

Public-Key Encryption Theory

Yu Zhang

Harbin Institute of Technology

Cryptography, Autumn, 2021

- 1** Definitions and Securities of Public-Key Encryption
- 2** Trapdoor Permutations
- 3** Security Against Chosen-Ciphertext Attacks
- 4** Public-Key Encryption from TDP in ROM

1 Definitions and Securities of Public-Key Encryption

2 Trapdoor Permutations

3 Security Against Chosen-Ciphertext Attacks

4 Public-Key Encryption from TDP in ROM

Limitations of Private-Key Cryptography

- The key-distribution need physically meeting.
- The number of keys for U users is $\Theta(U^2)$.
- Secure communication in open system:

Solutions that are based on private-key cryptography are not sufficient to deal with the problem of secure communication in open systems where parties cannot physically meet, or where parties have transient interactions.

Needham-Schroeder Protocol for Symmetric Key

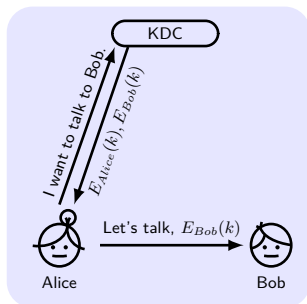
- Key Distribution Center (KDC) as Trusted Third Party (TTP), which has the shared key with Alice, and with Bob, respectively.
- $E_{Bob}(k)$ is a **ticket** to access Bob, k is **session key**.
- Used in MIT's Kerberos protocol (in Windows).

Strength:

- each one stores one key
- no updates

Weakness:

- single-point-of-failure



Merkle Puzzles (Key Exchange W/O TTP)

Alice prepares 2^{32} puzzles Puzzle_i , and sends to Bob.

$$\text{Puzzle}_i \leftarrow \text{Enc}_{(0^{96} \| p_i)}(\text{"Puzzle \#"} x_i \| k_i),$$

where Enc is 128-bit, $p_i \leftarrow \{0, 1\}^{32}$ and $x_i, k_i \leftarrow \{0, 1\}^{128}$.

Bob chooses Puzzle_j randomly, guesses p_j in 2^{32} time, obtains x_j, k_j and sends x_j to Alice.

Alice lookups puzzle with x_j , and uses k_j as secret key.

■ **Adversary** needs 2^{32+32} time.

Better Gap?

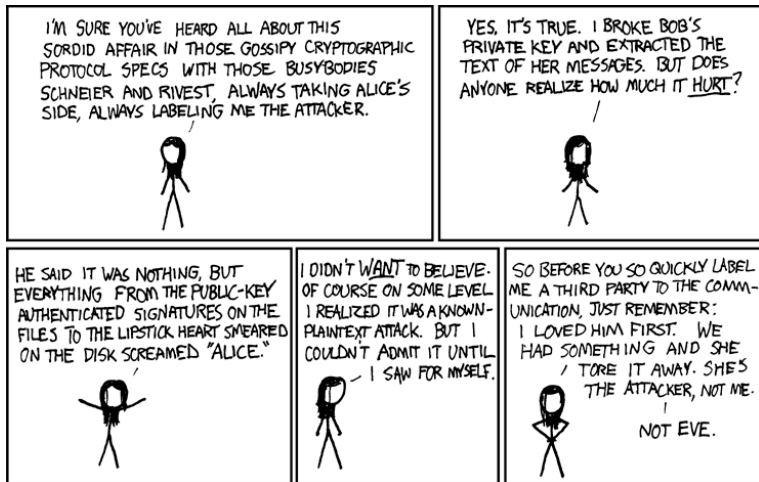
Quadratic gap is best possible if we treat cipher as a black box oracle.

Public-Key Revolution

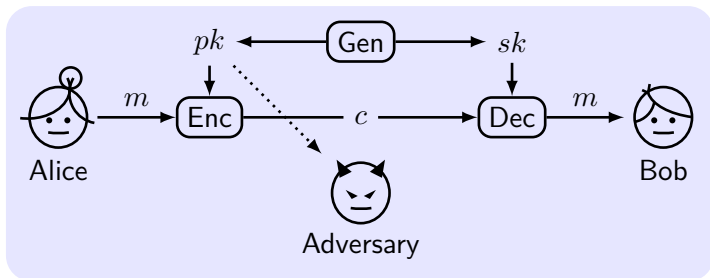
- In 1976, Whitfield Diffie and Martin Hellman published “*New Directions in Cryptography*”.
- **Asymmetric** or **public-key** encryption schemes:
 - **Public key** as the encryption key.
 - **Private key** as the decryption key.
- **Public-key primitives:**
 - Public-key encryption.
 - Digital signatures. (non-repudiation)
 - Interactive key exchange.
- **Strength:**
 - Key distribution over public channels.
 - Reduce the need to store many keys.
 - Enable security in open system.
- **Weakness:** 2 or 3 orders of magnitude slower than private-key encryptions, active attack on public key distribution.

Alice and Bob [xkcd:177]

Question: Who sends the message?



Definitions



- **Key-generation** algorithm: $(pk, sk) \leftarrow \text{Gen}$, key length $\geq n$.
- **Plaintext space** \mathcal{M} is associated with pk .
- **Encryption** algorithm: $c \leftarrow \text{Enc}_{pk}(m)$.
- **Decryption** algorithm: $m := \text{Dec}_{sk}(c)$, or outputs \perp .
- **Requirement**: $\Pr[\text{Dec}_{sk}(\text{Enc}_{pk}(m)) = m] \geq 1 - \text{negl}(n)$.

Security against Eavesdroppers = CPA

The eavesdropping indistinguishability experiment $\text{PubK}_{\mathcal{A},\Pi}^{\text{eav}}(n)$:

- 1 $(pk, sk) \leftarrow \text{Gen}(1^n)$.
- 2 \mathcal{A} is given input pk and so oracle access to $\text{Enc}_{pk}(\cdot)$, outputs m_0, m_1 of the same length.
- 3 $b \leftarrow \{0, 1\}$. $c \leftarrow \text{Enc}_{pk}(m_b)$ (challenge) is given to \mathcal{A} .
- 4 \mathcal{A} continues to have access to $\text{Enc}_{pk}(\cdot)$ and outputs b' .
- 5 If $b' = b$, \mathcal{A} succeeded $\text{PrivK}_{\mathcal{A},\Pi}^{\text{eav}} = 1$, otherwise 0.

Definition 1

Π is **CPA-secure** if \forall PPT \mathcal{A} , \exists negl such that

$$\Pr \left[\text{PubK}_{\mathcal{A},\Pi}^{\text{cpa}}(n) = 1 \right] \leq \frac{1}{2} + \text{negl}(n).$$

Security Properties of Public-Key Encryption

Symmetric ciphers are possible to encrypt a 32-bit message and obtain a 32-bit ciphertext (e.g. with the one time pad). Can the same be done with a public-key system?

Theorem 2

Q: Would a deterministic public-key encryption scheme be secure in the presence of an eavesdropper?

Proposition 3

Q: If Π is secure in the presence of an eavesdropper, is Π also CPA-secure? and is it secure for multiple encryptions?

Proposition 4

Q: Is perfectly-secret public-key encryption possible?

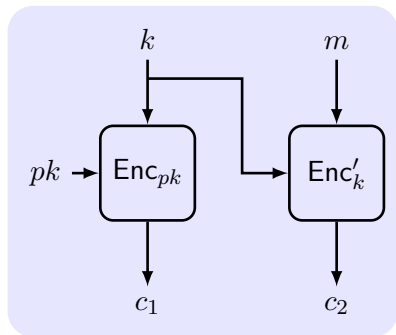
Key Size Comparison

NIST recommends the **key lengths** (in bits) with comparable security. NIST deems a 112-bit effective key length acceptable for security until the year 2030, but recommends 128-bit or higher key lengths for applications where security is required beyond then.

AES	RSA (N)/DH (p)	ECC (order q)
56	512	112
80	1024	160
112	2048	224
128	3072	256
192	7680	384
256	15360	512

Construction of Hybrid Encryption

To speed up the encryption, use private-key encryption Π' (data-encapsulation mechanism, DEM) in tandem with public-key encryption Π (key-encapsulation mechanism, KEM).



Construction 5

$\Pi^{\text{hy}} = (\text{Gen}^{\text{hy}}, \text{Enc}^{\text{hy}}, \text{Dec}^{\text{hy}})$:

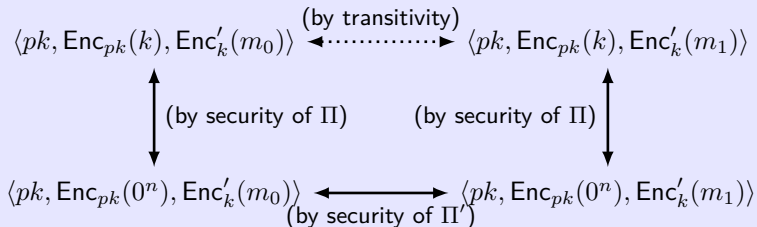
- Gen^{hy} :
 $(pk, sk) \leftarrow \text{Gen}(1^n)$.
- Enc^{hy} : pk and m .
 - 1 $k \leftarrow \{0, 1\}^n$.
 - 2 $c_1 \leftarrow \text{Enc}_{pk}(k)$,
 $c_2 \leftarrow \text{Enc}'_k(m)$.
- Dec^{hy} : sk and $\langle c_1, c_2 \rangle$.
 - 1 $k := \text{Dec}_{sk}(c_1)$.
 - 2 $m := \text{Dec}'_k(c_2)$.

Q: Is hybrid encryption a public-key enc. or a private-key enc. ?

Security of Hybrid Encryption

Theorem 6

If Π is a CPA-secure public-key encryption scheme and Π' is a private-key encryption scheme that has indistinguishable encryptions in the presence of an eavesdropper, then Π^{hy} is a CPA-secure public-key encryption scheme.



How do the following applications work by using hybrid encryption?

- **Sharing encrypted files:** In modern file systems, a user (Alice) can store encrypted files to which other users (Bob, Charlie) have read access.
- **Key escrow:** Consider a company that deploys an encrypted file system such as the one described above. The company runs a key escrow server which generates a public key pair. One day Alice is traveling, but her manager Bob needs to read one of her files to prepare for a meeting.

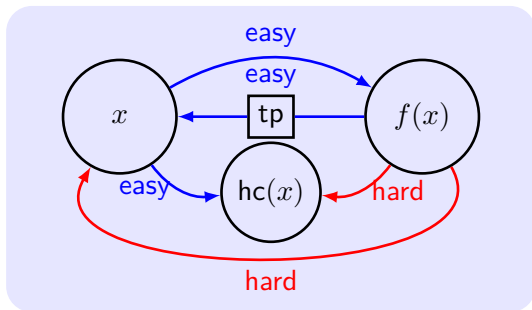
¹from BonehShuop v0.5

- 1 Definitions and Securities of Public-Key Encryption
- 2 Trapdoor Permutations**
- 3 Security Against Chosen-Ciphertext Attacks
- 4 Public-Key Encryption from TDP in ROM

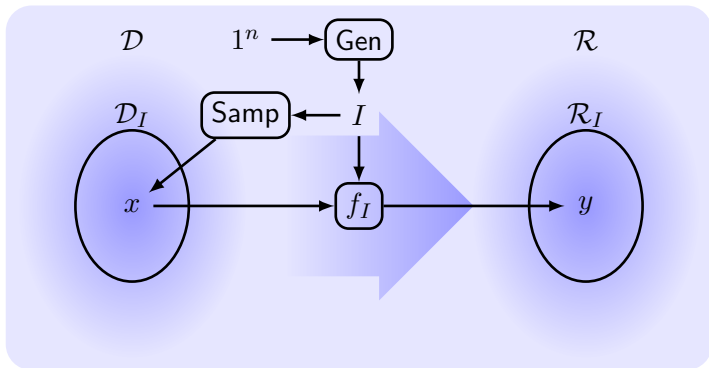
Overview

Trapdoor function: is easy to compute, yet difficult to find its inverse without special info., the “trapdoor”. (One Way Function with the “trapdoor”)

A public-key encryption scheme can be constructed from any trapdoor permutation. (*“Theory and Applications of Trapdoor Functions”*, [Yao, 1982])



Families of Functions



Definition 7

$\Pi = (\text{Gen}, \text{Samp}, f)$ is a **family of functions** if:

- 1 **Parameter-generation** algorithm: $I \leftarrow \text{Gen}(1^n)$.
- 2 **sampling** algorithm: $x \leftarrow \text{Samp}(I)$.
- 3 The deterministic **evaluation** algorithm: $y := f_I(x)$.

Definition of Families of Trapdoor Permutations

A tuple of polynomial-time algorithms $\Pi = (\text{Gen}, \text{Samp}, f, \text{Inv})$ is a **family of trapdoor permutations (TDP)** if:

- **parameter generation** algorithm Gen , on input 1^n , outputs (I, td) with $|I| \geq n$. (I, td) defines a set $\mathcal{D}_I = \mathcal{D}_{\text{td}}$.
- Gen_I outputs only I . $(\text{Gen}_I, \text{Samp}, f)$ is OWP.
- deterministic **inverting algorithm** Inv . $\forall (I, \text{td})$ and $\forall x \in \mathcal{D}_I$,

$$\text{Inv}_{\text{td}}(f_I(x)) = x.$$

Deterministic polynomial-time algorithm hc is a **hard-core predicate** of Π if \forall PPT \mathcal{A} , $\exists \text{negl}$ such that

$$\Pr[\mathcal{A}(I, f_I(x)) = \text{hc}_I(x)] \leq \frac{1}{2} + \text{negl}(n).$$

Let f with $\langle I, \text{td} \rangle$ be a TDP. Which of the following f' is also a TDP?

- $f'(x) = f(x) \parallel \text{td}$
- $f'(x \parallel x') = f(x) \parallel \text{Inv}_{\text{td}}(f(x'))$
- $f'(x \parallel x') = f(x) \parallel f(x')$
- $f'(x) = f(x) \parallel I$
- $f'(x) = \begin{cases} f(x) & \text{if } x[0, 1, 2, 3] \neq 1010 \\ x & \text{otherwise} \end{cases}$

Public-key Encryption Schemes from TDPs

Construction 8

- Gen: $(I, \text{td}) \leftarrow \widehat{\text{Gen}}$ output **public key** I and **private key** td .
- Enc: on input I and $m \in \{0, 1\}$, choose a random $x \leftarrow \mathcal{D}_I$ and output $\langle f_I(x), \text{hc}_I(x) \oplus m \rangle$.
- Dec: on input td and $\langle y, m' \rangle$, compute $x := f_I^{-1}(y)$ and output $\text{hc}_I(x) \oplus m'$.

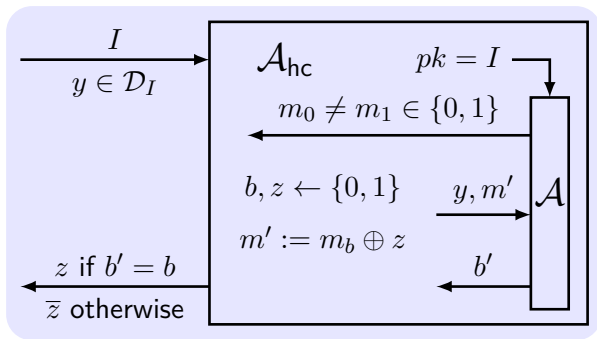
Theorem 9

If $\widehat{\Pi} = (\widehat{\text{Gen}}, f)$ is TDP, and hc is HCP for $\widehat{\Pi}$, then Construction Π is CPA-secure.

Is the following scheme is secure?

$$\text{Enc}_I(m) = f_I(m), \text{Dec}_{\text{td}}(c) = f_I^{-1}(c).$$

Idea: $\text{hc}_I(x)$ is pseudorandom. Reduce \mathcal{A}_{hc} for hc to \mathcal{A} for Π .



$$\Pr[\mathcal{A}_{\text{hc}}(I, f_I(x)) = \text{hc}_I(x)] = \frac{1}{2} \cdot (\Pr[b' = b | z = \text{hc}_I(x)] + \Pr[b' \neq b | z \neq \text{hc}_I(x)]).$$

$$\Pr[b' = b | z = \text{hc}_I(x)] = \Pr[\text{PubK}_{\mathcal{A}, \Pi}^{\text{eav}}(n) = 1] = \varepsilon(n).$$

If $z \neq \text{hc}_I(x)$, $m' = m_b \oplus \overline{\text{hc}_I(x)} = m_{\bar{b}} \oplus \text{hc}_I(x)$,
which means $m_{\bar{b}}$ is encrypted.

$$\Pr[b' = b | z \neq \text{hc}_I(x)] = \Pr[\text{PubK}_{\mathcal{A}, \Pi}^{\text{eav}}(n) = 0] = 1 - \varepsilon(n).$$

$$\Pr[b' \neq b | z \neq \text{hc}_I(x)] = \varepsilon(n).$$

$$\Pr[\mathcal{A}_{\text{hc}}(I, f_I(x)) = \text{hc}_I(x)] = \frac{1}{2} \cdot (\varepsilon(n) + \varepsilon(n)) = \varepsilon(n).$$

- 1 Definitions and Securities of Public-Key Encryption
- 2 Trapdoor Permutations
- 3 Security Against Chosen-Ciphertext Attacks**
- 4 Public-Key Encryption from TDP in ROM

Scenarios of CCA in Public-Key Setting

- 1 An adversary \mathcal{A} observes the ciphertext c sent by \mathcal{S} to \mathcal{R} .
- 2 \mathcal{A} send c' to \mathcal{R} in the name of \mathcal{S} or its own.
- 3 \mathcal{A} infer m from the decryption of c' to m' .

Scenarios

- **login to on-line bank with the password:** trial-and-error, learn info from the feedback of bank.
- **reply an e-mail with the quotation of decrypted text.**
- **malleability of ciphertexts:** e.g. doubling others' bids at an auction.

Definition of Security Against CCA/CCA2

The CCA/CCA2 indistinguishability experiment $\text{PubK}_{\mathcal{A}, \Pi}^{\text{cca}}(n)$:

- 1 $(pk, sk) \leftarrow \text{Gen}(1^n)$.
- 2 \mathcal{A} is given input pk and oracle access to $\text{Dec}_{sk}(\cdot)$, outputs m_0, m_1 of the same length.
- 3 $b \leftarrow \{0, 1\}$. $c \leftarrow \text{Enc}_{pk}(m_b)$ is given to \mathcal{A} .
- 4 \mathcal{A} have access to $\text{Dec}_{sk}(\cdot)$ except for c in **CCA2**² and outputs b' .
- 5 If $b' = b$, \mathcal{A} succeeded $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{cca}} = 1$, otherwise 0.

Definition 10

Π has **CCA/CCA2-secure** if \forall PPT \mathcal{A} , \exists negl such that

$$\Pr \left[\text{PubK}_{\mathcal{A}, \Pi}^{\text{cca}}(n) = 1 \right] \leq \frac{1}{2} + \text{negl}(n).$$

²CCA is also called Lunchtime attacks; CCA2 is also called Adaptive CCA.

Let (Gen, E, D) be CCA-secure on message space $\{0, 1\}^{128}$. Which of the following is also CCA-secure?

- $E'(pk, m) = (E(pk, m), 0^{128})$
$$D'(sk, (c_1, c_2)) = \begin{cases} D(sk, c_1) & \text{if } c_2 = 0^{128} \\ \perp & \text{otherwise} \end{cases}$$
- $E'(pk, m) = (E(pk, m), E(pk, 0^{128}))$
$$D'(sk, (c_1, c_2)) = D(sk, c_1)$$

State of the Art on CCA2-secure Encryption

- **Zero-Knowledge Proof**: complex, and impractical. (e.g., Dolev-Dwork-Naor)
- **Random Oracle** model: efficient, but not realistic (to consider CRHF as RO). (e.g., RSA-OAEP and Fujisaki-Okamoto)
- **DDH(Decisional Diffie-Hellman assumption) and UOWHF(Universal One-Way Hashs Function)**: $\times 2$ expansion in size, but security proved w/o RO or ZKP (e.g., Cramer-Shoup system).

CCA2-secure implies Plaintext-aware: an adversary cannot produce a valid ciphertext without “knowing” the plaintext.

Open problem

Constructing a CCA2-secure scheme based on RSA problem as efficient as “Textbook RSA”.

- 1 Definitions and Securities of Public-Key Encryption
- 2 Trapdoor Permutations
- 3 Security Against Chosen-Ciphertext Attacks
- 4 Public-Key Encryption from TDP in ROM**

Random Oracle Model (ROM) – Overview

- **Random oracle (RO):** a truly random function H answers every possible query with a random response.
 - **Consistent:** If H ever outputs y for an input x “on-the-fly”, then it always outputs the same answer given the same input.
 - No one “knows” the entire function H .
- **Random oracle model (ROM):** the existence of a public RO.
- **Methodology:** for constructing proven security in ROM.
 - 1 a scheme is designed and proven secure in ROM.
 - 2 Instantiate H with a hash function \hat{H} , such as SHA-1.
- No one seriously claims that a random oracle exists.³

With ROM, it is easy to achieve proven security, while keeping the efficiency by appropriate instantiation.

³There exists schemes that are proven secure in ROM but are insecure no matter how the random oracle is instantiated.

Simple Illustrations of ROM

An RO maps n_1 -bit inputs to n_2 -bit outputs.

- An RO as an OWF, experiment:

- 1 A random function H is chosen
- 2 A random $x \in \{0, 1\}^{n_1}$ is chosen, and $y := H(x)$ is evaluated
- 3 \mathcal{A} is given y , and succeeds if it outputs x' : $H(x') = y$

- An RO as a CRHF, experiment:

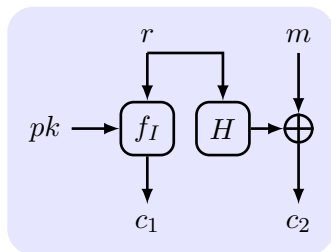
- 1 A random function H is chosen
- 2 \mathcal{A} succeeds if it outputs x, x' with $H(x) = H(x')$ but $x \neq x'$

- Constructing a PRF from an RO: $n_1 = 2n$, $n_2 = n$.

$$F_k(x) \stackrel{\text{def}}{=} H(k \| x), \quad |k| = |x| = n.$$

Security Against CPA

Idea: PubK CPA = PrivK + (Secret Key = TDP + RO)



Construction 11

- Gen: $pk = I$, $sk = td$
- Enc: $r \leftarrow \{0, 1\}^*$, output $\langle c_1 = f_I(r), c_2 = H(r) \oplus m \rangle$
- Dec: $r := f_{td}^{-1}(c_1)$, output $H(r) \oplus c_2$

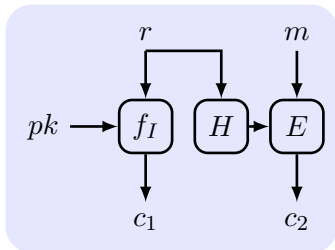
Theorem 12

If f is TPD and H is RO, Construction is CPA-secure.

H can not be replaced by PRG, since the partial info on r may be leaked by c_1 .

CCA-secure based on Private Key Encryption

Idea: PubK CCA = PrivK CCA + (Secret Key = TPD + RO).



Construction 13

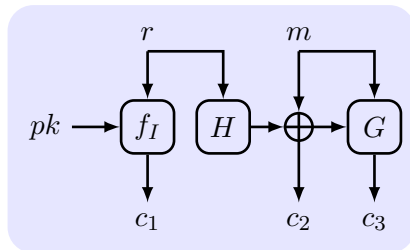
- Π' is PrivK
- Gen: $pk = I$, $sk = td$.
- Enc: $k := H(r)$, $r \leftarrow D_I$,
output $\langle c_1 = f_I(r), c_2 = \text{Enc}'_k(m) \rangle$.
- Dec: $r := f_{td}^{-1}(c_1)$,
 $k := H(r)$, output $\text{Dec}'_k(c_2)$.

Theorem 14

If f is TDP, Π' is CCA-secure, and H is RO, Construction is CCA-secure.

CCA-secure based on TPD in ROM

Idea: PubK CCA = TDP + 2 RO (one for enc, one for mac)



Construction 15

- Gen: $pk = I, sk = td$
- Enc: $r \leftarrow D_I$, output $\langle c_1 = f_I(r), c_2 = H(r) \oplus m, c_3 = G(c_2 \| m) \rangle$
- Dec: $r := f_{td}^{-1}(c_1)$, $m := H(r) \oplus c_2$. If $G(c_2 \| m) = c_3$ output m , otherwise \perp

Theorem 16

If f is TDP, G, H are ROs, Construction is CCA-secure.

Private Key Encryption vs. Public Key Encryption

	Private Key	Public Key
Secret Key	both parties	receiver
Weakest Attack	Eav	CPA
Probabilistic	CPA/CCA	always
Assumption against CPA	OWF	TDP
Assumption against CCA	OWF	TDP+RO
Efficiency	fast	slow