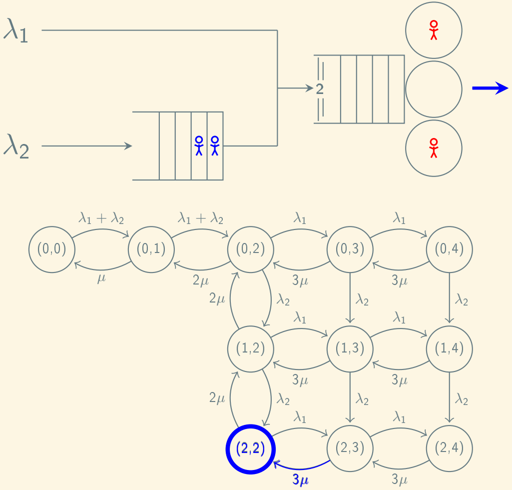
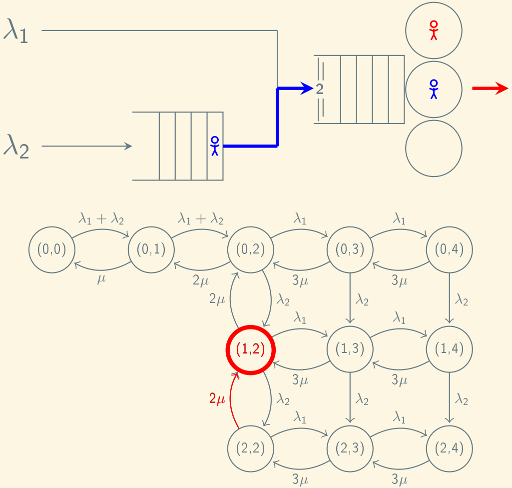




Markov Chain - Custom network

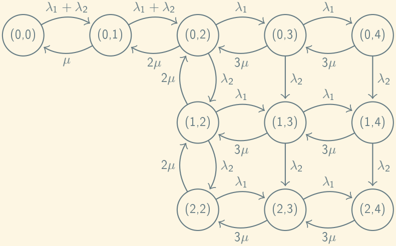


Markov Chain - Custom network



Steady state probabilities - Custom network

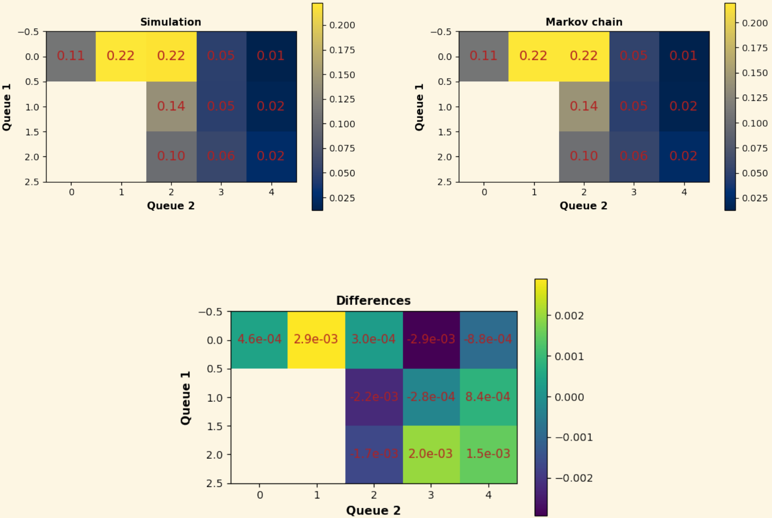
$$
Q = \begin{array}{c} \\ \\ \end{array}
\begin{array}{cccccc|c}
(0,0) & (0,1) & (0,2) & & (2,3) & (2,4) & \\
\hline
-\lambda_1-\lambda_2 & \lambda_1+\lambda_2 & 0 & \dots & 0 & 0 & (0,0) \\
\mu & -\mu-\lambda_1-\lambda_2 & \lambda_1+\lambda_2 & \dots & 0 & 0 & (0,1) \\
0 & 2\mu & -2\mu-\lambda_1-\lambda_2 & \dots & 0 & 0 & (0,2) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
0 & 0 & 0 & \dots & -\lambda_1-3\mu & \lambda_1 & (2,3) \\
0 & 0 & 0 & \dots & 3\mu & -3\mu & (2,4)
\end{array}
$$



$$\frac{d\pi}{dt} = \pi Q = 0, \qquad \sum \pi_{(u,v)} = 1$$

$$\pi = \begin{bmatrix} \pi_{(0,0)} \\ \pi_{(0,1)} \\ \pi_{(0,2)} \\ \vdots \\ \pi_{(2,3)} \\ \pi_{(2,4)} \end{bmatrix}$$

Steady state probabilities - Comparison



Performance Measures - Number of individuals

$$(u_i, v_i) \longrightarrow \pi_{(u_i, v_i)}$$

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$$(u_i, v_i) \longrightarrow \pi_{(u_i, v_i)}$$

$$L = \sum_{i=1}^{|\pi|} \pi_i (u_i + v_i)$$

$$L_1 = \sum_{i=1}^{|\pi|} \pi_i u_i$$

$$L_2 = \sum_{i=1}^{|\pi|} \pi_i v_i$$

Performance Measures - Waiting time

$$W = \frac{\lambda_1 P_{L'_1}}{\lambda_2 P_{L'_2} + \lambda_1 P_{L'_1}} W^{(1)} + \frac{\lambda_2 P_{L'_2}}{\lambda_2 P_{L'_2} + \lambda_1 P_{L'_1}} W^{(2)}$$

$$W^{(1)} = \frac{\sum_{\substack{(u,v) \in S_A^{(1)} \\ v \geq C}} \frac{1}{C\mu} \times (v - C + 1) \times \pi(u,v)}{\sum_{(u,v) \in S_A^{(1)}} \pi(u,v)}$$

$$W^{(2)} = \frac{\sum_{\substack{(u,v) \in S_A^{(2)} \\ min(v,T) \geq C}} \frac{1}{C\mu} \times (\min(v+1, T) - C) \times \pi(u,v)}{\sum_{(u,v) \in S_A^{(2)}} \pi(u,v)}$$

Performance Measures - Waiting time

$$W^{(1)} = \frac{\sum_{(u,v)\in S_A^{(1)}} w^{(1)}(u,v)\pi_{(u,v)}}{\sum_{(u,v)\in S_A^{(1)}} \pi_{(u,v)}}, \quad W^{(2)} = \frac{\sum_{(u,v)\in S_A^{(2)}} w^{(2)}(u,v)\pi_{(u,v)}}{\sum_{(u,v)\in S_A^{(2)}} \pi_{(u,v)}}$$

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$$S_A^{(1)} = \{(u,v) \in S \mid v < N\}, \quad S_A^{(2)} = \begin{cases} \{(u,v) \in S \mid u < M\} & \textbf{if } T \leq N \\ \{(u,v) \in S \mid v < N\} & \textbf{otherwise} \end{cases}$$

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$$S_W = \{(u,v) \in S \mid v > C\}$$

$$c^{(1)}(u,v) = \begin{cases} 0, & \text{if } u > 0 \text{ and } v = T \\ \frac{1}{\min(v,C)\mu}, & \text{otherwise} \end{cases}, \quad c^{(2)}(u,v) = \begin{cases} 0, & \text{if } u > 0 \\ \frac{1}{\min(v,C)\mu}, & \text{otherwise} \end{cases}$$

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$$w^{(i)}(u,v) = \begin{cases} 0, & \text{if } (u,v) \notin S_w \\ c^{(i)}(u,v) + w^{(i)}(u-1,v), & \text{if } u > 0 \text{ and } v = T \\ c^{(i)}(u,v) + w^{(i)}(u,v-1), & \text{otherwise} \end{cases}$$

Performance Measures - Blocking time

$$B = \frac{\sum_{(u,v)\in S_A^{(2)}} \pi_{(u,v)} \; b(u,v)}{\sum_{(u,v)\in S_A^{(2)}} \pi_{(u,v)}}$$

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$$b(u,v) = \begin{cases} 0, & \text{if } (u,v) \notin S_b \\ c(u,v) + b(u-1,v), & \text{if } v = N = T \\ c(u,v) + b(u,v-1), & \text{if } v = N \neq T \\ c(u,v) + p_s(u,v)b(u-1,v) + p_a(u,v)b(u,v+1), & \text{if } u > 0 \text{ and} \\ & \quad v = T \\ c(u,v) + p_s(u,v)b(u,v-1) + p_a(u,v)b(u,v+1), & \text{otherwise} \end{cases}$$

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$$S_b = \{(u,v) \in S \mid u > 0\}$$

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$$S_b = \{(u,v) \in S \mid u > 0\}$$

$$c(u,v) = \begin{cases} \frac{1}{\min(v,C)\mu}, & \text{if } v = N \\ \frac{1}{\lambda_1 + \min(v,C)\mu}, & \text{otherwise} \end{cases}$$

Performance Measures - Blocking time

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$$p_s(u,v) = \frac{\min(v,C)\mu}{\lambda_1 + \min(v,C)\mu}, \qquad p_a(u,v) = \frac{\lambda_1}{\lambda_1 + \min(v,C)\mu}$$

Performance Measures - Proportion within time

$$P(X < t) = \frac{\lambda_1 P_{L'_1}}{\lambda_2 P_{L'_2} + \lambda_1 P_{L'_1}} P(X^{(1)} < t) + \frac{\lambda_2 P_{L'_2}}{\lambda_2 P_{L'_2} + \lambda_1 P_{L'_1}} P(X^{(2)} < t)$$

$$P(X^{(1)} < t) = \frac{\sum_{(u,v) \in S_A^{(1)}} P(X_{(u,v)}^{(1)} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(1)}} \pi_{u,v}}$$

$$P(X^{(2)} < t) = \frac{\sum_{(u,v) \in S_A^{(2)}} P(X_{(u,v)}^{(2)} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(2)}} \pi_{u,v}}$$

Performance Measures - Proportion within time

$$P(X^{(i)} < t) = \frac{\sum_{(u,v)\in S_A^{(i)}} P(X_{u,v}^{(i)} < t)\pi_{u,v}}{\sum_{(u,v)\in S_A^{(i)}} \pi_{u,v}}, \quad \textbf{for } i = \{1, 2\}$$

Performance Measures - Proportion within time

$$P(X^{(i)} < t) = \frac{\sum_{(u,v) \in S_A^{(i)}} P(X_{u,v}^{(i)} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(i)}} \pi_{u,v}}, \quad \textbf{for } i = \{1, 2\}$$

$$P(X_{(u,v)}^{(1)} < t) = \begin{cases} 1 - \sum_{i=0}^{v-1} \frac{1}{i!} e^{-\mu t} (\mu t)^i, & \text{if } C = 1 \text{ and } v > 1 \\ 1 - (\mu C)^{v-C} \mu \sum_{k=1}^{|\vec{r}|} \sum_{l=1}^{r_k} \frac{\Psi_{k,l}(-\lambda_k) t^{r_k - l} e^{-\lambda_k t}}{(r_k - l)!(l-1)!}, & \text{if } C > 1 \text{ and } v > C \\ \quad \text{where } \vec{r} = (v - C, 1) \text{ and } \vec{\lambda} = (C\mu, \mu) & \\ 1 - e^{-\mu t}, & \text{if } v \leq C \end{cases}$$

$$P(X_{(u,v)}^{(2)} < t) = \begin{cases} 1 - \sum_{i=0}^{\min(v,T)-1} \frac{1}{i!} e^{-\mu t} (\mu t)^i, & \text{if } C = 1 \text{ and } v, T > 1 \\ 1 - \mu (\mu C)^{\min(v,T)-C} \times \sum_{k=1}^{|\vec{r}|} \sum_{l=1}^{r_k} \frac{\Psi_{k,l}(-\lambda_k) t^{r_k - l} e^{-\lambda_k t}}{(r_k - l)!(l-1)!}, & \text{if } C > 1 \text{ and } v, T > C \\ \quad \text{where } \vec{r} = (\min(v, T) - C, 1) \text{ and } \vec{\lambda} = (C\mu, \mu) & \\ 1 - e^{-\mu t}, & \text{if } v \leq C \text{ or } T \leq C \end{cases}$$

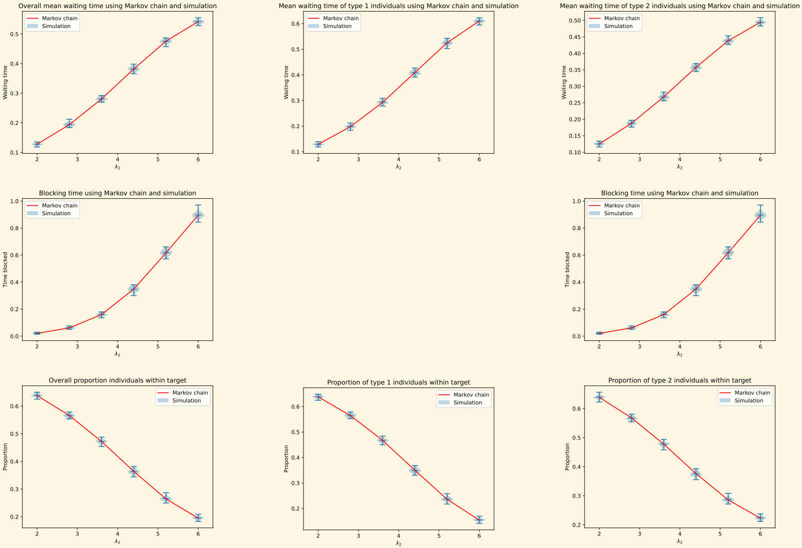
Performance Measures - Proportion within time

$$P(X^{(i)} < t) = \frac{\sum_{(u,v)\in S_A^{(i)}} P(X_{u,v}^{(i)} < t)\pi_{u,v}}{\sum_{(u,v)\in S_A^{(i)}} \pi_{u,v}}, \quad \textbf{for } i = \{1,2\}$$

$$X_{(u,v)}^{(1)} \sim \begin{cases} \textbf{Erlang}(v,\mu), & \textbf{if } C = 1 \textbf{ and } v > 1 \\ \textbf{Hypo}\left([v - C, 1], [C\mu, \mu]\right), & \textbf{if } C > 1 \textbf{ and } v > C \\ \textbf{Erlang}(1,\mu), & \textbf{if } v \leq C \end{cases}$$

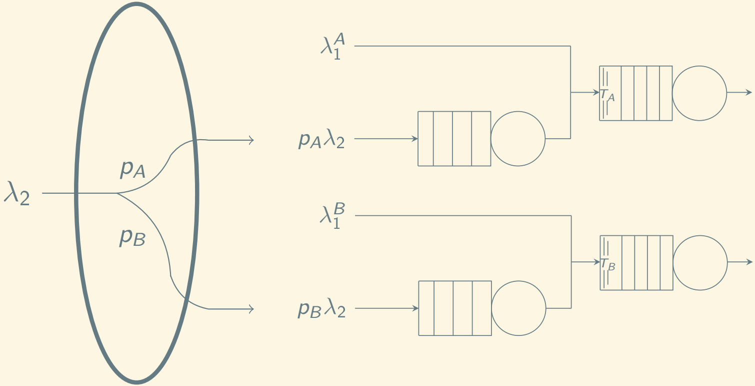
$$X_{(u,v)}^{(2)} \sim \begin{cases} \textbf{Erlang}(\min(v,T),\mu), & \textbf{if } C = 1 \textbf{ and } v, T > 1 \\ \textbf{Hypo}\left([\min(v,T) - C, 1], [C\mu, \mu]\right), & \textbf{if } C > 1 \textbf{ and } v, T > C \\ \textbf{Erlang}(1,\mu), & \textbf{if } v \leq C \textbf{ or } T \leq C \end{cases}$$

Comparisons



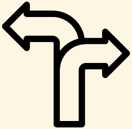
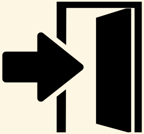
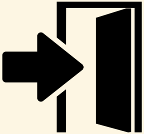
Game - Definition

Game - Players



Game - Strategies



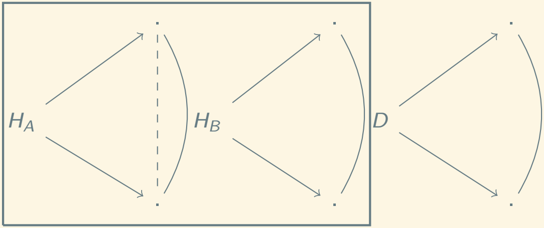
$p_A, p_B \in [0, 1]$

$T_A \in [1, N_A]$

$T_B \in [1, N_B]$

$p_A + p_B = 1$

Game - Formulation



$$A = \begin{pmatrix} U^A_{1,1} & U^A_{1,2} & \cdots & U^A_{1,N_B} \\ U^A_{2,1} & U^A_{2,2} & \cdots & U^A_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ U^A_{N_A,1} & U^A_{N_A,2} & \cdots & U^A_{N_A,N_B} \end{pmatrix}, \quad B = \begin{pmatrix} U^B_{1,1} & U^B_{1,2} & \cdots & U^B_{1,N_B} \\ U^B_{2,1} & U^B_{2,2} & \cdots & U^B_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ U^B_{N_A,1} & U^B_{N_A,2} & \cdots & U^B_{N_A,N_B} \end{pmatrix}$$

$$R = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,N_B} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N_A,1} & p_{N_A,2} & \cdots & p_{N_A,N_B} \end{pmatrix}$$

Solution concepts in games

$$A \in \mathbb{R}^{m \times n}, \qquad B \in \mathbb{R}^{m \times n}$$

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- Fudenberg, Drew, et al. The theory of learning in games. Vol. 2. MIT press, 1998.
- Elvio, Accinelli and Carrera, Edgar. 2011. Evolutionarily Stable Strategies and Replicator Dynamics in Asymmetric Two-Population Games. 10.1007/978-3-642-11456-4_3.

Inefficiency measure

$$PoA = \frac{\max_{s \in E} Cost(s)}{\min_{s \in S} Cost(S)}$$

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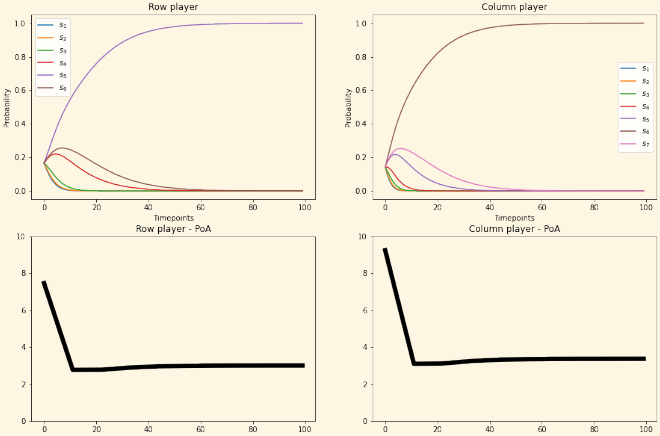
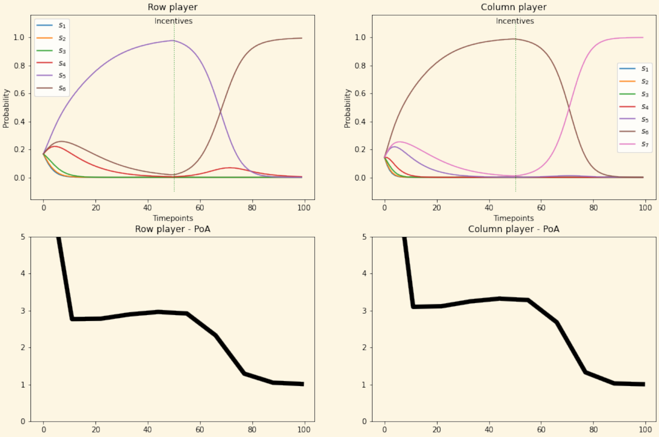
$$PoA_A(s_r) = \frac{Cost(s_r)}{\min_{s \in S} Cost(S)}, \qquad PoA_B(s_c) = \frac{Cost(s_c)}{\min_{s \in S} Cost(S)}$$

Learning algorithms - Asymmetric replicator dynamics



"Inefficiencies can be learned and emerged naturally in an interactive system"

Learning algorithms - Asymmetric replicator dynamics



"Targeted incentivisation of behaviours can help escape learned inefficiencies"

Thank you!

"Inefficiencies can be learned and emerged naturally in an interactive system"

"Targeted incentivisation of behaviours can help escape learned inefficiencies"

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https://github.com/11michalis11/AmbulanceDecisionGame