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《2015 数值分析》 期末考题.

一、设 $f^{n}(x)$ 在 $[a,b]$ 上连续，$f^{(n+1)}(x)$ 在 (a,b) 内存在，节点 $a\le x_0<x_1<\cdots<x_n\le b$，$L_n(x)$ 是满足条件 $L_n(x_j)=y_j$，$j=0,1,\cdots,n$ 的插值多项式，则对任何 $x\in[a,b]$，证明插值余项公式为

$$A_n(x)=f(x)-L_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}w_{n+1}(x)$$，其中 $\xi\in(a,b)$ 且依赖于 x.

$W_{n+1}(x)=(x-x_0)(x-x_1)\cdots(x-x_n)$.

证明：构造关于 t 的函数：

$$\varphi(t)=f(t)-L_n(t)-\frac{f(x)-L_n(x)}{W(x)}W(t)$$

则 $\varphi(x_0)=f(x_0)-L_n(x_0)-\frac{f(x)-L_n(x)}{W(x)}W(x_0)$

$\because W(x_0)=0$ $\therefore \varphi(x_0)=y_0-y_0=0$

$L_n(x_j)=y_j$

同理，$\varphi(x_1)=f(x_1)-L_n(x_1)-\frac{f(x)-L_n(x)}{W(x)}W(x_1)$

$=y_1-y_1=0$

$\vdots$

$\varphi(x_n)=0$

$$\varphi(x)=f(x)-L_n(x)-\frac{f(x)-L_n(x)}{W(x)}W(x)=0$$

综上所述，$\varphi(t)$ 有 $n+2$ 个零点.

由罗尔定理可知，$\varphi'(t)$ 有 $n+1$ 个零点.

同理，$\varphi^{(n+1)}(t)$ 有1个零点.

又 $\because$ 由前提可知 $f(x)$ 有 $n+1$ 阶导数，$W^{(n+1)}(t)=(n+1)!$ $L_n^{(n+1)}(t)=0$

$\therefore \varphi^{(n+1)}(t)=f^{(n+1)}(t)-0-\frac{f(x)-L_n(x)}{W(x)}(n+1)!$ 有1个零点.

$\therefore$ 存在一个 ξ 使 $\varphi^{(n+1)}(\xi)=0$

即：$f^{(n+1)}(\xi)-\frac{f(x)-L_n(x)}{W(x)}(n+1)!=0$

$$\Rightarrow A_n(x)=f(x)-L_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}W(x)$$

二. 求函数 $f(x)=x^2-2x+3$ 在区间 $[-1,1]$ 上权函数 $\rho(x)=1$ 的二次最佳逼近多项式.

即构造 $S^*(x)=a_0\varphi_0(x)+a_1\varphi_1(x)+a_2\varphi_2(x)$

其中, $\varphi_0(x)=1$, $\varphi_1(x)=x$, $\varphi_2(x)=x^2$

使 $S^*(x)$ 与 $f(x)$ 在二范数下相比较误差最小.

$$\|S^*(x)-f(x)\|_2^2=\int_a^b \rho(x)\left(\sum_{i=0}^{n} a_i\varphi_i(x)-f(x)\right)dx=I(a_0,a_1,\cdots a_k)$$

$$\frac{\partial I}{\partial a_k}=\rho(x)\int_a^b\left[\left(\sum_{i=0}^{n}a_i\varphi_i(x)-f(x)\right)\varphi_k(x)\right]dx=0$$

则有 $\int_a^b\sum_{i=0}^{n}a_i\varphi_i(x)\varphi_k(x)dx=\int_a^b f(x)\varphi_k(x)dx$

令 $\int_a^b\varphi_i(x)\varphi_k(x)dx=(\varphi_i,\varphi_k)$ $\quad\int_a^b f(x)\varphi_k(x)dx=(f,\varphi_k)$

$$\begin{pmatrix}(\varphi_0,\varphi_0) & (\varphi_0,\varphi_1) & \cdots & (\varphi_0,\varphi_n)\\ \vdots & & & \\ (\varphi_n,\varphi_0) & \cdots & & (\varphi_n,\varphi_n)\end{pmatrix}\begin{pmatrix}a_0\\ \vdots\\ a_n\end{pmatrix}=\begin{pmatrix}(f,\varphi_0)\\ \vdots\\ (f,\varphi_n)\end{pmatrix}$$

$$\therefore\begin{pmatrix}\rho(x)(\varphi_0,\varphi_0) & \rho(x)(\varphi_0,\varphi_1) & \rho(x)(\varphi_0,\varphi_2)\\ \rho(x)(\varphi_1,\varphi_0) & \rho(x)(\varphi_1,\varphi_1) & \rho(x)(\varphi_1,\varphi_2)\\ \rho(x)(\varphi_2,\varphi_0) & \rho(x)(\varphi_2,\varphi_1) & \rho(x)(\varphi_2,\varphi_2)\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ a_2\end{pmatrix}=\begin{pmatrix}(f,\varphi_0)\\ (f,\varphi_1)\\ (f,\varphi_2)\end{pmatrix}$$

本题中, $(\varphi_0,\varphi_0)=\int_{-1}^{1}1\,dx=2$ $\quad(f,\varphi_0)=\int_{-1}^{1}(x^2-2x+3)dx=\frac{20}{3}$

$(\varphi_0,\varphi_1)=(\varphi_1,\varphi_0)=\int_{-1}^{1}x\,dx=0$ $\quad(f,\varphi_1)=\int_{-1}^{1}(x^3-2x^2+3x)dx=-\frac{4}{3}$

$(\varphi_0,\varphi_2)=(\varphi_2,\varphi_0)=\int_{-1}^{1}x^2dx=\frac{2}{3}$ $\quad(f,\varphi_2)=\int_{-1}^{1}(x^4-2x^3+3x^2)dx=\frac{12}{5}$

$(\varphi_1,\varphi_1)=\int_{-1}^{1}x^2dx=\frac{2}{3}$

$(\varphi_1,\varphi_2)=(\varphi_2,\varphi_1)=\int_{-1}^{1}x^3dx=0$

$(\varphi_2,\varphi_2)=\int_{-1}^{1}x^4dx=\frac{2}{5}$

即 $\begin{pmatrix}2 & 0 & 2/3\\ 0 & 2/3 & 0\\ 2/3 & 0 & 2/5\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ a_2\end{pmatrix}=\begin{pmatrix}20/3\\ -4/3\\ 12/5\end{pmatrix}\Rightarrow\begin{cases}a_0=3\\ a_1=-2\\ a_2=1\end{cases}$

$\therefore S^*(x)=3-2x+x^2$

三、给出n阶牛顿差商定义，并计算下列已知数据差商.

i	x_i	$f(x_i)$	一阶	二阶	三阶
0	-2	3			
1	-1	5	2		
2	1	7	1	$-\frac{1}{3}$	
3	2	11	4	1	$\frac{1}{3}$

$$f[x_0, x_1, \cdots, x_k] = \frac{f[x_0, x_1, \cdots, x_{k-1}] - f[x_1, x_2, \cdots, x_k]}{x_0 - x_k}$$

一阶：$f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{3-5}{-2-(-1)} = \frac{-2}{-1} = 2$

$$f(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{5-7}{-1-1} = \frac{-2}{-2} = 1$$

$$f(x_2, x_3) = \frac{f(x_2) - f(x_3)}{x_2 - x_3} = \frac{7-11}{1-2} = \frac{-4}{-1} = 4$$

二阶 $f(x_0, x_1, x_2) = \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2} = \frac{2-1}{-2-1} = \frac{1}{-3} = -\frac{1}{3}$

$$f(x_1, x_2, x_3) = \frac{f[x_1, x_2] - f[x_2, x_3]}{x_1 - x_3} = \frac{1-4}{-1-2} = 1$$

三阶 $f[x_0, x_1, x_2, x_3] = \frac{f[x_0, x_1, x_2] - f[x_1, x_2, x_3]}{x_0 - x_3} = \frac{-\frac{1}{3} - 1}{-2-2} = \frac{-\frac{4}{3}}{-4} = \frac{1}{3}$

四、给出数值积分公式的m次代数精度概念.求下列求积公式中待定参数，使代数精度尽量高，并指明所构造的求积公式所具有的代数精度

$$\int_{-2}^{2} f(x)\,dx \approx A_{-1}f(-1) + A_0 f(0) + A_1 f(1)$$

如果一个数值积分公式I.它对于次数不超过m次的多项式恒精确成立，而对于m+1次多项式不能恒精确成立，则称该数值积分公式I有m次代数精度。

题中所给求积公式至少具有二次精度.

令 $f(x)=1$，则：$4 = A_{-1} + A_0 + A_1$

令 $f(x)=x$，则：$0 = -A_{-1} + A_1$

令 $f(x)=x^2$，则：$\frac{16}{3} = A_{-1} + A_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_{-1} \\ A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ \frac{16}{3} \end{pmatrix}$$

解得：$A_{-1} = \frac{8}{3}$　　$A_0 = -\frac{4}{3}$　　$A_1 = \frac{8}{3}$

$$\therefore \int_{-2}^{2} f(x)\,dx \approx \frac{8}{3}f(-1) + \left(-\frac{4}{3}\right)f(0) + \frac{8}{3}f(1)$$

当 $f(x)=x^3$ 时，左边 $=0$

右边 $= -\frac{8}{3} + \frac{8}{3} = 0$

左边 = 右边

当 $f(x)=x^4$ 时，左边 $= \frac{64}{5}$

右边 $= \frac{8}{3} + \frac{8}{3} = \frac{16}{3}$

左边 ≠ 右边

∴ 求积公式具有3次精度.

五. 利用LU分解法求

$$\begin{cases} x_1+2x_2+3x_3=10 \\ -x_1+x_3=-2 \\ x_1+2x_2+x_3=8 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

$\therefore U_{11}=1 \quad U_{12}=2 \quad U_{13}=3$

$$\begin{cases} U_{11}L_{21}=-1 \\ U_{12}L_{21}+U_{22}=0 \\ U_{13}L_{21}+U_{23}=1 \end{cases} \Rightarrow \begin{cases} L_{21}=-1 \\ U_{22}=2 \\ U_{23}=4 \end{cases} \quad \begin{cases} U_{11}L_{31}=1 \\ U_{12}L_{31}+U_{22}L_{32}=2 \\ U_{13}L_{31}+U_{23}L_{32}+U_{33}=1 \end{cases} \Rightarrow \begin{cases} L_{31}=1 \\ L_{32}=0 \\ U_{33}=-2 \end{cases}$$

则有：

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 8 \end{pmatrix}$$

令：

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \Rightarrow \begin{cases} y_1=10 \\ -y_1+y_2=-2 \\ y_1+y_3=8 \end{cases} \Rightarrow \begin{cases} y_1=10 \\ y_2=8 \\ y_3=-2 \end{cases}$$

即有：

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ -2 \end{pmatrix}$$

$$\begin{cases} x_1+2x_2+3x_3=10 \\ 2x_2+4x_3=8 \\ -2x_3=-2 \end{cases} \Rightarrow \begin{cases} x_1=3 \\ x_2=2 \\ x_3=1 \end{cases}$$

六、对于 $\begin{cases} 2x_1 - x_2 = 1 \\ -x_1 + 2x_2 + ax_3 = 0 \\ -x_2 + 2x_3 = -1 \end{cases}$ 求实数a的取值范围，使得用雅可比方法求解方程组时收敛。

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & a \\ 0 & -1 & 2 \end{pmatrix} = \overset{Q}{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}} - \overset{C}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -a \\ 0 & 1 & 0 \end{pmatrix}}$$

$B = Q^{-1}C$

$$Q^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & -a/2 \\ 0 & 1/2 & 0 \end{pmatrix}$$

$g = Q^{-1}b$

$$g = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}$$

使用雅可比方法求解方程组收敛时，$\rho(B)<1$. $\rho(B) = \max\{|\lambda_i|\} < 1$

$$|B-\lambda E| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1/2 & 0 \\ 1/2 & -\lambda & -a/2 \\ 0 & 1/2 & -\lambda \end{vmatrix} = -\lambda^3 - \frac{a}{4}\lambda + \frac{\lambda}{4} = 0 \Rightarrow \lambda^2 = \frac{1}{4} - \frac{a}{4} < 1$$

$\therefore \quad a > -3$

又 $\because \lambda^2 = \frac{1-a}{4}$

$\therefore \lambda = \sqrt{\frac{1-a}{4}}$

需要 $\frac{1-a}{4} \geq 0 \quad a \leq 1$

综上所述，$-3 < a \leq 1$

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