Integrals

Question1

If $\int_0^1 \frac{1}{\sqrt{3+x}+\sqrt{1+x}} dx = a + b\sqrt{2} + c\sqrt{3}$, where a, b, c are rational numbers, then $2a+3b-4c$ is equal to :

[27-Jan-2024 Shift 1]

Options:

A.

4

B.

10

C.

7

D.

8

Answer: D

Solution:

$$\int_0^1 \frac{1}{\sqrt{3+x}+\sqrt{1+x}} dx = \int_0^1 \frac{\sqrt{3+x}-\sqrt{1+x}}{(3+x)-(1+x)} dx$$

$$\frac{1}{2}\left[\int_0^1 \sqrt{3+x}\, dx - \int_0^1 (\sqrt{1+x})\, dx\right]$$

$$\frac{1}{2}\left[2\frac{(3+x)^{\frac{3}{2}}}{3} - \frac{2(1+x)^{\frac{3}{2}}}{3}\right]_0^1$$

$$\frac{1}{2}\left[\frac{2}{3}(8-3\sqrt{3}) - \frac{2}{3}\left(2^{\frac{3}{2}}-1\right)\right]$$

$$\frac{1}{3}[8-3\sqrt{3}-2\sqrt{2}+1]$$

$$= 3-\sqrt{3}-\frac{2}{3}\sqrt{2} = a+b\sqrt{2}+c\sqrt{3}$$

$$a=3, b=-\frac{2}{3}, c=-1$$

$$2a+3b-4c = 6-2+4 = 8$$

Question2

If (a, b) be the orthocentre of the triangle whose vertices are (1, 2), (2, 3) and (3, 1), and

$I_1 = \int_a^b x\sin(4x - x^2)\,dx$, $I_2 = \int_a^b \sin(4x - x^2)\,dx$, then $36\frac{I_1}{I_2}$ is equal to :

[27-Jan-2024 Shift 1]

Options:

A.

72

B.

88

C.

80

D.

66

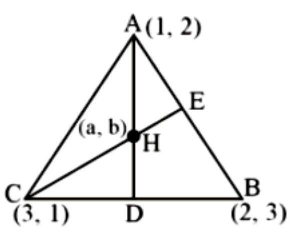
Answer: A

Solution:

Equation of CE

$y - 1 = -(x - 3)$

$x + y = 4$



orthocentre lies on the line $x + y = 4$

so, $a + b = 4$

$I_1 = \int_a^b x\sin(x(4 - x))\,dx$ ....(i)

Using king rule

$I_1 = \int_a^h (4 - x)\sin(x(4 - x))\,dx$ ....(ii)

(i) + (ii)

$2I_1 = \int_a^b 4\sin(x(4 - x))\,dx$

$2I_1 = 4I_2$

$I_1 = 2I_2$

$\frac{I_1}{I_2} = 2$

$\frac{36I_1}{I_2} = 72$

Question3

For $0 < a < 1$, the value of the integral $\int_0^{\pi} \frac{dx}{1-2a\cos x + a^2}$ is :

[27-Jan-2024 Shift 2]

Options:

A.

$$\frac{\pi^2}{\pi + a^2}$$

B.

$$\frac{\pi^2}{\pi - a^2}$$

C.

$$\frac{\pi}{1 - a^2}$$

D.

$$\frac{\pi}{1 + a^2}$$

Answer: C

Solution:

$$I = \int_0^{\pi} \frac{dx}{1 - 2a\cos x + a^2}; 0 < a < 1$$

$$I = \int_0^{\pi} \frac{dx}{1 + 2a\cos x + a^2}$$

$$2I = 2\int_0^{\pi/2} \frac{2(1+a^2)}{(1+a^2)^2 - 4a^2\cos^2 x} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{2(1+a^2)\cdot\ sec^2 x}{(1+a^2)^2\cdot\ sec^2 x - 4a^2} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{2\cdot(1+a^2)\cdot\ sec^2 x}{(1+a^2)^2\cdot\tan^2 x + (1-a^2)^2} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\frac{2\cdot\ sec^2 x}{1+a^2}\cdot dx}{\tan^2 x + \left(\frac{1-a^2}{1+a^2}\right)^2}$$

$$\Rightarrow I = \frac{2}{(1-a^2)}\left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{1-a^2}$$

Question4

The integral $\int \frac{(x^8 - x^2)\,dx}{(x^{12} + 3x^6 + 1)\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)}$ is equal to :

[27-Jan-2024 Shift 2]

Options:

A.

$\log_e\left(\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|\right)^{1/3} + C$

B.

$\log_e\left(\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|\right)^{1/2} + C$

C.

$\log_e\left(\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|\right) + C$

D.

$\log_e\left(\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|\right)^{3} + C$

Answer: A

Solution:

$$I = \int \frac{x^8 - x^2}{(x^{12} + 3x^6 + 1)\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)}\,dx$$

Let $\tan^{-1}\left(x^3 + \frac{1}{x^3}\right) = t$

$$\Rightarrow \frac{1}{1 + \left(x^3 + \frac{1}{x^3}\right)^2} \cdot \left(3x^2 - \frac{3}{x^4}\right) dx = dt$$

$$\Rightarrow \frac{x^6}{x^{12} + 3x^6 + 1} \cdot \frac{3x^6 - 3}{x^4}\,dx = dt$$

$$I = \frac{1}{3}\int \frac{dt}{t} = \frac{1}{3}\ln|t| + C$$

$$I = \frac{1}{3}\ln\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right| + C$$

$$I = \ln\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|^{1/3} + C$$

Hence option (1) is correct

Question5

For $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

if $y(x) = \int \frac{\operatorname{cosec} x + \sin x}{\operatorname{cosec} x \sec x + \tan x \sin^2 x} \mathrm{dx}$

and $\lim\limits_{x \to \left(\frac{\pi}{2}\right)^-} y(x) = 0$ then $y\left(\frac{\pi}{4}\right)$ is equal to

[29-Jan-2024 Shift 1]

Options:

A.

$\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

B.

$\frac{1}{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

C.

$-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

D.

$\frac{1}{\sqrt{2}}\tan^{-1}\left(-\frac{1}{2}\right)$

Answer: D

Solution:

$y(x) = \int \frac{(1 + \sin^2 x) \cos x}{1 + \sin^4 x} \mathrm{dx}$

Put $\sin x = t$

$= \int \frac{1 + t^2}{t^4 + 1} \mathrm{dt} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}} + C$

$x = \frac{\pi}{2}, t = 1 \quad \therefore C = 0$

$y\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\tan^{-1}\left(-\frac{1}{2}\right)$

Question6

If $\int \frac{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}{\sqrt{\sin^3 x\cos^3 x\sin(x-\theta)}}dx = A\sqrt{\cos\theta\tan x-\sin\theta}+B\sqrt{\cos\theta-\sin\theta\cot x}+C$,

where C is the integration constant, then AB is equal to

[29-Jan-2024 Shift 2]

Options:

A.

4 cosec(2θ)

B.

4 secθ

C.

2 secθ

D.

8 cosec(2θ)

Answer: D

Solution:

$$\int \frac{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}{\sqrt{\sin^3 x\cos^3 x\sin(x-\theta)}}dx$$

$$I=\int \frac{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}{\sqrt{\sin^3 x\cos^3 x(\sin x\cos\theta-\cos x\sin\theta)}}dx$$

$$=\int \frac{\sin^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x\cos^2 x\sqrt{\tan x\cos\theta-\sin\theta}}dx+\int \frac{\cos^{\frac{3}{2}}x}{\sin^2 x\cos^{\frac{3}{2}}x\sqrt{\cos\theta-\cot x\sin\theta}}dx$$

$$\int \frac{\sec^2 x}{\sqrt{\tan x\cos\theta-\sin\theta}}dx+\int \frac{\operatorname{cosec}_2 x}{\sqrt{\cos\theta-\cot^2\sin\theta}}dx$$

$I=I_1+I_2$ .... { Let }

For I_1, let $\tan x\cos\theta-\sin\theta=t^2$

$$\sec^2 x\,dx=\frac{2t\,dt}{\cos\theta}$$

For I_2, let $\cos\theta-\cot x\sin\theta=z^2$

$$\operatorname{cosec}^2 x\,dx=\frac{2\,zdz}{\sin\theta}$$

$I=I_1+I_2$

$$=\int\frac{2t\,dt}{\cos\theta\, t}+\int\frac{2zdz}{\sin\theta z}$$

$$=\frac{2t}{\cos\theta}+\frac{2z}{\sin\theta}$$

$$=2\sec\theta\sqrt{\tan x\cos\theta-\sin\theta}+2\operatorname{cosec}\theta\sqrt{\cos\theta-\cot x\sin\theta}$$

Comparing

$AB=8\operatorname{cosec}2\theta$

Question7

Let $f(x)=\int_0^x g(t)\log_e\left(\frac{1-t}{1+t}\right)$ dt, where g is a continuous odd function.

If $\int_{-\pi/2}^{\pi/2}\left(f(x)+\frac{x^2\cos x}{1+e^x}\right)\mathrm{dx}=\left(\frac{\pi}{\alpha}\right)^2-\alpha$, then α is equal to............

[27-Jan-2024 Shift 2]

Options:

Answer: 2

Solution:

$$f(x)=\int_0^x g(t)\ln\left(\frac{1-t}{1+t}\right)\mathrm{dt}$$

$$f(-x)=\int_0^{-x} g(t)\ln\left(\frac{1-t}{1+t}\right)\mathrm{dt}$$

$$f(-x)=-\int_0^x g(-y)\ln\left(\frac{1+y}{1-y}\right)dy$$

$$=-\int_0^x g(y)\ln\left(\frac{1-y}{1+y}\right)dy\ \text{(g is odd)}$$

$f(-x)=-f(x)\Rightarrow f$ is also odd

Now,

$$I=\int_{-\pi/2}^{\pi/2}\left(f(x)+\frac{x^2\cos x}{1+e^x}\right)\mathrm{dx}\ ....(1)$$

$$I=\int_{-\pi/2}^{\pi/2}\left(f(-x)+\frac{x^2e^x\cos x}{1+e^x}\right)\mathrm{dx}\ ....(2)$$

$$2I=\int_{-\pi/2}^{\pi/2}x^2\cos x\,\mathrm{dx}=2\int_0^{\pi/2}x^2\cos x\,\mathrm{dx}$$

$$I=(x^2\sin x)_0^{\pi/2}-\int_0^{\pi/2}2x\sin x\,\mathrm{dx}$$

$$=\frac{\pi^2}{4}-2(-x\cos x+\int\cos x\,\mathrm{dx})_0^{\pi/2}$$

$$=\frac{\pi^2}{4}-2(0+1)=\frac{\pi^2}{4}-2\Rightarrow\left(\frac{\pi}{2}\right)^2-2$$

$$\therefore\alpha=2$$

Question8

If the value of the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{x^2\cos x}{1+\pi^x}+\frac{1+\sin^2 x}{1+e^{\sin x^{2023}}}\right)\mathrm{dx}=\frac{\pi}{4}(\pi+a)-2,$$

then the value of a is

[29-Jan-2024 Shift 1]

Options:

A.

3

B.

$-\frac{3}{2}$

C.

2

D.

3/2

Answer: A

Solution:

$$I=\int_{-\pi/2}^{\pi/2}\left(\frac{x^2\cos x}{1+\pi^x}+\frac{1+sin^2x}{1+e^{sin\,x^{2023}}}\right)dx$$

$$I=\int_{-\pi/2}^{\pi/2}\left(\frac{x^2\cos x}{1+\pi^{-x}}+\frac{1+sin^2x}{1+e^{sin\,(-x)^{2023}}}\right)dx$$

On Adding, we get

$$2I=\int_{-\pi/2}^{\pi/2}(x^2\cos x+1+sin^2x)\,dx$$

On solving

$$I=\frac{\pi^2}{4}+\frac{3\pi}{4}-2$$

$a=3$

Question9

If $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{1-sin\,2x}\,dx=\alpha+\beta\sqrt{2}+\gamma\sqrt{3}$, where α, β and γ are rational numbers, then $3\alpha+4\beta-\gamma$ is equal to___

[29-Jan-2024 Shift 2]

Answer: 6

Solution:

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{1 - \sin 2x}\, dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} |\sin x - \cos x|\, dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cos x - \sin x)\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sin x - \cos x)\, dx$$

$$= -1 + 2\sqrt{2} - \sqrt{3}$$

$$= \alpha + \beta\sqrt{2} + \gamma\sqrt{3}$$

$$\alpha = -1, \beta = 2, \gamma = -1$$

$$3\alpha + 4\beta - \gamma = 6$$

Question10

The value of $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \frac{n^3}{(n^2 + k^2)(n^2 + 3k^2)}$ is :

[30-Jan-2024 Shift 1]

Options:

A.

$$\frac{(2\sqrt{3} + 3)\pi}{24}$$

B.

$$\frac{13\pi}{8(4\sqrt{3} + 3)}$$

C.

$$\frac{13(2\sqrt{3} - 3)\pi}{8}$$

D.

$$\frac{\pi}{8(2\sqrt{3} + 3)}$$

Answer: B

Solution:

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{n^3}{n^4\left(1+\frac{k^2}{n^2}\right)\left(1+\frac{3k^2}{n^2}\right)}$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{n^3}{\left(1+\frac{k^2}{n^2}\right)\left(1+\frac{3k^2}{n^2}\right)}$$

$$= \int_0^1 \frac{dx}{3(1+x^2)\left(\frac{1}{3}+x^2\right)}$$

$$= \int_0^1 \frac{1}{3} \times \frac{3}{2} \frac{(x^2+1)-\left(x^2+\frac{1}{3}\right)}{(1+x^2)\left(x^2+\frac{1}{3}\right)} \mathrm{dx}$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1}{x^2+\left(\frac{1}{\sqrt{3}}\right)^2} - \frac{1}{1+x^2}\right] \mathrm{dx}$$

$$= \frac{1}{2}[\sqrt{3}\tan^{-1}(\sqrt{3}x)]_0^1 - \frac{1}{2}(\tan^{-1}x)_0^1$$

$$= \frac{\sqrt{3}}{2}\left(\frac{\pi}{3}\right) - \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{2\sqrt{3}} - \frac{\pi}{8}$$

$$= \frac{13\pi}{8\cdot(4\sqrt{3}+3)}$$

Question11

The value $9\int_0^9 \left[\sqrt{\frac{10x}{x+1}}\right] \mathrm{dx}$, where $[t]$ denotes the greatest integer less than or equal to t, is

[30-Jan-2024 Shift 1]

Answer: 155

Solution:

$$\frac{10x}{x+1} = 1 \Rightarrow x = \frac{1}{9}$$

$$\frac{10x}{x+1} = 4 \Rightarrow x = \frac{2}{3}$$

$$\frac{10x}{x+1} = 9 \Rightarrow x = 9$$

$$I = 9\left(\int_0^{1/9} 0\,\mathrm{dx} + \int_{1/9}^{2/3} 1\,.\,\mathrm{dx} + \int_{2/3}^{9} 2\,\mathrm{dx}\right)$$

$$= 155$$

Question12

Let $y=f(x)$ be a thrice differentiable function in $(-5, 5)$. Let the tangents to the curve $y=f(x)$ at $(1, f(1))$ and $(3,$
angles $\frac{\pi}{6}$ and $\frac{\pi}{4}$, respectively with positive x-axis. If $27\int_1^3((f'(t))^2+1)f''(t)\,dt=\alpha+\beta\sqrt{3}$ where α, β are integers,
value of $\alpha+\beta$ equals

[30-Jar

Options:

A.

-14

B.

26

C.

-16

D.

36

Answer: B

Solution:

$y=f(x) \Rightarrow \frac{dy}{dx}=f'(x)$

$\left.\frac{dy}{dx}\right)_{(1,f(1))}=f'(1)=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}} \Rightarrow f'(1)=\frac{1}{\sqrt{3}}$

$\left.\frac{dy}{dx}\right)_{(3,f(3))}=f'(3)=\tan\frac{\pi}{4}=1 \Rightarrow f'(3)=1$

$27\int_1^3((f'(t))^2+1)f''(t)\,dt=\alpha+\beta\sqrt{3}$

$I=\int_1^3((f'(t))^2+1)f''(t)\,dt$

$f'(t)=z \Rightarrow f''(t)\,dt=dz$

$z=f'(3)=1$

$z=f'(1)=\frac{1}{\sqrt{3}}$

$I=\int_{1/\sqrt{3}}^{1}(z^2+1)dz=\left(\frac{z^3}{3}+z\right)_{1/\sqrt{3}}^{1}$

$=\left(\frac{1}{3}+1\right)-\left(\frac{1}{3}\cdot\frac{1}{3\sqrt{3}}+\frac{1}{\sqrt{3}}\right)$

$=\frac{4}{3}-\frac{10}{9\sqrt{3}}=\frac{4}{3}-\frac{10}{27}\sqrt{3}$

$\alpha+\beta\sqrt{3}=27\left(\frac{4}{3}-\frac{10}{27}\sqrt{3}\right)=36-10\sqrt{3}$

$\alpha=36, \beta=-10$

$\alpha+\beta=36-10=26$

Question13

Let $f : R \to R$ be defined $f(x) = ae^{2x} + be^{x} + cx$. If $f(0) = -1, f'(\log_e 2) = 21$ and $\int\limits_0 \log_e 4(f(x) - cx)\, dx = \frac{39}{2}$, then the valu

$|a + b + c|$ equals :

[30-Jan

Options:

A.

16

B.

10

C.

12

D.

8

Answer: D

Solution:

$f(x) = ae^{2x} + be^{x} + cx$

$f(0) = -1$

$a + b = -1$

$f'(x) = 2ae^{2x} + be^{x} + c$

$f'(\ln 2) = 21$

$8a + 2b + c = 21$

$\int\limits_0^{\ln 4} (ae^{2x} + be^{x})\, dx = \frac{39}{2}$

$\left[\frac{ae}{2x} + be^{x} \right]_0^{\ln 4} = \frac{39}{2} \Rightarrow 8a + 4b - \frac{a}{2} - b = \frac{39}{2}$

$15a + 6b = 39$

$15a - 6a - 6 = 39$

$9a = 45 \Rightarrow a = 5$

$b = -6$

$c = 21 - 40 + 12 = -7$

$a + b + c = -8$

$|a + b + c| = 8$

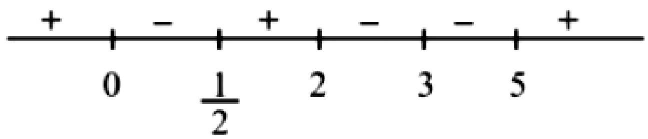
Question14

Let $S=(-1,\infty)$ and $f:S\rightarrow\mathbb{R}$ be defined as $f(x)=\int_{-1}^{x}(e^{t}-1)^{11}(2t-1)^{5}(t-2)^{7}(t-3)^{12}(2t-10)^{61}\,dt$ Let $p=$ Sum of squ
values of x, where f(x) attains local maxima on S. and q = Sum of the values of x, where $f(x)$ attains local minin
Then, the value of $p^{2}+2q$ is____ *[31-Jan*

Answer: 27

Solution:

$$f'(x)=(e^{x}-1)^{11}(2x-1)^{5}(x-2)^{7}(x-3)^{12}\,(2x-10)^{61}$$



Local minima at $x=\frac{1}{2}, x=5$

Local maxima at $x=0, x=2$

$\therefore p=0+4=4,\ q=\frac{1}{2}+5=\frac{11}{2}$

Then $p^{2}+2q=16+11=27$

Question15

If the integral $525\int_{0}^{\frac{\pi}{2}} \sin 2x\cos^{\frac{11}{2}}x\left(1+\cos^{\frac{5}{2}}x\right)^{\frac{1}{2}}dx$ is equal to $(n\sqrt{2}-64)$, then n is equal to______

[31-Jan-2024 Shift 1]

Answer: 176

Solution:

$$I=\int_0^{\frac{\pi}{2}} \sin 2x\cdot(\cos x)^{\frac{11}{2}}\left(1+(\cos x)^{\frac{5}{2}}\right)^{\frac{1}{2}}dx$$

Put $\cos x=t^2 \Rightarrow \sin x\,dx=-2t\,dt$

$$\therefore I=4\int_0^1 t^2\cdot t^{11}\sqrt{(1+t^5)(t)}\,dt$$

$$I=4\int_0^1 t^{14}\sqrt{1+t^5}\,dt$$

Put $1+t^5=k^2$

$\Rightarrow 5t^4\,dt=2k\,dk$

$$\therefore I=4\cdot\int_1^{\sqrt{2}}(k^2-1)^2\cdot k\,\frac{2k}{5}dk$$

$$I=\frac{8}{5}\int_1^{\sqrt{2}} k^6-2k^4+k^2\,dk$$

$$I=\frac{8}{5}\left[\frac{k^7}{7}-\frac{2k^5}{5}+\frac{k^3}{3}\right]_1^{\sqrt{2}}$$

$$I=\frac{8}{5}\left[\frac{8\sqrt{2}}{7}-\frac{8\sqrt{2}}{5}+\frac{2\sqrt{2}}{3}-\frac{1}{7}+\frac{2}{5}-\frac{1}{3}\right]$$

$$I=\frac{8}{5}\left[\frac{22\sqrt{2}}{105}-\frac{8}{105}\right]$$

$$\therefore 525\cdot I=176\sqrt{2}-64$$

Question16

Let $f,g:(0,\infty)\longrightarrow R$ be two functions defined by $f(x)=\int_{-x}^{x}(|t|-t^2)e^{-t^2}dt$ and $g(x)=\int_0^{x^2}t^{1/2}e^{-t}dt$. Then the value of $(f(\sqrt{\log_e 9})+g(\sqrt{\log_e 9}))$ is equal to

[31-Jan

Options:

A.

6

B.

9

C.

8

D.

10

Answer: C

Solution:

$$f(x)=\int_{-x}^{x}(|t|-t^2)e^{-t^2}\,dt$$

$$\Rightarrow f'(x)=2\cdot(|x|-x^2)e^{-x^2}\ldots\ldots\ldots\ldots(1)$$

$$g(x)=\int_{0}^{x^2}t^{\frac{1}{2}}e^{-t}\,dt$$

$$g'(x)=xe^{-x^2}(2x)-0$$

$$f'(x)+g'(x)=2xe^{-x^2}-2x^2e^{-x^2}+2x^2e^{-x^2}$$

Integrating both sides w.r.t. x

$$f(x)+g(x)=\int_{0}^{\alpha}2xe^{-x^2}\,dx$$

$$x^2=t$$

$$\Rightarrow\int_{0}^{\sqrt{\alpha}}e^{-t}\,dt=[-e^{-t}]_0^{\sqrt{\alpha}}$$

$$=-e^{(\log_e(9)^{-1})+1}$$

$$\Rightarrow 9(f(x)+g(x))=\left(1-\frac{1}{9}\right)9=8$$

Question17

$\left|\frac{120}{\pi^3}\int_{0}^{\pi}\frac{x^2\sin x\cos x}{\sin^4 x+\cos^4 x}dx\right|$ is equal to___

[31-Jan-2024 Shift 2]

Answer: 15

Solution:

$$\int_0^{\pi} \frac{x^2 \sin x \cdot \cos x}{\sin^4 x+\cos^4 x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x \cdot \cos x}{\sin^4 x+\cos^4 x}(x^2-(\pi-x)^2) dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x \cdot \cos x(2\pi x-\pi^2)}{\sin^4 x+\cos^4 x}$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x+\cos 4x} dx - \pi^2 \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x+\cos 4x} dx$$

$$= 2\pi \cdot \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos^4 x}{\sin^4 x+\cos^4 x} dx - \pi^2 \int_0^{\frac{\pi}{2}} \frac{\sin x \cos^4 x}{\sin^4 x+\cos^4 x} dx$$

$$= -\frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x+\cos^4 x} dx$$

$$= -\frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x \, dx}{1-2\sin^2 x \times \cos^2 x}$$

$$= -\frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2-\sin^2 2x} dx$$

$$= -\frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+\cos^2 2x} dx$$

Let $\cos 2x = t$

Question18

The value of the integral $\int_0^{\frac{\pi}{4}} \frac{x \, dx}{\sin^4(2x)+\cos^4(2x)}$ equals :

[1-Feb-2024 Shift 1]

Options:

A.

$$\frac{\sqrt{2}\pi^2}{8}$$

B.

$$\frac{\sqrt{2}\pi^2}{16}$$

C.

$$\frac{\sqrt{2}\pi^2}{32}$$

D.

$$\frac{\sqrt{2}\pi^2}{64}$$

Answer: C

Solution:

$$\int_0^{\frac{\pi}{4}} \frac{x\,dx}{\sin^4(2x)+\cos^4(2x)}$$

Let $2x = t$ then $dx = \frac{1}{2}dt$

$$I = \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{t\,dt}{\sin^4 t+\cos^4 t}$$

$$I = \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-t\right)dt}{\sin^4\left(\frac{\pi}{2}-t\right)+\cos^4\left(\frac{\pi}{2}-t\right)}$$

$$I = \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}dt}{\sin^4 t+\cos^4 t} - I$$

$$2I = \frac{\pi}{8}\int_0^{\frac{\pi}{2}} \frac{dt}{\sin^4 t+\cos^4 t}$$

$$2I = \frac{\pi}{8}\int_0^{\frac{\pi}{2}} \frac{\sec^4 t\,dt}{\tan^4 t+1}$$

Let $\tan = y$ then $\sec^2 t dt = dy$

$$2I = \frac{\pi}{8}\int_0^{\infty} \frac{(1+y^2)dy}{1+y^4}$$

$$= \frac{\pi}{16}\int_0^{\infty} \frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}dy$$

Put $y-\frac{1}{y} = p$

$$I = \frac{\pi}{16}\int_{-\infty}^{\infty} \frac{dp}{p^2+(\sqrt{2})^2}$$

$$= \frac{\pi}{16\sqrt{2}}\left[\tan^{-1}\left(\frac{p}{\sqrt{2}}\right)\right]_{-\infty}^{\infty}$$

$$I = \frac{\pi^2}{16\sqrt{2}}$$

Question19

If $\int_{-\pi/2}^{\pi/2} \frac{8\sqrt{2}\cos x\,dx}{(1+e^{\sin x})(1+\sin^4 x)} = \alpha\pi+\beta\log_e(3+2\sqrt{2})$, where α, β are integers, then $\alpha^2+\beta^2$ equals

[1-Feb-2024 Shift 1]

Answer: None

Solution:

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{8\sqrt{2}\cos x}{(1+e^{\sin x})(1+\sin^4 x)}\,dx \text{ ....(1)}$$

Apply king

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{8\sqrt{2}\cos x(e^{\sin x})}{(1+e^{\sin x})(1+\sin^4 x)}\,dx \text{ ....(2)}$$

adding (1) & (2)

$$2I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{8\sqrt{2}\cos x}{1+\sin^4 x}\,dx$$

$$I=\int_{0}^{\frac{\pi}{2}}\frac{8\sqrt{2}\cos x}{1+\sin^4 x}\,dx,$$

$\sin x=t$

$$I=\int_0^1\frac{8\sqrt{2}}{1+t^4}\,dx$$

$$I=4\sqrt{2}\int_0^1\left(\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}-\frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\right)dt$$

$$I=4\sqrt{2}\int_0^1\frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}-\frac{\left(1-\frac{1}{t^2}\right)}{\left(t+\frac{1}{t}\right)^2-2}\,dt$$

Let $t-\frac{1}{t}=z\&t+\frac{1}{t}=k$

$$=4\sqrt{2}\left[\int_{-\infty}^{0}\frac{dz}{z^2+2}-\int_{\infty}^{2}\frac{dk}{k^2-2}\right]$$

$$=4\sqrt{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}\right]_{-\infty}^{0}-\left[\frac{1}{2\sqrt{2}}\ln\left(\frac{k-\sqrt{2}}{k+\sqrt{2}}\right)\right]_{\infty}^{2}$$

$$=4\sqrt{2}\left[\frac{\pi}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}\left[\ln\frac{2-\sqrt{2}}{2+\sqrt{2}}\right]\right]$$

$$=2\pi+2\ln(3+2\sqrt{2})$$

$\alpha=2$

$\beta=2$

Question20

The value of $\int_0^1(2x^3-3x^2-x+1)^{\frac{1}{3}}dx$ is equal to:

[1-Feb-2024 Shift 2]

Options:

A.

0

B.

1

C.

2

D.

-1

Answer: A

Solution:

$$I = \int_0^1 (2x^3 - 3x^2 - x + 1)^{\frac{1}{3}} \, dx$$

Using $\int_0^{2a} f(x)\,dx$ where $f(2a - x) = -f(x)$

Here $f(1 - x) = f(x)$

$\therefore$ I = 0

Question21

If $\int_0^{\frac{\pi}{3}} \cos^4 x \, dx = a\pi + b\sqrt{3}$, where a and b are rational numbers, then $9a + 8b$ is equal to :

[1-Feb-2024 Shift 2]

Options:

A.

2

B.

1

C.

3

D.

3/2

Answer: A

Solution:

$$\int_0^{\pi/3} \cos^4 x \, dx$$

$$= \int_0^{\pi/3} \left(\frac{1+\cos 2x}{2} \right)^2 dx$$

$$= \frac{1}{4} \int_0^{\pi/3} (1 + 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4} \left[\int_0^{\pi/3} dx + 2 \int_0^{\pi/3} \cos 2x \, dx + \int_0^{\pi/3} \frac{1+\cos 4x}{2} dx \right]$$

$$= \frac{1}{4} \left[\frac{\pi}{3} + (\sin 2x)_0^{\pi/3} + \frac{1}{2} \left(\frac{\pi}{3} \right) + \frac{1}{8} (\sin 4x)_0^{\pi/3} \right]$$

$$= \frac{1}{4} \left[\frac{\pi}{3} + (\sin 2x)_0^{\pi/3} + \frac{1}{2} \left(\frac{\pi}{3} \right) + \frac{1}{8} (\sin 4x)_0^{\pi/3} \right]$$

$$= \frac{1}{4} \left[\frac{\pi}{2} + \frac{\sqrt{3}}{2} + \frac{1}{8} \times \left(-\frac{\sqrt{3}}{2} \right) \right]$$

$$= \frac{\pi}{2} + \frac{7\sqrt{3}}{64}$$

$$\therefore a = \frac{1}{8}; b = \frac{7}{64}$$

$$\therefore 9a + 8b = \frac{9}{8} + \frac{7}{8} = 2$$

Question22

Let $f : (0, \infty) \longrightarrow R$ and $F(x) = \int_0^x t f(t) \, dt$. If $F(x^2) = x^4 + x^5$, then $\sum_{r=1}^{12} f(r^2)$ is equal to :

[1-Feb-2024 Shift 2]

Answer: 219

Solution:

$F(x) = \int_0^x t \cdot f(t)\,dt$

$F^1(x) = xf(x)$

Given $F(x^2) = x^4 + x^5$, let $x^2 = t$

$F(t) = t^2 + t^{5/2}$

$F'(t) = 2t + 5/2t^{3/2}$

$t \cdot f(t) = 2t + 5/2t^{3/2}$

$f(t) = 2 + 5/2r^{1/2}$

$\sum_{r=1}^{12} f(r^2) = \sum_{r=1}^{12} 2 + \frac{5}{2}r$

$= 24 + 5/2\left[\frac{12(13)}{2}\right]$

$= 219$

Question23

The value of $\frac{8}{\pi}\int_0^{\frac{\pi}{2}} \frac{(\cos x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}}\,dx$ is____
[24-Jan-2023 Shift 1]

Answer: 2

Solution:

$$I = \frac{8}{\pi}\int_0^{\frac{\pi}{2}} \frac{(\cos x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}}\,dx \ldots\ldots (1)$$

Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \frac{8}{\pi}\int_0^{\frac{\pi}{2}} \frac{(\sin x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}}\,dx \ldots\ldots (2)$$

Adding (1)&(2)

$$II = \frac{8}{\pi}\int_0^{\frac{\pi}{2}} 1\,dx$$

Question24

$\int_{\frac{3\sqrt{2}}{4}}^{\frac{3\sqrt{3}}{4}} \frac{48}{\sqrt{9-4x^2}}\,dx$ is equal to
[24-Jan-2023 Shift 2]

Options:

A. $\frac{\pi}{3}$

B. $\frac{\pi}{2}$

C. $\frac{\pi}{6}$

D. 2π

Answer: D

Solution:

Solution:

$$\int_{\frac{3\sqrt{2}}{4}}^{\frac{3\sqrt{3}}{4}} \frac{48}{\sqrt{9-4x^2}}\,dx$$

We have $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

Hence $\int_{\frac{3\sqrt{2}}{4}}^{\frac{3\sqrt{3}}{4}} \frac{48}{\sqrt{9-4x^2}}\,dx = \frac{48}{2} \times \left[\sin^{-1}\frac{2x}{3}\right]_{\frac{3\sqrt{2}}{4}}^{\frac{3\sqrt{3}}{4}}$

$= 24 \times \left[\sin^{-1}\left(\frac{2}{3} \times \frac{3\sqrt{3}}{4}\right) - \sin^{-1}\left(\frac{2}{3} \times \frac{3\sqrt{2}}{4}\right)\right]$

$= 24 \times \left[\sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{\sqrt{2}}\right]$

$= 24 \times \left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= 24 \times \frac{\pi}{12} = 2\pi$

Question25

Let f be a differentiable function defined on $\left[0, \frac{\pi}{2}\right]$ such that f(x) > 0 and

$f(x) + \int_0^x f(t)\sqrt{1-(\log_e f(t))^2}\,dt = e,\ \forall x \in \left[0, \frac{\pi}{2}\right]$.

Then $\left(6\log_e f\left(\frac{\pi}{6}\right)\right)^2$ is equal to___

[24-Jan-2023 Shift 2]

Answer: 27

Solution:

$$f(x) + \int_0^x f(t)\sqrt{1-(\log_e f(t))^2}\,dt = e$$

$\Rightarrow f(0) = e$

$f'(x) + f(x)\sqrt{1-(\ln f(x))^2} = 0$

$f(x) = y$

$\frac{dy}{dx} = -y\sqrt{1-(\ln y)^2}$

$\int \frac{dy}{y\sqrt{1-(\ln y)^2}} = -\int dx$

Put $\ln y = t$

$\int \frac{dt}{\sqrt{1-t^2}} = -x + C$

$\sin^{-1} t = -x + C \Rightarrow \sin^{-1}(\ln y) = -x + C$

$\sin^{-1}(\ln f(x)) = -x + C$

$f(0) = e$

$\Rightarrow \frac{\pi}{2} = C$

$\Rightarrow \sin^{-1}(\ln f(x)) = -x + \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\left(\ln f\left(\frac{\pi}{6}\right)\right) = \frac{-\pi}{6} + \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\left(\ln f\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{3}$

$\Rightarrow \ln f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, we need $\left(6 \times \frac{\sqrt{3}}{2}\right)^2 = 27.$

Question26

The minimum value of the function $f(x) = \int_0^2 e^{|x-t|} dt$ is is
[25-Jan-2023 Shift 1]

Options:

A. $2(e-1)$

B. $2e-1$

C. 2

D. $e(e-1)$

Answer: A

Solution:

Solution:
For $x \leq 0$
$f(x) = \int_0^2 e^{t-x} dt = e^{-x}(e^2 - 1)$
For $0 < x < 2$
$f(x) = \int_0^x e^{x-t} dt + \int_x^2 e^{t-x} dt = e^x + e^{2-x} - 2$
For $x \geq 2$
$f(x) = \int_0^2 e^{x-t} dt = e^{x-2}(e^2 - 1)$
For $x \leq 0$, $f(x)$ is ↓ and $x \geq 2$, $f(x)$ is ↑
∴ Minimum value of $f(x)$ lies in $x \in (0, 2)$
Applying A.M ≥ G.M
minimum value of $f(x)$ is $2(e-1)$

Question27

The value of
$$\lim_{n \to \infty} \frac{1+2-3+4+5-6+\ldots+(3n-2)+(3n-1)-3n}{\sqrt{2n^4+4n+3} - \sqrt{n^4+5n+4}}$$
is :
[25-Jan-2023 Shift 1]

Options:

A. $\frac{\sqrt{2}+1}{2}$

B. $3(\sqrt{2}+1)$

C. $\frac{3}{2}(\sqrt{2}+1)$

D. $\frac{3}{2\sqrt{2}}$

Answer: C

Solution:

Solution:

$$\lim_{n\to\infty} \frac{0+3+6+9+\dots n \text{ terms}}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}}$$

$$\lim_{n\to\infty} \frac{3n(n-1)}{2\left(\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}\right)}$$

$$= \frac{3}{2(\sqrt{2}-1)} = \frac{3}{2}(\sqrt{2}+1)$$

Question28

The integral $16\int_1^2 \frac{dx}{x^3(x^2+2)^2}$ is equal to [25-Jan-2023 Shift 2]

Options:

A. $\frac{11}{6}+\log_e 4$

B. $\frac{11}{12}+\log_e 4$

C. $\frac{11}{12}-\log_e 4$

D. $\frac{11}{6}-\log_e 4$

Answer: D

Solution:

Solution:

$$I = 16\int_1^2 \frac{dx}{x^3(x^2+2)^2}$$

$$= 16\int_1^2 \frac{dx}{x^3x^4\left(1+\frac{2}{x^2}\right)^2}$$

Let, $1+\frac{2}{x^2}=t \Rightarrow \frac{-4}{x^3}dx = dt$

$$I = -4\int_3^{\frac{3}{2}} \frac{dt}{\left(\frac{2}{t-1}\right)^2 t^2}$$

$$I = -4\int_3^{\frac{3}{2}} \left(\frac{t-1}{2}\right)^2 \frac{dt}{t^2}$$

$$I = -\frac{4}{4}\int_3^{\frac{3}{2}} \left(1-\frac{2}{t}+\frac{1}{t^2}\right) dt$$

$$I = -1\left[t-2\ln|t|-\frac{1}{t}\right]_3^{\frac{3}{2}}$$

$$I = -1\left[\left(\frac{3}{2}-2\ell\ln\frac{3}{2}-\frac{2}{3}\right)-\left(3-2\ell\ln 3-\frac{1}{3}\right)\right]$$

$I = -1\left[2\ell n2 - \frac{11}{6}\right]$

$I = \frac{11}{6} - \ell n4$

Question29

If $\int_{\frac{1}{3}}^{3} \left| \log_e x \right| dx = \frac{m}{n} \log_e \left(\frac{n^2}{e} \right)$, where m and n are coprime natural numbers, then $m^2 + n^2 - 5$ is equal to ________.

[25-Jan-2023 Shift 2]

Answer: 20

Solution:

$$\int_{\frac{1}{3}}^{3} |\ell nx| dx = \int_{\frac{1}{3}}^{1} (-\ell nx) dx + \int_{1}^{3} (\ell nx) dx$$

$$= -[x\ell nx - x]_{1/3}^{1} + [x\ell nx - x]_{1}^{3}$$

$$= -\left[-1 - \left(\frac{1}{3}\ell n \frac{1}{3} - \frac{1}{3} \right) \right] + [3\ell n3 - 3 - (-1)]$$

$$= \left[-\frac{2}{3} - \frac{1}{3}\ell n \frac{1}{3} \right] + [3\ell \ln 3 - 2]$$

$$= -\frac{4}{3} + \frac{8}{3}\ell nn 3$$

$$= \frac{4}{3}(2\ell n3 - 1)$$

$$= \frac{4}{3}\left(\ell n \frac{9}{e} \right)$$

$\therefore m = 4, n = 3$

Now, $m^2 + n^2 - 5 = 16 + 9 - 5 = 20$

Question30

Let [x] denote the greatest integer ≤x. Consider the function $f(x) = \max\{x^2, 1 + [x]\}$. Then the value of the integral $\int_0^2 f(x)\,dx$ is:

[29-Jan-2023 Shift 1]

Options:

A. $\frac{5 + 4\sqrt{2}}{3}$

B. $\frac{8 + 4\sqrt{2}}{3}$

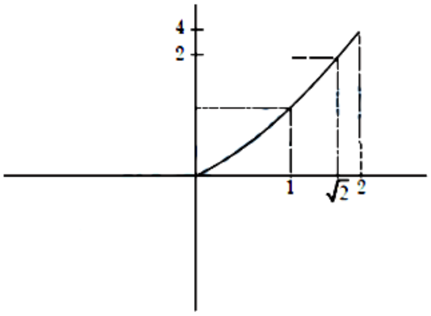
C. $\frac{1 + 5\sqrt{2}}{3}$

D. $\frac{4 + 5\sqrt{2}}{3}$

Answer: A

Solution:

Solution:



$$A = \int_0^1 1 \cdot dx + \int_1^{\sqrt{2}} 2\, dx + \int_{\sqrt{2}}^{2} x^2\, dx$$
$$= 1 + 2\sqrt{2} - 2 + \frac{8}{3} - \frac{2\sqrt{2}}{3}$$
$$= \frac{5}{3} + \frac{4\sqrt{2}}{3}$$

Question31

Let $f(x) = x + \frac{a}{\pi^2 - 4}\sin x + \frac{b}{\pi^2 - 4}\cos x, x \in \mathbb{R}$ be a function which satisfies $f(x) = x + \int_0^{\pi/2} \sin(x + y) f(y) dy$. Then $(a + b)$ is equal to
[29-Jan-2023 Shift 1]

Options:

A. $-\pi(\pi + 2)$

B. $-2\pi(\pi + 2)$

C. $-2\pi(\pi - 2)$

D. $-\pi(\pi - 2)$

Answer: B

Solution:

Solution:

$f(x) = x + \int_0^{\pi/2} (\sin x \cos y + \cos x \sin y) f(y) dy$

$f(x) = x + \int_0^{\pi/2} ((\cos y f(y) d\ y) \sin x + (\sin y f(y) d y) \cos x)$

On comparing with

$f(x) = x + \frac{a}{\pi^2 - 4}\sin x + \frac{b}{\pi^2 - 4}\cos x, x \in \mathbb{R}$ then

$\Rightarrow \frac{a}{\pi^2 - 4} = \int_0^{\pi/2} \cos y f(y) d\ y$ . . . (2)

$\Rightarrow \frac{b}{\pi^2 - 4} = \int_0^{\pi/2} \sin y f(y) d y$ . . . (3)

Add (2) and (3)

$\frac{a + b}{\pi^2 - 4} = \int_0^{\pi/2} (\sin y + \cos y) f(y) d y$ . . . (4)

$\frac{a + b}{\pi^2 - 4} = \int_0^{\pi/2} (\sin y + \cos y) f\left(\frac{\pi}{2} - y\right) d y$ . . . (5)

Add (4) and (5)

$\frac{2(a + b)}{\pi^2 - 4} = \int_0^{x/2} (\sin y + \cos y)\left(\frac{\pi}{2} + \frac{(a + b)}{\pi^2 - 4}(\sin y + \cos y)\right) d y$

$= \pi + \frac{a + b}{\pi^2 - 4}\left(\frac{\pi}{2} + 1\right)$

$(a+b)=-2\pi(\pi+2)$

Question32

The value of the integral $\int_{1/2}^{2} \frac{\tan^{-1}x}{x} dx$ is equal to [29-Jan-2023 Shift 2]

Options:

A. $\pi \log_e 2$

B. $\frac{1}{2}\log_e 2$

C. $\frac{\pi}{4}\log_e 2$

D. $\frac{\pi}{2}\log_e 2$

Answer: D

Solution:

Solution:

$I = \int_{1/2}^{2} \frac{\tan^{-1}x}{x} dx \ldots$ (i)

Put $x = \frac{1}{t}$ $dx = -\frac{1}{t^2}dt$

$I = -\int_{2}^{1/2} \frac{\tan^{-1}\frac{1}{t}}{\frac{1}{t}} \cdot \frac{1}{t^2} dt = -\int_{2}^{1/2} \frac{\tan^{-1}\frac{1}{t}}{t} dt$

$I = \int_{1/2}^{2} \frac{\cot^{-1}t}{t} dt = \int_{1/2}^{2} \frac{\cot^{-1}x}{x} dx \ldots$ (ii)

Add both equation

$2I = \int_{1/2}^{2} \frac{\tan^{-1}x + \cot^{-1}x}{x} dx = \frac{\pi}{2}\int_{1/2}^{2} \frac{dx}{x} = \frac{\pi}{2}(\ell n2)_{1/2}{}^{2}$

$= \frac{\pi}{2}\left(\ln 2 - \ell \ln \frac{1}{2}\right) = \pi \ell n2$

$I = \frac{\pi}{2}\ell \ln 2$

Question33

The value of the integral $\int_{1}^{2}\left(\frac{t^4+1}{t^6+1}\right) dt$ is :

[29-Jan-2023 Shift 2]

Options:

A. $\tan^{-1}\frac{1}{2} + \frac{1}{3}\tan^{-1}8 - \frac{\pi}{3}$

B. $\tan^{-1}2 - \frac{1}{3}\tan^{-1}8 + \frac{\pi}{3}$

C. $\tan^{-1}2 + \frac{1}{3}\tan^{-1}8 - \frac{\pi}{3}$

D. $\tan^{-1}\frac{1}{2} - \frac{1}{3}\tan^{-1}8 + \frac{\pi}{3}$

Answer: C

Solution:

$$I = \int_1^2 \left(\frac{t^4+1}{t^6+1} \right) dt$$
$$= \int_1^2 \frac{(t^4+1-t^2)+t^2}{(t^2+1)(t^4-t^2+1)} dt$$
$$= \int_1^2 \left(\frac{1}{t^2+1} + \frac{t^2}{t^6+1} \right) dt$$
$$= \int_1^2 \left(\frac{1}{t^2+1} + \frac{1}{3} \frac{3t^2}{(t^3)^2+1} \right) dt$$
$$= \tan^{-1}(t) + \frac{1}{3}\tan^{-1}(t^3)\Big|_1^2$$
$$= (\tan^{-1}(2) - \tan^{-1}(1)) + \frac{1}{3}(\tan^{-1}(2^3) - \tan^{-1}(1^3))$$
$$= \tan^{-1}(2) + \frac{1}{3}\tan^{-1}(8) - \frac{\pi}{3}$$

Question34

If [t denotes the greatest integer ≤1, then the value of of $\frac{3(e-1)^2}{e}\int_1^2 x^2 e^{[x]+[x^3]} dx$ is :
[30-Jan-2023 Shift 1]

Options:

A. $e^9 - e$

B. $e^8 - e$

C. $e^7 - 1$

D. $e^8 - 1$

Answer: B

Solution:

$$\int_1^2 x^2 e^{[x^3]+1} dx$$
$$x^3 = t$$
$$3x^2\, dx = dt$$
$$= \frac{e}{3}\int_1^8 e^{[t]} dt$$
$$= \frac{e}{3}\left\{ \int_1^2 e\, dt + \int_2^3 e^2\, dt + \ldots\ldots\ldots + \int_7^8 e^7\, dt \right\}$$
$$= \frac{e}{3}(e + e^2 + \ldots\ldots\ldots + e^7)$$
$$= \frac{e^2}{3}(1 + e + \ldots\ldots\ldots + e^6) = \frac{e^2}{3}\frac{(e^7-1)}{(e-1)}$$
$$\frac{3(e-1)}{e}\int_1^2 x^2 \times e^{[x]+[x^3]} dx = \frac{3}{e}(e-1) \times \frac{e^2}{3}\frac{(e^7-1)}{(e-1)}$$
$$= e(e^7-1)$$
$$= e^8 - e$$

Question35

$\lim_{x \to 0} \frac{48}{x^4} \int_0^x \frac{t^3}{t^6+1}$ **dt is equal to _______.**

[30-Jan-2023 Shift 1]

Answer: 12

Solution:

$$48 \lim_{x \to 0} \frac{\int_0^x \frac{t^3}{t^6+1} dt}{x^4} \left(\frac{0}{0}\right)$$

Applying L'Hospitals Rule

$$48 \lim_{x \to 0} \frac{x^3}{x^6+1} \times \frac{1}{4x^3}$$

$$= 12$$

Question36

$\lim_{n \to \infty} \frac{3}{n} \left\{ \mathbf{4} + \left(\mathbf{2} + \frac{1}{n}\right)^{\mathbf{2}} + \left(\mathbf{2} + \frac{2}{n}\right)^{\mathbf{2}} + \ldots + \left(\mathbf{3} - \frac{1}{n}\right)^{\mathbf{2}} \right\}$

is equal to

[30-Jan-2023 Shift 2]

Options:

A. 12

B. $\frac{19}{3}$

C. 0

D. 19

Answer: D

Solution:

$$\lim_{n \to \infty} \frac{3}{n} \sum_{r=0}^{n-1} \left(2 + \frac{r}{n}\right)^2$$

$$= 3 \int_0^1 (2+x)^2 dx = 27 - 8 = 19$$

Question37

The value of $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(2+3\sin x)}{\sin x(1+\cos x)}$ **dx is equal to**

[31-Jan-2023 Shift 1]

Options:

A. $\frac{7}{2} - \sqrt{3} - \log_e \sqrt{3}$

B. $-2+3\sqrt{3}+\log_e\sqrt{3}$

C. $\frac{10}{3}-\sqrt{3}+\log_e\sqrt{3}$

D. $\frac{10}{3}-\sqrt{3}-\log_e\sqrt{3}$

Answer: C

Solution:

Solution:

$$\int_{\pi/3}^{\pi/2}\left(\frac{2+3\sin x}{\sin x(1+\cos x)}\right)dx = 2\int_{\pi/3}^{\pi/2}\frac{dx}{\sin x+\sin x\cos x}+3$$
$$3\int_{\pi/3}^{\pi/2}\frac{dx}{1+\cos x}$$
$$\int_{\pi/3}^{\pi/2}\frac{dx}{1+\cos x}=\int_{\pi/3}^{\pi/2}\frac{1-\cos x}{\sin^2 x}dx$$
$$=\int_{\pi/3}^{\pi/2}(\operatorname{cosec}^2 x-\cot x\operatorname{cosec} x)\,dx$$
$$=(\operatorname{cosec}x-\cot x)\int_{\pi/3}^{\pi/2}=(1)-\left(\frac{2}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right)=1-\frac{1}{\sqrt{3}}$$
$$\int_{\pi/3}^{\pi/2}\frac{dx}{\sin x(1+\cos x)}=$$
$$\int\frac{dx}{(2\tan x/2)(1+1-\tan^2 x/2)}$$
$$=\int\frac{(1+\tan^2 x/2)\sec^2 x/2\,dx}{2\tan x/22}$$
$$\tan x/2=t$$
$$\frac{1}{2}\int\left(\frac{1+t^2}{t}\right)dt=\frac{1}{2}\left[\ell nt+\frac{t^2}{2}\right]_{\frac{1}{\sqrt{3}}}^{1}$$
$$=\frac{1}{2}\left[\left(0+\frac{1}{2}\right)-\left(\ell n\frac{1}{\sqrt{3}}+\frac{1}{6}\right)\right]=\left(\frac{1}{3}+\ell n\sqrt{3}\right)\frac{1}{2}$$
$$=\left(\frac{1}{6}+\frac{1}{2}\ell n\sqrt{3}\right)dt$$
$$2\left(\frac{1}{6}+\frac{1}{2}\ell n\sqrt{3}\right)+3\left(1-\frac{1}{\sqrt{3}}\right)$$
$$=\frac{1}{3}+\ell n\sqrt{3}+3-\sqrt{3}=\frac{10}{3}+\ell n\sqrt{3}-\sqrt{3}$$

Question38

Let a differentiable function f satisfy $f(x)+\int_3^x \frac{f(t)}{t}dt=\sqrt{x+1}$, $x\geq 3$. Then $12f(8)$ is equal to: [31-Jan-2023 Shift 1]

Options:

A. 34

B. 19

C. 17

D. 1

Answer: C

Solution:

Differentiate w.r.t. x
$$f'(x)+\frac{f(x)}{x}=\frac{1}{2\sqrt{x+1}}$$

$$\text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$xf(x) = \int \frac{x}{2\sqrt{x+1}} dx$$

$$x + 1 = t^2$$

$$= \int \frac{t^2 - 1}{2t} 2t\, dt$$

$$xf(x) = \frac{t^3}{3} - t + c$$

$$xf(x) = \frac{(x+1)^{3/2}}{3} - \sqrt{x+1} + c$$

Also putting $x = 3$ in given equation $f(3) + 0 = \sqrt{4}\ f(3) = 2$

$$\Rightarrow C = 8 - \frac{8}{3} = \frac{16}{3}$$

$$f(x) = \frac{\frac{(x+1)^{3/2}}{3} - \sqrt{x+1} + \frac{16}{3}}{x}$$

$$f(8) = \frac{9 - 3 + \frac{16}{3}}{8} = \frac{34}{24}$$

$$\Rightarrow 12f(8) = 17$$

Question39

Let $\alpha \in (0, 1)$ and $\beta = \log_e(1 - \alpha)$. Let $P_n(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots.. + \frac{x^n}{n}$, $x \in (0, 1)$. Then the integral $\int_0^\alpha \frac{t^{50}}{1-t} dt$ is equal to
[31-Jan-2023 Shift 1]

Options:

A. $\beta - P_{50}(\alpha)$

B. $-(\beta + P_{50}(\alpha))$

C. $P_{50}(\alpha) - \beta$

D. $\beta + P_{50}(\alpha)$

Answer: B

Solution:

$$\int_0^\alpha \frac{t^{50} - 1 + 1}{1 - t} = -\int_0^\alpha (1 + t + \ldots.. + t^{49}) + \int_0^\alpha \frac{1}{1-t} dt$$

$$= -\left(\frac{\alpha^{50}}{50} + \frac{\alpha^{49}}{49} + \ldots.. + \frac{\alpha^1}{1} \right) + \left(\frac{\ln(1-f)}{-1} \right)_0^\alpha$$

$$= -P_{50}(\alpha) - \ln(1 - \alpha)$$

$$= -P_{50}(\alpha) - \beta$$

Question40

Let $\alpha > 0$. If $\int_0^\alpha \frac{x}{\sqrt{x+\alpha} - \sqrt{x}} dx = \frac{16 + 20\sqrt{2}}{15}$, then α is equal to :
[31-Jan-2023 Shift 2]

Options:

A. 2

B. 4

C. $\sqrt{2}$

D. $2\sqrt{2}$

Answer: A

Solution:

Solution:
After rationalising

$$\int_0^{\alpha} \frac{x}{\alpha}(\sqrt{x+\alpha}+\sqrt{x})$$

$$\frac{1}{\alpha}\left[\frac{2}{5}(x+\alpha)^{5/2}-\alpha\frac{2}{3}(x+\alpha)^{3/2}+\frac{2}{5}x^{5/2}\right]\Big|_0^{\alpha}$$

$$=\frac{1}{\alpha}\left(\frac{5}{2}(2\alpha)^{5/2}-\frac{2\alpha}{3}(2\alpha)^{3/2}+\frac{2}{5}\alpha^{5/2}-\frac{2}{5}\alpha^{5/2}+\frac{2}{3}\alpha^{5/2}\right)$$

$$=\frac{1}{\alpha}\left(\frac{2^{7/2}\alpha^{5/2}}{5}\ \frac{2^{5/2}\alpha^{5/2}}{3}+\frac{2}{3}\alpha^{5/2}\right)$$

$$=\alpha^{3/2}\left(\frac{2^{7/2}}{5}-\frac{2^{5/2}}{3}+\frac{2}{3}\right)$$

$$=\frac{\alpha^{3/2}}{15}(24\sqrt{2}-20\sqrt{2}+10)=\frac{\alpha^{3/2}}{15}(4\sqrt{2}+10)$$

Now,

$$\frac{\alpha^{3/2}}{15}(4\sqrt{2}+10)=\frac{16+20\sqrt{2}}{15}$$

$$\Rightarrow \alpha=2$$

Question41

If $\varphi(x)=\frac{1}{\sqrt{x}}\int_{\frac{\pi}{4}}^{x}(4\sqrt{2}\sin t-3\varphi'(t))\,dt,\ x>0$ then $\varphi'\left(\frac{\pi}{4}\right)$ is equal to :

[31-Jan-2023 Shift 2]

Options:

A. $\frac{8}{\sqrt{\pi}}$

B. $\frac{4}{6+\sqrt{\pi}}$

C. $\frac{8}{6+\sqrt{\pi}}$

D. $\frac{4}{6-\sqrt{\pi}}$

Answer: C

Solution:

Solution:

$$\varphi'(x)=\frac{1}{\sqrt{x}}[(4\sqrt{2}\sin x-3\varphi'(x))\cdot 1-0]-\frac{1}{2}x^{-3/2}$$

$$\int_{\frac{\pi}{4}}^{x}(4\sqrt{2}\sin t-3\varphi'(t))\,dt$$

$$\varphi'\left(\frac{\pi}{4}\right)=\frac{2}{\sqrt{\pi}}\left[4-3\varphi'\left(\frac{\pi}{4}\right)\right]+0$$

$$\left(1+\frac{6}{\sqrt{\pi}}\right)\varphi'\left(\frac{\pi}{4}\right)=\frac{8}{\sqrt{\pi}}$$

$$\varphi'\left(\frac{\pi}{4}\right)=\frac{8}{\sqrt{\pi}+6}$$

Question42

Let $f(x) = \int \frac{2x}{(x^2+1)(x^2+3)} dx$.
If $f(3) = \frac{1}{2}(\log_e 5 - \log_e 6)$, then $f(4)$ is equal to
[25-Jan-2023 Shift 1]

Options:

A. $1/2(\log_e 17 - \log_e 19)$

B. $\log_e 17 - \log_e 18$

C. $\frac{1}{2}(\log_e 19 - \log_e 17)$

D. $\log_e 19 - \log_e 20$

Answer: A

Solution:

Solution:
Put $x^2 = t$

$$\int \frac{dt}{(t+1)(t+3)} = \frac{1}{2}\int\left(\frac{1}{t+1} - \frac{1}{t+3}\right)dt$$

$$f(x) = \frac{1}{2}\ln\left(\frac{x^2+1}{x^2+3}\right) + C$$

$$f(3) = \frac{1}{2}(\ln 10 - \ln 12) + C$$

$$\Rightarrow C = 0$$

$$f(4) = \frac{1}{2}\ln\left(\frac{17}{19}\right)$$

Question43

If $\int \sqrt{\sec 2x - 1}\, dx = \alpha \log_e \left| \cos 2x + \beta + \sqrt{\cos 2x\left(1 + \cos \frac{1}{\beta}x\right)} \right|$

+ constant, then $\beta - \alpha$ is equal to _______.
[30-Jan-2023 Shift 2]

Answer: 1

Solution:

$$\int \sqrt{\sec 2x - 1}\, dx = \int \sqrt{\frac{1 - \cos 2x}{\cos 2x}}\, dx$$

$$= \sqrt{2}\int \frac{\sin x}{\sqrt{2\cos^2 x - 1}}\, dx$$

put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$= -\sqrt{2}\int \frac{dt}{\sqrt{2t^2 - 1}}$$

$$= -\ln|\sqrt{2}\cos x + \sqrt{\cos 2x}| + c$$

$$= -\frac{1}{2}\ln\left|2\cos^2 x + \cos 2x + 2\sqrt{\cos 2x}\cdot\sqrt{2}\cos x\right| + c$$

$$= -\frac{1}{2}\ln\left|\cos^{2x} + \frac{1}{2} + \sqrt{\cos 2x}\cdot\sqrt{1+\cos 2x}\right| + c$$
$$\because \beta = \frac{1}{2}, \alpha = -\frac{1}{2} \Rightarrow \beta - \alpha = 1$$

Question44

$\lim_{n\to\infty}\left(\frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \ldots + \frac{1}{2n}\right)$ **is equal to :-**
[1-Feb-2023 Shift 1]

Options:

A. 0

B. $\log_e 2$

C. $\log_e\left(\frac{3}{2}\right)$

D. $\log_e\left(\frac{2}{3}\right)$

Answer: B

Solution:

$$\lim_{n\to\infty}\left(\frac{1}{1+n} + \ldots + \frac{1}{n+n}\right) = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}$$
$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left(\frac{1}{1+\frac{r}{n}}\right)$$
$$= \int_0^1 \frac{1}{1+x}dx = [\ln(1+x)]_0^{\,1} = \ln 2.$$

Question45

If $\int_0^1 (x^{21}+x^{14}+x^7)(2x^{14}+3x^7+6)^{1/7}dx = \frac{1}{l}(11)^{m/n}$
where 1, m, n $\in \mathbb{N}$**, m and n are coprime then** $l+m+n$ **is equal to ________.**
[1-Feb-2023 Shift 1]

Answer: 63

Solution:

$$\int (x^{20}+x^{13}+x^6)(2x^{21}+3x^{14}+6x^7)^{1/7}dx$$
$$2x^{21}+3x^{14}+6x^7 = t$$
$$42(x^{20}+x^{13}+x^6)\,dx = dt$$
$$\frac{1}{42}\int_0^{11} t^{\frac{1}{7}}dt = \left(\frac{t^{\frac{8}{7}}}{\frac{8}{7}}\times\frac{1}{42}\right)_0^{11}$$
$$= \frac{1}{48}\left(t^{\frac{8}{7}}\right)_0^{11} = \frac{1}{48}(11)^{8/7}$$

$l = 48, m = 8, n = 7$
$l + m + n = 63$

Question46

The value of the integral $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx$ is:

[1-Feb-2023 Shift 2]

Options:

A. $\frac{\pi^2}{6}$

B. $\frac{\pi^2}{12\sqrt{3}}$

C. $\frac{\pi^2}{3\sqrt{3}}$

D. $\frac{\pi^2}{6\sqrt{3}}$

Answer: D

Solution:

Solution:

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx \ldots (1)$$

$x \to -x$

$$I = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{-x + \frac{\pi}{4}}{2 - \cos 2x} dx \ldots (2)$$

$(1) + (2)$

$$2I = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\frac{\pi}{2}}{2 - \cos 2x} dx$$

$$I = \frac{\pi}{4} \cdot 2 \int_{0}^{\frac{\pi}{4}} \frac{dx}{2 - \cos 2x} dx$$

$$I = \frac{\pi}{4} \cdot 2 \int_{0}^{\frac{\pi}{4}} \frac{(1 + \tan^2 x)\, dx}{2(1 + \tan^2 x) - (1 - \tan^2 x)}$$

$$I = \frac{\pi}{4} \int_{0}^{1} \frac{dt}{3t^2 + 1}$$

$$\Rightarrow I = \frac{\pi}{2\sqrt{3}} \tan^{-1}\sqrt{3}$$

$$I = \frac{\pi^2}{6\sqrt{3}}$$

Question47

If $\int_{0}^{\pi} \frac{5^{\cos x}(1 + \cos x \cos 3x + \cos^2 x + \cos^3 x \cos 3x)\, dx}{1 + 5^{\cos x}} = \frac{k\pi}{16}$, then k is equal to ______.

[1-Feb-2023 Shift 2]

Answer: 13

Solution:

$$I = \int_0^{\pi} \frac{5^{\cos x}(1+\cos x\cos 3x+\cos^2 x+\cos^3 x\cos 3x)}{1+5^{\cos x}}\,dx$$

$$I = \int_0^{\pi} \frac{5^{-\cos x}(1+\cos x\cos 3x+\cos^2 x+\cos^3 x\cos 3x)}{1+5^{\cos x}}\,dx$$

$$2I = \int_0^{\pi}(1+\cos x\cos 3x+\cos^2 x+\cos^3 x\cos 3x)\,dx$$

$$\text{not}\, I = \text{not}\, 2\int_0^{\frac{\pi}{2}}(1+\cos x\cos 3x+\cos^2 x+\cos^3 x\cos 3x)\,dx$$

$$I = \int_0^{\frac{\pi}{2}}(1+\sin x(-\sin 3x)+\sin^2 x-\sin^3 x\sin 3x)\,dx$$

$$2I = \int_0^{\frac{\pi}{2}}(3+\cos 4x+\cos^3 x\cos 3x-\sin^3 x\sin 3x)\,dx$$

$$2I = \int_0^{\frac{\pi}{2}} 3+\cos 4x+\left(\frac{\cos 3x+3\cos x}{4}\right)\cos 3x-\sin 3x\left(\frac{3\sin x-\sin 3x}{4}\right)dx$$

$$2I = \int_0^{\frac{\pi}{2}}\left(3+\cos 4x+\frac{1}{4}+\frac{3}{4}\cos 4x\right)dx$$

$$2I = \frac{13}{4}\times\frac{\pi}{2}+\frac{7}{4}\left(\frac{\sin 4x}{4}\right)_0^{\frac{\pi}{2}} \Rightarrow I = \frac{13\pi}{16}$$

Question48

Let f(x) be a function satisfying $f(x)+f(\pi-x)=\pi^2$, $\forall x\in\mathbb{R}$. Then $\int_0^{\pi} f(x)\sin x\,dx$ is equal to :
[6-Apr-2023 shift 2]

Options:

A. $\frac{\pi^2}{2}$

B. π^2

C. $2\pi^2$

D. $\frac{\pi^2}{4}$

Answer: B

Solution:

$$I = \int_0^{\pi} f(x)\sin x\,dx \;...\;(1)$$

Apply king property

$$I = \int_0^{\pi} f(\pi-x)\sin(\pi-x)\,dx \;...\;(1)$$

Add

$$2I = \int_0^{\pi} f(x)+f(\pi-x)\sin x\,dx$$

$2I = \int_0^{\pi} \pi^2 \sin x\, dx$

$2I = \pi^2(\text{not})$

$I = \pi^2$

Ans. Option 2

Question49

Let $f(x) = \frac{x}{(1+x^n)^{\frac{1}{n}}}$, $x \in \mathbb{R} - \{-1\}$, $n \in \mathbb{N}$, $n > 2$. If $f^n(x) = n$ (f of of ..... upto n times) (x), then $\lim_{n \to \infty} \int_0^1 x^{n-2}(f^n(x))\, dx$ is equal to ________.

[6-Apr-2023 shift 2]

Answer: 0

Solution:

Let $f(x) = \frac{x}{1+x^{n\frac{1}{n}}}$, $x \in \mathbb{R} - \{-1\}$, $n \in \mathbb{N}$, $n > 2$.

If $f^n(x) = n$ (fofof..... upto n times) (x)

then $\lim_{n \to \infty} \int_0^1 x^{n-2}(f^n(x))\, dx$

$f(f(x)) = \frac{x}{(1+2x^n)^{1/n}}$

$f(f(f(x))) = \frac{x}{(1+3x^n)^{1/n}}$

Similarly $f^n(x) = \frac{x}{(1+n \cdot x^n)^{1/n}}$

Now $\lim_{n \to \infty} \int \frac{x^{n-2} \cdot x\, dx}{(1+n \cdot x^n)^{1/n}} = \lim_{n \to \infty} \int \frac{x^{n-1} \cdot dx}{(1+n \cdot x^n)^{1/n}}$

Now $1 + nx\, x^n = t$

$n^2 \cdot x^{n-1} dx = dt$

$x^{n-1} dx = \frac{dt}{n^2}$

$\Rightarrow \lim_{n \to \infty} \frac{1}{n^2} \int_1 1 + n \frac{dt}{t^{1/n}}$

$\Rightarrow \lim_{n \to \infty} \frac{1}{n^2} \left[\frac{t^{1-\frac{1}{n}}}{1-\frac{1}{n}} \right]_1^{1+n}$

$\Rightarrow \lim_{n \to \infty} \frac{1}{n(n-1)} \left((1+n)^{\frac{n-1}{n}} - 1 \right)$ Now let $n = \frac{1}{h}$

$\Rightarrow \lim_{h \to 0} \frac{\left(1+\frac{1}{h}\right)^{1-h} - 1}{\frac{1}{h} \frac{(1-h)}{h}}$

Using series expansion

$\Rightarrow 0$

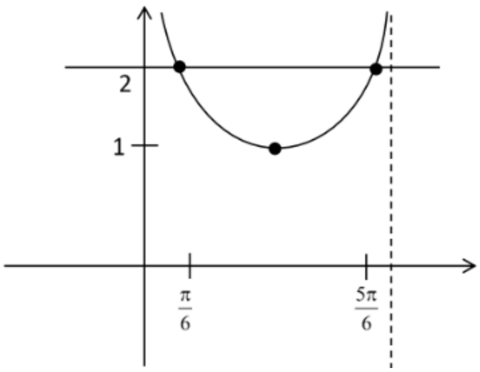
Question50

Let [t] denote the greatest integer $\le$t. The $\frac{2}{\pi} \int_{\pi/6}^{5\pi/6} (8[\operatorname{cosec} x] - 5[\cot x])\, dx$ is equal to

[8-Apr-2023 shift 1]

Answer: 14

Solution:



$$8\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}[\operatorname{cosec} x]\,dx$$

$$8\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}dx = \frac{16\pi/3}{16\pi/3}$$

$$I = \int_{\pi}^{\frac{5\pi}{6}}[\cot x]\,dx$$

$x \to \pi - x$

$$I = \int_{\frac{5\pi}{6}}^{\frac{\pi}{6}}[-\cot x]\,dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}([\cot x] + [-\cot x])\,dx$$

$$I = -\frac{1}{2}\int_{\pi/6}^{5\pi/6}dx \Rightarrow -\frac{1}{2}\left(\frac{4\pi}{6}\right)$$

$$= -\pi/3$$

$$\therefore \frac{2}{\pi}\left[\frac{16\pi}{3} + \frac{5\pi}{3}\right] = \frac{2}{\pi}\left(\frac{21\pi}{3}\right)$$

$$= 14$$

Question51

Let [t] denote the greatest integer function. If $\int_0^{2.4}[x^2]\,dx = \alpha + \beta\sqrt{2} + \gamma\sqrt{3} + \delta\sqrt{5}$, then $\alpha + \beta + \gamma + \delta$ is equal to _______.

[8-Apr-2023 shift 2]

Answer: 6

Solution:

$$\int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^{2} 3\,dx + \int_2^{\sqrt{5}} 4\,dx + \int_{\sqrt{5}}^{2\cdot 4} 5\,dx$$
$$\sqrt{2} - 1 + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3}) + 4(\sqrt{5} - 2) + 5((2\cdot 4) - \sqrt{5})$$
$$= 9 - \sqrt{2} - \sqrt{3} - \sqrt{5}$$
$$\alpha + \beta + \gamma + \delta = 9 - 1 - 1 - 1 = 6$$

Question52

Let f be a continuous function satisfying $\int_0^{t^2} (f(x) + x^2)\,dx = \frac{4}{3}t^3$, $\forall t > 0$. Then $f\left(\frac{\pi^2}{4}\right)$ is is equal to
[10-Apr-2023 shift 2]

Options:

A. $-\pi^2\left(1 + \frac{\pi^2}{16}\right)$

B. $\pi\left(1 - \frac{\pi^3}{16}\right)$

C. $-\pi\left(1 + \frac{\pi^3}{16}\right)$

D. $\pi^2\left(1 - \frac{\pi^3}{16}\right)$

Answer: B

Solution:

$$\int_0 t^2(f(x) + x^2)\,dx = \frac{4}{3}t^3, \forall t > 0$$
$$(f(t^2) + t^4) = 2t$$
$$f(t^2) = 2t - t^4$$
$$t = \frac{\pi}{2} \Rightarrow f\left(\frac{\pi^2}{4}\right) = \frac{2\pi}{2} - \frac{\pi^4}{16}$$
$$= \pi - \frac{\pi^4}{16} = \pi\left(1 - \frac{\pi^3}{16}\right)$$

Question53

The value of the integral $\int_{-\log_e 2}^{\log_e 2} e^x\left(\log_e\left(e^x + \sqrt{1 + e^{2x}}\right)\right)$ is equal to :
[11-Apr-2023 shift 1]

Options:

A. $\log_e\left(\frac{(2+\sqrt{5})^2}{\sqrt{1+\sqrt{5}}}\right) + \frac{\sqrt{5}}{2}$

B. $\log_e\left(\frac{2(2+\sqrt{5})^2}{\sqrt{1+\sqrt{5}}}\right) - \frac{\sqrt{5}}{2}$

C. $\log_e\left(\frac{\sqrt{2}(3-\sqrt{5})^2}{\sqrt{1+\sqrt{5}}}\right) + \frac{\sqrt{5}}{2}$

D. $\log_e\left(\frac{\sqrt{2}(2+\sqrt{5})^2}{\sqrt{1+\sqrt{5}}}\right) - \frac{\sqrt{5}}{2}$

Answer: D

Solution:

Solution:

$I = \int_{-\ln 2}^{\ln 2} e^x\left(\ln\left(e^x + \sqrt{1+e^{2x}}\right)\right) dx$

Put $e^x = t \Rightarrow e^x\, dx = dt$

$I = \int_{1/2}^{2} \ln\left(t + \sqrt{1+t^2}\right) dt$

Applying integration by parts.

$= \left[t\ln\left(t + \sqrt{1+t^2}\right)\right]_{\frac{1}{2}}^{2} - \int_{1/2}^{2} \frac{t}{t+\sqrt{1+t^2}}\left(1 + \frac{2t}{2\sqrt{1+t^2}}\right) dt$

$= 2\ln(2+\sqrt{5}) - \frac{1}{2}\ln\left(\frac{1+\sqrt{5}}{2}\right) - \int_{1/2}^{2} \frac{t}{\sqrt{1+t^2}} dt$

$= 2\ln(2+\sqrt{5}) - \frac{1}{2}\ln\left(\frac{1+\sqrt{5}}{2}\right) - \frac{\sqrt{5}}{2}$

$= \ln\left(\frac{(2+\sqrt{5})^2}{\left(\left(\frac{\sqrt{5}+1}{2}\right)^{\frac{1}{2}}\right.}\right) - \frac{\sqrt{5}}{2}$

Question54

For m, n > 0, let $\alpha(m, n) = \int_0^2 t^m(1+3t)^n\, dt$. If $11\alpha(10, 6) + 18\alpha(11, 5) = p(14)^6$, then p is equal to _______. [11-Apr-2023 shift 1]

Answer: 32

Solution:

Solution:

$\alpha(m, n) = \int_0^2 t^m(1+3t)^n\, dt$

If $11\alpha(10, 6) + 18\alpha(11, 5) = p(14)^6$ then P

$= 11\int_0^2 \underset{\text{II}}{t^{10}}\ \underset{\text{I}}{(1+3t)^6} + 10\int^2 t^{11}(1+3t)^5\, dt$

$= 11\left[(1+3t)^6 \cdot \frac{t^{11}}{11} - \int 6(1+3t)^5 \cdot 3\,\frac{t^{11}}{11}\right]_0^2 + 18\int_0^2 t^{11}(1+3t)^5\, dt$

$= (t^{11}(1+3t)^6)_0^{\ 2}$

$= 2^{11}(7)^6$

$= 2^5(14)^6$

$= 32(14)^6$

Question55

Let the function $f : [0, 2] \rightarrow R$ be defined as

$$f(x) = \begin{cases} e^{\min\{x^2,\, x-[x]\}} & x \in [0, 1) \\ e^{[x - \log_e x]} & x \in [1, 2) \end{cases}.$$

where [t] denotes the greatest integer less than or equal to t. Then the value of the integral $\int_0^2 xf(x)\,dx$ is [11-Apr-2023 shift 2]

Options:

A. $(e-1)\left(e^2+\frac{1}{2}\right)$

B. $1+\frac{3e}{2}$

C. $2e-\frac{1}{2}$

D. $2e-1$

Answer: A

Solution:

$F[0,2]\to R$

$$F(x)=\begin{cases}\min\{x^2,\{x\}\};\ x\in[0,1)\\ [x-\log_e x]=1;\ x\in[1,2)\end{cases}.$$

$$F(x)=\begin{cases}e^{x^2}:x\in[0,1)\\ e\ \ x\in[1,2)\end{cases}.$$

$$\int_0^2 xf(x)\,dx=\int_0^1 x\cdot e^{x^2}dx+\int_1^2 x\cdot e\,dx$$

$$=\frac{1}{2}(e-1)+\frac{1}{2}(4-1)e$$

$$\Rightarrow 2e-\frac{1}{2}$$

Question56

If $f:R\to R$ be a continuous function satisfying $\int_0^{\frac{\pi}{2}} f(\sin 2x)\sin x\,dx+\alpha\int_0^{\frac{\pi}{4}} f(\cos 2x)\cos x\,dx=0$, then the value of α is
[11-Apr-2023 shift 2]

Options:

A. $-\sqrt{3}$

B. $\sqrt{3}$

C. $-\sqrt{2}$

D. $\sqrt{2}$

Answer: C

Solution:

$F:R\to R$

$$\Rightarrow\int_0^{\frac{\pi}{2}} F(\sin 2x)\sin dx+\alpha\int_0^{\frac{\pi}{4}} F(\cos 2x)\cdot\cos x\,dx=0$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} F(\sin 2x)\sin x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} F(\sin 2x)\cdot\sin x\,dx + \alpha\int_0^{\frac{\pi}{4}} F(\cos 2x)\cdot\cos x\,dx = 0$$

$$\int_0^a F(x)\,dx = \int_0^a F(a-x)\,dx$$

Let $x = t + \frac{\pi}{4}$

$$\int_0^{\frac{\pi}{4}} F(\cos 2x)\sin\left(\frac{\pi}{4}-x\right)dx + \int_0^{\frac{\pi}{4}} F(\cos 2t)\sin\left(t+\frac{\pi}{4}\right) + \alpha\int_0^{\frac{\pi}{4}} F(\cos 2x)\cos x\,dx = 0$$

$$\int_0^{\frac{\pi}{4}} F(\cos 2x)\left\{\sin\left(\frac{\pi}{4}-x\right) + \sin\left(x+\frac{\pi}{4}\right) + \alpha\cos x = 0.\right.$$

$$\int_0^{\frac{\pi}{4}} F(\cos 2x)\{(\sqrt{2}+\alpha)\cos x\}\,dx = 0$$

$$(\sqrt{2}+\alpha)\int_0^{\frac{\pi}{4}} F(\cos 2x)\cos x\,dx = 0$$

$\because$ in interval $\left(0, \frac{\pi}{4}\right) \Rightarrow F(\cos 2x)\,\&\,\cos x$ is NOT Zero.

$\therefore \sqrt{2}+\alpha = 0$

$\alpha = -\sqrt{2}$

Question57

If $\int_{-0.15}^{0.15} |100x^2-1|\,dx = \frac{k}{3000}$, then k is equal to ______.
[12-Apr-2023 shift 1]

Answer: 575

Solution:

$$\int_{-0.15}^{0.15} |100x^2-1|\,dx = 2\int_0^{0.15} |100x^2-1|\,dx$$

Now $100x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{100} \Rightarrow x = 0.1$

$$I = 2\left[\int_0^{0.1}(1-100x^2)\,dx + \int_{0.1}^{0.15}(100x^2-1)\,dx\right]$$

$$I = 2\left[x - \frac{100}{3}x^3\right]_0^{0.1} + 2\left[\frac{100x^3}{3} - x\right]_{0.1}^{0.15}$$

$$= 2\left[0.1 - \frac{0.1}{3}\right] + 2\left[\frac{0.3375}{3} - 0.15 - \frac{0.1}{3} + 0.1\right]$$

$$= 2\left[0.2 - \frac{0.2}{3} + 0.1125 - 0.15\right]$$

$$= 2\left[\frac{5}{100} - \frac{2}{30} + \frac{1125}{10000}\right] = 2\left(\frac{1500-2000+3375}{30000}\right)$$

$$= \frac{575}{3000} \Rightarrow k = 575$$

Question58

The value of $\dfrac{e^{-\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}} e^{-x}\tan^{50}x\,dx}{\int_0^{\frac{\pi}{4}} e^{-x}(\tan^{49}x+\tan^{51}x)\,dx}$ **is**

[13-Apr-2023 shift 2]

Options:

A. 25

B. 51

C. 50

D. 49

Answer: C

Solution:

$$\text{let } I_1 = e^{-\pi/4}+\int_0^{\pi/4} e^{-x}\tan^{50}x\,dx$$

$$I_2 = \int_0^{\pi/4} e^{-x}(\tan^{49}x+\tan^{51}x)\,dx$$

$$= \int_0^{\pi/4} e^{-x}\tan^{49}x(\sec^2x)\,dx$$

$$= \left|e^{-x}\frac{\tan^{50}x}{50}\right|_0^{\pi/4} + \frac{1}{50}\int_0^{\pi/4} e^{-x}\tan^{50}x\,dx$$

$$= \frac{e^{-\pi/4}}{50}+\frac{1}{50}\int_0^{\pi/4} e^{-x}\tan^{50}x\,dx = \frac{I_1}{50}$$

$$\text{then } \frac{I_1}{I_2}=50$$

Question59

Let $f_n = \int_0^{\frac{\pi}{2}}\left(\sum_{k=1}^{n}\sin^{k-1}x\right)\left(\sum_{k=1}^{n}(2k-1)\sin^{k-1}x\right)\cos x\,dx,\ n\in N$**. Then** $f_{21}-f_{20}$ **is equal to _________**

[13-Apr-2023 shift 2]

Answer: 41

Solution:

$$f_n = \int_0^{\pi/2}\left(\sum_{k=1}^{n}\sin^{k-1}x\right)\left(\sum_{k=1}^{n}(2k-1)\sin^{k-1}x\right)\cos x\,dx$$

$$\sin x = t$$
$$\cos x\,dx = d\,t$$

$$f_n\int_0^1\left(\sum_{k=1^n t}^{k-1}\right)\left(\sum_{k=1^n(2k-1)t}^{k-1}\right)dt$$

$$=\int_0^1(1+t+t^2\ldots t^n)(1+3t+5t^2+\ldots.+(2n+1)t^n)\,dt$$

$$=\int_0^1(1+t+t^2+\ldots t^{n-1})(1+3t+5t^2+\ldots+(2n-1)t^{n-1})\,dt$$

$$+\int_0^1(1+3t+5t^2+\ldots(2n+1))t^n\,dt$$

$+\int_0^1 (1+t+t^2+\ldots+t^{n-1})(2n+1)t^n\,dt$

$f_{n+1}-f_n=\int_0^1 (1+3t+5t^2+\ldots+(2n+1)t^n)t^n\,dt$

$+\int_0^1 (1+t+t^2+\ldots t^{n+1})((2n+1)t^n)\,dt$

put $n=20$

$f_{21}-f_{20}=\int_0^1 (1+3t+5t^2\ldots 41\cdot t^{20})t^{20}\,dt+\int_0^1 (1+t+t^2\ldots t^{19})(41\cdot t^{20})\,dt$

$=\left(\frac{1}{21}+\frac{3}{22}+\frac{5}{23}+\ldots+\frac{39}{40}+\frac{41}{41}\right)+\left(\frac{41}{21}+\frac{41}{22}+\frac{41}{40}\right)$

$=\frac{1+41}{21}+\frac{3+41}{22}+\ldots+\frac{39+41}{40}+1=40+1=41$

Question60

If $\int_0^1 \frac{1}{(5+2x-2x^2)(1+e^{(2-4x)})}dx = \frac{1}{\alpha}\log_e\left(\frac{\alpha+1}{\beta}\right)$, $\alpha, \beta > 0$, then $\alpha^4-\beta^4$ is equal to [15-Apr-2023 shift 1]

Options:

A. 19

B. -21

C. 21

D. 0

Answer: C

Solution:

Solution:

$I=\int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}\ldots$ (i)

$x\to 1-x$

$I=\int_0^1 \frac{e^{2-4x}dx}{(5+2x-2x^2)(1+e^{2-4x})}\ldots$ (ii)

Add (i) and (ii)

$2I\int_0^1 \frac{dx}{5+2x-2x^2}=\int_0^1 \frac{dx}{2\left(\frac{11}{4}-\left(x-\frac{1}{2}\right)^2\right)}$

$I=\frac{1}{\sqrt{11}}\ln\left(\frac{\sqrt{11}+1}{\sqrt{10}}\right)$

$\alpha=\sqrt{11}$

$\beta=\sqrt{10}$

$\alpha^4-\beta^4=121-100=21$

Question61

Let $I(x)=\int \frac{x^2(x\sec^2x+\tan x)}{(x\tan x+1)^2}dx$. If $I(0)=0$, then $I\left(\frac{\pi}{4}\right)$ is equal to : [6-Apr-2023 shift 1]

Options:

A. $\log_e\frac{(\pi+4)^2}{16}+\frac{\pi^2}{4(\pi+4)}$

B. $\log_e\frac{(\pi+4)^2}{32}-\frac{\pi^2}{4(\pi+4)}$

C. $\log_e \frac{(\pi+4)^2}{16} - \frac{\pi^2}{4(\pi+4)}$

D. $\log_e \frac{(\pi+4)^2}{32} + \frac{\pi^2}{4(\pi+4)}$

Answer: B

Solution:

Solution:

$I(x) = \int \frac{x^2(x\sec^2 x + \tan x)}{(x\tan x + 1)^2} dx$

Let $x\tan x + 1 = t$

$I = x^2\left(\frac{-1}{x\tan x + 1}\right) + \int \frac{2x}{x\tan x + 1} dx$

$I = x^2\left(\frac{-1}{x\tan x + 1}\right) + 2\int \frac{2x}{x\tan x + 1} dx$

$I = x^2\left(\frac{-1}{x\tan x + 1}\right) + 2\ln\left|x\sin x + \cos x\right| + C$

As $I(0) = 0 \Rightarrow C = 0$

$I\left(\frac{\pi}{4}\right) = \ln\left(\frac{(\pi+4)^2}{32}\right) - \frac{\pi^2}{4(\pi+4)}$

Question62

Let $I(x) = \int \frac{(x+1)}{x(1+xe^x)^2} dx$, $x > 0$. If $\lim_{x\to\infty} I(x) = 0$, then $I(1)$ is equal to
[8-Apr-2023 shift 1]

Options:

A. $\frac{e+1}{e+2} - \log_e(e+1)$

B. $\frac{e+2}{e+1} + \log_e(e+1)$

C. $\frac{e+2}{e+1} - \log_e(e+1)$

D. $\frac{e+1}{e+2} + \log_e(e+1)$

Answer: D

Question63

The integral $\int \left(\left(\frac{x}{2}\right)^x + \left(\frac{2}{x}\right)^x\right)\log_2 x \, dx$ is equal to
[8-Apr-2023 shift 2]

Options:

A. $\left(\frac{x}{2}\right)^x \log_2\left(\frac{2}{x}\right) + C$

B. $\left(\frac{x}{2}\right)^x - \left(\frac{2}{x}\right)^x + C$

C. $\left(\frac{x}{2}\right)^{x}\log_{2}\left(\frac{x}{2}\right)+C$

D. $\left(\frac{x}{2}\right)^{x}+\left(\frac{2}{x}\right)^{x}+C$

Answer: B

Solution:

Solution:

$\int(x^{x}2^{-x}+2^{x}x^{-x})\log_{2}{}^{x}\,dx$

$\int(e^{x\ln x}\cdot e^{-x\ln 2}+e^{x\ln 2}\cdot e^{-x\ln x})\,dx$

$\int(e^{x\ln x-x\ln 2}+e^{x\ln 2-x\ln x})\frac{\ln x}{\ln 2}\,dx$

let $x\ln x-x\ln 2=t$

$(\ln x+1-\ln 2)\,dx=d\,t$

Question64

If $I(x)=\int e^{\sin^{2}x}(\cos x\sin 2x-\sin x)\,dx$ and $I(0)=1$, then $I\left(\frac{\pi}{3}\right)$ is equal to :

[10-Apr-2023 shift 1]

Options:

A. $e^{\frac{3}{4}}$

B. $-e^{\frac{3}{4}}$

C. $\frac{1}{2}e^{\frac{3}{4}}$

D. $-\frac{1}{2}e^{\frac{3}{4}}$

Answer: C

Solution:

Solution:

$I=\int\underbrace{e^{\sin^{2}x}}_{II}\sin 2x\underbrace{\cos x}_{I}\,dx-\int e^{\sin^{2}x}\sin x\,dx$

$=\cos x\int e^{\sin^{2}x}\sin 2x\,dx-\int\left((-\sin x)\int e^{\sin^{2}x}\sin 2x\,dx\right)dx-\int e^{\sin^{2}x}\sin x\,dx$

$\sin^{2}x=t$

$\sin 2x\,dx=d\,t$

$=\cos x\int e^{t}\,dt+\int(\sin x\int e^{t}\,dt)\,dx-\int e^{\sin^{2}x}\sin x\,dx$

$=e^{\sin^{2}x}\cos x+\int e^{\sin^{2}x}\sin x\,dx-\int e^{\sin^{2}x}\sin x\,dx$

$I=e^{\sin^{2}x}\cos x+C$

$I(0)=1$

$\Rightarrow 1=1+C$

$\Rightarrow C=0$

$\therefore I=e^{\sin^{2}x}\cos x$

$I\left(\frac{\pi}{3}\right)=e^{\sin^{2}\frac{\pi}{3}}\cos\frac{\pi}{3}$

$=\frac{e^{\frac{3}{4}}}{2}$

Question65

For $\alpha, \beta, \gamma, \delta \in N$, if $\int \left(\left(\frac{x}{e}\right)^{2x} + \left(\frac{e}{x}\right)^{2x}\right)\log_e x dx = \frac{1}{\alpha}\left(\frac{x}{e}\right)^{\beta x} - \frac{1}{\gamma}\left(\frac{e}{x}\right)^{\delta x} + C$, where $e = \sum_{n=0}^{\infty}\frac{1}{n!}$ and C is constant of integration, then $\alpha + 2\beta + 3\gamma - 4\delta$ is equal to
[10-Apr-2023 shift 2]

Options:

A. 4

B. -4

C. -8

D. 1

Answer: A

Solution:

Solution:

$x = e^{\ln x}$

$\int \left(\left(\frac{x}{e}\right)^{2x} + \left(\frac{e}{x}\right)^{2x}\right)\log_e x\, dx = \int [e^{2(x\ln x - x)} + e^{-2(x\ln x - x)}]\ln x\, dx$

$x\ln x - x = t$

$\ln x \cdot dx = dt$

$\int (e^{2t} + e^{-2t})\, dt$

$\frac{e^{2t}}{2} - \frac{e^{-2t}}{2} + C$

$= \frac{1}{2}\left(\frac{x}{e}\right)^{2x} - \frac{1}{2}\left(\frac{e}{x}\right)^{2x} + C$

$\alpha = B = \gamma = \delta = 2$

$\alpha + 2B + 3\gamma - 4\delta = 4$

Question66

$\int_0^{\infty} \frac{6}{e^{3x} + 6e^{2x} + 11e^{x} + 6}$ **dx=**
[13-Apr-2023 shift 1]

Options:

A. $\log_e\left(\frac{32}{27}\right)$

B. $\log_e\left(\frac{256}{81}\right)$

C. $\log_e\left(\frac{512}{81}\right)$

D. $\log_e\left(\frac{64}{27}\right)$

Answer: A

Solution:

$$I = \int_0^\infty \frac{6}{(e^x+1)(e^x+2)(e^x+3)}\,dx$$
$$= 6\int_0^\infty \left(\frac{\frac{1}{2}}{e^x+1} + \frac{-1}{e^x+2} + \frac{\frac{1}{2}}{e^x+3} \right) dx$$
$$= 3\int_0^\infty \frac{e^{-x}}{1+e^{-x}}\,dx - 6\int_0^\infty \frac{e^{-x}\,dx}{1+2e^{-x}} + 3\int_0^\infty \frac{e^{-x}}{1+3e^{-x}}\,dx$$
$$= 3[-\ln(1+e^{-x})]_0^\infty + 6\,\frac{1}{2}[\ln(1+2e^{-x})]_0^\infty$$
$$- \frac{3}{3}[\ln(1+3e^{-x})]_0^\infty$$
$$= 3\ln 2 - 3\ln 3 + \ln 4$$
$$= 3\ln\frac{2}{3} + \ln 4$$
$$= \ln\frac{32}{27}$$

Question67

Let $f(x) = \int \frac{dx}{(3+4x^2)\sqrt{4-3x^2}}, \left|x\right| < \frac{2}{\sqrt{3}}$. If $f(0) = 0$ and $f(1) = \frac{1}{\alpha\beta}\tan^{-1}\left(\frac{\alpha}{\beta}\right)$ $\alpha, \beta > 0$, then $\alpha^2 + \beta^2$ is equal to

[15-Apr-2023 shift 1]

Answer: 28

Solution:

Solution:

$$f(x) = \int \frac{d\,x}{(3+4x^2)\sqrt{4-3x^2}}$$
$$x = \frac{1}{t}$$
$$= \int \frac{\frac{-1}{t^2}\,dt}{\frac{(3t^2+4)}{t^2}\,\frac{\sqrt{4t^2-3}}{t}}$$
$$= \int \frac{-d\,t \cdot t}{(3t^2+4)\sqrt{4t^2-3}} : \quad \text{Put } 4t^2-3 = z^2$$
$$= -\frac{1}{4}\int \frac{z\,dx}{\left(3\left(\frac{z^2+3}{4}\right)+4\right)z}$$
$$= \int \frac{-d\,z}{3z^2+25} = -\frac{1}{3}\int \frac{d\,z}{z^2+\left(\frac{5}{\sqrt{3}}\right)^2}$$
$$= -\frac{1}{3}\,\frac{\sqrt{3}}{5}\tan^{-1}\left(\frac{\sqrt{3}z}{5}\right) + C$$
$$= -\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{5}\sqrt{4t^2-3}\right) + C$$
$$f(x) = -\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{5}\sqrt{\frac{4-3x^2}{x^2}}\right) + C$$
$$\because f(0) = 0 \because c = \frac{\pi}{10\sqrt{3}}$$
$$f(1) = -\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{5}\right) + \frac{\pi}{10\sqrt{3}}$$
$$f(1) = \frac{1}{5\sqrt{3}}\cot^{-1}\left(\frac{\sqrt{3}}{5}\right) = \frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{5}{\sqrt{3}}\right)$$
$$\alpha = 5 : \beta = \sqrt{3} \therefore \alpha^2 + \beta^2 = 28$$

Question68

The value of $12\int_0^3 |x^2 - 3x + 2|\, dx$ is
[24-Jan-2023 Shift 1]

Answer: 22

Solution:

$$12\int_0^3 \left| x^2 - 3x + 2 \right| dx$$

$$= 12\int_0^3 \left| \left(x - \frac{3}{2} \right)^2 - \frac{1}{4} \right| dx$$

If $x - \frac{3}{2} = t$

$dx = dt$

$$= 24\int_0^{3/2} \left| t^2 - \frac{1}{4} \right| dt$$

$$= 24\left[-\int_0^{1/2} \left(t^2 - \frac{1}{4} \right) dt + \int_{1/2}^{3/2} \left(t^2 - \frac{1}{4} \right) dt \right] = 22$$

Question69

If $\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}\, d\,x = g(x) + c$, $g(1) = 0$, then $g\left(\frac{1}{2}\right)$ is equal to:

[26-Jun-2022-Shift-2]

Options:

A. $\log_e\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) + \frac{\pi}{3}$

B. $\log_e\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) + \frac{\pi}{3}$

C. $\log_e\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) - \frac{\pi}{3}$

D. $\frac{1}{2}\log_e\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) - \frac{\pi}{6}$

Answer: A

Solution:

$\because \int_x^1 \sqrt{\frac{1-x}{1+x}} dx = g(x) + c$

$\int_1^{\frac{1}{2}} \frac{1}{x} \sqrt{\frac{1-x}{1-x}} dx = g\left(\frac{1}{2}\right) - g(1)$

$\therefore g\left(\frac{1}{2}\right) = \int_1^{\frac{1}{2}} \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx$

$\cot x = \cos 2\theta$

$= \int_0^{\frac{\pi}{6}} \frac{1}{\cos 2\theta} \cdot \frac{\sin\theta}{\cos\theta}(-2\sin 2\theta) d\theta$

$= -\int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\cos 2\theta} d\theta$

$= 2\int_0^{\frac{\pi}{6}} \frac{(1-2\sin^2\theta)-1}{\cos 2\theta} d\theta$

$= 2\int_0^{\frac{\pi}{6}} (1-\sec 2\theta) d\theta$

$= \frac{\pi}{3} - 2 \cdot \frac{1}{2}[\ln|\sec 2\theta + \tan 2\theta|]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{3} - [\ln|2+\sqrt{3}| - \ln 1]$

$= \frac{\pi}{3} + \ln\left(\frac{1}{2-\sqrt{3}}\right)$

$= \frac{\pi}{3} + \ln\left|\frac{\sqrt{3}-1}{\sqrt{3}+1}\right|$

Question70

If $\int \frac{(x^2+1)e^x}{(x+1)^2} dx = f(x)e^x + C$, where C is a constant, then $\frac{d^3 f}{dx^3}$ at $x = 1$ is equal to :
[27-Jun-2022-Shift-1]

Options:

A. $-\frac{3}{4}$

B. $\frac{3}{4}$

C. $-\frac{3}{2}$

D. $\frac{3}{2}$

Answer: B

Solution:

Solution:

$I = \int \frac{e^x(x^2+1)}{(x+1)^2} dx = f(x)e^x + c$

$= \int \frac{e^x(x^2-1+1+1)}{(x+1)^2} dx$

$= \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right] dx$

$= e^x\left(\frac{x-1}{x+1}\right) + c$

$\therefore f(x) = \frac{x-1}{x+1}$

$f(x) = 1 - \frac{2}{x+1}$

$f'(x) = 2\left(\frac{1}{x+1}\right)^2$

$f''(x) = -4\left(\frac{1}{x+1}\right)^3$

$f'''(x) = \frac{12}{(x+1)^4}$

for $x = 1$

$f'''(1) = \frac{12}{2^4} = \frac{12}{16} = \frac{3}{4}$

Question71

The value of the integral $\int_{-\pi/2}^{\pi/2} \frac{dx}{(1+e^x)(\sin^6 x + \cos^6 x)}$ is equal to [24-Jun-2022-Shift-2]

Options:

A. 2π

B. 0

C. π

D. $\frac{\pi}{2}$

Answer: C

Solution:

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{(1+e^x)(\sin^6 x + \cos^6 x)} \ldots\ldots \text{ (i)}$$

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{(1+e^{-x})(\sin^6 x + \cos^6 x)} \ldots\ldots \text{ (ii)}$$

(i) and (ii)

From equation (i) \& (ii)

$$2I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\sin^6 x + \cos^6 x}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{dx}{\sin^6 x + \cos^6 x} = \int_0^{\frac{\pi}{2}} \frac{dx}{1 - \frac{3}{4}\sin^2 2x}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{4\sec^2 2x dx}{4 + \tan^2 2x} = 2\int_0^{\frac{\pi}{4}} \frac{4\sec^2 2x}{4 + \tan^2 2x} dx$$

when $x = 0$, $t = 0$Now, $\tan 2x = t$when $x = \frac{\pi}{4}$, $t \to \infty$

$2\sec^2 2x dx = dt$

$$\therefore I = 2\int_0^{\infty} \frac{2dt}{4+t^2} = 2\left(\tan^{-1}\frac{t}{2}\right)_0^{\infty}$$

$$= 2\,\frac{\pi}{2} = \pi$$

Question72

$$\lim_{n \to \infty} \left(\frac{n^2}{(n^2+1)(n+1)} + \frac{n^2}{(n^2+4)(n+2)} + \frac{n^2}{(n^2+9)(n+3)} + \dots + \frac{n^2}{(n^2+n^2)(n+n)} \right)$$

is equal to :
[24-Jun-2022-Shift-2]

Options:

A. $\frac{\pi}{8} + \frac{1}{4}\log_e 2$

B. $\frac{\pi}{4} + \frac{1}{8}\log_e 2$

C. $\frac{\pi}{4} - \frac{1}{8}\log_e 2$

D. $\frac{\pi}{8} + \log_e \sqrt{2}$

Answer: A

Solution:

$$\lim_{n \to \infty} \left(\frac{n^2}{(n^2+1)(n+1)} + \frac{n^2}{(n^2+4)(n+2)} + \dots + \frac{n^2}{(n^2+n^2)(n+n)} \right)$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{n^2}{(n^2+r^2)(n+r)}$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \frac{1}{\left[1+\left(\frac{r}{n}\right)^2\right]\left[1+\left(\frac{r}{n}\right)\right]}$$

$$= \int_0^1 \frac{1}{(1+x^2)(1+x)} dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1}{1+x} - \frac{(x-1)}{(1+x^2)} \right] dx$$

$$= \frac{1}{2}\left[\ln(1+x) - \frac{1}{2}\ln(1+x^2) + \tan^{-1}x \right]_0^1$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\ln 2 \right] = \frac{\pi}{8} + \frac{1}{4}\ln 2$$

Question73

The value of $\int_0^{\pi} \frac{e^{\cos x}\sin x}{(1+\cos^2 x)(e^{\cos x}+e^{-\cos x})}$ dx is equal to:
[25-Jun-2022-Shift-1]

Options:

A. $\frac{\pi^2}{4}$

B. $\frac{\pi^2}{2}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{2}$

Answer: C

Solution:

$$\int_0^{\pi} \frac{e^{\cos x} \sin x}{(1+\cos^2 x)(e^{\cos x} - e^{-\cos x})} dx$$

Let $\cos x = t$

$\sin x \, d x = d t$

$$= \int_1^{-1} \frac{-e^t dt}{(1+t^2)(e^t + e^{-t})}$$

$$I = \int_{-1}^{1} \frac{e^t}{(1+t^2)(e^t - e^{-t})} dt \ldots\ldots \text{ (i)}$$

$$I = \int_{-1}^{1} \frac{e^{-t}}{(1+t^2)(e^{-t} + e^t)} dt \ldots \text{ (ii)}$$

Adding (i) and (ii)

Adding (i) and (ii)

$$2I = \int_{-1}^{1} \frac{dt}{1-t^2}$$

$$2I = .\tan^{-t}|_{-1}^{1}$$

$$2I = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

Question74

Let g : (0, ∞) → R be a differentiable function such that

$$\int \left(\frac{x(\cos x - \sin x)}{e^x + 1} + \frac{g(x)(e^x + 1 - xe^x)}{(e^x + 1)^2} \right) d x = \frac{xg(x)}{e^x + 1} + c$$

, for all x > 0, where c is an arbitrary constant. Then :
[25-Jun-2022-Shift-1]

Options:

A. g is decreasing in $\left(0, \frac{\pi}{4}\right)$

B. g' is increasing in $\left(0, \frac{\pi}{4}\right)$

C. $g + g'$ is increasing in $\left(0, \frac{\pi}{2}\right)$

D. $g - g'$ is increasing in $\left(0, \frac{\pi}{2}\right)$

Answer: D

Solution:

$\int\left(\frac{x(\cos x-\sin x)}{e^x+1}+\frac{g(x)(e^x+1-xe^x)}{(e^x-1)^2}\right)dx=\frac{xg(x)}{e^x+1}+c$

On differentiating both sides w.r.t. x, we get

$\left(\frac{x(\cos x-\sin x)}{e^x+1}+\frac{g(x)(e^x+1-xe^x.)}{(e^x+1)^2}\right)$

$=\frac{(e^x+1)(g(x)+xg'(x))-e^x\cdot x\cdot g(x)}{(e^x+1)^2}$

$(e^x+1)x(\cos x-\sin x)+g(x)(e^x-1-xe^x)$

$=(e^x+1)(g(x)+xg'(x))-e^x\cdot x\cdot g(x)$

$\Rightarrow g'(x)=\cos x-\sin x$

$\Rightarrow g(x)=\sin x-\cos x+C$

g(x) is increasing in $(0,\pi/4)$

$g''(x)=-\sin x-\cos x<0$

$\Rightarrow g'(x)$ is decreasing functionlet $h(x)=g(x)+g'(x)=2\cos x+C\Rightarrow h'(x)=g'(x)+g''(x)=-2\sin x<0$

$\Rightarrow$h is decreasing let $\varphi(x)=g(x)-g'(x)=2\sin x+C\Rightarrow\varphi'(x)=g'(x)-g''(x)=2\cos x>0\Rightarrow\varphi$ is increasing

---

Question75

If $b_n=\int_0^{\frac{\pi}{2}}\frac{\cos^2 nx}{\sin x}dx,\ n\in N$, then
[25-Jun-2022-Shift-2]

Options:

A. b_3-b_2,b_4-b_3,b_5-b_4 are in A.P. with common difference -2

B. $\frac{1}{b_3-b_2},\frac{1}{b_4-b_3},\frac{1}{b_5-b_4}$ are in an A.P. with common difference 2

C. b_3-b_2,b_4-b_3,b_5-b_4 are in a G.P.

D. $\frac{1}{b_3-b_2},\frac{1}{b_4-b_3},\frac{1}{b_5-b_4}$ are in an A.P. with common difference -2

Answer: D

Solution:

$b_n-b_{n-1}=\int_0^{\frac{\pi}{2}}\frac{\cos^2 nx-\cos^2(n-1)x}{\sin x}dx$

$=\int_0^{\frac{\pi}{2}}\frac{-\sin(2n-1)x\cdot\sin x}{\sin x}dx$

$=.\frac{\cos(2n-1)x}{2n-1}\Big|_0^{\pi/2}=-\frac{1}{2n-1}$

So, b_3-b_2,b_4-b_3,b_5-b_4 are in H.P.

$\Rightarrow\frac{1}{b_3-b_2},\frac{1}{b_4-b_3},\frac{1}{b_5-b_4}$ are in A.P. with common difference -2.

Question76

The value of b > 3 for which

$12\int_3^b\frac{1}{(x^2-1)(x^2-4)}dx=\log_e\left(\frac{49}{40}\right)$

, is equal to
[25-Jun-2022-Shift-2]

Answer: 6

Solution:

$$I = \int \frac{1}{(x^2-1)(x^2-4)} d x = \frac{1}{3} \int \left(\frac{1}{x^2-4} - \frac{1}{x^2-1} \right) d x$$
$$= \frac{1}{3} \left(\frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| - \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| \right) + C$$
$$12I = \ln \left| \frac{x-2}{x+2} \right| + 2 \ln \left| \frac{x-1}{x+1} \right| + C$$
$$12 \int_3^b \frac{d x}{(x^2-4)(x^2-1)}$$
$$= \ln \left(\frac{b-2}{b+2} \right) - 2 \ln \left(\frac{b-1}{b+1} \right) - \left(\ln \left(\frac{1}{5} \right) - 2 \ln \left(\frac{1}{2} \right) \right)$$
$$= \ln \left(\left(\frac{b-2}{b+2} \right) \cdot \frac{(b+1)^2}{(b-1)^2} \right) - \left(\ln \frac{4}{5} \right)$$
$$\text{So,} \quad \frac{49}{40} = \frac{(b-2)}{(b+2)} \frac{(b+1)^2}{(b-1)^2} \cdot \frac{5}{4}$$
$$\Rightarrow b = 6$$

Question77

Let $f(x) = \max\{|x+1|, |x+2|, \ldots, |x+5|\}$. Then $\int_{-6}^{0} f(x) d x$ is equal to___
[26-Jun-2022-Shift-1]

Answer: 21

Solution:

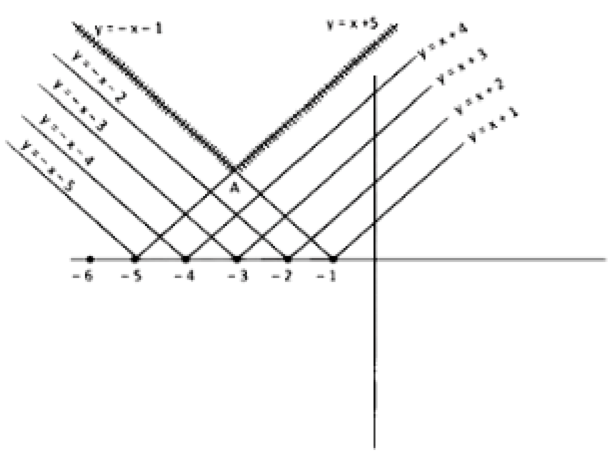
For $|x+1|$ critical point, $x+1=0 \Rightarrow x=-1$

For $|x+2|$ critical point, $x+2=0 \Rightarrow x=-2$

For $|x+3|$ critical point, $x+3=0 \Rightarrow x=-3$

For $|x+4|$ critical point, $x+4=0 \Rightarrow x=-4$

For $|x+5|$ critical point, $x+5=0 \Rightarrow x=-5$



Here maximum function is represent by the dotted line.

$\therefore$ Point of intersection A of line $y=-x-1$ and $y=x+5$:

$-x-1=x+5$

$\Rightarrow 2x=-6$

$\Rightarrow x=-3$

$\therefore y=-3+5=2$

$\therefore$ Point $A=(-3,2)$

$\therefore \int_{-6}^{0} f(x)dx$

$=\int_{-6}^{-3}(-x-1)dx+\int_{-3}^{0}(x+5)dx$

$=\left(-\frac{x^2}{2}-x\right)_{-6}^{-3}+\left[\frac{x^2}{2}+5x\right]_{-3}^{0}$

$=\left[\left(-\frac{9}{2}+3\right)-\left(-\frac{36}{2}+6\right)\right]+\left[0-\left(\frac{9}{2}-15\right)\right]$

$=\left(-\frac{3}{2}+12\right)+\frac{21}{2}$

$=\frac{21}{2}=21$

Question78

The value of the integral

$\frac{48}{\pi^4}\int_0^{\pi}\left(\frac{3\pi x^2}{2}-x^3\right)\frac{\sin x}{1+\cos^2 x}dx$ is equal to ____
[26-Jun-2022-Shift-1]

Answer: 6

Solution:

$I=\frac{48}{\pi^4}\int_0^{\pi}\left[\left(\frac{\pi}{2}-x\right)^3-\frac{3\pi^2}{4}\left(\frac{\pi}{2}-x\right)+\frac{\pi^3}{4}\right]\frac{\sin x\,dx}{1+\cos^2 x}$

Using $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

$$I=\frac{48}{\pi^4}\int_0^{\pi}\left[-\left(\frac{\pi}{2}-x\right)^3+\frac{3\pi^4}{4}\left(\frac{\pi}{2}-x\right)+\frac{\pi^3}{4}\right]\frac{\sin x\,d\,x}{1+\cos^2 x}$$

Adding these two equations, we get

$$2I=\frac{48}{\pi^4}\int_0^{\pi}\frac{\pi^3}{2}\cdot\frac{\sin x\,d\,x}{1+\cos^2 x}$$

$$\Rightarrow I=\frac{12}{\pi}[-\tan^{-1}(\cos x)]_0^{\pi}=\frac{12}{\pi}\cdot\frac{\pi}{2}=6$$

Question79

The integral $\frac{24}{\pi}\int_0^{\sqrt{2}}\frac{(2-x^2)dx}{(2+x^2)\sqrt{4+x^4}}$ is equal to
[26-Jun-2022-Shift-2]

Answer: 6

Solution:

$$I=\frac{24}{\pi}\int_0^{\sqrt{2}}\frac{2-x^2}{(2+x^2)\sqrt{4+x^4}}dx$$

Let $x=\sqrt{2}t\Rightarrow dx=\sqrt{2}dt$

$$I=\frac{24}{\pi}\int_0^{1}\frac{(2-2t^2)\cdot\sqrt{2}dt}{(2+2t^2)\sqrt{4+4t^4}}$$

$$=\frac{12\sqrt{2}}{\pi}\int_0^{1}\frac{\left(\frac{1}{t^2}-1\right)dt}{\left(t+\frac{1}{t}\right)\sqrt{\left(t+\frac{1}{t}\right)^2-2}}$$

Let $t+\frac{1}{t}=u$

$$\Rightarrow\left(1-\frac{1}{t^2}\right)dt=du$$

$$=\frac{12\sqrt{2}}{\pi}\int_{\infty}^{2}\frac{-du}{u\sqrt{4^2-2}}$$

$$=\frac{12\sqrt{2}}{\pi}\int_2^{\infty}\frac{du}{u^2\sqrt{-\left(\frac{\sqrt{2}}{u}\right)^2}}$$

$$=\frac{12\sqrt{2}}{\pi}\int_{\frac{1}{\sqrt{2}}}^{0}\frac{-\frac{1}{\sqrt{2}}dp}{\sqrt{1-p^2}}$$

$$=\frac{12}{\pi}[\sin^{-1}p]_0^{\frac{1}{\sqrt{2}}}$$

$$=\frac{12}{\pi}\cdot\frac{\pi}{4}=3$$

Question80

The value of the integral
$\int_{-2}^{2} \frac{x^3+x|}{(e^{x|x|}+1)} d x$ **is equal to:**
[27-Jun-2022-Shift-1]

Options:

A. $5e^2$

B. $3e^{-2}$

C. 4

D. 6

Answer: D

Solution:

$I = \int_{-2}^{2} \frac{|x^3+x|}{e^{x|x|}+1} d x \dots (i)$
$I = \int_{-2}^{2} \frac{|x^3+x|}{e^{-x|x|}+1} d x \dots (ii)$
$2I = \int_{-2}^{2} \left| x^3+x \right| d x$
$2I = 2\int_{0}^{2} (x^3+x) d x$
$I = \int_{0}^{2} (x^3+x) d x$
$. = \frac{x^4}{4} + \frac{x^2}{2}]_0^2$
$= \left(\frac{16}{4} + \frac{4}{2} \right) - 0$
$= 4 + 2 = 6$

Question81

Let f be a differentiable function in $\left(0, \frac{\pi}{2}\right)$. If $\int_{\cos x}^{1} t^2 f(t) d t = \sin^3 x + \cos x$, then $\frac{1}{\sqrt{3}} f'\left(\frac{1}{\sqrt{3}}\right)$ is equal to

[27-Jun-2022-Shift-2]

Options:

A. $6 - 9\sqrt{2}$

B. $6 - \frac{9}{\sqrt{2}}$

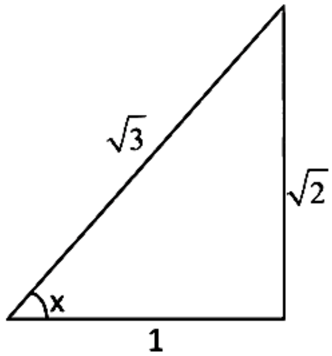
C. $\frac{9}{2} - 6\sqrt{2}$

D. $\frac{9}{\sqrt{2}} - 6$

Answer: B

Solution:

$\int_{\cos x}^{1} t^2 f(t) dt = \sin^3 x + \cos x$
$\Rightarrow \sin x \cos^2 x f(\cos x) = 3\sin^2 x \cos x - \sin x$
$\Rightarrow f(\cos x) = 3\tan x - \sec^2 x$
$\Rightarrow f'(\cos x) \cdot (-\sin x) = 3\sec^2 x - 2\sec^2 x \tan x$



Put $\cos x = \frac{1}{\sqrt{3}}$,

$\therefore f'\left(\frac{1}{\sqrt{3}}\right)\left(-\frac{\sqrt{2}}{\sqrt{3}}\right) = 9 - 6\sqrt{2}$

$\frac{1}{\sqrt{3}} f'\left(\frac{1}{\sqrt{3}}\right) = 6 - \frac{9}{\sqrt{2}}$

Question82

The integral $\int_0^1 \frac{1}{7^{\left[\frac{1}{x}\right]}} d\,x$, where [.] denotes the greatest integer function, is equal to

[27-Jun-2022-Shift-2]

Options:

A. $1 + 6\log_e\left(\frac{6}{7}\right)$

B. $1 - 6\log_e\left(\frac{6}{7}\right)$

C. $\log_e\left(\frac{7}{6}\right)$

D. $1 - 7\log_e\left(\frac{6}{7}\right)$

Answer: A

Solution:

$\int_0^1 \frac{1}{7^{\left[\frac{1}{x}\right]}} dx, \quad \text{let } \frac{1}{x} = t$

$\frac{-1}{x^2} dx = d\,t$

$= \int_\infty^1 \frac{1}{-t^2 7^{[t]}} dt = \int_1^\infty \frac{1}{t^2 7^{[t]}} dt$

$= \int_1^2 \frac{1}{7t^2} dt + \int_2^3 \frac{1}{7^2 t^2} dt + \ldots$

$= \frac{1}{7}\left[-\frac{1}{t}\right]_1^2 + \frac{1}{7^2}\left[\frac{-1}{t}\right]_2^3 + \frac{1}{7^3}\left[-\frac{1}{t}\right]_2^3 + \ldots$

$= \sum_{n=1}^{\infty} \frac{1}{7^n}\left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= \sum_{n=1}^{\infty} \frac{\left(\frac{1}{7}\right)^n}{n} - 7\sum_{n=1}^{\infty} \frac{\left(\frac{1}{7}\right)^{n+1}}{n+1}$

$= -\log\left(1-\frac{1}{7}\right)+7\log\left(1-\frac{1}{7}\right)+1$
$= 1+6\log\frac{6}{7}$

Question83

Let [t] denote the greatest integer less than or equal to t. Then, the value of the integral $\int_0^1 [-8x^2+6x-1]dx$ is equal to
[28-Jun-2022-Shift-1]

Options:

A. -1

B. $\frac{-5}{4}$

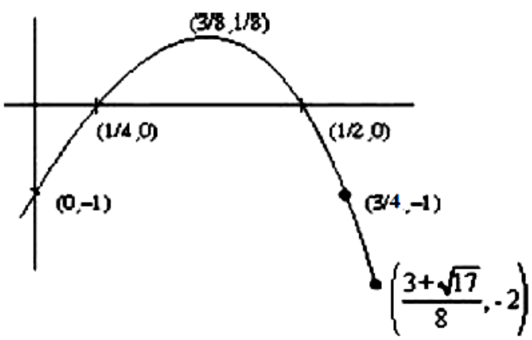
C. $\frac{\sqrt{17}-13}{8}$

D. $\frac{\sqrt{17}-16}{8}$

Answer: C

Solution:

$\int_0^1 [-8x^2+6x-1]\,dx$
$= \int_0^{1/4} -1\,dx + \int_{1/4}^{1/2} 0\,dx + \int_{1/2}^{3/4} -1\,dx$



$+\int_{3/4}^{\frac{3+\sqrt{17}}{8}} -2\,dx + \int_{\frac{3+\sqrt{17}}{8}}^{1-3} dx$

$= -[x]_0^{1/4}+0-[x]_{1/2}^{3/4}+-2[x]_{3/4}^{\frac{3+\sqrt{17}}{8}}-3[x]_{\frac{3+\sqrt{17}}{8}}^{1}$

$= -\left(\frac{1}{4}-0\right)-\left(\frac{3}{4}-\frac{1}{2}\right)-2\left(\frac{3+\sqrt{17}}{8}-\frac{3}{4}\right)-3\left(1-\frac{3+\sqrt{17}}{8}\right)$
$= -\frac{1}{4}-\frac{1}{4}+\frac{-6-2\sqrt{17}}{8}+\frac{3}{2}-3+\frac{9+3\sqrt{17}}{8}$
$= \frac{\sqrt{17}-13}{8}$

Question84

Let $f : R \rightarrow R$ be a differentiable function such that $f\left(\frac{\pi}{4}\right) = \sqrt{2}, f\left(\frac{\pi}{2}\right) = 0$ and $f'\left(\frac{\pi}{2}\right) = 1$ and let $g(x) = \int_x^{\pi/4}(f'(t)\sec t + \tan t \sec t f(t))dt$ for $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then $\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} g(x)$ is equal to :

[28-Jun-2022-Shift-2]

Options:

A. 2

B. 3

C. 4

D. −3

Answer: B

Solution:

Solution:

Given: $f\left(\frac{\pi}{4}\right) = \sqrt{2}, f\left(\frac{\pi}{2}\right) = 0$ and $f'\left(\frac{\pi}{2}\right) = 1$

$$g(x) = \int_x^{\frac{\pi}{4}} (f'(t)\sec t + \tan t\ \sec t f(t))\,dt$$

$$= [\sec t + f(t)]_x^{\frac{\pi}{4}} = 2 - \sec x f(x)$$

Now, $\lim_{x \rightarrow \frac{\pi}{2}^-} g(x) = \lim_{h \rightarrow 0} g\left(\frac{\pi}{2} - h\right)$

$$= \lim_{h \rightarrow 0} 2 - (\operatorname{cosec} h) f\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h \rightarrow 0}\left[2 - \frac{f\left(\frac{\pi}{2} - h\right)}{\sin h}\right]$$

$$= \lim_{h \rightarrow 0}\left[2 + \frac{f'\left(\frac{\pi}{2} - h\right)}{\cos h}\right]$$

$= 3$

Question85

Let $f : R \rightarrow R$ be a continuous function satisfying $f(x) + f(x + k) = n$, for all $x \in R$ where $k > 0$ and n is a positive integer. If $I_1 = \int_0^{4nk} f(x)dx$ and $I_2 = \int_{-k}^{3k} f(x)dx$, then :

[28-Jun-2022-Shift-2]

Options:

A. $I_1 + 2I_2 = 4nk$

B. $I_1 + 2I_2 = 2nk$

C. $I_1 + nI_2 = 4n^2k$

D. $I_1 + nI_2 = 6n^2k$

Answer: C

Solution:

Solution:
$f: R \to R$ and $f(x) + f(x+k) = n \quad \forall x \in R$
$x \to x + k$
$f(x+k) + f(x+2k) = n$
$\therefore\ f(x+2k) = f(x)$
So, period of $f(x)$ is $2k$
Now, $I_1 = \int_0^{4nk} f(x)\,dx = 2n \int_0^{2k} f(x)\,dx$

$= 2n\left[\int_0^{k} f(x)\,dx + \int_k^{2k} f(x)\,dx\right]$
$x = t + k \Rightarrow dx = dt$ (in second integral)

$= 2n\left[\int_0^{k} f(x)\,dx + \int_0^{k} f(t+k)\,dt\right]$
$= 2n^2k$
Now, $I_2 = \int_{-k}^{3k} f(x)\,dx = 2\int_0^{2k} f(x)\,dx$
$I_2 = 2(nk)$
$\therefore\ I_1 + nI_2 = 4n^2k$

Question86

$$\int_0^5 \cos\left(\pi\left(x - \left[\frac{x}{2}\right]\right)\right) dx$$

where [t] denotes greatest integer less than or equal to t, is equal to:
[29-Jun-2022-Shift-1]

Options:

A. −3

B. −2

C. 2

D. 0

Answer: D

Solution:

We know,
$\left[\frac{x}{2}\right]$ is discontinuous at $1, 2, 3, 4\ldots\ldots$
$\therefore [x]$ is discontinuous at $2, 4, 6, 8\ldots\ldots$
In between 0 to 5 it is discontinuous at 2 and 4.
Break the integration into 3 parts
(1) 0 to 2
(2) 2 to 4
(3) 4 to 5

$\therefore \int_0^5 \cos\left(\pi\left(x - \left[\frac{x}{2}\right]\right)\right) dx$
$= \int_0^2 \cos(\pi(x-0))\,dx + \int_2^4 \cos(\pi(x-1))\,dx + \int_4^5 \cos(\pi(x-2))\,dx$
$= \int_0^2 \cos\pi x\,dx + \int_2^4 \cos(\pi x - \pi)\,dx + \int_4^5 \cos(\pi x - 2\pi)\,dx$
$= \int_0^2 \cos\pi\,dx - \int_2^4 \cos\pi x\,dx + \int_4^5 \cos\pi x\,dx$
$= \left[\frac{\sin \pi x}{\pi}\right]_0^2 - \left[\frac{\sin \pi x}{\pi}\right]_2^4 + \left[\frac{\sin \pi x}{\pi}\right]_4^5$
$= 0 - 0 + 0$
$= 0$

Question87

Let f be a real valued continuous function on [0, 1] and $f(x) = x + \int_0^1 (x - t)f(t)dt$
Then, which of the following points (x, y) lies on the curve $y = f(x)$?
[29-Jun-2022-Shift-2]

Options:

A. (2, 4)

B. (1, 2)

C. (4, 17)

D. (6, 8)

Answer: D

Solution:

$f(x) = \left(1 + \int_0^1 f(t)\,dt\right)x - \int_0^1 tf(t)\,dt$

$f(x) = Ax - B$

$A = 1 + \int_0^1 f(t)\,dt = 1 + \int_0^1 (At - B)\,dt$

$\Rightarrow A = 2(1 - B)$

Also $B = \int_0^1 tf(t)\,dt = \int_0^1 (At^2 - Bt)\,dt$

$A = \frac{9}{2}B$

From (2), (3)

$A = \frac{18}{13}, B = \frac{4}{13}$

so $f(6) = 8$

Question88

If $\int_0^2 \left(\sqrt{2x} - \sqrt{2x - x^2}\right)dx = \int_0^1 \left(1 - \sqrt{1 - y^2} - \frac{y^2}{2}\right)dy + \int_1^2 \left(2 - \frac{y^2}{2}\right)dy + I$, then I equals
[29-Jun-2022-Shift-2]

Options:

A. $\int_0^1 \left(1 + \sqrt{1 - y^2}\right)dy$

B. $\int_0^1 \left(\frac{y^2}{2} - \sqrt{1 - y^2} + 1\right)dy$

C. $\int_0^1 \left(1 - \sqrt{1 - y^2}\right)dy$

D. $\int_0^1 \left(\frac{y^2}{2} + \sqrt{1 - y^2} + 1\right)dy$

Answer: C

Solution:

LHS $= \int_0^2 \left(\sqrt{2x} - \sqrt{2x - x^2}\right) dx = \frac{8}{3} - \frac{\pi}{2}$

RHS $= \int_0^1 \left(1 - \sqrt{1 - y^2} - \frac{y^2}{2}\right) dy + \int_1^2 \left(2 - \frac{y^2}{2}\right) dy + I$

$I + \frac{5}{3} - \frac{\pi}{4}$

So, $I = 1 - \frac{\pi}{4} = \int_0^1 \left(1 - \sqrt{1 - y^2}\right) dy$

Question89

Let $f(\theta) = \sin\theta + \int_{-\pi/2}^{\pi/2} (\sin\theta + t\cos\theta) f(t)\, dt$. Then the value of $\left| \int_0^{\pi/2} f(\theta) d\theta \right|$ is

[24-Jun-2022-Shift-1]

Answer: 1

Solution:

$f(\theta) = \sin\theta \left(1 + \int_{-\pi/2}^{\pi/2} f(t) dt\right) + \cos\theta \left(\int_{-\pi/2}^{\pi/2} t f(t) dt\right)$

Clearly $f(\theta) = a\sin\theta + b\cos\theta$

Where $a = 1 + \int_{-\pi/2}^{\pi/2} (a\sin t + b\cos t) dt \Rightarrow a = 1 + 2b \quad \ldots(i)$ and $b = \int_{-\pi/2}^{\pi/2} (at\sin t + bt\cos t) dt \Rightarrow b = 2a \quad \ldots(ii)$ from (i) and (ii) we get

$a = -\frac{1}{3}$ and $b = -\frac{2}{3}$

So $f(\theta) = -\frac{1}{3}(\sin\theta + 2\cos\theta)$

$\Rightarrow \left| \int_0^{\pi/2} f(\theta) d\theta \right| = \frac{1}{3}(1 + 2 \times 1) = 1$

Question90

Let $\underset{0 \le x \le 2}{\text{Max}} \left\{ \frac{9 - x^2}{5 - x} \right\} = \alpha$ and $\underset{0 \le x \le 2}{\text{Min}} \left\{ \frac{9 - x^2}{5 - x} \right\} = \beta$.

If $\int_{\beta - \frac{8}{3}}^{2\alpha - 1} \text{Max} \left\{ \frac{9 - x^2}{5 - x}, x \right\} dx = \alpha_1 + \alpha_2 \log_e \left(\frac{8}{15} \right)$ then $\alpha_1 + \alpha_2$ is equal to

[24-Jun-2022-Shift-1]

Answer: 34

Solution:

Let $f(x)=\frac{x^2-9}{x-5} \Rightarrow f'(x)=\frac{(x-1)(x-9)}{(x-5)^2}$

So, $\alpha=f(1)=2$ and $\beta=\min(f(0),f(2))=\frac{5}{3}$

Now, $\int_{-1}^{3}\max\left\{\frac{x^2-9}{x-5},x\right\}dx=\int_{-1}^{9/5}\frac{x^2-9}{x-5}dx+\int_{9/5}^{3}xdx$

$=\int_{-1}^{9/5}\left(x+5+\frac{16}{x-5}\right)dx+.\frac{x^2}{2}\Big|_{9/5}^{3}$

$=\frac{28}{25}+14+16\ln\left(\frac{8}{15}\right)+\frac{72}{25}=18+16\ln\left(\frac{8}{15}\right)$

Clearly $\alpha_1=18$ and $\alpha_2=16$, so $\alpha_1+\alpha_2=34$.

Question91

The integral $\int\frac{\left(1-\frac{1}{\sqrt{3}}\right)(\cos x-\sin x)}{\left(1+\frac{2}{\sqrt{3}}\sin 2x\right)}dx$ is equal to

[26-Jul-2022-Shift-2]

Options:

A. $\frac{1}{2}\log_e\left|\frac{\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)}{\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)}\right|+C$

B. $\frac{1}{2}\log_e\left|\frac{\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)}{\tan\left(\frac{x}{2}+\frac{\pi}{3}\right)}\right|+C$

C. $\log_e\left|\frac{\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)}{\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)}\right|+C$

D. $\frac{1}{2}\log_e\left|\frac{\tan\left(\frac{x}{2}-\frac{\pi}{12}\right)}{\tan\left(\frac{x}{2}-\frac{\pi}{6}\right)}\right|+C$

Answer: A

Solution:

$$=\int\frac{\left(1-\frac{1}{\sqrt{3}}\right)(\cos x-\sin x)}{\left(1+\frac{2}{\sqrt{3}}\sin 2x\right)}dx$$

$$=\int\frac{\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)\sqrt{2}\sin\left(\frac{\pi}{4}-x\right)}{\left(\frac{2}{\sqrt{3}}\right)\left(\sin\frac{\pi}{3}+\sin 2x\right)}dx$$

$$=\int\frac{\frac{(\sqrt{3}-1)}{\sqrt{2}}\sin\left(\frac{\pi}{4}-x\right)}{\left(\sin\frac{\pi}{3}+\sin 2x\right)}dx$$

$$= \int \frac{\frac{\sqrt{3}-1}{2\sqrt{2}} \sin\left(\frac{\pi}{4} - x\right)}{\sin\left(\frac{\pi}{6} + x\right)\cos\left(\frac{\pi}{6} - x\right)} dx$$

$$= \frac{1}{2}\int \frac{2\sin\frac{\pi}{12}\sin\left(\frac{\pi}{4} - x\right)}{\sin\left(\frac{\pi}{6} + x\right)\cos\left(\frac{\pi}{6} - x\right)} dx$$

$$= \frac{1}{2}\int \frac{\cos\left(\frac{\pi}{6} - x\right) - \cos\left(\frac{\pi}{3} - x\right)}{\sin\left(\frac{\pi}{6} + x\right)\cos\left(\frac{\pi}{6} - x\right)} dx$$

$$= \frac{1}{2}\left[\int \operatorname{cosec}\left(\frac{\pi}{6} + x\right) d\ x - \int \sec\left(\frac{\pi}{6} - x\right) d\ x\right]$$

$$= \frac{1}{2}\left[\ln\left|\tan\left(\frac{\pi}{12} + \frac{x}{2}\right)\right| - \int \operatorname{cosec}\left(\frac{\pi}{3} - x\right) d\ x\right]$$

$$= \frac{1}{2}\left[\ln\left|\tan\left(\frac{\pi}{12} + \frac{x}{2}\right)\right| - \ln\left|\frac{\pi}{6} + \frac{x}{2}\right|\right] + C$$

$$= \frac{1}{2}\ln\left|\frac{\tan\left(\frac{\pi}{2} + \frac{x}{2}\right)}{\tan\left(\frac{\pi}{6} + \frac{x}{2}\right)}\right| + C$$

Question92

For $I(x) - \int \frac{\sec^2 x - 2022}{\sin^{2022} x} d\ x$, if $I\left(\frac{\pi}{4}\right) - 2^{1011}$, then [29-Jul-2022-Shift-2]

Options:

A. $3^{1010} I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) = 0$

B. $3^{1010} I\left(\frac{\pi}{6}\right) - I\left(\frac{\pi}{3}\right) = 0$

C. $3^{1011} I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) = 0$

D. $3^{1011} I\left(\frac{\pi}{6}\right) - I\left(\frac{\pi}{3}\right) = 0$

Answer: A

Solution:

Solution:
Given,

$$I(x) = \int \frac{\sec^2 x - 2022}{\sin^{2022} x} dt$$

$$= \int \frac{\sec^2 x}{\sin^{2022} x} dt - \int \frac{2022}{\sin^{2022} x} dt$$

$$= \int \frac{1}{\sin^{2022} x} \cdot \sec^2 x\, dt - \int \frac{2022}{\sin^{202} x} dt$$

$$= \frac{1}{\sin^{2022} x} \cdot \tan x - \int\left(\frac{-2022}{\sin^{2023} x} \cdot \cos x \cdot \tan x\right) dt - \int \frac{2022}{\sin^{2022} x} dt + C$$

$$= \frac{\tan x}{\sin^{2022} x} + \int\left(\frac{2022}{\sin^{2023} x} \cdot \cos x \cdot \frac{\sin x}{\cos x}\right) dt - \int \frac{2022}{\sin^{2022} x} dt + C$$

$$= \frac{\tan x}{\sin^{2022} x} + \int \frac{2022}{\sin^{2022} x} dt - \int \frac{2022}{\sin^{2022} x} dt$$

$$= \frac{\tan x}{\sin^{2022} x} + C$$

Given, $I\left(\frac{\pi}{4}\right)=2^{1011}\therefore I\left(\frac{\pi}{4}\right)=\dfrac{\tan\left(\frac{\pi}{4}\right)}{\left(\sin\frac{\pi}{4}\right)^{2022}}+C$

$\Rightarrow 2^{1011}=\dfrac{1}{\left(\frac{1}{\sqrt{2}}\right)^{2022}}+C$

$\Rightarrow C=2^{1011}-2^{1011}=0$

$\therefore I(x)=\dfrac{\tan x}{\sin^{2022}x}$

$\therefore I\left(\frac{\pi}{3}\right)=\dfrac{\tan\frac{\pi}{3}}{\left(\sin\frac{\pi}{3}\right)^{2022}}=\dfrac{\sqrt{3}}{\left(\frac{\sqrt{3}}{2}\right)^{2022}}$

$I\left(\frac{\pi}{6}\right)=\dfrac{\frac{1}{\sqrt{3}}}{\left(\frac{1}{2}\right)^{2022}}=\frac{1}{\sqrt{3}}\times(2)^{2022}$

From option (A),

$3^{1010}\cdot I\left(\frac{\pi}{3}\right)-I\left(\frac{\pi}{6}\right)$

$=3^{1010}\cdot\sqrt{3}\cdot\left(\frac{2}{\sqrt{3}}\right)^{2022}-\frac{(2)^{2022}}{\sqrt{3}}$

$=3^{1010}\cdot\sqrt{3}\times\frac{2^{2022}}{3^{1011}}-\frac{2^{2022}}{\sqrt{3}}$

$=\frac{2^{2022}}{\sqrt{3}}-\frac{2^{2022}}{\sqrt{3}}=0$

Question93

For any real number x , let [x] denote the largest integer less than equal to x. Let f be a real valued function defined on the interval [−10, 10] by $f(x)=\begin{cases}x-[x], & \text{if } [x] \text{ is odd}\\ 1+[x]-x, & \text{if } [x] \text{ is even}\end{cases}$ **.**

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos\pi x\,dt$ **is :**
[25-Jul-2022-Shift-1]

Options:

A. 4

B. 2

C. 1

D. 0

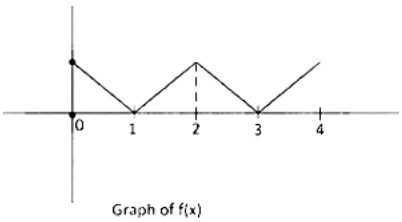
Answer: A

Solution:

Solution:
Case 1:
Let $0\le x<1$
then $[x]=0$, which is even
$\therefore f(x)=1+[x]-x$
$=1+0-x$
$=1-x$
Case 2 :
Let $1\le x<2$
then $[x]=1$, which is odd
$\therefore f(x)=x-[x]$
$=x-1$
Case 3 :
Let $2\le x<3$ then $[x]=2$, which is even

$\therefore f(x) = 1 + [x] - x$
$= 1 + 2 - x$
$= 3 - x$
Case 4 :
Let $3 \le x < 4$
then $[x] = 3$, which is odd
$\therefore f(x) = x - [x]$
$= x - 3$

$$\therefore f(x) = \begin{cases} 1 - x & 0 \le x < 1 \\ x - 1 & 1 \le x < 2 \\ 3 - x & 2 \le x < 3 \\ x - 3 & 3 \le x < 4. \end{cases}$$



Graph of f(x)

$\therefore f(x)$ is periodic and period of $f(x) = 2$
And period of $\cos \pi x = \frac{2\pi}{\pi} = 2$
$\therefore$ Period of $f(x)\cos \pi x = 2$
Now,

$I = \frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, d\,x$
$= \frac{\pi^2}{10} \int_{-10}^{-10+10\times 2} f(x) \cos \pi x \, d\,x$
$= \frac{\pi^2}{10} \int_0^{10\times 2} f(x) \cos \pi x \, d\,x$
$= \frac{\pi^2}{10} \times 10 \int_0^2 f(x) \cos \pi x \, d\,x$
$= \pi^2 \int_0^2 f(x) \cos \pi x \, d\,x$
$\therefore I = \pi^2 \left[\int_0^1 f(x) \cos \pi x \, d\,x + \int_1^2 f(x) \cos \pi x \, d\,x \right]$
$= \pi^2 \left[\int_0^1 (1 - x) \cos \pi x \, d\,x + \int_1^2 (x - 1) \cos \pi x \, d\,x \right]$
$= \pi^2 \left[\int_0^1 \cos \pi x \, d\,x - \int_0^1 x \cos \pi x \, d\,x + \int_1^2 x \cos \pi x \, d\,x - \int_1^2 \cos \pi x \, d\,x \right]$
$= \pi^2 \left[\frac{1}{\pi}[\sin \pi x]_0^{\,1} - \int_0^1 x \cos \pi x \, d\,x + \int_1^2 x \cos \pi x \, d\,x - \frac{1}{\pi}[\sin \pi x]_1^{\,2} \right]$
$= \pi^2 \left[0 - \int_0^1 x \cos \pi x \, d\,x + \int_1^2 x \cos \pi x \, d\,x - 0 \right]$
$= \pi^2 \left[- \left[x \frac{\sin \pi x}{\pi} + \frac{1}{\pi^2} \cos \pi x \right]_0^1 + \left[x \frac{\sin \pi x}{\pi} + \frac{1}{\pi^2} \cos \pi x \right]_1^2 \right]$ [As $.\int x \cos \pi x \, d\,x = x \cdot \int \cos \pi x - \int \left(1. \frac{\sin \pi x}{\pi} \right) d\,x = x \cdot \frac{\sin \pi x}{\pi} + \frac{1}{\pi^2} \cos \pi x + c \right]$
$= \pi^2 \left[- \left[\left(1 \cdot \frac{\sin \pi}{\pi} + \frac{1}{\pi^2} \cdot \cos \pi \right) - \left(0 + \frac{1}{\pi^2} \cdot \cos 0 \right) \right] + \left[\left(2 \cdot \frac{\sin 2\pi}{\pi} + \frac{1}{\pi^2} \cos 2\pi \right) - \left(1. \frac{\sin \pi}{\pi} + \frac{1}{\pi^2} \cos \pi \right) \right] \right]$
$= \pi^2 \left[- \left\{ \left(- \frac{1}{\pi^2} \right) - \left(\frac{1}{\pi^2} \right) \right\} + \left(\left(+ \frac{1}{\pi^2} \right) - \left(- \frac{1}{\pi^2} \right) \right) \right].$
$= \pi^2 \left[- \left(- \frac{2}{\pi^2} \right) + \frac{2}{\pi^2} \right]$
$= \pi^2 \left[\frac{2}{\pi^2} + \frac{2}{\pi^2} \right]$
$= \pi^2 \times \frac{4}{\pi^2}$
$= 4$

Question94

$$\lim_{n \to \infty} \frac{1}{2^n} \left(\frac{1}{\sqrt{1 - \frac{1}{2^n}}} + \frac{1}{\sqrt{1 - \frac{2}{2^n}}} + \frac{1}{\sqrt{1 - \frac{3}{2^n}}} + \ldots + \frac{1}{\sqrt{1 - \frac{2^n - 1}{2^n}}} \right)$$

is equal to
[25-Jul-2022-Shift-2]

Options:

A. $\frac{1}{2}$

B. 1

C. 2

D. −2

Answer: C

Solution:

Solution:

$$I = \lim_{n \to \infty} \frac{1}{2^n} \left(\frac{1}{\sqrt{1 - \frac{1}{2^n}}} + \frac{1}{\sqrt{1 - \frac{2}{2^n}}} + \frac{1}{\sqrt{1 - \frac{3}{2^n}}} + \ldots\ldots + \frac{1}{\sqrt{1 - \frac{2^n - 1}{2^n}}} \right)$$

Let $2^n = t$ and if $n \to \infty$ then $t \to \infty$

$$I = \lim_{n \to \infty} \frac{1}{t} \left(\sum_{r=1}^{t-1} \frac{1}{\sqrt{1 - \frac{r}{t}}} \right)$$

$$I = \int_0^1 \frac{dx}{\sqrt{1-x}} = \int_0^1 \frac{dx}{\sqrt{x}} \quad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx. \right.$$

$$= \left[2x^{\frac{1}{2}} \right]_0^1 = 2$$

Question95

Let |t| denote the greatest integer less than or equal to t. Then the value of the integral $\int_{-3}^{101} ([\sin(\pi x)] + e^{[\cos(2\pi x)]}) dx$ is equal to
[25-Jul-2022-Shift-2]

Options:

A. $\frac{52(1-e)}{e}$

B. $\frac{52}{e}$

C. $\frac{52(2+e)}{e}$

D. $\frac{104}{e}$

Answer: B

Solution:

Solution:

$$I = \int_{-3}^{101} ([\sin(\pi x)] + e^{[\cos(2\pi x)]}) dx$$

$[\sin \pi x]$ is periodic with period 2 and $e^{[\cos(2\pi x)]}$ is periodic with period 1 .
So,

$$I = 52 \int_0^2 ([\sin \pi x] + e^{[\cos 2\pi x]}) dx$$

$$= 52\left\{\int_1^2 - 1\,dx + \int_{\frac{1}{4}}^{\frac{3}{4}} e^{-1}dx + \int_{\frac{5}{4}}^{\frac{7}{4}} e^{-1}dx + \int_0^{\frac{1}{4}} e^0 dx + \int_{\frac{3}{4}}^{\frac{5}{4}} e^0 dx + \int_{\frac{7}{4}}^{2} e^0 dx\right\}$$

$$= \frac{52}{e}$$

Question96

Let f be a twice differentiable function on C. If $f'(0) = 4$ and $f(x) + \int_0^x (x - t)f'(t)dt = (e^{2x} + e^{-2x})\cos 2x + \frac{2}{a}x$, then $(2a + 1)^5 a^2$ is equal to ___
[25-Jul-2022-Shift-2]

Answer: 8

Question97

Let $a_n = \int_{-1}^{n}\left(1 + \frac{x}{2} + \frac{x^2}{3} + ... + \frac{x^{n-1}}{n}\right)dx$ for every $n \in N$. Then the sum of all the elements of the set $\{n \in N : a_n \in (2, 30)\}$ is___
[25-Jul-2022-Shift-2]

Answer: 5

Solution:

Solution:

$\because a_n = \int_{-1}^{n}\left(1 + \frac{x}{2} + \frac{x^2}{3} + ... + \frac{x^{n-1}}{n}\right)dx$

$= \left[x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + ... + \frac{x^n}{n^2}\right]_{-1}^{n}$

$a_n = \frac{n+1}{1^2} + \frac{n^2-1}{2^2} + \frac{n^3+1}{3^2} + \frac{n^4-1}{4^2} + ... + \frac{n^n + (-1)^{n+1}}{n^2}$

Here, $a_1 = 2$, $a_2 = \frac{2+1}{1} + \frac{2^2-1}{2} = 3 + \frac{3}{2} = \frac{9}{2}$

$a_3 = 4 + 2 + \frac{28}{9} = \frac{100}{9}$

$a_4 = 5 + \frac{15}{4} + \frac{65}{9} + \frac{255}{16} > 31$

$\therefore$ The required set is $\{2, 3\}$. $\because a_n \in (2, 30)$

$\therefore$ Sum of elements $= 5$.

Question98

If $a = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{2n}{n^2+k^2}$ and $f(x) = \sqrt{\frac{1-\cos x}{1+\cos x}}$, $x \in (0, 1)$, then :
[26-Jul-2022-Shift-1]

Options:

A. $2\sqrt{2} f\left(\frac{a}{2}\right) = f'\left(\frac{a}{2}\right)$

B. $f\left(\frac{a}{2}\right) f'\left(\frac{a}{2}\right) = \sqrt{2}$

C. $\sqrt{2} f\left(\frac{a}{2}\right) = f'\left(\frac{a}{2}\right)$

D. $f\left(\frac{a}{2}\right) = \sqrt{2} f'\left(\frac{a}{2}\right)$

Answer: C

Solution:

$$a = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{2n}{n^2+k^2}$$
$$= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{2}{1+\left(\frac{k}{n}\right)^2}$$
$$a = \int_0^1 \frac{2}{1+x^2} dx = 2\tan^{-1} x \Big|_0^1 = \frac{\pi}{2}$$
$$f(x) = \sqrt{\frac{1-\cos x}{1+\cos x}}, x \in (0, 1)$$
$$f(x) = \frac{1-\cos x}{\sin x} = \operatorname{cosec} x - \cot x$$
$$f'(x) = \operatorname{cosec}^2 x - \operatorname{cosec} x \cot x$$

$$\left.\begin{array}{l} f\left(\frac{a}{2}\right) = f\left(\frac{\pi}{4}\right) = \sqrt{2}-1 \\ f'\left(\frac{a}{2}\right) = f'\left(\frac{\pi}{4}\right) = 2-\sqrt{2} \end{array}\right\} f'\left(\frac{a}{2}\right) = \sqrt{2} \cdot f\left(\frac{a}{2}\right)$$

Question99

If $n(2n+1)\int_0^1 (1-x^n)^{2n} dx = 1177 \int_0^1 (1-x^n)^{2n+1} dx$, then $n \in N$ is equal to ________.
[26-Jul-2022-Shift-1]

Answer: 24

Solution:

$$\int_0^1 (1-x^n)^{2n+1} dx = \int_0^1 1 \cdot (1-x^n)^{2n+1} dx$$
$$= [(1-x^n)^{2n+1} \cdot x]_0^1 - \int_0^1 x \cdot (2n+1)(1-x^n)^{2n} \cdot -nx^{n-1} dx$$
$$= n(2n+1)\int_0^1 (1-(1-x^n))(1-x^n)^{2n} dx$$
$$= n(2n+1)\int_0^1 (1-x^n)^{2n} dx - n(2n+1)\int_0^1 (1-x^n)^{2n+1} dx$$

$(1+n(2n+1))\int_0^1(1-x^n)^{2n+1}dx = n(2n+1)\int_0^1(1-x^n)^{2n}dx$

$(2n^2+n+1)\int_0^1(1-x^n)^{2n+1}dx = 1177\int_0^1(1-x^n)^{2n+1}dx$

$\therefore 2n^2+n+1=1177$

$2n^2+n-1176=0$

$\therefore n=24$ or $-\frac{49}{2}$

$\therefore n=24$

Question100

$\int_0^{20\pi}(|\sin x|+|\cos x|)^2 dx$ is equal to
[26-Jul-2022-Shift-2]

Options:

A. $10(\pi+4)$

B. $10(\pi+2)$

C. $20(\pi-2)$

D. $20(\pi+2)$

Answer: D

Solution:

$$I=\int_0^{20\pi}(|\sin x|+|\cos x|)^2 dx$$

$$=20\int_0^{\pi}(1+|\sin 2x|)dx$$

$$=40\int_0^{\frac{\pi}{2}}(1+\sin 2x)dx$$

$$=.40\left(x-\frac{\cos 2x}{2}\right)\Big|_0^{\frac{\pi}{2}}$$

$$=40\left(\frac{\pi}{2}+\frac{1}{2}+\frac{1}{2}\right)=20(\pi+2)$$

Question101

Let f $: \mathbb{R} \to \mathbb{R}$ be a function defined as

$f(x) = a\sin\left(\frac{\pi[x]}{2}\right) + [2-x]$, $a \in \mathbb{R}$ where [t] is the greatest integer less than or equal to t. If $\lim_{x \to -1} f(x)$ exists, then the value of $\int_0^4 f(x)dx$ is equal to [27-Jul-2022-Shift-1]

Options:

A. -1

B. -2

C. 1

D. 2

Answer: B

Solution:

Solution:

$f(x) = a\sin\left(\frac{\pi[x]}{2}\right) + [2-x] a \in R$

Now,

$\because \lim_{x \to -1} f(x)$ exist

$\therefore \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$

$\Rightarrow a\sin\left(\frac{-2\pi}{2}\right) + 3 = a\sin\left(\frac{-\pi}{2}\right) + 2$

$\Rightarrow -a = 1 \Rightarrow a = -1$

Now, $\int_0^4 f(x)dx = \int_0^4 \left(-\sin\left(\frac{\pi[x]}{2}\right) + [2-x]\right)dx$

$= \int_0^1 1dx + \int_1^2 -1dx + \int_2^3 -1dx + \int_3^4 (1-2)dx$

$= 1 - 1 - 1 - 1 = -2$

Question102

Let $f(x) = 2 + |x| - |x-1| + |x+1|, x \in R$.
Consider

(S1): $f'\left(-\frac{3}{2}\right) + f'\left(-\frac{1}{2}\right) + f'\left(\frac{1}{2}\right) + f'\left(\frac{3}{2}\right) = 2$

(S2) : $\int_{-2}^{2} f(x)dx = 12$

Then,
[27-Jul-2022-Shift-2]

Options:

A. both (S1) and (S2) are correct

B. both (S1) and (S2) are wrong

C. only (S1) is correct

D. only (S2) is correct

Answer: D

Solution:

Solution:
$f(x) = 2 + |x| - |x-1| + |x+1|, x \in R$

$$\therefore f(x) = \begin{cases} -x & , \quad x < -1 \\ x+2 & , \quad -1 \le x < 0 \\ 3x+2 & , \quad 0 \le x < 1 \\ x+4 & , \quad x \ge 1. \end{cases}$$

$$\therefore f'\left(-\frac{3}{2}\right) + f'\left(-\frac{1}{2}\right) + f'\left(\frac{1}{2}\right) + f'\left(\frac{3}{2}\right) = -1 + 1 + 3 + 1 = 4$$

$$\text{and } \int\limits_{-2}^{2} f(x)dx = \int\limits_{-2}^{-1} f(x)dx + \int\limits_{-1}^{0} f(x)dx + \int\limits_{0^{1}f(x)dx+\int\limits_{1}}^{2} f(x)dx$$

$$= \left[-\frac{x^2}{2}\right]_{2}^{-1} + \left[\frac{(x+2)^2}{2}\right]_{-1}^{0} + \left[\frac{(3x+2)^2}{6}\right]_{0}^{1} + \left[\frac{(x+4)^2}{2}\right]_{1}^{2}$$

$$= \frac{3}{2} + \frac{3}{2} + \frac{7}{2} + \frac{11}{2} = \frac{24}{2} = 12$$

$\therefore$ Only (S2) is correct

Question103

$\int_0^2 \left(|2x^2 - 3x| + \left[x - \frac{1}{2} \right] \right) dx$**, where [t] is the greatest integer function, is equal to:**
[27-Jul-2022-Shift-2]

Options:

A. $\frac{7}{6}$

B. $\frac{19}{12}$

C. $\frac{31}{12}$

D. $\frac{3}{2}$

Answer: B

Solution:

Solution:

$$\int_0^2 \left| 2x^2 - 3x \right| dx + \int_0^2 \left[x - \frac{1}{2} \right] dx$$
$$= \int_0^{3/2} (3x - 2x^2) dx + \int_{3/2}^2 (2x^2 - 3x) dx + \int_0^{1/2} -1 dx + \int_{1/2}^{3/2} 0 dx + \int_{3/2}^2 1 dx$$
$$= \left(\frac{3x^2}{2} - \frac{2x^3}{3} \right) \Big|_0^{3/2} +. \left(\frac{2x^3}{3} - \frac{3x^2}{2} \right) \Big|_{3/2}^2 - \frac{1}{2} + \frac{1}{2}$$
$$= \left(\frac{27}{8} - \frac{27}{12} \right) + \left(\frac{16}{3} - 6 - \frac{27}{12} + \frac{27}{8} \right)$$
$$= \frac{19}{12}$$

Question104

Let $f(x) = \min\{[x - 1], [x - 2], \ldots, [x - 10]\}$ **where [t] denotes the greatest integer ≤t. Then** $\int_0^{10} f(x) dx + \int_0^{10} (f(x))^2 dx + \int_0^{10} \left| f(x) \right| dx$ **is equal to ________.**

[27-Jul-2022-Shift-2]

Answer: 385

Solution:

Solution:

$\because f(x) = \min\{[x-1], [x-2], \ldots\ldots, [x-10]\} = [x-10]$

Also $|f(x)| = \begin{cases} -f(x), & \text{if } x \le 10 \\ f(x), & \text{if } x \ge 10. \end{cases}$

$\therefore \int_0^{10} f(x)dx + \int_0^{10} (f(x))^2 dx + \int_0^{10} (-f(x))dx$

$= \int_0^{10} (f(x))^2 dx$

$= 10^2 + 9^2 + 8^2 + \ldots\ldots + 1^2$

$= \dfrac{10 \times 11 \times 21}{6} = 385$

Question105

Let f be a differentiable function satisfying $f(x) = \frac{2}{\sqrt{3}} \int_0^{\sqrt{3}} f\left(\frac{\lambda^2 x}{3}\right) d\lambda$, $x > 0$ and $f(1) = \sqrt{3}$. If $y = f(x)$ passes through the point $(\alpha, 6)$, then α is equal to ________.

[27-Jul-2022-Shift-2]

Answer: 12

Solution:

Solution:

$\because f(x) = \dfrac{2}{\sqrt{3}} \int_0^{\sqrt{3}} f\left(\dfrac{\lambda^2 x}{3}\right) d\lambda,\ x > 0$

On differentiating both sides w.r.t., x, we get

$f'(x) = \dfrac{2}{\sqrt{3}} \int_0^{\sqrt{3}} \dfrac{\lambda^2}{3} f'\left(\dfrac{\lambda^2 x}{3}\right) d\lambda$

$f'(x) = \dfrac{1}{\sqrt{3}} \int_0^{\sqrt{3}} \lambda \cdot \dfrac{2\lambda}{3} f'\left(\dfrac{\lambda^2 x}{3}\right) d\lambda$

$\therefore \sqrt{3} f'(x) = \left[\dfrac{\lambda}{x} \cdot f\left(\dfrac{\lambda^2 x}{3}\right)\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \dfrac{1}{x} f\left(\dfrac{\lambda^2 x}{3}\right) dx$

$\sqrt{3} x f'(x) = \sqrt{3} f(x) - \dfrac{\sqrt{3}}{2} f(x)$

$x f'(x) = \dfrac{f(x)}{2}$

On integrating we get : $\ln y = \frac{1}{2}\ln x + \ln c \because f(1) = \sqrt{3}$ then $c = \sqrt{3}$

$\therefore(\alpha, 6)$ lies on

$\therefore y = \sqrt{3x}$

$\therefore 6 = \sqrt{3\alpha} \Rightarrow \alpha = 12$

Question106

If $\int_0^{\sqrt{3}} \frac{15x^3}{\sqrt{1 + x^2 + \sqrt{(1 + x^2)^3}}} dx - \alpha\sqrt{2} + \beta\sqrt{3}$, where α, β are integers, then $\alpha + \beta$ is equal to ________.

[28-Jul-2022-Shift-1]

Answer: 10

Solution:

Solution:

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$\Rightarrow I = \int_0^{\frac{\pi}{3}} \frac{15\tan^3\theta \cdot \sec^2\theta d\theta}{\sqrt{1 + \tan^2\theta + \sqrt{\sec^6\theta}}}$

$\Rightarrow I = \int_0^{\frac{\pi}{3}} \frac{15\tan^2\theta \sec^2\theta d\theta}{\sec\theta\sqrt{1 + \sec\theta}}$

$\Rightarrow I = \int_0^{\frac{\pi}{3}} \frac{15(\sec^2\theta - 1)\sec\theta\tan\theta d\theta}{(\sqrt{1 + \sec\theta})}$

Now put $1 + \sec\theta = t^2$

$\Rightarrow \sec\theta\tan\theta d\theta = 2tdt$

$\Rightarrow I = \int_{\sqrt{2}}^{\sqrt{3}} \frac{15((t^2 - 1)^2 - 1)2tdt}{t}$

$\Rightarrow I = 30\int_{\sqrt{2}}^{\sqrt{3}} (t^4 - 2t^2 + 1 - 1)dt$

$\Rightarrow I = 30\int_{\sqrt{2}}^{\sqrt{3}} (t^4 - 2t^2)dt$

$\Rightarrow I = 30\int_{\sqrt{2}}^{\sqrt{3}} (t^4 - 2t^2)dt$

$\Rightarrow I = .30\left(\frac{t^5}{5} - \frac{2t^3}{3}\right)\Big|_{\sqrt{2}}^{\sqrt{3}}$

$= 30\left[\left(\frac{9}{5}\sqrt{3} - 2\sqrt{3}\right) - \left(\frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3}\right)\right]$

$= (54\sqrt{3} - 60\sqrt{3}) - (24\sqrt{2} - 40\sqrt{2})$

$= 16\sqrt{2} - 6\sqrt{3}$
$\therefore \alpha = 16$ and $\beta = -6$
$\alpha + \beta = 10$

Question107

Let $I_n(x) = \int_0^x \frac{1}{(t^2+5)^n} dt$, n = 1, 2, 3, ... Then : [28-Jul-2022-Shift-2]

Options:

A. $50I_6 - 9I_5 - xI_5'$

B. $50I_6 - 11I_5 - xI_5'$

C. $50I_6 - 9I_5 - I_5'$

D. $50I_6 - 11I_5 - I_5'$

Answer: A

Solution:

Solution:

$I_n(x) = \int_0^x \frac{1}{(t^2+5)^n} dt$

$= \int_0^x \frac{1}{(\underbrace{\ }\, 2_I + 5)^n} \times \int_{II}^{I} dt$

$= .\ \frac{t}{(t^2+5)^n} \Big|_0^x - \int_0^x \frac{-2nt}{(t^2+5)^{n+1}} \times t dt$

$= \frac{x}{(x^2+5)^n} + \int_0^x 2n\left(\frac{t^2+5-5}{(t^2+5)^{n+1}}\right) dt$

$I_n(x) = \frac{x}{(x^2+5)^n} + 2nI_n(x) - 10nI_{n+1}(x)$

$10nI_{n+1}(x) - (2n-1)I_n(x) = xI_n'(x)$

For $n = 5$

$50I_6(x) - 9I_5(x) = xI_5'(x)$

Question108

The value of the integral $\int_0^{\frac{\pi}{2}} 60 \frac{\sin(6x)}{\sin x} dx$ is equal to________.
[28-Jul-2022-Shift-2]

Answer: 104

Solution:

Solution:

$$I = \int_0^{\frac{\pi}{2}} 60 \cdot \frac{\sin 6x}{\sin x} dx$$

$$= 60 \cdot 2 \int_0^{\frac{\pi}{2}} (3 - 4\sin^2 x)(4\cos^2 x - 3)\cos x\, dx$$

$$= 120 \int_0^{\frac{\pi}{2}} (3 - 4\sin^2 x)(1 - 4\sin^2 x)\cos x\, dx$$

Let $\sin x = t \Rightarrow \cos x\, dx = dt$

$$= 120 \int_0^1 (3 - 4t^2)(1 - 4t^2) dt$$

$$= 120 \int_0^1 (3 - 16t^2 + 16t^4) dt$$

$$= 120 \left[3t - \frac{16t^3}{3} + \frac{16t^5}{5} \right]_0^1$$

$$= 104$$

Question109

The integral $\int_0^{\frac{\pi}{2}} \frac{1}{3 + 2\sin x + \cos x} dx$ is equal to:
[29-Jul-2022-Shift-1]

Options:

A. $\tan^{-1}$

B. $\tan^{-1}(2) - \frac{\pi}{4}$

C. $\frac{1}{2}\tan^{-1}(2) - \frac{\pi}{8}$

D. $\frac{1}{2}$

Answer: B

Solution:

Solution:

$I = \int_0^{\pi/2} \frac{1}{3+2\sin x+\cos x} dx$

$= \int_0^{\pi/2} \frac{(1+\tan^2 x/2) dx}{3(1+\tan^2 x/2)+2(2\tan x/2)+(1-\tan^2 x/2)}$

Let $\tan x/2 = t \Rightarrow \sec^2 x/2 dx = 2dt$

$I = \int_0^1 \frac{2dt}{4+2t^2+4t}$

$= \int_0^1 \frac{dt}{t^2+2t+2} = \int_0^1 \frac{dt}{(t+1)^2+1}$

$= .\tan^{-1}(t+1)|_0^1 = \tan^{-1}2 - \frac{\pi}{4}$

Question110

If $f(\alpha) = \int_1^{\alpha} \frac{\log_{10} t}{1+t} dt, \alpha > 0$, then $f(e^3) + f(e^{-3})$ is equal to : [29-Jul-2022-Shift-1]

Options:

A. 9

B. $\frac{9}{2}$

C. $\frac{9}{\log_e(10)}$

D. $\frac{9}{2\log_a(10)}$

Answer: D

Solution:

$$f(\alpha)=\int_1^{\alpha}\frac{\log_{10}t}{1+t}dt\ldots\ldots(i)$$

$$f\left(\frac{1}{\alpha}\right)=\int_1^{\frac{1}{\alpha}}\frac{\log_{10}t}{1+t}dt$$

Substituting $t\rightarrow\frac{1}{p}$

$$f\left(\frac{1}{\alpha}\right)=\int_1^{\alpha}\frac{\log_{10}\left(\frac{1}{p}\right)}{1+\frac{1}{p}}\left(\frac{-1}{p^2}\right)dp$$

$$=\int_1^{\alpha}\frac{\log_{10p}p}{p(p+1)}dp=\int_1^{\alpha}\left(\frac{\log_{10}t}{t}-\frac{\log_{10}t}{t+1}\right)dt\ldots\ldots\text{(ii)}$$

By $(i)+(ii)$

$$f(\alpha)+f\left(\frac{1}{\alpha}\right)=\int_1^{\alpha}\frac{\log_{10}t}{t}dt=\int_1^{\alpha}\frac{\ln t}{t}\cdot\log_{10}e\,dt$$

$$=\frac{(\ln\alpha)^2}{2\log_e 10}$$

$$\alpha=e^3\Rightarrow f(e^3)+f(e^{-3})=\frac{9}{2\log_e 10}$$

Question111

If [t] denotes the greatest integer $\leq t$, then the value of $\int_0^1[2x-|3x^2-5x+2|+1]dx$ is:

[29-Jul-2022-Shift-2]

Options:

A. $\frac{\sqrt{37}+\sqrt{13}-4}{6}$

B. $\frac{\sqrt{37}-\sqrt{13}-4}{6}$

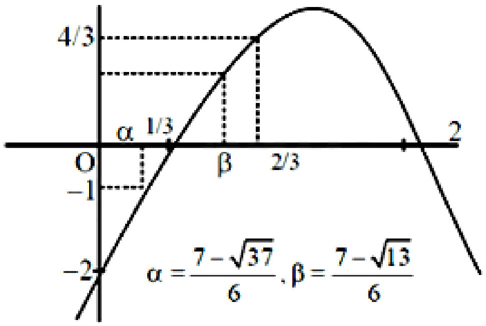
C. $\frac{-\sqrt{37}-\sqrt{13}+4}{6}$

D. $\frac{-\sqrt{37}+\sqrt{13}+4}{6}$

Answer: A

Solution:

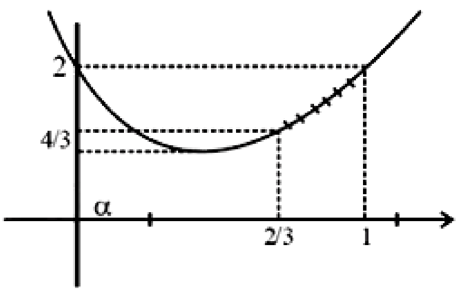
$$I=\int_0^1 [2x-|3x^2-3x-2x+2|+1]\,dt$$
$$I=\int_0^1 [2x-|(3x-2)(x-1)|]\,dt+\int_0^1 1\,dt$$
$$I=\int_0^{2/3} [(2x-(3x^2-5x+2))]\,dt+\int_{2/3}^{1} (2x+(3x^2-5x+2))\,dt+1$$
$$I=\int_0^{2/3} [-3x^2+7x-2]\,dt+\int_{2/3}^{1} (3x^2-3x+2)\,dt+1$$



$$\int_0^{\alpha}(-2)\,dt+\int_{\alpha}^{1/3}(-1)\,dt+\int_{1/3}^{\beta}0\,dt+\int_{\beta}^{2/3}1\,.\,dt$$

$$=-2\alpha-\left(\frac{1}{3}-\alpha\right)+\frac{2}{3}-\beta=-\alpha-\beta+\frac{1}{3}$$

$y=3x^2-3x+2$



When $x\in\left(\frac{2}{3},1\right)$

$3x^2-3x+2\in\left(\frac{4}{3},2\right)$

$[3x^2-3x+2]=1$

$$\therefore \int_{2/3}^{1}[3x^2-3x+2]\,dt=1\left(1-\frac{2}{3}\right)=\frac{1}{3}$$

Hence $I=\left(\frac{1}{3}-(\alpha+\beta)\right)+\left(\frac{1}{3}\right)+1$

$$=\frac{5}{3}-\left(\frac{7-\sqrt{37}}{6}+\frac{7-\sqrt{13}}{6}\right)$$
$$=\frac{-2}{3}+\frac{\sqrt{37}+\sqrt{13}}{6}$$
$$=\frac{\sqrt{37}+\sqrt{13}-4}{6}$$

Question112

The integral $\int \frac{e^{3\log_e 2x}+5e^{2\log_e 2x}}{e^{4\log_e x}+5e^{3\log_e x}-7e^{2\log_e x}}dx$, x > 0, is equal to (where, c is a constant of integration)
[25 Feb 2021 Shift 2]

Options:

A. $\log_e x^2+5x-7|+c$

B. $4\log_e x^2+5x-7|+c$

C. $\frac{1}{4}\log_e x^2+5x-7\Big|+c$

D. $\log_e\sqrt{x^2+5x-7}+c$

Answer: B

Solution:

Solution:

$I=\int \frac{e^{3\log_e(2x)}+5e^{2\log_e(2x)}}{e^{4\log_e(x)}+5e^{3\log_e(x)}-7e^{2\log_e(x)}}dx$

$=\int \frac{e^{\log_e(2x)^3}+5e^{\log_e(2x)^2}}{e^{\log_e x^4}+5e^{\log_e(x)^3}-7e^{\log_e(x)^2}}dx$

[using property $a\log x=\log x^a$]

$=\int \frac{8x^3+5(2x)^2}{x^4+5(x)^3-7x^2}dx$ [using property $a^{\log_a x}=x$]

$=\int \frac{8x^3+20x^2}{x^4+5x^3-7x^2}dx=\int \frac{4x^2(2x+5)}{x^2(x^2+5x-7)}dx$

$=\int \frac{4(2x+5)}{x^2+5x+7}dx$

Let $x^2+5x-7=t$, then $(2x+5)dx=dt$

$I=\int \frac{4dt}{t}=4\log_e t+c$

Put $t=x^2+5x-7$

$I=4\log_e|x^2+5x-7|+C$

Question113

The value of the integral

$\int \frac{\sin\theta\sin 2\theta(\sin^6\theta+\sin^4\theta+\sin^2\theta)\sqrt{2\sin^4\theta+3\sin^2\theta+6}}{1-\cos 2\theta}d\theta$

is (where, c is a constant of integration)
[25 Feb 2021 Shift 1]

Options:

A. $\frac{1}{18}[11-18\sin^2\theta+9\sin^4\theta-2\sin^6\theta]^{\frac{3}{2}}+c$

B. $\frac{1}{18}[9-2\cos^6-\theta 3\cos^4\theta-6\cos^2\theta]^{\frac{3}{2}}+c$

C. $\frac{1}{18}[9-2\sin^6\theta-3\sin^4\theta-6\sin^2\theta]^{\frac{3}{2}}+c$

D. $\frac{1}{18}[11-18\cos^2\theta+9\cos^4\theta-2\cos^6\theta]^{\frac{3}{2}}+c$

Answer: D

Solution:

Solution:
Let
$\int\left[\frac{\sin\theta\cdot\sin 2\theta(\sin^6\theta+\sin^4\theta+\sin^2\theta)\sqrt{2\sin^4\theta+3\sin^2\theta+6}}{1-\cos 2\theta}\right]d\theta$
$\because\ \sin 2A=2\sin A\cos A$
and $1-\cos 2A=2\sin^2 A\ \sin\theta\cdot 2\sin\theta(\sin^6\theta+\sin^4\theta+\sin^2\theta)$
$I=\int\frac{\sqrt{2\sin^4\theta+3\sin^2\theta+6}}{2\sin^2\theta}d\theta$
$I=\int\cos\theta(\sin^6\theta+\sin^4\theta+\sin^2\theta)$
$\sqrt{2\sin^4\theta+3\sin^2\theta+6}\,d\theta$
$=\int(t^6+t^4+t^2)\sqrt{2t^4+3t^2+6}\,dt$
$=\int(t^5+t^3+t)\sqrt{2t^6+3t^4+6t^2}\,dt$
Let $2t^6+3t^4+6t^2=z$
$\therefore\ dz=(12t^5+12t^3+12t)dt$
$\therefore\ dz=12(t^5+t^3+t)dt$
Now, $\frac{1}{12}\int\sqrt{z}\,dz=\frac{1}{12}\times\frac{z^{3/2}}{3/2}+c$
$=\frac{1}{18}z^{3/2}+c$
$=\frac{1}{18}[2t^6+3t^4+6t^2]^{3/2}+c$
$=\frac{1}{18}[2\sin^6\theta+3\sin^4\theta+6\sin^2\theta]^{3/2}+c$
$=\frac{1}{18}[(1-\cos^2\theta)\{2(1-\cos^2\theta)^3+3-3\cos^2\theta+6\}]^{3/2}+c$

$= \frac{1}{18}[(1-\cos^2\theta)(2\cos^4\theta - 7\cos^2\theta + 11)]^{3/2} + c$
$= \frac{1}{18}[-2\cos^6\theta + 9\cos^4\theta - 18\cos^2\theta + 11]^{3/2} + c$
$= \frac{1}{18}[11 - 18\cos^2\theta + 9\cos^4\theta - 2\cos^6\theta]^{3/2}$

Question114

For x > 0, if $f(x) = \int_1^x \frac{\log_e t}{(1+t)} dt$, then $f(e) + f\left(\frac{1}{e}\right)$ is equal to [26 Feb 2021 Shift 2]

Options:

A. 1

B. −1

C. $\frac{1}{2}$

D. 0

Answer: C

Solution:

Solution:
$f(x) = \int_1^x \frac{\log_e t}{(1+t)} dt$
Then, $f(e) = \int_1^{e \log_e t} \frac{dt}{1+t} \ldots$ (i)
and $f\left(\frac{1}{e}\right) = \int_1^{\frac{1}{e}} \frac{\log_e t}{1+t} dt \ldots$(ii)
Let $t = \frac{1}{u}$, $dt = \frac{-1}{u^2} du$ and put in Eq. (ii), we get
$f\left(\frac{1}{e}\right) = \int_1^e \frac{\log\left(\frac{1}{u}\right)}{1+\frac{1}{u}} \cdot \frac{-1}{u^2} du = \int_1^e \frac{\log u}{u(u+1)} du$
Using change of variable
$f\left(\frac{1}{e}\right) = \int_1^e \frac{\log t}{t(t+1)} dt \ldots$ (iii)
From Eqs. (i) and (iii), we get
$f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\log t}{1+t} dt + \int_1^e \frac{\log t}{t(1+t)} dt = \int_1^{e \log t} \frac{gt}{t}$
Take $\log t = v$, then $\frac{1}{t} dt = dv$

$f(e)+f\left(\frac{1}{e}\right)=\int_0^1 v\,dv=\left[\frac{v^2}{2}\right]_0^1=\frac{1}{2}$

$\therefore\ f(e)+f\left(\frac{1}{e}\right)=\frac{1}{2}$

Question115

If $I_{m\cdot n}=\int_0^1 x^{m-1}(1-x)^{n-1}dx$, for $m, n\geq 1$ and $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx=\alpha I_{m\cdot n}$, $\alpha\in R$, then α equals ______.
[26 Feb 2021 Shift 2]

Answer: 1

Solution:

Solution:

Given, $I_{mn}=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

Using substitution put $x=\frac{1}{t+1}$

Then, $dx=\frac{-1}{(t+1)^2}dt$

$I_{mn}=\int_\infty^0(-1)\frac{1}{(t+1)^{m-1}}\cdot\frac{t^{n-1}}{(t+1)^{n-1}}\cdot\frac{1}{(t+1)^2}dt$

$=-\int_\infty^0\frac{t^{n-1}}{(t+1)^{m+n}}dt\ldots$ (i)

Similarly,

$I_{mn}=\int_0^1 x^{n-1}(1-x)^{m-1}dx\ldots$ (ii)

$\Rightarrow\ I_{mn}=\int_0^\infty\frac{t^{m-1}}{(t+1)^{m+n}}$

From Eqs. (i) and (ii), we get

$2I_{mn}=\int_0^\infty\frac{t^{n-1}+t^{m-1}}{(t+1)^{m+n}}dt$

$2I_{mn}=\int_0^1\frac{t^{n-1}+t^{m-1}\,dt}{(t+1)^{m+n}}+\int_1^\infty\frac{t^{n-1}+t^{m-1}}{(t+1)^{m+n}}dt$ Let $I_1=\int_1^\infty\frac{t^{n-1}+t^{m-1}}{(t+1)^{m+n}}dt$

Let $t=\frac{1}{z}$, then $dt=\frac{-1}{z^2}dz$

$I_1=\int_1^0(-1)\frac{\left(\frac{1}{z}\right)^{n-1}+\left(\frac{1}{z}\right)^{m-1}}{\left(\frac{1}{z}+1\right)^{m+n}}\cdot\frac{1}{z^2}dz=-\int_1^0\frac{z^{n-1}+z^{m-1}}{(z+1)^{m+n}}dz$

$$2I_{mn}=\int_0^1\frac{t^{n-1}+t^{m-1}}{(t+1)^{m+n}}dt-\int_1^0\frac{z^{n-1}+z^{m-1}}{(z+1)^{m+n}}dz$$
$$=2\int_0^1\frac{t^{m-1}+t^{n-1}}{(t+1)^{m+n}}dt$$
$\Rightarrow\alpha=1$

Question116

The value of $\sum_{n=1}^{100}\int_n^n e^{x-[x]}dx$, where [x] is the greatest integer $\le x$, is [26 Feb 2021 Shift 1]

Options:

A. $100(e-1)$

B. $100(1-e)$

C. $100e$

D. $100(1+e)$

Answer: A

Solution:

Solution:

Let ' x ' be any real number, then $x=[x]+\{x\}$, where $[x]$ is integer part of x and $\{x\}$ is fractional part of x.

Then, $x-[x]=\{x\}$, Also period of $\{x\}=1$

Now, $\sum_{n=1}^{100}\int_{n-1}^{n}e^{x-[x]}dx=\sum_{n=1}^{100}\int_{n-1}^{n}e^{[x]}dx$

[Difference between upper and lower limit is 1 unit]

$$=\int_0^1 e^{[x]}dx+\int_1^2 e^{\{x\}}dx+\ldots+\int_{99}^{100}e^{[x]}dx$$
$$=e^x]_0^1+e^{(x-1)}]_1^2+\ldots+e^{(x-99)}]_{99}^{100}$$
$$=(e-1)+(e-1)+\ldots+(e-1)=100(e-1)$$

Question117

The value of $\int_{-\pi/2}^{\pi/2}\frac{\cos^2x}{1+3^x}dx$ is [26 Feb 2021 Shift 1]

Options:

A. $\frac{\pi}{4}$

B. 4π

C. $\frac{\pi}{2}$

D. 2π

Answer: A

Solution:

Solution:

Let $I = \int_{-\pi/2}^{\pi} \frac{\cos^2 x}{1+3^x} d x$ ... (i)

Using the property, $\int_a^b f(x) d x = \int_a^b f(a+b-x) d x$

$I = \int_{-\pi/2}^{\pi/2} \frac{\cos^2(\pi/2-\pi/2}{1+3^{\pi/2-\pi/2}}$

$= \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+3^{-x}} d x \quad [\because \cos(-x)=\cos x]$

$I = \int_{-\pi/2}^{\pi/2} \frac{3^x \cos^2 x}{(1+3^x)} d x$. . . (ii)

Adding Eqs. (i) and (ii),

$2I = \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+3^x} d x + \int_{-\pi/2}^{\pi/2} \frac{3^x \cos^2 x}{1+3^x} d x$

$= \int_{-\pi/2}^{\pi/2} \frac{(1+3^x)\cos^2 x}{1+3^x} d x = \int_{-\pi/2}^{\pi/2} \cos^2 x d x$

$= \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2x}{2} d x$

$= \frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_{-\pi/2}^{\pi/2} = \frac{1}{2}[\pi]$

$\Rightarrow 2I = \pi/2 \Rightarrow I = \frac{\pi}{4}$

Question118

The value of the integral $\int_0^{\pi} \left|\sin 2x\right| d x$ is ....

[26 Feb 2021 Shift 1]

Answer: 2

Solution:

Solution:

Let $I = \int_0^{\pi} \left| \sin 2x \right| dx$

$= 2 \int_0^{\pi/2} \left| \sin 2x \right| dx$ [$\because \sin 2x$ is periodic function]

$= 2 \int_0^{\pi/2} \sin 2x\, dx$ [$\sin 2x$ is positive in range $(0, \pi/2)$]

$= 2\left[\frac{-\cos 2x}{2} \right]_0^{\pi/2}$

$= -[\cos \pi - \cos 0] = -(-1 - 1) = 2$

$I = 2$

Question119

The value of $\int_{-2}^{2} \left| 3x^2 - 3x - 6 \right| dx$ is ....

[25 Feb 2021 Shift 2]

Answer: 19

Solution:

Solution:

$\int_{-2}^{2} \left| 3x^2 - 3x - 6 \right| dx = I$ (say)

$I = 3 \int_{-2}^{2} \left| x^2 - x - 2 \right| dx$

$= 3\left[\int_{-2}^{-1} (x^2 - x - 2) dx + \int_{-1}^{2} (-x^2 + x + 2) dx \right]$

$= 3\left[\left(\frac{x^3}{3} - \frac{x^2}{2} - 2x \right)^{-1} - \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x \right)^{2} \right]$

$= 19$

Question120

If $I_n = \int_{\pi/4}^{\pi/2} \cot^n x \, dx$, then
[25 Feb 2021 Shift 2]

Options:

A. $\frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}$ are in AP

B. $I_2+I_4, I_3+I_5, I_4+I_6$ are in AP

C. $\frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}$ are in GP

D. $I_2+I_4, (I_3+I_5)^2, I_4+I_6$ are in GP

Answer: A

Solution:

Solution:

$I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x \, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x (\cot^2 x) \, dx$

$I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x \operatorname{cosec}^2 x \, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x \, dx$

$I_n + I_{n-2} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x \cdot \operatorname{cosec}^2 x \, dx$

Now, let $\cot x = t$, then $\operatorname{cosec}^2 x \, dx = -dt$, limit will be

$I_n + I_{n-2} = \int_1^0 -t^{n-2} \, dt$

$= \frac{-(t)^{n-1}}{n-1}]_1^0 = -\left\{ \frac{0}{n-1} - \frac{(1)^{n-1}}{n-1} \right\}$

$I_n + I_{n-2} = \frac{1}{n-1}$

Now, put $n = 4$

$\Rightarrow I_2 + I_4 = \frac{1}{3}$, then $\frac{1}{I_2+I_4} = 3 \ldots$ (i)

Put $n = 5$

$\Rightarrow I_5 + I_3 = \frac{1}{4}$, then $\frac{1}{I_3+I_5} = 4 \ldots$ (ii)

Put $n = 6$

$\Rightarrow I_6 + I_4 = \frac{1}{5}$, then $\frac{1}{I_4+I_6} = 5 \ldots$ (iii)

Here, from Eqs. (i), (ii) and (iii), we conclude
$\frac{1}{I_2+I_4}$, $\frac{1}{I_3+I_5}$ and $\frac{1}{I_4+I_6}$ are in AP with common difference 1.

Question121

The value of $\int_{-1}^{1} x^2 e^{[x^3]} dx$, where [t] denotes the greatest integer $\le$t, is [25 Feb 2021 Shift 1]

Options:

A. $\frac{e-1}{3e}$

B. $\frac{e+1}{3}$

C. $\frac{e+1}{3e}$

D. $\frac{1}{3e}$

Answer: C

Solution:

Solution:
Given, $\int_{-1}^{1} x^2 e^{[x^3]} dx$, where $[t]$ is greatest integer function.
$\because\ [x^3] = 0\ \forall x \in (0, 1)$
and $[x^3] = -1\ \forall x \in (-1, 0)$
So, $\int_{-1}^{1} x^2 e^{[x^3]} dx = \int_{-1}^{0} x^2 e^{-1} dx + \int_{0}^{1} x^2 e^{0} dx$
$= \frac{1}{e}\int_{-1}^{0} x^2 dx + \int_{0}^{1} x^2 dx = \frac{1}{e} \times \left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{0}^{1}$
$= \frac{1}{e} \times \left[0 + \frac{1}{3}\right] + \left[\frac{1}{3}\right] = \frac{1}{3e} + \frac{1}{3} = \left(\frac{1+e}{3e}\right)$

Question122

Let $f(x)$ be a differentiable function defined on [0, 2], such that $f'(x) = f'(2-x)$, for all $x \in (0, 2)$, $f(0) = 1$ and $f(2) = e^2$. Then, the value of

$\int_0^2 f(x)dx$ **is**

[24 Feb 2021 Shift 2]

Options:

A. $1-e^2$

B. $1+e^2$

C. $2(1-e^2)$

D. $2(1+e^2)$

Answer: B

Solution:

Solution:

Given, $f(0)=1$ . . . (i)

$f(2) = e^2$ . . . (ii)

$f'(x) = f'(2-x)$

Integrating w.r.t. x,

$f(x)=-f(2-x)+C$

Put $x=0$

$f(0) = -f(2)+C$

$\Rightarrow 1 = -e^2+C$ [from Eqs. (i) and (ii)]

$\Rightarrow C = 1+e^2$

$\therefore f(x) = -f(2-x)+1+e^2$

or $f(x)+f(2-x)=1+e^2$ . . . (iii)

Let $I = \int_0^2 f(x)dx$ . . . (iv)

Also, $I = \int_0^2 f(2-x)dx$. . . (v)

Now, adding Eqs. (iv) and (v),

$2I = \int_0^2 [f(x)+f(2-x)]dx$

$2I = \int_0^2 (1+e^2)dx$ [from Eq. (iii)]

$2I = 2(1+e^2)$

$\therefore I = (1+e^2)$

Question123

The value of the integral $\int_1^3 [x^2-2x-2]dx$, where $[x]$ denotes the greatest integer less than or equal to x, is

[24 Feb 2021 Shift 2]

Options:

A. $-\sqrt{2}-\sqrt{3}+1$

B. $-\sqrt{2}-\sqrt{3}-1$

C. -5

D. -4

Answer: B

Solution:

Solution:

Let $I = \int_1^3 [x^2-2x-2]dx$

$= \int_1^3 [x^2-2x+1-3]dx = \int_1^3 (x-1)^2-3 \Big] dx$

$= \int_1^3 [(x-1)^2]dx + \int_1^3 -3dx$

Put $x-1=t$; $dx=dt$, when $x=1, t=0$ and $x=3, t=2$

$\therefore I = -3[x]_1{}^3 + \int_0^2 [t^2]dt$

$= -6 + \int_0^1 0dt + \int_1^{\sqrt{2}} 1dt + \int_{\sqrt{2}}^{\sqrt{3}} 2dt + \int_{\sqrt{3}}^{2} 3dt$

$= -6+(0)+(\sqrt{2}-1)+2(\sqrt{3}-\sqrt{2})+3(2-\sqrt{3})$

$= -6+\sqrt{2}-1+2\sqrt{3}-2\sqrt{2}+6-3\sqrt{3}$

$I = -1-\sqrt{2}-\sqrt{3}$

Question124

$\lim\limits_{x\to 0} \frac{\int_0^{x^2}(\sin\sqrt{t})dt}{x^3}$ **is equal to :**

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Options:

A. $\frac{2}{3}$

B. $\frac{3}{2}$

C. 0

D. $\frac{1}{15}$

Answer: A

Solution:

Solution:

$$\lim_{x \to 0^+} \frac{\int_0^{x^2} \sin \sqrt{t}\, d\,t}{x^3} = \lim_{x \to 0^+} \frac{(\sin x)2x}{3x^2}$$

$$= \lim_{x \to 0^+} \left(\frac{\sin x}{x} \right) \times \frac{2}{3} = \frac{2}{3}$$

Question125

If $\int_{-a}^{a} (|x| + |x-2|)d\,x = 22$, (a > 2) and [x] denotes the greatest integer ≤x, then $\int_{a}^{-a} (x + [x])d\,x$ is equal to __
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Answer: 3

Solution:

Solution:

$$\int_{-a}^{0} (-2x+2)d\,x + \int_{0}^{2} (x+2-x)d\,x + \int_{2}^{a} (2x-2)d\,x = 22$$

$$\Rightarrow x^2 - 2x \,|_0^{-a} + 2x \,|_0^{2} + x^2 - 2x \,|_2^{a} = 22$$
$$\Rightarrow a^2 + 2a + 4 + a^2 - 2a - (4-4) = 22$$
$$\Rightarrow 2a^2 = 18 \Rightarrow a = 3$$
$$\therefore \int_{3}^{-3} (x + [x])d\,x = -(-3-2-1+1+2) = 3$$

Question126

The integral $\int \frac{(2x-1)\cos\sqrt{(2x-1)^2+5}}{\sqrt{4x^2-4x+6}}dx$ is equal to (where, c is a constant of integration)
[18 Mar 2021 Shift 1]

Options:

A. $\frac{1}{2}\sin\sqrt{(2x-1)^2+5}+c$

B. $\frac{1}{2}\cos\sqrt{(2x+1)^2+5}+c$

C. $\frac{1}{2}\cos\sqrt{(2x-1)^2+5}+c$

D. $\frac{1}{2}\sin\sqrt{(2x+1)^2+5}+c$

Answer: A

Solution:

Solution:

Let $I=\int \frac{(2x-1)\cos\sqrt{(2x-1)^2+5}}{\sqrt{4x^2-4x+6}}dx$

$=\int \frac{(2x-1)\cos\sqrt{(2x-1)^2+5}}{\sqrt{(2x-1)^2+5}}dx$

Putting $(2x-1)^2+5=z^2$

$\Rightarrow\ 2(2x-1)\times 2\cdot dx=2zdz$

$\Rightarrow\ (2x-1)dx=\frac{1}{2}zdz$

$\therefore\ I=\int \frac{\cos z}{z}\cdot\frac{1}{2}z\cdot dz=\frac{1}{2}\int\cos z\,dz=\frac{1}{2}\sin z+C$

$=\frac{1}{2}\sin\sqrt{(2x-1)^2+5}+c$

[Note You can also substitute $\sqrt{(2x-1)^2+5}=z$ and then proceed.]

Question127

If $f(x)=\int \frac{5x^8+7x^6}{(x^2+1+2x^7)^2}dx$, $(x\geq 0)$, $f(0)=0$ and $f(1)=\frac{1}{K}$, then the value of K is
[18 Mar 2021 Shift 1]

Answer: 4

Solution:

Solution:

Let $I = \int \frac{5x^8 + 7x^6}{(x^2 + 1 + 2x^7)^2} dx$

$= \int \frac{5x^8 + 7x^6}{x^{14}(x^{-5} + x^{-7} + 2)^2} dx$

$= \int \frac{\frac{5x^8 + 7x^6}{x^{14}}}{\frac{x^{14}(x^{-5} + x^{-7} + 2)^2}{x^{14}}} dx$

$\Rightarrow \ I = \int \frac{5x^{-6} + 7x^{-8}}{(x^{-5} + x^{-7} + 2)^2} dx$

Putting $x^{-5} + x^{-7} + 2 = z$

$\Rightarrow \ -(5x^{-6} + 7x^{-8}) dx = dz$

$\therefore I = -\int \frac{dz}{z^2} = -\left(\frac{1}{-z} \right) + c$

$\Rightarrow \Rightarrow \ I = \frac{1}{x^{-5} + x^{-7} + 2} + c$

$\Rightarrow f(x) = \frac{x^7}{x^2 + 1 + 2x^7} + c$

Given, $f(0) = 0$

$\Rightarrow \ c = 0$

$\therefore \ f(x) = \frac{x^7}{x^2 + 1 + 2x^7}$

$\therefore \ f(1) = \frac{1}{1 + 1 + 2}$

$= \frac{1}{4} = \frac{1}{K}$

Hence, $K = 4$.

Question128

For real numbers α, β, γ and δ, if $\int \frac{(x^2 - 1) + \tan^{-1}\left(\frac{x^2 + 1}{x}\right)}{(x^4 + 3x^2 + 1)\tan^{-1}\left(\frac{x^2 + 1}{x}\right)} dx$

$= \ \alpha \log_e \left[\tan^{-1}\left(\frac{x^2 + 1}{x} \right) \right] + \beta \tan^{-1}\left(\frac{\gamma(x^2 - 1)}{x} \right) + \delta \tan^{-1}\left(\frac{x^2 + 1}{x} \right) + C$

where C is an arbitrary constant, then the value of $10(\alpha + \beta\gamma + \delta)$ is equal to..........

[16 Mar 2021 Shift 2]

Answer: 6

Solution:

Solution:

Let $I = \int \dfrac{(x^2-1)+\tan^{-1}\left(\dfrac{x^2+1}{x}\right)}{(x^4+3x^2+1)\tan^{-1}\left(\dfrac{x^2+1}{x}\right)}dx$

$\Rightarrow \; I = \int \dfrac{x^2-1}{(x^4+3x^2+1)\tan^{-1}\left(\dfrac{x^2+1}{x}\right)}dx + \int \dfrac{1}{x^4+3x^2+1}dx$

Again let $I_1 = \int \dfrac{x^2-1}{(x^4+3x^2+1)\tan^{-1}\left(\dfrac{x^2+1}{x}\right)}dx$

and $\; I_2 = \int \dfrac{dx}{x^4+3x^2+1}$

$\therefore \; I = I_1 + I_2$......(i)

Now, $\; I_1 = \int \dfrac{(x^2-1)}{(x^4+3x^2+1)\tan^{-1}\left(\dfrac{x^2+1}{x}\right)}dx$

Let $\tan^{-1}\left(\dfrac{x^2+1}{x}\right) = t$

$\Rightarrow \dfrac{x^2-1}{(x^4+3x^2+1)}dx = dt$

$\therefore I_1 = \int \dfrac{dt}{t} = \log\left|t\right| + C_1 = \log\left|\tan^{-1}\left(\dfrac{x^2+1}{x}\right)\right| + C_1$

$I_2 = \int \dfrac{1}{x^4+3x^2+1}dx = \dfrac{1}{2}\int \dfrac{(x^2+1)-(x^2-1)}{x^4+3x^2+1}dx$

$= \dfrac{1}{2}\int \dfrac{x^2+1}{x^4+3x^2+1}dx - \dfrac{1}{2}\int \dfrac{x^2-1}{x^4+3x^2+1}dx$

$= \dfrac{1}{2}\int \dfrac{1+1/x^2}{x^2+3+\dfrac{1}{x^2}}dx - \dfrac{1}{2}\int \dfrac{1-1/x^2}{x^2+3+\dfrac{1}{x^2}}dx$

$I_2 = \dfrac{1}{2}\int \dfrac{1+\dfrac{1}{x^2}}{\left(x-\dfrac{1}{x}\right)^2+5}dx - \dfrac{1}{2}\int \dfrac{1-1/x^2}{\left(x+\dfrac{1}{x}\right)^2+1}dx$

$= \dfrac{1}{2\sqrt{5}}\tan^{-1}\left(\dfrac{x^2-1}{\sqrt{5}x}\right) - \dfrac{1}{2}\tan^{-1}\left(\dfrac{x^2+1}{x}\right) + C_2$

$I = \log\left|\tan^{-1}\left(\dfrac{x^2+1}{x}\right)\right| + \dfrac{1}{2\sqrt{5}}\tan^{-1}\left(\dfrac{x^2-1}{\sqrt{5}x}\right) - \dfrac{1}{2}\tan^{-1}\left(\dfrac{x^2+1}{x}\right) + C$

$= \alpha\log_e\left(\tan^{-1}\left(\dfrac{x^2+1}{x}\right)\right) + \beta\tan^{-1}\left(\dfrac{\gamma(x^2-1)}{x}\right) + \delta\tan^{-1}\left(\dfrac{x^2+1}{x}\right) + C$ (given)

$\therefore\ \alpha = 1, \beta = \frac{1}{2\sqrt{5}} \gamma = \frac{1}{\sqrt{5}}$ and $\delta = -\frac{1}{2}$

$\therefore$ Required value of $10(\alpha + \beta\gamma + \delta)$

$= 10\left(1 + \frac{1}{10} - \frac{1}{2}\right)$

$= 10\left(\frac{10 + 1 - 5}{10}\right)$

$= 6$

Question129

Let $f(x)$ and $g(x)$ be two functions satisfying $f(x^2) + g(4 - x) = 4x^3$ and $g(4 - x) + g(x) = 0$ then the value of $\int_{-4}^{4}\int_{-1}^{2} (x)^2 d x$ is ....
[18 Mar 2021 Shift 1]

Answer: 512

Solution:

Solution:

Given, $f(x^2) + g(4 - x) = 4x^3$

and $g(4 - x) + g(x) = 0$

Let $I = \int_{-4}^{4} f(x^2) d x$

$= 2\int_{0}^{4} f(x^2) d x$

$\Rightarrow\ I = 2 \cdot \int_{0}^{4} [4x^3 - g(4 - x)] d x$

$= 8\int_{0}^{4} x^3 d x - 2\int_{0}^{4} g(4 - x) d x$

$= 8\left(\frac{x^4}{4}\right)_{0}^{4} - 2I_1 = 2(4^4 - 0^4) - 2I_1$

$= 2^9 - 2I_1$

where, $I_1 = \int_{0}^{4} g(4 - x) d x$

Now, $I_1 = \int_{0}^{4} g(4 - x) d x$.....(i)

$\Rightarrow\ I_1 = \int_{0}^{4} g\left[4 - (0 + 4 - x] d x\left[\because \int_{a}^{b} f(x) d x = \int_{a}^{b} f(a + b - x) d x\right]\right.$

$\Rightarrow\ I_1 = \int_{0}^{4} g(x) = d x$...(ii)

Adding Eqs. (i) and (ii),

$$2I_1 = \int_0^4 [g(x) + g(4-x)]dx$$
$\Rightarrow \ 2I_1 = 0$
$\Rightarrow \ I_1 = 0$ ($\because g(x) + g(4-x) = 0$, given)
$\therefore I = 2^9 - 2I_1$
$\Rightarrow I = 2^9 = 512$

Question130

Let P(x) be a real polynomial of degree 3 which vanishes at $x = -3$. Let P(x) have local minima at $x = 1$, local maxima at $x = -1$ and $\int_{-1}^{1} P(x)dx = 18$, then the sum of all the coefficients of the polynomial P(x) is equal to
[18 Mar 2021 Shift 2]

Answer: 8

Solution:

Solution:
Let $P'(x) = a(x-1)(x+1)$
$\Rightarrow \ P'(x) = a(x^2 - 1)$
$\therefore \ P(x) = a\int(x^2 - 1)dx \Rightarrow P(x) = a\left(\frac{x^3}{3} - x\right) + C$
According to the question, $P(-3) = 0$
$a\left(-\frac{27}{3} + 3\right) + C = 0$
$\Rightarrow \ -6a + C = 0$...(i)
Now, $\int_{-1}^{1}\left(a\left(\frac{x^3}{3} - x\right) + C\right)dx = 18$ (given)
$\Rightarrow \ 2C = 18$
$\Rightarrow \ C = 9$...(ii)
From Eqs. (i) and (ii),
$-6a + 9 = 0 \Rightarrow a = \frac{3}{2}$
$\therefore \ P(x) = \frac{3}{2}\left(\frac{x^3}{3} - x\right) + 9$
$\therefore$ Sum of the all coefficient $= \frac{1}{2} - \frac{3}{2} + 9 = 8$

Question131

Which of the following statements is correct for the function g(α) for α ∈ R, such that $g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\sin^{\alpha} x}{\cos^{\alpha} x + \sin^{\alpha} x} dx$
[17 Mar 2021 Shift 1]

Options:

A. g(α) is a strictly increasing function

B. g(α) has an inflection point at $\alpha = -\frac{1}{2}$

C. g(α) is a strictly decreasing function

D. g(α) is an even function

Answer: D

Solution:

Solution:

$g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\sin^{\alpha} x}{\cos^{\alpha} x + \sin^{\alpha} x} dx$...(i)

Applying $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\sin^{\alpha}(\pi/2 - x)}{\cos^{\alpha}\left(\frac{\pi}{2} - x\right) + \sin^{\alpha}\left(\frac{\pi}{2} - x\right)} dx$

$g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\cos^{\alpha} x}{\cos^{\alpha} x + \sin^{\alpha} x} dx$.....(ii)

Adding Eqs. (i) and (ii),

$2g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\sin^{\alpha} x + \cos^{\alpha} x}{\sin^{\alpha} x + \cos^{\alpha} x} dx$

$2g(\alpha) = \int_{\pi/6}^{\pi/3} 1 \cdot dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

$\therefore\ g(\alpha) = \frac{\pi}{12}$

g(α) is constant function.

∴ It is even function.

Question132

Let $f: R \to R$ be defined as $f(x) = e^{-x} \sin x$. If $F: [0, 1] \to R$ is a differentiable function, such that $F(x) = \int_0^x f(t) dt$, then the value of $\int_0^1 [F'(x) + f(x)] e^x dx$ lies in the interval
[17 Mar 2021 Shift 2]

Options:

A. $\left[\frac{327}{360}, \frac{329}{360}\right]$

B. $\left[\frac{330}{360}, \frac{331}{360}\right]$

C. $\frac{331}{360}, \frac{334}{360}$

D. $\left[\frac{335}{360}, \frac{336}{360}\right]$

Answer: B

Solution:

Solution:

Given, $f(x) = e^{-x} \cdot \sin x$

and $F(x) = \int_0^x f(t) d t$

$\because F(x)$ is differentiable function.

$\therefore F'(x) = f(x) \times 1 - f(0) \times 0$ (using Newton-Leibnitz rule)

$\Rightarrow F'(x) = f(x)$...(i)

Let $I = \int_0^1 [F'(x) + f(x)] e^x d x$

$= \int_0^1 [f(x) + f(x)] e^x d x = \int_0^1 2 \cdot f(x) \cdot e^x d x$[from Eq. (i)]

$\Rightarrow I = 2 \cdot \int_0^1 f(x) \cdot e^x d x$

$= 2 \cdot \int_0^1 e^{-x} \sin x \cdot e^x d x$

$= 2 \int_0^1 \sin x d x = 2[-\cos x]_0^1$

$= 2[-(\cos 1 - \cos 0)] = 2(1 - \cos 1)$

$\Rightarrow 1 = 2 \cdot \left[1 - \left(1 - \frac{(1)^2}{2!} + \frac{(1)^4}{4!} - \frac{(1)^6}{6!} + \frac{(1)^8}{8!} - \ldots\right]\right.$

[using expansion of $\cos x$ i.e.,

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots 1$

$I = 2\left[1 - 1 + \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} - \frac{1}{8!} + \ldots\right]$

$\Rightarrow I = 2\left(\frac{1}{2} - \frac{1}{24} + \frac{1}{720} - \ldots\right)$

Now, $I < 2\left(\frac{1}{2} - \frac{1}{24} + \frac{1}{720}\right)$

$\Rightarrow I < \left(1 - \frac{1}{12} + \frac{1}{360}\right) \Rightarrow I < \frac{360 - 30 + 1}{360}$

$\Rightarrow I < \frac{331}{360}$...(ii)

Also, $I > 2\left(\frac{1}{2} - \frac{1}{24}\right) \Rightarrow I > \left(1 - \frac{1}{12}\right)$

$\Rightarrow I > \frac{11}{12} \Rightarrow I > \frac{11 \times 30}{12 \times 30}$

$\Rightarrow I > \frac{330}{360}$ ....(iii)

From Eqs. (ii) and (iii), we get

$\frac{330}{360} < 1 < \frac{331}{360}$

Question133

If the integral $\int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \gamma$, where α, β, γ are integers and [x] denotes the greatest integer less than or equal to x, then the value of $\alpha + \beta + \gamma$ is equal to
[17 Mar 2021 Shift 2]

Options:

A. 0

B. 20

C. 25

D. 10

Answer: A

Solution:

Solution:

Let $I = \int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx$

$\Rightarrow I = \int_0^{10} \frac{[\sin 2\pi x]}{e^{\{x\}}} dx \quad [\because x - [x] = \{x\}]$

From above integrand, we observe that $\frac{[\sin 2\pi x]}{e^{[x]}}$ is a periodic function with period ' 1 '.

$\therefore I = 10 \int_0^1 \frac{[\sin 2\pi x]}{e^{[x]}}$

[by the property of definite integral,

$\int_0^{nT} f(x) dx = n \int_0^T f(x) dx$

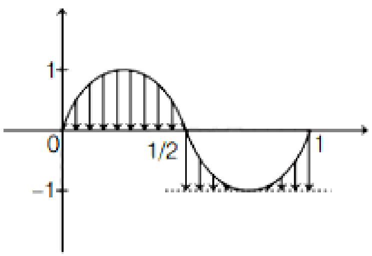
where $f(x)$ is a periodic function with period $= T$]

$\Rightarrow \ I = 10 \cdot \int_0^1 \frac{[\sin 2\pi x]}{e^x} d\,x \ \ [\because \{x\} = x, 0 \le x < 1]$

$\Rightarrow \ I = 10\left(\int_0^{1/2} \frac{[\sin 2\pi x]}{e^x} d\,x + \int_{1/2}^{1} \frac{[\sin 2\pi x]}{e^x} d\,x \right)$

$\Rightarrow \ I = 10\left(\int_0^{1/2} \frac{0}{e^x} d\,x + \int_{1/2}^{1} \frac{(-1)}{e^x} d\,x \right)$

$\Rightarrow \ I = 10\left(0 - \int_{1/2}^{1} e^{-x} d\,x \right) \Rightarrow I = -10\left[\frac{e^{-x}}{-1} \right]_{1/2}^{1}$



$\Rightarrow \ I = 10[e^{-1} - e^{-1/2}] \Rightarrow I = 10e^{-1} - 10e^{\frac{-1}{2}}$

$\Rightarrow \ I = 10e^{-1} + (-10) \cdot e^{\frac{-1}{2}} + 0 \ldots$(i)

Comparing Eq. (i) by $\alpha e^{-1} + \beta e^{\frac{-1}{2}} + \gamma$, we get

$\alpha = 10, \beta = -10$ and $\gamma = 0$

Hence, $\alpha + \beta + \gamma = 10 - 10 + 0$

$\Rightarrow \ \alpha + \beta + \gamma = 0$

Question134

Let $I_n = \int_1^e x^{19}(\log|x|)^n d\,x$, where $n \in N$. If $(20)\ I_{10} = \alpha I_9 + \beta I_8$, for natural numbers α and β, then $\alpha - \beta$ is equal to ....
[17 Mar 2021 Shift 2]

Answer: 1

Solution:

Solution:

Given, $I_n = \int_1^e x^{19}(\log|x|)^n_{d\,x}$

$\Rightarrow \ I_n = \left[\frac{x^{20}}{20}(\ln|x|^n] \right]_1^e - \int_1^e n \cdot \frac{(\ln|x|)^{n-1}}{x} \cdot \frac{x^{20}}{20} d\,x$

(using integration by parts)

$\Rightarrow I_n = \frac{e^{20}}{20} - \frac{n}{20}\int_1^e (\ln|x|)^{n-1} \cdot x^{19} dx$

$\Rightarrow I_n = \frac{e^{20}}{20} - \frac{n}{20} \cdot I_{n-1} \Rightarrow 20I_n + nI_{n-1} = e^{20}$

Put $n = 10$ and $n = 9$, we get

$20I_{10} + 10I_9 = e^{20}$...(i)

and $20I_9 + 9I_8 = e^{20}$.....(ii)

From Eqs. (i) and (ii),

$20I_{10} - 10I_9 - 9I_8 = 0$

$\Rightarrow 20l_{10} = 10l_9 + 9I_8$ comparing this to

$20(I_{10}) = \alpha I_9 + \beta I_8$, we get

$\alpha = 10, \beta = 9$

$\therefore \alpha - \beta = 1$

Question135

Let f : (0, 2) → R be defined as $f(x) = \log_2\left[1 + \tan\left(\frac{\pi x}{4}\right)\right]$ Then,

$\lim_{n \to \infty} \frac{2}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \ldots + f(1)\right]$ is equal to..........

[16 Mar 2021 Shift 1]

Answer: 1

Solution:

Solution:

$f(x) = \log_2\left[1 + \tan\left(\frac{\pi x}{4}\right)\right]$

Then, $= \lim_{n \to \infty} \frac{2}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \ldots + f(1)\right] = 2\lim_{n \to \infty} \sum_{r=1}^{n}\left(\frac{1}{n}\right) f\left(\frac{r}{n}\right)$

Let $l = \frac{2}{\log_n 2}\int_0^1 \log_n\left[1 + \tan\left(\frac{\pi x}{4}\right)\right] dx$ .....(i)

as, $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$

So, $x \to 1 - x$

$l = \frac{2}{\log_n 2}\int_0^1 \log_n\left[1 + \tan\frac{\pi}{4}(1 - x)\right] dx$

$= \frac{2}{\log_n 2}\int_0^1 \log_n\left[1 + \tan\left(\frac{\pi}{4} - \frac{\pi x}{4}\right)\right] dx$

$$= \frac{2}{\log_n 2}\int_0^1 \log_n\left[1+\left(\frac{1-\tan\pi x/4}{1+\tan\pi x/4}\right)\right]dx$$

$$= \frac{2}{\log_n 2}\int_0^1 \log_n\left(\frac{2}{1+\tan\frac{\pi x}{4}}\right)dx$$

$$= \frac{2}{\log_n 2}\int_0^1 \log_n 2-\log_n\left(1+\tan\frac{\pi x}{4}\right)dx\text{.....(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2l = \frac{2}{\log_n 2}\int_0^1 \log_n 2 dx$$

$l = 1$

Question136

If the normal to the curve $y(x) = \int_0^x (2t^2 - 15t + 10)dt$ at a point (a, b) is parallel to the line $x + 3y = -5$, $a > 1$, then the value of $|a + 6b|$ is equal to.........
[16 Mar 2021 Shift 1]

Answer: 406

Solution:

Solution:

Given, $y(x) = \int_0^x (2t^2 - 15t + 10)dt$

$\Rightarrow\ y'(x) = 2x^2 - 15x + 10$

Since equation of normal is parallel to $x + 3y = -5$

$\therefore$ Slope of normal to $y(x) =$ Slope of lime

$$\Rightarrow\ \frac{-1}{[y'(x)]_{a,b}} = \frac{-1}{3}$$

or $[y'(x)]_{a,b} = 3$

$2a^2 - 15a + 10 = 3$

$\Rightarrow\ 2a^2 - 15a + 7 = 0$

$\Rightarrow\ (2a - 1)(a - 7) = 0$

$a = \frac{1}{2}$ or 7

As, $a > 1$, so, $a = 7$

Now, $(7, b)$ lies on $y(x)$,

$\therefore\ b = \int_0^a (2t^2 - 5t + 10)dt$

$\Rightarrow\ b = \frac{2}{3}a^3 - \frac{15}{2}a^2 + 10a$

$\Rightarrow \ b = \frac{2}{3}(7)^3 - \frac{15}{2}(7)^2 + 10(7)$

$\Rightarrow \ b = \frac{-413}{6}$

So, $a + 6b = 7 - 6\left(\frac{413}{6}\right) = -406$

$\therefore \ |a + 6b| = 406$

Question137

Let $f : R \to R$ be a continuous function such that $f(x) + f(x+1) = 2$, for all $x \in R$. If $I_1 = \int_0^8 f(x)dx$ and $I_2 = \int_{-1}^3 f(x)dx$, then the value of $I_1 + 2I_2$ is equal to.......... [16 Mar 2021 Shift 1]

Answer: 16

Solution:

Solution:

Given, $f(x) + f(x+1) = 2$....(i)

$I_1 = \int_0^8 f(x)$

and $I_2 = \int_{-1}^3 f(x)dx$

Let $f(0) = a$

Put $x = 0$ in Eq. (i)

$f(0) + f(1) = 2$

$f(1) = 2 - a$

Put $x = 1$ in Eq. (i)

$f(1) + f(2) = 2$

$f(2) = a$ and so on

So, $f(0) = f(2) = f(4) \ldots = a$

$f(1) = f(3) = f(5) \ldots = 2 - a$

Clearly, $f(x)$ is periodic with its period 2 units.

So, $I_1 = \int_0^{24} f(x)dx$

$\Rightarrow I_1 = 4\int_0^2 f(x)dx$

Now, $I_2 = \int_{-1}^3 f(x)dx$

$x \to x + 1$

$I_2 = \int_0^4 f(x+1)dx = \int_0^4 [2 - f(x)]dx$

$\Rightarrow \; I_2 = 8 - 2\int_0^2 f(x)dx$

$\Rightarrow \; 2I_2 = 16 - 4\int_0^2 f(x)dx$

$\Rightarrow 2I_2 = 16 - I_1$

$\therefore \; I_1 + 2I_2 = 16$

Question138

Consider the integral $I = \int_0^{10} \frac{[x]e^{[x]}}{e^{x-1}}dx$ where [x] denotes the greatest integer less than or equal to x. Then, the value of / is equal to [16 Mar 2021 Shift 2]

Options:

A. $9(e-1)$

B. $45(e+1)$

C. $45(e-1)$

D. $9(e+1)$

Answer: C

Solution:

Solution:

We have, $\int_0^{10} \frac{[x]e^{[x]}}{e^{x-1}}dx$

$= e\int_0^{10} \frac{[x]e^{[x]}}{e^x}dx$

$= e\int_0^1 \frac{0}{e^x}dx + e\int_1^2 \frac{e}{e^x}dx + e\int_2^3 \frac{2e^2}{e^x}dx + \dots$

$\Rightarrow \int_a^b e^{-x}dx = \frac{e^{-x}}{-1}\Big|_a^b \Rightarrow (e^{-a} - e^{-b})$

$\Rightarrow \; e^2\left(\frac{1}{e} - \frac{1}{e^2}\right) + 2e^3\left(\frac{1}{e^2} - \frac{1}{e^3}\right) + 3e^4\left(\frac{1}{e^3} - \frac{1}{e^4}\right) + \dots + 9e^{10}\left(\frac{1}{e^9} - \frac{1}{e^{10}}\right)$

$= (e-1) + 2(e-1) + 3(e-1) + \dots + 9(e-1)$

$= (1+2+3+\dots+9)(e-1) = \left(\frac{9 \times 10}{2}\right)(e-1)$

$= 45(e-1)$

Question139

Let the domain of the function $f(x) = \log_4(\log_5(\log_3(18x - x^2 - 77)))$ be (a, b)Then the value of the integral $\int_a^b \frac{\sin^3 x}{(\sin^3 x + \sin^3(a + b - x))} dx$ is equal to ________.
[27 Jul 2021 Shift 1]

Answer: 1

Solution:

Solution:
For domain
$\log_5(\log_3(18x - x^2 - 77)) > 0$
$\log_3(18x - x^2 - 77) > 1$
$18x - x^2 - 77 > 3$
$x^2 - 18x + 80 < 0$
$x \in (8, 10)$
$\Rightarrow a = 8$ and $b = 10$
$I = \int_a^b \frac{\sin^3 x}{\sin^3 x + \sin^3(a + b - x)} dx$
$I = \int_a^b \frac{\sin^3(a + b - x)}{\sin^3 x + \sin^3(a + b - x)}$
$2I = (b - a) \Rightarrow I = \frac{b - a}{2}$ ($\because a = 8$ and $b = 10$)
$I = \frac{10 - 8}{2} = 1$

Question140

The value of $\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \frac{(2j - 1) + 8n}{(2j - 1) + 4n}$ is equal to :
[27 Jul 2021 Shift 1]

Options:

A. $5 + \log_e\left(\frac{3}{2}\right)$

B. $2 - \log_e\left(\frac{2}{3}\right)$

C. $3+2\log_e\left(\frac{2}{3}\right)$

D. $1+2\log_e\left(\frac{3}{2}\right)$

Answer: D

Solution:

Solution:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{\left(\frac{2j}{n}-\frac{1}{n}+8\right)}{\left(\frac{2j}{n}-\frac{1}{n}+4\right)}$$

$$\int_0^1\frac{2x+8}{2x+4}dx=\int_0^1 dx+\int_0^1\frac{4}{2x+4}dx$$

$$=1+4\frac{1}{2}(\ln|2x+4|)\Big|_0^1$$

$$=1+2\ln\left(\frac{3}{2}\right)$$

Question141

The value of the definite integral $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dx}{(1+e^{x\cos x})(\sin^4x+\cos^4x)}$ is equal to:

[27 Jul 2021 Shift 1]

Options:

A. $-\frac{\pi}{2}$

B. $\frac{\pi}{2\sqrt{2}}$

C. $-\frac{\pi}{4}$

D. $\frac{\pi}{\sqrt{2}}$

Answer: B

Solution:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1+e^{x\cos x})(\sin^4 x+\cos^4 x)} \quad \text{.........(1)}$$

Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1+e^{-x\cos x})(\sin^4 x+\cos^4 x)} \quad \text{........(2)}$$

Add (1) and (2)

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\sin^4 x+\cos^4 x}$$

$$2I = 2\int_0^{\frac{\pi}{4}} \frac{dx}{\sin^4 x+\cos^4 x}$$

$$I = \int_0^{\frac{\pi}{4}} \frac{(1+\tan^2 x)\sec^2 x}{\tan^4 x+1} dx$$

$$I = \int_0^{\frac{\pi}{4}} \left(1+\frac{1}{\tan^2 x}\right)\sec^2 \frac{x}{\left(\tan x-\frac{1}{\tan x}\right)^2+2} dx$$

$$\tan x-\frac{1}{\tan x} = t$$

$$\left(1+\frac{1}{\tan^2 x}\right)\sec^2 x dx = dt$$

$$I = \int_{-\infty}^{0} \frac{dt}{t^2+2} = \left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_{-\infty}^{0}$$

$$I = 0-\frac{1}{\sqrt{2}}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2\sqrt{2}}$$

Question142

The value of the definite integral $\int_{\pi/24}^{5\pi/24} \frac{dx}{1+\sqrt[3]{\tan 2x}}$ is [25 Jul 2021 Shift 1]

Options:

A. $\frac{\pi}{3}$

B. $\frac{\pi}{6}$

C. $\frac{\pi}{12}$

D. $\frac{\pi}{18}$

Answer: C

Solution:

Solution:

Let $I = \int_{\pi/24}^{5\pi/24} \frac{(\cos 2x)^{1/3}}{(\cos 2x)^{1/3} + (\sin 2x)^{1/3}} dx$ ....(i)

$$\Rightarrow I = \int_{\pi/24}^{5\pi/24} \frac{\left(\cos\left\{2\left(\frac{\pi}{4} - x\right)\right\}\right)^{\frac{1}{3}}}{\left(\cos\left\{2\left(\frac{\pi}{4} - x\right)\right\}\right)^{\frac{1}{3}} + \left(\sin\left\{2\left(\frac{\pi}{4} - x\right)\right\}\right)^{\frac{1}{3}}} dx$$

$\left\{\int_a^b f(x) dx = \int_a^b f(a + b - x) dx\right\}$

So $I = \int_{\pi/24}^{5\pi/24} \frac{(\sin 2x)^{1/3}}{(\sin 2x)^{1/3} + (\cos 2x)^{1/3}} dx$. ....(ii)

Hence $2I = \int_{\pi/24}^{5\pi/24} dx$

$[(i) + (ii)]$

$\Rightarrow 2I = \frac{4\pi}{24} \Rightarrow I = \frac{\pi}{12}$

Question143

The value of the integral $\int_{-1}^{1} \log\left(x + \sqrt{x^2 + 1}\right) d\ x$ is: [25 Jul 2021 Shift 2]

Options:

A. 2

B. 0

C. -1

D. 1

Answer: B

Solution:

Solution:

Let $I = \int_{-1}^{1} \log\left(x + \sqrt{x^2+1}\right) d\,x$

$\because \log\left(x + \sqrt{x^2+1}\right)$ is an odd function

$\therefore I = 0$

Question144

If $\int_0^{100\pi} \frac{\sin^2 x}{e^{\left(\frac{x}{\pi}\left[\frac{x}{\pi}\right]\right)}} d\,x = \frac{\alpha\pi^3}{1+4\pi^2}$, $\alpha \in R$ where [x] is the greatest integer less than or equal to x, then the value of α is :
[22 Jul 2021 Shift 2]

Options:

A. $200(1-e^{-1})$

B. $100(1-e)$

C. $50(e-1)$

D. $150(e^{-1}-1)$

Answer: A

Solution:

Solution:

$I = \int_0^{100\pi} \frac{\sin^2 x}{e^{(x/\pi\}}} d\,x = 100\int_0^{\pi} \frac{\sin^2 x}{e^{x/\pi}} d\,x$

$100\int_0^{\pi} e^{-x/\pi} \frac{(1-\cos 2x)}{2} d\,x$

$= 50\left\{\int_0^{\pi} e^{-x/\pi} d\,x - \int_0^{\pi} e^{-x/\pi} \cos 2x\,d\,x\right\}$

$I_1 = \int_0^{\pi} e^{-x/\pi} d\,x = [-\pi e^{-x/\pi}]_0^{\pi} = \pi(1-e^{-1})$

$I_2 = \int_0^{\pi} e^{-x/\pi} \cos 2x\,d\,x$

$= -\pi e^{-x/\pi} \cos 2x\,]_0^{\pi} - \int -\pi e^{-x/\pi}(-2\sin 2x) d\,x$

$= \pi(1-e^{-1}) - 2\pi\int_0^{\pi} e^{-x/\pi} \sin 2x\,d\,x$

$= \pi(1-e^{-1}) - 2\pi\{-\pi e^{-x/\pi} \sin 2x]_0^{\pi} - \int_0^{\pi} -\pi e^{-x/\pi} 2\cos 2x d\,x\Big\} = \pi(1-e^{-1}) - 4\pi^2 I_2$

$\Rightarrow I_2 = \frac{\pi(1-e^{-1})}{1+4\pi^2}$

$\therefore I = 50\left\{ \pi(1-e^{-1}) - \frac{\pi(1-e^{-1})}{1+4\pi^2} \right\}$

$= \frac{200(1-e^{-1})\pi^3}{1+4\pi^2}$

Question145

The value of the integral $\int_{-1}^{1} \log_e(\sqrt{1-x}+\sqrt{1+x})dx$ is equal to : [20 Jul 2021 Shift 1]

Options:

A. $\frac{1}{2}\log_e 2 + \frac{\pi}{4} - \frac{3}{2}$

B. $2\log_e 2 + \frac{\pi}{4} - 1$

C. $\log_e 2 + \frac{\pi}{2} - 1$

D. $2\log_e 2 + \frac{\pi}{2} - \frac{1}{2}$

Answer: B

Solution:

Solution:

Let $I = 2\int_0^1 \ln(\sqrt{1-x}+\sqrt{1+x})\, 1\, dx$

$\therefore I = 2[(x \,.\, \ln(\sqrt{1-x}+\sqrt{1-x}))_0^{\,1} - \int_0^1 x \,.\, \left(\frac{1}{\sqrt{1-x}+\sqrt{1+x}}\right) \cdot \left(\frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{1-x}}\right) dx$

$= 2(\ln\sqrt{2} - 0) - \frac{2}{2}\int_0^1 \frac{x\sqrt{1-x}-\sqrt{1+x}\,dx}{(\sqrt{1-x}+\sqrt{1+x})\sqrt{1-x^2}}$

$= (\log_e 2) - \int_0^1 \frac{x \,.\, \left(2-2\sqrt{1-x^2}\right)}{-2x\sqrt{1-x^2}} dx$ (After rationalisation)

$= (\log_e 2) + \int_0^1 \left(\frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}}\right) dx$

$= (\log_e 2) + (\sin^{-1}x)_0^{\,1} - 1$

$= \log_e 2 + \left(\frac{\pi}{2} - 0\right) - 1$

$\therefore I = (\log_e 2) + \frac{\pi}{2} - 1$

$\Rightarrow$ Option (3) is correct.

Question146

Let a be a positive real number such that $\int_0^a e^{x-[x]} dx = 10e - 9$ where [x] is the greatest integer less than or equal to x. Then a is equal to :
[20 Jul 2021 Shift 1]

Options:

A. $10 - \log_e(1 + e)$

B. $10 + \log_e 2$

C. $10 + \log_e 3$

D. $10 + \log_e(1 + e)$

Answer: B

Solution:

Solution:
$a > 0$
Let $n \le a < n + 1, n \in W$

$$\therefore a = \underset{\text{G.I.F}}{\underset{\Downarrow}{[a]}} + \underset{\text{Fractional part}}{\underset{\Downarrow}{\{a\}}}$$

Here $[a] = n$Now, $\int_0^a e^{x-[x]} dx = 10e - 9$

$\Rightarrow \int_0^n e^{\{x\}} dx + \int_n^a e^{x-[x]} dx = 10e - 9$

$\therefore n \int_0^1 e^x dx + \int_n^a e^{x-n} dx = 10e - 9$

$\Rightarrow n(e - 1) + (e^{a-n} - 1) = 10e - 9$

$\therefore n = 0$ and $\{a\} = \log_e 2$

So, $a = [a] + \{a\} = (10 + \log_e 2)$

⇒ Option (2) is correct.

Question147

If [x] denotes the greatest integer less than or equal to x, then the value of the integral $\int_{-\pi/2}^{\pi/2} [[x] - \sin x] dx$ is equal to :

[20 Jul 2021 Shift 2]

Options:

A. $-\pi$

B. π

C. 0

D. 1

Answer: A

Solution:

Solution:

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} ([x] + [-\sin x]) dx \text{ ....(1)}$$

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} ([-x] + [\sin x]) dx \text{ ......(2)}$$

(King property)

$$2I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\underbrace{[x] + [-x]}_{-1}\right) + \left(\underbrace{[\sin x] + [-\sin x]}_{-1}\right) dx$$

$$2I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (-2) dx = -2(\pi)$$

$$I = -\pi$$

Question148

If $f : R \to R$ is given by $f(x) = x + 1$, then the value of $\lim_{n \to \infty} \frac{1}{n}\left[f(0) + f\left(\frac{5}{n}\right) + f\left(\frac{10}{n}\right) + \text{........} + f\left(\frac{5(n-1)}{n}\right)\right]$ is:

[20 Jul 2021 Shift 2]

Options:

A. $\frac{3}{2}$

B. $\frac{5}{2}$

C. $\frac{1}{2}$

D. $\frac{7}{2}$

Answer: D

Solution:

Solution:

$I = \sum_{r=0}^{n-1} f\left(\frac{5r}{n}\right)\frac{1}{n}$

$I = \int_0^1 f(5x)dx$

$I = \int_0^1 (5x+1)dx$

$I = \left[\frac{5x^2}{2} + x\right]_0^1$

$I = \frac{5}{2} + 1 = \frac{7}{2}$

Question149

Let $g(t) = \int_{-\pi/2}^{\pi/2} \cos\left(\frac{\pi}{4}t + f(x)\right) d\ x$, where $f(x) = \log_e\left(x + \sqrt{x^2+1}\right)$, $x \in R$. Then which one of the following is correct?
[20 Jul 2021 Shift 2]

Options:

A. $g(1) = g(0)$

B. $\sqrt{2}g(1) = g(0)$

C. $g(1) = \sqrt{2}g(0)$

D. $g(1) + g(0) = 0$

Answer: B

Solution:

$$g(t) = \int_{-\pi/2}^{\pi/2} \left(\cos\frac{\pi}{4}t + f(x)\right) dx$$
$$g(t) = \pi\cos\frac{\pi}{4}t + \int_{-\pi/2}^{\pi/2} f(x)dx$$
$$g(t) = \pi\cos\frac{\pi}{4}t$$
$g(1) = \frac{\pi}{\sqrt{2}}$, $g(0) = pi$

Question150

Let $F : [3, 5] \to R$ be a twice differentiable function on $(3, 5)$ such that $F(x) = e^{-x}\int_{3}^{x}(3t^2 + 2t + 4F'(t))dt$ If $F'(4) = \frac{\alpha e^{\beta} - 224}{(e^{\beta} - 4)^2}$, then $\alpha + \beta$ is equal to ________. [27 Jul 2021 Shift 1]

Answer: 16

Solution:

Solution:
$F(3) = 0$
$$e^{x}F(x) = \int_{3}^{x}(3t^2 + 2t + 4F'(t))dt$$
$$e^{x}F(x) + e^{x}F'(x) = 3x^2 + 2x + 4F'(x)$$
$$(e^{x} - 4)\frac{dy}{dx} + e^{x}y = (3x^2 + 2x)$$
$$\frac{dy}{dx} + \frac{e^{x}}{(e^{x} - 4)}y = \frac{(3x^2 + 2x)}{(e^{x} - 4)}$$
$$ye^{\int\frac{e^{x}}{(e^{x} - 4)}dx} = \int\frac{(3x^2 + 2x)}{(e^{x} - 4)}e^{\int\frac{e^{x}}{e^{x} - 4}dx}dx$$
$$y.(e^{x} - 4) = \int(3x^2 + 2x)dx + c$$
$$y(e^{x} - 4) = x^3 + x^2 + c$$
Put $x = 3 \Rightarrow c = -36$
$$F(x) = \frac{(x^3 + x^2 - 36)}{(e^{x} - 4)}$$
$$F'(x) = \frac{(3x^2 + 2x)(e^{x} - 4) - (x^3 + x^2 - 36)e^{x}}{(e^{x} - 4)^2}$$
Now put value of $x = 4$ we will get $\alpha = 12$ & $\beta = 4$

Question151

If $\int_0^{\pi}(\sin^3 x)e^{-\sin^2 x}d\,x = \alpha - \frac{\beta}{e}\int_0^1 \sqrt{t}e^t d\,t$, then $\alpha + \beta$is equal to _____ .
[27 Jul 2021 Shift 2]

Answer: 5

Solution:

Solution:

$I = \int_0^{\pi}(\sin^3 x)e^{-\sin^2 x}d\,x$

$= 2\int_0^{\pi/2} \sin x\, e^{-\sin^2 x} d\,x + \int_0^{\pi/2} \cos x\, e^{-\sin^2 x}(-\sin 2x)\,d\,x$

$= 2\int_0^{\pi/2} \sin x\, e^{-\sin^2 x} d\,x + \left[\cos x\, e^{-\sin^2 x}\right]_0^{\pi/2} + \int_0^{\pi/2} \sin x\, e^{-\sin^2 x} d\,x$

$= 3\int_0^{\pi/2} \sin x\, e^{-\sin^2 x} d\,x - 1$

$= \frac{3}{2}\int_{-1}^{0} \frac{e^{\alpha}d\,\alpha}{\sqrt{1+\alpha}} - 1$(Put $-\sin^2 x = t$)

$= \frac{3}{2e}\int_0^1 \frac{e^x}{\sqrt{x}}d\,x - 1$(put $1+\alpha = x$)

$= \frac{3}{2e}\int_0^1 e^x \frac{1}{\sqrt{x}}d\,x - 1$

$= 2 - \frac{3}{e}\int_0^1 e^x\sqrt{x}\,d\,x$

Hence, $\alpha + \beta = 5$

Question152

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\sin^2 x}{1+\pi^{\sin x}}\right) d\,x$ is

[26 Aug 2021 Shift 2]

Options:

A. $\frac{\pi}{2}$

B. $\frac{5\pi}{2}$

C. $\frac{3\pi}{4}$

D. $\frac{3\pi}{2}$

Answer: C

Solution:

Solution:

$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\sin^2 x}{1+\pi^{\sin x}} d x$ ...(i)

$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\sin^2(-x)}{1+\pi^{\sin(-x)}} \left[\because \int_a^b f(x) d x = \int_a^b f(a+b-x) d x \right]$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\sin x}{1+\pi^{-\sin x}} d x$

$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi^{\sin x}(1+\sin^2 x)}{1+\pi^{\sin x}} d x$ ...(ii)

Adding Eqs. (i) and (ii), we get

$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+\pi^{\sin x})(1+\sin^2 x)}{(1+\pi^{\sin x})} d x$

$\Rightarrow 2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\sin^2 x) d x$

$\Rightarrow 2I = [x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 2 \int_0^{\frac{\pi}{2}} \sin^2 x d x$

[$\because \sin^2 x$ is an even function, so $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 d x = 2 \int_0^{\frac{\pi}{2}} \sin^2 x d x$]

$\Rightarrow I = \frac{1}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) + \frac{1}{2} \int_0^{\frac{\pi}{2}} (1-\cos 2 x) d x$

$$\Rightarrow I = \frac{\pi}{2} + \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$
$$\frac{\pi}{2} + \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right]$$
$$I = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

Question153

If $\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx = a\sin^{-1}\left(\frac{\sin x + \cos x}{b}\right) + c$, where c is a constant of integration, then the ordered pair (a, b) is equal to
[2021]

Options:

A. $(-1, 3)$

B. $(3, 1)$

C. $(1, 3)$

D. $(1, -3)$

Answer: C

Solution:

Solution:
$$\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx$$
$$\int \frac{\cos x - \sin x}{\sqrt{9 - (\sin x + \cos x)^2}} dx$$
Let $\sin x + \cos x = t$
$$\int \frac{dt}{\sqrt{9 - t^2}} = \sin^{-1}\frac{t}{3} + c$$
$$= \sin^{-1}\left(\frac{\sin x + \cos x}{3}\right) + c$$
So $a = 1, b = 3$.

Question154

The integral $\int \frac{1}{\sqrt[4]{(x-1)^3(x+2)^5}} dx$ is equal to (where C is a constant of integration) [31 Aug 2021 Shift 1]

Options:

A. $\frac{3}{4}\left(\frac{x+2}{x-1}\right)^{\frac{1}{4}}+C$

B. $\frac{3}{4}\left(\frac{x+2}{x-1}\right)^{\frac{5}{4}}+C$

C. $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{\frac{1}{4}}+C$

D. $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{\frac{5}{4}}+C$

Answer: C

Solution:

Solution:

$\int \frac{1}{(x-1)^{\frac{3}{4}}(x+2)^{\frac{5}{4}}} dx = \int \frac{dx}{\left(\frac{x+2}{x-1}\right)^{\frac{5}{4}}(x-1)^2}$

$\frac{x+2}{x-1}=t$

$\Rightarrow \left(\frac{(x-1)-(x+2)}{(x-1)^2}\right) dx = dt$

$\Rightarrow -\frac{3}{(x-1)^2} dx = dt$

$\Rightarrow -\frac{1}{3}\int \frac{dt}{t^{5/4}} = \frac{4}{3}\cdot\frac{1}{t^{1/4}}+C$

$= \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{\frac{1}{4}}+C$

Question155

If

$\int \frac{\sin x}{\sin^3 x+\cos^3 x} dx = \alpha \log_e\left|1+\tan x\right| + \beta \log_e\left|1-\tan x+\tan^2 x\right| + \gamma \tan^{-1}\left(\frac{2\tan x-1}{\sqrt{3}}\right)+C,$

when C is constant of integration, then the value of $18(\alpha+\beta+\gamma^2)$ is [31 Aug 2021 Shift 2]

Answer: 3

Solution:

Solution:

Let $I=\int\frac{\sin x}{\sin^3x+\cos^3x}dx=$

$=\frac{\tan x\sec^2x}{\tan^3x+1}dx$

Put tan x = t

$\Rightarrow \sec^2xdx=dt$

$I=\int\frac{tdt}{t^3+1}=\int\frac{t}{(t+1)(t^2-t+1)}dt$

Now, $\frac{t}{(t+1)(t^2-t+1)}=\frac{A}{t+1}+Bt+Ct^2-t+1$

$\Rightarrow t=A(^2-t+1)+(Bt+C)(t+1)$

Comparing coefficients to both the sides and solving them for A, B, C, we have

$A=-\frac{1}{3}, B=\frac{1}{3}, C=\frac{1}{3}$

Hence, $I=-\frac{1}{3}\int\frac{1}{t+1}dt+\frac{1}{3}\int\frac{t+1}{t^2-t+1}dt$

$=-\frac{1}{3}\ln(t+1)+\frac{1}{3}\int\frac{\frac{1}{2}(2t-1)+\frac{3}{2}}{t^2-t+1}dt$

$=-\frac{1}{3}\ln(t+1)+\frac{1}{6}\ln(t^2-t+1)+\frac{1}{2}\int\frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$=-\frac{1}{3}\ln(t+1)+\frac{1}{6}\ln(t^2-t+1)+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)+C$

$=-\frac{1}{3}\ln(\tan x+1)+\frac{1}{6}\ln(\tan^2x-\tan x+1)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x-1}{\sqrt{3}}\right)+C$

$\Rightarrow \alpha=\frac{-1}{3}, \beta=\frac{1}{6}, \gamma=\frac{1}{\sqrt{3}}$

So, $18(\alpha+\beta+\gamma^2)=18\left(\frac{-1}{3}+\frac{1}{6}+\frac{1}{3}\right)=3$

Question156

If $\int \frac{dx}{(x^2+x+1)^2} = a\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + b\left(\frac{2x+1}{x^2+x+1}\right) + C$, $x > 0$ where C is the constant of integration, then the value of $9(\sqrt{3}a + b)$ is equal to
[27 Aug 2021 Shift 1]

Answer: 15

Solution:

Solution:

$\int \frac{dx}{(x^2+x+1)^2} = \int \frac{dx}{\left[\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right]^2}$

Let $x + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$

$\Rightarrow dx = \frac{\sqrt{3}}{2}\sec^2\theta d\theta$

$\therefore \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta d\theta}{\frac{9}{16}(\tan^2\theta+1)^2} = \frac{8}{3\sqrt{3}}\int \frac{\sec^2\theta d\theta}{\sec^4\theta}$

$= \frac{8}{3\sqrt{3}}\int \cos^2\theta d\theta$

$= \frac{8}{3\sqrt{3}}\int \frac{1+\cos 2\theta}{2} d\theta$

$= \frac{4}{3\sqrt{3}}(\theta + \sin 2\theta 2) + C$

$= \frac{4}{2\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{4}{3\sqrt{3}}\frac{\frac{2x+1}{\sqrt{3}}}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2} + C$

$= \frac{4}{3\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{3}\right) + \frac{1}{3}\frac{2x+1}{(x^2+x+1)} + C$

$\therefore a = \frac{4}{3\sqrt{3}}, b = \frac{1}{3}$

Hence, $9(\sqrt{3}a+b) = 9\left(\frac{4}{3}+\frac{1}{3}\right) = 15$

Question157

If $\int \frac{2e^x+3e^{-x}}{4e^x+7e^{-x}} dx = \frac{1}{14}(ux + v\log_e(4e^x+7e^{-x})) + C$,

where C is a constant of integration, then u + v is equal to
[27 Aug 2021 Shift 2]

Answer: 7

Solution:

Solution:

$I=\int \frac{2e^{x}+3e^{-x}}{4e^{x}+7e^{-x}}dx=\int \frac{2e^{2x}+3}{4e^{2x}+7}dx$

Let $2e^{2x}+3=A\frac{d}{dx}(4e^{2x}+7)+B(4e^{2x}+7)$

$\Rightarrow 2e^{2x}+3=(8A+4B)e^{2x}+7B$

Comparing both sides

$B=\frac{3}{7}$ and $A=\frac{1}{28}$

$\therefore I=\int \frac{\frac{1}{28}(8e^{2x})+\frac{3}{7}(4e^{2x}+7)}{4e^{2x}+7}dx$

$=\frac{1}{28}\ln\left|4e^{2x}+7\right|+\frac{3}{7}x+C$

$=\frac{1}{28}\ln\left|e^{x}(4e^{x}+7e^{-x})\right|+\frac{3}{7}x+C$

$=\frac{1}{28}x+\frac{1}{28}\ln\left|4e^{x}+7e^{-x}\right|+\frac{3}{7}x+C$

$=\frac{1}{14}\left(\frac{13}{2}x+\frac{1}{2}\ln\left|4e^{x}+7e^{-x}\right|\right)+C$

$\Rightarrow u=\frac{13}{2}$ and $v=\frac{1}{2}$

$\therefore u+v=\frac{13}{2}+\frac{1}{2}=7$

Question158

Section B : Numerical Type Questions

Let [t] denote the greatest integer $\leq$ t. Then the value of $8\cdot\int_{-\frac{1}{2}}^{1}([2x]+|x|)dx$ is

[31 Aug 2021 Shift 1]

Answer: 5

Solution:

Solution:

$8\int_{-\frac{1}{2}}^{1}([2x]+|X|)dx$

$=-\frac{1}{2}\leq x0$

$\Rightarrow[2x]=-1$

$0\leq x<\frac{1}{2}$

$\Rightarrow[2x]=0$

$\frac{1}{2}\leq x<1$

$\Rightarrow[2x]=1$

$I=\int_{-\frac{1}{2}}^{0}(-1-x)dx+\int_{0}^{\frac{1}{2}}(0+x)dx+\int_{-\frac{1}{2}}^{1}(1+x)dx$

$=\left[-x-\frac{x^2}{2}\right]_{-\frac{1}{2}}^{0}+\left[\frac{x^2}{2}\right]_{0}^{\frac{1}{2}}+[x+x^22]_{\frac{1}{2}}^{1}$

$=-\left(\frac{1}{2}-\frac{1}{8}\right)+\left(\frac{1}{8}\right)+\left(1+\frac{1}{2}\right)-\left(\frac{1}{2}+\frac{1}{8}\right)\ =\frac{5}{8}$

$\therefore 8I=8\cdot\frac{5}{8}=5$

Question159

If $x\phi(x)=\int_{5}^{x}(3t^2-2\phi'(t)dt$, $x>-2$, and $\phi(0)=4$, then $\phi(2)$ is [31 Aug 2021 Shift 1]

Answer: 4

Solution:

Solution:

$x\phi(x)=\int_{5}^{x}(3t^2-2\phi'(t)dt$

$\Rightarrow x\phi(x)=[t^3-2\phi(t)]_5^x$

$\Rightarrow x\phi(x)=(x^3-125)-2[\phi(x)-\phi(5)]$

Now, $\phi(0)=4$

$\Rightarrow 0 = -125 - 2[4 - \phi(5)]$
$\Rightarrow \phi(5) = \frac{133}{2}$
For $\phi(2)$,
$\Rightarrow 2\phi(2) = (8 - 125) - 2\left[\phi(2) - \frac{133}{2}\right]$
$\Rightarrow 4\phi(2) = 16$
$\Rightarrow \phi(2) = 4$

Question160

If [x] is the greatest integer ≤x, then $\pi^2 \int_0^2 \left(\sin \frac{\pi x}{2}\right)(x - |x|)^{[x]} dx$ is equal to [31 Aug 2021 Shift 2]

Options:

A. $2(\pi - 1)$

B. $4(\pi - 1)$

C. $4(\pi + 1)$

D. $2(\pi + 1)$

Answer: B

Solution:

Solution:

$\pi^2 \int_0^2 \sin\left(\frac{\pi x}{2}\right)(x - |x|)^{[x]} d x$
$= \pi^2 \int_0^1 \sin\left(\frac{\pi x}{2}\right) x^0 d x + \pi^2 \int_1^2 \sin\left(\frac{\pi x}{2}\right)(x - 1) d x$
$= \pi^2 \left[\frac{-2}{\pi} \cos \frac{\pi x}{2}\right]_0^1 + \pi^2 \left[(x - 1)\frac{2}{\pi}\left(-\cos \frac{\pi x}{2}\right)\right]_1^2 + \pi^2 \int_1^2 \frac{2}{\pi} \cos \frac{\pi x}{2} d x$
$= \pi^2 \left(\frac{2}{\pi}\right) + \frac{2\pi^2}{\pi}(1 - 0) + 2\pi . \frac{2}{\pi}\left(\sin \frac{\pi x}{2}\right)\Big|_1^2$
$= 2\pi + 2\pi + 4(0 - 1) = 4\pi - 4 = 4(\pi - 1)$

Question161

If $U_n = \left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)^2 \ldots \left(1+\frac{n^2}{n^2}\right)^n$, then $\lim\limits_{n\to\infty}(U_n)^{\frac{-4}{n^2}}$ is equal to

[27 Aug 2021 Shift 1]

Options:

A. $\frac{e^2}{16}$

B. $\frac{4}{e}$

C. $\frac{16}{e^2}$

D. $\frac{4}{e^2}$

Answer: A

Solution:

Solution:

Let $y = \lim\limits_{n\to\infty}(U_n)^{\frac{-4}{n^2}}$

$y = \lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)^{\frac{-4}{n^2}}\left(1+\frac{2^2}{n^2}\right)^{\frac{-4}{n^2}.2}\left(1+\frac{3^2}{n^2}\right)^{\frac{-4}{n^2}.3}\ldots\right]$

Taking log on both sides, we get

$\ln y = \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\left[\frac{-4}{n^2}.r\ln\left(1+\frac{r^2}{n^2}\right)\right]$

Now, replace $\lim\limits_{n\to\infty}\Sigma \to \int$

$\frac{r}{n}\to x, \frac{1}{n}\to dx$

Lower limit = 0

Upper limit = 1

$\therefore \ln y = \int_0^1 -4x\ln(1+x^2)\,dx$

Let $1+x^2 = t$

$\Rightarrow x\,dx = \frac{dt}{2}$

When x→ 0, t → 1

and x → 1, t → 2

$\therefore \ln y = \int_1^2 -2\ln t\,dt$

$= -2(t\ln t - t)_1^2$

$= -2(2\ln 2 - 2 + 1)$

$= -2(2\ln 2 - 1)$

$\Rightarrow \ln y = \ln \frac{1}{16} + 2$
$\Rightarrow y = \frac{1}{16} e^2$

Question162

$\int_6^{16} \frac{\log_e x^2}{\log_e x^2 + \log_e (x^2 - 44x + 484)} dx$ is equal to
[27 Aug 2021 Shift 1]

Options:

A. 6

B. 8

C. 5

D. 10

Answer: C

Solution:

Solution:

Let $I = \int_6^{16} \frac{\ln_e(x^2)}{\ln_e(x^2) + \ln_e(484 - 44x + x^2)} dx$
$= \int_6^{16} \frac{\ln_e(x^2)}{\ln_e(x^2) + \ln_e(22 - x)^2} dx$
$= \int_6^{16} \frac{2\ln_e x dx}{2\ln_e x + 2\ln_e(22 - x)}$
$I = \int_6^{16} \frac{\ln_e x dx}{\ln_e x + \ln_e(22 - x)}$ ...(i)
$\because \int_a^b f(x) dx = \int_a^b f(a + b - x) dx$
$\therefore I = \int_6^{16} \frac{\ln_e(22 - x)}{\ln_e(22 - x) + \ln_e x} dx$ ...(ii)
Adding Eqs. (i) and (ii), we get
$2I = \int_6^{16} \frac{\ln_e x + \ln_e(22 - x)}{\ln_e x + \ln_e(22 - x)} dx$
$2I = \int_6^{16} dx = x |_6^{16} = 10$
or I = 5

Question163

The value of the integral $\int_0^1 \frac{\sqrt{x}\,dx}{(1+x)(1+3x)(3+x)}$ is
[27 Aug 2021 Shift 2]

Options:

A. $\frac{\pi}{8}\left(1-\frac{\sqrt{3}}{2}\right)$

B. $\frac{\pi}{4}\left(1-\frac{\sqrt{3}}{6}\right)$

C. $\frac{\pi}{8}\left(1-\frac{\sqrt{3}}{6}\right)$

D. $\frac{\pi}{4}\left(1-\frac{\sqrt{3}}{2}\right)$

Answer: A

Solution:

Solution:

$$\int_0^1 \frac{\sqrt{x}\,dx}{(1+x)(1+3x)(3+x)}$$

Put $\sqrt{x}=t$

$\Rightarrow x=t^2$

or dx = 2t dt

$$\therefore I=\int_0^1 \frac{2t^2 dt}{(t^2+1)(3t^2+1)(t^2+3)}$$

$$=\int_0^1 \frac{(3t^2+1)-(t^2+1)}{(t^2+1)(3t^2+1)(t^2+3)}dt$$

$$=\int_0^1 \left[\frac{1}{(t^2+3)(t^2+1)}-\frac{1}{(t^2+3)(3t^2+1)}\right]dt$$

$$=\int_0^1 \left[\frac{1}{2(t^2+1)}-\frac{1}{2(t^2+3)}-\frac{3}{8(3t^2+1)}+\frac{1}{8(t^2+3)}\right]dt$$

$$=\int_0^1 \frac{dt}{2(t^2+1)}=\int_0^1 \frac{3}{8}\frac{dt}{(3t^2+1)}-\int_0^1 \frac{3}{8}\frac{dt}{(t^2+3)}$$

$$=\left(\frac{1}{2}\tan^{-1}t\right)_0^1-\left(\frac{3}{8}\frac{\sqrt{3}}{3}\tan^{-1}\sqrt{3}t\right)_0^1-\left(\frac{3}{8\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right)_0^1$$

$$=\frac{\pi}{8}-\frac{\sqrt{3}}{8}\frac{\pi}{3}-\frac{\sqrt{3}}{8}\frac{\pi}{6}=\frac{\pi}{8}-\frac{\sqrt{3}\pi}{16}=\frac{\pi}{8}\left(1-\frac{\sqrt{3}}{2}\right)$$

Question164

The value of $\int_{-1\sqrt{2}}^{1/\sqrt{2}}\left(\left(\frac{x+1}{x-1}\right)^2+\left(\frac{x-1}{x+1}\right)^2-2\right)^{1/2}dx$ **is**

[26 Aug 2021 Shift 1]

Options:

A. $\log_e 4$

B. $\log_e 16$

C. $2\log_e 16$

D. $4\log_e(3+2\sqrt{2})$

Answer: B

Solution:

Solution:

Let $I=\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\left(\left(\frac{x+1}{x-1}\right)^2+\left(\frac{x-1}{x+1}\right)^2-2\right)^{1/2}dx$

$$I=\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\left[\left(\frac{x+1}{x-1}-\frac{x-1}{x+1}\right)^2\right]^{\frac{1}{2}}dx=\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\left|\frac{x+1}{x-1}-\frac{x-1}{x+1}\right|dx$$

$$=\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\left|\frac{(x+1)^2-(x-1)^2}{(x-1)(x+1)}\right|dx=\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\left|\frac{4x}{(x-1)(x+1)}\right|dx$$

$$=2.4\int_{0}^{\frac{1}{\sqrt{2}}}\left|\frac{x}{(x-1)(x+1)}\right|dx=4\int_{0}^{\frac{1}{\sqrt{2}}}\frac{-2x}{x^2-1}dx$$

$$=-4[\log(x^2-1)]_0^{\frac{1}{\sqrt{2}}}$$

$$=-4\left[\log\left(\frac{1}{2}-1\right)-\log\left|-1\right|\right]$$

$$=-4\log\left(\frac{1}{2}\right)=4\ln 2=\ln 16$$

Question165

The value of $\lim\limits_{n \to \infty} \frac{1}{n} \sum\limits_{r=0}^{2n-1} \frac{n^2}{n^2+4r^2}$ is
[26 Aug 2021 Shift 1]

Options:

A. $\frac{1}{2}\tan^{-1}(2)$

B. $\frac{1}{2}\tan^{-1}(4)$

C. $\tan^{-1}(4)$

D. $\frac{1}{4}\tan^{-1}(4)$

Answer: B

Solution:

Solution:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{n^2}{n^2+4r^2}$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{1}{1+4\left(\frac{r}{n}\right)^2} = \int_0^2 \frac{1}{1+4x^2} d\,x$$

$$= \frac{1}{2}[\tan^{-1}2x]_0^2 = \frac{1}{2}\tan^{-1}4$$

Question166

If the value of the integral $\int_0^5 \frac{x+[x]}{e^{x-[x]}} d\,x = \alpha e^{-} + \beta$, where $\alpha, \beta \in R$, $5\alpha + 6\beta = 0$ and [x] denotes the greatest integer less than or equal to x, then the value of $(\alpha+\beta)^2$ is equal to:
[26 Aug 2021 Shift 2]

Options:

A. 100

B. 25

C. 16

D. 36

Answer: B

Solution:

Solution:

$I=\int_0^5 \frac{x+[x]}{e^{x-[x]}}dx=\alpha e^{-}+\beta,$

$I=\int_0^1 \frac{x}{e^x}dx+\int_1^2 \frac{x+1}{e^{x-1}}dx+\int_2^3 \frac{x+2}{e^{x-2}}dx+\int_3^4 \frac{x+3}{e^{x-3}}dx+\int_4^5 \frac{x+4}{e^{x-4}}dx$

Let $I=I_1+I_2+I_3+I_4+I_5$

Here, $,I_2=\int_1^2 \frac{x+1}{e^{x-1}}dx$ Put $x=t+1$

$\Rightarrow dx=dt$

$=\int_0^1 \frac{t+2}{e^t}dt=\int_0^1 \frac{t}{e^t}dt+\int_0^1 \frac{2}{e^t}dt$

$I_2=I_1+2\int_0^1 e^{-t}dt=I_1+2(1-e^{-1})$

Similarly,

$I_3=I_1+4(1-e^{-1})$

$I_4=I_1+6(1-e^{-1})$

$I_5=I_1+8(1-e^{-1})$

$I=I_1+I_2+I_3+I_4+I_5=5I_1+(2+4+6+8)(1-e^{-1})$

$=5l_1+20(1-e^{-1})$

$I_1=\int_0^1 xe^{-1}dx=-[e^{-x}(x+1)]_0^{\ 1}=1-2e^{-1}$

$\therefore 5l_1+20(1-e^{-1})=5(1-2e^{-1})+20(1-e^{-1})=25-30e^{-1}$

$\therefore \alpha=-30,\ \beta=25$

Also it satisfy $5\beta+6\alpha=0$

Now, $(\alpha+\beta)^2=(-30+25)^2=(-5)^2=25$

Question167

Let $J_{n,m}=\int_0^{\frac{1}{2}} \frac{x^n}{x^m-1}dx,\ \forall n>m$ and $n, m\in N$.
Consider a matrix $A=[a_{ij}]_{3\times 3}$ where

$$a_{ii}=\begin{cases} J_6+i\quad ,3-J_i+3,\ \ 3 & i\le j \\ 0 & i>j \end{cases}$$

then $|\text{adj } A^{-1}|$ is
[1 Sep 2021 Shift 2]

Options:

A. $(15)^2 \times 2^{42}$

B. $(15)^2 \times 2^{34}$

C. $(105)^2 \times 2^{38}$

D. $(105)^2 \times 2^{36}$

Answer: C

Solution:

Solution:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$\Rightarrow |A| = a_{11}a_{22}a_{33}$

$\Rightarrow |A| = (J_{7,3} - J_{4,3})\,(J_{8,3} - J_{5,3})\,(J_{9,3} - J_{6,3})$

$$= \int_0^{\frac{1}{2}} \frac{x^7 - x^4}{x^3 - 1} d\,x \,.\, \int_0^{\frac{1}{2}} \frac{x^8 - x^5}{x^3 - 1} d\,x \,.\, \int_0^{\frac{1}{2}} \frac{x^9 - x^6}{x^3 - 1} d\,x$$

$$= \int_0^{\frac{1}{2}} x^4 d\,x \int_0^{\frac{1}{2}} x^5 d\,x \,.\, \int_0^{\frac{1}{2}} x^6 d\,x$$

$$= \left.\frac{x^5}{5}\right|_0^{\frac{1}{2}} . \left.\frac{x^6}{6}\right|_0^{\frac{1}{2}} . \left.\frac{x^7}{7}\right|_0^{\frac{1}{2}} = \frac{1}{(210)2^{18}}$$

Now, $|\text{adj } A^{-1}| = \frac{1}{|A|^2} = ((210).2^{18})^2 = 105^2.2^{38}$

Question168

The function f(x), that satisfies the condition $f(x) = x + \int\limits_0^{\pi/2} \sin x \,.\, \cos y\, f(y)\, d\,\ y$, is [1 Sep 2021 Shift 2]

Options:

A. $x + \frac{2}{3}(\pi - 2)\sin x$

B. $x + (\pi + 2)\sin x$

C. $x + \frac{\pi}{2}\sin x$

D. $x+(\pi-2)\sin x$

Answer: D

Solution:

Solution:

$f(x)=x+\int_0^{\frac{\pi}{2}}\sin x\,.\cos y\, f(y)\,d\,y$

Let $K=\int_0^{\frac{\pi}{2}}\cos y\, f(y)\,d\,y$ ...(i)

Then, $f(x)=x+K\sin x$... (ii)

From Eqs. (i) and (ii),

$f(x)=x+\int_0^{\frac{\pi}{2}}\sin x\cos y(y+k\sin y)\,d\,y$

$=x+\sin x\int_0^{\frac{\pi}{2}}y\cos y\,d\,y+\frac{k}{2}\sin x\int_0^{\frac{\pi}{2}}\sin 2y\,d\,y$

$f(x)=x+\sin x\,.\,\frac{\pi.2}{2}+\frac{k\sin x}{2}$ ...(iii)

$k=\frac{\pi-2}{2}+\frac{k}{2}$

$\Rightarrow k=\pi-2$

$\therefore f(x)=x+(\pi-2)\sin x$

Question169

The integral $\int\frac{dx}{(x+4)^{8/7}(x-3)^{6/7}}$ is equal to: (where C is a constant of integration) [Jan. 9, 2020 (I)]

Options:

A. $\left(\frac{x-3}{x+4}\right)^{1/7}+C$

B. $-\left(\frac{x-3}{x+4}\right)^{1/7}+C$

C. $\frac{1}{2}\left(\frac{x-3}{x+4}\right)^{3/7}+C$

D. $-\frac{1}{13}\left(\frac{x-3}{x+4}\right)^{-13/7}+C$

Answer: A

Solution:

Solution:

$I=\int\frac{dx}{(x+4)^{8/7}(x-3)^{6/7}}$

$=\int\left(\frac{x-3}{x+4}\right)^{\frac{-6}{7}}\frac{1}{(x+4)^2}dx$

Let $\frac{x-3}{x+4}=t^7$

Differentiate on both sides, we get

$\frac{7}{(x+4)^2}dx=7t^6dt$

Hence, $I=\int t^{-6}t^6dt=t+C=\left(\frac{x-3}{x+4}\right)^{\frac{1}{7}}+C$

Question170

If $\int\frac{d\theta}{\cos^2\theta(\tan 2\theta+\sec 2\theta)}=\lambda\tan\theta+2\log_e|f(\theta)|+C$ whereC is a constant of integration, then the ordered pair $(\lambda, f(\theta))$ is equal to:
[Jan. 9,2020 (II)]

Options:

A. $(1, 1-\tan\theta)$

B. $(-1, 1-\tan\theta)$

C. $(-1, 1+\tan\theta)$

D. $(1, 1+\tan\theta)$

Answer: C

Solution:

Solution:

$I=\int\frac{d\theta}{\cos^2\theta(\tan 2\theta+\sec 2\theta)}$

$=\int\frac{\sec^2\theta}{\frac{1+\tan^2\theta}{1-\tan^2\theta}+\frac{2\tan\theta}{1-\tan^2\theta}}d\theta$

$=\int\frac{\sec^2\theta(1-\tan^2\theta)}{(1+\tan\theta)^2}d\theta$

$=\int\frac{\sec^2\theta(1-\tan\theta)}{1+\tan\theta}d\theta$

Let $\tan\theta = t \Rightarrow \sec^2\theta d\theta = dt$, then

$I = \int\left(\frac{1-t}{1+t}\right)dt = \int\left(-1+\frac{2}{1+t}\right)dt$

$= -t + 2\log(1+t) + C$

$= -\tan\theta + 2\log(1+\tan\theta) + C$

Hence, by comparison $\lambda = -1$ and $f(x) = 1+\tan\theta$

Question171

If $\int \frac{\cos x\, dx}{\sin^3 x(1+\sin^6 x)^{2/3}} = f(x)(1+\sin^6 x)^{1/\lambda} + c$ where c is a constant of integration, then $\lambda f\left(\frac{\pi}{3}\right)$ is equal to:

[Jan. 8, 2020 (II)]

Options:

A. $-\frac{9}{8}$

B. 2

C. $\frac{9}{8}$

D. -2

Answer: D

Solution:

Solution:

Let $I = \int \frac{\cos x\, dx}{\sin^3 x(1+\sin^6 x)^{2/3}}$

$= f(x)(1+\sin^6 x)^{1/\lambda} + c$ ......(i)

If $\sin x = t$

then, $\cos x\, dx = dt$

$I = \int \frac{dt}{t^3(1+t^6)^{\frac{2}{3}}} = \int \frac{dt}{t^7\left(1+\frac{1}{t^6}\right)^{\frac{2}{3}}}$

Put $1+\frac{1}{t^6} = r^3 \Rightarrow \frac{dt}{t^7} = \frac{-1}{2}r^2 dr - \frac{1}{2}\int\frac{r^2 dr}{r^2} = -\frac{1}{2}r + c$

$= -\frac{1}{2}\left(\frac{\sin^6 x + 1}{\sin^6 x}\right)^{\frac{1}{3}} + c = -\frac{1}{2\sin^2 x}(1+\sin^6 x)^{\frac{1}{3}} + c$

$f(x)=-\frac{1}{2}\operatorname{cosec}^2 x$ and $\lambda=3$[from eqn. (i)]

$\therefore \lambda f\left(\frac{\pi}{3}\right)=-2$

Question172

If for all real triplets (a, b, c), $f(x)=a+bx+cx^2$; then $\int_0^1 f(x)dx$ is equal to: [Jan. 9, 2020 (I)]

Options:

A. $2\left\{3f(1)+2f\left(\frac{1}{2}\right)\right\}$

B. $\frac{1}{2}\left\{f(1)+3f\left(\frac{1}{2}\right)\right\}$

C. $\frac{1}{3}\left\{f(0)+f\left(\frac{1}{2}\right)\right\}$

D. $\frac{1}{6}\left\{f(0)+f(1)+4f\left(\frac{1}{2}\right)\right\}$

Answer: D

Solution:

Solution:

$\int_0^1 (a+bx+cx^2)dx = ax+\frac{bx^2}{2}+\frac{cx^3}{3}\Big|_0^1 = a+\frac{b}{2}+\frac{c}{3}$

Now, $f(1)=a+b+c$, $f(0)=a$ and $f\left(\frac{1}{2}\right)=a+\frac{b}{2}+\frac{c}{4}$

Now, $\frac{1}{6}\left(f(1)+f(0)+4f\left(\frac{1}{2}\right)\right)$

$=\frac{1}{6}\left(a+b+c+a+4\left(a+\frac{b}{2}+\frac{c}{4}\right)\right)$

$=\frac{1}{6}(6a+3b+2c)=a+\frac{b}{2}+\frac{c}{3}$

Hence, $\int_0^1 f(x)=\frac{1}{6}\left\{f(0)+f(1)+4f\left(\frac{1}{2}\right)\right\}$

Question173

The value of $\int_0^{2\pi} \frac{x\sin^8 x}{\sin^8 x+\cos^8 x} dx$ is equal to:
[Jan. 9, 2020 (I)]

Options:

A. 2π

B. $2\pi^2$

C. π^2

D. 4π

Answer: C

Solution:

$$\int_0^{2\pi} \frac{x\sin^8 x}{\sin^8 x+\cos^8 x} dx$$
$$= \int_0^{\pi} \left[\frac{x\sin^8 x}{\sin^8 x+\cos^8 x} + \frac{(2\pi-x)\sin^8 x}{\sin^8 x+\cos^8 x} \right] dx \quad \left[\because \int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx \right]$$
$$= \int_0^{\pi} \frac{2\pi\sin^8 x}{\sin^8 x+\cos^8 x} dx$$
$$= 2\pi \int_0^{\pi/2} \left[\frac{\sin^8 x}{\sin^8 x+\cos^8 x} + \frac{\cos^8 x}{\sin^8 x+\cos^8 x} \right] dx$$
$$= 2\pi \int_0^{\pi/2} 1dx = 2\pi \times \frac{\pi}{2} = \pi^2$$

Question174

If I $= \int_1^2 \frac{dx}{\sqrt{2x^3-9x^2+12x+4}}$, then:
[Jan. 8, 2020 (II)]

Options:

A. $\frac{1}{8} < I^2 < \frac{1}{4}$

B. $\frac{1}{9} < I^2 < \frac{1}{8}$

C. $\frac{1}{16} < I^2 < \frac{1}{9}$

D. $\frac{1}{6} < I^2 < \frac{1}{2}$

Answer: B

Solution:

Solution:

$f(x) = \frac{1}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$

$'(x) = \frac{-1}{2}\left(\frac{(6x^2 - 18x + 12)}{(2x^3 - 9x^2 - 12x + 4)^{3/2}}\right)$

$= \frac{-6(x-1)(x-2)}{2(2x^3 - 9x^2 + 12x + 4)^{3/2}}$

$f(1) = \frac{1}{3}$ and $f(2) = \frac{1}{\sqrt{8}}$

It is increasing function

$\frac{1}{3} < I < \frac{1}{\sqrt{8}}$

$\frac{1}{9} < I^2 < \frac{1}{8}$

Question175

If $f(a + b + 1 - x) = f(x)$, for all x, where a and b are fixed positive real numbers, then $\frac{1}{a+b}\int_a^b x(f(x) + f(x+1))dx$ is equal to:
[Jan. 7, 2020 (I)]

Options:

A. $\int_{a+1}^{b+1} f(x)dx$

B. $\int_{a-1}^{b-1} f(x)dx$

C. $\int_{a-1}^{b-1} f(x+1)dx$

D. $\int_{a+1}^{b+1} f(x+1)dx$

Answer: C

Solution:

$I = \frac{1}{(a+b)}\int_a^b x[f(x)+f(x+1)]dx$ ....(i)

$x \to a+b-x$

$I = \frac{1}{(a+b)}\int_a^b (a+b-x)[f(a+b-x)+f(a+b+1-x)]\,dx$

$I = \frac{1}{(a+b)}\int_a^b (a+b-x)\,[f(x+1)+f(x)]dx$ ......(ii)

[$\because$ put $x \to x+1$ in $f(a+b+1-x) = f(x)$]

Add (i) and (ii)

$2I = \int_a^b f(x+1)+f(x)\Big]\,dx$

$2I = \int_a^b f(x+1)dx + \int_a^b f(x)dx$

$= \int_a^b f(a+b+1-x)dx + \int_a^b f(x)dx$

$2I = 2\int_a^b f(x)dx$

$\therefore \int_{a-1}^{b-1} f(x+1)dx$ [$\because$ Put $x \to x+1$]

Question176

The value of α for which $4\alpha\int_{-1}^{2} e^{-\alpha|x|}dx = 5$, is : [Jan. 7, 2020 (II)]

Options:

A. $\log_e 2$

B. $\log_e\left(\frac{3}{2}\right)$

C. $\log_e\sqrt{2}$

D. $\log_e\left(\frac{4}{3}\right)$

Answer: A

Solution:

$$4\alpha\left\{\int_{-1}^{0} e^{\alpha x} dx + \int_{0}^{2} e^{-\alpha x} dx\right\} = 5$$

$$\Rightarrow 4\alpha\left\{\left.\frac{e^{\alpha x}}{\alpha}\right|_{-1}^{0} + \left.\frac{e^{-\alpha x}}{-\alpha}\right|_{0}^{2}\right\} = 5$$

$$\Rightarrow 4\alpha\left\{\left(\frac{1-e^{-\alpha}}{\alpha}\right) - \left(\frac{e^{-2\alpha}-1}{\alpha}\right)\right\} = 5$$

$$\Rightarrow 4(2 - e^{-\alpha} - e^{-2\alpha}) = 5$$

Put $e^{-\alpha} = t$

$$\Rightarrow 4t^2 + 4t - 3 = 0 \Rightarrow (2t+3)(2t-1) = 0$$

$$\Rightarrow e^{-\alpha} = \frac{1}{2} \Rightarrow \alpha = \log_e 2$$

Question177

If θ_1 and θ_2 be respectively the smallest and the largest values of theta in $(0, 2\pi) - \{\pi\}$ which satisfy the equation, $2\cot^2\theta - \frac{5}{\sin\theta} + 4 = 0$, then $\int_{\theta_1}^{\theta_2} \cos^2 3\theta d\theta$ is equal to:
[Jan. 7, 2020 (II)]

Options:

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. $\frac{\pi}{3} + \frac{1}{6}$

D. $\frac{\pi}{9}$

Answer: A

Solution:

Solution:

$$2\cot^2\theta - \frac{5}{\sin\theta} + 4 = 0$$

$$\frac{2\cos^2\theta}{\sin^2\theta} - \frac{5}{\sin\theta} + 4 = 0$$

$$\Rightarrow 2\cos^2\theta - 5\sin\theta + 4\sin^2\theta = 0,\ \sin\theta \neq 0$$

$$\Rightarrow 2\sin^2\theta - 5\sin\theta + 2 = 0$$

$$\Rightarrow (2\sin\theta - 1)(\sin\theta - 2) = 0$$

$\therefore \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore \int_{\pi/6}^{5\pi/6} \cos^2 3\theta d\theta = \int_{\pi/6}^{5\pi/6} \frac{1+\cos 6\theta}{2} d\theta$

$= \frac{1}{2}\left[\theta + \frac{\sin 6\theta}{6}\right]_{\pi/6}^{5\pi/6} = \frac{1}{2}\left[\frac{5\pi}{6} - \frac{\pi}{6} + \frac{1}{6}(0-0)\right]$

$= \frac{1}{2} \cdot \frac{4\pi}{6} = \frac{\pi}{3}$

Question178

Let a function $f : [0, 5] \rightarrow R$ be continuous, $f(1) = 3$ and F be defined as: $F(x) = \int_1^x t^2 g(t) dt$, where $g(t) = \int_1^t f(u) du$ Then for the function F, the point $x = 1$ is: [Jan. 9, 2020 (II)]

Options:

A. a point of local minima.

B. not a critical point.

C. a point of local maxima.

D. a point of inflection.

Answer: A

Solution:

Solution:

$F(x) = \int_1^x t^2 g(t) dt$

Differentiate by using Leibnitz's rule, we get

$F(x) = x^2 g(x) = x^2 \int_1^x f(u) du$ ....(i)

At $x = 1$

$F'(1) = 1 \int_1^1 f(u) du = 0$

Now, differentiate eqn (i)

$F''(x) = x^2 f(x) - 2x \int_1^x f(u) du$

At $x = 1$,

$F''(1) = 1 . f(1) - 2 \times 1 . \int_1^1 f(u) du$

$= f(1) - 2 \times 0 = f(1)$

$F''(1) = 3$

Then, for $F'(1) = 0$, $F''(1) = 3 > 0$
Hence, $x = 1$ is a point of local minima.

Question179

$$\lim_{x \to 1} \left(\frac{\int_0^{(x-1)^2} t\cos(t^2)\, dt}{(x-1)\sin(x-1)} \right)$$

[Sep. 06, 2020 (I)]

Options:

A. is equal to $\frac{1}{2}$

B. is equal to 1

C. is equal to $-\frac{1}{2}$

D. (Bonus)

Answer: D

Solution:

Solution:

$$\lim_{x \to 1} \frac{\frac{1}{2}\sin(x-1)^4}{(x-1)\sin(x-1)}$$

Let $x - 1 = h$ when $x \to 1$ then $h \to 0$

$$\lim_{h \to 0} \frac{\sin h^4}{h^4} \times \frac{h}{\sin h} \times h^2 = 1 \times 1 \times 0 = 0$$

(No any option is correct)

Question180

If $\int (e^{2x} + 2e^{x} - e^{-x} - 1)e^{(e^{x} + e^{-x})} dx = g(x)e^{(e^{x} + e^{-x})} + c$, where c is a constant of integration, then g(0) is equal to:
[Sep. 05, 2020 (I)]

Options:

A. e

B. e^2

C. 1

D. 2

Answer: D

Solution:

Solution:

$\int (e^{2x} + 2e^x - e^{-x} - 1) \cdot e^{(e^x + e^{-x})} dx$

$I = \int (e^{2x} + e^x - 1) \cdot e^{(e^x + e^{-x})} dx + \int (e^x - e^{-x}) e^{(e^x + e^{-x})} dx$

$= \int e^x (e^x + 1 - e^{-x}) \cdot e^{(e^x + e^{-x})} dx + e^{(e^x + e^{-x})}$

$= \int (e^x - e^{-x} + 1) e^{(e^x + e^{-x} + x)} dx + e^{(e^x + e^{-x})}$

Let $e^x + e^{-x} + x = t \Rightarrow (e^x + e^{-x} + 1) dx = dt$

$= \int e^t dt + e^{(e^x + e^{-x})} = e^t + e^{(e^x + e^{-x})} + C$

$= e^{(e^x + e^{-x} + x)} + e^{(e^x + e^{-x})} + C$

$= (e^x + 1) \cdot e^{(e^x + e^{-x})} + C$

So, $g(x) = 1 + e^x$ and $g(0) = 2$

Question181

If $\int \frac{\cos\theta}{5 + 7\sin\theta - 2\cos^2\theta} d\theta = A\log_e |B(\theta)| + C$, where C is a constant of integration, then $\frac{B(\theta)}{A}$ can be: [Sep. 05, 2020 (II)]

Options:

A. $\frac{2\sin\theta + 1}{\sin\theta + 3}$

B. $\frac{2\sin\theta + 1}{5(\sin\theta + 3)}$

C. $\frac{5(\sin\theta + 3)}{2\sin\theta + 1}$

D. $\frac{5(2\sin\theta + 1)}{\sin\theta + 3}$

Answer: D

Solution:

Solution:

Let $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

$\int \frac{\cos\theta}{5+7\sin\theta-2\cos^2\theta} d\theta = \frac{dt}{5+7t-2+2t^2}$

$\Rightarrow \frac{1}{2}\int \frac{dt}{\left(t+\frac{7}{4}\right)^2-\left(\frac{5}{4}\right)^2} = \frac{1}{5}\ln\left|\frac{t+\frac{1}{2}}{t+3}\right| + C$

$= \frac{1}{5}\ln\left|\frac{2t+1}{t+3}\right| + C = \frac{1}{5}\ln\left|\frac{2\sin\theta+1}{\sin\theta+3}\right| + C$

$\therefore B(\theta) = \frac{2\sin\theta+1}{2(\sin\theta+3)}$ and $A = \frac{1}{5}$

$\Rightarrow \frac{B(\theta)}{A} = \frac{5(2\sin\theta+1)}{(\sin\theta+3)}$

Question182

The integral $\int \left(\frac{x}{x\sin x+\cos x}\right)^2 dx$ is equal to(where C is a constant of integration): [Sep. 04, 2020 (I)]

Options:

A. $\tan x - \frac{x\sec x}{x\sin x+\cos x} + C$

B. $\sec x + \frac{x\tan x}{x\sin x+\cos x} + C$

C. $\sec x - \frac{x\tan x}{x\sin x+\cos x} + C$

D. $\tan x + \frac{x\sec x}{x\sin x+\cos x} + C$

Answer: A

Solution:

Solution:

$\int \frac{x^2}{(x\sin x+\cos x^2)} dx$

$\because \frac{d}{dx}(x\sin x+\cos x) = x\cos x$

$= \int \frac{x\cos x}{(x\sin x+\cos x)^2}\left(\frac{x}{\cos x}\right) dx$

$$= \frac{x}{\cos x}\left[\frac{-1}{x\sin x+\cos x}\right] - \int \frac{x\sin x+\cos x}{\cos^2 x}\left[\frac{-1}{x\sin x+\cos x}\right] dx$$
$$= \frac{x}{\cos x}\left[\frac{-1}{x\sin x+\cos x}\right] + \int \sec^2 x dx$$
$$= \frac{-x\sec x}{x\sin x+\cos x} + \tan x + C$$

Question183

Let $f(x) = \int \frac{\sqrt{x}}{(1+x)^2} dx (x \geq 0)$. Then $f(3) - f(1)$ is equal to : [Sep. 04, 2020 (I)]

Options:

A. $-\frac{\pi}{12} + \frac{1}{2} + \frac{\sqrt{3}}{4}$

B. $\frac{\pi}{6} + \frac{1}{2} - \frac{\sqrt{3}}{4}$

C. $-\frac{\pi}{6} + \frac{1}{2} + \frac{\sqrt{3}}{4}$

D. $\frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4}$

Answer: D

Solution:

Solution:

$\int \frac{\sqrt{x}}{(1+x)^2} dx (x > 0)$

Put $x = \tan^2\theta \Rightarrow 2x dx = 2\tan\theta \sec^2\theta d\theta$

$I = \int \frac{2\tan^2\theta . \sec^2\theta}{\sec^4\theta} d\theta = \int 2\sin^2\theta d\theta$

$= \theta - \frac{\sin 2\theta}{2} + C$

$\Rightarrow f(x) = \theta - \frac{1}{2} \times \frac{2\tan\theta}{1+\tan^2\theta} + C$

$f(x) = \theta - \frac{\tan\theta}{1+\tan^2\theta} + C = \tan^{-1}\sqrt{x} - \frac{\sqrt{x}}{1+x} + C$

Now $f(3) - f(1) = \tan^{-1}(\sqrt{3}) - \frac{\sqrt{3}}{1+3} - \tan^{-1}(1) + \frac{1}{2}$

$= \frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4}$

Question184

If $\int \sin^{-1}\left(\sqrt{\frac{x}{1+x}}\right)dx = A(x)\tan^{-1}(\sqrt{x}) + B(x) + C$, wher e C is a constant of integration, then the ordered pair (A(x), B(x)) can be :
[Sep. 03, 2020 (II)]

Options:

A. $(x+1, -\sqrt{x})$

B. $(x+1, \sqrt{x})$

C. $(x-1, -\sqrt{x})$

D. $(x-1, \sqrt{x})$

Answer: A

Solution:

Solution:

$I = \int \sin^{-1}\left(\frac{\sqrt{x}}{\sqrt{1+x}}\right)dx = \int \tan^{-1}\sqrt{x}\,.\,1dx$

$= x\tan^{-1}\sqrt{x} - \int \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}\,.\,xdx + C$

$= x\tan^{-1}\sqrt{x} - \frac{1}{2}\int \frac{t\,.\,2tdt}{1+t^2} + C$ (Put $x = t^2 \Rightarrow dx = 2tdt$)

$= x\tan^{-1}\sqrt{x} - \int \frac{t^2}{1+t^2}dt + C$

$= x\tan^{-1}\sqrt{x} - t + \tan^{-1}t + C$

$= x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}\sqrt{x} + C$

$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$

$\Rightarrow A(x) = x+1 \Rightarrow B(x) = -\sqrt{x}$

Question185

The integral $\int_1^2 e^x\,.\,x^x(2+\log_e x)dx$ equals:
[Sep. 06, 2020 (II)]

Options:

A. $e(4e+1)$

B. $4e^2-1$

C. $e(4e-1)$

D. $e(2e-1)$

Answer: C

Solution:

Solution:

$I=\int_1^2 e^x x^x(2+\log_e x)dx$

$I=\int_1^2 e^x x^x[1+(1+\log_e x)]dx$

$=\int_1^2 e^x[x^x+x^x(1+\log_e x)]dx$

$\because \int e^x(f(x)+f'(x))dx=e^x f(x)+c$

$\therefore I=[e^x x^x]_1^2$

$=e^2\times 4-e\times 1=4e^2-e\ =e(4e-1)$

Question186

If $I_1=\int_0^1(1-x^{50})^{100}dx$ and $I_2=\int_0^1(1-x^{50})^{101}dx$ such that $I_2=\alpha I_1$ then α equals to :
[Sep. 06, 2020 (I)]

Options:

A. $\frac{5049}{5050}$

B. $\frac{5050}{5049}$

C. $\frac{5050}{5051}$

D. $\frac{5051}{5050}$

Answer: C

Solution:

$I_2=\int_0^1(1-x^{50})^{101}dx=\int_0^1(1-x^{50})(1-x^{50})^{100}dx$

$I_2=\int_0^1(1-x^{50})^{100}dx-\int_0^1 x\,.\,x^{49}(1-x^{50})^{100}dx$

$I_2=I_1+\left[\frac{x}{5050}(1-x^{50})^{101}\right]_0^1-\int_0^1\frac{(1-x^{50})^{101}}{5050}dx$

$I_2=I_1+0-\frac{I_2}{5050}$

$\Rightarrow\frac{5051}{5050}I_2=I_1\Rightarrow I_2=\frac{5050}{5051}I_1$

$\Rightarrow\alpha=\frac{5050}{5051}$

Question187

The value of $\int_{-\pi/2}^{\pi/2}\frac{1}{1+e^{\sin x}}dx$ is:

[Sep. 05, 2020 (I)]

Options:

A. $\frac{\pi}{4}$

B. π

C. $\frac{\pi}{2}$

D. $\frac{3\pi}{2}$

Answer: C

Solution:

Solution:

$I=\int_{-\pi/2}^{\pi/2}\frac{1}{1+e^{\sin x}}dx$

$=\int_{-\pi/2}^{0}\frac{1}{1+e^{\sin x}}dx+\int_0^{\pi/2}\frac{1}{1+e^{\sin x}}dx$

$=\int_0^{\pi/2}\left(\frac{1}{1+e^{\sin x}}+\frac{1}{1+e^{-\sin x}}\right)dx$

$=\int_0^{\pi/2}\frac{1+e^{\sin x}}{1+e^{\sin x}}dx=\frac{\pi}{2}$

Question188

Let $f(x) = |x-2|$ and $g(x) = f(f(x)), x \in [0, 4]$ Then $\int_0^3 (g(x) - f(x))dx$ is equal to:

[Sep. 04, 2020 (I)]

Options:

A. 1

B. 0

C. $\frac{1}{2}$

D. $\frac{3}{2}$

Answer: A

Solution:

Solution:

$$f(x) = |x-2| = \begin{cases} 2-x, & x < 2 \\ x-2, & x \geq 2 \end{cases}$$

$$g(x) = f(f(x)) = \begin{cases} 2-f(x), & f(x) < 2 \\ f(x)-2, & f(x) \geq 2 \end{cases}$$

$$= \begin{cases} 2-(2-x), & 2-x < 2, \quad x < 2 \\ (2-x)-2, & 2-x \geq 2, \quad x < 2 \\ 2-(x-2), & x-2 < 2, \quad x \geq 2 \\ (x-2)-2, & x-2 \geq 2, \quad x \geq 2 \end{cases}$$

$$= \begin{cases} -x, & 0 < x \leq 0 \\ x, & 0 < x < 2 \\ 4-x, & 2 \leq x < 4 \\ x-4, & x \geq 4 \end{cases}$$

$$\therefore \int_0^3 [g(x) - f(x)]dx$$

$$= \int_0^2 x dx + \int_2^3 (4-x)dx - \int_0^3 |x-2| dx = 1$$

Question189

The integral $\int_{\pi/6}^{\pi/3} \tan^3 x \,.\, \sin^2 3x (2\sec^2 x \,.\, \sin^2 3x + 3\tan x \,.\, \sin 6x)\, dx$ is equal to :
[Sep. 04, 2020 (II)]

Options:

A. $\frac{7}{18}$

B. $-\frac{1}{9}$

C. $-\frac{1}{18}$

D. $\frac{9}{2}$

Answer: C

Solution:

Solution:

$$\int_{\pi/6}^{\pi/3} \left[\frac{1}{2} \frac{d(\tan^4 x)}{dx} \,.\, \sin^4 3x + \frac{1}{2} \tan^4 x \,.\, \frac{d(\sin^4 3x)}{dx} \right] dx$$

$$= \frac{1}{2} \int_{\pi/6}^{\pi/3} d(\tan^4 x \,.\, \sin^4 3x) dx$$

$$= \left[\frac{\tan^4 x \sin^4 3x}{2} \right]_{\pi/6}^{\pi/3} = \frac{9 \,.\, 0}{2} - \frac{\frac{1}{9} \,.\, 1}{2} = \frac{-1}{18}$$

Question190

Let {x} and [x] denote the fractional part of x and the greatest integer leq x respectively of a real number x. If $\int_0^n \{x\} dx$, $\int_0^n [x] dx$ and $10(n^2 - n)$, $(n \in N, n > 1)$ are three consecutive terms of a G.P., then n is equal to_______.
[NA Sep. 04, 2020 (II)]

Answer: 21

Solution:

$\int_0^n \{x\} d x = n \int_0^1 x . d x = \frac{n}{2}$

$\Rightarrow \int_0^n [x] d x = \int_0^n (x - \{x\}) d x = \frac{n^2}{2} - \frac{n}{2}$

According to the questions,

$\frac{n}{2}, \frac{n^2 - n}{2}, 10(n^2 - n)$ are in GP

$\therefore \left(\frac{n^2 - n}{2}\right)^2 = \frac{n}{2} \times 10(n^2 - n)$

$\Rightarrow n^2 = 21n \Rightarrow n = 21$

Question191

$\int_{-\pi}^{\pi} |\pi - |x|| d x$ is equal to :
[Sep. 03, 2020 (I)]

Options:

A. $\sqrt{2}\pi^2$

B. $2\pi^2$

C. π^2

D. $\frac{\pi^2}{2}$

Answer: C

Solution:

Solution:

$I = \int_{-\pi}^{\pi} |\pi - |x|| d x$ [$\because$ | $\pi -$ | x|| is even]

$= 2 \int_0^{\pi} |\pi - |x|| d x$

$= 2 \int_0^{\pi} (\pi - x) d x$

$= 2 \left[\pi x - \frac{x^2}{2} \right]_0^{\pi} = 2 \left(\pi^2 - \frac{\pi^2}{2} \right) = \pi^2.$

Question192

If the value of the integral $\int_{0}^{1/2} \frac{x^2}{(1-x^2)^{3/2}} dx$ is $\frac{k}{6}$, then k is equal to:
[Sep. 03, 2020 (II)]

Options:

A. $2\sqrt{3}-\pi$

B. $2\sqrt{3}_{\pi}$

C. $3\sqrt{2}+\pi$

D. $3\sqrt{2}-\pi$

Answer: A

Solution:

Solution:

$\frac{k}{6} = \int_{0}^{\frac{1}{2}} \frac{x^2}{(1-x^2)^{3/2}} dx$

Let $x = \sin\theta$; $dx = \cos\theta\, d\theta$

then $\int_{0}^{\frac{1}{2}} \frac{x^2}{(1-x^2)^{3/2}} dx = \int_{0}^{\frac{\pi}{6}} \frac{\sin^2\theta\cos\theta}{\cos^3\theta} d\theta$

$\therefore \frac{k}{6} = \int_{0}^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos^3\theta} . \cos\theta\, d\theta$

$\Rightarrow \frac{k}{6} = \int_{0}^{\frac{\pi}{6}} \tan^2\theta d\theta = \int_{0}^{\frac{\pi}{6}} (\sec^2\theta - 1) d\theta$

$\Rightarrow \frac{k}{6} = (\tan\theta - \theta)_0^{\pi/6} = \left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right) = \frac{2\sqrt{3}-x}{6}$

$\Rightarrow k = 2\sqrt{3} - \pi$

Question193

The integral $\int_{0}^{2} ||x-1|-x| dx$ is equal to _______.
[NA Sep. 02, 2020 (I)]

Answer: 1.50

Solution:

Solution:

$\int_0^2 \|x-1\|-x|dx = \int_0^1 |1-x-x|dx$ \$ +\∫↙{1}↖{2}{||
x-1-x \|}d x\$

$= \int_0^1 (1-2x)dx + \int_{1/2}^1 (2x-1)dx + \int_1^2 dx$

$= [x-x^2]_0^{\frac{1}{2}} + [x^2-x]_{\frac{1}{2}}^1 + [x]_1^2$

$= \frac{1}{2} - \frac{1}{4} + (1-1) - \left(\frac{1}{4} - \frac{1}{2}\right) + 2 - 1 \;= \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2}$

Question194

Let [t] denote the greatest integer less than or equal to t. Then the value of $\int_1^2 |2x-[3x]|dx$ is ________.
[NA Sep. 02, 2020 (II)]

Answer: 1

Solution:

Solution:

$\int_1^2 |2x-[3x]|dx$

$= \int_1^2 |3x-[3x]-x|dx$

$= \int_1^2 |\{3x\}-x|dx = \int_1^2 (x-\{3x\})dx$

$= \int_1^2 xdx - \int_1^2 \{3x\}dx$

$= \left[\frac{x^2}{2}\right]_1^2 - 3\int_0^{1/3} 3xdx$

$= \frac{(4-1)}{2} - 9\left[\frac{x^2}{2}\right]_0^{1/3} \;= \frac{3}{2} - \frac{1}{2} = 1$

Question195

For $x^2 \neq n\pi + 1,\ n \in N$ (the set of natural numbers), the integral $\int x \sqrt{\frac{2\sin(x^2-1) - \sin 2(x^2-1)}{2\sin(x^2-1) + \sin 2(x^2-1)}}\, dx$ is equal to:

[Jan. 09, 2019(I)]

Options:

A. $\log_e \left| \frac{1}{2}\sec^2(x^2-1) \right| + c$

B. $\frac{1}{2}\log_e \left| \sec^2(x^2-1) \right| + c$

C. $\frac{1}{2}\log_e \left| \sec^2\left(\frac{x^2-1}{2}\right) \right| + c$

D. $\log_e \left| \sec^2\left(\frac{x^2-1}{2}\right) \right| + c$

Answer: 0

Solution:

Solution:
Consider the given integral

$I = \int x \sqrt{\frac{2\sin(x^2-1) - 2\sin(x^2-1)\cos(x^2-1)}{2\sin(x^2-1) + 2\sin(x^2-1)\cos(x^2-1)}}\, dx$

$(\because \sin 2\theta = 2\sin\theta\cos\theta)$

$\Rightarrow I = \int x \sqrt{\frac{1-\cos(x^2-1)}{1+\cos(x^2-1)}}\, dx$

Now let $\frac{x^2-1}{2} = t \Rightarrow \frac{2x}{2} dx = dt$

$\therefore I = \int |\tan(t)|\, dt = \ln|\sec t| + C$

or $I = \ln\left|\sec\left(\frac{x^2-1}{2}\right)\right| + c = \frac{1}{2}\ln\left|\sec^2\left(\frac{x^2-1}{2}\right)\right| + c$

Question196

If $f(x) = \int \frac{5x^8 + 7x^6}{(x^2 + 1 + 2x^7)^2} dx$, $(x \geq 0)$ and $f(0) = 0$, then the value of $f(1)$ is:
[Jan. 09, 2019 (II)]

Options:

A. $-\frac{1}{2}$

B. $-\frac{1}{4}$

C. $\frac{1}{2}$

D. $\frac{1}{4}$

Answer: D

Solution:

Solution:

$f(x) = \int \frac{5x^8 + 7x^6}{(x^2 + 1 + 2x^7)^2} dx, x \geq 0$

$= \int \frac{5x^8 + 7x^6}{x^{14}(x^{-5} + x^{-7} + 2)^2} dx$

$= \int \frac{5x^{-6} + 7x^{-8}}{\left(2 + x^{-7} + x^{-5}\right)^2 dx}$

Let $2 + x^{-7} + x^5 = t$

$\Rightarrow (-7x^{-8} - 5x^{-6}) dx = dt$

$\Rightarrow f(x) = \int \frac{-dt}{t^2} = \int -t^{-2} dt = t^{-1} + c$

$\Rightarrow f(x) = \frac{1}{2 + x^{-7} + x^{-5}} + c$, $f(0) = 0 \Rightarrow c = 0$

$\therefore f(1) = \frac{1}{4}$

Question197

The integral $\int \cos(\log_e x)\, d\ x$ is equal to : (where C is a constant of integration) [Jan. 12,2019 (I)]

Options:

A. $\frac{x}{2}[\sin(\log_e x) - \cos(\log_e x)] + C$

B. $x[\cos(\log_e x)+\sin(\log_e x)]+C$

C. $\frac{x}{2}[\cos(\log_e x)+\sin(\log_e x)]+C$

D. $x[\cos(\log_e x)-\sin(\log_e x)]+C$

Answer: C

Solution:

Solution:

Let the integral, $I = \text{int}\cos(\ln x)\,d\,x$

$\Rightarrow I = \cos(\ln x)\,x - \int \frac{-\sin(\ln x)}{x} x d\,x$

$= x\cos(\ln x) + \int \sin(\ln x)\,d\,x$

$= x\cos(\ln x) + \sin(\ln x)\,x - \int \frac{\cos(\ln x)}{x} x d\,x$

$= x\cos(\ln x) + \sin(\ln x).x - I$

$\Rightarrow 2I = x(\cos(\ln x) + \sin(\ln x)) + C$

$\Rightarrow I = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + C$

Question198

The integral $\int \frac{3x^{13}+2x^{11}}{(2x^4+3x^2+1)^4} d\,x$ is equal to:
(where C is a constant of integration)
[Jan.12,2019 (II)]

Options:

A. $\frac{x^4}{6(2x^4+3x^2+1)^3}+C$

B. $\frac{x^{12}}{6(2x^4+3x^2+1)^3}+C$

C. $\frac{x^4}{(2x^4+3x^2+1)^3}+C$

D. $\frac{x^{12}}{(2x^4+3x^2+1)^3}+C$

Answer: B

Solution:

$$I = \int \frac{3x^{13}+2x^{11}}{(2x^4+3x^2+1)^4}dx = \int \frac{3x^{13}+2x^{11}}{x^{16}\left(2+\frac{3}{x^2}+\frac{1}{x^4}\right)^4}dx$$

$$I = \int \frac{\frac{3}{x^3}+\frac{2}{x^5}}{\left(2+\frac{3}{x^2}+\frac{1}{x^4}\right)^4}dx$$

Let $2+\frac{3}{x^2}+\frac{1}{x^4}=t, -2\left(\frac{3}{x^3}+2x^5\right)dx = dt$

Then, $I = \int \frac{-\frac{dt}{2}}{t^4} = -\frac{1}{2}\frac{t^{-4+1}}{-4+1}+C$

$$I = \frac{-1}{2}\times\frac{1}{(-3)}\frac{1}{\left(2+\frac{3}{x^2}+\frac{1}{x^4}\right)^3}+C$$

$$I = \frac{1}{6}\frac{x^{12}}{(2x^4+3x^2+1)^3}+C$$

Question199

If $\int \frac{\sqrt{1-x^2}}{x^4}dx = A(x)\left(\sqrt{1-x^2}\right)^m + C$, for a suitable chosen integer m and a function A(x), where C is a constant of integration, then $(A(x))^m$ equals: [Jan. 11, 2019 (I)]

Options:

A. $\frac{-1}{27x^9}$

B. $\frac{-1}{3x^3}$

C. $\frac{1}{27x^6}$

D. $\frac{1}{9x^4}$

Answer: A

Solution:

$$A(x)\left(\sqrt{1-x^2}\right)^m+C=\int\frac{\sqrt{1-x^2}}{x^4}dx$$

$$=\int\frac{\sqrt{\frac{1}{x^2}-1}}{x^3}dx$$

Let $\frac{1}{x^2}-1=u^2$

$$\Rightarrow-\frac{2}{x^3}=\frac{2udu}{dx}$$

$$\frac{dx}{x^3}=-udu$$

$$A(x)\left(\sqrt{1-x^2}\right)^m+C=\int(-u^2)du=-\frac{u^3}{3}+C$$

$$=-\frac{1}{3}\left(\frac{1}{x^2}-1\right)^{\frac{3}{2}}+C$$

$$=-\frac{1}{3}\cdot\frac{1}{x^3}\cdot(1-x^2)^{\frac{3}{2}}+C$$

$$=\frac{-1}{3x^3}\left(\sqrt{1-x^2}\right)^3+C$$

Compare both sides,

$\Rightarrow A(x)=-\frac{1}{3x^3}$ and $m=3$

$$\Rightarrow(A(x))^3=\frac{-1}{27x^9}$$

Question200

If $\int\frac{x+1}{\sqrt{2x-1}}dx=f(x)\sqrt{2x-1}+C$, where C is a constant of integration, then $f(x)$ is equal to:
[Jan. 11, 2019 (II)]

Options:

A. $\frac{1}{3}(x+1)$

B. $\frac{2}{3}(x+2)$

C. $\frac{2}{3}(x-4)$

D. $\frac{1}{3}(x+4)$

Answer: D

Solution:

Let $I = \int \frac{x+1}{\sqrt{2x-1}} dx$

Put $\sqrt{2x-1} = t$

$\therefore 2x - 1 = t^2 \Rightarrow dx = t dt$

$I = \int \frac{(t^2+3)}{2} dt = \frac{t^3}{6} + \frac{3t}{2} + C$

$= \frac{(2x-1)^{\frac{3}{2}}}{6} + \frac{3}{2}(2x-1)^{\frac{1}{2}} + C$

$= \sqrt{2x-1}\left(\frac{x+4}{3}\right) + C$

$= f(x) . \sqrt{2x-1} + C$

Hence, $f(x) = \frac{x+4}{3}$

Question201

Let $n \geq 2$ be a natural number and $0 < \theta < \frac{\pi}{2}$

Then $\int \frac{(\sin^n\theta + \sin\theta)^{\frac{1}{n}} \cos\theta}{\sin^{n+1}\theta} d\theta$ is equal to:

[Jan 10, 2019(I)]

Options:

A. $\frac{n}{n^2-1}\left(1-\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+C$

B. $\frac{n}{n^2+1}\left(1-\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+C$

C. $\frac{n}{n^2-1}\left(1+\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+C$

D. $\frac{n}{n^2-1}\left(1-\frac{1}{\sin^{n+1}\theta}\right)^{\frac{n+1}{n}}+C$

Answer: A

Solution:

Solution:

Let, $I=\int\frac{(\sin^n\theta-\sin\theta)^{\frac{1}{n}}\cos\theta}{\sin^{n+1}\theta}d\theta$

Let $\sin\theta=u$

$\Rightarrow\cos\theta\, d\theta=du$

$\therefore I = \int \frac{(u^n - u)^{\frac{1}{n}}}{u^{n+1}} du$

$= \int \frac{\left(1 - \frac{1}{u^{n-1}}\right)^{\frac{1}{n}}}{u^n} du = \int u^{-n}(1 - u^{1-n})^{\frac{1}{n}} du$

Let $1 - u^{1-n} = v$

$\Rightarrow -(1-n)u^{-n} du = dv$

$\Rightarrow u^n du = \frac{dv}{n-1}$

$\therefore I = \int v^{\frac{1}{n}} \cdot \frac{dv}{n-1} = \frac{1}{n-1} \cdot \frac{\frac{v^n+1}{1} - 1}{n}$

$= nn^2 - 1v^{\frac{n+1}{n}} + C = \frac{n}{n^2-1}\left(1 - \frac{1}{u^{n-1}}\right)^{\frac{n+1}{n}} + C$

$= \frac{n}{n^2-1}\left(1 - \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C$

Question202

$\lim_{n \to \infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \ldots\ldots + \frac{1}{5n}\right)$ **is equal to :**

[Jan. 12, 2019 (II)]

Options:

A. $\frac{\pi}{4}$

B. $\tan^{-1}(3)$

C. $\frac{\pi}{2}$

D. $\tan^{-1}(2)$

Answer: D

Solution:

Solution:

Let $L = \lim_{n \to \infty} \sum_{r=1}^{2n} \frac{n}{n^2 + r^2} = \int_0^2 \frac{dx}{1 + x^2}$

$\left[\because \frac{r}{n} \to x, \frac{1}{r} \to dx\right]$

$= [\tan^{-1} x]_0^2$

$= \tan^{-1} 2$

Question203

Let f and g be continuous functions on $[0, a]$ such that $f(x) = f(a - x)$ and $g(x) + g(a - x) = 4$, then $\int_0^a f(x)g(x)dx$ is equal to:
[Jan. 12, 2019 (I)]

Options:

A. $4\int_0^a f(x)dx$

B. $\int_0^a f(x)dx$

C. $2\int_0^a f(x)dx$

D. $-3\int_0^a f(x)dx$

Answer: C

Solution:

Solution:

$f(x) = f(a - x)$

$g(x) + g(a - x) = 4$

Let, the integral,

$I = \int_0^a f(x)g(x)dx$

$= \int_0^a f(a - x) . g(a - x)dx$

$\left[\because \int_a^b f(x)dx = \int_a^b f(a + b - x)dx\right]$

$\Rightarrow I = \int_0^a f(x)[4 - g(x)]dx$

$\Rightarrow I = \int_0^a 4f(x)dx - \int_0^a f(x) . g(x)dx$

$\Rightarrow I = \int_0^a 4f(x)dx - I$

$\Rightarrow 2I = \int_0^a 4f(x)dx$

$\Rightarrow I = 2\int_0^a f(x)dx$

Question204

The integral $\int_1^e \left\{ \left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^x \right\} \log_e x dx$ is equal to : [Jan. 12, 2019 (II)]

Options:

A. $\frac{1}{2} - e - \frac{1}{e^2}$

B. $-\frac{1}{2} + \frac{1}{e} - \frac{1}{2e^2}$

C. $\frac{3}{2} - \frac{1}{e} - \frac{1}{2e^2}$

D. $\frac{3}{2} - e - \frac{1}{2e^2}$

Answer: D

Solution:

Solution:

$I = \int_1^e \left\{ \left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^x \right\} \log_e x dx$

Let $\left(\frac{x}{e}\right)^x = t$

$\Rightarrow x \ln\left(\frac{x}{e}\right) = \ln t$

$\Rightarrow x(\ln x - 1) = \ln t$

On differentiating both sides w.r. $t x$ we get

$\ln x \, . \, d x = \frac{d t}{t}$

When $x = e$ then $t = 1$ and when $x = 1$ then $t = \frac{1}{e}$.

$$I = \int_{\frac{1}{e}}^{1} \left(t^2 - \frac{1}{t}\right) . \frac{d t}{t} = \int_{\frac{1}{e}}^{1} \left(t - \frac{1}{t^2}\right) d t$$

$$= \left(\frac{t^2}{2} + \frac{1}{t}\right)_{\frac{1}{e}}^{1} = \left(\frac{1}{2} + 1\right) - \left(\frac{1}{2e^2} + e\right) = \frac{3}{2} - e - \frac{1}{2e^2}$$

Question205

The value of the integral $\int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} d x$

(where [x] denotes the greatest integer less than or equal to x) is: [Jan. 11, 2019 (I)]

Options:

A. 0

B. $\sin 4$

C. 4

D. $4 - \sin 4$

Answer: A

Solution:

Solution:

Let $f(x) = \frac{\sin^2 x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}$

So, $f(-x) = \frac{\sin^2(-x)}{\left[\frac{-x}{\pi}\right] + \frac{1}{2}}$ $\because [-x] = -1 - [x]$

$$\Rightarrow f(-x) = \frac{\sin^2 x}{-1 - \left[\frac{x}{\pi}\right] + \frac{1}{2}} = \frac{\sin^2 x}{-\frac{1}{2} - \left[\frac{x}{\pi}\right]} = -f(x)$$

$\Rightarrow f(x)$ is odd function

Hence, $\int_{-2}^{2} f(x) dx = 0$

Question206

The integral $\int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x(\tan^5 x + \cot^5 x)}$ equals: [Jan. 11, 2019 (II)]

Options:

A. $\frac{1}{20}\tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)$

B. $\frac{1}{10}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)\right)$

C. $\frac{\pi}{40}$

D. $\frac{1}{5}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{3\sqrt{3}}\right)\right)$

Answer: B

Solution:

Solution:

$$I = \int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x(\tan^5 x + \cot^5 x)}$$

$$= \int_{\pi/6}^{\pi/4} \frac{\tan^5 x \,.\, \sec^2 x}{2\frac{\sin x}{\cos x}((\tan^5 x)^2 + 1)}$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/4} \frac{\tan^4 x \,.\, \sec^2 x}{(\tan^5 x)^2 + 1} dx$$

Let $\tan^4 x = t$

$5\tan^4 x \,.\, \sec^2 x dx = dt$

When $x \to \frac{\pi}{4}$ then $t \to 1$

and $x \to \frac{\pi}{6}$ then $t \to \left(\frac{1}{\sqrt{3}}\right)^5$

$\therefore I = \frac{1}{10} \int_{\left(\frac{1}{\sqrt{3}}\right)^5}^{1} \frac{d t}{t^2+1}$

$= \frac{1}{10}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)\right)$

Question207

Let $I = \int_a^b (x^4 - 2x^2) d x$. If I is minimum then the ordered pair (a, b) is: [Jan 10, 2019 (I)]

Options:

A. $(0, \sqrt{2})$

B. $(-\sqrt{2}, 0)$

C. $(\sqrt{2}, -\sqrt{2})$

D. $(-\sqrt{2}, \sqrt{2})$

Answer: D

Solution:

Solution:

$I = \int_a^b (x^4 - 2x^2) d x$

$\Rightarrow \frac{d I}{d x} = x^4 - 2x^2 = 0$(for minimum)

$\Rightarrow x = 0, \pm\sqrt{2}$

Also, $I = \left[\frac{x^5}{5} - \frac{2x^3}{3}\right]_a^b$

For $a = 0, b = \sqrt{2}$

$I = \frac{-8\sqrt{2}}{15}$

For $a = -\sqrt{2}, b = 0$

$I = \frac{-8\sqrt{2}}{15}$

For $a = \sqrt{2}, b = -\sqrt{2}$

$I = \frac{16\sqrt{2}}{15}$.

For $a = -\sqrt{2}, b = \sqrt{2}$

$I = \frac{-16\sqrt{2}}{15}$

$\therefore I$ is minimum when $(a, b) = (-\sqrt{2}, \sqrt{2})$

Question208

If $\int_0^x f(t)dt = x^2 + \int_x^1 t^2 f(t)dt$, then $f'(1/2)$ is: [Jan. 10, 2019 (II)]

Options:

A. $\frac{24}{25}$

B. $\frac{18}{25}$

C. $\frac{4}{5}$

D. $\frac{6}{25}$

Answer: A

Solution:

Solution:

$\int_0^x f(t)dt = x^2 + \int_x^1 t^2 f(t)dt$

$\Rightarrow f(x) = 2x - x^2 f(x)$

$\Rightarrow f(x) = \frac{2x}{1 + x^2}$

$\Rightarrow f'(x) = \frac{2(1 - x^2)}{(1 + x^2)^2}$

Then,

$$f'(1/2)=\frac{2\left(1-\frac{1}{4}\right)}{\left(1+\frac{1}{4}\right)^2}=\frac{3}{2}\times\frac{16}{25}=\frac{24}{25}$$

Question209

The value of $\int_{-\pi/2}^{\pi/2}\frac{dx}{[x]+[\sin x]+4}$, where [t] denotes the greatest integer less than or equal to t, is:
[Jan. 10, 2019 (II)]

Options:

A. $\frac{1}{12}(7\pi+5)$

B. $\frac{1}{12}(7\pi-5)$

C. $\frac{3}{20}(4\pi-3)$

D. $\frac{3}{10}(4\pi-3)$

Answer: C

Solution:

Solution:

$$I=\int_{-\pi/2}^{\pi/2}\frac{dx}{[x]+[\sin x]+4}$$

$$=\int_{\frac{-\pi}{2}}^{-1}\frac{dx}{-2-1+4}+\int_{-1}^{0}\frac{dx}{-1-1+4}+\int_{0}^{1}\frac{dx}{0+0+4}+\int_{1}^{\frac{\pi}{2}}\frac{dx}{1+0+4}$$

$$=\left(-1+\frac{\pi}{2}\right)+\frac{1}{2}(0+1)+\frac{1}{4}(1-0)+\frac{1}{5}\left(\frac{\pi}{2}-1\right)$$

$$=\frac{3\pi}{5}-\frac{9}{20}=\frac{3}{20}(4\pi-3)$$

Question210

The value of $\int_0^{\pi} |\cos x|^3 d x$ is:
[Jan 9, 2019 (I)]

Options:

A. 0

B. $\frac{4}{3}$

C. $\frac{2}{3}$

D. $-\frac{4}{3}$

Answer: B

Solution:

Solution:

$I = \int_0^{\pi} |\cos x|^3 d x$

$= 2 \int_0^{\pi/2} \cos^3 x d x$

$= \frac{2}{4} \int_0^{\pi/2} (3 \cos x + \cos 3 x) d x \ [\because \cos 3\theta = 4\cos^3\theta - 3 \cos\theta]$

$= \frac{1}{2} \left[3 \sin x + \frac{\sin 3 x}{3} \right]_0^{\pi/2}$

$= \frac{1}{2} \left(3 - \frac{1}{3} \right) = \frac{4}{3}$

Question211

Let f be a differentiable function from R to R such that $|f(x) - f(y)| \le 2 |x - y|^{3/2}$, for all $x, y, \in R$. If $f(0) = 1$ then $\int_0^1 f^2(x) d x$ is equal to :
[Jan. 09, 2019 (II)]

Options:

A. 1

B. 2

C. $\frac{1}{2}$

D. 0

Answer: A

Solution:

Solution:

$\because f : R \rightarrow R$

and $|f(x) - f(y)| \leq 2 . x - y|^{3/2}$

$\Rightarrow \left|\frac{f(x) - f(y)}{x - y}\right| \leq 2\sqrt{x - y}$

$\Rightarrow \lim\limits_{x \rightarrow y}\left|\frac{f(x) - f(y)}{x - y}\right| \leq \lim\limits_{x \rightarrow y} 2\sqrt{x - y}$

$\Rightarrow f'(x) | = 0$

$\therefore f(x)$ is a constant function.

Given $f(0) = 1 \Rightarrow f(x) = 1$

Hence, the integral

$\int_0^1 f^2(x) d x = \int_0^1 1 d x = [x]_0{}^1 = 1$

Question212

If $\int_0^{\pi/3} \frac{\tan\theta}{\sqrt{2k\sec\theta}} d\theta = 1 - \frac{1}{\sqrt{2}}$, (k > 0) then the value of kis: [Jan. 09, 2019 (II)]

Options:

A. 4

B. $\frac{1}{2}$

C. 1

D. 2

Answer: D

Solution:

Solution:

Let, $I = \int_0^{\pi/3} \frac{\tan\theta}{\sqrt{2k\sec\theta}} d\theta$

$= \frac{1}{\sqrt{2k}} \int_0^{\pi/3} \frac{\sin\theta}{\sqrt{\cos\theta}} d\theta$

Let $\cos\theta = t^2$

$\therefore \sin\theta\, d\theta = -2t\, dt$

Hence, integral becomes,

$I = \frac{1}{\sqrt{2k}} \int_1^{\sqrt{\frac{1}{2}}} \frac{-2t\, dt}{t}$

$= \sqrt{\frac{2}{k}} \int_{\frac{1}{\sqrt{2}}}^{1} dt$

$= \sqrt{\frac{2}{k}} \left(1 - \frac{1}{\sqrt{2}}\right)$

$= \frac{\sqrt{2} - 1}{\sqrt{k}}$

$= 1 - \frac{1}{\sqrt{2}}$ (Given)

$\therefore k = 2$

Question213

If $\int x^5 e^{-4x^3} dx = \frac{1}{48} e^{-4x^3} f(x) + C$, where C is a constant of integration, then $f(x)$ is equal to: [Jan. 10, 2019 (II)]

Options:

A. $-2x^3 - 1$

B. $-4x^3 - 1$

C. $-2x^3+1$

D. $4x^3+1$

Answer: B

Solution:

Solution:

$I=\int x^5 e^{-4x^3} dx$

Put $-4x^3=\theta$

$\Rightarrow -12x^2 dx = d\theta$

$\Rightarrow x^2 dx = -\frac{d\theta}{12}$

$I=\int \frac{1}{48}\theta e^{\theta} d\theta = \frac{1}{48}[\theta e^{\theta}-e^{\theta}]+C$

$I=\frac{1}{48}e^{-4x^3}(-4x^3-1)+C$

Then, by comparison

$f(x)=-4x^3-1$

Question214

A value of α such that $\int_{\alpha}^{\alpha+1} \frac{dx}{(x+\alpha)(x+\alpha+1)} = \log_e\left(\frac{9}{8}\right)$ is:

[April 12, 2019 (II)]

Options:

A. -2

B. $\frac{1}{2}$

C. $-\frac{1}{2}$

D. 2

Answer: A

Solution:

$$\int_{\alpha}^{\alpha+1} \frac{dx}{(x+\alpha)(x+\alpha+1)}$$

$= \int_{\alpha}^{\alpha+1} \left[\frac{1}{x+\alpha} - \frac{1}{x+\alpha+1}\right] dx$ [Using partial fraction]

$= \log\left(\frac{(x+\alpha)}{(x+\alpha+1)}\right)\Big|_{\alpha}^{\alpha+1} = \log\left(\frac{2\alpha+1}{2\alpha+2} \cdot \frac{2\alpha+1}{2\alpha}\right)$

$= \log\frac{9}{8}\Big($ Given $)$

So, $\frac{(2\alpha+1)^2}{\alpha(\alpha+1)} = \frac{9}{2} \Rightarrow 8\alpha^2 + 8\alpha + 2 = 9\alpha^2 + 9\alpha$

$\Rightarrow \alpha^2 + \alpha - 2 = 0 \Rightarrow \alpha = 1, -2$

Question215

The integral $\int \frac{2x^3 - 1}{x^4 + x} dx$ is equal to :
(Here C is a constant of integration)
[April 12, 2019 (I)]

Options:

A. $\frac{1}{2}\log_e \frac{|x^3+1|}{x^2} + C$

B. $\frac{1}{2}\log_e \frac{(x^3+1)^2}{|x^3|} + C$

C. $\log_e \left|\frac{x^3+1}{x}\right| + C$

D. $\log_e \frac{|x^3+1|}{x^2} + C$

Answer: C

Solution:

Solution:

Given integral, $I = \int \frac{(2x^3-1)dx}{x^4+x} = \int \frac{(2x - x^{-2})dx}{x^2 + x^{-1}}$

Put $x^2 + x^{-1} = u \Rightarrow (2x - x^{-2})dx = du$

$\Rightarrow I = \int \frac{du}{u} = \log|u| + c = \log|x^2 + x^{-1}| + c$

$= \log\left|\frac{x^3+1}{x}\right| + c$

Question216

Let $\alpha \in (0, \pi/2)$ be fixed. If the integral $\int \frac{\tan x + \tan \alpha}{\tan x - \tan \alpha} dx$ $= A(x)\cos 2\alpha + B(x)\sin 2\alpha + C$, where C is a constant of integration, then the functions A(x) and B(x) are respectively:
[April 12, 2019 (II)]

Options:

A. $x + \alpha$ and $\log_e|\sin(x+\alpha)|$

B. $x - \alpha$ and $\log_e|\sin(x-\alpha)|$

C. $x - \alpha$ and $\log_e|\cos(x-\alpha)|$

D. $x + \alpha$ and $\log_e|\sin(x-\alpha)|$

Answer: B

Solution:

Solution:
Given integral
$\int \frac{\tan x + \tan \alpha}{\tan x - \tan \alpha} dx = \int \frac{\sin(x+\alpha)}{\sin(x-\alpha)} dx$
Let $x - \alpha = t \Rightarrow dx = dt$
$= \int \frac{\sin(t+2\alpha)}{\sin t} dt = \int [\cos 2\alpha + \sin 2\alpha . \cot t] dt$
$= \cos 2\alpha \, cdot \, t + \sin 2\alpha \; .\log|\sin t| + c$
$= (x-\alpha) . \cos 2\alpha + \sin 2\alpha \; .\log|\sin(x-\alpha)| + c$

Question217

If $\int \frac{dx}{(x^2-2x+10)^2} = A\left(\tan^{-1}\left(\frac{x-1}{3}\right)+\frac{f(x)}{x^2-2x+10}\right)$ +C where C is a constant of integration, then :
[April 10, 2019 (I)]

Options:

A. $A=\frac{1}{54}$ and $f(x)=3(x-1)$

B. $A=\frac{1}{81}$ and $f(x)=3(x-1)$

C. $A=\frac{1}{27}$ and $f(x)=9(x-1)$

D. $A=\frac{1}{54}$ and $f(x)=9(x-1)^2$

Answer: A

Solution:

Solution:
Let $I=\int \frac{dx}{(x^2-2x+10)^2}=\int \frac{dx}{((x-1)^2+9)^2}$
Let $(x-1)^2=9\tan^2\theta$ .....(i)
$\Rightarrow \tan\theta=\frac{x-1}{3}$
After differentiating equation ...(i), we get
$2(x-1)dx=18\tan\theta\sec^2\theta\, d\theta$
$\therefore I=\int \frac{18\tan\theta\sec^2\theta\, d\theta}{2\times 3\tan\theta\times 81\sec^4\theta}$
$I=\frac{1}{27}\int \cos^2\theta d\theta=\frac{1}{27}\times\frac{1}{2}\int(1+\cos 2\theta)d\theta$
$I=\frac{1}{54}\left\{\theta+\frac{\sin 2\theta}{2}\right\}+c$
$$I=\frac{1}{54}\left[\tan^{-1}\left(\frac{x-1}{3}\right)+\frac{1}{2}\times\frac{2\left(\frac{x-1}{3}\right)}{1+\left(\frac{x-1}{3}\right)^2}\right]+c$$
$$I=\frac{1}{54}\left[\tan^{-1}\left(\frac{x-1}{3}\right)+\frac{3(x-1)}{x^2-2x+10}\right]+c$$

Compare it with $A\left[\tan^{-1}\left(\frac{x-1}{b}\right)+\frac{f(x)}{x^2-2x+10}\right]+c$,

we get: $A=\frac{1}{54}$ and $f(x)=3(x-1)$

Question218

If $f(x)$ is a non-zero polynomial of degree four, having local extreme points at $x=-1, 0, 1$; then the set $S=\{x \in R : f(x)=f(0)\}$ contains exactly:
[April 09, 2019 (I)]

Options:

A. four irrational numbers.

B. four rational numbers.

C. two irrational and two rational numbers.

D. two irrational and one rational number

Answer: D

Solution:

Solution:
Since, function $f(x)$ have local extreem points at $x=-1, 0, 1$. Then
$f'(x)=K(x+1)x(x-1)$
$=K(x^3-x)$
$\Rightarrow f(x)=K\left(\frac{x^4}{4}-\frac{x^2}{2}\right)+C$ (using integration)
$\Rightarrow f(0)=C$
$\because f(x)=f(0)\Rightarrow K\left(\frac{x^4}{4}-\frac{x^2}{2}\right)=0$
$\Rightarrow \frac{x^2}{2}\left(\frac{x^2}{2}-1\right)=0 \Rightarrow x=0, 0, \sqrt{2}, -\sqrt{2}$
$\therefore S=\{0, -\sqrt{2}, \sqrt{2}\}$

Question219

The integral $\int \sec^{2/3}x\operatorname{cosec}^{4/3}x\,dx$ is equal to:
(Here C is a constant of integration)
[April 09, 2019 (I)]

Options:

A. $-3\tan^{-1/3}x+C$

B. $-\frac{3}{4}\tan^{-4/3}x+C$

C. $-3\cot^{-1/3}x+C$

D. $3\tan^{-1/3}x+C$

Answer: A

Solution:

Solution:

$I=\int \sec^{\frac{2}{3}}x\,.\operatorname{cosec}^{\frac{4}{3}}dx$

$I=\int \frac{\sec^2 x\,dx}{\tan^{\frac{4}{3}}x}$

Put $\tan x=z$

$\Rightarrow \sec^2 x\,dx=dz$

$\Rightarrow I=\int z^{-\frac{4}{3}}\,.dz=\frac{z^{\frac{-1}{3}}}{\left(\frac{-1}{3}\right)}+C \Rightarrow I=-3(\tan x)^{\frac{-1}{3}}+C$

Question220

If $\int e^{\sec x}(\sec x\tan x\,f(x)+(\sec x\tan x+\sec^2 x))dx=e^{\sec x}f(x)+C$, then a possible choice of $f(x)$ is:
[April 09, 2019(II)]

Options:

A. $\sec x+\tan x+\frac{1}{2}$

B. $\sec x-\tan x-\frac{1}{2}$

C. $\sec x+x\tan x-\frac{1}{2}$

D. $x\sec x+\tan x+\frac{1}{2}$

Answer: A

Solution:

Solution:
Given,
$\int e^{\sec x}(\sec x\tan x\, f(x)+(\sec x\tan x+\sec^2 x))\,dx = e^{\sec x}f(x)+C$, ...(i)
$\because \int e^{g(x)}((g'(x)f(x))+f'(x))dx = e^{g(x)}\times f(x)+C$
Our comparing above equation by equation (i),
$f(x)=\int((\sec x\tan x)+\sec^2 x)dx$
$\therefore f(x)=\sec x+\tan x+C$

Question221

$\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}}dx$ **is equal to :**

(where c is a constant of integration.)
[April 08, 2019 (I)]

Options:

A. $x+2\sin x+2\sin 2x+c$

B. $2x+\sin x+2\sin 2x+c$

C. $x+2\sin x+\sin 2x+c$

D. $2x+\sin x+\sin 2x+c$

Answer: C

Solution:

Solution:

$$\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}}dx = \int \frac{2\cos\frac{x}{2}\,.\,\sin\frac{5x}{2}}{2\cos\frac{x}{2}\,.\,\sin\frac{x}{2}}dx$$

$$= \int \frac{\sin 3x + \sin 2x}{\sin x}dx$$

$$= \int (3 - 4\sin^2 x + 2\cos x)dx$$

$[\because \sin 2x = 2\sin x\cos x$ and $\sin 3x = 3\sin x - 4\sin^3 x]$

$$= \int (3 - 2(1 - \cos 2x) + 2\cos x)dx$$

$$= \int (1 + 2\cos x + 2\cos 2x)dx$$

$$= x + 2\sin x + \sin 2x + c$$

Question222

If $\int \frac{dx}{x^3(1+x^6)^{\frac{2}{3}}} = xf(x)(1+x^6)^{\frac{1}{3}} + C$, where C is a constant of integration, then the function $f(x)$ is equal to :
[April 08,2019 (II)]

Options:

A. $\frac{3}{x^2}$

B. $-\frac{1}{6x^3}$

C. $-\frac{1}{2x^2}$

D. $-\frac{1}{2x^3}$

Answer: D

Solution:

Solution:

Let, $\int \frac{dx}{x^3(1+x^6)^{\frac{2}{3}}} = \int \frac{dx}{x^7(1+x^{-6})^{\frac{2}{3}}}$

Put $1+x^{-6}=t^3 \Rightarrow -6^{-7}dx = 3t^2dt \Rightarrow \frac{dx}{x^7} = \left(-\frac{1}{2}\right)t^2dt$

Now, $I = \int\left(-\frac{1}{2}\right)\frac{t^2dt}{t^2} = -\frac{1}{2}t + C$

$= -\frac{1}{2}(1+x^{-6})^{\frac{1}{3}} + C \ = -\frac{1}{2}\frac{(1+x^6)^{\frac{1}{3}}}{x^2} + C$

$= -\frac{1}{2x^3}x(1+x^6)^{\frac{1}{3}} + C$

Hence, $f(x) = -\frac{1}{2x^3}$

Question223

If $\int x^5 e^{-x^2} dx = g(x)e^{-x^2} + c$, where c is a constant of integration, then g(−1) is equal to : [April 10,2019 (II)]

Options:

A. -1

B. 1

C. $-\frac{5}{2}$

D. $-\frac{1}{2}$

Answer: C

Solution:

Solution:

Let, $1 = \int x^2 . e^{-x^2} dx$

Put $-x^2 = t \Rightarrow -2xdx = dt$

$1 = \int \frac{t^2 . e^t d\,t}{(-2)} = \frac{-1}{2} e^t (t^2 - 2t + 2)c$

$\therefore g(x) = \frac{-1}{2}(x^4 + 2x^2 + 2) \Rightarrow g(-1) = \frac{-5}{2}$

Question224

Let $f : R \rightarrow R$ be a continuously differentiable function such that $f(2) = 6$ and $f'(2) = \frac{1}{48}$.

If $\int_{6}^{f(x)} 4t^3 d\,t = (x-2)g(x)$, then $\lim_{x \to 2} g(x)$ is equal to:

[April 12, 2019 (I)]

Options:

A. 18

B. 24

C. 12

D. 36

Answer: A

Solution:

Solution:

Given, $\int_{6}^{f(x)} 4t^3 d\,t = (x-2)g(x)$

Differentiating both sides,

$4(f(x))^3 . f'(x) = g'(x)(x-2) + g(x)$

Putting $x = 2$, $\frac{4(6)^3 . 1}{48} = g(2) \Rightarrow \lim_{x \to 2} g(x) = 18$

Question225

If $\int_0^{\frac{x}{2}} \frac{\cot x}{\cot x+\operatorname{cosec} x} d x = m(\pi+n)$, then m . n is equal to :
[April 12, 2019 (I)]

Options:

A. $-\frac{1}{2}$

B. 1

C. $\frac{1}{2}$

D. -1

Answer: D

Solution:

Solution:

$\int_0^{\frac{x}{2}} \frac{\cot x}{\cot x+\operatorname{cosec} x} d x$

$= \int_0^{\frac{x}{2}} \frac{\cot x\, d\, x}{1+\cos x} = \int_0^{\frac{x}{2}} \left(1-\frac{1}{1+\cos x}\right) d x$

$= [x]_0^{\pi \overline{2}} - \int_0^{\frac{x}{2}} \frac{1}{2 \cos^2 \frac{x}{2}} d x = \frac{\pi}{2}-\frac{1}{2} \int_0^{\frac{x}{2}} \sec^2 \frac{x}{2} d x$

$= \frac{\pi}{2}-\left(\tan \frac{x}{2}\right)_0^{\pi / 2} = \frac{\pi}{2}-[1] = \left(\frac{\pi}{2}-1\right) = m\pi+mn$

$\therefore m=, n=-2$, Hence, $mn=-1$

Question226

The value of $\int_0^{2\pi} [\sin 2 x(1+\cos 3 x)] d x$, where [t] denotes the greatest integer function, is:
[April 10, 2019 (I)]

Options:

A. π

B. $-\pi$

C. -2π

D. 2π

Answer: B

Solution:

Solution:

$I = \int_0^{2\pi} [\sin 2x(1+\cos 3x)]dx$ ....(1)

$\because \int_0^a f(x) = \int_0^a f(a-x)dx$

$\therefore I = \int_0^{2\pi} [-\sin 2x(1+\cos 3x)]dx$ ....(2)

From $(1)+(2)$, we get;

$2I = \int_0^{2\pi} (-1)dx \Rightarrow 2I = -(x)_0^{2\pi} \Rightarrow I = -\pi$

Question227

The integral $\int_{\pi/6}^{\pi/3} \sec^{2/3}x \operatorname{cosec}^{4/3} x\, dx$ is equal to: [April 10, 2019(II)]

Options:

A. $3^{5/6} - 3^{2/3}$

B. $3^{4/3} - 3^{1/3}$

C. $3^{7/6} - 3^{5/6}$

D. $3^{5/3} - 3^{1/3}$

Answer: C

Solution:

Let, $I = \int_{\pi/6}^{\pi/3} \sec^{\frac{2}{3}} x \cdot \operatorname{cosec}^{\frac{4}{3}} x\, dx = \int_{\pi/6}^{\pi/3} \frac{1 \cdot dx}{\cos^{\frac{2}{3}} x \cdot \sin^{\frac{4}{3}} x}$

$$= \int_{\pi/6}^{\pi/3} \frac{1\,dx}{\cos^2 x \cdot \tan^{\frac{4}{3}} x} = \int_{\pi/6}^{\pi/3} \frac{\sec^2 x\, dx}{\tan^{\frac{4}{3}} x}$$

Let $\tan x = u$

$$I = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} u^{-\frac{4}{3}} du = \frac{3\left[u^{-\frac{1}{3}}\right]_{\frac{1}{\sqrt{3}}}}{-1}$$

$$= -3\left[3^{-\frac{1}{6}} - \frac{1}{3^{\frac{-1}{6}}}\right] = -3\left(3^{\frac{-1}{6}} - 3^{\frac{1}{6}}\right)$$

$$= 3\left(3^{\frac{1}{6}} - 3^{\frac{1}{6}}\right) = \left(3^{\frac{7}{6}} - 3^{\frac{5}{6}}\right)$$

Question228

The value of $\int_0^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x} dx$ is: [April 9, 2019 (I)]

Options:

A. $\frac{\pi - 2}{8}$

B. $\frac{\pi - 1}{4}$

C. $\frac{\pi - 2}{4}$

D. $\frac{\pi - 1}{2}$

Answer: B

Solution:

Let $I=\int_{0}^{\pi/2}\frac{\sin^{3}x}{\sin x+\cos x}dx$ ....(1)

Use the property $\int_{0}^{a}f(x)dx=\int_{0}^{a}f(a-x)dx$

$I=\int_{0}^{\pi/2}\frac{\cos^{3}x\,dx}{\sin x+\cos x}$ ....(2)

Adding equation (1) and (2), we get

$\Rightarrow 2I=\int_{0}^{\pi/2}\left(1-\frac{1}{2}\sin(2x)\right)dx$

$\Rightarrow I=\frac{1}{2}\left[x+\frac{1}{4}\cos 2x\right]_{0}^{\pi/2}$

$\Rightarrow I=\frac{\pi-1}{4}$

Question229

The value of the integral $\int_{0}^{1}x\cot^{-1}(1-x^{2}+x^{4})dx$ is: [April 09, 2019 (II)]

Options:

A. $\frac{\pi}{2}-\frac{1}{2}\log_{e}2$

B. $\frac{\pi}{4}-\log_{e}2$

C. $\frac{\pi}{2}-\log_{e}2$

D. $\frac{\pi}{4}-\frac{1}{2}\log_{e}2$

Answer: D

Solution:

Solution:

$\int_{0}^{1}x\cot^{-1}(1-x^{2}+x^{4})dx=\int_{0}^{1}x\tan^{-1}\left(\frac{1}{1+x^{4}-x^{2}}\right)$

$=\int_{0}^{1}x\tan^{-1}\left(\frac{x^{2}-(x^{2}-1)}{1+x^{2}(x^{2}-1)}\right)dx$

$=\frac{1}{2}\int_{0}^{1}1\tan^{-1}t^{2}dt-\frac{1}{2}\int_{-1}^{0}1\tan^{-1}k\,dk$

Put $x^2 = t \Rightarrow 2x\,dx = dt$ in the first integral and
$x^2 - 1 = k \Rightarrow 2x\,dx = dk$ in the second integral.

$$= \frac{1}{2}\int_0^1 1\tan^{-1}t\,dt - \frac{1}{2}\int_0^1 1\tan^{-1}k\,dk$$

$$= \frac{1}{2}\left(t\tan^{-1}t\Big|_0^1 - \int_0^1 \frac{t}{1+t^2}dt\right) - \frac{1}{2}\left(k\tan^{-1}k\Big|_0^1 - \int_{-1}^0 \frac{k}{1+k^2}dk\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \left(\frac{1}{2}\ln(1+t^2)\Big|_0^1\right) - \frac{1}{2}\left(-\frac{\pi}{4} - \left(\frac{1}{2}\ln(1+k^2)\Big|_{-1}^0\right)\right)\right).$$

$$= \left(\frac{\pi}{8} - \frac{1}{4}\ln 2\right) - \left(\frac{-\pi}{8} - \frac{1}{4}10 - \ln 2\right) \;= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Question230

If $f : R \to R$ is a differentiable function and $f(2) = 6$, then $\lim\limits_{x \to 2} \int_6^{f(x)} \frac{2t\,dt}{(x-2)}$ is:
[April 09, 2019 (II)]

Options:

A. $24f'(2)$

B. $2f'(2)$

C. 0

D. $12f'(2)$

Answer: D

Solution:

Solution:
Using L' Hospital rule and Leibnitz theorem, we get

$$\lim_{x \to 2} \frac{\int_6^{f(x)} 2t\,dt}{(x-2)} = \lim_{x \to 2} \frac{2f(x)f'(x) - 0}{1}$$

Putting $x = 2,\ 2f(2)f'(2) = 12f'(2)\ [\because f(2) = 6]$

Question231

If $f(x)=\frac{2-x\cos x}{2+x\cos x}$ and $g(x)=\log_e x$, $(x>0)$ then the value of the integral $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} g(f(x))dx$ is :

[April 8, 2019 (I)]

Options:

A. $\log_e 2$

B. $\log_e 3$

C. $\log_e e$

D. $\log_e 1$

Answer: D

Solution:

Solution:

$g(f(x))=\log\left(\frac{2-x\cos x}{2+x\cos x}\right), x>0$

Let $I=\int_{-\pi/4}^{\pi/4}\log\left(\frac{2-x\cos x}{2+x\cos x}\right)dx$ ....(i)

Use the property $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

Then, equation (i) becomes,

$I=\int_{-\pi/4}^{\pi/4}\log\left(\frac{2+x\cos x}{2-x\cos x}\right)dx$ ....(ii)

Adding (i) and (ii)

$2I=\int_{-\pi/4}^{\pi/4}\log\left(\frac{2-x\cos x}{2+x\cos x}\cdot\frac{2+x\cos x}{2-x\cos x}\right)dx$

$2I=\int_{-\pi/2}^{\pi/2}\log(1)\,dx=0$

$\Rightarrow I=0=\log 1$

Question232

Let $f(x)=\int_0^x g(t)dt$, where g is a non-zero even function.
If $f(x+5)=g(x)$, then $\int_0^x f(t)dt$ equals :
[April 08, 2019(II)]

Options:

A. $\int_{x+5}^{5} g(t)dt$

B. $\int_{5}^{x+5} g(t)dt$

C. $2\int_{5}^{x+5} g(t)dt$

D. $5\int_{x+5}^{5} g(t)dt$

Answer: A

Solution:

Solution:
$f(x)=\int_0^x g(g)dt$, ...(i)
$\because g$ is a non-zero even function.
$\therefore g(-x)=g(x)$.......(ii)
Given, $f(x+5)=g(x)$ ....(iii)
From (i) $f'(x)=g(x)$
Let, $I=\int_0^x f(t)dt$,
Put $t=\lambda-5 \Rightarrow I=\int_5^{x+5} f(\lambda-5)d\lambda$
$\because f(x+5)=g(x)$
$\Rightarrow f(-x+5)=g(-x)=g(x)$ ....(iv)
$I=\int_5^{x+5} f(\lambda-5)d\lambda$
$f(0)=0$, $g(x)$ is even $\Rightarrow f(x)$ is odd
$\therefore I=\int_5^{x+5} -f(5-\lambda)d\lambda$
$\Rightarrow I=\int_5^{x+5} g(\lambda)d\lambda=\int_{x+5}^{5} g(t)dt($ from

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Question233

$\lim_{n\to\infty}\left(\frac{(n+1)^{1/3}}{n^{4/3}}+\frac{(n+2)^{1/3}}{n^{4/3}}+\ldots\ldots+\frac{(2n)^{1/3}}{n^{4/3}}\right)$ is equal to :

[April 10, 2019 (I)]

Options:

A. $\frac{3}{4}(2)^{4/3}-\frac{3}{4}$

B. $\frac{4}{3}(2)^{4/3}$

C. $\frac{3}{2}(2)^{4/3}-\frac{4}{3}$

D. $\frac{4}{3}(2)^{3/4}$

Answer: A

Solution:

Solution:

$\lim_{n\to\infty}\left(\frac{(n+1)^{1/3}}{n^{4/3}}+\frac{(n+2)^{1/3}}{n^{4/3}}+\ldots\ldots+\frac{(2n)^{1/3}}{n^{4/3}}\right)$

$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{(n+r)^{\frac{1}{3}}}{n\,.\,n^{\frac{1}{3}}}$

$=\int_0^1(1+x)^{\frac{1}{3}}dx\quad\left[\because\frac{r}{n}\to x\text{ and }\frac{1}{n}\to\frac{d}{x}\right]$

$=\left[\frac{3}{4}(1+x)^{\frac{4}{3}}\right]_0^1=\frac{3}{4}(2)^{\frac{4}{3}}-\frac{3}{4}$

Question234

The integral

$$\int\frac{\sin^2x\cos^2x}{(\sin^5x+\cos^3x\sin^2x+\sin^3x\cos^2x+\cos^5x)^2}\mathbf{dx}$$

is equal to:
[2018]

Options:

A. $\frac{-1}{3(1+\tan^3 x)}+C$

B. $\frac{1}{1+\cot^3 x}+C$

C. $\frac{-1}{1+\cot^3 x}+C$

D. $\frac{1}{3(1+\tan^3 x)}+C$

Answer: A

Solution:

Solution:
Let I

$\int \frac{\sin^2 x\cos^2 x}{(\sin^5 x+\cos^3 x\sin^2 x+\sin^3 x\cos^2 x+\cos^5 x)^2}dx$

$=\int \frac{\sin^2 x\cos^2 x}{[(\sin^2 x+\cos^2 x)(\sin^3 x+\cos^3 x)]^2}dx$

$=\int \frac{\sin^2 x\cos^2 x}{(\sin^3 x+\cos^3 x)^2}dx \ =\int \frac{\tan^2 x\,.\,\sec^2 x}{(1+\tan^3 x)^2}dx$

Now, put $(1+\tan^3 x)=t$

$\Rightarrow 3\tan^2 x\sec^2 x dx=dt$

$\therefore I \ =\frac{1}{3}\int \frac{dt}{t^2}=-\frac{1}{3t}+C \ =\frac{-1}{3(1+\tan^3 x)}+C$

Question235

If $\int \frac{\tan x}{1+\tan x+\tan^2 x}dx \ =x-\frac{K}{\sqrt{A}}\tan^{-1}\left(\frac{K\ \tan x+1}{\sqrt{A}}\right)$ +C,

(C is a constant of integration), then the ordered pair (K , A) is equal to
[Online April 16, 2018]

Options:

A. (2,3)

B. (2,1)

C. (-2,1)

D. (-2,3)

Answer: A

Solution:

Solution:

Let $I = \int \frac{\tan x}{1+\tan x+\tan^2 x} d x$

$\Rightarrow I = \int \frac{\tan x+1+\tan^2 x}{\tan x+1+\tan^2 x} d x - \int \frac{(1+\tan^2 x)}{1+\tan x+\tan^2 x}$

$\Rightarrow I = x - \int \frac{\sec^2 x d x}{1+\tan x+\tan^2 x}$

Put $\tan x = t \Rightarrow \sec^2 x . d x = d t$

$\therefore I = x - \int \frac{d t}{t^2+t+\frac{1}{4}+1-\frac{1}{4}}$

$= x - \int \frac{d t}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$\Rightarrow I = x - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$

$\Rightarrow I = x - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2 \tan x+1}{\sqrt{3}}\right) + C$

$\therefore A = 3$ and $K = 2$

Question236

If $f\left(\frac{x-4}{x+2}\right) = 2x+1, (x \in R = \{1, -2\})$, then int $f(x)\, d\, x$ is equal to (where C is a constant of integration)
[Online April 15, 2018]

Options:

A. $12\log_e|1-x|-3x+c$

B. $-12\log_e|1-x|-3x+c$

C. $-12\log_e|1-x|+3x+c$

D. $12\log_e|1-x|+3x+c$

Answer: B

Solution:

Solution:

Suppose, $\frac{x-4}{x+2}=y \Rightarrow x-4=y(x+2)$

$\Rightarrow x(1-y)=2y+4 \Rightarrow x=\frac{2y+4}{1-y}$

So, $f(y)=2\left(\frac{2y+4}{1-y}\right)+1$

Now, $f(x)=2\left(\frac{2x+4}{1-x}\right)+1 \;=\frac{3x+9}{1-x}$

$=\frac{3(x+3)}{1-x}=\frac{3(x-1+4)}{1-x} \;=-3+\frac{12}{1-x}$

$\therefore \int f(x)dx=-12\log_e|1-x|-3x+c$

Question237

$\int \frac{2x+5}{\sqrt{7-6x-x^2}}dx \;= A\sqrt{7-6x-x^2}+B\sin^{-1}\left(\frac{x+3}{4}\right)+C$

(where C is a constant of integration), then the ordered pair (A, B) is equal to
[Online April 15, 2018]

Options:

A. (-2,-1)

B. (2,-1)

C. (-2,1)

D. (2,1)

Answer: A

Solution:

Solution:

$\because 7-6x-x^2 = 16-(x+3)^2$

and $\frac{d}{dx}(7-6x-x^2) = -2x-6$

So, $\int \frac{2x+5}{\sqrt{7-6x-x^2}}dx = \int \frac{2x+6}{\sqrt{7-6x-x^2}}dx$

$-\int \frac{1}{\sqrt{16-(x+3)^2}}dx$

$= -2\sqrt{7-6x-x^2} - \sin^{-1}\left(\frac{x+3}{4}\right) + C$

Therefore, $A = -2,\ \& B = -1$

Question238

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x} dx$ is

[2018]

Options:

A. $\frac{\pi}{2}$

B. 4π

C. $\frac{\pi}{4}$

D. $\frac{\pi}{8}$

Answer: C

Solution:

Solution:

Let, $I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$ ....(i)

Using ,$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$, we get :

$I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^{-x}} dx$ ....(ii)

Adding (i) and (ii), we get;

$2I = \int_{-\pi/2}^{\pi/2} \sin^2 x dx \Rightarrow 2I = 2 . \int_0^{x/2} \sin^2 x dx$

$\Rightarrow 2I = 2 \times \frac{\pi}{4} \Rightarrow I = \frac{\pi}{4}$

Question239

If $f(x) = \int_0^x t(\sin x - \sin t) dt$ then [Online April 16, 2018]

Options:

A. $f'''(x) + f'(x) = \cos x - 2x \sin x$

B. $f'''(x) + f''(x) - f'(x) = \cos x$

C. $f'''(x) - f''(x) = \cos x - 2x \sin x$

D. $f'''(x) + f''(x) = \sin x$

Answer: A

Solution:

Solution:

$f(x) = \int_0^x t(\sin x - \sin t) . dt$

$= \sin x \int_0^x t . dt - \int_0^x t \sin t \, cdot \, d \ t$

$= \frac{x^2}{2} \sin x + [t \cos t]_0^x + \sin x$

$$\Rightarrow f(x) = \frac{x^2}{2}\sin x + x\cos x - \sin x$$

$$f'(x) = \frac{x^2}{2}\cos x + 2\cos x$$

$$f''(x) = x\cos x - \frac{x^2}{2}\sin x - 2\sin x$$

$$f'''(x) = \cos x - 2x\sin x - \frac{x^2}{2}\cos x - 2\cos x$$

$$\therefore f'''(x) + f'(x) = \cos x - 2x\sin x$$

Question240

The value of integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{x}{1+\sin x} dx$ is

[Online April 15, 2018]

Options:

A. $\frac{\pi}{2}(\sqrt{2}+1)$

B. $\pi(\sqrt{2}-1)$

C. $2\pi(\sqrt{2}-1)$

D. $\pi\sqrt{2}$

Answer: A

Solution:

Solution:

Let $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{x}{1+\sin x} dx$

also let $K = \frac{x}{1+\sin x}$

Multiplying numerator and denominator by $(1-\sin x)$, we get;

$$K = \frac{x(1-\sin x)}{1-(\sin x)^2} = \frac{x(1-\sin x)}{(\cos x)^2}$$

$= x(1 - \sin x)\sec^2 x$

$= x\sec^2 x - x\sin x\sec^2 x \ = x\sec^2 x - x\tan x\sec x$

Now, $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x\sec^2 x\,dx - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x\sec x\tan x\,dx$

$= \left[x\tan x - \int \frac{dx}{dx}\tan x\,dx\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} - \left[x\sec x - \int \frac{dx}{dx}\sec x\,dx\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$

$= [x\tan x - \ln|\sec x|]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$

$-[x\sec x - \ln|\sec x + \tan x|]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + c$

$\Rightarrow I = \left\{\left[\frac{3\pi}{4}\tan\frac{3\pi}{4} - \ln\left|\frac{3\pi}{4}\right| - \left[\frac{3\pi}{4}\sec\frac{3\pi}{4} - \ln\left|\sec\frac{3\pi}{4} + \tan\frac{3\pi}{4}\right|\right]\right\} - \left\{\left[\frac{\pi}{4}\tan\frac{\pi}{4} - \ln\left|\frac{\pi}{4}\right|\right.\right.$

$\left.\left. -\left[\frac{\pi}{4}\sec\frac{\pi}{4} - \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right|\right]\right\}$

$= \frac{\pi}{2}(\sqrt{2} + 1)$

Question241

If $I_1 = \int_0^1 e^{-x}\cos^2 x\,dx$; $I_2 = \int_0^1 e^{-x^2}\cos^2 x\,dx$ and $I_3 = \int_0^1 e^{-x^3}\,dx$; then [Online April 15, 2018]

Options:

A. $I_2 > I_3 > I_1$

B. $I_3 > I_1 > I_2$

C. $I_2 > I_1 > I_3$

D. $I_3 > I_2 > I_1$

Answer: D

Solution:

Given: $I_1 = \int_0^1 e^{-x} \cos^2 x \, dx$;

$I_2 = \int_0^1 e^{-x^2} \cos^2 x \, dx$

$I_3 = \int_0^1 e^{-x^3} dx$;

For $x \in (0, 1)$

$\Rightarrow x > x^2$ or $-x < -x^2$

and $x^2 > x^3$ or $-x^2 < -x^3$

$\therefore e^{-x^2} < e^{-x^3}$ and $e^{-x} < e^{-x^2}$

$\Rightarrow e^{-x} < e^{-x^2} < e^{-x^3}$

$\Rightarrow e^{-x^3} > e^{-x^2} > e^{-x}$

$\Rightarrow I_3 > I_2 > I_1$

Question242

The value of the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x \left(1 + \log\left(\frac{2 + \sin x}{2 - \sin x}\right)\right) dx$ is

[Online April 15, 2018]

Options:

A. $\frac{3}{16}\pi$

B. 0

C. $\frac{3}{8}\pi$

D. $\frac{3}{4}$

Answer: C

Solution:

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x\left(1+\log\left(\frac{2+\sin x}{2-\sin x}\right)\right)dx \text{ ....(1)}$$

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4(-x)\left(1+\log\left(\frac{2+\sin x}{2-\sin x}\right)\right).dx$$

$$= \left[\because \int_a^b f(x).dx = \int_a^b f(a+b-x).dx\right]$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^4 x)\left(1+\log\left(\frac{2-\sin x}{2+\sin x}\right)\right).dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x\left(1-\log\left(\frac{2+\sin x}{2-\sin x}\right)\right).dx \text{ ......(2)}$$

After adding equation (1) and (2) we get,

$$2I = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x.dx$$

$$2I = 4\int_0^{\frac{\pi}{2}} \sin^4 x.dx$$

$$I = 2\int_0^{\frac{\pi}{2}} \sin^4 x.dx = \frac{2\times\frac{3}{2}\times\frac{1}{2}\times\pi}{2\times 2} = \frac{3\pi}{8}$$

[By Gamma function]

Question243

If $f\left(\frac{3x-4}{3x+4}\right) = x+2,\ x \neq -\frac{4}{3}$, and $\int f(x)dx = A\log|1-x| + Bx + C$, then the ordered pair(A, B) is equal to:
(where C is a constant of integration)
[Online April 9, 2017]

Options:

A. $\left(\frac{8}{3}, \frac{2}{3}\right)$

B. $\left(-\frac{8}{3}, \frac{2}{3}\right)$

C. $\left(-\frac{8}{3}, -\frac{2}{3}\right)$

D. $\left(\frac{8}{3}, -\frac{2}{3}\right)$

Answer: B

Solution:

Solution:

$f\left(\frac{3x-4}{3x+4}\right) = x+2, x \neq -\frac{4}{3}$

Consider $\frac{3x-4}{3x+4} = t$

$\Rightarrow 3x-4 = 3tx+4t$

$\Rightarrow x = \frac{4t+4}{3-3t} + 2$

$\Rightarrow f(t) = \frac{10-2t}{3-3t}$

$\Rightarrow f(x) = \frac{2x-10}{3x-3}$

$\therefore \int f(x)dx = \int \frac{2x-10}{3x-3}dx$

$= \int \frac{2x}{3x-3}dx - 10\int \frac{dx}{3x-3}$

$= \frac{2}{3}\int \frac{x-1}{x-1}dx + \frac{2}{3}\int \frac{dx}{x-1} - \frac{10}{3}\int \frac{dx}{x-1}$

$= \frac{2x}{3} - \frac{8}{3}\ln(x-1) + C$

Here, $A = -\frac{8}{3}, B = \frac{2}{3}$

$\therefore (A, B) = \left(-\frac{8}{3}, \frac{2}{3}\right)$

Question244

The integral $\int \sqrt{1+2\cot x(\operatorname{cosec} x+\cot x)}\,dx \left(0<x<\frac{\pi}{2}\right)$ is equal to:

(where C is a constant of integration)
[Online April 8, 2017]

Options:

A. $2\log\left|\sin\frac{x}{2}\right|+C$

B. $4\log\left|\sin\frac{x}{2}\right|+C$

C. $2\log\left|\cos\frac{x}{2}\right|+C$

D. $4\log\left|\cos\frac{x}{2}\right|+C$

Answer: A

Solution:

Solution:

Let, $I=\int\sqrt{1+2\cot x\operatorname{cosec} x+2\cot^2 x}\,.dx$

$\Rightarrow I=\int\sqrt{\frac{\sin^2 x+2\cos x+2\cos^2 x}{\sin^2 x}}\,.dx$

$\Rightarrow I=\int\sqrt{\frac{1+2\cos x+\cos^2 x}{\sin x}}dx$

$\Rightarrow I=\int\left|\frac{1+\cos x}{\sin x}\right|.dx$

$\Rightarrow I=\int|\operatorname{cosec} x+\cot x|\,.dx$

$\Rightarrow I=\log|\operatorname{cosec} x-\cot x|+\log|\sin x|+C$

$\Rightarrow I=\log|1-\cos x|+C$

$\Rightarrow I=\log\left|2\sin^2\frac{x}{2}\right|+C$

$\Rightarrow I=\log\left|\sin^2\frac{x}{2}\right|+\log 2+C$

$\Rightarrow I=2\log\left|\sin\frac{x}{2}\right|+C_1$

Question245

The integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x}$ is equal to :

[2017]

Options:

A. -1

B. -2

C. 2

D. 4

Answer: C

Solution:

Solution:

$$I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x} \text{ ......(i)}$$

$$I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1-\cos x} \text{ ....(ii)}$$

Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Adding (i) and (ii)

$$2I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{2}{\sin^2 x} dx;\ I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \operatorname{cosec}^2 x dx$$

$$I = -(\cot x)_{\pi/4}^{3\pi/4} = -\left[\cot\frac{3\pi}{4} - \cot\frac{\pi}{4}\right] = 2$$

Question246

Let $I_n = \int \tan^n x\,dx,\ (n > 1)$. $I_4 + I_6 = a\tan^5 x + bx^5 + C$
where C is constant of integration, then the ordered pair (a, b) is equal to :
[2017]

Options:

A. $\left(-\frac{1}{5}, 0\right)$

B. $\left(-\frac{1}{5}, 1\right)$

C. $\left(\frac{1}{5}, 0\right)$

D. $\left(\frac{1}{5}, -1\right)$

Answer: C

Solution:

Solution:
$I_n = \int \tan^n x\,dx,\ n > 1$
Let $I = I_4 + I_6$
$= \int (\tan^4 x + \tan^6 x)\,dx = \int \tan^4 x \sec^2 x\,dx$
Let $\tan x = t$
$\Rightarrow \sec^2 x\,dx = dt$
$\therefore I = \int t^4\,dt = \frac{t^5}{5} + C$
$= \frac{1}{5}\tan^5 x + C \Rightarrow$ On comparing, we have
$a = \frac{1}{5},\ b = 0$

Question247

If $\int_{1}^{2} \frac{dx}{(x^2-2x+4)^{\frac{3}{2}}} = \frac{k}{k+5}$ then k is equal to :

[Online April 9, 2017]

Options:

A. 1

B. 2

C. 3

D. 4

Answer: A

Solution:

Solution:

Let $I = \int_{1}^{2} \frac{dx}{((x-1a)^2+3)^{3/2}}$

Let; $x-1 = \sqrt{3}\tan\theta$

$\Rightarrow dx = \sqrt{3}\sec^2 . d\theta$

$\Rightarrow I = \int_{0}^{\pi/6} \frac{\sqrt{3}\sec^2\theta d\theta}{((\sqrt{3}\tan\theta)^2+(\sqrt{3})^2)^{3/2}}$

$= \frac{1}{3}\int_{0}^{\pi/6} \frac{\sec^2\theta}{\sec^3\theta} d\theta = \frac{1}{3}\int_{0}^{\pi/6} \cos\theta \, d\theta$

$= \frac{1}{3}[\sin\theta]_0^{\pi/6} = \frac{1}{3}\times\frac{1}{2} = \frac{1}{6}$

$= \frac{1}{6} = \frac{k}{k+5} \Rightarrow k+5 = 6k$

$\Rightarrow k = 1$

Question248

The integral $\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8\cos 2x}{(\tan x + \cot x)^3} dx$ equals:

[Online April 8, 2017]

Options:

A. $\frac{15}{128}$

B. $\frac{15}{64}$

C. $\frac{13}{32}$

D. $\frac{15}{256}$

Answer: A

Solution:

Solution:

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{\cos 2x}{\left(\frac{1}{\sin 2x}\right)^3} = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \cos 2x \times \sin 2x \,.\, \sin^2(2x) dx$$

$$= \frac{1}{4} \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sin 4x \,.\, (1 - \cos 4x) dx$$

$$= \frac{1}{4} \left[\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sin 4x - \frac{1}{2} \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sin 8x \right]$$

$$= \frac{1}{4}\left[-\frac{\cos 4x}{4} + \frac{\cos 8x}{16}\right]_{\pi/12}^{\pi/4} = \frac{1}{4}\left[\frac{15}{32}\right] = \frac{15}{128}$$

Question249

If

$$\lim_{n \to \infty} \frac{1^a + 2^a + \ldots\ldots + n^a}{(n+1)^{a-1}[(na+2) + \ldots\ldots.(na+n)]} = \frac{1}{60}$$

for some positive real number a, then a is equal to : [Online April 9, 2017]

Options:

A. 7

B. 8

C. $\frac{15}{2}$

D. $\frac{17}{2}$

Answer: A

Solution:

Solution:

$$\lim_{n \to \infty} \frac{\frac{1}{(a+1)} \cdot n^{a+1} + a_1 n^a + a_2 n^{a-1} + \ldots..}{(n+1)^{a-1} \cdot n^2 \left(a + \frac{1 + \frac{1}{n}}{2} \right)} = \frac{1}{60}$$

$$\Rightarrow \lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + \ldots\ldots. + \left(\frac{n}{n}\right)^a}{(n+1)^{a-1}\left[n^2 a + \frac{n(n+1)}{2}\right]} = \frac{1}{60}$$

$$= \frac{\lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n}\right)^a}{\left(1 + \frac{1}{n}\right)^{a-1}\left[a + \frac{1}{2}\left(1 + \frac{1}{n}\right)\right]} = \frac{1}{60}$$

$$= \frac{\int_0^1 x^a d\, x}{\left(a + \frac{1}{2}\right)} = \frac{1}{60} = \frac{\frac{1}{a+1}}{a+12} = \frac{1}{60}$$

$$\Rightarrow \frac{\frac{1}{a+1}}{\left(a + \frac{1}{2}\right)} = \frac{1}{60}$$

$$\Rightarrow (a+1)(2a+1) = 120$$

$$\Rightarrow 2a^2 + 3a - 119 = 0$$

$$\Rightarrow 2a^2 + 17a - 14a - 119 = 0$$

$$\Rightarrow (a-7)(2a+17) = 0$$

$$\Rightarrow a = 7, -\frac{17}{2}$$

Question250

If $\int \frac{dx}{\cos^3 x\sqrt{2\sin 2x}} = (\tan x)^A + C(\tan x)^B + k$, where k is a constant of integration, then $A + B + C$ equals:
[Online April 9, 2016]

Options:

A. $\frac{16}{5}$

B. $\frac{27}{10}$

C. $\frac{7}{10}$

D. $\frac{21}{5}$

Answer: A

Solution:

Solution:
$\int \frac{dx}{\cos^3 x\sqrt{4\sin x\cos x}} = \int \frac{dx}{2\cos^4 x\sqrt{\tan x}}$
Let $\tan x = t^2 \Rightarrow \sec^2 x = 1 + t^4$
$\sec^2 x dx = 2t dt$
$= \int \frac{\sec^4 x dx}{2\sqrt{\tan x}} = \int \frac{\sec^2 x(\sec^2 x dx)}{2\sqrt{\tan x}}$
$= \int \frac{(1+t^4)2t dt}{2t} = \int (1+t^4)dt = t + \frac{t^5}{5} + k$
$= \sqrt{\tan x} + \frac{1}{5}\tan^{5/2} x + k \; [t = \sqrt{\tan x}]$
$A = \frac{1}{2}, B = \frac{5}{2}, C = \frac{1}{5}$
$A + B + C = \frac{16}{5}$

Question251

The integral $\int \frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$ is equal to:

(where C is a constant of integration)
[Online April 10, 2016]

Options:

A. $-2\sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}}+C$

B. $-\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}+C$

C. $-2\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}+C$

D. $2\sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}}+C$

Answer: C

Solution:

Solution:

$I=\int \frac{dx}{(1+\sqrt{x}).\sqrt{x}\sqrt{1-x}}$

Put $1+\sqrt{x}=t \Rightarrow \frac{1}{2\sqrt{x}}dx=dt$

$\Rightarrow I=\int \frac{2dt}{t\sqrt{2t-t^2}}$

Again put $t=\frac{1}{z} \Rightarrow dt=\frac{-1}{2^2}dz$

$\Rightarrow I=2\int \frac{\frac{-1}{z^2}dz}{\frac{1}{z}\sqrt{\frac{2}{z}-\frac{1}{z^2}}}=2\int \frac{-dz}{\sqrt{2z-1}}$

$=-2\sqrt{2z-1}+c=-2\sqrt{\frac{2}{t}-1}+c$

$=-2\sqrt{\frac{2-t}{t}}+c=-2\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}+c$

Question252

The integral $\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$ is equal to:

[2016]

Options:

A. $\frac{x^5}{2(x^5+x^3+1)^2}+C$

B. $\frac{-x^{10}}{2(x^5+x^3+1)^2}+C$

C. $\frac{-x^5}{(x^5+x^3+1)^2}+C$

D. $\frac{x^{10}}{2(x^5+x^3+1)^2}+C$

Answer: D

Solution:

Solution:

$\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$

Dividing by x^{15} in numerator and denominator

$\int \frac{\frac{2}{x^3}+\frac{5}{x^6}dx}{\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^3}$

Let $1+\frac{1}{x^2}+\frac{1}{x^5}=t$

$\Rightarrow \left(\frac{-2}{x^3}-\frac{5}{x^6}\right)dx=dt \Rightarrow \left(\frac{2}{x^3}+\frac{5}{x^6}\right)dx=-dt$

This gives,

$$\int \frac{\frac{2}{x^3}+\frac{5}{x^6}dx}{\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^3} = \int \frac{-dt}{t^3} = \frac{1}{2t^2}+C$$

$$= \frac{1}{2\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^2}+C = \frac{x^{10}}{2(x^5+x^3+1)^2}+C$$

Question253

For $x \in R$, $x \neq 0$, if $y(x)$ is a differentiable function such that $x\int_1^x y(t)dt = (x+1)\int_1^x ty(t)dt$, then $y(x)$ equals:
(where C is a constant)
[Online April 10, 2016]

Options:

A. $Cx^3e^{\frac{1}{x}}$

B. $\frac{C}{x^2}e^{-\frac{1}{x}}$

C. $\frac{C}{x}e^{-\frac{1}{x}}$

D. $\frac{C}{x^3}e^{-\frac{1}{x}}$

Answer: D

Solution:

$x\int_1^x y(t)dt = x\int_1^x ty(t)dt + \int_1^x ty(t)dt$

Differentiate $w\,cdot\,r$ to x.

$\int_1^x y(t)dt + x[y(x) - y(1)]$

$$= \int_1^x ty(t)dt + x[xy(x) - y(1)] + xy(x) - y(1)$$

$$\int_1^x y(t)dt = \int_1^x ty(t)dt + x^2y(x) - y(1)$$

Diff. again w . r. to x

$$y(x) - y(a) = xy(x) - y(a) + 2xy(x) + x^2y^1(x)$$

$$(1-3x)y(x) = x^2y^1(x)$$

$$\frac{y^1(x)}{y(x)} = \frac{1-3x}{x^2}$$

$$\frac{1dy}{ydx} = \frac{1-3x}{x^2} \Rightarrow \ln y = -\frac{1}{x} - 3\ln x$$

$$\ln(yx^3) = -\frac{1}{x}$$

$$yx^3 = -e^{-1/x}$$

$$y = \frac{e^{-1x}}{x^3} \text{ or } y = \frac{ce^{-\frac{1}{x}}}{x^3}$$

Question254

The value of the integral $\int_4^{10} \frac{[x^2]dx}{[x^2 - 28x + 196] + [x^2]}$, where [x]denotes the greatest integer less than or equal to x, is :
[Online April 10, 2016]

Options:

A. $\frac{1}{3}$

B. 6

C. 7

D. 3

Answer: D

Solution:

$$I=\int_4^{10}\frac{[x^2]dx}{[x^2-28x+196]+[x^2]}dx \text{ ....(a)}$$

Use $\int_a^b f(a+b-x)dx=\int_a^b f(x)dx$

$$I=\int_4^{10}\frac{[(x-14)^2]}{[x^2]+[(x-14)^2]}dx \text{ ....(b)}$$

(a) +(b)

$$2I=\int_4^{10}\frac{[(x-14)^2]+[x^2]}{[x^2]+[(x-14)^2]}dx$$

$$2I=\int_4^{10}dx \Rightarrow 2I=6 \Rightarrow I=3$$

Question255

If $2\int_0^1 \tan^{-1}x\,dx=\int_0^1 \cot^{-1}(1-x+x^2)dx$, then $\int_0^1 \tan^{-1}(1-x+x^2)dx$ is equal to:

[Online April 9, 2016]

Options:

A. $\frac{\pi}{2}+\log 2$

B. $\log 2$

C. $\frac{\pi}{2}-\log 4$

D. $\log 4$

Answer: B

Solution:

Solution:

$$2\int_0^1 \tan^{-1}x\,dx=\int_0^1\left(\frac{\pi}{2}-\tan^{-1}(1-x+x^2)\right)dx$$

$$2\int_0^1 \tan^{-1}x\,dx=\int_0^1\frac{\pi}{2}dx-\int_0^1 \tan^{-1}(1-x+x^2)dx$$

$$\int_0^1 \tan^{-1}(1-x+x^2)dx=\frac{\pi}{2}-2\int_0^1 \tan^{-1}x\,dx \text{ ....(a)}$$

Let, $I_1=\int_0^1 \tan^{-1}x\,dx$

$$= [(\tan^{-1}x)x]_0^{\ 1} - \int_0^1 \frac{1}{1+x^2} x\,d\,x$$
$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} d\,x = \frac{\pi}{4} - \frac{1}{2}\log 2$$
By equation (a)
$$\frac{\pi}{2} - 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right] = \log 2$$

Question256

$\lim\limits_{n\to\infty}\left(\frac{(n+1)(n+2)\ldots..3n}{n^{2n}}\right)^{\frac{1}{n}}$ **is equal to:**

[2016]

Options:

A. $\frac{9}{e^2}$

B. $3\log 3 - 2$

C. $\frac{18}{e^4}$

D. $\frac{27}{e^2}$

Answer: D

Solution:

Solution:

$$y = \lim_{n\to\infty}\left(\frac{(n+1)(n+2)\ldots..3n}{n^{2n}}\right)^{\frac{1}{n}}$$
$$\ln y = \lim_{n\to\infty}\frac{1}{n}\ln\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots.\left(1+\frac{2n}{n}\right)$$
$$\ln y = \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\ldots\ldots.+\ln\left(1+\frac{2n}{n}\right)\right]$$
$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\ln\left(1+\frac{r}{n}\right) = \int_0^2 \ln(1+x)\,d\,x$$
Let $1+x=t \Rightarrow d\,x = d\,t$

when $x = 0, t = 1$
$x = 2, t = 3$
$\ln y = \int_1^3 \ln t \, d t = [t \ln t - t]_1^3 = \ln(3^3 e^2) = \ln\left(\frac{27}{e^2}\right)$
$\Rightarrow y = \frac{27}{e^2}$

Question257

If $\int \frac{\log\left(t + \sqrt{1+t^2}\right)}{\sqrt{1+t^2}} d t = \frac{1}{2}(g(t))^2 + C$, where C is a constant, then g(b) is equal to :
[Online April 11, 2015]

Options:

A. $\frac{1}{\sqrt{5}} \log(2 + \sqrt{5})$

B. $\frac{1}{2} \log(2 + \sqrt{5})$

C. $2 \log(2 + \sqrt{5})$

D. $\log(2 + \sqrt{5})$

Answer: D

Solution:

Solution:
Let $I = \int \frac{\log\left(t + \sqrt{1+t^2}\right)}{\sqrt{1+t^2}} d t$
put $u = \log\left(t + \sqrt{1+t^2}\right)$
$d u = \frac{1}{t + \sqrt{1+t^2}} \cdot \left[\frac{t + \sqrt{1+t^2}}{\sqrt{1+t^2}}\right] = \frac{1}{\sqrt{1+t^2}} d t$
$\therefore I = \int u \, d u = \frac{u^2}{2} + c$
Since, $I = \frac{1}{2}[g(t)]^2 + c$

$\therefore g(t) = \log\left(t + \sqrt{1+t^2}\right)$
Put $t = 2$
$g(b) = \log(2 + \sqrt{5})$

Question258

The integral $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ equals: [2015]

Options:

A. $-(x^4+1)^{\frac{1}{4}} + c$

B. $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$

C. $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$

D. $(x^4+1)^{\frac{1}{4}} + c$

Answer: B

Solution:

Solution:

$$I = \int \frac{dx}{x^2[x^4+1]^{\frac{3}{4}}}$$
$$= \int \frac{dx}{x^2\left[(x^4)^{\frac{3}{4}}\left(1+\frac{1}{x^4}\right)^{\frac{3}{4}}\right]}$$
$$= \int \frac{dx}{x^5\left[1+\frac{1}{x^4}\right]^{\frac{3}{4}}}$$

Substitute: $1 + \frac{1}{x^4} = t$

Differentiating w.r.t. x

$0-4\frac{1}{x^5}dx=dt$

$\Rightarrow \frac{dx}{x^5}=-\frac{dt}{4}$

$I=\int\frac{\left(-\frac{dt}{4}\right)}{t^{\frac{3}{4}}}$

$=-\frac{1}{4}\int t^{-\frac{3}{4}}dt$

$=-\frac{1}{4}\frac{t^{-\frac{3}{4}+1}}{\left(-\frac{3}{4}+1\right)}+C$

$=-t^{\frac{1}{4}}+C$

$=-\left[1+\frac{1}{x^4}\right]^{\frac{1}{4}}+C$

$I=-\left[\frac{x^4+1}{x^4}\right]^{\frac{1}{4}}+c$

Question259

The integral $\int\frac{dx}{(x+1)^{\frac{3}{4}}(x-2)^{\frac{5}{4}}}$ is equal to:

[Online April 10, 2015]

Options:

A. $-\frac{4}{3}\left(\frac{x+1}{x-2}\right)^{\frac{1}{4}}+C$

B. $4\left(\frac{x+1}{x-2}\right)^{\frac{1}{4}}+C$

C. $4\left(\frac{x-2}{x+1}\right)^{\frac{1}{4}}+C$

D. $-\frac{4}{3}\left(\frac{x-2}{x+1}\right)^{\frac{1}{4}}+C$

Answer: B

Solution:

Solution:

$\int \frac{dx}{(x+1)^{3/4}(x-2)^{5/4}}$

$\int \frac{dx}{\left(\frac{x+1}{x-2}\right)^{3/4}(x-2)^{2}}$, put $\frac{x+1}{x-2}=t$

$\frac{-3}{(x-2)^2}=\frac{dt}{dx}$

$\frac{dx}{(x-2)^2}=-\frac{dt}{3}=\frac{-1}{3}\int\frac{dt}{t^{\frac{3}{4}}}\ =-\frac{1}{3}\int t^{\frac{-3}{4}}1t$

$=\frac{1}{3}\left[\frac{\frac{-3}{4}+1}{\frac{-3}{4}+1}\right]\ =\frac{-4}{3}\left[\frac{x+1}{x-2}\right]^{1/4}+c$

Question260

The integral $\int_{2}^{4}\frac{\log x^2}{\log x^2+\log(36-12x+x^2)}dx$ is equal to [2015]

Options:

A. 1

B. 6

C. 2

D. 4

Answer: A

Solution:

$$I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)} dx$$

$$I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(6 - x)^2} dx \text{ ......(i)}$$

$$I = \int_2^4 \frac{\log(6 - x)^2}{\log(6 - x)^2 + \log x^2} dx \text{ ....(ii)}$$

Adding (i) and (ii)

$$2I = \int_2^4 dx = [x]_2^4 = 2$$

$$I = 1$$

Question261

Let $f : R \to R$ be a function such that $f(2 - x) = f(2 + x)$ and $f(4 - x) = f(4 + x)$, for all $x \in R$ and $\int_0^2 f(x) dx = 5$. Then the value of $\int_{10}^{50} f(x) dx$ is :
[Online April 11, 2015]

Options:

A. 125

B. 80

C. 100

D. 200

Answer: C

Solution:

Solution:

Let $f : R \to R$ be a function such that $f(2 - x) = f(e + x)$

Put $x = 2 + x$ we get

$f(-x) = f(4 + x) = f(4 - x)$

$\Rightarrow f(x) = f(x + 4)$

Hence period is 4

Consider $\int_{10}^{50} f(x) dx = 10 \int_{10}^{14} f(x) dx = 10[5 + 5] = 100$

Question262

Let $f:(-1,1)\rightarrow R$ be a continuous function. If $\int_0^x f(t)dt=\frac{\sqrt{3}}{2}x$, then $f\left(\frac{\sqrt{3}}{2}\right)$ is equal to :

[Online April 11, 2015]

Options:

A. $\frac{1}{2}$

B. $\frac{\sqrt{3}}{2}$

C. $\sqrt{\frac{3}{2}}$

D. $\sqrt{3}$

Answer: D

Solution:

Solution:

Let $f:(-1,1)\rightarrow R$ be a continuous function

Let $\int_0^{\sin x} f(t)dt=\frac{\sqrt{3}}{2}x$

$f(\sin x).\frac{d}{dx}(\sin x)=\frac{\sqrt{3}}{2}$

$\Rightarrow f(\sin x).\cos x=\frac{\sqrt{3}}{2}$

put $x=\frac{\pi}{3}$

$f\left(\sin\frac{\pi}{3}\right).\cos\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

$f\left(\frac{\sqrt{3}}{2}\right).\frac{1}{2}=\frac{\sqrt{3}}{2}$

$f\left(\frac{\sqrt{3}}{2}\right)=\sqrt{3}$

Question263

For $x > 0$, let $f(x) = \int\limits_1^x \frac{\log t}{1+t} dt$. Then $f(x) + f\left(\frac{1}{x}\right)$ is equal to:
[Online April 10, 2015]

Options:

A. $\frac{1}{4}(\log x)^2$

B. $\log x$

C. $\frac{1}{2}(\log x)^2$

D. $\frac{1}{4} \log x^2$

Answer: C

Solution:

Solution:
$f\left(\frac{1}{x}\right) = \int\limits_1^{1/x} \frac{1nt}{1+t} dt$
Let $t = \frac{1}{z}$
$dt = -\frac{1}{z^2} dz$
$f(x) = \int\limits_1^x \frac{\ln z}{z(z+1)} dz$
$f(x) + f\left(\frac{1}{x}\right) = \int\limits_1^x \frac{\ln x}{z} dz = \left[\frac{(\ln z)^2}{2}\right]_1^X = \frac{(\ln x)^2}{2}$

Question264

The integral $\int \left(1 + x - \frac{1}{x}\right) e^{x + \frac{1}{x}} dx$ is equal to
[2014]

Options:

A. $(x+1)e^{x+\frac{1}{x}}+c$

B. $-xe^{x+\frac{1}{x}}+c$

C. $(x-1)e^{x+\frac{1}{x}}+c$

D. $xe^{x+\frac{1}{x}}+c$

Answer: D

Solution:

Solution:

Let $I=\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$

$=\int e^{x+\frac{1}{x}}dx+\int\left(x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$

$=x.e^{x+\frac{1}{x}}-\int x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx+\int\left(x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$

$=x.e^{x+\frac{1}{x}}-\int\left(x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx+\int(x-1x)e^{x+\frac{1}{x}}dx$

$=xe^{x+\frac{1}{x}}+C$

Question265

The integral $\int\frac{\sin^2x\cos^2x}{(\sin^3x+\cos^3x)^2}dx$ is equal to: [Online April 12, 2014]

Options:

A. $\frac{1}{(1+\cot^3x)}+c$

B. $-\dfrac{1}{3(1+\tan^3 x)}+c$

C. $\dfrac{\sin^3 x}{(1+\cos^3 x)}+c$

D. $-\dfrac{\cos^3 x}{3(1+\sin^3 x)}+c$

Answer: B

Solution:

Solution:

Let $I=\int \dfrac{\sin^2 x\cos^2 x}{(\sin^3 x+\cos^3 x)^2}dx$

$I=\int \left(\dfrac{\sin x \,.\, \cos x}{\sin^3 x+\cos^3 x}\right)^2 dx$

$I=\int \left(\dfrac{\sin x \,.\, \cos x}{\cos^2 x(1+\tan^3 x)}\right)^2 dx$

$=\int \left(\dfrac{\sin x \,.\, \sec^2 x}{(1+\tan^3 x)}\right)^2 dx$

Put $1+\tan^3 x=t$

$dt=3\tan^2 x\sec^2 x dx$ or $dx=\dfrac{dt}{3\tan^2 x\sec^2 x}$

$\therefore I=\int \dfrac{\sin^2 x \,.\, \sec^4 x}{t^2}\times\dfrac{dt}{3\tan^2 x\sec^2 x}$

$I=\dfrac{1}{3}\int \dfrac{\sin^2 x \,.\, \sec^4 x}{t^2}\times\dfrac{dt}{\dfrac{\sin^2 x}{\cos^2 x}\times\sec^2 x}$

$=\dfrac{1}{3}\int \dfrac{\sin^2 x \,.\, \sec^4 x}{t^2}\times\dfrac{dt}{\sin^2 x\sec^4 x}$

$\therefore I=\dfrac{1}{3}\int \dfrac{dt}{t^2}=\dfrac{1}{3}\int t^{-2}dt$

$I=\dfrac{1}{3}\left[\dfrac{t^{-2+1}}{-2+1}\right]+c=\dfrac{-1}{3}\left[\dfrac{1}{t}\right]+c$

or $I=-\dfrac{1}{3(1+\tan^3 x)}+c$

Question266

The integral $\int x\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx (x > 0)$ is equal to:

[Online April 11, 2014]

Options:

A. $-x+(1+x^2)\tan^{-1}x+c$

B. $x-(1+x^2)\cot^{-1}x+c$

C. $-x+(1+x^2)\cot^{-1}x+c$

D. $x-(1+x^2)\tan^{-1}x+c$

Answer: A

Solution:

Solution:

Let $I=\int x\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$

$\therefore I=2\int x\,.\tan^{-1}x dx$

Applying Integration by parts

$$I=2\left[\tan^{-1}x\int x dx-\int\left(\frac{d}{dx}(\tan^{-1}x)\int x dx\right)dx\right]$$

$$I=2\left[\frac{x^2}{2}\tan^{-1}x-\int\frac{1}{1+x^2}\times\frac{x^2}{2}dx\right]+c$$

$$I=2\left[\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2+1-1}{x^2+1}dx\right]+c$$

$$I=2\left[\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2+1}{x^2+1}dx+\frac{1}{2}\int\frac{1}{1+x^2}dx\right]+c$$

$$I=2\left[\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int 1\,.dx+\frac{1}{2}\tan^{-1}x\right]+c$$

$$I=2\left[\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{1}{2}\tan^{-1}x\right]+c$$

$I=x^2\tan^{-1}x+\tan^{-1}x-x+c$

or $I=-x+(x^2+1)\tan^{-1}x+c$

Question267

$\int \frac{\sin^8 x - \cos^8 x}{(1 - 2\sin^2 x\cos^2 x)}$**d x is equal to:**

[Online April 9, 2014]

Options:

A. $\frac{1}{2}\sin 2x + c$

B. $-\frac{1}{2}\sin 2x + c$

C. $-\frac{1}{2}\sin x + c$

D. $-\sin^2 x + c$

Answer: B

Solution:

Solution:

Let $I = \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x\cos^2 x} dx$

$= \int \frac{(\sin^4 x)^2 - (\cos^4 x)^2}{1 - 2\sin^2 x\cos^2 x} dx$

$= \int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{1 - 2\sin^2 x\cos^2 x} dx$

$= \int \frac{[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x]\,[(\sin^2 x + \cos^2 x][\sin^2 x - \cos^2 x]}{1 - 2\sin^2 x\cos^2 x} dx$

$= -\int \cos 2x\, dx = \frac{-\sin 2x}{2} + c = -\frac{1}{2}\sin 2x + c$

Question268

If m is a non-zero number and $\int \frac{x^{5m-1} + 2x^{4m-1}}{(x^{2m} + x^m + 1)^3} dx = f(x) + c$, then $f(x)$ is:

[Online April 19, 2014]

Options:

A. $\frac{x^{5m}}{2m(x^{2m} + x^m + 1)^2}$

B. $\dfrac{x^{4m}}{2m(x^{2m}+x^{m}+1)^{2}}$

C. $\dfrac{2m(x^{5m}+x^{4m})}{(x^{2m}+x^{m}+1)^{2}}$

D. $\dfrac{(x^{5m}-x^{4m})}{2m(x^{2m}+x^{m}+1)^{2}}$

Answer: B

Solution:

Solution:

$\int \dfrac{x^{5m-1}+2x^{4m-1}}{(x^{2m}+x^{m}+1)^{3}}dx \quad = \int \dfrac{x^{5m-1}+2x^{4m-1}}{x^{6m}(1+x^{-m}+x^{-2m})^{3}}dx$

$= \int \dfrac{x^{-m-1}+2x^{-2m-1}}{(1+x^{-m}+x^{-2m})^{3}}dx$

Put $t = 1+x^{-m}+x^{-2m}$

$\therefore \dfrac{dt}{dx} = -mx^{-m-1}-2mx^{-2m-1}$

$\Rightarrow \dfrac{dt}{-m} = (x^{-m-1}+2x^{-2m-1})dx$

$\therefore \int \dfrac{x^{5m-1}+2x^{4m-1}}{(x^{2m}+x^{m}+1)^{3}}dx \ = \dfrac{1}{-m}\int t^{-3}dt \ = \dfrac{1}{2mt^{2}}+C$

$= \dfrac{1}{2m(1+x^{-m}+x^{-2m})^{2}}+C$

$= \dfrac{x^{4m}}{2m(x^{2m}+x^{m}+1)^{2}}+C$

$\therefore f(x) = \dfrac{x^{4m}}{2m(x^{2m}+x^{m}+1)^{2}}$

Question269

The integral $\int_{0}^{\pi} \sqrt{1+4\sin^{2}\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals:

[2014]

Options:

A. $4\sqrt{3}-4$

B. $4\sqrt{3}-4-\frac{\pi}{3}$

C. $\pi-4$

D. $\frac{2\pi}{3}-4-4\sqrt{3}$

Answer: B

Solution:

Solution:

Let $I=\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,d x=\int_0^{\pi}\left|2\sin\frac{x}{2}-1\right|d x$

$=\int_0^{\pi/3}\left(1-2\sin\frac{x}{2}\right)d x+\int_{\pi/3}^{\pi}\left(2\sin\frac{x}{2}-1\right)d x$

$\left[\because\sin\frac{x}{2}=\frac{1}{2}\Rightarrow\frac{x}{2}=\frac{\pi}{6}\Rightarrow x=\frac{\pi}{3},\frac{x}{2}=\frac{5\pi}{6}\Rightarrow x=\frac{5\pi}{3}>\pi\right]$

$=\left[x+4\cos\frac{x}{2}\right]_0^{\pi/3}+\left[-4\cos\frac{x}{2}-x\right]_{\pi/3}^{\pi}$

$=\frac{\pi}{3}+4\frac{\sqrt{3}}{2}-4+\left(0-\pi+4\frac{\sqrt{3}}{2}+\frac{\pi}{3}\right)$

$=4\sqrt{3}-4-\frac{\pi}{3}$

Question270

Let function F be defined as $F(x)=\int_1^x\frac{e^t}{t}d t$, $x>0$ then the value of the integral $\int_1^x\frac{e^t}{t+a}d t$, where $a>0$, is:
[Online April 19, 2014]

Options:

A. $e^a[F(x)-F(1+a)]$

B. $e^{-a}[F(x+a)-F(a)]$

C. $e^a[F(x+a)-F(1+a)]$

D. $e^{-a}[F(x+a)-F(1+a)]$

Answer: D

Solution:

Solution:

$F(x)=\int_1^x \frac{e^t}{t} dt, x>0$

Let $I=\int_1^x \frac{e^t}{t+a} dt$

Put $t+a=z \Rightarrow t=z-a; dt=dz$

for $t=1, z=1+a$

for $t=x, z=x+a$

$\therefore I=\int_{1+a}^{x+a} \frac{e^{z-a}}{z} dz$

$=e^{-a}\int_{1+a}^{x+a} \frac{e^z}{z} dz \equiv e^{-a}\int_{1+a}^{x+a} \frac{e^t}{t} dt$

$I=e^{-a}\left[\int_{1+a}^{1} \frac{e^t}{t} dt+\int_1^{x+a} \frac{e^t}{t} dt\right]$

$=e^{-a}\left[-\int_1^{1+a} \frac{e^t}{t} dt+\int_1^{x+a} \frac{e^t}{t} dt\right]$

$=e^{-a}[-F(1+a)+F(x+a)]$

(By the definition of $F(x)$)

$=e^{-a}[F(x+a)-F(1+a)]$

Question271

If for a continuous function $f(x)$, $\int_{-\pi}^{t}(f(x)+x)dx=\pi^2-t^2$, for all $t\geq -\pi$, then $f\left(-\frac{\pi}{3}\right)$ is equal to:

[Online April 12, 2014]

Options:

A. π

B. $\frac{\pi}{2}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{6}$

Answer: A

Solution:

Solution:

Let $\int_{-\pi}^{t}(f(x)+x)dx = \pi^2 - t^2$

$\Rightarrow \int_{-\pi}^{t} f(x)dx + \int_{-\pi}^{t} xdx = \pi^2 - t^2$

$\Rightarrow \int_{-\pi}^{t} f(x)dx + \left(\frac{t^2}{2} - \frac{\pi^2}{2}\right) = \pi^2 - t^2$

$\Rightarrow \int_{-\pi}^{t} f(x)dx = \frac{3}{2}(\pi^2 - t^2)$

differentiating with respect to t

$\frac{d}{dt}\left[\int_{-\pi}^{t} f(x)dx\right] = \frac{3}{2}\frac{d}{dt}(\pi^2 - t^2)$

$f(t).\frac{dt}{dt} - f(-\pi)\frac{d}{dt}(-\pi) = -3t$

$f(t) = -3t$

$f\left(-\frac{\pi}{3}\right) = -3\left(-\frac{\pi}{3}\right) = \pi$

Question272

If [ ]denotes the greatest integer function, then the integral $\int_{0}^{\pi}[\cos x]dx$ is equal to:
[Online April 12, 2014]

Options:

A. $\frac{\pi}{2}$

B. 0

C. -1

D. $-\frac{\pi}{2}$

Answer: D

Solution:

Let $I = \int_0^{\pi} [\cos x] dx$ ......(1)

$I = \int_0^{\pi} [\cos(\pi - x)] dx = \int_0^{\pi} [-\cos x] dx$ ......(2)

On adding (1) and (2), we get

$2I = \int_0^{\pi} [\cos x] dx + \int_0^{\pi} [-\cos x] dx$

$2I = \int_0^{\pi} [\cos x] + [-\cos x] dx$

$2I = \int_0^{\pi} -1 dx \quad (\because [x] + [-x] = -1 \text{ if } x \notin Z)$

$2I = -x |_0^{\pi} = -\pi$

$\Rightarrow I = \frac{-\pi}{2}$

Question273

If for $n \geq 1$, $P_n = \int_1^e (\log x)^n dx$, then $P_{10} - 90P_8$ is equal to: [Online April 11, 2014]

Options:

A. -9

B. 10e

C. $-9e$

D. 10

Answer: C

Solution:

$P_n = \int_1^e (\log x)^n d\,x$

put $\log x = t$ then $x = e^t$ and $d\,x = e^t d\,t$
Also, when $x = 1$, then $t = \log 1 = 0$
and when $x = e$, then $t = \log_e e = 1$

$\therefore P_n = \int_0^1 t^n . e^t d\,t$

$\therefore P_{10} = \int_0^1 t^{10} e^t d\,t$ and $P_8 = \int_0^1 t^8 e^t d\,t$

Now, $P_{10} - 90P_8 = \int_0^1 t^{10} e^t d\,t - 90\int_0^1 t^8 e^t d\,t$

$P_{10} - 90P_8 = [t^{10}e^t]_0{}^1 - 10\int_0^1 t^9 e^t d\,t - 90\int_0^1 t^8 e^t d\,t$

$P_{10} - 90P_8 \ = e - 10\left[t^9\int_0^1 e^t d\,t - \int_0^1 \frac{d}{d\,t}(t^9)\int e^t d\,t\right] - 90\int_0^1 t^8 e^t d\,t$

$P_{10} - 90P_8 = e - 10\left[e - 9\int_0^1 t^8 e^t d\,t\right] - 90\int_0^1 t^8 e^t d\,t$

$P_{10} - 90P_8 = e - 10e + 90\int t^8 e^t d\,t - 90\int_0^1 t^8 e^t d\,t$

$\therefore P_{10} - 90P_8 = -9e$

Question274

The integral $\int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} d\,x$, equals:
[Online April 9, 2014]

Options:

A. $\frac{\pi}{4}\ln 2$

B. $\frac{\pi}{8}\ln 2$

C. $\frac{\pi}{16}\ln 2$

D. $\frac{\pi}{32}\ln 2$

Answer: C

Solution:

Solution:

Let $I = \int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx$ or $= \int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+(2x)^2} dx$

Put $2x = \tan\theta$

$\therefore \frac{2dx}{d\theta} = \sec^2\theta$ or $dx = \frac{\sec^2\theta d\theta}{2}$

also when $x = 0 \Rightarrow \theta = 0$

and when $x = \frac{1}{2} \Rightarrow \theta = 45°$ or $\frac{\pi}{4}$

$$\therefore I = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta} \times \frac{\sec^2\theta d\theta}{2}$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta} \times \sec^2\theta d\theta \text{ (because } 1+\tan^2\theta = \sec^2\theta)$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\, d\theta \text{ ......(i)}$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right] d\theta \text{(Using the property of definite integral)}$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\left[1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\times\tan\theta}\right] d\theta$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right] d\theta$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\left[\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right] d\theta$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\left[\frac{2}{1+\tan\theta}\right] d\theta$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{4}} [\ln 2 - \ln(1+\tan\theta)] d\theta$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln 2\,.\, d\theta - \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\, d\theta$$

$$I = \frac{1}{2}\ln 2\,\theta\,\Big|_0^{\pi/4} - I \text{ (from eq. (i))}$$

$I + I = \frac{1}{2}\ln 2\left(\frac{\pi}{4} - 0\right)$

$2I = \frac{1}{2} \times \frac{\pi}{4} \times \ln 2$

$2I = \frac{\pi}{8}\ln 2$ or $I = \frac{\pi}{16}\ln 2$

Question275

If $\int f(x)dx = \psi(x)$, then $\int x^5 f(x^3)dx$ is equal to [2013]

Options:

A. $\frac{1}{3}[x^3\psi(x^3) - \int x^2\psi(x^3)dx] + C$

B. $\frac{1}{3}x^3\psi(x^3) - 3\int x^3\psi(x^3)dx + C$

C. $\frac{1}{3}x^3\psi(x^3) - \int x^2\psi(x^3)dx + C$

D. $\frac{1}{3}[x^3\psi(x^3) - \int x^3\psi(x^3)dx] + C$

Answer: C

Solution:

Solution:

Let $\int f(x)dx = \psi(x)$

Let $I = \int x^5 f(x^3)dx$

put $x^3 = t$

$\Rightarrow 3x^2 dx = dt$

$I = \frac{1}{3}\int 3 \,.\, x^2 \,.\, x^3 \,.\, f(x^3) \,.\, dx$

$= \frac{1}{3}\int tf(t)dt = \frac{1}{3}[t\int f(t)dt - \int f(t)dt]$

$= \frac{1}{3}[t\psi(t) - \int \psi(t)dt]$

$= \frac{1}{3}[x^3\psi(x^3) - 3\int x^2\psi(x^3)dx] + c$

$= \frac{1}{3}x^3\psi(x^3) - \int x^2\psi(x^3)dx + c$

Question276

If the integral$\int \frac{\cos 8x + 1}{\cot 2x - \tan 2x} dx = A\cos 8x + k$
where k is an arbitrary constant, then A is equal to :
[Online April 25, 2013]

Options:

A. $-\frac{1}{16}$

B. $\frac{1}{16}$

C. $\frac{1}{8}$

D. $-\frac{1}{8}$

Answer: A

Solution:

Solution:

Let $I = \int \frac{\cos 8x + 1}{\cot 2x - \tan 2x} dx$

Now, $D^r = \cot 2x - \tan 2x = \frac{\cos 2x}{\sin 2x} - \frac{\sin 2x}{\cos 2x}$

$= \frac{\cos^2 2x - \sin^2 2x}{\sin 2x \cos 2x} = \frac{2\cos 4x}{\sin 4x}$

$\therefore I = \int \frac{2\cos^2 4x}{\frac{2\cos 4x}{\sin 4x}} dx = \int \frac{2\cos^2 4x \,.\, \sin 4x}{2\cos 4x} dx$

$= \frac{1}{2}\int \sin 8x \, dx = -\frac{1}{2}\frac{\cos 8x}{8} + k = -\frac{1}{16} . \cos 8x + k$

Now, $-\frac{1}{16} . \cos 8x + k = A\cos 8x + k$

$\Rightarrow A = -\frac{1}{16}$

Question277

The integral $\int \frac{x\,dx}{2-x^2+\sqrt{2-x^2}}$ equals:
[Online April 23, 2013]

Options:

A. $\log\left|1+\sqrt{2+x^2}\right|+c$

B. $-\log\left|1+\sqrt{2-x^2}\right|+c$

C. $x\log\left|1-\sqrt{2+x^2}\right|+c$

D. $-x\log\left|1-\sqrt{2-x^2}\right|+c$

Answer: B

Solution:

Solution:

$I=\int \frac{x\,dx}{2-x^2+\sqrt{2-x^2}}$

Put $t=\sqrt{2-x^2}, \frac{dt}{dx}=\frac{1}{2\sqrt{2-x^2}}.(-2x)$

$\Rightarrow -t\,dt=x\,dx$

$\therefore I=\int \frac{(-t)dt}{t^2+t}=-\int \frac{1}{t+1}dt=-\log|t+1|$

$=-\log\left|1+\sqrt{2-x^2}\right|+c$

Question278

If $\int \frac{x^2-x+1}{x^2+1}e^{\cot^{-1}x}dx=A(x)e^{\cot^{-1}x}+C$, then A(x) is equal to :
[Online April 22, 2013]

Options:

A. $-x$

B. x

C. $\sqrt{1-x}$

D. $\sqrt{1+x}$

Answer: B

Solution:

Solution:

Let $I = \int \frac{x^2 - x + 1}{x^2 + 1} \cdot e^{\cot^{-1} x} dx$

Put $x = \cot t \Rightarrow -\operatorname{cosec}^2 t dt = dx$

Now, $1 + \cot^2 t = \operatorname{cosec}^2 t$

$\therefore I = \int \frac{e^t(\cot^2 t - \cot t + 1)}{(1 + \cot^2 t)} (-\operatorname{cosec}^2 t) dt$

$= -\int e^t(\operatorname{cosec}^2 t - \cot t) dt$

$= \int e^t(\cot t - \operatorname{cosec}^2 t) dt = e^t \cot t + C$

$= e^{\cot^{-1} x}(x) + C \equiv A(x) \cdot e^{\cot^{-1} x} + C$

$\Rightarrow A(x) = x$

Question279

If $\int d\, xx + x^7 = p(x)$ then $\int \frac{x^6}{x + x^7} d\, x$ is equal to:

[Online April 9, 2013]

Options:

A. $\ln|x| - p(x) + c$

B. $\ln|x| + p(x) + c$

C. $x - p(x) + c$

D. $x + p(x) + c$

Answer: A

Solution:

$$\int \frac{x^6}{x+x^7}dx = \int \frac{x^6}{x(1+x^6)}dx = \int \frac{(1+x^6)-1}{x(1+x^6)}dx$$
$$= \int \frac{1}{x}dx - \int \frac{1}{x+x^7}dx \;= \ln|x| - p(x) + c$$

Question280

The intercepts on x -axis made by tangents to the curve,y = $\int_0^x |t|dt$, x ∈ R, which are parallel to the line y = 2x, are equal to :
[2013]

Options:

A. ±1

B. ±2

C. ±3

D. ±4

Answer: A

Solution:

Solution:

Since, $y = \int_0^x |t|dt, x \in R$

therefore $\frac{dy}{dx} = |x|$

But from $y = 2x, \therefore \frac{dy}{dx} = 2$

$\Rightarrow |x| = 2 \Rightarrow x = \pm 2$

Points $y = \int_0^{\pm 2} |t|dt = \pm 2$

∴ Equation of tangent is

$y - 2 = 2(x - 2)$ or $y + 2 = 2(x + 2)$

$\Rightarrow x$ -intercept $= \pm 1$

Question281

Statement-1 : The value of the integral $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ is equal to $\pi/6$

Statement-2 : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

[2013]

Options:

A. Statement-1 is true; Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

B. Statement-1 is true; Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

C. Statement-1 is true; Statement-2 is false.

D. Statement-1 is false; Statement-2 is true.

Answer: D

Solution:

Solution:

Let, $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\,dx}{1+\sqrt{\tan x}}$ ....(i)

Also, given

$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\,dx}{1+\sqrt{\tan x}}$ ....(ii)

By adding (i) and (ii), we get

$2I = \int_{\pi/6}^{\pi/3} dx$

$\Rightarrow I = \frac{1}{2}\left[\frac{\pi}{3}-\frac{\pi}{6}\right] = \frac{\pi}{12}$

Statement-1 is false

$\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

It is fundamental property.

Statement -2 is true.

Question282

For $0 \le x \le \frac{\pi}{2}$, the value of $\int_0^{\sin^2 x} \sin^{-1}(\sqrt{t})dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t})dt$ equals : [Online April 25, 2013]

Options:

A. $\frac{\pi}{4}$

B. 0

C. 1

D. $-\frac{\pi}{4}$

Answer: A

Solution:

Solution:
Consider
$\int_0^{\sin^2 x} \sin^{-1}(\sqrt{t})dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t})dt$
Let I = f (x) after integrating and putting the limits.
$f'(x) = \sin^{-1}\sqrt{\sin^2 x}(2\sin x\cos x) - 0 + \cos^{-1}\sqrt{\cos^2 x}(-2\cos x\sin x) - 0$
$\therefore f'(x) = 0 \Rightarrow f(x) = C$ (constant)
Now, we find $f(x)$ at $x = \frac{\pi}{4}$
$\therefore I = \int_0^{1/2} \sin^{-1}\sqrt{t}\,dt + \int_0^{1/2} \cos^{-1}\sqrt{t}\,dt$
$= \int_0^{1/2} (\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})dt = \int_0^{1/2} \frac{\pi}{2}dt = \frac{\pi}{4} = C$
$\therefore f(x) = \frac{\pi}{4}$
$\therefore$ Required integration $= \frac{\pi}{4}$

Question283

The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$ **is :**
[Online April 23, 2013]

Options:

A. π

B. $\frac{\pi}{2}$

C. 4π

D. $\frac{\pi}{4}$

Answer: D

Solution:

Solution:
$I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$ ....(i)
$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^{-x}} dx$, by replacing x by $\left(\frac{\pi}{2} - \frac{\pi}{2} - x\right)$
$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \frac{2^x . \sin^2 x}{1+2^x} dx$ ....(ii)
Adding equations (i) and (ii), we get
$2I = \int_{-\pi/2}^{\pi/2} \sin^2 x dx = \frac{1}{2} \int_{-\pi/2}^{\pi/2} (1 - \cos 2x) dx$
$\Rightarrow I = \frac{1}{4} \left[x + \frac{\sin 2x}{2} \right]_{-\pi/2}^{\pi/2}$
$= \frac{1}{4} \left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left(-\frac{\pi}{2} + \frac{\sin(-\pi)}{2} \right) \right]$
$\Rightarrow I = \frac{1}{4} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = \frac{\pi}{4}$

Question284

The integral $\int_{7\pi/4}^{7\pi/3} \sqrt{\tan^2 x} dx$ **is equal to :**
[Online April 22, 2013]

Options:

A. $\log 2\sqrt{2}$

B. $\log 2$

C. $2 \log 2$

D. $\log \sqrt{2}$

Answer: D

Solution:

Solution:

Let $I = \int_{7\pi/4}^{7\pi/3} \sqrt{\tan^2 x}\, dx$

$= \int_{7\pi/4}^{7\pi/3} \tan x\, dx = -\log \cos x \Big|_{7\pi/4}^{7\pi/3}$

$= -\left[\log \cos \frac{7\pi}{3} - \log \cos \frac{7\pi}{4}\right]$

$= \log \cos \frac{7\pi}{4} - \log \cos \frac{7\pi}{3}$

$= \log \left[\frac{\cos \frac{7\pi}{4}}{\cos \frac{7\pi}{3}}\right] = \log \left[\frac{\cos\left(2\pi - \frac{\pi}{4}\right)}{\cos\left(2\pi + \frac{\pi}{3}\right)}\right]$

$= \log \left(\frac{\cos \frac{\pi}{4}}{\cos \frac{\pi}{3}}\right) = \log \left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{2}}\right)$

$= \log\left(\frac{2}{\sqrt{2}}\right) = \log \sqrt{2}$

Question285

If $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$, then $\frac{d^2y}{dx^2}$ is equal to:

[Online April 9, 2013]

Options:

A. y

B. $\sqrt{1+y^2}$

C. $\frac{x}{\sqrt{1+y^2}}$

D. y^2

Answer: A

Solution:

Solution:

$x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$

$\Rightarrow 1 = \frac{1}{\sqrt{1+y^2}} \cdot \frac{dy}{dx}$

$\left[\because \text{If } I(x) = \int_{\phi(x)}^{\Psi(x)} f(t)\,dt, \text{ then } \frac{dI(x)}{dx} = f\{\Psi(x)\} \left\{ \frac{d}{dx} psi(x) \right\} - f\{\phi(x)\} \cdot \left\{ \frac{d}{dx}\phi(x) \right\} \right]$

$\frac{dy}{dx} = \sqrt{1-y^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{1+y^2}} \cdot 2y \cdot \frac{dy}{dx} = \frac{y}{\sqrt{1+y^2}} \cdot \sqrt{1+y^2} = y$

Question286

If the $\int \frac{5\tan x}{\tan x - 2} dx = x + a \ln|\sin x - 2\cos x| + k$, then a is equal to : [2012]

Options:

A. -1

B. -2

C. 1

D. 2

Answer: D

Solution:

Solution:

$$\int \frac{5\tan x}{\tan x-2}dx=\int \frac{5\frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x}-2}dx$$

$$=\int\left(\frac{5\sin x}{\cos x}\times\frac{\cos x}{\sin x-2\cos x}\right)dx$$

$$=\int\frac{5\sin x\,dx}{\sin x-2\cos x}$$

$$=\int\left(\frac{4\sin x+\sin x+2\cos x-2\cos x}{\sin x-2\cos x}\right)dx$$

$$=\int\frac{(\sin x-2\cos x)+(4\sin x+2\cos x)}{\sin x-2\cos x}dx$$

$$=\int\frac{(\sin x-2\cos x)+2(\cos x+2\sin x)}{(\sin x-2\cos x)}dx$$

$$=\int\frac{\sin x-2\cos x}{\sin x-2\cos x}dx+2\int\left(\frac{\cos x+2\sin x}{\sin x-2\cos x}\right)dx$$

$$=\int dx+2\int\frac{\cos x+2\sin x}{\sin x-2\cos x}dx \;=I_1+I_2$$

where, $I_1=\int dx$ and $I_2=2\int\frac{\cos x+2\sin x}{\sin x-2\cos x}dx$

Let $\sin x-2\cos x=t$

$\Rightarrow(\cos x+2\sin x)dx=dt$

$\therefore I_2=2\int\frac{dt}{t}=2\ln t+C\;=2\ln(\sin x-2\cos x)+C$

Hence, $I_1+I_2=\int dx+2\ln(\sin x-2\cos x)+c$

$=x+2\ln|(\sin x-2\cos x)|+k\Rightarrow a=2$

Question287

If $f(x)=\int\left(\frac{x^2+\sin^2x}{1+x^2}\right)\sec^2x\,dx$ and $f(0)=0$, then $f(1)$ equals

[Online May 19, 2012]

Options:

A. $\tan 1-\frac{\pi}{4}$

B. $\tan 1+1$

C. $\frac{\pi}{4}$

D. $1-\frac{\pi}{4}$

Answer: A

Solution:

Solution:

Let $f(x)=\int\left(\frac{x^2+\sin^2 x}{1+x^2}\right)\sec^2 x\,dx$

$=\int\frac{x^2\sec^2 x+\frac{\sin^2 x}{\cos^2 x}}{1+x^2}dx$

$=\int\frac{x^2\sec^2 x+\tan^2 x}{1+x^2}dx$

$=\int\frac{x^2(1+\tan^2 x)+\tan^2 x}{1+x^2}dx$

$=\int\frac{x^2+\tan^2 x(1+x^2)}{1+x^2}dx$

$=\int\frac{x^2}{1+x^2}dx+\int\tan^2 x\,dx$

$=\int\frac{x^2+1-1}{1+x^2}dx+\int(\sec^2 x-1)dx$

$=\int 1dx-\int\frac{dx}{1+x^2}+\int\sec^2 x\,dx-\int dx$

$=-\tan^{-1}x+\tan x+c$

Given : $f(0)=0$

$\Rightarrow f(0)\ =-\tan^{-1}0+\tan 0+c\Rightarrow c=0$

$\therefore f(x)=-\tan^{-1}x+\tan x$

Now ,$f(1)=-\tan^{-1}(1)+\tan 1\ =\tan 1-\frac{\pi}{4}$

Question288

The integral of $\frac{x^2-x}{x^3-x^2+x-1}$ w.r.t. x is

[Online May 12, 2012]

Options:

A. $\frac{1}{2}\log(x^2+1)+C$

B. $\frac{1}{2}\log x^2-1\Big|+C$

C. $\log(x^2+1)+C$

D. $\log x^2-1\,|+C$

Answer: A

Solution:

Solution:

Let $I=\int\frac{x^2-x}{x^3-x^2+x-1}dx$

$=\int\frac{x(x-1)}{x^2(x-1)+(x-1)}dx=\int\frac{xdx}{x^2+1}=\frac{1}{2}\int\frac{2xdx}{(x^2+1)}$

Let $x^2+1=t\Rightarrow 2xdx=dt$

$\therefore I=\frac{1}{2}\int\frac{dt}{t}=\frac{1}{2}\log t+c$

$=\frac{1}{2}\log(x^2+1)+c$

where ‘c’ is the constant of integration.

Question289

Let f (x) be an indefinite integral of $\cos^3x$.
Statement 1: f (x) is a periodic function of period π.
Statement 2: $\cos^3x$ is a periodic function.
[Online May 7, 2012]

Options:

A. Statement 1 is true, Statement 2 is false.

B. Both the Statements are true, but Statement 2 is not the correct explanation of Statement 1.

C. Both the Statements are true, and Statement 2 is correct explanation of Statement 1.

D. Statement 1 is false, Statement 2 is true.

Answer: D

Solution:

Solution:
Statement -2 : $\cos^3 x$ is a periodic function.
It is a true statement.
Statement -1

Given $f(x) = \int \cos^3 x d x = \int \left(\frac{\cos 3 x}{4} + \frac{3 \cos x}{4}\right) d x$

$= \frac{1}{4} \frac{\sin 3 x}{3} + \frac{3}{4} \sin x = \frac{1}{12} \sin 3 x + \frac{3}{4} \sin x$

Now, period of $\frac{1}{12} \sin 3 x = \frac{2\pi}{3}$

Period of $\frac{3}{4} \sin x = 2\pi$

Hence period of $f(x) = \frac{\text{L.C.M. } (2\pi, 2\pi)}{\text{HCF of } (1, 3)} = \frac{2\pi}{1} = 2\pi$

Thus, $f(x)$ is a periodic function of period 2π.
Hence, Statement - 1 is false.

Question290

If $g(x) = \int_0^x \cos 4 t \, d t$, then $g(x + \pi)$ equals [2012]

Options:

A. $\frac{g(x)}{g(\pi)}$

B. $g(x) + g(\pi)$

C. $g(x) - g(\pi)$

D. $g(x) . g(\pi)$

Answer: 0

Solution:

$g(x+\pi)=\int_0^{x+\pi}\cos 4t\,dt$

$=\int_0^x \cos 4t\,dt+\int_0^{x+\pi}\cos 4t\,dt=g(x)+\int_0^{\pi}\cos 4t\,dt$

(it is clear from graph of $\cos 4t$)

$\int_0^{x+\pi}\cos 4t\,dt=\int_0^{\pi}\cos 4t\,dt\ =g(x)+g(\pi)=g(x)-g(\pi)$

($\because$ From graph of $\cos 4t,\ g(\pi)=0$)

Question291

If [x] is the greatest integer ≤x, then the value of the integral $\int_{-0.9}^{0.9}\left([x^2]+\log\left(\frac{2-x}{2+x}\right)\right)dx$ is

[Online May 26, 2012]

Options:

A. 0.486

B. 0.243

C. 1.8

D. 0

Answer: D

Solution:

Solution:

$\int_{-0.9}^{0.9}\left\{[x^2]+\log\left(\frac{2-x}{2+x}\right)\right\}dx$

$=\int_{-0.9}^{0.9}[x^2]dx+\int_{-0.9}^{0.9}\log\left(\frac{2-x}{2+x}\right)dx$

$=0+\int_{-0.9}^{0.9}\log\left(\frac{2-x}{2+x}\right)dx$

Put $x=-x\Rightarrow f(x)=\log\frac{2-x}{2+x}$

and $f(-x)=\log\frac{2+x}{2-x}\ =-\log\frac{(2-x)}{2+x}=-f(x)$

So, it is an odd function,
hence Required integral $= 0$

Question292

The value of the integral $\int_0^{0.9} [x - 2[x]] d x$, where [.] denotes the greatest integer function is
[Online May 19, 2012]

Options:

A. 0.9

B. 1.8

C. -0.9

D. 0

Answer: D

Solution:

Solution:

Since $\int_0^a [x] = 0$ where $0 \leq a \leq 1$

$\therefore \int_0^{0.9} [x - 2[x]] d x = 0$

Question293

If $\frac{d}{d x} G(x) = \frac{e^{\tan x}}{x}$, $x \in (0, \pi / 2)$, then $\int_{1/4}^{1/2} \frac{2}{x} . e^{\tan(\pi x^2)} d x$ is equal to
[Online May 12, 2012]

Options:

A. $G(\pi / 4) - G(\pi / 16)$

B. $2[G(\pi/4)-G(\pi/16)]$

C. $\pi[G(1/2)-G(1/4)]$

D. $G(1/\sqrt{2})-G(1/2)$

Answer: A

Solution:

Solution:

Let $\frac{d}{dx}G(x)=\frac{e^{\tan x}}{x}, x \in \left(0, \frac{\pi}{2}\right)$

Now, $I=\int_{1/4}^{1/2} \frac{2}{x} e^{\tan \pi x^2} . dx=\int_{1/4}^{1/2} \frac{2\pi x}{\pi x^2} e^{\tan \pi x^2} dx$

Let $\pi x^2=t \Rightarrow 2\pi x dx=dt$

When $x=\frac{1}{2}, t=\frac{\pi}{4}$ and $x=\frac{1}{4}, t=\frac{\pi}{16}$

$\therefore I=\int_{\pi/16}^{\pi/4} \frac{e^{\tan t}}{t} dt=G(t)\Big|_{\frac{\pi}{16}}^{\frac{\pi}{4}}=G\left(\frac{\pi}{4}\right)-G\left(\frac{\pi}{16}\right)$

Question294

If $\int_e^x tf(t)dt=\sin x-x\cos x-\frac{x^2}{2}$, for all $x \in R-\{0\}$,then the value of $f\left(\frac{\pi}{6}\right)$ is

[Online May 7, 2012]

Options:

A. $1/2$

B. 1

C. 0

D. $-1/2$

Answer: D

Solution:

Solution:

Let $\int_{e}^{x} tf(t)dt = \sin x - x\cos x - \frac{x^2}{2}$

By using Leibnitz rule, we get

$$\frac{d}{dx}\left[\int_{e}^{x} tf(t)dt\right] = \frac{d}{dx}\left[\sin x - x\cos x - \frac{x^2}{2}\right]$$

$\Rightarrow xf(x) - ef(e).0 = x\sin x - x$

Now, put $x = \frac{\pi}{6}$, we get

$$\frac{\pi}{6}.f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}.\sin\frac{\pi}{6} - \frac{\pi}{6}$$

$$\Rightarrow f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

Question295

$f(x) = \int \frac{dx}{\sin^6 x}$ is a polynomial of degree
[Online May 26, 2012]

Options:

A. 5 in $\cot x$

B. 5 in $\tan x$

C. 3 in $\tan x$

D. 3 in $\cot x$

Answer: A

Solution:

Solution:

Let $f(x) = \int \frac{dx}{\sin^6 x}$

$f(x) = \int \text{cosec}^6 x dx$

From reduction formula, we have

$$I_n = \int \text{cosec}^n x dx = -\frac{\text{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$\therefore f(x) = -\frac{\operatorname{cosec}^4 x \cot x}{5} + \frac{4}{5}\left[\frac{-\operatorname{cosec}^2 x \cot x}{3} + \frac{2}{3} I_2\right]$$

$$= -\frac{\operatorname{cosec}^4 x \cot x}{5} - \frac{4}{15}\operatorname{cosec}^2 x \, . \cot x + \frac{8}{15}[-\cot x]$$

$$= \frac{-(1+\cot^2 x)^2 \, . \cot x}{5} - \frac{4}{15}(1+\cot^2 x)\cot x - \frac{8}{15}(-\cot x)(\because \operatorname{cosec}^2 x = 1+\cot^2 x)$$

$$= \frac{-1}{5}[1+\cot^4 x + 2\cot^2 x]\cot x - \frac{4}{15}[\cot x + \cot^3 x] - \frac{8}{15}\cot x$$

$$= \frac{-1}{5}[\cot x + \cot^5 x + 2\cot^3 x]\frac{-4}{15}\cot x - \frac{4}{15}\cot^3 x - \frac{8}{15}\cot x$$

$$= \frac{-15}{15}\cot x - \frac{\cot^5 x}{5} - \frac{10}{15}\cot^3 x$$

$$= \frac{-\cot^5 x}{5} - \frac{2}{3}\cot^3 x - \cot x$$

It is a polynomial of degree 5 in $\cot x$.

Question296

Let [.] denote the greatest integer function then the value of $\int_0^{1.5} x[x^2] dx$ is
[2011 RS]

Options:

A. 0

B. $\frac{3}{2}$

C. $\frac{3}{4}$

D. $\frac{5}{4}$

Answer: C

Solution:

$$\int_0^{1.5} x[x^2]dx = \int_0^1 x[x^2]dx + \int_1^{\sqrt{2}} x[x^2]dx + \int_{\sqrt{2}}^{1.5} x[x^2]dx = \int_0^1 x \cdot 0 dx + \int_1^{\sqrt{2}} x dx + \int_{\sqrt{2}}^{1.5} 2x dx = 0 + \left[\frac{x^2}{2}\right]_1^{\sqrt{2}} + [x^2]_{\sqrt{2}}^{1.5}$$

$$= \frac{1}{2}(2-1) + (2.25-2) = \frac{1}{2} + 0.25$$

$$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

Question297

The value of $\int_0^1 \frac{8\log(1+x)}{1+x^2}dx$ is

[2011]

Options:

A. $\frac{\pi}{8}\log 2$

B. $\frac{\pi}{2}\log 2$

C. $\log 2$

D. $\pi \log 2$

Answer: D

Solution:

Solution:

$\int_0^1 \frac{8\log(1+x)}{1+x^2}dx$

Put $x = \tan\theta$

$\therefore dx = \sec^2\theta d\theta$

$\therefore I = 8\int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta} \cdot \sec^2\theta d\theta$

$I = 8\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$ ....(i)

Applying $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$= 8\int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right] d\theta$

$= 8\int_0^{\pi/4} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right] d\theta = 8\int_0^{\pi/4} \log\left[\frac{2}{1+\tan\theta}\right] d\theta$

$= 8 \int_0^{\pi/4} [\log 2 - \log(1 + \tan\theta)] d\theta$

$= 8 \log 2 \int_0^{\pi/4} 1 d\theta - 8 \int_0^{\pi/4} \log(1 + \tan\theta)\, d\theta$

$I = 8 . (\log 2)[x]_0^{\pi/4} - 8 \int_0^{\pi/4} \log(1 + \tan\theta)\, d\theta$

$I = 8 . \frac{\pi}{4} . \log 2 - I$ [From equation (i)]

$\Rightarrow 2I = 2\pi \log 2$

$\therefore I = \pi \log 2$

Question298

Let p(x) be a function defined on R such that $p'(x) = p'(1 - x)$, for all $x \in [0, 1]$, $p(0) = 1$ and $p(1) = 41$Then $\int_0^1 p(x) d x$ equals [2010]

Options:

A. 21

B. 41

C. 42

D. $\sqrt{41}$

Answer: A

Solution:

$p'(x) = p'(1 - x)$

$\Rightarrow p(x) = -p(1 - x) + c$

at $x = 0$

$p(0) = -p(1) + c \Rightarrow 42 = c$

Now, $p(x) = -p(1 - x) + 42$

$\Rightarrow p(x) + p(1 - x) = 42$

Let $I = \int_0^1 p(x) d x$ ....(i)

$\Rightarrow I = \int_0^1 p(1 - x) d x$ ....(ii)

Adding eqn. (i) and (ii),

$$2I = \int_0^1 (42)dx \Rightarrow I = 21$$

Question299

$\int_0^{\pi} [\cot x]dx$, where [.] denotes the greatest integer function, is equal to : [2009]

Options:

A. 1

B. -1

C. $-\frac{\pi}{2}$

D. $\frac{\pi}{2}$

Answer: C

Solution:

Solution:

Let $I = \int_0^{\pi} [\cot x]dx$ ......(i)

$= \int_0^{\pi} [\cot(\pi - x)]dx = \int_0^{\pi} [-\cot x]dx$ ....(ii)

Adding eq ^{n}s(i)& (ii),

We get

$$2I = \int_0^{\pi} ([\cot x] + [-\cot x])dx$$

$$= \int_0^{\pi} (-1)dx$$

$[\because [x] + [-x] = -1$, if $x \notin z$ and $[x] + [-x] = 0$, if $x \in z]$

$= [-x]_0^{\pi} = -\pi \Rightarrow I = -\frac{\pi}{2}$

Question300

The value of $\sqrt{2}\int \frac{\sin x\, dx}{\sin\left(x-\frac{\pi}{4}\right)}$ is

[2008]

Options:

A. $x+\log\left|\cos\left(x-\frac{\pi}{4}\right)\right|+c$

B. $x-\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$

C. $x+\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$

D. $x-\log\left|\cos\left(x-\frac{\pi}{4}\right)\right|+c$

Answer: C

Solution:

Solution:

Let $I = \sqrt{2}\int \frac{\sin x\, dx}{\sin\left(x-\frac{\pi}{4}\right)}$

Let $x-\frac{\pi}{4} = t \Rightarrow dx = dt$

$\Rightarrow I = \sqrt{2}\int \frac{\sin\left(t+\frac{\pi}{4}\right)}{\sin t}dt = \frac{\sqrt{2}}{\sqrt{2}}\int\left(\frac{\sin t+\cos t}{\sin t}\right)dt$

$\Rightarrow I = \int(1+\cot t)dt = t+\log|\sin t|+c_1$

$= x-\frac{\pi}{4}+\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c_1$

$= x+\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c\left(\text{where } c = c_1-\frac{\pi}{4}\right)$

Question301

Let I $=\int_0^1 \frac{\sin x}{\sqrt{x}} d x$ and J $=\int_0^1 \frac{\cos x}{\sqrt{x}} d x$. Then which one of the following is true?
[2008]

Options:

A. I $>\frac{2}{3}$ and J >2

B. I $<\frac{2}{3}$ and J <2

C. I $<\frac{2}{3}$ and J >2

D. I >23 and J <2

Answer: B

Solution:

Solution:
We know that $\frac{\sin x}{x}<1$, for $x \in(0,1)$
$\Rightarrow \frac{\sin x}{\sqrt{x}}<\sqrt{x}$ on $x \in(0,1)$
$\Rightarrow \int_0^1 \frac{\sin x}{\sqrt{x}} d x<\int_0^1 \sqrt{x} d x=\left[\frac{2 x^{3/2}}{3}\right]_0^1$
$\Rightarrow \int_0^1 \frac{\sin x}{\sqrt{x}} d x<\frac{2}{3} \Rightarrow I<\frac{2}{3}$
Also $\frac{\cos x}{\sqrt{x}}<\frac{1}{\sqrt{x}}$ for $x \in(0,1)$

$\Rightarrow \int_0^1 \frac{\cos x}{\sqrt{x}} d x<\int_0^1 x^{-1/2} d x=[2 \sqrt{x}]_0{ }^1=2$
$\Rightarrow \int_0^1 \frac{\cos x}{\sqrt{x}} d x<2 \Rightarrow J<2$

Question302

$\int \frac{dx}{\cos x + \sqrt{3}\sin x}$ **equals**
[2007]

Options:

A. $\log \tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + C$

B. $\log \tan\left(\frac{x}{2} - \frac{\pi}{12}\right) + C$

C. $\frac{1}{2}\log \tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + C$

D. $\frac{1}{2}\log \tan\left(\frac{x}{2} - \frac{\pi}{12}\right) + C$

Answer: C

Solution:

$$I = \int \frac{dx}{\cos x + \sqrt{3}\sin x}$$

$$\Rightarrow I = \int \frac{dx}{2\left[\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right]}$$

$$= \frac{1}{2}\int \frac{dx}{\left[\sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x\right]} = \frac{1}{2}.\int \frac{dx}{\sin\left(x + \frac{\pi}{6}\right)}$$

$$\Rightarrow I = \frac{1}{2}.\int \operatorname{cosec}\left(x + \frac{\pi}{6}\right) d\ x$$

We know that

$\int \operatorname{cosec} x \, dx = \log |(\tan x/2)| + C$

$\therefore I = \frac{1}{2} . \log \tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + C$

Question303

The solution for x of the equation $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2 - 1}} = \frac{\pi}{2}$ is

[2007]

Options:

A. $\frac{\sqrt{3}}{2}$

B. $2\sqrt{2}$

C. 2

D. None of these

Answer: D

Solution:

Solution:

$\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2 - 1}} = \frac{\pi}{2}$

$\therefore [\sec^{-1} t]_{\sqrt{2}}^{x} = \pi/2 \quad \left[\because \int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x\right]$

$\Rightarrow \sec^{-1} x - \sec^{-1} \sqrt{2} = \frac{\pi}{2}$

$\Rightarrow \sec^{-1} x - \frac{\pi}{4} = \frac{\pi}{2} \Rightarrow \sec^{-1} x = \frac{\pi}{2} + \frac{\pi}{4}$

$\Rightarrow \sec^{-1} x = \frac{3\pi}{4} \Rightarrow x = \sec \frac{3\pi}{4} = \sec\left(\pi - \frac{\pi}{4}\right)$

$\Rightarrow x = -\sec \frac{\pi}{4} \Rightarrow x = -\sqrt{2}$

Question304

Let $F(x) = f(x) + f\left(\frac{1}{x}\right)$, where $f(x) = \int_1^x \frac{\log t}{1+t} dt$, ThenF (e) equals [2007]

Options:

A. 1

B. 2

C. $1/2$

D. 0

Answer: C

Solution:

Solution:

Given that $F(x) = f(x) + f\left(\frac{1}{x}\right)$, where

$f(x) = \int_1^x \frac{\log t}{1+t} dt$

$\therefore F(e) = f(e) + f\left(\frac{1}{e}\right)$

$\Rightarrow F(e) = \int_1^e \frac{\log t}{1+t} dt + \int_1^{1/e} \frac{\log t}{1+t} dt$ ....(1)

Let $I = \int_1^{1/e} \frac{\log t}{1+t} dt$

$\therefore$ Put $\frac{1}{t} = z \Rightarrow -\frac{1}{t^2} dt = dz \Rightarrow dt = -\frac{dz}{z^2}$

when $t = 1 \Rightarrow z = 1$ and when $t = \frac{1}{e}$

$\Rightarrow z = e$

$\therefore I = \int_1^e \frac{\log\left(\frac{1}{z}\right)}{1+\frac{1}{z}} \left(-\frac{dz}{z^2}\right)$

$= \int_1^e \frac{(\log 1 - \log z) . z}{z+1} \left(-\frac{dz}{z^2}\right)$

$= \int_1^e - \frac{\log z}{(z+1)} \left(-\frac{dz}{z}\right.$ [$\because$ log 1 = 0]

$= \int_1^e \frac{\log z}{z(z+1)} dz$

$\therefore I = \int_1^e \frac{\log t}{t(t+1)} dt$

[By property $\int_a^b f(t)dt = \int_a^b f(x)dx$]
Now from eqn. (1)
$F(e) = \int_1^e \frac{\log t}{1+t}dt + \int_1^e \frac{\log t}{t(1+t)}dt$
$= \int_1^e \frac{t.\log t + \log t}{t(1+t)}dt \ = \int_1^e \frac{(\log t)(t+1)}{t(1+t)}dt$
$\Rightarrow F(e) = \int_1^e \frac{\log t}{t}dt$
Let $\log t = x \therefore \frac{1}{t}dt = dx$
[when $t = 1, x = 0$ and when $t = e, x = \log e = 1$]
$\therefore F(e) = \int_0^1 x dx F(e) = \left[\frac{x^2}{2}\right]_0^1$
$\Rightarrow F(e) = \frac{1}{2}$

Question305

The value of $\int_1^a [x]f'(x)dx$, $a > 1$ where $[x]$ denotes the greatest integer not exceeding x is
[2006]

Options:

A. $af(a) - \{f(1) + f(2) + \ldots\ldots. f([a])\}$

B. $[a]f(a) - \{f(1) + f(2) + \ldots\ldots f([a])\}$

C. $[a]f([a]) - \{f(1) + f(2) + \ldots\ldots f(a)\}$

D. $af([a]) - \{f(1) + f(2) + \ldots\ldots. f(a)\}$

Answer: B

Solution:

Solution:
Let $a = k + h$ where k is an integer such that and $0 \le h < 1$
$\Rightarrow [a] = k$
$\therefore \int_1^a [x]f'(x)dx = \int_1^2 1f'(x)dx + \int_2^3 2f'(x)dx + \ldots. \int_{k-1}^{k} (k-1)f'(x)dx + \int_k^{k+h} kf'(x)dx$

$= \{f(2) - f(1)\} + 2\{f(3) - f(2)\} + 3\{f(4) - f(3)\} + \ldots.. + (k-1)\{f(k) - f(k-1)\} + k\{f(k+h) - f(k)\}$
$= -f(1) - f(2) - f(3) \ldots\ldots - f(k) + kf(k+h)$
$= [a]f(a) - \{f(1) + f(2) + f(3) + \ldots . f([a])\}$

Question306

$\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} [(x+\pi)^3 + \cos^2(x+3\pi)]dx$ **is equal to**

[2006]

Options:

A. $\frac{\pi^4}{32}$

B. $\frac{\pi^4}{32} + \frac{\pi}{2}$

C. $\frac{\pi}{2}$

D. $\frac{\pi}{4} - 1$

Answer: C

Solution:

Solution:

$$\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} [(x+\pi)^3 + \cos^2(x+3\pi)]dx$$

Put $x + \pi = t$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [t^3 + \cos^2 t]dt = 2\int_{0}^{\frac{\pi}{2}} \cos^2 t dt$$

[$\because t^3$ is odd and $\cos^2 t$ is even function]

$$= \int_0^{\frac{\pi}{2}} (1+\cos 2t)dt = \frac{\pi}{2} + 0$$

Question307

$\int_0^{\pi} xf(\sin x)dx$ is equal to
[2006]

Options:

A. $\pi \int_0^{\pi} f(\cos x)dx$

B. $\pi \int_0^{\pi} f(\sin x)dx$

C. $\frac{\pi}{2} \int_0^{\pi/2} f(\sin x)dx$

D. $\pi \int_0^{\pi/2} f(\cos x)dx$

Answer: D

Solution:

Solution:

$I = \int_0^{\pi} xf(\sin x)dx = \int_0^{\pi} (\pi - x)f(\sin x)dx$

$= \pi \int_0^{\pi} f(\sin x)dx - I \Rightarrow 2I = \pi \int_0^{\pi} f(\sin x)dx$

$I = \frac{\pi}{2} \int_0^{\pi} f(\sin x)dx = \pi \int_0^{\pi/2} f(\sin x)dx$ [$\because \sin(\pi - x) = \sin x$]

$= \pi \int_0^{\pi/2} f(\cos x)dx$

Question308

The value of integral, :$\int_{3}^{6} \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}} dx$ is
[2006]

Options:

A. $\frac{1}{2}$

B. $\frac{3}{2}$

C. 2

D. 1

Answer: B

Solution:

Solution:
$I = \int_{3}^{6} \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}} dx \ldots$ (1)
$I = \int_{3}^{6} \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{x}} dx \ldots$ (2)
$\left[\because \int_{a}^{b} f(x) dx = \int_{a}^{b} f(a+b-x) dx \right]$
Adding equation (1) and (2)
$2I = \int_{3}^{6} dx = [x]_{3}^{6} = 3 \Rightarrow I = \frac{3}{2}$

Question309

$\int \left\{ \frac{(\log x - 1)}{1 + (\log x)^2} \right\}^2$ **d x is equal to**
[2005]

Options:

A. $\frac{\log x}{(\log x)^2 + 1} + C$

B. $\dfrac{x}{x^2+1}+C$

C. $\dfrac{xe^x}{1+x^2}+C$

D. $\dfrac{x}{(\log x)^2+1}+C$

Answer: D

Solution:

Solution:

$\int \dfrac{(\log x-1)^2}{(1+(\log x)^2)^2}dx = \int \dfrac{1+(\log x)^2-2\log x}{[1+(\log x)^2]^2}dx$

$=\int \left[\dfrac{1}{(1+(\log x)^2)}-\dfrac{2\log x}{(1+(\log x)^2)^2}\right]dx$

$\therefore I = \int \left[\dfrac{e^t}{1+t^2}-\dfrac{2te^t}{(1+t^2)^2}\right]dt$

$=\int e^t\left[\dfrac{1}{1+t^2}-\dfrac{2t}{(1+t^2)^2}\right]dt$

[Which is of the form $\int e^x(f(x)+f'(x))dx = f(x).e^x+c$]

$=\dfrac{e^t}{1+t^2}+c=\dfrac{x}{1+(\log x)^2}+c$

Question310

The value of $\int_{-\pi}^{\pi}\dfrac{\cos^2 x}{1+a^x}dx$, a > 0, is [2005]

Options:

A. $a\pi$

B. $\dfrac{\pi}{2}$

C. $\dfrac{\pi}{a}$

D. 2π

Answer: B

Solution:

Solution:

Let $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$ ....(1)

$I = \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^x} dx$ Using $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx \Big]$

$= \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x} dx$ ....(2)

Adding equations (1) and (2) we get

$2I = \int_{-\pi}^{\pi} \cos^2 x \left(\frac{1+a^x}{1+a^x}\right) dx = \int_{-\pi}^{\pi} \cos^2 x dx$

$= 2\int_0^{\pi} \cos^2 x dx [\because f(\pi - x) = f(x)]$

$= 2 \times 2 \int_0^{\frac{\pi}{2}} \cos^2 x dx = 4 \int_0^{\frac{\pi}{2}} \sin^2 x dx \left[\because f\left(\frac{\pi}{2} - x\right) = f(x)\right]$

$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \sin^2 x dx = 2\int_0^{\frac{\pi}{2}} (1 - \cos^2 x) dx$

$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} dx - 2\int_0^{\frac{\pi}{2}} \cos^2 x dx$

$\Rightarrow I + I = 2\left(\frac{\pi}{2}\right) = \pi \Rightarrow I = \frac{\pi}{2}$

Question311

If $I_1 = \int_0^1 2^{x^2} dx$, $I_2 = \int_0^1 2^{x^3} dx$, $I_3 = \int_1^2 2^{x^2} dx$ and $I_4 = \int_1^2 2^{x^3} dx$ then [2005]

Options:

A. $I_2 > I_1$

B. $I_1 > I_2$

C. $I_3 = I_4$

D. $I_3 > I_4$

Answer: B

Solution:

Solution:

$I_1 = \int_0^1 2^{x^2} dx, I_2 = \int_0^1 2^{x^3} dx, I_3$
$= \int_1^2 2^{x^2} dx, I_4 = \int_1^2 2^{x^3} dx$
$\because 2^{x^3} < 2^{x^2}, 0 < x < 1$
$\Rightarrow \int_0^1 2^{x^2} dx > \int_0^1 2^{x^3} dx \Rightarrow I_1 > I_2$
and $2^{x^3} > 2^x, x > 1$
$\Rightarrow I_4 > I_3$

Question312

Let $f : R \to R$ be a differentiable function having $f(2) = 6$, $f'(2) = \left(\frac{1}{48}\right)$. Then $\lim_{x \to 2} \int_6^{f(x)} \frac{4t^3}{x-2} dt$ equals [2005]

Options:

A. 24

B. 36

C. 12

D. 18

Answer: D

Solution:

$$\lim_{x \to 2} \int_0^{f(x)} \frac{4t^3}{x-2} dt = \lim_{x \to 2} \frac{\int_0^{f(x)} 4t^3 dt}{x-2}$$

Applying L Hospital rule

$$\lim_{x \to 2} \frac{[4f(x)^3 f'(x)]}{1} = 4(f(2))^3 f'(2)$$

$$= 4 \times 6^3 \times \frac{1}{48} = 18$$

Question313

$$\lim_{n \to \infty} \left[\frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{4}{n^2} \ldots\ldots. + \frac{1}{n} \sec^2 1 \right]$$

equals

[2005]

Options:

A. $\frac{1}{2}\sec 1$

B. $\frac{1}{2}\operatorname{cosec} 1$

C. $\tan 1$

D. $\frac{1}{2}\tan 1$

Answer: D

Solution:

Solution:

$\lim_{n \to \infty} \left[\frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{4}{n^2} + \frac{3}{n^2} \sec^2 \frac{9}{n^2} \ldots\ldots. + \frac{1}{n} \sec^2 1 \right]$ is equal to

$$\lim_{n \to \infty} \frac{r}{n^2} \sec^2 \frac{r^2}{n^2} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{r}{n} \sec^2 \frac{r^2}{n^2}$$

⇒ Given limit is equal to value of integral $\int_0^1 x \sec^2 x^2 dx$

or $\frac{1}{2}\int_0^1 2x\sec x^2\, dx = \frac{1}{2}\int_0^1 \sec^2 t\, dt$ [put $x^2 = t$]

$= \frac{1}{2}(\tan t)_0^{\ 1} = \frac{1}{2}\tan 1$

Question314

$\int \frac{dx}{\cos x - \sin x}$ is equal to
[2004]

Options:

A. $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}+\frac{3\pi}{8}\right)\right| + C$

B. $\frac{1}{\sqrt{2}}\log\left|\cot\left(\frac{x}{2}\right)\right| + C$

C. $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}-\frac{3\pi}{8}\right)\right| + C$

D. $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}-\frac{\pi}{8}\right)\right| + C$

Answer: A

Solution:

Solution:

$\int \frac{dx}{\cos x - \sin x} = \int \frac{dx}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)}$

$= \int \frac{dx}{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}}\int \sec\left(x+\frac{\pi}{4}\right) dx$

$= \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}+\frac{\pi}{8}\right)\right| + C \left[\because \int \sec x\, dx = \log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|\right]$

$= \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}+\frac{3\pi}{8}\right)\right| + C$

Question315

If $\int \frac{\sin x}{\sin(x-\alpha)} d\,x = Ax + B \log \sin(x-\alpha), + C$, then value of (A, B) is [2004]

Options:

A. $(-\cos\alpha, \sin\alpha)$

B. $(\cos\alpha, \sin\alpha)$

C. $(-\sin\alpha, \cos\alpha)$

D. $(\sin\alpha, \cos\alpha)$

Answer: B

Solution:

Solution:

$\int \frac{\sin x}{\sin(x-\alpha)} d\,x = \int \frac{\sin(x-\alpha+\alpha)}{\sin(x-\alpha)} d\,x$

$= \int \frac{\sin(x-\alpha)\cos\alpha + \cos(x-\alpha)\sin\alpha}{\sin(x-\alpha)} d\,x$

$= \int \{\cos\alpha + \sin\alpha\cot(x-\alpha)\} d\,x$

$= (\cos\alpha)x + (\sin\alpha)\log\sin(x-\alpha) + C$

Comparing with $Ax + B\log\sin(x-\alpha) + c$

$\therefore A = \cos\alpha,\ B = \sin\alpha$

Question316

If $f(x) = \frac{e^x}{1+e^x}$, $I_1 = \int_{f(-a)}^{f(a)} xg\{x(1-x)\} d\,x$ and $I_2 = \int_{f(-a)}^{f(a)} g\{x(1-x)\} d\,x$, then the value of $\frac{I_2}{I_1}$ is [2004]

Options:

A. 1

B. –3

C. –1

D. 2

Answer: D

Solution:

Solution:

$f(x)=\frac{e^{x}}{1+e^{x}} \Rightarrow f(-x)=\frac{e^{-x}}{1+e^{-x}}=\frac{1}{e^{x}+1}$

$\therefore f(x)+f(-x)=1\ \forall x \in R$

Now $I_1=\int_{f(-a)}^{f(a)} xg\{x(1-x)\}dx$

$=\int_{f(-a)}^{f(a)}(1-x)g\{x(1-x)\}dx\left[\text{using} \int_a^b f(x)dxa=\int_a^b f(a+b-x)dx\right]$

$\Rightarrow \int_{f(-a)}^{f(a)} g\{x(1-x)\}dx-\int_{f(-a)}^{f(a)} xg(x(1-x)dx$

$=I_2-I_1 \Rightarrow 2I_1=I_2$

Question317

If $\int_0^{\pi} xf(\sin x)dx=A\int_0^{\pi/2} f(\sin x)dx$, then A is [2004]

Options:

A. 2π

B. π

C. $\frac{\pi}{4}$

D. 0

Answer: B

Solution:

Solution:

Let $I = \int_0^{\pi} x f(\sin x) dx$ ......(i)

We know that

$\int_0^a f(x) dx = \int_0^a f(a-x) dx = \int_0^{\pi} (\pi - x) f(\sin x) dx$ ....(ii)

Adding (i) and (ii)

$\therefore 2I = \pi \int_0^{\pi} f(\sin x) dx = \pi . 2 \int_0^{\pi/2} f(\sin x) dx \; [\because \sin(\pi - x) = \sin x]$

$\therefore I = \pi \int_0^{\pi/2} f(\sin x) dx \Rightarrow A = \pi$

Let $\log x = t \Rightarrow e^t = x$

$\Rightarrow \frac{1}{x} dx = dt \Rightarrow dx = x dt \Rightarrow e^t dt$

Question318

The value of $I = \int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{1 + \sin 2x}} dx$ is [2004]

Options:

A. 3

B. 1

C. 2

D. 0

Answer: C

Solution:

Solution:

$I = \int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{1 + \sin 2x}} dx$

$\int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{(\sin x + \cos x)^2}}$

$I = \int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{(\sin x + \cos x)} dx = \int_0^{\pi/2} (\sin x + \cos x) dx$ [$\because \sin x + \cos x > 0$ if $0 < x < \frac{\pi}{2}$]

or $I = [-\cos x + \sin x]_0^{\frac{\pi}{2}} = 2$

Question319

The value of $\int_{-2}^{3} |1 - x^2| dx$ is
[2004]

Options:

A. $\frac{1}{3}$

B. $\frac{14}{3}$

C. $\frac{7}{3}$

D. $\frac{28}{3}$

Answer: D

Solution:

Solution:

$\int_{-2}^{3} |1 - x^2| dx = \int_{-2}^{3} |x^2 - 1| dx$

Now $|x^2 - 1| = \begin{cases} x^2 - 1 & \text{if} \quad x \leq -1 \\ 1 - x^2 & \text{if} \quad -1 \leq x \leq 1 \\ x^2 - 1 & \text{if} \quad x \geq 1 \end{cases}$

$\therefore$ Integral is $\int_{-2}^{-1} (x^2 - 1) dx + \int_{-1}^{1} (1 - x^2) dx + \int_{1}^{3} (x^2 - 1) dx$

$= \left[\frac{x^3}{3} - x\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3} - x\right]_{1}^{3}$

$= \left(-\frac{1}{3} + 1\right) - \left(-\frac{8}{3} + 2\right) + \left(2 - \frac{2}{3}\right) + \left(\frac{27}{3} - 3\right) - \left(\frac{1}{3} - 1\right)$

$= \frac{2}{3} + \frac{2}{3} + \frac{4}{3} + 6 + \frac{2}{3} = \frac{28}{3}$

Question320

$\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} e^{\frac{r}{n}}$ **is**

[2004]

Options:

A. $e+1$

B. $e-1$

C. $1-e$

D. e

Answer: B

Solution:

Solution:

$\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} e^{\frac{r}{n}}$ [Using definite integrals as limit of sum]

$= \int_0^1 e^x dx = e-1$

Question321

The value of the integral $I = \int_0^1 x(1-x)^n dx$ is

[2003]

Options:

A. $\frac{1}{n+1} + \frac{1}{n+2}$

B. $\frac{1}{n+1}$

C. $\frac{1}{n+2}$

D. $\frac{1}{n+1}-\frac{1}{n+2}$.

Answer: D

Solution:

Solution:

$$I=\int_0^1 x(1-x)^n dx=\int_0^1 (1-x)(1-1+x)^n dx$$
$$=\int_0^1 (1-x)x^n dx=\int_0^1 (x^n-x^{n+1})dx$$
$$=\left[\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right]_0^1=\frac{1}{n+1}-\frac{1}{n+2}$$

Question322

Let $f(x)$ be a function satisfying $f'(x)=f(x)$ with $f(0)=1$ and $g(x)$ be a function that satisfies $f(x)+g(x)=x^2$. Then the value of the integral $\int_0^1 f(x)g(x)dx$, is [2003]

Options:

A. $e+\frac{e^2}{2}+\frac{5}{2}$

B. $e-\frac{e^2}{2}-\frac{5}{2}$

C. $e+\frac{e^2}{2}-\frac{3}{2}$

D. $e-\frac{e^2}{2}-\frac{3}{2}$.

Answer: D

Solution:

Solution:

Given that $f'(x) = f(x) \Rightarrow \frac{f'(x)}{f(x)} = 1$

Integrating both side we get

$\log f(x) = x + c \Rightarrow f(x) = e^{x+c}$

$f(0) = 1 \Rightarrow f(x) = e^x$

$\therefore g(x) = x^2 - f(x) = x^2 - e^x$

$\therefore \int_0^1 f(x)g(x)dx = \int_0^1 e^x(x^2 - e^x)dx$

$= \int_0^1 x^2e^x dx - \int_0^1 e^{2x} dx$

$= [x^2e^x]_0^1 - 2[xe^x - e^x]_0^1 - \frac{1}{2}[e^{2x}]_0^1$

$= e - \left[\frac{e^2}{2} - \frac{1}{2}\right] - 2[e - e + 1] = e - \frac{e^2}{2} - \frac{3}{2}$

Question323

If $f(a+b-x) = f(x)$ then $\int_a^b xf(x)dx$ is equal to [2003]

Options:

A. $\frac{a+b}{2}\int_a^b f(a+b+x)dx$

B. $\frac{a+b}{2}\int_a^b f(b-x)dx$

C. $\frac{a+b}{2}\int_a^b f(x)dx$

D. $\frac{b-a}{2}\int_a^b f(x)dx.$

Answer: C

Solution:

Solution:

$I = \int_a^b xf(x)dx = \int_a^b (a+b-x)f(a+b-x)dx$

We know that

because $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$= (a+b)\int_a^b f(a+b-x)dx - \int_a^b xf(a+b-x)dx$

$= (a+b)\int_a^b f(x)dx - \int_a^b xf(x)dx$ [$\because$ Given that $f(a+b-x) = f(x)$]

$2I = (a+b)\int_a^b f(x)dx$

$\Rightarrow I = \dfrac{(a+b)}{2}\int_a^b f(x)dx$

Question324

The value of $\lim\limits_{x \to 0} \dfrac{\int_0^{x^2} \sec^2 t\,dt}{x \sin x}$ **is**

[2003]

Options:

A. 0

B. 3

C. 2

D. 1

Answer: D

Solution:

$\lim\limits_{x \to 0} \dfrac{\frac{d}{dx}\int_0^{x^2} \sec^2 t\,dt}{\frac{d}{dx}(x \sin x)} = \lim\limits_{x \to 0} \dfrac{\sec^2 x^2 \,.\, 2x}{\sin x + x \cos x}$ (by L' Hospital rule)

$\lim\limits_{x \to 0} \dfrac{2\sec^2 x^2}{\left(\frac{\sin x}{x} + \cos x\right)} = \dfrac{2 \times 1}{1+1} = 1$

Question325

If $f(y) = e^y$, $g(y) = y$; $y > 0$ and $F(t) = \int_0^t f(t-y)g(y)dy$, then [2003]

Options:

A. $F(t) = te^{-t}$

B. $F(t) = 1 - te^{-t}(1+t)$

C. $F(t) = e^t - (1+t)$

D. $F(t) = te^t$

Answer: C

Solution:

Solution:

$F(t) = \int_0^t f(t-y)g(y)dy$

$= \int_0^t e^{t-y} y\,dy = e^t \int_0^t e^{-y} y\,dy$

$= e^t[-ye^{-y} - e^{-y}]_0^t = -e^t[ye^{-y} + e^{-y}]_0^t$

$= -e^t[te^{-t} + e^{-t} - 0 - 1] = -e^t\left[\frac{t+1-e^t}{e^t}\right]$

$= e^t - (1+t)$

Question326

$\lim_{n\to\infty} \frac{1+2^4+3^4+\ldots n^4}{n^5} - \lim_{n\to\infty} \frac{1+2^3+3^3+\ldots n^3}{n^5}$

[2003]

Options:

A. $\frac{1}{5}$

B. $\frac{1}{30}$

C. Zero

D. $\frac{1}{4}$

Answer: A

Solution:

Solution:

$$\lim_{n\to\infty}\frac{1+2^4+3^4+\ldots.n^4}{n^5}-\lim_{n\to\infty}\frac{1+2^3+3^3+\ldots.n^3}{n^5}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^4-\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{1}{n}\left(\frac{r}{n}\right)^3$$

$$=\int_0^1 x^4dx-\lim_{n\to\infty}\frac{1}{n}\times\int_0^1 x^3dx=\left[\frac{x^5}{5}\right]_0^1-0=\frac{1}{5}$$

Question327

f (x) and g(x) are two differentiable functions on [0,2] such that $f''(x)-g''(x)=0$, $f'(1)=2g'(1)=4f(2)=3g(2)=9$ then $f(x)-g(x)$ at $x=3/2$ is [2002]

Options:

A. 0

B. 2

C. 10

D. 5

Answer: D

Solution:

$\because f''(x) - g''(x) = 0$
Integrating, $f'(x) - g'(x) = c$;
$\Rightarrow f'(1) - g'(1) = c \Rightarrow 4 - 2 = c \Rightarrow c = 2$
$\therefore f'(x) - g'(x) = 2$
Integrating, $f(x) - g(x) = 2x + c_1$
$\Rightarrow f(2) - g(2) = 4 + c_1 \Rightarrow 9 - 3 = 4 + c_1$
$\Rightarrow c_1 = 2 \therefore f(x) - g(x) = 2x + 2$
At $x = 3/2,\ f(x) - g(x) = 3 + 2 = 5.$

Question328

$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}$**d x is**
[2002]

Options:

A. $\frac{\pi^2}{4}$

B. π^2

C. zero

D. $\frac{\pi}{2}$

Answer: B

Solution:

$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$
$= \int_{-\pi}^{\pi} \frac{2x\,dx}{1+\cos^2 x} + 2\int_{-\pi}^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$
$= 0 + 4\int_{0}^{\pi} \frac{x \sin x\, dx}{1+\cos^2 x}$
We know that
$\because \int_{-a}^{a} f(x)dx = 0$, if $f(x)$ is odd.

$= 2\int_0^a f(x)dx$, if $f(x)$ is even

$I = 4\int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}dx$

$I = 4\int_0^\pi \frac{(\pi - x)\sin x}{1 + \cos^2 x}dx$

$\Rightarrow I = 4\pi\int_0^\pi \frac{\sin x\, dx}{1 + \cos^2 x} - 4\int \frac{x \sin x\, dx}{1 + \cos^2 x}$

$\Rightarrow 2I = 4\pi\int_0^\pi \frac{\sin x}{1 + \cos^2 x}dx$

put $\cos x = t \Rightarrow -\sin x\, dx = dt$

when $x = 0, t = 1$ and when $x = \pi, t = -1$

$\therefore I = -2\pi\int_1^{-1} \frac{1}{1 + t^2}dt = 2\pi\int_{-1}^{1} \frac{1}{1 + t^2}dt$

$= 2\pi[\tan^{-1}t]_{-1}^{1} = 2\pi[\tan^{-1}1 - \tan^{-1}(-1)]$

$= 2\pi\left[\frac{\pi}{4} - \left(\frac{-\pi}{4}\right)\right] = 2\pi \cdot \frac{\pi}{2} = \pi^2$

Question329

$\int_0^2 [x^2]dx$ is

[2002]

Options:

A. $2 - \sqrt{2}$

B. $2 + \sqrt{2}$

C. $\sqrt{2} - 1$

D. $-\sqrt{2} - \sqrt{3} + 5$

Answer: D

Solution:

Solution:

We know that [x] is greatest integer function less than equal to x

$\therefore \int_0^2 [x^2]dx = \int_0^1 [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2] + \int_{\sqrt{3}}^{2} [x^2]dx$

$$= \int_0^1 0dx + \int_1^{\sqrt{2}} 1dx + \int_{\sqrt{2}}^{\sqrt{3}} 2dx + \int_{\sqrt{3}}^{2} 3dx$$
$$= [x]_1^{\sqrt{2}} + [2x]_{\sqrt{2}}^{\sqrt{3}} + [3x]_{\sqrt{3}}^{2}$$
$$= \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3}$$
$$= 5 - \sqrt{3} - \sqrt{2}$$

Question330

$I_n = \int_0^{\pi/4} \tan^n x dx$ then $\lim_{n \to \infty} n[I_n + I_{n+2}]$ equals [2002]

Options:

A. $\frac{1}{2}$

B. 1

C. of

D. zero

Answer: B

Solution:

Solution:

$$I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x(1 + \tan^2 x)dx$$
$$= \int_0^{\pi/4} \tan^n x \sec^2 x dx = \left[\frac{\tan^{n+1} x}{n+1}\right]_0^{\pi/4} \left[\because \int x^n dx = \frac{x^{n+1}}{n+1}\right]$$
$$= \frac{1-0}{n+1} = \frac{1}{n+1}$$
$$\therefore I_n + I_{n+2} = \frac{1}{n+1} \Rightarrow \lim_{n \to \infty} n[I_n + I_{n+2}]$$
$$= \lim_{n \to \infty} n \cdot \frac{1}{n+1} = \lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{n}{n\left(1 + \frac{1}{n}\right)} = 1$$

Question331

$\int_{0}^{10\pi} |\sin x| d x$ is
[2002]

Options:

A. 20

B. 8

C. 10

D. 18

Answer: A

Solution:

Solution:
$I = \int_{0}^{10\pi} |\sin x| d x = 10 \int_{0}^{\pi} |\sin x| d x \; [\because \sin(10\pi - x) = \sin x]$
$= \int_{0}^{10\pi} |\sin x| d x$
$\because \sin x > 0$, for $0 < x < \pi$
as $\sin(\pi - x) = \sin x$
$I = 20 \int_{0}^{\pi/2} \sin x \, d x = 20[-\cos x]_{0}^{\pi/2} = 20$

Question332

$\lim_{n \to \infty} \frac{1^{p} + 2^{p} + 3^{p} + ... + n^{p}}{n^{p+1}}$ is
[2002]

Options:

A. $\frac{1}{p+1}$

B. $\frac{1}{1-p}$

C. $\frac{1}{p} - \frac{1}{p-1}$

D. $\frac{1}{p+2}$

Answer: A

Solution:

Solution:

We have $\lim\limits_{n \to \infty} \frac{1^p + 2^p + 3^p + ... + n^p}{n^{p+1}}$;

$\lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} \frac{r^p}{n^p . n} = \int_0^1 x^p d\,x = \left[\frac{x^{p+1}}{p+1}\right]_0^1 = \frac{1}{p+1}$
