开学考试（数学部分）

一.(填空题, 每题9分)

1.已知$\beta=\dfrac{\sqrt{5}+1}{2}$, 计算$\left[\beta^{12}\right]$($[\bullet]$为$gauss$函数).

解: $\beta=\dfrac{\sqrt{5}+1}{2}$是方程$x^2-x-1=0$的根, 故: $x^2=x+1, x^4=(x+1)^2=x^2+2x+1=3x+2,$

$x^{12}=(3x+2)^3=27x^3+54x^2+36x+8=27x(x+1)+54(x+1)+36x+8=27x^2+117x+62$

$=27(x+1)+117x+62=144x+89\Rightarrow\beta^{12}=72\sqrt{5}+161\in(321,322)$, 故: $\left[\beta^{12}\right]=321.$

注释: 考察对偶结构$\left(\dfrac{\sqrt{5}+1}{2}\right)^n-\left(\dfrac{1-\sqrt{5}}{2}\right)^n$也是一个重要思想,

分析: 熟知$Fibonacci$数列通项: $a_n=\dfrac{1}{\sqrt{5}}\left[\left(\dfrac{\sqrt{5}+1}{2}\right)^n-\left(\dfrac{1-\sqrt{5}}{2}\right)^n\right]$,

$$\left(\begin{array}{l}a_{n+2}=a_{n+1}+a_n, a_1=a_2=1, \text{则特征方程}: x^2=x+1, x_1=\dfrac{\sqrt{5}+1}{2}, x_2=\dfrac{1-\sqrt{5}}{2},\\ \text{则}: a_n=C_1\left(\dfrac{\sqrt{5}+1}{2}\right)^n+C_2\left(\dfrac{1-\sqrt{5}}{2}\right)^n, \text{代入初值条件得}: a_n=\dfrac{1}{\sqrt{5}}\left[\left(\dfrac{\sqrt{5}+1}{2}\right)^n-\left(\dfrac{1-\sqrt{5}}{2}\right)^n\right]\end{array}\right),$$

解: 令$a_n=\dfrac{1}{\sqrt{5}}\left[\left(\dfrac{\sqrt{5}+1}{2}\right)^n-\left(\dfrac{1-\sqrt{5}}{2}\right)^n\right]$, 则: $a_{12}=144=\dfrac{1}{\sqrt{5}}\left[\left(\dfrac{\sqrt{5}+1}{2}\right)^{12}-\left(\dfrac{1-\sqrt{5}}{2}\right)^{12}\right]$,

$\left(\dfrac{\sqrt{5}+1}{2}\right)^{12}=144\sqrt{5}+\left(\dfrac{1-\sqrt{5}}{2}\right)^{12}$, 注意到: $144\sqrt{5}=\sqrt{103680}\in(321.5,322)$,

$\left(\dfrac{1-\sqrt{5}}{2}\right)^{12}\approx(0.618)^{12}\approx\dfrac{1}{2^{12}}$, 故: $\left[\beta^{12}\right]=321.$□

2.已知: $\beta_1=3, \beta_{n+1}=\beta_n^2-3\beta_n+4$, 计算: $\sum\limits_{i=1}^{\infty}\dfrac{1}{\beta_i-1}$.

解: $\beta_{n+1}=\beta_n^2-3\beta_n+4\Rightarrow\beta_{n+1}-2=\beta_n^2-3\beta_n+2=(\beta_n-2)(\beta_n-1)$,

则: $\dfrac{1}{\beta_{n+1}-2}=\dfrac{1}{(\beta_n-2)(\beta_n-1)}=\dfrac{1}{\beta_n-2}-\dfrac{1}{\beta_n-1}$, 即: $\dfrac{1}{\beta_n-1}=\dfrac{1}{\beta_n-2}-\dfrac{1}{\beta_{n+1}-2}$,

$\sum\limits_{i=1}^{\infty}\dfrac{1}{\beta_i-1}=1-\lim\limits_{n\to\infty}\dfrac{1}{\beta_{n+1}-2}$,

下证: β_n单调递增趋于$+\infty$, 注意到: $\beta_{n+1}-\beta_n=\beta_n^2-4\beta_n+4=(\beta_n-2)^2>0$,

假设$\lim\limits_{n\to\infty}\beta_n$存在, 记为$A$, 两边对$n\to+\infty$取极限知: $A^2-3A+3=0$, 矛盾,

故: $\lim\limits_{n\to\infty}\beta_n=+\infty$, 即: $\sum\limits_{i=1}^{\infty}\dfrac{1}{\beta_i-1}=1.$□

3.已知:$a^2+b^2=5$,求$2a+3b$的最大值.

解:$2a+3b\le\sqrt{2^2+3^2}\cdot\sqrt{a^2+b^2}\left(cauchy\right)=\sqrt{65}.\square$

4.已知:$\begin{cases}x+\sin x\cos x-1=0\\2\cos y-2y+\pi+4=0\end{cases}$,求$\sin\left(2x-y\right)$.

解:$\begin{cases}x+\sin x\cos x-1=0\\2\cos y-2y+\pi+4=0\end{cases}\Rightarrow x+\frac{1}{2}\sin 2x-1=0,$

注意到同构式:令$2x=y-\frac{\pi}{2}$,有:$\frac{y}{2}-\frac{\pi}{4}-\frac{1}{2}\cos y-1=0,$

即:$2\cos y-2y+\pi+4=0$,注意到:$\left(2\cos y-2y+\pi+4\right)'=-2\sin y-2\le 0,$

故:$2x-y=-\frac{\pi}{2},\sin\left(2x-y\right)=-1.\square$

5.已知:$f\left(1\right)=2022,\forall n>1,n\in\mathbb{N}$,有:$\sum_{k=1}^{n}f\left(k\right)=n^2f\left(n\right)$,求:$f\left(2022\right)$.

解:$\begin{cases}\sum_{k=1}^{n}f\left(k\right)=n^2f\left(n\right)\\\sum_{k=1}^{n+1}f\left(k\right)=\left(n+1\right)^2f\left(n+1\right)\end{cases}\Rightarrow f\left(n+1\right)=\left(n+1\right)^2f\left(n+1\right)-n^2f\left(n\right),$

即:$\left(n^2+2n\right)f\left(n+1\right)=n^2f\left(n\right)\Rightarrow\frac{f\left(n+1\right)}{f\left(n\right)}=\frac{n}{n+2},$

故:$f\left(2022\right)=f\left(1\right)\prod_{k=1}^{2021}\frac{k}{k+2}=2022\cdot\frac{2}{2022\cdot 2023}=\frac{2}{2023}.\square$

6.定义$\mathbb{C}$为复数集,集合$A=\left\{z\middle|z^{18}=1,z\in\mathbb{C}\right\},B=\left\{w\middle|w^{48}=1,z\in\mathbb{C}\right\},$

求:$card\left[\left\{zw\middle|z\in A,w\in B\right\}\right]$.

解:根据复平面下的单位根分解知:$z=e^{\frac{2k\pi}{18}i}=\cos\frac{2k\pi}{18}+i\sin\frac{2k\pi}{18},w=e^{\frac{2k'\pi}{48}i}=\cos\frac{2k'\pi}{48}+i\sin\frac{2k'\pi}{48},$

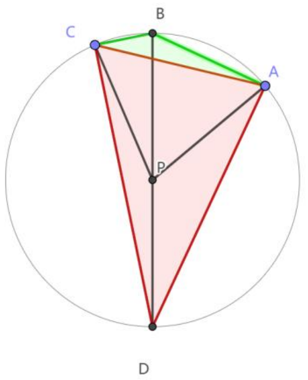
$zw=e^{\frac{2\left(8k+3k'\right)}{144}i\pi}$,注意到8与3互质,则根据裴蜀定理,$8k+3k'$可以取得任意整数值,$k,k'\in\mathbb{Z}$,

$8k+3k'\in\mathbb{Z}$,且$8k+3k'$可以取到$\left[0,143\right]$的任意值,故:zw一共有144个.$\square$

7.在一个凸四边形$ABCD$中存在点P,满足$S\Delta PAB=S\Delta PBC=S\Delta PCD=S\Delta PDA$,

$S\Delta ABC=\alpha S\Delta ADC$,求:$\alpha$.

注释:本题有误,考察如下情况,可以发现B在圆弧AC上运动时,α的值也会随之改变$\square$



8.三位数中任意两个数字之和都能被第三个数整除,求这样的三位数的个数

解:$case1$:该三位数各位数字相等,显然有9个,

$case2$:该三位数各位数字不等,有:$(1,2,3),(2,4,6),(3,6,9)$,一共$3A_3^3=18$个;

$case3$:该三位数有且仅有2位数相同,有:$(1,1,2),(2,2,4),(3,3,6),(4,4,8)$,一共$4C_3^1=12$个;

故这样的三位数有:$9+18+12=39$个.□

二.解答题(12分)

请用3种方法证明:$\left(\dfrac{\alpha+\beta+\gamma}{3}\right)^3\geq\alpha\cdot\beta\cdot\gamma\left(\alpha,\beta,\gamma\in\mathbb{R}^+\right)$.

解:$(method1)$根据基本不等式,$\dfrac{\alpha+\beta}{2}\geq\sqrt{\alpha\beta}$,

故:$\dfrac{\alpha+\beta+\gamma+\eta}{4}=\dfrac{\dfrac{\alpha+\beta}{2}+\dfrac{\gamma+\eta}{2}}{2}\geq\sqrt{\sqrt{\alpha\beta}\sqrt{\gamma\eta}}=\sqrt[4]{\alpha\cdot\beta\cdot\gamma\cdot\eta}$,

令$\eta=\dfrac{\alpha+\beta+\gamma}{3}$,有:$\dfrac{\alpha+\beta+\gamma}{3}\geq\sqrt[3]{\alpha\cdot\beta\cdot\gamma}$,证毕!

$(method2)$令$\alpha=x^3,\beta=y^3,\gamma=z^3$,则:

$$\frac{x^3+y^3+z^3}{3}-xyz=\frac{1}{3}\left(x^3+y^3+z^3-3xyz\right)=\frac{1}{3}\left\{\left[(x+y)^3-3x^2y-3xy^2\right]+z^3-3xyz\right\}$$

$$=\frac{1}{3}\left\{\left[(x+y)^3+z^3\right]-3x^2y-3xy^2-3xyz\right\}=\frac{1}{6}(x+y+z)\left[(x-y)^2+(y-z)^2+(z-x)^2\right]\geq0,$$ 证毕!

$(method3)$令$f(x)=\ln x$,则:$f'(x)=\dfrac{1}{x},f''(x)=-\dfrac{1}{x^2}<0$,

故:$f(x)$是上凸函数,根据$jensen$不等式,$\ln\dfrac{\alpha+\beta+\gamma}{3}\geq\dfrac{1}{3}(\ln\alpha+\ln\beta+\ln\gamma)$,证毕!□

注释:$AM-GM$不等式的重要证明方法之一:反向数学归纳法.

下图节选自裴礼文老师的《数学分析中的典型问题与方法》,可做了解.

☆**例 1.1.7**（平均值不等式） 任意 n 个非负实数的几何平均值小于或等于它们的算术平均值．即 $\forall a_i \geqslant 0\ (i=1,2,\cdots,n)$，恒有

$$\sqrt[n]{a_1 \cdot a_2 \cdot \cdots \cdot a_n} \leqslant \frac{a_1 + a_2 + \cdots + a_n}{n}, \tag{1}$$

且其中的等号当且仅当 $a_1 = a_2 = \cdots = a_n$ 时成立.

该定理有许多巧妙的证明方法．这里采用反向归纳法.

证 1°（证明命题对一切 $n = 2^k\ (k=1,2,\cdots)$ 成立.）首先，有

$$\sqrt{a_1 a_2} = \sqrt{\left(\frac{a_1 + a_2}{2}\right)^2 - \left(\frac{a_1 - a_2}{2}\right)^2} \leqslant \frac{a_1 + a_2}{2} \tag{2}$$

（等号当且仅当 $a_1 = a_2$ 时成立）.

其次，

$$\sqrt[4]{a_1 a_2 a_3 a_4} = \sqrt{\sqrt{a_1 a_2}\sqrt{a_3 a_4}} \leqslant \frac{\left(\frac{a_1 + a_2}{2}\right) + \left(\frac{a_3 + a_4}{2}\right)}{2} = \frac{a_1 + a_2 + a_3 + a_4}{4}$$

（利用(2)，等号当且仅当 $a_1 = a_2 = a_3 = a_4$ 时成立）.

类似，$\forall k \in \mathbf{N}$，重复上述方法 k 次，

$$\sqrt[2^k]{a_1 a_2 \cdots a_{2^k}} \leqslant \sqrt[2^{k-1}]{\frac{a_1 + a_2}{2}\,\frac{a_3 + a_4}{2} \cdots \frac{a_{2^k - 1} + a_{2^k}}{2}} \leqslant \cdots \leqslant \frac{a_1 + a_2 + \cdots + a_{2^k}}{2^k}$$

（等号当且仅当 $a_1 = a_2 = \cdots = a_{2^k}$ 时成立）.

2° 记 $A = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$，则 $nA = a_1 + a_2 + \cdots + a_n$．假设不等式对 $n+1$ 成立，则

$$A = \frac{nA + A}{n+1} = \frac{a_1 + a_2 + \cdots + a_n + A}{n+1} \geqslant \sqrt[n+1]{a_1 a_2 \cdots a_n A},$$

故 $A^{n+1} \geqslant a_1 a_2 \cdots a_n A$，$A^n \geqslant a_1 a_2 \cdots a_n$，$A \geqslant (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$．这表明不等式对 n 成立．跟 $n+1$ 时一样，等号当且仅当 $a_1 = a_2 = \cdots = a_n$ 时成立.

三.解答题$(16分)$

已知:$f(x)=\beta x-\ln x-1$,

(i):若$f(x)\geq 0$恒成立,求β的最小值;

(ii):求证:$\dfrac{1}{xe^{x}}+x+\ln x\geq 1$;

(iii):若$\alpha\left(e^{-x}+x^{2}\right)\geq x-x\ln x$恒成立,求$\alpha$的取值范围

解:(i): $f(x)$的定义域是$(0,+\infty)$, $f(x)\geq 0\Rightarrow \beta x-\ln x-1\geq 0$,

$\Rightarrow \beta\geq\dfrac{\ln x+1}{x}$,令$g(x)=\dfrac{\ln x+1}{x}$, $g'(x)=-\dfrac{\ln x}{x^{2}}$,

当$x\in(0,1)$时, $g'(x)>0$, $g(x)$单调增,当$x\in(1,+\infty)$时, $g'(x)<0$, $g(x)$单调减,

故$g_{\max}(x)=g(1)=1$,即: $\beta\geq g_{\max}(x)=1$, β的最小值为1.

证明:(ii):当$\beta=1$时,有: $\ln x\leq x-1$,

令$\dfrac{1}{xe^{x}}=t$,则: $-x-\ln x=\ln t\leq t-1=\dfrac{1}{xe^{x}}-1$,

即: $\dfrac{1}{xe^{x}}+x+\ln x\geq 1$.

解:(iii): $\alpha\left(e^{-x}+x^{2}\right)\geq x-x\ln x\Rightarrow \alpha\left(\dfrac{1}{xe^{x}}+x\right)\geq 1-\ln x$,

注意到: $\dfrac{1}{xe^{x}}+x>0$,即: $\alpha\geq\dfrac{1-\ln x}{\dfrac{1}{xe^{x}}+x}$,由$(ii)$知: $\dfrac{1-\ln x}{\dfrac{1}{xe^{x}}+x}\leq 1$,

且$xe^{x}=1$时, $\dfrac{1-\ln x}{\dfrac{1}{xe^{x}}+x}=1$,故: $\alpha\geq 1$, α的取值范围是$[1,+\infty)$.□