Public-Key Encryption Theory

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Outline

1. Definitions and Securities of Public-Key Encryption
2. Trapdoor Permutations
3. Security Against Chosen-Ciphertext Attacks
4. Public-Key Encryption from TDP in ROM

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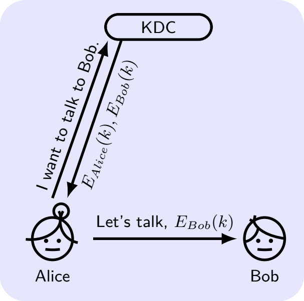
Limitations of Private-Key Cryptography

- The key-distribution need physically meeting.
- The number of keys for U users is $\Theta(U^2)$.
- Secure communication in open system:

Solutions that are based on private-key cryptography are not sufficient to deal with the problem of secure communication in open systems where parties cannot physically meet, or where parties have transient interactions.

Needham-Schroeder Protocol for Symmetric Key

- Key Distribution Center (KDC) as Trusted Third Party (TTP), which has the shared key with Alice, and with Bob, respectively.
- $E_{Bob}(k)$ is a **ticket** to access Bob, k is **session key**.
- Used in MIT's Kerberos protocol (in Windows).



Strength:

- each one stores one key
- no updates

Weakness:

- single-point-of-failure

Merkle Puzzles (Key Exchange W/O TTP)

Alice prepares 2^{32} puzzles Puzzle_i, and sends to Bob.

$$\mathsf{Puzzle}_i \leftarrow \mathsf{Enc}_{(0^{96}\|p_i)}(\text{“Puzzle \#”}x_i\|k_i),$$

where Enc is 128-bit, $p_i \leftarrow \{0,1\}^{32}$ and $x_i, k_i \leftarrow \{0,1\}^{128}$.

Bob chooses Puzzle_j randomly, guesses p_j in 2^{32} time, obtains x_j, k_j and sends x_j to Alice.

Alice lookups puzzle with x_j, and uses k_j as secret key.

- **Adversary** needs 2^{32+32} time.

Better Gap?

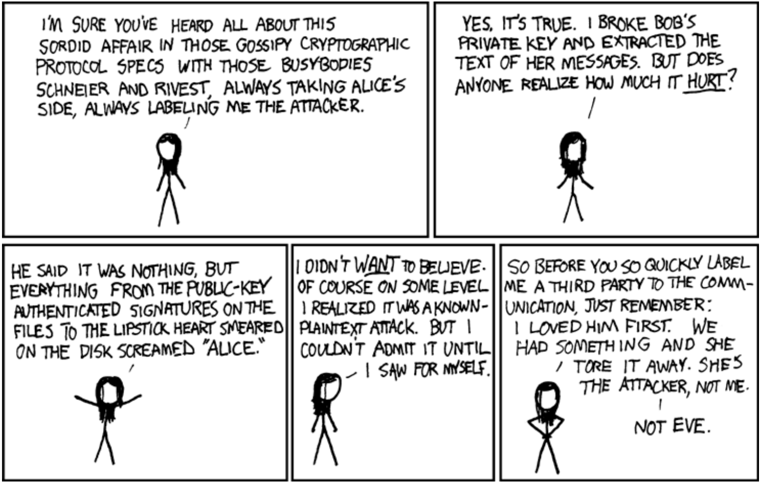
Quadratic gap is best possible if we treat cipher as a black box oracle.

Public-Key Revolution

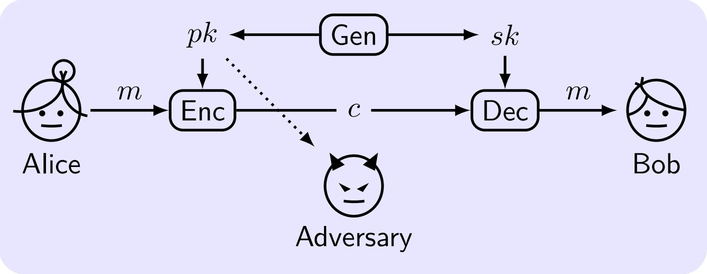
- In 1976, Whitfield Diffie and Martin Hellman published "*New Directions in Cryptography*".
- **Asymmetric** or **public-key** encryption schemes:
  - **Public key** as the encryption key.
  - **Private key** as the decryption key.
- **Public-key primitives**:
  - Public-key encryption.
  - Digital signatures. (non-repudiation)
  - Interactive key exchange.
- **Strength**:
  - Key distribution over public channels.
  - Reduce the need to store many keys.
  - Enable security in open system.
- **Weakness**: 2 or 3 orders of magnitude slower than private-key encryptions, active attack on public key distribution.

Alice and Bob [xkcd:177]

Question: Who sends the message?


I'M SURE YOU'VE HEARD ALL ABOUT THIS SORDID AFFAIR IN THOSE GOSSIPY CRYPTOGRAPHIC PROTOCOL SPECS WITH THOSE BUSYBODIES SCHNEIER AND RIVEST, ALWAYS TAKING ALICE'S SIDE, ALWAYS LABELING ME THE ATTACKER.
YES, IT'S TRUE. I BROKE BOB'S PRIVATE KEY AND EXTRACTED THE TEXT OF HER MESSAGES. BUT DOES ANYONE REALIZE HOW MUCH IT HURT?
HE SAID IT WAS NOTHING, BUT EVERYTHING FROM THE PUBLIC-KEY AUTHENTICATED SIGNATURES ON THE FILES TO THE LIPSTICK HEART SMEARED ON THE DISK SCREAMED "ALICE."
I DIDN'T WANT TO BELIEVE. OF COURSE ON SOME LEVEL I REALIZED IT WAS A KNOWN-PLAINTEXT ATTACK. BUT I COULDN'T ADMIT IT UNTIL I SAW FOR MYSELF.
SO BEFORE YOU SO QUICKLY LABEL ME A THIRD PARTY TO THE COMM-UNICATION, JUST REMEMBER: I LOVED HIM FIRST. WE HAD SOMETHING AND SHE TORE IT AWAY. SHE'S THE ATTACKER, NOT ME.
NOT EVE.

Definitions



- **Key-generation** algorithm: $(pk, sk) \leftarrow$ Gen, key length $\geq n$.
- **Plaintext space** $\mathcal{M}$ is associated with pk.
- **Encryption** algorithm: $c \leftarrow \mathsf{Enc}_{pk}(m)$.
- **Decryption** algorithm: $m := \mathsf{Dec}_{sk}(c)$, or outputs $\perp$.
- **Requirement**: $\Pr[\mathsf{Dec}_{sk}(\mathsf{Enc}_{pk}(m)) = m] \geq 1 - \mathsf{negl}(n)$.

Security against Eavesdroppers = CPA

The eavesdropping indistinguishability experiment $\mathsf{PubK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n)$:

1. $(pk, sk) \leftarrow \mathsf{Gen}(1^n)$.
2. $\mathcal{A}$ **is given input** $\mathbf{pk}$ **and so oracle access to** $\mathsf{Enc}_{\mathbf{pk}}(\cdot)$, outputs m_0, m_1 of the same length.
3. $b \leftarrow \{0,1\}$. $c \leftarrow \mathsf{Enc}_{pk}(m_b)$ (challenge) is given to $\mathcal{A}$.
4. $\mathcal{A}$ **continues to have access to** $\mathsf{Enc}_{\mathbf{pk}}(\cdot)$ and outputs b'.
5. If $b' = b$, $\mathcal{A}$ succeeded $\mathsf{PubK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1$, otherwise 0.

Definition 1

Π is **CPA-secure** if $\forall$ PPT $\mathcal{A}$, $\exists$ negl such that

$$\Pr\left[\mathsf{PubK}^{\mathsf{cpa}}_{\mathcal{A},\Pi}(n) = 1\right] \leq \frac{1}{2} + \mathsf{negl}(n).$$

Security Properties of Public-Key Encryption

Theorem 2

Q: Would a deterministic public-key encryption scheme be secure in the presence of an eavesdropper?

Proposition 3

Q: If Π is secure in the presence of an eavesdropper, is Π also CPA-secure? and is it secure for multiple encryptions?

Proposition 4

Q: Is perfectly-secret public-key encryption possible?

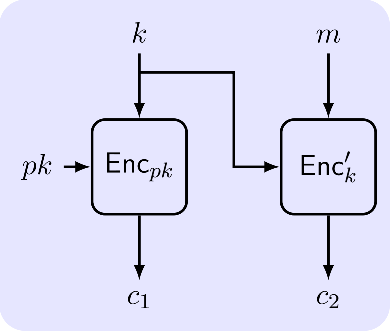
Key Size Comparison

NIST recommends the **key lengths** (in bits) with comparable security. NIST deems a 112-bit effective key length acceptable for security until the year 2030, but recommends 128-bit or higher key lengths for applications where security is required beyond then.

AES	RSA (N)/DH (p)	ECC (order q)
56	512	112
80	1024	160
112	2048	224
128	3072	256
192	7680	384
256	15360	512

Construction of Hybrid Encryption

To speed up the encryption, use private-key encryption Π' (data-encapsulation mechanism, DEM) in tandem with public-key encryption Π (key-encapsulation mechanism, KEM).



Construction 5

$\Pi^{\mathsf{hy}} = (\mathsf{Gen}^{\mathsf{hy}}, \mathsf{Enc}^{\mathsf{hy}}, \mathsf{Dec}^{\mathsf{hy}})$:

- $\mathsf{Gen}^{\mathsf{hy}}$: $(pk, sk) \leftarrow \mathsf{Gen}(1^n)$.
- $\mathsf{Enc}^{\mathsf{hy}}$: pk *and* m.
  1. $k \leftarrow \{0,1\}^n$.
  2. $c_1 \leftarrow \mathsf{Enc}_{pk}(k)$, $c_2 \leftarrow \mathsf{Enc}'_k(m)$.
- $\mathsf{Dec}^{\mathsf{hy}}$: sk *and* $\langle c_1, c_2 \rangle$.
  1. $k := \mathsf{Dec}_{sk}(c_1)$.
  2. $m := \mathsf{Dec}'_k(c_2)$.

Q: Is hybrid encryption a public-key enc. or a private-key enc. ?

Security of Hybrid Encryption

Theorem 6

If Π is a CPA-secure public-key encryption scheme and Π' is a private-key encryption scheme that has indistinguishable encryptions in the presence of an eavesdropper, then Π^{hy} is a CPA-secure public-key encryption scheme.

$$\begin{tikzcd}
\langle pk, \mathsf{Enc}_{pk}(k), \mathsf{Enc}'_k(m_0)\rangle \arrow[r, leftrightarrow, dotted, "\text{(by transitivity)}"] \arrow[d, leftrightarrow, "\text{(by security of } \Pi \text{)}"] & \langle pk, \mathsf{Enc}_{pk}(k), \mathsf{Enc}'_k(m_1)\rangle \arrow[d, leftrightarrow, "\text{(by security of } \Pi \text{)}"'] \\
\langle pk, \mathsf{Enc}_{pk}(0^n), \mathsf{Enc}'_k(m_0)\rangle \arrow[r, leftrightarrow, "\text{(by security of } \Pi' \text{)}"'] & \langle pk, \mathsf{Enc}_{pk}(0^n), \mathsf{Enc}'_k(m_1)\rangle
\end{tikzcd}$$

Applications of Hybrid Encryption[1]

How do the following applications work by using hybrid encryption?

- **Sharing encrypted files:** In modern file systems, a user (Alice) can store encrypted files to which other users (Bob, Charlie) have read access.
- **Key escrow:** Consider a company that deploys an encrypted file system such as the one described above. The company runs a key escrow server which generates a public key pair. One day Alice is traveling and unreachable, but her manager Bob needs to read one of her files to prepare for a meeting.

[1] from BonehShuop v0.5

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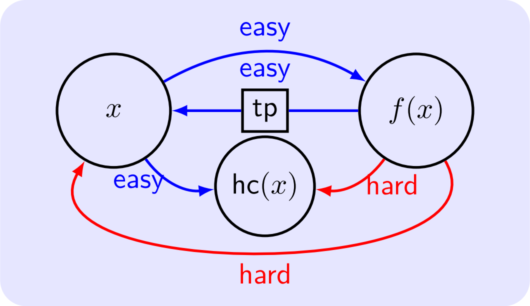
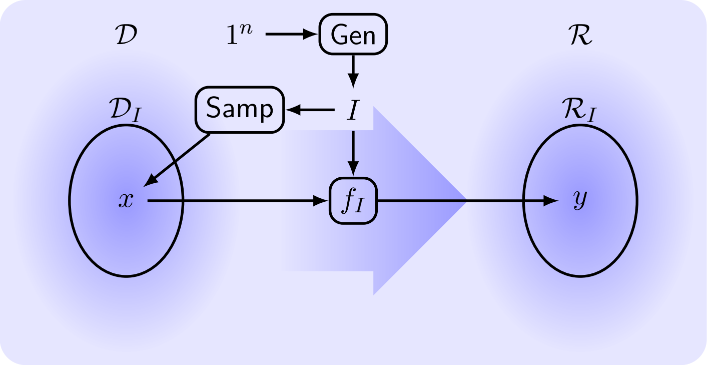
Overview

Trapdoor function: is easy to compute, yet difficult to find its inverse without special info., the "trapdoor". (One Way Function with the "trapdoor")

A public-key encryption scheme can be constructed from any trapdoor permutation. ("*Theory and Applications of Trapdoor Functions*", [Yao, 1982])



Families of Functions



Definition 7

$\Pi = (\mathsf{Gen}, \mathsf{Samp}, f)$ is a **family of functions** if:

1. **Parameter-generation** algorithm: $I \leftarrow \mathsf{Gen}(1^n)$.
2. **sampling** algorithm: $x \leftarrow \mathsf{Samp}(I)$.
3. The deterministic **evaluation** algorithm: $y := f_I(x)$.

Definition of Families of Trapdoor Permutations

A tuple of polynomial-time algorithms $\Pi = (\mathsf{Gen}, \mathsf{Samp}, f, \mathsf{Inv})$ is a **family of trapdoor permutations (TDP)** if:

- **parameter generation** algorithm Gen, on input 1^n, outputs (I, td) with $|I| \geq n$. (I, td) defines a set $\mathcal{D}_I = \mathcal{D}_{\mathsf{td}}$.
- Gen_I outputs only I. $(\mathsf{Gen}_I, \mathsf{Samp}, f)$ is OWP.
- deterministic **inverting algorithm** Inv. $\forall (I, \mathsf{td})$ and $\forall x \in \mathcal{D}_I$,

$$\mathsf{Inv}_{\mathsf{td}}(f_I(x)) = x.$$

Deterministic polynomial-time algorithm hc is a **hard-core predicate** of Π if $\forall$ PPT $\mathcal{A}$, $\exists$ negl such that

$$\Pr[\mathcal{A}(I, f_I(x)) = \mathsf{hc}_I(x)] \leq \frac{1}{2} + \mathsf{negl}(n).$$

Examples

Let f with $< I, \mathrm{td} >$ be a TDP. Which of the following f' is also a TDP?

- $f'(x) = f(x) \| \mathsf{td}$
- $f'(x) = f(x) \| I$
- $f'(x \| x') = f(x) \| \mathsf{Inv}_{\mathsf{td}}(f(x'))$
- $f'(x \| x') = f(x) \| f(x')$
- $f'(x) = \begin{cases} f(x) & \text{if } x[0,1,2,3] \neq 1010 \\ x & \text{otherwise} \end{cases}$

Is the following public-key encryption scheme from any TDP is secure?

$\mathsf{Enc}_I(m) = f_I(m)$, $\mathsf{Dec}_{\mathsf{td}}(c) = f_I^{-1}(c)$.

Public-key Encryption Schemes from TDPs

Construction 8

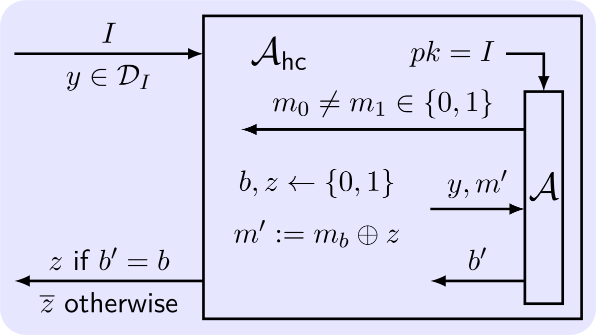
- Gen*:* $(I, \mathsf{td}) \leftarrow \widehat{Gen}$ *output* ***public key*** I *and* ***private key*** td*.*
- Enc*: on input* I *and* $m \in \{0,1\}$*, choose a random* $x \leftarrow \mathcal{D}_I$ *and output* $\langle f_I(x), \mathsf{hc}_I(x) \oplus m \rangle$*.*
- Dec*: on input* td *and* $\langle y, m' \rangle$*, compute* $x := f_I^{-1}(y)$ *and output* $\mathsf{hc}_I(x) \oplus m'$*.*

Theorem 9

If $\widehat{\Pi} = (\widehat{Gen}, f)$ *is TDP, and* hc *is HCP for* $\widehat{\Pi}$*, then Construction* Π *is CPA-secure.*

Proof

Idea: $\mathsf{hc}_I(x)$ is pseudorandom. Reduce $\mathcal{A}_{\mathsf{hc}}$ for hc to $\mathcal{A}$ for Π.



$$\Pr[\mathcal{A}_{\mathsf{hc}}(I, f_I(x)) = \mathsf{hc}_I(x)] =$$

$$\frac{1}{2} \cdot (\Pr[b' = b | z = \mathsf{hc}_I(x)] + \Pr[b' \neq b | z \neq \mathsf{hc}_I(x)]).$$

Proof (Cont.)

$$\Pr[b' = b | z = \mathsf{hc}_I(x)] = \Pr[\mathsf{PubK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1] = \frac{1}{2} + \varepsilon(n).$$

If $z \neq \mathsf{hc}_I(x)$, $m' = m_b \oplus \overline{\mathsf{hc}}_I(x) = m_{\overline{b}} \oplus \mathsf{hc}_I(x)$,
which means $m_{\overline{b}}$ is encrypted.

$$\Pr[b' = b | z \neq \mathsf{hc}_I(x)] = \Pr[\mathsf{PubK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 0] = \frac{1}{2} - \varepsilon(n).$$

$$\Pr[b' \neq b | z \neq \mathsf{hc}_I(x)] = \frac{1}{2} + \varepsilon(n).$$

$$\Pr[\mathcal{A}_{\mathsf{hc}}(I, f_I(x)) = \mathsf{hc}_I(x)] = \frac{1}{2} + \varepsilon(n).$$

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Scenarios of CCA in Public-Key Setting

1. An adversary $\mathcal{A}$ observes the ciphertext c sent by $\mathcal{S}$ to $\mathcal{R}$.
2. $\mathcal{A}$ send c' to $\mathcal{R}$ in the name of $\mathcal{S}$ or its own.
3. $\mathcal{A}$ infer m from the decryption of c' to m'.

Scenarios

- **login to on-line bank with the password**: trial-and-error, learn info from the feedback of bank.
- **reply an e-mail with the quotation of decrypted text**.
- **malleability of ciphertexts**: e.g. doubling others' bids at an auction.

Definition of Security Against CCA/CCA2

The CCA/CCA2 indistinguishability experiment $\mathsf{PubK}^{\mathsf{cca}}_{\mathcal{A},\Pi}(n)$:

1. $(pk, sk) \leftarrow \mathsf{Gen}(1^n)$.
2. $\mathcal{A}$ **is given input** pk **and oracle access to** $\mathsf{Dec}_{sk}(\cdot)$, outputs m_0, m_1 of the same length.
3. $b \leftarrow \{0,1\}$. $c \leftarrow \mathsf{Enc}_{pk}(m_b)$ is given to $\mathcal{A}$.
4. $\mathcal{A}$ **have access to** $\mathsf{Dec}_{sk}(\cdot)$ **except for** c **in CCA2**[2] and outputs b'.
5. If $b' = b$, $\mathcal{A}$ succeeded $\mathsf{PubK}^{\mathsf{cca}}_{\mathcal{A},\Pi} = 1$, otherwise 0.

Definition 10

Π has **CCA/CCA2-secure** if $\forall$ PPT $\mathcal{A}$, $\exists$ negl such that

$$\Pr\left[\mathsf{PubK}^{\mathsf{cca}}_{\mathcal{A},\Pi}(n) = 1\right] \leq \frac{1}{2} + \mathsf{negl}(n).$$

[2] CCA is also called Lunchtime attacks; CCA2 is also called Adaptive CCA.

Examples

Let (Gen, E, D) be CCA-secure on message space $\{0,1\}^{128}$. Which of the following is also CCA-secure?

- $E'(pk, m) = (E(pk, m), 0^{128})$
  $$D'(sk, (c_1, c_2)) = \begin{cases} D(sk, c_1) & \text{if } c_2 = 0^{128} \\ \perp & \text{otherwise} \end{cases}$$
- $E'(pk, m) = (E(pk, m), E(pk, 0^{128}))$
  $D'(sk, (c_1, c_2)) = D(sk, c_1)$

State of the Art on CCA2-secure Encryption

- **Zero-Knowledge Proof**: complex, and impractical. (e.g., Dolev-Dwork-Naor)
- **Random Oracle** model: efficient, but not realistic (to consider CRHF as RO). (e.g., RSA-OAEP and Fujisaki-Okamoto)
- **DDH(Decisional Diffie-Hellman assumption) and UOWHF(Universal One-Way Hashs Function)**: x2 expansion in size, but security proved w/o RO or ZKP (e.g., Cramer-Shoup system).

CCA2-secure implies Plaintext-aware: an adversary cannot produce a valid ciphertext without "knowing" the plaintext.

Open problem

Constructing a CCA2-secure scheme based on RSA problem as efficient as "Textbook RSA".

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Random Oracle Model (ROM) – Overview

- **Random oracle (RO)**: a truly random function H answers every possible query with a random response.
  - **Consistent**: If H ever outputs y for an input x "on-the-fly", then it always outputs the same answer given the same input.
  - No one "knows" the entire function H.
- **Random oracle model (ROM)**: the existence of a public RO.
- **Methodology**: for constructing proven security in ROM.
  1. a scheme is designed and proven secure in ROM.
  2. Instantiate H with a hash function $\hat{H}$, such as SHA-1.
- No one seriously claims that a random oracle exists.[3]

With ROM, it is easy to achieve proven security, while keeping the efficiency by appropriate instantiation.

[3] There exists schemes that are proven secure in ROM but are insecure no matter how the random oracle is instantiated.

Simple Illustrations of ROM

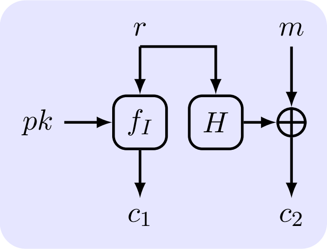
An RO maps n_1-bit inputs to n_2-bit outputs.

- An RO as an OWF, experiment:
  1. A random function H is chosen
  2. A random $x \in \{0,1\}^{n_1}$ is chosen, and $y := H(x)$ is evaluated
  3. $\mathcal{A}$ is given y, and succeeds if it outputs x': $H(x') = y$
- An RO as a CRHF, experiment:
  1. A random function H is chosen
  2. $\mathcal{A}$ succeeds if it outputs x, x' with $H(x) = H(x')$ but $x \neq x'$
- Constructing a PRF from an RO: $n_1 = 2n$, $n_2 = n$.

  $F_k(x) \overset{\text{def}}{=} H(k \| x), \;\; |k| = |x| = n.$

Security Against CPA

Idea: PubK CPA = PrivK + (Secret Key = TDP + RO)



Construction 11

- Gen*: $pk = I$, $sk =$ td*
- Enc*: $r \leftarrow \{0,1\}^*$, output $\langle c_1 = f_I(r), c_2 = H(r) \oplus m \rangle$*
- Dec*: $r := f_{\textsf{td}}^{-1}(c_1)$, output $H(r) \oplus c_2$*

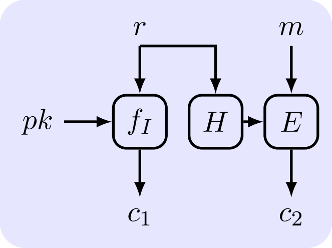
Theorem 12

If f is TPD and H is RO, Construction is CPA-secure.

H can not be replaced by PRG, since the partial info on r may be leaked by c_1.

CCA-secure based on Private Key Encryption

Idea: PubK CCA = PrivK CCA + (Secret Key = TPD + RO).



Construction 13

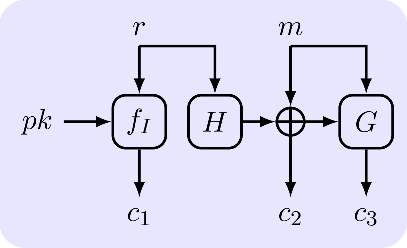
- Π' *is PrivK*
- Gen*:* $pk = I$*,* $sk =$ td*.*
- Enc*:* $k := H(r), r \leftarrow D_I$*, output* $\langle c_1 = f_I(r), c_2 =$ $\text{Enc}'_k(m)\rangle$*.*
- Dec*:* $r := f_{\text{td}}^{-1}(c_1)$*,* $k := H(r)$*, output* $\text{Dec}'_k(c_2)$*.*

Theorem 14

If f is TDP, Π' is CCA-secure, and H is RO, Construction is CCA-secure.

CCA-secure based on TPD in ROM

Idea: PubK CCA = TDP + 2 RO (one for enc, one for mac)



Construction 15

- Gen: $pk = I$, $sk = \mathsf{td}$
- Enc: $r \leftarrow D_I$, *output* $\langle c_1 = f_I(r), c_2 = H(r) \oplus m, c_3 = G(c_2 \| m)\rangle$
- Dec: $r := f_{\mathsf{td}}^{-1}(c_1)$, $m := H(r) \oplus c_2$. *If* $G(c_2 \| m) = c_3$ *output* m, *otherwise* $\perp$

Theorem 16

If f is TDP, G, H are ROs, Construction is CCA-secure.

Private Key Encryption vs. Public Key Encryption

	Private Key	**Public Key**
Secret Key	both parties	receiver
Weakest Attack	Eav	CPA
Probabilistic	CPA/CCA	always
Assumption against CPA	OWF	TDP
Assumption against CCA	OWF	TDP+RO
Efficiency	fast	slow