华中科技大学 2021-2022 转专业数学回忆版

1、填空题

（1）求极限 $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\arctan(\dfrac{1}{k^2+k+1})=$________.

解:$\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}[\arctan(k+1)-\arctan k]$

$=\lim\limits_{n\to\infty}[\arctan(n+1)-\arctan 0]$

$=\dfrac{\pi}{2}$

（2）已知 $\lim\limits_{x\to0}\dfrac{\ln(1+\dfrac{f(x)}{\tan x})}{2^x-1}=8$，则极限 $\lim\limits_{x\to0}\dfrac{f(x)}{x^2}=$________.

解：上式易化为:$\dfrac{f(x)}{\ln 2\cdot x^2}=8$，故 $\lim\limits_{x\to0}\dfrac{f(x)}{x^2}=8\ln 2.$

（3）求极限 $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n^4}+2\sqrt[n]{n^2}-3}{\sqrt[n]{n^2}-3\sqrt[n]{n}+2}=$________.

解:$\sqrt[n]{n}=t,(t\to1),LHS=\dfrac{t^4+2t^2-3}{t^2-3t+2}=\dfrac{(t^2-1)(t^2+3)}{(t-1)(t-2)}=\dfrac{(t+1)(t^2+3)}{t-2}=-8(t\to1)$

（4）设 $g(x)$ 在 $(-\infty,+\infty)$ 上连续且满足方程 $g(x)+g(y)=g(\sqrt{x^2+y^2})$，$g(x)=$________.

解：由 $g(x)+g(y)=g(\sqrt{x^2+y^2})$，令 $x=y=0$，可得 $g(0)=0$，再令 $y=0$,可得 $g(x)=g(|x|)$,可得偶函数.

下面考虑 $x,y>0$ 的情况：

$g(\sqrt{x^2})+g(\sqrt{y^2})=g(\sqrt{x^2+y^2})$，令 $h(x)=g(\sqrt{x})\ (x>0)$

$h(x^2)+h(y^2)=h(x^2+y^2)$，令 $x'=x^2,y'=y^2$

则 $h(x')+h(y')=h(x'+y')\to h(x')=kx'$，即 $h(x)=kx.$

$g(\sqrt{x})=kx\to g(x)=kx^2\ (x>0).$

由偶函数可知，$g(x)=kx^2\ (x\in\mathbb{R})$.其中 $k=g(1).$

（5）求 $x=a\cos t+b\sin t$，$y=a\sin t-b\cos t$，则 $\dfrac{d^mx}{dt^m}\cdot\dfrac{d^ny}{dt^n}-\dfrac{d^nx}{dt^n}\cdot\dfrac{d^my}{dt^m}=$________.

解：令 $x=\sqrt{a^2+b^2}(\dfrac{a}{\sqrt{a^2+b^2}}\cos t+\dfrac{b}{\sqrt{a^2+b^2}}\sin t)=\sqrt{a^2+b^2}\cos(t-\alpha)$,

同理令 $y=\sqrt{a^2+b^2}(\dfrac{a}{\sqrt{a^2+b^2}}\sin t-\dfrac{b}{\sqrt{a^2+b^2}}\cos t)=\sqrt{a^2+b^2}\sin(t-\alpha).$

用Leibniz公式可得答案：$(a^2+b^2)\sin\dfrac{n-m}{2}\pi.$

2、设$\{\theta_n\}\neq 0$，且满足$\theta_n{}^2-\theta_{n-1}\cdot\theta_{n+1}=1(n=1,2,3\cdots)$，证明存在一个实数$\lambda$，使得对所有$n\geq 1$，有$\theta_{n+1}=\lambda\theta_n-\theta_{n-1}$.

证明：由题$\theta_{n+1}{}^2-\theta_n\cdot\theta_{n+2}=1$，$\theta_n{}^2-\theta_{n-1}\cdot\theta_{n+1}=1$，

作差得：$\theta_{n+1}{}^2+\theta_{n-1}\cdot\theta_{n+1}-\theta_n{}^2-\theta_n\cdot\theta_{n+2}=0\to\theta_{n+1}(\theta_{n-1}+\theta_{n+1})=\theta_n(\theta_n+\theta_{n+2})$

$\to\dfrac{\theta_{n+1}}{\theta_n+\theta_{n+2}}=\dfrac{\theta_n}{\theta_{n-1}+\theta_{n+1}}=\lambda=\dfrac{\theta_1}{\theta_0+\theta_2}$.

3、设$f(x)$定义在$x=0$的某个邻域上，且$\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}\dfrac{f(x)-f(\frac{x}{2})}{x}=0$, 证明$\lim\limits_{x\to 0}\dfrac{f(x)}{x}=0$.

证明：已知$\forall\varepsilon>0$，$\exists\delta>0$，当$|x|<\delta$时，有$\left|\dfrac{f(x)-f(\frac{x}{2})}{x}\right|<\dfrac{\varepsilon}{3}$.亦即

$-\dfrac{\varepsilon}{3}|x|<f(x)-f(\dfrac{x}{2})<\dfrac{\varepsilon}{3}|x|$.

将x替换为$\dfrac{x}{2^k}$,得

$-\dfrac{\varepsilon}{3}\cdot\dfrac{1}{2^k}|x|<f(\dfrac{x}{2^k})-f(\dfrac{x}{2^{k+1}})<\dfrac{\varepsilon}{3}\cdot\dfrac{1}{2^k}|x|(k=0,1,2,\cdots,n)$.

相加（注意到$\sum\limits_{k=0}^{n}[f(\dfrac{x}{2^k})-f(\dfrac{x}{2^{k+1}})]=f(x)-f(\dfrac{x}{2^{n+1}})$）得

$-\dfrac{\varepsilon}{3}\sum\limits_{k=0}^{n}\dfrac{1}{2^k}|x|<f(\dfrac{x}{2^k})-f(\dfrac{x}{2^{n+1}})<\dfrac{\varepsilon}{3}\sum\limits_{k=0}^{n}\dfrac{1}{2^k}|x|$.

令$n\to\infty$,取极限得 $-\dfrac{2\varepsilon}{3}|x|\leq f(x)\leq\dfrac{2\varepsilon}{3}|x|$(因$\sum\limits_{k=0}^{n}\dfrac{1}{2^k}=2$)，故$\left|\dfrac{f(x)}{x}\right|\leq\dfrac{2\varepsilon}{3}<\varepsilon$

4、已知函数$f(x)$在$(-1,1)$内有二阶导数，且$f(0)=f'(0)=0$,$|f''(x)|\leq|f(x)|+|f'(x)|$，证明：存在$\delta>0$，使得在$(-\delta,\delta)$内$f(x)\equiv 0$.

法一：证明：考察区间$[-\frac{1}{4},\frac{1}{4}]$上的函数$|f(x)|+|f'(x)|$，并假定它在$x_0\in[-\frac{1}{4},\frac{1}{4}]$处取到最大值$M$.

$f(x_0)=f''(\xi_0)\dfrac{x_0{}^2}{2}$，$f'(x_0)=f''(\eta_0)x_0$，其中$\xi_0$，$\eta_0$位于$x_0$和0之间. 从而有：

$M=|f(x_0)|+|f'(x_0)|=|f''(\xi_0)|\dfrac{x_0{}^2}{2}+|f''(\eta_0)x_0|$

$\leq\dfrac{|f''(\xi_0)|+|f''(\eta_0)|}{4}$

$\leq\dfrac{|f(\xi_0)|+|f'(\xi_0)|+|f(\eta_0)|+|f'(\eta_0)|}{4}$

$\leq\dfrac{M}{2}$

故$M=0$，得证.

法二：取$\delta=\frac{1}{2}$，$\because f(x),f'(x)\in C\left[-\frac{1}{2},\frac{1}{2}\right]$，由闭区间连续函数的有界性可知：$\exists M_1,M_2>0$，

$s.t.|f(x)|<M_1,|f'(x)|<M_2,\therefore|f''(x)|\le|f(x)|+|f'(x)|<M_1+M_2$，即$|f''(x)|$有上界.

由确界原理可知，$|f''(x)|$有上确界，记为$S(S\ge 0)$.

由上确界的定义可知：$\exists x_1\in\left[-\frac{1}{2},\frac{1}{2}\right],s.t.|f''(x_1)|>S-\varepsilon,(\forall\varepsilon>0)$.

下面用反证法说明$S>0$不成立：若$S>0$，取$\varepsilon=\frac{1}{8}S>0$，则$|f''(x_1)|>\frac{7}{8}S$，

$\because f(x_1)\underline{\underline{Taylor}}f(0)+f'(0)x_1+\frac{1}{2}f''(\xi)x_1^2$，$\xi$介于0，$x_1$之间，

$f'(x_1)\underline{\underline{Lagrange}}f'(0)+x_1f''(\eta)$，$\eta$介于0，$x_1$之间，

故$\frac{7}{8}S<|f''(x_1)|\le|f(x_1)|+|f'(x_1)|=\frac{1}{2}|f''(\xi)|x_1^2+|x_1||f''(\eta)|\le\frac{5}{8}S\Rightarrow S<0$，与假设相悖，不可能，舍去.

$\because S\ge 0,\therefore S=0$即$|f''(x)|=0,\therefore f''(x)=0,\therefore f'(x)\equiv f'(0)=0\Rightarrow f(x)\equiv f(0)=0,x\in\left[-\frac{1}{2},\frac{1}{2}\right]$.

错解：考虑区间$\left(0,\frac{1}{2}\right)$，在该区间上有$|f''(x)|\le|f(x)|+|f'(x)|,x\in\left(0,\frac{1}{2}\right)$，

则$|f'(x)|=|\int_0^x f''(t)dt|\le\int_0^x|f''(t)|dt\le\int_0^x(|f(t)|+|f'(t)|)dt$

又因为$|f(x)|=\left|\int_0^x f'(t)dt\right|\le\int_0^x|f'(t)|dt,\int_0^x|f(t)|dt+\int_0^x|f'(t)|dt=|f(\theta)|x+\int_0^x|f'(t)|dt$（积分中值定理）

$=x\int_0^\theta|f'(t)|dt+\int_0^x|f'(t)|dt\le x\int_0^x|f'(t)|dt+\int_0^x|f'(t)|dt=(x+1)\int_0^x|f'(t)|dt$

记$f'(x)$在$(0，x)$上的最大值点为ε，$|f'(\varepsilon)|\le(\varepsilon+1)\int_0^\varepsilon|f'(t)|dt\le(\varepsilon+1)\varepsilon|f'(\varepsilon)|\le\frac{3}{4}|f'(\varepsilon)|$

$|f'(\varepsilon)|=0,|f'(x)|=0,(x\in\left(0,\frac{1}{2}\right)),f(x)=0$，同理可将此情况推广$(-1,1)$上.

错误原因在于使用牛顿—莱布尼茨公式时要求$f''(t)$在$t\in(0，x)$上连续，此处由已知条件无法得出.

5、设可微函数$f(x)$在$(0,+\infty)$上严格单调减少，如果当$x\in(0,+\infty)$时$0<|f(x)|<|f'(x)|$成立，

证明：当$x\in(0,1)$时，必有$x\cdot f(x)>\frac{1}{x}f(\frac{1}{x})$.

首先我们给出$f(x)<0$的证明，（由介值定理，$f(x)$在$(0,+\infty)$上不变号）

$g(x)\triangleq xf(x)-\frac{1}{x}f(\frac{1}{x}),g'(x)=xf'(x)+f(x)+\frac{1}{x^2}f(\frac{1}{x})+\frac{1}{x^3}f'(\frac{1}{x})<0$

$g(x)>g(1)=0$，原命题成立。

下面给出$f(x)>0$的证明：

$\Leftrightarrow$ 证明：$x\in(0,1),\frac{f(\frac{1}{x})}{f(x)}<x^2$ 或 $\ln\frac{f(\frac{1}{x})}{f(x)}<2\ln x$.

因$f(x)$严格递减，$f'(x)<0$，有$f'(x)=-|f'(x)|$，

$$\ln\frac{f(\frac{1}{x})}{f(x)}=\ln f(\frac{1}{x})-\ln f(x)\underline{\underline{Lagrange定理}}\frac{f'(\xi)}{f(\xi)}(\frac{1}{x}-x).$$

注意到$0<f(x)<|f'(x)|=-f'(x),\frac{f'(x)}{f(x)}<-1,\frac{1}{x}-x>0(0<x<1).$

接下来只需证：$x-\frac{1}{x}<2\ln x(0<x<1)$.【求导，显然】

故$\ln\frac{f(\frac{1}{x})}{f(x)}<x-\frac{1}{x}<2\ln x(0<x<1).$

错解：由题可知，$f'(x)<0,0<f(x)<-f'(x)$，故$\frac{-f'(x)}{f(x)}>1$，所以

$$\ln\frac{f(x)}{f(\frac{1}{x})}=-\int_x^{\frac{1}{x}}\frac{f'(t)}{f(t)}dt>\int_x^{\frac{1}{x}}dt=\frac{1}{x}-x,x\in(0,1)$$

故$\frac{f(x)}{f(\frac{1}{x})}>e^{\frac{1}{x}-x}$，又$e^{\frac{1}{x}-x}>\frac{1}{x^2}$

可得$x\cdot f(x)>\frac{1}{x}f(\frac{1}{x}),x\in(0,1).$

错误原因在于使用牛顿—莱布尼茨公式时要求$\frac{f'(t)}{f(t)}$在$t\in\left(x,\frac{1}{x}\right)$上连续，此处由已知条件无法得出.