ASEN 5044, Fall 2017

Statistical Estimation for Dynamical Systems

Lecture 27:
The DT Linearized KF and Extended KF (EKF)

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Announcements

- Midterm 2 grading should be done very soon (sorry for delay)

- **HW 8 [due 12/6: TOMORROW]** = group assignment
  - Final project report **to be due Wed 12/20**: non-linear filtering and analysis

- Prof. Ahmed in Australia next week 12/11 - 12/15 → guest lectures
  - Ben Bercovici, Tues 12/12
  - Prof. Jay McMahon, Thurs 12/14 (last class, wrap up)

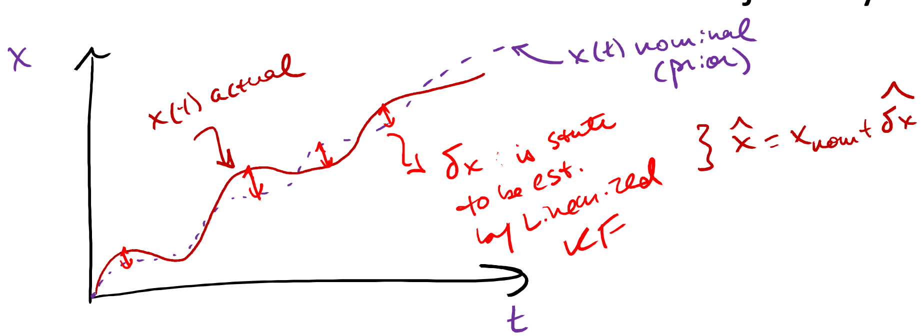
Last Time…

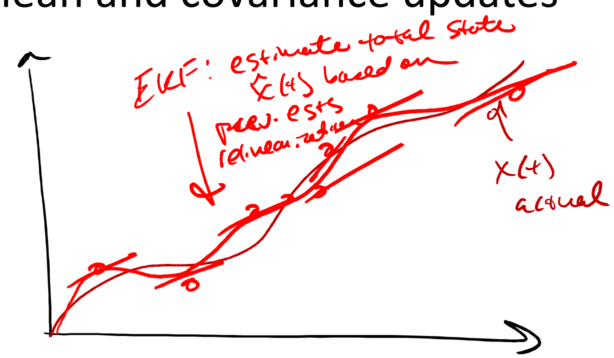
- Some useful basic facts for DT nonlinear dynamical state estimation
  - **Useful fact #1: the KF works for DT LTV systems also! (minor/obvious changes)**
  - **Useful fact #2: Calculating DT Jacobians from CT nonlinear models (for small ΔT)**

READ SIMON BOOK, CHAPTERS 13.1-13.2 (nonlinear KFs)

Today…

- Approximately optimal DT state estimators based on linearization
  - **Linearized KF:** estimate perturbations around known nominal state trajectory:
    - Uses linearization about nominal trajectory for <u>both</u> mean and covariance updates





  - **Extended KF (EKF):** estimate total state using linearization about estimated trajectory:
    - Uses full nonlinear model for predicted mean and predicted sensor measurements
    - Uses <u>linearization only to approximate matrix quantities</u> (Kalman gain and covariances)

The Linearized KF

- Suppose nonlinear system stays near a nominal trajectory x*(t) for some u*(t) with 0 process noise input (desired equilibrium, or offline-calculated solution to nonlinear ODE)

$$\dot{x}(t) = \mathcal{F}(x,u) + \Gamma\tilde{w}(t), \qquad \text{where } \tilde{w}, \tilde{v} \text{ are AWGN}$$

$\Gamma(t)$

$$y(t) = h(x) + \tilde{v}(t),$$

$\rightarrow$ Nominal state satisfies: $\dot{x}^*(t) = \mathcal{F}(x^*(t), u^*(t))$ (deterministic solution **with no** process noise)

$\rightarrow$ Now consider actual state evolution **with** process noise present:

$$x(t) = x^*(t) + \delta x(t), \qquad \delta x(t) = x(t) - x^*(t) \text{ (perturbation from } x^*(t))$$
$$u(t) = u^*(t) + \delta u(t), \qquad \delta u(t) = u(t) - u^*(t) \text{ (perturbation from } u^*(t))$$

$\rightarrow$ Plug into dynamics and measurement equation:

$$(\dot{x}^* + \dot{\delta x}) = \mathcal{F}(x^* + \delta x, u^* + \delta u) + \Gamma\tilde{w}(t),$$

$$y(t) = h(x^* + \delta x) + \tilde{v}(t),$$

Linearization via Vector Taylor Series

- Now consider Taylor Series expansion of dynamics and measurement models near x* & u*
- Using results of CT linearization from beginning of the course, we have that

for **small** δx and δu perturbations,

$$(\dot{x}^* + \dot{\delta x}) \approx \mathcal{F}(x^*, u^*) + \frac{\partial \mathcal{F}}{\partial x}|_{(x^*,u^*)}\delta x(t) \ + \ \frac{\partial \mathcal{F}}{\partial u}|_{(x^*,u^*)}\delta u(t) \ + \ \Gamma \tilde{w}(t),$$

$$y(t) \overset{\approx}{\cancel{=}} h(x^*(t)) + \frac{\partial h}{\partial x}|_{(x^*,u^*)}\delta x(t) \ + \ \tilde{v}(t),$$

$\rightarrow$ simplify using fact that $\dot{x}^*(t) = \mathcal{F}(x^*, u^*)$ and $\delta y(t) = y(t) - h(x^*)$:

$$\dot{\delta x}(t) \approx \frac{\partial \mathcal{F}}{\partial x}|_{(x^*,u^*)}\delta x(t) \ + \ \frac{\partial \mathcal{F}}{\partial u}|_{(x^*,u^*)}\delta u(t) \ + \ \Gamma \tilde{w}(t)$$

$$\delta y(t) \approx \frac{\partial h}{\partial x}|_{(x^*,u^*)}\delta x(t) + \tilde{v}(t)$$

$$\rightarrow \dot{\delta x}(t) \approx \tilde{A}|_{(x^*,u^*)}\delta x(t) + \tilde{B}|_{(x^*,u^*)}\delta u(t) + \Gamma \tilde{w}(t),$$

$$\rightarrow \delta y(t) \approx \tilde{C}|_{(x^*,u^*)}\delta x(t) + \tilde{v}(t),$$

The Linearized KF Model

- Thus:

$$
\left.
\begin{aligned}
\delta\dot{x}(t) &\approx \tilde{A}|_{(x^*,u^*)}\delta x(t) + \tilde{B}|_{(x^*,u^*)}\delta u(t) + \Gamma\tilde{w}(t),\\
\delta y(t) &\approx \tilde{C}|_{(x^*,u^*)}\delta x(t) + \tilde{v}(t),
\end{aligned}
\right\} \text{CT perturbation dynamics model}
$$

where $\tilde{A}, \tilde{B}, \tilde{C}$ are the CT Jacobian matrices evaluated at (x^*, u^*)

$\rightarrow$ now convert CT perturb. model into DT model (for suff. small ΔT):

$$\delta x(k+1) \approx \tilde{F}_k|_{nom[k]}\delta x(k) \; + \; \tilde{G}_k|_{nom[k]}\delta u(k) \; + \; \tilde{\Omega}_k w(k),$$

$$\delta y(k+1) \approx \tilde{H}_{k+1}|_{nom[k+1]}\delta x(k+1) + v(k+1)$$

where from Lec 26 we showed that:

$$\tilde{F}_k|_{nom[k]} \approx I + \Delta T \cdot \tilde{A}|_{(x^*,u^*,t=t_k)}, \qquad \tilde{G}_k|_{nom[k]} \approx \Delta T \cdot \tilde{B}|_{(x^*,u^*,t=t_k)},$$

$$\tilde{\Omega}_k|_{nom[k]} \approx \Delta T \cdot \Gamma(t)|_{(t=t_k)}, \qquad \tilde{H}_{k+1}|_{nom[k+1]} = \tilde{C}|_{(x^*,u^*,t=t_{k+1})} = \frac{\partial h}{\partial x}|_{(x^*,u^*,t=t_{k+1})}$$

The Linearized KF Algorithm

- So now we can estimate the total state as follows:

$$\hat{x}_k^+ \approx \underbrace{x^*_{k+1}}_{\text{deterministic}} + \underbrace{\hat{\delta x}^+_{k+1}}_{\text{random}}$$

where $x^*_{k+1} = x^*(t = t_{k+1})$ and $\hat{\delta x}_{k+1}$ is estimated using LTV KF for δx_{k+1} and δy_{k+1}:

Time update/prediction step for time k+1:

$$\hat{\delta x}^-_{k+1} = \tilde{F}_k \hat{\delta x}^+_k + \tilde{G}_k \delta u_k$$

$$P^-_{k+1} = \tilde{F}_k P^+_k \tilde{F}^T_k + \tilde{\Omega}_k Q_k \tilde{\Omega}^T_k$$

$$\delta u_{k+1} = u_{k+1} - u^*_{k+1}$$

Measurement update/correction step for time k+1:

$$\hat{\delta x}^+_{k+1} = \hat{\delta x}^-_{k+1} + K_{k+1}(\delta y_{k+1} - \underbrace{\tilde{H}_{k+1}\hat{\delta x}^-_{k+1}}_{\text{predicted perturbed meas. } \delta y^-_{k+1}})$$

$$P^+_{k+1} = (I - K_{k+1}\tilde{H}_{k+1})P^-_{k+1}$$

$$K_{k+1} = P^-_{k+1}\tilde{H}^T_{k+1}[\tilde{H}_{k+1}P^-_{k+1}\tilde{H}^T_{k+1} + R_{k+1}]^{-1}$$

$$\delta y_{k+1} = \underbrace{y_{k+1}}_{\substack{\text{Actual received sensor}\\ \text{measurement}\\ \text{at time k+1}}} - \underbrace{y^*_{k+1}}_{} = y_{k+1} - \underbrace{h(x^*_{k+1})}_{\substack{\text{Computed nominal sensor}\\ \text{measurement}\\ \text{at time k+1}}}$$

where $\tilde{F}_k, \tilde{G}_k, \tilde{\Omega}_k, \tilde{H}_k$ eval'd along (x^*, u^*) nom. sol'n at each time step k

Pros/Cons of the Linearized KF

- Pros:
  - easy to program
  - numerically fast
    [can compute all required Jacobians offline]
  - good for predictable systems w/ small/low process noise inputs
- Cons:
  - will break if actual system $x(t)$ trajectory deviates too far from nominal $x^*(t)$
    (i.e. if $\delta x(t)$ & $\delta u(t)$ gets too big $\Rightarrow$ $\delta\hat{x}(t)$ will have large errors, $\rightarrow$ possibly unrecoverable!)
- Alternative: what if we kept estimating total state (not just perturbation) using most recent online state estimate as prior (instead of fixed nominal trajectory)?

The Extended Kalman Filter (EKF) Algorithm

- Step 1: Initialization: start with some initial estimate of total state and covariance

$$\hat{x}^{+}(0), \;\; \hat{P}^{+}(0)$$

- Step 2: set k=0

- Step 3: **Time update/prediction step for time k+1**:

$$\hat{x}_{k+1}^{-} = f[\hat{x}_k^{+}, u_k, w_k = 0] \;\approx\; \hat{x}_k^{+} + \Delta T \cdot \mathcal{F}[\hat{x}_k^{+}, u_k], \qquad \text{(deterministic nonlinear DT dyn. fxn eval. on } \hat{x}_k^{+}\text{)}$$

$$P_{k+1}^{-} = \tilde{F}_k P_k^{+} \tilde{F}_k^{T} + \tilde{\Omega}_k Q_k \tilde{\Omega}_k^{T}, \qquad \text{(approx. predicted covar. via dyn. linearization about } \hat{x}_k^{+}\text{)}$$

$$\text{where} \qquad \tilde{F}_k|_{\hat{x}_k^{+}, u_k, w_k=0} \approx I + \Delta T \cdot \tilde{A}|_{(\hat{x}_k^{+}, u(t_k), w(t_k)=0)},$$

$$\tilde{\Omega}_k| \approx \Delta T \cdot \Gamma(t)|_{(t=t_k)},$$

The Extended Kalman Filter (EKF)

- Step 4: **Measurement update/correction step for time k+1:**

Compute:

$$\hat{y}_{k+1}^{-} = h[\hat{x}_{k+1}^{-}, v_{k+1} = 0] \quad \text{(deterministic nonlinear fxn evaluation)}$$

$$\tilde{H}_{k+1} = \frac{\partial h}{\partial x}\Big|_{\hat{x}_{k+1}^{-}} \quad \text{(meas. fxn Jacobian at predicted state)}$$

$$\tilde{e}_{y_{k+1}} = y_{k+1} - \hat{y}_{k+1}^{-} \quad \text{(nonlinear meas. innovation: actual data minus predicted)}$$

$$\tilde{K}_{k+1} = P_{k+1}^{-}\tilde{H}_{k+1}^{T}[\tilde{H}_{k+1}P_{k+1}^{-}\tilde{H}_{k+1}^{T} + R_{k+1}]^{-1} \quad \text{(approx. KF gain from meas. linearization)}$$

$$\Rightarrow \quad \hat{x}_{k+1}^{+} = \hat{x}_{k+1}^{-} + \tilde{K}_{k+1}\tilde{e}_{y_{k+1}} \quad \text{(updated total state estimate)}$$

$$P_{k+1}^{+} = (I - \tilde{K}_{k+1}\tilde{H}_{k+1})P_{k+1}^{-} \quad \text{(approx. updated covar. via linearization)}$$

- Step 5: recursion: go back to step 3 and repeat cycle for next time step…

The "1st Order" EKF Algorithm: Important Features

Useful to remember some key ideas for the EKF:

- Finding approx. Gaussian joint pdf for state and measurements from "best available guess" of total nonlinear system behavior/state at each time k
- Only use best available estimate of state at any point in time to compute required Jacobian matrices and nonlinear function evaluations at that time

  → do not need to know nominal trajectory in advance!!! (figuring it out online)

- We only need 1st order Taylor series/linearization of dynamics and measurements to get predicted covariance P^-_{k+1}, updated covariance P^+_{k+1}, and EKF gain $\tilde{K}_{k+1}$

  → **all of these matrix quantities are obtained via Jacobians**

  (similar to vanilla KF, except now matrices are time-varying and depend on $\hat{x}^+_k$!)

- **DO NOT use linearization/Jacobians to get predicted state $\hat{x}^-_{k+1}$ or measurement $\hat{y}_{k+1}$**

  → **predicted vectors come directly from integrating/evaluating nonlinear CT fxns!**