PRACTICAL - 01

TOPIC - Limits and Continuity.

1. $\lim_{x\to 0}\left[\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]$

$\lim_{x\to 0}\left[\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \times \dfrac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}} \times \dfrac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}\right.$

$\lim_{x\to 0} \dfrac{(a+2x-3x)(\sqrt{3a+x}+2\sqrt{x})}{(3a+x-4x)(\sqrt{a+2x}+\sqrt{3x})}$

$= \dfrac{1}{3}\dfrac{\sqrt{3a+a}+2\sqrt{a}}{\sqrt{a+2a}+\sqrt{3a}}$

$= \dfrac{1}{3}\dfrac{\sqrt{4a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$

$= \dfrac{1}{3}\dfrac{2\sqrt{a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$

$= \dfrac{1}{3} \times \dfrac{4\sqrt{a}}{2\sqrt{3}\sqrt{a}}$

$= \dfrac{2}{3\sqrt{3}}$

2. $\lim_{y\to 0}\left[\dfrac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\right]$

$= \lim_{y\to 0}\left[\dfrac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}} \times \dfrac{\sqrt{a+y}+\sqrt{a}}{\sqrt{a+y}+\sqrt{a}}\right]$

$$= \lim_{y \to 0} \frac{a+y-a}{y\sqrt{a+y}(\sqrt{a+y}+\sqrt{a}}$$

$$= \lim_{y \to 0} \frac{y}{y\sqrt{a+y}(\sqrt{a+y}+\sqrt{a})}$$

$$= \lim_{y \to 0} \frac{1}{\sqrt{a+0}\,(\sqrt{a+0}+\sqrt{a}}$$

$$= \frac{1}{\sqrt{a}(\sqrt{a}+\sqrt{a})}$$

$$= \frac{1}{\sqrt{a}(2\sqrt{a})} = \frac{1}{2a}$$

8. $\lim_{x \to \pi/6} \frac{\cos x - \sqrt{3}\sin x}{\pi - 6x}$

By substituting $x - \frac{\pi}{6} = h$

$x = h + \frac{\pi}{6}$ where $h \to 0$

$$\lim_{h \to 0} \frac{\cos(h+\frac{\pi}{6}) - \sqrt{3}\sin(h+\frac{\pi}{6})}{\pi - 6(h+\frac{\pi}{6})}$$

(using $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$]

$$\lim_{h \to 0} \frac{\cos h - \cos\frac{\pi}{6} - \sin h \cdot \sin\frac{\pi}{6} - \sqrt{3}\sin h\cos\frac{\pi}{6} + \cos h \sin\frac{\pi}{6}}{\pi - 6(\frac{6h+\pi}{6})}$$

$\cos\frac{\pi}{6} = \cos 30^\circ = \sqrt{3}/2$

$\sin\frac{\pi}{6} = \sin 30 = \frac{1}{2}$

$$\lim_{h \to 0} \frac{\cos h \cdot \frac{\sqrt{3}}{2} - \sin h\frac{1}{2} - \sqrt{3}(\sin h\frac{\sqrt{3}}{2} + \cos h \cdot \frac{1}{2})}{\pi - 6h + \pi}$$

$$= \lim_{h\to 0} \frac{\cos \frac{\sqrt{3}}{2}h - \sin\frac{h}{2} - \sin\frac{3h}{2} - \cos\frac{\sqrt{3}}{2}h}{-6h}$$

$$= \lim_{h\to 0} \frac{-\sin\frac{4h}{2}}{-6h}$$

$$= \lim_{h\to 0} \frac{\sin 4h}{3+2h}$$

$$= \frac{1}{3}\lim_{h\to 0}\frac{\sin h}{h} = \frac{1}{3}\times 1 = \frac{1}{3}$$

4.

$$\lim_{x\to\infty}\left[\frac{\sqrt{x^2+5}-\sqrt{x^2-2}}{\sqrt{x^2+3}-\sqrt{x^2+1}}\right]$$

By rationalizing numerator and denominator both.

$$\lim_{x\to\infty}\left[\frac{\sqrt{x^2+5}-\sqrt{x^2-2}}{\sqrt{x^2+3}-\sqrt{x^2+1}} \times \frac{\sqrt{x^2+5}+\sqrt{x^2-3}}{\sqrt{x^2+5}+\sqrt{x^2-3}} \times \frac{\sqrt{x^2+3}+\sqrt{x^2+1}}{\sqrt{x^2+3}-\sqrt{x^2+1}}\right]$$

$$\lim_{x\to\infty}\left(\frac{(x^2+5-x^2+3)\,(\sqrt{x^2+3}+\sqrt{x^2+1})}{(x^2+3-x^2-1)\,(\sqrt{x^2+5}+\sqrt{x^2-3})}\right)$$

$$\lim_{x\to\infty}\frac{\overset{4}{\cancel{8}}(\sqrt{x^2+3}+\sqrt{x^2+1})}{2(\sqrt{x^2+3}+\sqrt{x^2-3})}$$

$$= 4\lim_{x\to\infty}\frac{\sqrt{x^2\left(1+\frac{3}{x^2}\right)}+\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{\sqrt{x^2\left(1+\frac{5}{x^2}\right)}+\sqrt{x^2\left(1-\frac{3}{x^2}\right)}}$$

After applying limit we get,

$= 4$

5. $f(x) = \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$, for $0 < x \le \dfrac{\pi}{2}$

$= \dfrac{\cos x}{x - 2\pi}$, for $\dfrac{\pi}{2} < x < \pi$ $\Big\}$ at $x = \dfrac{\pi}{2}$

$f\left(\dfrac{\pi}{2}\right) = \dfrac{\sin 2(\pi/2)}{\sqrt{1-\cos 2(\pi/2)}}$ $\therefore f(\pi/2) = 0$]

f at $x = \dfrac{\pi}{2}$ define

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\pi/2^+} \frac{\cos x}{\pi - 2x}$$

By substitling method,

$$x - \frac{\pi}{2} = h$$

$$x = h + \pi/2$$

where $h \to 0$

$$\lim_{h\to 0} \frac{\cos\left(h + \frac{\pi}{2}\right)}{\pi - 2\left(h + \frac{\pi}{2}\right)}$$

$$\lim_{h\to 0} \frac{\cos(h + \pi/2)}{x - 2\left(\frac{2h+\pi}{2}\right)}$$

$$\lim_{h\to 0} \frac{\cos(\cancel{h} + \pi/2)}{\cancel{x}-2\left(\frac{2h+\pi}{2}\right)}$$

$$\lim_{h\to 0} \frac{\cos(h+\pi/2)}{-2h}$$ using $\cos(A+B) = \cos A \cos B - \sin A \cdot \sin B$

$$\lim_{h\to 0} \frac{\cos h \cdot \cos \pi/2 - \sin h \cdot \sin \pi/2}{-2h}$$

$$\lim_{h\to 0} \frac{\cos h \cdot 0 - \sin h}{-2h}$$

$$\lim_{h\to 0} \frac{\cancel{-}\sin h}{\cancel{-}2h}$$

$$\frac{1}{2} \lim_{h\to 0} \frac{\sin h}{h} = \frac{1}{2}$$

(ii) $\lim_{x\to\pi/2^-} f(x) = \lim_{x\to\pi/2} = \frac{\sin 2x}{\sqrt{1-\cos 2x}}$

using $\sin 2x = 2\sin x \cdot \cos x$

$$\lim_{x\to\pi/2^-} \frac{2\sin x \cdot \cos x}{\sqrt{2\sin^2 x}}$$

$$\lim_{x\to\pi/2^-} \frac{2\cancel{\sin x} \cdot \cos x}{\sqrt{2} \cdot \cancel{\sin x}}$$

$$\lim_{x\to\pi/2^-} \frac{2\cos x}{\sqrt{2}}$$

$$\frac{2}{\sqrt{2}} \lim_{x\to\pi/2} -\cos x$$

$\because$ LHL $\neq$ RHL

$\therefore$ f is not continuous at $x = \pi/2$

ii) $f(x) = \frac{x^2-9}{x-3}$, $0 < x < 3$

$= x+3$, $3 \le x \le 6$, } at $x=3$

$= \frac{x^2-9}{x+3}$, $6 \le x < 9$ } & $x = 6$

at $x = 3$

$f(3) = \frac{x^2-9}{x-3} = 0$

f at $x = 3$ define

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x+3$

$f(3) = x+3 = 3+3 = 6$

f is define at $x=3$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x+3) = 6$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3} \frac{x^2-9}{x-3} = (x+3)$

LHL = RHL

for $x=6$ f is continuous at $x=3$

$f(6) = \frac{x^2-9}{x+3} = \frac{36-9}{6+3} = \frac{27}{9} = 3$

$\lim_{x \to 6^+} \frac{x^2-9}{x+3} = \lim_{x \to 6^+} ~~36-9~~ \frac{(x-3)(x+3)}{\cancel{(x+3)}}$

$$\lim_{x \to 6^+} (x-3) = 6-3 = 3$$

$$\lim_{x \to 6^-} (x+3) = 3+6 = 9$$

$\therefore$ LHL $\neq$ RHL

f^n is not continuous

6. $f(x) = \dfrac{1-\cos 4x}{x^2}$, $x < 0$
$= K$, $x = 0$ } at $x=0$

f is continuous at $x = 0$

$$\lim_{x \to 0} f(x) = f(0)$$

$$\lim_{x \to 0} \frac{1-\cos 4x}{x^2} = K$$

$$\lim_{x \to 0} \frac{2\sin^2 2x}{x^2} = K$$

$$2\lim_{x \to 0} \frac{\sin^2 2x}{x^2} = K$$

$$2\lim_{x \to 0} \left(\frac{\sin 2x}{x}\right)^2 = K$$

$$2(2)^2 = K$$

$\therefore K = 8$

ii) $f(x) = (\sec^2 x)^{\cot^2 x}$

Using $\tan^2 x = 1 + \sec^2 x$

$\therefore \sec^2 x = 1 + \tan^2 x$

& $\cot^2 x = \frac{1}{\tan^2 x}$

$\lim_{x \to 0} (\sec^2 x)^{\cot^2 x}$

$\lim_{x \to 0} (1 + \tan^2 x)^{\frac{1}{\tan^2 x}}$

Wkt,

$\lim_{x \to 0} (1 + px)^{1/px} = e$

$= e$

$K = e$

iii) $f(x) = \frac{\sqrt{3} - \tan x}{\pi - 3x}$, $x \neq \pi/3$
 $= K$, $x = \pi/3$ } at $x = \frac{\pi}{3}$

$x - \pi/3 = h$

$x = h + \pi/3$

when $h \to 0$

$f(\pi/3 + h) = \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$

$$\lim_{h\to 0} \frac{\sqrt{3} - \tan \cancel{\sqrt{3}}\,(\pi/3+h)}{\pi - 3(\pi/3+h)}$$

$\lim_{h\to 0}$ $\cancel{\sqrt{3} - t}$ · using $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

$$\lim_{h\to 0} \frac{\sqrt{3} - \dfrac{\tan \pi/3 + \tan h}{1 - \tan \pi/3 \cdot \tan h}}{\pi - 3h}$$

$$\lim_{h\to 0} \frac{\dfrac{\sqrt{3}\,(1 - \tan \pi/3 \cdot \tan h) - (\tan \pi/3 + \tan h)}{1 - \tan \pi/3 \cdot \tan h}}{-3h}$$

using $\tan \pi/3 = \tan 60^\circ = \sqrt{3}$

$$\lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - \sqrt{3} \times \sqrt{3} \cdot \tan h) - (\sqrt{3} + \tan h)}{1 - \tan \pi/3 \cdot \tan h}}{-3h}$$

$$= \lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - 3\tan h - \sqrt{3} - \tan h)}{1 - \sqrt{3}\tan h}}{-3h}$$

$$\frac{4}{3} \lim_{h\to 0} \frac{\tan h}{h} \lim_{h\to 0} \frac{1}{(1 - \sqrt{3}\tan h)}$$

$$= \frac{4}{3}\left(\frac{1}{1}\right) = \frac{4}{3}$$

7. $f(x) = \dfrac{1-\cos 3x}{x \cdot \tan x}$, $x \neq 0$ } at $x=0$

$= 9$, $x = 0$

$f(x) = \dfrac{1-\cos 3x}{x \tan x}$

$\lim\limits_{x\to 0} \dfrac{2\sin^2 3/2x}{x \cdot \tan x}$

$= \lim\limits_{x\to 0} \dfrac{\dfrac{2\sin^2 2x}{x^2} \cdot x^2}{\dfrac{x \cdot \tan x}{x^2} \cdot x^2}$

$= 2\lim\limits_{x\to 0} \dfrac{(3/2)^2}{1} = 2 \times \dfrac{9}{4} = \dfrac{9}{2}$

$\lim\limits_{x\to 0} f(x) = \dfrac{9}{2}$, $9 = f(6)$

$\therefore f$ is not continuous at $x=0$

Redefine function

$f(x) = \begin{cases} \dfrac{1-\cos 3x}{x \tan x} , & x \neq 0 \\ = \dfrac{9}{2} , & x = 0 \end{cases}$

Now, $\lim\limits_{x\to 0} f(x) = f(0)$

f has removable discontinuity at $x=0$

ii) $f(x) = (e^{3x}-1)\sin x^{\circ}/x^2 \; ; \; x \neq 0$ } at $x=0$
$= \pi/6 \; , \; x=0$

$$\lim_{x\to0} \frac{(e^{x}-1)\sin(\pi x/180)}{x^2}$$

$$\lim_{x\to0} 3\cdot\frac{e^{3x}-1}{3x} \lim_{x\to0} \sin\left(\frac{\pi x}{180}\right)/x$$

$$3\cdot\lim_{x\to0}\left(\frac{e^{3x}-1}{3x}\right)\lim_{x\to0}\sin(x/180)/x$$

$$3\log(\pi/180) = \pi/60 = f(0)$$

f is continuous at $x=0$

8. $f(x) = \dfrac{e^{x^2}-\cos x}{x^2} \; , \; x=0$

is continuous at $x=0$

$\therefore$ given, f is cts at $x=0$

$$\lim_{x\to0} f(x) = f(0)$$

$$= \lim_{x\to0}\frac{e^{x^2}-\cos x}{x^2} = f(0)$$

$$\lim_{x\to 0} \frac{e^{x^2}-1}{x} + \lim_{x\to 0} \frac{1-\cos x}{x^2}$$

$$\log e + \lim_{x\to 0} \left(\frac{2 \sin^2 x/2}{x}\right)^2$$

Multiply with 2 on Nm & dm

$$= 1 + 2 \times \frac{1}{4} \times \frac{3}{2} = f(0)$$

9. $f(x) = \dfrac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x}$, $x \neq \pi/2$

$f(0)$ is continuous at $x = \pi/2$

$$\lim_{x\to \pi/2} \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \times \frac{\sqrt{2} + \sqrt{1+\sin x}}{\sqrt{2} + \sqrt{1+\sin x}}$$

$$\lim_{x\to \pi/2} \frac{1 + \sin x}{1 - \sin^2 x\,(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \lim_{x\to \pi/2} \frac{1 + \sin x}{(1-\sin x)(1+\sin x)(\sqrt{2} + (\sqrt{1-\sin x})}$$

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$$= \lim_{x\to \pi/2} \frac{1}{(1-\sin x)(\sqrt{2} + (\sqrt{1+\sin x})}$$

$$= \frac{1}{2(\sqrt{2}+\sqrt{2})} = \frac{1}{2(2\sqrt{2})} = \frac{1}{4\sqrt{2}}$$

$$\therefore f(\pi/2) = \frac{1}{4\sqrt{2}}$$

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TOPIC - Derivative

Q1 Show that the following function defined from $\mathbb{R}$ to $\mathbb{R}$ are differentiable.

i) $\cot x$

$f(x) = \cot x$

$Df(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

$= \lim_{x\to a} \frac{\cot x - \cot a}{x-a}$

$= \lim_{x\to a} \frac{\frac{1}{\tan x} - \frac{1}{\tan a}}{x-a}$

$= \lim_{x\to 0} \frac{\tan a - \tan x}{(x-a)\tan x \tan a}$

Put $x-a=h$

$x=a+h$

as $x\to a$, $h\to 0$

$Df(h) = \lim_{h\to 0} \frac{\tan a - \tan(a+h)}{(a+h-a)\tan(a+h)\tan a}$

$= \lim_{h\to 0} \frac{\tan a - \tan(a+h)}{h\times\tan(a+h)\tan a}$

$\left[\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\cdot\tan B} \therefore \tan A\cdot\tan B = \tan(A-B)(1+\tan A\tan B)\right]$

$= \lim_{h\to 0} \frac{\tan(a-a-h)-(1+\tan a.\tan(a+h))}{h\times\tan(a+h)\tan a}$

$= \lim_{h\to 0} \frac{-\tan h}{h} \times \frac{1+\tan a\tan(a+h)}{\tan(a+h)\tan a}$

$= -1\times\frac{1+\tan^2 a}{\tan^2 a}$

$= \frac{-\sec^2 a}{\tan^2 a}$

$= \frac{-1}{\cos^2 a}\times\frac{\cos^2 a}{\sin^2 a}$

$= \frac{-\sec^2 a}{\tan^2 a}$

$= \frac{-1}{\cos^2 a}\times\frac{\cos^2 a}{\sin^2 a}$

$= -\operatorname{cosec}^2 a$

$\therefore Df(a) = -\operatorname{cosec}^2 a$

$\therefore f$ is differentiable $\forall a\in\mathbb{R}$

ii) $\operatorname{cosec} x$

$f(x) = \operatorname{cosec} x$

$Df(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

$= \lim_{x\to a}\frac{\operatorname{cosec} x - \operatorname{cosec} a}{x-a}$

$= \lim_{x\to a}\frac{\frac{1}{\sin x}-\frac{1}{\sin a}}{x-a}$

$= \lim_{x\to a}\frac{\sin a - \sin x}{(x-a)\sin a\sin x}$

put $x-a=h$

$x=a+h$

as $x\to a$, $h\to 0$

$Df(h) = \lim_{h\to 0}\frac{\sin a - \sin(a+h)}{(a+h-a)\sin a\sin(a+h)}$

$\left[\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a-h}{2}\right)}{h\times \sin a \sin(a+h)}$$

$$= \lim_{h\to 0} \frac{-\sin\frac{h}{2}}{\frac{h}{2}} \times \frac{1}{2} \times \frac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a \sin(a+h)}$$

$$= -\frac{1}{2} \times \frac{2\cos\left(\frac{2a+0}{2}\right)}{\sin(a+0)}$$

$$= -\frac{\cos a}{\sin^2 a} = -\cot a \operatorname{cosec} a$$

iii) $\sec x$

$f(x) = \sec x$

$$Df(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\sec x - \sec a}{x-a}$$

$$= \lim_{x\to a} \frac{\frac{1}{\cos x} - \frac{1}{\cos a}}{x-a}$$

$$= \lim_{x\to a} \frac{\cos a - \cos x}{(x-a)\cos a \cos x}$$

Put $x - a = h$

$x = a+h$

as $x \to a$, $h \to 0$

$$Df(h) = \lim_{h\to 0} \frac{\cos a - \cos(a+h)}{h\times \cos a \cos(a+h)}$$

$$\left[-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a-h}{2}\right)}{h\times \cos a \cos(a+h)}$$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{2a+h}{2}\right)\sin -\frac{h}{2}}{\cos a \cos(a+h) \times -\frac{h}{2}} \times -\frac{1}{2}$$

$$= \frac{-\frac{1}{2} \times -2\sin\left(\frac{2a+0}{2}\right)}{\cos a \cos(a+0)}$$

$$= -\frac{1}{2} \times -2\frac{\sin a}{\cos a \times \cos a}$$

$$= \tan a \sec a$$

Q.2 If $f(x) = 4x+1$, $x \leq 2$

$= x^2+5$, $x > 0$ at $x = 2$, then find function is differentiable or not.

Solution :

LHD:

$$Df(2^-) = \lim_{x\to 2^-} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x\to 2^-} \frac{4x+1 - (4\times 2+1)}{x-2}$$

$$= \lim_{x\to 2^-} \frac{4x+1-9}{x-2}$$

$$= \lim_{x\to 2^-} \frac{4x-8}{x-2}$$

$$= \lim_{x\to 2^-} \frac{4(x-2)}{(x-2)} = 4$$

$Df(2^-) = 4$

RHD :

$Df(2^+) = \lim_{x\to 2^+} \frac{x^2+5-9}{x-2}$

$= \lim_{x\to 2^+} \frac{x^2-4}{x-2}$

$= \lim_{x\to 2^+} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$

$= 2+2 = 4$

$Df(2^+) = 4$

$\therefore$ RHD = LHD

f is differentiable at $x=2$

Q3 If $f(x) = 4x+7$, $x<3$
$= x^2+3x+1$, $x \geq 3$ at $x=3$, then

find f is differentiable or not ?

Solution :

RHD :

$Df(3^+) = \lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3}$

$= \lim_{x\to 3^+} \frac{x^2+3x+1-(3^2+3\cdot3+1)}{x-3}$

$= \lim_{x\to 3^+} \frac{x^2+3x+1-19}{x-3}$

$= \lim_{x\to 3^+} \frac{x^2+3x-18}{x-13}$

$= \lim_{x\to 3^+} \frac{x^2+6x-3x-18}{x-3}$

$= \lim_{x\to 3^+} \frac{x(x+6)-3(x+6)}{x-3}$

$= \lim_{x\to 3^+} \frac{(x+6)\cancel{(x-3)}}{\cancel{(x-3)}} = 3+6 = 9$

$Df(3^+) = 9$

LHD = $Df(3^-)$

$= \lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3}$

$= \lim_{x\to 3^-} \frac{4x-12}{x-3}$

$= \lim_{x\to 3^-} \frac{4\cancel{(x-3)}}{\cancel{(x-3)}}$

$Df(3^+) = 4$

RHD = LHD

f is not differentiable at $x=3$

Q4> If $f(x) = 8x-5$, $x \leq 2$
$= 3x^2-4x+7$, $x>2$ at $x=2$, then

find ~~f is~~ differentiable or not

Sol^n :

$f(2) = 8\times2-5 = 16-5 = 11$

RHD :

$$Df(2^+) = \lim_{x\to 2^+} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x\to 2^+} \frac{3x^2 - 4x + 7 - 11}{x - 2}$$

$$= \lim_{x\to 2^+} \frac{3x^2 - 4x - 4}{x-2}$$

$$= \lim_{x\to 2^+} \frac{3x^2 - 6x + 2x - 4}{x-2}$$

$$= \lim_{x\to 2^+} \frac{3x(x-2) + 2(x-2)}{x-2}$$

$$= \lim_{x\to 2^+} \frac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= \cancel{\lim_{x\to 2^+}} \; 3\times 2 + 2 = 8$$

$$Df(2^+) = 8$$

LHD :

$$Df(2^-) = \lim_{x\to 2^-} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x\to 2} \frac{x - 5 - 11}{x-2}$$

$$= \lim_{x\to 2^-} \frac{8x - 16}{x-2}$$

$$= \lim_{x\to 2^-} \frac{8(x-2)}{(x-2)}$$

$$= 8$$

$$Df(2^-) = 8$$

LHD = RHD

f is differentiable at $x = 3$

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Practical-03

TITLE - Application of derivatives

Q1] Find the intervals in which function is increasing or decreasing.

a $f(x) = x^3 - 5x - 11$

Sol^n : f is increasing iff $f'(x) > 0$

$\therefore f(x) = x^3 - 5x - 11$

$\therefore f'(x) = 3x^2 - 5$

$\therefore 3x^2 - 5 > 0$

$\therefore x = \pm\sqrt{\frac{5}{3}}$

$$-\infty \quad -\sqrt{\tfrac{5}{3}} \quad \sqrt{\tfrac{5}{3}} \quad \infty$$

$\therefore x \in \left(-\infty, -\sqrt{\frac{5}{3}}\right) \cup \left(\sqrt{\frac{5}{3}}, \infty\right)$

Now f is decreasing if & only if

$f'(x) < 0$

$\therefore 3x^2 - 5 < 0$

$\therefore x = \pm\sqrt{\frac{5}{3}}$

$$-\infty \quad -\sqrt{\tfrac{5}{3}} \quad \sqrt{\tfrac{5}{3}} \quad \infty$$

$\therefore x \in \left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$

b. $f(x) = x^2 - 4x$

sol^n : f is increasing iff $f'(x) > 0$

$\because$ $f(x) = x^2 - 4x$

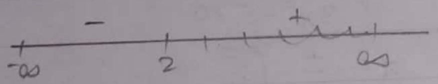
$\therefore$ $f'(x) = 2x - 4$

$\therefore$ $2x - 4 > 0$

$\therefore$ $2(x-2) > 0$

$\therefore$ $x - 2 > 0$

$\therefore$ $x = 2$



$\therefore$ $x \in (2, \infty)$

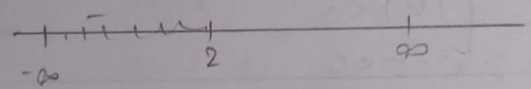
Now f is decreasing iff $f'(x) < 0$

$\therefore$ $2x - 4 < 0$

$\therefore$ $2(x-2) < 0$

$x - 2 < 0$

$\therefore$ $x = 2$



$\therefore$ $x \in (-\infty, 2)$

c. $f(x) = 2x^3 + x^2 - 20x + 4$

sol^n : f is ~~d~~ increasing iff $f'(x) > 0$

$\therefore$ $f(x) = 2x^3 + x^2 - 20x + 4$

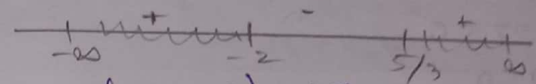
$\therefore$ $f'(x) = 6x^2 + 2x - 20$

$\therefore$ $6x^2 + 2x - 20 > 0$

$\therefore$ $6x^2 + 12x - 10x - 20 > 0$

$\therefore$ $6x(x+2) - 10(x+2) > 0$

$\therefore$ $x = -2, \frac{5}{3}$



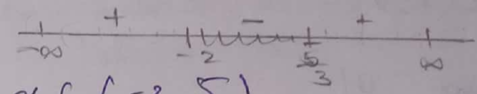
$\therefore$ $x \in (-\infty, -2) \cup (\frac{5}{3}, \infty)$

now f is decreasing if & only if $f'(x) < 0$

$\therefore$ $6x^2 + 2x - 20 < 0$

$\therefore$ $(x+2)(6x-10) < 0$

$\therefore$ $x = -2, 5/3$



$\therefore$ $x \in \left(-2, \frac{5}{3}\right)$

d. $f(x) = x^3 - 27x + 5$

sol^n : f is increasing if & only if $f'(x) > 0$

$\therefore$ $f(x) = x^3 - 27x + 5$

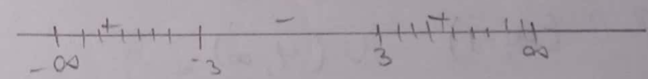
$\therefore$ $f'(x) = 3x^2 - 27$

$\therefore$ $3x^2 - 27 > 0$

$\therefore$ $3(x^2 - 9) > 0$

$\therefore$ $x^2 - 9 > 0$

$x = 3, -3$



$\therefore$ $x \in (-\infty, -3) \cup (3, \infty)$

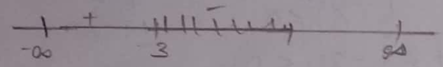
Now f is decreasing iff

$$f'(x) < 0$$

$$\therefore 3x^2 - 27 < 0$$

$$\therefore x^2 - 9 < 0$$

$$\therefore x = 3, -3$$



$$\therefore x \in (-3, 3)$$

c. $f(x) = 69 - 24x - 9x^2 + 2x^3$

sol^n : f is increasing iff

$$f'(x) > 0$$

$$\therefore f(x) = 69 - 24x - 9x^2 + 2x^3$$

$$\therefore f'(x) = -24 - 18x + 6x^2$$

$$\therefore -24 - 18x + 6x^2 > 0$$

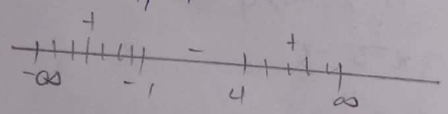
$$\therefore -6(-4 - 3x + x^2) > 0$$

$$x^2 - 3x - 4 > 0$$

$$x^2 - 4x + x - 4 > 0$$

$$\cancel{x}(x-4)\cancel{+}(x+1) > 0$$

$$\therefore x = 4, -1$$



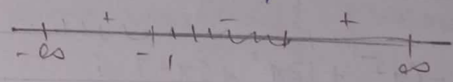
$$\therefore x \in (-\infty, -1) \cup (4, \infty)$$

Now f is decreasing if & only if $f'(x) < 0$

$$\therefore -24 - 18x + 6x^2 < 0$$

$$\therefore (x-4)(x+1) < 0$$

$$\therefore x = 4, -1$$



$$\therefore x \in (-1, 4)$$

Q.2 Find the intervals in which function is concave upwards & concave downwards.

a. $y = 3x^2 - 2x^3$

sol : $y = f(x)$

$$\therefore f(x) = 3x^2 - 2x^3$$

$$\therefore f'(x) = 6x - 6x^2$$

$$\therefore f''(x) = 6 - 12x$$

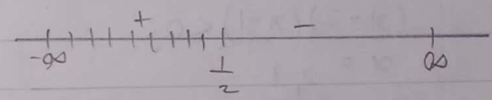
$\therefore$ f is concave upward iff $f''(x) > 0$

$$\therefore 6 - 12x > 0$$

$$\therefore 6(1 - 2x) > 0$$

$$\therefore 1 - 2x > 0$$

$$\therefore -(2x - 1) > 0$$

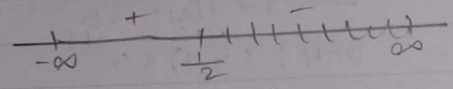


$$\therefore x \in \left(-\infty, \frac{1}{2}\right)$$

$\because$ f is concave downward iff $f''(x) < 0$

$\therefore 6(1-2x) < 0$

$\therefore -(2x-1) < 0$



$\therefore x \in \left(\frac{1}{2}, \infty\right)$

b. $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

sol^n: $\therefore y = f(x)$

$\therefore f(x) = x^4 - 6x^3 + 12x^2 + 5x + 7$

$\therefore f'(x) = 4x^3 - 18x^2 + 24x + 5$

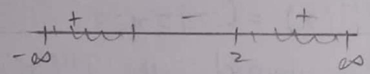
$\therefore f''(x) = 12x^2 - 36x + 24$

$\therefore$ f is concave upward iff $f''(x) > 0$

$\therefore 12(x^2 - 3x + 2) > 0$

$\therefore x^2 - 3x + 2 > 0$

$\therefore (x-2)(x-1) > 0$



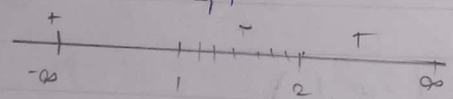
$x \in (-\infty, 1) \cup (2, \infty)$

$\therefore$ f is concave downward iff $f''(x) < 0$

$\therefore x^2 - 3x + 2 < 0$

$\therefore (x-2)(x-1) < 0$

$\therefore x = 2, 1$



$\therefore x \in (1, 2)$

c) $y = x^3 - 27x + 5$

sol^n: $\therefore y = f(x)$

$\therefore f(x) = x^3 - 27x + 5$

$\therefore f'(x) = 3x^2 - 27$

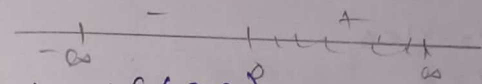
$\therefore f''(x) = 6x$

$\therefore$ f is concave upward iff $f''(x) > 0$

$\therefore 6x > 0$

$\therefore x > 0$

$\therefore x = 0$



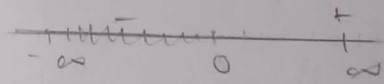
$\therefore x \in (0, \infty)$

$\therefore$ f is concave downward iff $f''(x) < 0$

$\therefore 6x < 0$

$\therefore x < 0$

$\therefore x = 0$



$\therefore x \in (-\infty, 0)$

d] $y = 69 - 24x - 9x^2 + 2x^3$

sol^n: $\therefore y = f(x)$

$\therefore f(x) = 69 - 24x - 9x^2 + 2x^3$

$\therefore f'(x) = -24 - 18x + 6x^2$

$\therefore f''(x) = -18 + 12x$

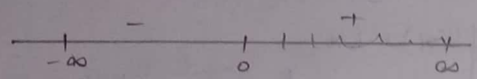
$\therefore$ f is concave upward iff $f''(x) > 0$

$\therefore -18 + 12x > 0$

$\therefore 6(2x-3) > 0$

$\therefore 2x - 3 > 0$

$\therefore x = 3/2$



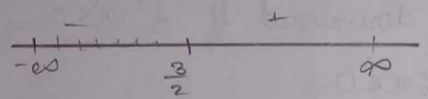
$\therefore x \in \left(+\frac{3}{2}, \infty\right)$

$\therefore f$ is concave upward iff $f''(x) < 0$

$\therefore -18 + 12x < 0$

$\therefore 6(2x-3) < 0$

$\therefore x = \frac{3}{2}$



$\therefore x \in \left(-\infty, \frac{3}{2}\right)$

e] $y = 2x^3 + x^2 - 20x + 4$

sol^n :

$y = f(x)$

$\therefore f(x) = 2x^3 + x^2 - 20x + 4$

$\therefore f'(x) = 6x^2 + 2x - 20$

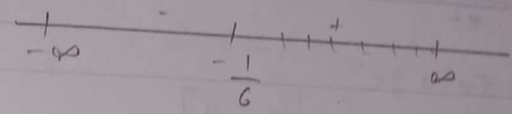
$\therefore f''(x) = 12x + 2$

$\therefore f$ is concave upwards iff $f''(x) > 0$

$\therefore 12x + 2 > 0$

$\therefore 2(6x+1) > 0$

$x = -\frac{1}{6}$



$\therefore x \in \left(-\frac{1}{6}, \infty\right)$

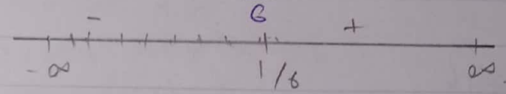
$\therefore f$ is concave downward iff $f''(x) < 0$

$\therefore 12x + 2 < 0$

$\therefore 2(6x+1) < 0$

$\therefore 6x + 1 < 0$

$\therefore x = -\frac{1}{6}$



$\therefore x \in \left(-\infty, -\frac{1}{6}\right)$

A | 18/12/19

Application of Derivatives & Newton's method

1] Find maximum & minimum value of following

i) $f(x) = x^2 + \frac{16}{x^2}$

ii) $f(x) = 3 - 5x^3 + 3x^5$

iii) $f(x) = x^3 - 3x^2 + 1 \quad [-1/2, 4]$

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1 \quad [-2, 3]$

2] Find the root of the following equation by newton's (Take 4 iteration only correct upto 4 decimal)

i) $f(x) = x^3 - 3x^2 - 55x + 95$ (take $x_0 = 0$)

ii) $f(x) = x^3 - 4x - 9$ in $[2, 3]$

iii) $f(x) = x^3 - 18x^2 - 10x + 17$ in $[1, 2]$

Solution

Q.1]

i) $f(x) = x^2 + \frac{16}{x^2}$

$f'(x) = 2x - 32/x^3$

Now consider $f'(x) = 0$

$2x - 32/x^3 = 0$

$2x = 32/x^3$

$x^4 = 32/2$

$x^4 = 16$

$x = \pm 2$

$f''(x) = 2 + 96/x^4$

$f''(x) = 2 + 96/x^4$

$= 2 + 96/16$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f has maximum value at $x = 2$

$f(2) = 2^2 + 16/2^2$

$= 4 + 16/4$

$= 4 + 4$

$= 8$

$\therefore f''(-x) = 2 + 96/(-2)^4$

$= 2 + 96/16$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f has maximum value at $x = 2$

Function reaches min value at $x = 2$, & $x = -2$

ii) $f(x) = 3 - 5x^3 + 3x^5$

$f'(x) = -15x^2 + 15x^4$

Consider, $f'(x) = 0$

$\therefore -15x^2 + 15x^4 = 0$

$\therefore 15x^4 = 15x^2$

$\therefore x^2 = 1$

$\therefore x = \pm 1$

$\therefore f''(x) = -30x + 60x^3$

$f(1) = -30 + 60$

$= 30 > 0$

$f(1) = 3 - 5(1)^3 + 3(1)^5$

$= 6 - 5 = 1$

$\therefore$ f has maximum

$f''(-1) = 3 - 5(-1)^2 + 3(-1)$

$= 3+5-3 = 5$

$\therefore$ f has maximum value 5 at $x=-1$ & has maximum value 1 at $x=1$

iii) $f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x$

Consider, $f'(x) = 0$

$\therefore 3x^2 - 6x = 0$

$\therefore 3x(x-2) = 0$

$\therefore 3x = 0$ or $x-2=0$

$\therefore x=0$ or $x=2$

$\therefore f''(x) = 6x - 6$

$\therefore f''(0) = 6(0) - 6$

$= -6 < 0$

$\therefore$ f has maximum value at $x=0$

$\therefore f(0) = (0)^3 - 3(0)^2 + 1 = 1$

$f''(2) = 6(2) - 6$

$= 12 - 6$

$= 6 > 0$

$\therefore$ f has minimum value at $x=2$

$\therefore f(x) = (2)^3 - 3(2)^2 + 1$

$= 8 - 3(4) + 1$

$= 9 - 12$

$= -3$

$\therefore$ f has maximum value 1 at $x=0$ &

f has minimum value -3 at $x=2$

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$\therefore f'(x) = 6x^2 - 6x - 12$

Consider, $f'(x) = 0$

$\therefore 6x^2 - 6x - 12 = 0$

$\therefore 6(x^2 - x - 2) = 0$

$x^2 - x - 2 = 0$

$x^2 + x - 2x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$

$\therefore f''(x) = 12x - 6$

$\therefore f''(2) = 12(2) - 6$

$= 24 - 6$

$= 18 > 0$

$\therefore$ f has minimum value at $x = +2$

$\therefore f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 2(8) - 3(4) - 24 + 1$

$= 16 - 12 - 24 + 1$

$= -19$

$\therefore f''(-1) = 12(-1) - 6$

$= -12 - 6$

$= -18 < 0$

$\therefore$ f has maximum value at $x = -1$

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(1) + 1$

$= -2 - 3 + 12 + 1$

$= 8$

$\therefore$ f has maximum value 8 at $x = -1$ &

f has minimum value ~~at~~ -19 at $x=2$

Scanned with CamScanner

Q.2 i) $f(x) = x^3 - 3x^2 - 55x + 95$, $x_0 = 0 \Rightarrow$ given

$f'(x) = 3x^2 - 6x - 55$

By Newton's method,

$x_{n+1} = x_n - f(x_n) / f'(x_n)$

$x_1 = x_0 - f(x_0) / f'(x_0)$

$x_1 = 0 + 9.5/55$

$x_1 = 0.1727$

$f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1725) + 9.5$

$= 0.0051 - 0.0895 - 9.4985 + 9.5$

$= .0.0829$

$f'(x_1) = 3(0.1727)^2 - 6(0.1727) - 55$

$= 0.0895 - 1.0362 - 55$

$= -55.9467$

$\therefore x_2 = x_1 - f(x_1) / f'(x_1)$

$= 0.1727 - 0.0829 / 55.9467$

$= 0.1712$

$\therefore f(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712)^2$

$= 0.0050 - 0.0879 - 9.416 + 9.5$

$= 0.0011$

$f'(x_2) = 3(0.1412)^2 - 6(0.1712) - 55$

$= 0.0879 - 10272 - 55$

$= -55.9393$

$x_3 = x_2 - f(x_2) / f'(x_2)$

$= 0.1712 + 0.0011 / 55.9393$

$= 0.1712$

$\therefore$ The ~~root~~ of the eq^n is 0.1712

ii) $f(x)$

$f'(x) = 3x^2 - 4$

$f(2) = 23 - 4(3) - 9$

$= 8 - 8 - 9$

$= -9$

$f(13) = 33 - 4(3) - 9$

$= 27 - 12 - 9$

$= 6$

Let $x_0 = 3$ be the initial approximation by Newton's method

$x_{n+1} = x_n - f(x_n) / f'(x_n)$

$x_1 = x_0\ f(x_0) / f'(x_0)$

$= 3 - 6/23$

$= 2.7392$

$f(x_1) = (2.7392)^3 - 4(-2.7392) - 9$

$= 0.5528 - 10.9568 - 9$

$= 0.596$

$f'(x_1) = 3(2.7392)^2 - 4$

$= 22.5096 - 4$

$= 18.5096$

$x_2 = x_1 - f(x_1) / f'(x_1)$

$= 2.7392 - 6.596 / 18.5096$

$= 2.7071$

$f(x_2) = (2.7071)^3 - 4(2.7071)$

$= 19.8386 - 10.8284$

$= 0.0162$

$f'(x_2) = 3(2.7071)^2 - 4$

$= 21.9851 - 4$

$= 17.9851$

$2.7071 - \frac{0.0102}{17.9851} = 2.7071 - 0.0056 = 2.7015$

$f(x_3) = (2.7015)^3 - 4(2.7015) - 9$

$= (19.7)58 - 10.806 - 9 = -0.0901$

$f(3) = 3(2.7015)^2 - 4 = 2.18943 - 4 = 17.8943$

$x_4 = 2.7015 + 0.0901 / 17.8943 = 2.7015 + 0.0030$

$= 2.7065$

3) $f(x) = x^3 - 18x^2 - 10x + 17 \quad [1, 2]$

$f'(x) = 3x^2 - 3.6x - 10$

$f(1) = (1)^3 - 1.8(2)^2 - 10(1) + 17$

$= 1.8 - 10 + 17$

$= 6.2$

$f(2) = (2)^3 - 1.8(2)^2 - 10(2) + 17$

$= 8 - 7.2 - 20 + 17 = 2.2$

Let $x_0 = 2$ be initial approximation by Newton's Method.

$x_{n+1} = x_n - f(x_n) / f'(x_n)$

$x_1 = x_0 - f(x_2) / f'(x_2)$

$= 2 - 2.2/5.2$

$= 2 - 0.4230 = 1.897$

$f(x_1) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$

$= 3.9219 - 4.4764 - 15.77 + 17$

$= 0.6755$

$f'(x) = 3(1.577)^2 - 3.6(1.577) - 10$

$= 2.4608 - 5.6772 - 10$

$= -8.2164$

$x_2 = x_1 - f(x_1) / f'(x_1)$

$= 1.577 + 0.6755 / 8.2164$

$= 1.577 + 0.0822$

$= 1.6592$

$f(x_2) = (1.6592)^3 - 18(1.6592)^2 - 10(1.6592) + 17$

$= 4.5677 - 4.9553 - 16.592 + 17$

$= 0.0204$

$f'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10$

$= 8.2588 - 5.97312 - 10$

$= -7.7143$

$x_3 = x_2 - f(x_2) / f'(x_2)$

$= 1.6592 / 0.0204 / 7.7143$

$= 1.6592 + 0.0026$

$= 1.6618$

$f(x_3) = (1.6618)^2 - 1.8(1.6618)^2 - 10(1.6618) + 17$

$= 4.5892 - 4.9708 - 16.618 + 17$

$= 0.0004$

$f'(x_3) = 3(1.6618)^2 - 3.6(1.6618) - 10$

$= 8.2841 - 5.9824 - 10$

$= -7.6977$

$x_4 = x_3 - f(x_3) / f'(x_3)$

$= 1.8618 + \frac{0.0004}{7.6977}$

$= 1.6618$

PRACTICAL - 5

Integration

1] solve the following integration

i) $\int \frac{1}{x^2+2x-3} \, dx$

$= \int \frac{1}{\sqrt{x^2+2x-5}} dx$

$= \int \frac{1}{\sqrt{x^2+2x+1-4}}$

$= a^2 + 2ab + b^2 = (a+b)^2$

$= \int \frac{1}{\sqrt{(x+1)^2-4}} dx$

substitute put $x+1 = t$

$\underline{dx} = \frac{1}{t} \cdot dt$ where $t=1$, $t = x+1$

$= \int \frac{1}{\sqrt{t^2-4}} dt$

using $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln(|x + \sqrt{x^2-a^2}|)$

$= \ln(|t + \sqrt{t^2-4}|)$

$= \ln(|x+1 + \sqrt{(x+1)^2-4}|)$

$= \ln(|x+1 + \sqrt{x^2+2x-3}|)$

$= \ln|x+1 + \sqrt{x^2+2x-3}| + C$

2) $\int(4e^{3x}+1)\,dx$

$= \int 4e^{3x} dx + \int 1 \, dx$

$= 4\int e^{3x} dx + \int 1 . dx$ # $\int e^{dx} dx = \frac{1}{d} e^{dx}$

$= \frac{4e^{3x}}{3} + x + C$

3) $\int 2x^2 - 3\sin(x) + 5\sqrt{x} \, dx$

$= \int 2x^2 - 3\sin(x) + 5x^{1/2} dx$ # $\sqrt[n]{a^m} = a^{m/n}$

$= \int 2x^2 dx - \int 3\sin(x) dx + \int 5x^{1/2} dx$

$= \frac{2x^3}{3} + 3\cos x + \frac{\cos\sqrt{x}}{3} + c$

$= \frac{2x^3 + \cos\sqrt{x}}{3} + 3\cos x + C$

4) $\int \frac{x^3+3x+4}{\sqrt{x}} dx$

$= \int \frac{x^3+3x+4}{x^{1/2}} dx$

$= \int \frac{x^3}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}} dx$

$= \int x^{5/2} + 3x^{1/2} + 4/x^{1/2} dx$

$= \int x^{5/2} dx + \int 3x^{1/2} dx + \int 4/x^{1/2} dx$

$= \cancel{\frac{x^{5/2+1}}{5/2+1}}$

$= \frac{2x^3 \sqrt{x}}{7} + 2x\sqrt{x} + 8\sqrt{x} + C$

v) $\int t^7 \cdot \sin(2t^4)\, dt$

put $u = 2t^4$

$du = 2 \times 4t^3$

$= \int t^7 \cdot \sin(2t^4) \cdot \frac{1}{2 \times 4t^3}\, du$

$= \int t^4 \sin(2t^4) \cdot \frac{1}{2 \times 4}\, du$

$= \int t^4 \sin(2t^4)\, \frac{1}{8}\, du = \frac{t^4 \sin(2t^4)\, du}{8}$

substitute t^4 with $u/2$

$= \int \frac{u/2 \times \sin(u)\, du}{8}$

$= \int \frac{u \times \sin(u)}{2} / 8\, du$

$= \int \frac{u \times \sin(u)}{16}\, du$

$= 1/16 \int u \times \sin(u)\, du$

$\int u\,dv = uv - \int v\, du$

where $u = u$

$dv = \sin(u) \times du$

$\cancel{du} = 1\, du \quad v = -\cos(u)$

$= \frac{1}{16} (u \times (-\cos(u)) \cdot \int -\cos(u)\, du$

$= \frac{1}{16} \times (u \times (-\cos u)) + \int \cos(u)\, du$

$\int \cos x\, dx = \sin(x)$

$= \frac{1}{16} \times (u \times (-\cos(u)) + \sin(u))$

Return the substitution $u = 2t^4$

$= \frac{1}{16} \times (2t^4 \times (-\cos(2t^4)) + \sin(2t^4))$

$= \frac{-t^4 \times \cos(2t^4)}{8} + \frac{\sin(2t^4)}{16} + C$

vi) $\int \sqrt{x}\, (x^2 - 1)\, dx$

$= \int \sqrt{x}\, dx^2 - \sqrt{x}\, dx$

$= \int x^{1/2} \times x^2 - x^{1/2}\, dx$

$= \int x^{5/2} - x^{1/2}\, dx$

$= \int x^{5/2}\, dx - \int x^{1/5}\, dx$

$= x_1 = \frac{x^{5/2+1}}{5/2+1} = \frac{x^{7/2}}{7/2} = \frac{2x^{7/2}}{7} = \frac{2\sqrt{x}}{7} = \frac{2x^3\sqrt{x}}{7}$

$= x_2 = \frac{x^{1/2+1}}{1/2+1} = \frac{x^{3/2}}{3/2} = \frac{2x^{3/2}}{3/2} = \frac{2\sqrt{x^3}}{3}$

$= \frac{2x^3\sqrt{x}}{7} + \frac{2\sqrt{x^3}}{3} + C$

vii) $\int \frac{\cos x}{3\sqrt{\sin(x)^2}} dx$

$= \int \frac{\cos x}{\sin x^{2}/3} dx$

put $t = \sin(x)$

$t = \cos x$

$= \int \frac{\cos(x)}{\sin(x)^{3/2}} \times \frac{1}{\cos(x)} dt$

$= \frac{1}{\sin x^{3/2}} dt$

$= \frac{1}{t^{2/3}} dt$

$I = \int \frac{1}{t^{2/3}} dt = \frac{-1}{(2/3^{-1}) t^{2/3-1}}$

$= \frac{-1}{-1/3 \, t^{2/3-1}}$

$= \frac{1}{1/3 \, t^{-1/3}} = \frac{t^{1/3}}{1/3} = 3t^{1/3}$

$= 3 \sqrt[3]{t}$

Return substitution $t = \sin(x)$

$= 3 \sqrt[3]{\sin(x)} + C$

viii) $\int \frac{1}{x^2} \sin\left(\frac{1}{x^2}\right) dx$

$\frac{1}{x^2} = 7$

$\therefore -\frac{2}{x^2} = \frac{dt}{dx}$

$I = \frac{-1}{2} \int \sin t \, dt$

$= \frac{-1}{2} \times -\cot + C$

$= \frac{\cos t}{2} + C$

$I = \frac{\cos(1/x^2)}{2} + C$

ix) $I = \int e^{\cos^2 x} \sin 2x \, dx$

put $\cos^2 x = 1$

$-2\cos x \sin x = \frac{dt}{dx}$

$\sin^2 x \, dx = -dt$

$\because I = -\int e^t dt$

$= et + C$

$I = -e^{\cos^2 x} + C$

x) $I = \int \left(\frac{x^2 - 2x}{x^2 - 3x^2 + 1} \right) dx$

$x^3 - 3x^2 + 1 = 7$

$3x^2 - 6x = \frac{dt}{dx}$

$\because (x^2 - 2x) dx = \frac{dt}{3}$

$I = \frac{1}{3} \int \frac{1}{7} dt$

$= \frac{1}{3} \log |t| + c$

$I = \frac{1}{3} \log |x^2 - 3x^2 + 1| + c$

PRACTICAL-06

Q.1 $y = \sqrt{4 - x^2}$

$\because \frac{dy}{dx} = \frac{1}{2\sqrt{4 \cdot x^2}} \times (-2x)$

$= \frac{-x}{\sqrt{4 - x^2}}$

$t = \int_{0.2}^{2} \sqrt{1 + \frac{x^2}{4 - x^2}} \, dx$

$= \int_{-2}^{2} \sqrt{\frac{\cancel{1+x^2}\,4}{4 - x^2}} \, dx$

$= 2 \int \frac{1}{\sqrt{2^2 - x^2}} \, dx$

$= 2 \left[\frac{\cancel{\pi}}{2} - \left(-\frac{\pi}{2} \right) \right]$ $\quad L = 2\pi$

$= 2 \left[\sin^{-1} \left(\frac{x}{2} \right) \right]^2$

$= 2 \left[\sin^{-1}(1) - \sin^{-1}(-1) \right]$

1.3: $y = x^{3/2}$ at $[0, 4]$

$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$

$$x = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^4 \sqrt{1 + 9/4x}\, dx$$

$$= \frac{1}{2} \int_0^4 \frac{(4+9x)^{3/2}}{3/2} \times \frac{1}{9}$$

$$= \frac{1}{27} \left[(4+9x)^{3/2} \times \frac{1}{9} \right]$$

$$= \frac{-1}{21} \left[(40)^{3/2} - (4 \times 36)^{3/2} \right]$$

$$= \frac{1}{27} \left(40^{3/8} - 8 \right) \text{ units}$$

6. $x = 3x \sin t \quad y = 3\cos t$

$\frac{dx}{dt} \; 3 \cos t \quad \frac{dy}{dx} = 3 \sin t$

$$L = \int_0^{2x} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$$= \int_0^2 \sqrt{(3\cos t)^2 + 3(\sin t)^2}\, dt$$

$$= \int_0^2 \sqrt{9\sin^2 t + 9\cos^2 + dt}$$

$$= \int_0^2 3 \sqrt{1}\, dt$$

$$= 3 \int_0^{2x} dt$$

$$= 3(x)^{2x}$$

$$= 3(2x - 0)$$

$L = 6x$ units

Find the length of the following curve

$x = t \sin t, \quad y = 1 - \cos t \quad t(0, 2\pi)$

arc length $= \int_0^4 \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^3 dt$

$$= \int_0^{2\pi} \sqrt{(1-\cos t)^2 - (\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + 1}\, dt$$

$$= \int_0^{2\pi} \sqrt{2}\, \sqrt{\sin^{\pi/2}}\, dt$$

$$= \int_0^{2\pi} 2 \sin \frac{1}{2}\, dt$$

$$= \left[-4 \cos\left(\frac{1}{2}\right) \right]_0^{2\pi}$$

$$= [-4 \cos \pi] - (-4 \cos 0)$$

$= 4 + 4 = 8$

$x = \frac{1}{6} y^3 + \frac{1}{2y}$ on $y \in (1, 2)$

$\therefore \frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2}$

$\frac{dx}{dy} = \frac{y^4 - 1}{2y^2}$

$L = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$= \int_1^2 \sqrt{1 + \frac{(y^4-1)}{4y^2}}\, dy$

$= \int_1^2 \sqrt{\frac{(y^4-1) + 4 \times y^3 \times 1}{4y}}\, dy$

$= \int_1^2 \frac{y^5 + 1}{(2y^2)^2}\, dy$

$= \int_1^2 \frac{y^4+1}{2y^2}\, dy$

$= \frac{1}{2} \int_1^2 y^2\, dy + \frac{1}{2} \int_1^2 y^{2} \cdot dy$

$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$

$= \frac{1}{2}\left[\frac{7}{3} - \frac{1}{2}\right]$

$= \frac{1}{2}\left[\frac{17}{6}\right]$

$L = \frac{17}{12}$ units

Q.2

i) $\int_0^2 e^{x^2}\, dx$ with $n = 4$

$h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

x	0	0.5	1	1.5	2
y	1	1.284	2.7183	5.4	0.5982

$\int_0^2 e^{x^2}\, dx = \frac{2}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + y_2\right]$

$= \frac{0.5}{3}\left[(1 + 54.5982) + 5(1.285 + 0.487) + 2x\right]$

$= \frac{0.5}{3}\left[55.5982 + 43.0868 + 5.416\right]$

$= \int_0^2 e^{x^2}\, dx = 17.3535$

ii) $\int_0^4 x^2\, dx$ $\quad n = 4$

$\cancel{L = \frac{4-0}{4}} = 1$

x	0	1	2	3	4
y	0	1	4	9	16

$$\int_0^1 x^2\,dx = \frac{1}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\right)$$

$$= \frac{1}{3}\left[0 + 16 + 4(1+9) + 2\times 4\right]$$

$$= \frac{1}{3}\left[16 + 4(10) + 8\right]$$

$$= \frac{64}{3}$$

$$\int_0^4 x^2\,dx = 21.3333$$

$\int_0^{\pi/3} \sqrt{\sin x}\,dx$ with $n = 6$

$$L = \frac{\pi(3-0)}{6} = \frac{\pi}{18}$$

x	0	$\pi/18$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{12}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$
y	0	0.4167	0.5948	0.7071	0.8017	0.8752	0.4306

$\pi/3$

$$\int_0^{\pi/3} \sqrt{\sin x}\,dx = \frac{1}{3}\left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_3)\right]$$

$$= \frac{\frac{\pi}{18}}{3}\left[0.4167 + 0.8306 + 2\right]$$

$$(0.4167 + 0.7071 + 0.8752) + 2(0.5848 + 0.847)\Big]$$

$$= \frac{\pi}{54}\left[1.3473 + 4(1.999) + 2(1.3865)\right]$$

$$= \frac{\pi}{54}\left[1.3473 + 7.476 + 2.773\right]$$

$$= \frac{\pi}{54} \times 12.1163$$

$$\int_0^{\pi/3} \sqrt{\sin x}\,dx = 0.7049$$

AK
08/01/2020

PRACTICAL-7

DIFFERENTIAL EQUATIONS

Q1

1) $x\frac{dy}{dx} + y = e^x$

$\frac{dy}{dx} + 1y = \frac{e^x}{x}$

$P(x) = \frac{1}{x}$ $\quad Q = (x) = \frac{e^x}{x}$

$IF = e^{\int 1/x}\, dx$

$= e^{\int 1/x} dx$

$= e^{\ln} dx$

$IF = x$

$y(IF) = \int Q(x)\ (IF)\ dx + c$

$= \int \frac{e^x}{x} \cdot x\, dx + c$

$= \int e^x dx + C$

$= xy = e^x + C$

Q.2 $e^x \frac{dy}{dx} + 2e^x \cdot y = 1$

$\frac{dy}{dx} + \frac{2e^x y}{e^x} = \frac{1}{e^x}$ $\quad (\div \text{by } e^x)$

$\frac{dy}{dx} + 2y = \frac{1}{e^x}$

$P(x) = 2$ $\quad Q(x) = e^{-x}$

$\int P(x) = dx$

$IF = e^{\int 2 dx}$

$= e^{2x}$

$y = (I.F) = \int Q(x)\ (I.F)\ dx + C$

$y = e^{2x} \int e^{-x} + 2x\, dx + C$

$= \int e^x\, dx + C$

$y = e^{2x} = e^x + C$

Q3 $x\frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$x\frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$\therefore \frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2}$

$P(x) = 2(x)$ $\quad Q(x = \frac{\cos x}{x^2}$

Scanned with CamScanner

$$I.F = e^{\int P(x)\,dx}$$

$$= e^{\int 2/x\,dx}$$

$$= e^{2/x}$$

$$= \ln x^2$$

$$y\,(I.F) = \int Q(x)\,(I.F)\,dx + C$$

$$\int \frac{\cos x}{x^3} - x^2\,dx + C$$

$$= \int \cos x + C$$

$$x^2 y = \sin x + C$$

iv) $$x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$$

$$= x\frac{dy}{dx} + 3y = \frac{\sin x}{x^3} \quad (\div \text{ by } x \text{ on both sides})$$

$$= P(x) = 3/x \qquad Q(x) = \sin x / x^3$$

$$= e^{\int P(x)\,dx}$$

$$= e^{\int 3/x\,dx}$$

$$= e^{3/x}\,dx$$

$$= e^{\ln x^3}$$

$$IF = x^3$$

Scanned with CamScanner

$$y(IF) = \int Q(x)(I\cdot F)\,dx + C$$

$$= \int \frac{\sin x}{x^3} \cdot x^3\,dx + C$$

$$= \int \sin x\,dx + C$$

$$= x^3 y - \cos x + C$$

v) $$e^{2x}\frac{dy}{dx} + 2e^{2x}y - 2x$$

$$\frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}$$

$$P(x) = 2 \quad Q(x) = 2x/e^2x = 2xe^{-2x}$$

$$IF = e^{\int P(x)\,dx}$$

$$= e^{\int 2\,dx}$$

$$= e^{2x}$$

$$y(IF) = \int Q(x)(IF)\,dx + C$$

$$= \int 2xe^{-2x}e^{2x}\,dx + C$$

$$= \int 2x\,dx + C$$

$$ye^{2x} = x^2 + C$$

vii) $\sec^2 x \cdot \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$\sec^2 x \cdot \tan y \cdot dx = -\sec^2 y \cdot \tan x \, dy$

$\frac{\sec^2 x \, dx}{\tan x} = -\frac{\sec^2 y \, dx}{\tan x}$

$\int \frac{\sec^2 x \, dx}{\tan x} = -\int \frac{\sec^2 y \, dy}{\tan y}$

$\log|\tan x| = -\log|\tan y| + c$

$\log|\tan x \cdot \tan y| = c$

$\tan x \cdot \tan y = e^c$

viii) $\frac{dy}{dx} = \sin^2(x-y+1)$

Put $x-y+1 = v$

Differentiating on b.s

$x = y+1 = v$

$1 - \frac{dy}{dx} = \frac{dy}{dx}$

$\frac{1-dy}{dx} v - \frac{dy}{dx}$

$1 - \frac{dv}{dx} = \sin^2 x$

$\frac{dv}{dx} = 1 - \sin^2 v$

$\frac{dv}{dx} = \cos^2 v$

$\frac{dv}{\cos^2 v} = dx$

$\int \sec^2 v \, dv = \int dx$

$\tan v = x + c$

$\tan(x+y-1) = x + C$

$\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$

Put $2x+3y = v$

$2 + 3\frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\frac{(v-1)}{(v+2)}$

$\frac{dv}{dx} = \frac{v-1}{v+2} + 2$

$$\frac{dv}{dx} = \frac{v-1+2v+4}{v+2}$$

$$= \frac{3v+3}{v+2}$$

$$= \frac{3(v+1)}{v+2}$$

$$\int \frac{(v+2)}{(v+1)}\, dv = 3dx$$

$$= \int \frac{v+1}{v}\, dx + \int \frac{1}{v+1}\, dv = 3x$$

$$= v + \log|x| + 3x + c$$

$$= 2x + 3y + \log|2x+3y+1| = 3x + c$$

$$= 3y = x - \log|2x+3y+1| + c$$

PRACTICAL - 8

TOPIC - Euler's Method

Q1 $\frac{dy}{dx} = y + e^x - 2$ $y(0) = 2$ $h = 0.5$ find $y(2) = ?$

Sol^n - $f(x) = y + e^x - 2$ $x_0 = 0, y(0) = 2, h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.5743
2	1	3.5743	4.2925	5.7205
3	1.5	5.7205	8.2021	9.8215
4	2	9.8215		

$\therefore y(2) = 9.8215$

Q.2 $\frac{dy}{dx} = 1 + y^2$, $y(0) = 1$ $h = 0.2$ find $y(1) = ?$

$x_0 = 0$, $y_0 = 0$, $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.408
2	0.4	0.408	1.1664	0.6412
3	0.6	0.6412	1.4111	0.9234
4	0.8	0.9234	1.8526	1.2939
5	1	1.2939		

$y(1) = 1.2939$

Q.3 $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$ $y(0) = 1$, $h = 0.2$ find $y(1) = ?$

$x_0 = 0$ $y(0) = 1$ $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	1	0	1
1	0.2	1	0.4472	1.0894
2	0.4	1.0894	0.6059	1.2105
3	0.6	1.2105	0.7040	1.3513
4	0.8	1.3513	0.7696	1.5051
5	1	1.5051		

$y(1) = 1.5051$

Q.4 $\frac{dy}{dx} = 3x^2 + 1$ $y(1) = 2$ find $y(2)$ $h = 0.5$

$y_0 = 2$ $x_0 = 1$ $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	4
1	1.5	4	7.75	7.875
2	2	7.875		

Q.4 $y_0 = 2$ $x_0 = 1$ $h = 0.25$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.6875	4.4218
2	1.5	4.4218	89.6569	19.3360
3	1.75	19.3360	1122.6426	299.9960
4	2	299.9960		

$y(2) = 299.9960$

Q5 $\frac{dy}{dx} = \sqrt{xy} + 2$ $y(1) = 1$ $h = 0.2$

$x_0 = 1$ $y_0 = 1$ $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	3.6
1	1.2	3.6		

$y(1.2) = 3.6$

PRACTICAL - 9

TOPIC - Limits & Partial order derivatives

Q.1 Evaluate the following limits

1.) $\lim_{(x,y)\to(-4,-1)} \frac{x^3-3x+y^2-1}{xy+5}$

$= \lim_{(x,y)\to(-4,-1)} \frac{x^3-3x+y^2-1}{xy+5}$

Apply limit

$= \frac{(-4)^3-3(-4)+(-1)^2-1}{(-4)(-1)+5}$

$= \frac{-64+12+1-1}{4+5} = \frac{-52}{9}$

2. $\lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2+y^2-4x)}{x+3y}$

Apply limit

$= \frac{(0+1)((2)^2+(0)^2-4(2))}{2+3(0)}$

$= \frac{1(4+0-8)}{2}$

$= \frac{4-8}{2} = \frac{-4}{2} = -2$

3] $\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-y^2x^2y^2}$

$= \lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2y^2}$

Apply limit

$= \frac{(1)^2-(1)^2(1)^2}{(1)^3-(1)^2(1)(1)} = \frac{1-1}{1-1} = 0$

$\therefore$ limit does not exist

Q.2] Find f_x, f_y for each of the following of

1.) $f(x,y) = xye^{x^2}+y^2$

$f_x = y(1\,e^{x}+y^2) + xy(e^{x^2}+y^2.2x)$

$= ye^{x^2}+y^2 + 2x^2ye^{x^2}+y^2$

$f_y = x(1.e^{x^2}+y^2) + xy(e^{x^2}+y^2.2y)$

$= x.e^{x^2}+y^2 + 2xy^2 e^{x^2}+y^2$

$\therefore f_x = ye^{x^2}+y^2 + 2x^2y.e^{x^2}+y^2$

$\therefore f_y = xe^{x^2}+y^2 + 2xy^2e^{x^2}+y^2$

ii) $f(x,y) = e^x \cos y$

$f_x = \cos y\, e^x$

$f_y = e^x - \sin y$

$\therefore f_y = -\sin y\, e^x$

iii) $f(x,y) = x^3 y^2 - 3x^2 y + y^3 + 1$

$fx = y^2\, 3x^2 - 3y^2 x + 6 + 0$

$= 3x^2 y^2 - 6xy$

$fy = x^3\, 2y - 3x^2 + 3y^2$

$= 2x^3 y - 3x^2 + 3y^2$

Q.3] Using definition find values of fx, fy at $(0,0)$ for $f(x,y) = \dfrac{2x}{1+y^2}$

$fx(a,b) = \lim_{h \to 0} \dfrac{f(a+h,b) - f(a,b)}{h}$

$fy(a,b) = \lim_{h \to 0} \dfrac{f(a,b+h) - f(a,b)}{h}$

According to given $(a,b) = (0,0)$

$fx(0,0) = \lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h}$

$= \lim_{h \to 0} \dfrac{2h - 0}{h} = 2$

$fy(0,0) = \lim_{h \to 0} \dfrac{f(0,h) - f(0,0)}{h}$

$= \lim_{h \to 0} \dfrac{0 - 0}{h} = 0$

$\therefore fx = 2, \; fy = 0$

$fx(0,0) = \lim_{h \to 0} \dfrac{f(0,h) - f(0,0)}{h}$

$= \lim_{h \to 0} \dfrac{0 - 0}{h} = 0$

$\therefore fx = 2, \; fy = 0$

Q.4] Find all second order partial derivatives of the f. Also verify whether $fxy = fyx$

i) $f(x,y) = \dfrac{y^2 - xy}{x^2}$

$\therefore fxx = \dfrac{d^2 f}{dx^2}$

$fyy = \dfrac{d^2 f}{dy^2}$

Applying $\dfrac{u}{v}$ rule.

$fx = \dfrac{x^2(0 - y) - (y^2 - xy)2x}{x^4}$

$= \dfrac{-x^2 y - 2xy^2 + 2x^2 y}{x^4}$

$\therefore fx = \dfrac{x^2 y - 2xy^2}{x^4}$

$$f_{xx} = \frac{x^4(2xy-2y^2) - (x^2y - 2xy^2)(4x^3)}{x^8}$$

$$= \frac{2x^5y - 2x^4y^2 - (4x^5y - x^4y^2)}{x^8}$$

$$= \frac{2x^5y - 2x^4y^2 - 4x^5y + 8x^4y^2}{x^8}$$

$$= \frac{-2x^5y + 6x^4y^2}{x^8}$$

$$= \frac{6x^4y^2 - 2x^5y}{x^8}$$

$$f_{xx} = \frac{6y^2 - 2xy}{x^4}$$

$$f_y = \frac{1}{x^2}(2y-x)$$

$$\therefore f_y = \frac{2x}{x^2}$$

$$f_{yy} = \frac{1}{x^2} \cdot 2 = \frac{2}{x^2}$$

$$\therefore f_{xy} = \frac{2y-x}{x^2}$$

$$= \frac{x^2(-1) - (2y-x)(2x)}{x^4}$$

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$$= \frac{-x^{\cancel{2}} - 4xy + 2x^2}{x^4}$$

$$= \frac{x^2 - 4xy}{x^4}$$

$$= \frac{x - 4y}{x^3}$$

$$\therefore f_{xy} = \frac{x - 4y}{x^3}$$

$$\therefore f_{yx} = \frac{x^2 y - 2xy}{x^4}$$

$$= \frac{x^2 - 4xy}{x^4}$$

$$= \frac{x - 4y}{x^3}$$

$$= f_{xy}$$

$$\therefore f_{yx} = f_{xy}$$

Hence verified.

Q.5] Find the linearization of (x, y) at given point.

i) $f(x, y) = \sqrt{x^2 + y^2}$ at $(1, 1)$

$$f_x = \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2x \qquad f_y = \frac{2y}{2\sqrt{y^2 + x^2}}$$

$$= \frac{x}{\sqrt{x^2 - y^2}} \qquad = \frac{y}{\sqrt{x^2 + y^2}}$$

$$f_x(1,1) = \frac{1}{\sqrt{2}} \qquad f_y(1,1) = \frac{1}{\sqrt{2}}$$

$\therefore L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= \sqrt{2} + \frac{x-1}{\sqrt{2}} + \frac{y-1}{\sqrt{2}}$

$= \sqrt{2} + \frac{x+y-2}{\sqrt{2}}$

$= \frac{x+y}{\sqrt{2}}$

ii) $f(x,y) = 1-x+y\sin x$ at $\frac{\pi}{2}$

$f(\pi/2, 0) = 1-\frac{\pi}{2}+0+\sin\frac{\pi}{2}$

$f(\frac{\pi}{2}, 0) = \frac{2-\pi}{2}$

$f_x = -1+y\cos x \quad \therefore f_y = \sin x$

$f_x(\frac{\pi}{2}, 0) = -1+0 - \sin\frac{\pi}{2}$

$f_y(\frac{\pi}{2}, 0) = \sin\frac{\pi}{2}$

$= 1$

$L(x,y) = f(\frac{\pi}{2}, 0) + f_x(\frac{\pi}{2}, 0)(x-\frac{\pi}{2})$
$+ f_y(\frac{\pi}{2}, 0)(y-0)$

$= \frac{2-\pi}{2} + (-1)(x-\frac{\pi}{2}) + 1(y)$

$= 1-\frac{\pi}{2} - x + \frac{\pi}{2} + y$

$L(x,y) = 1-x+y$

iii) $f(x,y) = \log x + \log y$
$f(1,1) = \log 1 + \log 1$
$= 0$

$f_x = \frac{1}{x} \qquad f_y = \frac{1}{y}$

$f_x(1,1) = 1 \qquad f_y(1,1) = 1$

$\therefore L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$
$= 0 + 1[x-1] + 1[y-1]$

$\therefore L(x,y) = x+y-2$

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PRACTICAL - 10

Q.1 Find the directional derivative of the following function at given points & in the direction of given vector.

i) $f(x,y) = x + 2y - 3$ $a = (1,-1)$ $u = 3i - j$

Here, $u = 3i - j$ is not a unit vector.

$|\bar{u}| = \sqrt{(3)^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$

Unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{10}}(3,-1)$

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a) = f(1,-1) = (1) + 2(-1) - 3$
$= 1 - 2 - 3$
$= -4$

$f(a+hu) = \cancel{f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)}$

$= f\left(\left(1 + \frac{3}{\sqrt{10}}\right), \left(-1, \frac{-h}{\sqrt{10}}\right)\right)$

$f(a+hu) = \left(1 + \frac{3}{\sqrt{10}}\right) + 2\left(-1 - \frac{h}{\sqrt{10}}\right) - 3$

$= 1 + \frac{3}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$

$f(a+hu) = -4 + \frac{h}{\sqrt{10}}$

$D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$

$= \lim_{h \to 0} \frac{-4 + h/\sqrt{10} + 4}{h}$

$D_u f(a) = \frac{1}{\sqrt{10}}$

ii. $f(x) = y^2 - 4x + 1$ $a = (3,4)$ $u = i + 5j$

Here, $u = i + 5j$ is not a unit vector

$|\bar{u}| = \sqrt{(1)^2 + (5)^2} = \sqrt{26}$

Unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{26}}(1,5)$

$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$f(a) = f(3,4) = (4)^2 - 4(3) + 1 = 5$

$f(a+hu) = f(3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$= f\left(3+\frac{h}{\sqrt{26}}, 4+\frac{5h}{\sqrt{26}}\right)$

$f_{x,y}(a+hu) = \left(4+\frac{5h}{\sqrt{26}}\right)^2 - 4\left(3+\frac{h}{\sqrt{26}}\right)+1$

$= 16 + \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$

$= \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - \frac{4h}{\sqrt{26}} + 5$

$= \frac{25h^2}{26} + \frac{40h - 4h}{\sqrt{26}} + 5$

$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5$

$\therefore D_u f(a) = \frac{25h}{26} + \frac{36}{\sqrt{26}}$

iii. $2x+3y \quad a = (1,2) \quad , u = (3i, 4j)$

$|\vec{u}| = \sqrt{(3)^2+(4)^2}$

$= \sqrt{25}$

$= 5$

Unit vector ~~along~~ u is $\frac{u}{|u|} = \frac{1}{5}(3,4)$

$= \left(\frac{3}{5}, \frac{4}{5}\right)$

$f(a) = f(1,2) = 2(1) + 3(2) = 8$

$f(a+hu) = f(1,2) + h\left(\frac{3}{5}, \frac{4}{5}\right)$

$= f\left(1+\frac{3h}{5}, 2+\frac{4}{5}\right)$

$f(a+hu) = 2\left(1+\frac{3h}{5}\right) + 3\left(3+\frac{4h}{5}\right)$

$= 2 + \frac{6h}{5} + 6 + \frac{12h}{5}$

$= \frac{18h}{5} + 8$

$D_u f(a) = \lim_{h\to 0} \frac{\frac{18h}{5} + 8 - 8}{h}$

$= \frac{18}{5}$

Q2 Find gradient vector for the following function at given point.

i) $f(x,y) = x^y + y^x \quad a = (1,1)$

$f_x = y \cdot x^{y-1} + y^x \log y$

$f_y = x^y \log x + xy^{x-1}$

$f(x,y) = (f_x, f_y)$

$= (yx^{y-1} + y^x \log y, x^y \log x + xy^{x-1})$

$f(1,1) = (1+0,\ 1+0)$
$= (1,1)$

$f(x,y) = (\tan^{-1}x) \cdot y^2 \quad a = (1,-1)$

$f_x = \frac{1}{1+x^2} \cdot y^2$

$f_y = 2y \tan^{-1} x$

$\nabla f(x,y) = (f_x, f_y)$
$= \left(\frac{y^2}{1+x^2},\ 2y \tan^{-1} x\right)$

$\cancel{\nabla f(x,y) = (f_x, f_y)}$
$\cancel{= \left(\frac{y^2}{}\right)}$

$f(1,-1) = \left(\frac{1}{2},\ \tan^{-1}(1)(-2)\right)$

$= \left(\frac{1}{2},\ \frac{\pi}{4_2}(-2)\right)$

$= \left(\frac{1}{2},\ \frac{-\pi}{2}\right)$

iii) $f(x,y,z) = xyz - e^{x+y+z} \quad a = (1,-1,0)$

$f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

$\nabla f(x,y,z) = f_x, f_y, f_z$
$= yz - e^{x+y+z},\ xz - e^{x+y+z},\ xy - e^{x+y+z}$

$f(1,-1,0) = \left((-1)(0) - e^{1+(-1)+0},\ (1)(0) - e^{1+(-1)+0},\ (1)(-1) - e^{1+(-1)+0}\right)$

$= (0 - e^0,\ 0 - e^0,\ -1 - e^0)$

$= (-1,\ -1,\ -2)$

Q.3 Find the equation of tangent & normal to each of the following using curves at given points.

i. $x^2 \cos y + e^{xy} = 2$ at $(1,0)$

$f_x = \cos y\, 2x + e^{xy}\, y$

$f_y = x^2(-\sin y) + e^{xy} \cdot x$

$(x_0, y_0) = (1,0)$

$\therefore x_0 = 1,\ y_0 = 0$

eq^n of tangent

$f_x(x - x_0) + f_y(y - y_0) = 0$

$f_x(x_0, y_0) = \cos 0 \, 2(1) + e^0 . 0$

$= 1(2) + 0$

$= 2$

$f_y(x_0, y_0) = (1)^2 (-\sin 0) + e^0 . 1$

$= 0 + 1 . 1$

$= 1$

$2(x-1) + 1(y-0) = 0$

$2x - 2 + y = 0$

$2x + y - 2 = 0$ It is the required eqn of tangent

eqn of Normal

$= ax + by + c = 0$

$= bx + ay + d = 0$

$1(1) + 2(y) + d = 0$

$1 + 2y + d = 0$ at $(1, 0)$

$= 1 + 2(0) + d = 0$

$d + 1 = 0$

$\therefore d = -1$

i) $x^2 + y^2 - 2x + 3y + 2 = 0$ at $(2, -2)$

$f_x = 2x + 0 - 2 + 0 + 0$

$= 2x - 2$

$f_y = 0 + 2y - 0 + 3 + 0$

$= 2y + 3$

$(x_0, y_0) = (2, -2)$

$\therefore x_0 = 2, \; y_0 = -2$

$f_x(x_0, y_0) = 2(2) - 2 = 2$

$f_y(x_0, y_0) = 2(-2) + 3 = -1$

eqn of tangent

$f_x(x - x_0) + f_y(y - y_0) = 0$

$2(x-2) + (-1(y+2)) = 0$

$2x - 2 - y - 2 = 0$

$2x - y - 4 = 0$

It is required eqn of tangent.

eqn of Normal

$= ax + by + c = 0$

$= bx + ay + d = 0$

$= -1(x) + 2(y) + d = 0$

$-x + 2y + d = 0$ at $(2, -2)$

$-2 + 2(-2) + d = 0$

$-2 - 4 + d = 0$

$-6 + d = 0$

$\therefore d = 6$

Q4 Find the eq^n of tangent & normal line to each of the following surface.

i) $x^2 - 2yz + 3y + xz = 7$ at $(2,1,0)$

$fx = 2x - 0 + 0 + z$

$fx = 2x + z$

$fy = 0 - 2z + 3 + 0$

$= 2z + 3$

$fz = 0 - 2y + 0 + x$

$= -2y + x$

$(x_0, y_0, z_0) = (2,1,0)$ $\therefore x_0 = 2,\ y_0 = 1,\ z_0 = 0$

$fx\,(x_0, y_0, z_0) = 2(2) + 0 = 4$

$fy\,(x_0, y_0, z_0) = 2(0) + 3 = 3$

$fz\,(x_0, y_0, z_0) = -2(1) + 2 = 0$

eq^n of tangent

$fx(x_0 - x_0) + fy(y_0 - y_0) + fz(z_0 - z_0) = 0$

$= 4(x-2) + 3(y-1) + 0(z-0) = 0$

$= 4x - 8 + 3y - 3 = 0$

$4x + 3y - 11 = 0$ → This is required eq^n of tangent

eq^n of normal at $(4, 3, -11)$

$$\frac{x - x_0}{fx} = \frac{y - y_0}{fy} = \frac{z - z_0}{fz}$$

$$= \frac{x-2}{4} = \frac{y-1}{3} = \frac{z+11}{0}$$

ii) $3xyz - x - y + z = -4$ at $(1,-1,2)$

$3xyz - x - y + z + 4 = 0$ at $(1,-1,2)$

$fx = 3yz - 1 - 0 + 0 + 0$

$= 3yz - 1$

$fy = 3xz - 0 - 1 + 0 + 0$

$= 3xz - 1$

$fz = 3xy - 0 - 0 + 1 + 0$

$= 3xy + 1$

$(x_0, y_0, z_0) = (1,-1,2)$ $\therefore x_0 = 1,\ y = -1,\ z_0 = 2$

$fx\,(x_0, y_0, z_0) = 3(-1)(2) - 1 = -7$

$fy\,(x_0, y_0, z_0) = 3(+1)(2) - 1 = 5$

$fz\,(x_0, y_0, z_0) = 3(1)(-1) + 1 = -2$

eq^n of tangent

$-7(x-1) + 5(y+1) - 2(z-2) = 0$

$-7x + 7 + 5y + 5 - 2z + 4 = 0$

$-7x + 5y - 2z + 16 = 0$

This is required eq^n of tangent

Eqⁿ of normal at $(-7, 5, -2)$

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$= \frac{x-1}{-7} = \frac{y+1}{\sqrt{5}} = \frac{z-2}{-2}$$

Q5 Find the local maxima & minima for the following functions

i. $f(x,y) = 3x^2 + y^2 - 3xy + 6x - 4y$

$f_x = 6x + 0 - 3y + 6 - 0$

$= 6x - 3y + 6$

$f_y = 0 + 2y - 3x + 0 - 4$

$= 2y - 3x - 4$

$f_x = 0$

$6x - 3y + 6 = 0$

$3(2x - y + 2) = 0$

$2x - y + 2 = 0$

$2x - y = -2$ — (1)

$f_y = 0$

$2y - 3x - 4 = 0$

$2y - 3x = 4$ — (2)

Multiply eqⁿ 1 with 2

$4x - \cancel{2y} = -4$

$\cancel{2y} - 3x = 4$

$x = 0$

substitute value of x in eqⁿ (1)

$2(0) - y = -2$

$-y = -2$

$\therefore y = 2$

$\therefore$ Critical points are $(0, 2)$

$r = f_{xx} = 6$

$t = f_{yy} = 2$

$s = f_{xy} = -3$

Here $r > 0$

$= rt - s^2$

$= 6(2) - (-3)^2$

$= 12 - 9$

$= 3 > 0$

$\therefore f$ has maximum at $(0, 2)$

$3x^2 + y^2 - 3xy + 6x - 4y$ at $(0, 2)$

$3(0)^2 + (2)^2 - 3(0)(2) + 6(0) - 4(2)$

$0 + 4 - 0 + 0 - 8$

$= -4$

ii. $f(x,y) = 2x^4 + 3x^2 y - y^2$

$f_x = 8x^3 + 6xy$

$f_y = 3x^2 - 2y$

$f_x = 0$

$\therefore 8x^3 + 6xy = 0$

$2x(4x^2 + 3y) = 0$

$4x^2 + 3y = 0 \rightarrow (1)$

$f_y = 0$

$3x^2 - 2y = 0 \rightarrow (2)$

Multiply eq^n (1) with 3
(2) with 4

$12x^2 + 9y = 0$

$-\,12x^2 - 8y = 0$

$11y = 0$

$\therefore y = 0$

Substitute value of y in eq^n (1)

$4x^2 + 3(0) = 0$

$4x^2 = 0$

$x = 0$

Critical point is (0,0)

$r = f_{xx} = 24x^2 + 6x$

$t = f_{yy} = 0 - 2 = -2$

$s = f_{xy} = 6x - 0 = 6x = 6(0) = 0$

r at (0,0)

$= 24(0) + 6(0) = 0$

$\therefore r = 0$

$rt - s^2 = 0(-2) - (0)^2$

$= 0 - 0 = 0$

$r = 0$ & $rt - s^2 = 0$

(nothing to say)

f(x,y) at (0,0)

$2(0)^4 + 3(0)^2(0) - 0$

$= 0 + 0 - 0$

$= 0$

iii) $f(x,y) = x^2 - y^2 + 2x + 8y - 70$

$f_x = 2x + 2$

$f_y = -2y + 8$

$f_x = 0$

$\therefore 2x + 2 = 0$

$x = \frac{-2}{2}$

$\therefore x = -1$

$f_y = 0$

$-2y + 8 = 0$

$y = \frac{-8}{-2} = 4$

$\therefore y = 4$

$\therefore$ Critical point is (-1,4)

$r = f_{xx} = 2$

$t = f_{yy} = -2$

$s = f_{xy} = 0$

$r > 0$

$rt - s^2 = 2(-2) - 0^2$

$= -4 - 0$

$= -4 < 0$

$f(x,y)$ at $(-1,4)$

$(-1)^2 - (4)^2 + 2(-1) + 8(4) - 70$

$= 1 + 16 - 2 + 32 - 70$

$= 17 + 30 - 70$

$= 37 - 70$

$= 33$

AB

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