

Message Authentication Codes and Collision-Resistant Hash Functions

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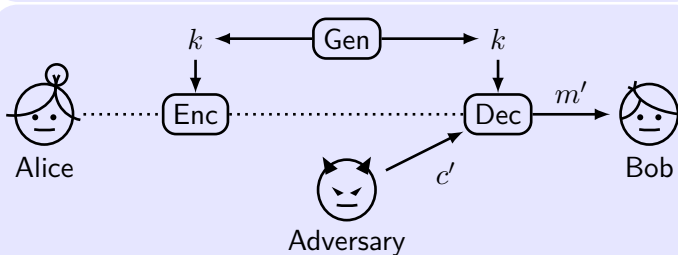
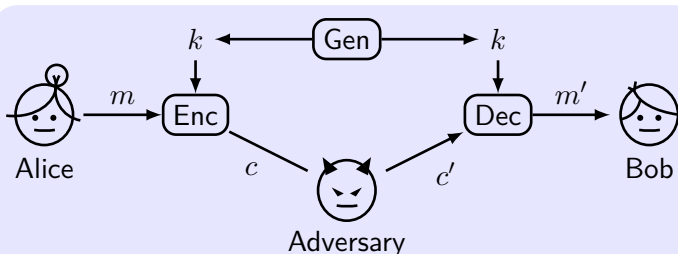
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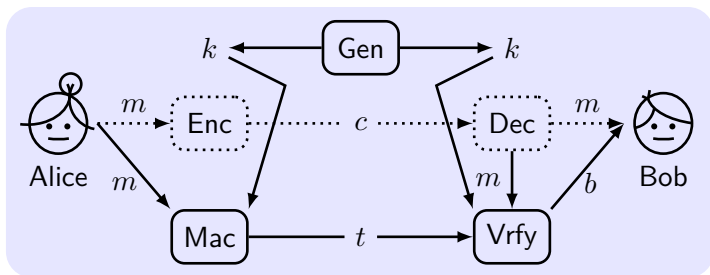
- 1** Message Authentication Codes (MAC) – Definitions
- 2** Constructing Secure MAC
- 3** CBC-MAC
- 4** Collision-Resistant Hash Functions
- 5** Hash-based MAC
- 6** Information-Theoretic MACs

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Integrity and Authentication



The Syntax of MAC



- key k , tag t , a bit b means valid if $b = 1$; invalid if $b = 0$.
- **Key-generation** algorithm $k \leftarrow \text{Gen}(1^n)$, $|k| \geq n$.
- **Tag-generation** algorithm $t \leftarrow \text{Mac}_k(m)$.
- **Verification** algorithm $b := \text{Vrfy}_k(m, t)$.
- **Message authentication code**: $\Pi = (\text{Gen}, \text{Mac}, \text{Vrfy})$.
- **Basic correctness requirement**: $\text{Vrfy}_k(m, \text{Mac}_k(m)) = 1$.

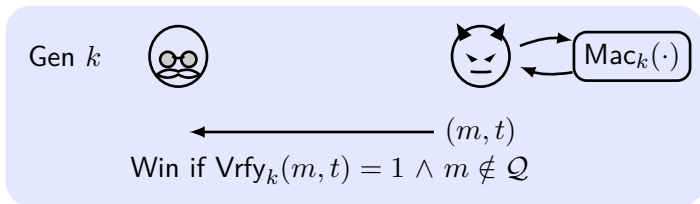
- **Intuition:** No adversary should be able to generate a **valid** tag on any “**new**” message¹ that was not previously sent.
- **Replay attack:** Copy a message and tag previously sent. (excluded by only considering “**new**” message)
 - Sequence numbers: receiver must store the previous ones.
 - Time-Stamps: sender/receiver maintain synchronized clocks.
- **Existential unforgeability:** **Not** be able to forge a valid tag on **any** message.
 - **Existential forgery:** *at least one* message.
 - **Selective forgery:** message chosen *prior* to the attack.
 - **Universal forgery:** *any* given message.
- **Adaptive chosen-message attack (CMA):** be able to obtain tags on *any* message chosen adaptively *during* its attack.

¹A stronger requirement is concerning *new message/tag pair*.

Definition of MAC Security

The message authentication experiment $\text{Macforge}_{\mathcal{A}, \Pi}(n)$:

- 1 $k \leftarrow \text{Gen}(1^n)$.
- 2 \mathcal{A} is given input 1^n and oracle access to $\text{Mac}_k(\cdot)$, and outputs (m, t) . \mathcal{Q} is the set of queries to its oracle.
- 3 $\text{Macforge}_{\mathcal{A}, \Pi}(n) = 1 \iff \text{Vrfy}_k(m, t) = 1 \wedge m \notin \mathcal{Q}$.



Definition 1

A MAC Π is **existentially unforgeable under an adaptive CMA** if \forall PPT \mathcal{A} , \exists negl such that: $\Pr[\text{Macforge}_{\mathcal{A}, \Pi}(n) = 1] \leq \text{negl}(n)$.

The 802.11b Insecure MAC²

Consider a variant of WiFi encryption in 802.11b WEP (Wired Equivalent Privacy). Let F be a PRF with a 32-bit length output. Let CRC32 be an error-detecting code outputting a 32-bit string. Define the following MAC scheme:

$$S(k, m) := (r \leftarrow \{0, 1\}^n, t \leftarrow F(k, r) \oplus \text{CRC32}(m))$$

$$V(k, m, (r, t)) := 1 \quad \text{if} \quad t = F(k, r) \oplus \text{CRC32}(m)$$

- Different messages may have the same CRC32 output.
- Attacker can learn $F(k, r)$ from a valid tag, and then output $(m', (r, F(k, r) \oplus \text{CRC32}(m')))$.

²from BonehShoup v0.5 p.234

Suppose $\langle S, V \rangle$ are CMA-secure, are $\langle S', V' \rangle$ secure?

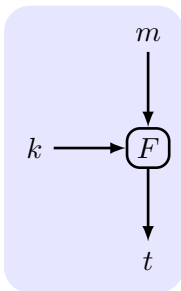
- $S'_k(m) = (S_k(m), m)$, $V'_k(m, (t_1, t_2)) = V_k(m, t_1) \wedge t_2 = m$
- $S'_{k_1, k_2}(m) = (S_{k_1}(m), S_{k_2}(m))$
 $V'_{k_1, k_2}(m, (t_1, t_2)) = V_{k_1}(m, t_1) \wedge V_{k_2}(m, t_2)$
- $S'_k(m) = (S_k(m), S_k(m))$
 $V'_k(m, (t_1, t_2)) = \begin{cases} V_k(m, t_1) & \text{if } t_1 = t_2 \\ 0 & \text{otherwise} \end{cases}$
- $S'_k(m) = (S_k(m), S_k(0^n))$
 $V'_k(m, (t_1, t_2)) = V_k(m, t_1) \wedge V_k(0^n, t_2)$
- $S'_k(m) = S_k(m)$, $V'_k(m, t) = \begin{cases} V_k(m, t) & \text{if } m \neq 0^n \\ 1 & \text{otherwise} \end{cases}$
- $S'_k(m) = S_k(m)$ without the LSB
 $V'_k(m, t) = V_k(m, t \parallel 0) \vee V_k(m, t \parallel 1)$

- **Browser cookie:** includes a MAC tag of the user's account generated by the web server, against the attacker forging others' cookies.
- **TCP SYN cookie:** The server's initial sequence number includes a MAC tag of the client's IP address, port number and some other values generated by the server, against "half-open" DDoS attack in TCP handshake.
- **Timed one-time passwords:** $p = \text{Mac}_k(T)$, where k is the key shared between the user and the service provider, T is the current date + time (usually rounded to the nearest 30 seconds.) The attacker who learns the current p can not gain access to your account in the future.

³from The Joy of Cryptography

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Constructing Secure MAC



Construction 2

- F is PRF. $|m| = n$.
- $\text{Gen}(1^n): k \leftarrow \{0, 1\}^n$ u.a.r.
- $\text{Mac}_k(m): t := F_k(m)$.
- $\text{Vrfy}_k(m, t): 1 \iff t \stackrel{?}{=} F_k(m)$.

Theorem 3

If F is a PRF, Construction is a secure fixed-length MAC.

Lemma 4

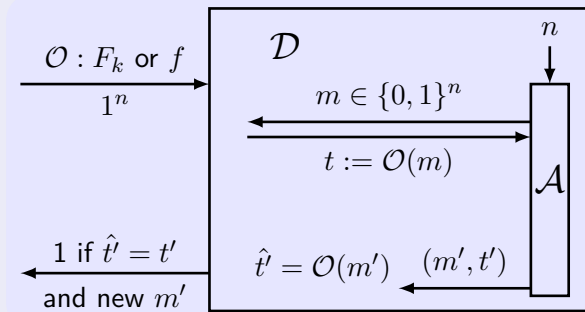
Truncating MACs based on PRFs: If F is a PRF, so is $F_k^t(m) = F_k(m)[1, \dots, t]$.

Proof of Secure MAC from PRF

Idea: Show Π is secure unless F_k is not PRF by reduction.

Proof.

D distinguishes F_k ; \mathcal{A} attacks Π .



Proof of Secure MAC from PRF (Cont.)

Proof.

(1) If true random f is used, $t = f(m)$ is uniformly distributed.

$$\Pr[D^{f(\cdot)}(1^n) = 1] = \Pr[\text{Macforge}_{\mathcal{A}, \tilde{\Pi}}(n) = 1] \leq 2^{-n}.$$

(2) If F_k is used, conduct the experiment $\text{Macforge}_{\mathcal{A}, \Pi}(n)$.

$$\Pr[D^{F_k(\cdot)}(1^n) = 1] = \Pr[\text{Macforge}_{\mathcal{A}, \Pi}(n) = 1] = \varepsilon(n).$$

According to the definition of PRF,

$$\left| \Pr[D^{F_k(\cdot)}(1^n) = 1] - \Pr[D^{f(\cdot)}(1^n) = 1] \right| \geq \varepsilon(n) - 2^{-n}.$$



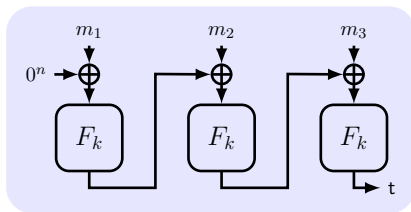
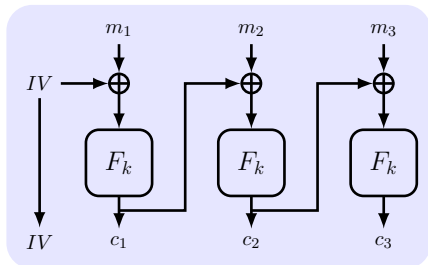
Extension to Variable-Length Messages

For variable-length messages, would the following suggestions be secure?

- **Suggestion 1:** XOR all the blocks together and authenticate the result. $t := \text{Mac}'_k(\oplus_i m_i)$.
- **Suggestion 2:** Authenticate each block separately. $t_i := \text{Mac}'_k(m_i)$.
- **Suggestion 3:** Authenticate each block along with a sequence number. $t_i := \text{Mac}'_k(i \| m_i)$.

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Constructing Fixed-Length CBC-MAC



Modify CBC encryption into CBC-MAC:

- Change random IV to encrypted fixed 0^n , *otherwise*:
Q: query m_1 and get (IV, t_1) ; output $m'_1 = IV' \oplus IV \oplus m_1$ and $t'_1 = \underline{\hspace{1cm}}$.
- Tag only includes the output of the final block, *otherwise*:
Q: query m_i and get t_i ; output $m'_i = t'_{i-1} \oplus t_{i-1} \oplus m_i$ and $t'_i = \underline{\hspace{1cm}}$.

Constructing Fixed-Length CBC-MAC (Cont.)

Construction 5

- a PRF F and a length function ℓ . $|m| = \ell(n) \cdot n$. $\ell = \ell(n)$.
 $m = m_1, \dots, m_\ell$.
- $\text{Gen}(1^n)$: $k \leftarrow \{0, 1\}^n$ u.a.r.
- $\text{Mac}_k(m)$: $t_i := F_k(t_{i-1} \oplus m_i)$, $t_0 = 0^n$. Output $t = t_\ell$.
- $\text{Vrfy}_k(m, t)$: $1 \iff t \stackrel{?}{=} \text{Mac}_k(m)$.

Theorem 6

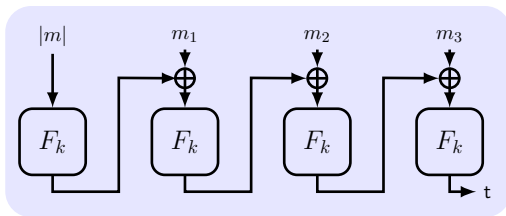
If F is a PRF, Construction is a secure **fixed-length** MAC.

Not for **variable-length** message:

Q: For one-block message m with tag t , adversary can append a block ____ and output tag t .

Secure Variable-Length MAC

- **Input-length key separation:** $k_\ell := F_k(\ell)$, use k_ℓ for CBC-MAC.
- **Length-prepend:** Prepend m with $|m|$, then use CBC-MAC.



- **Encrypt last block (ECBC-MAC):** Use two keys k_1, k_2 . Get t with k_1 by CBC-MAC, then output $\hat{t} := F_{k_2}(t)$.

Q: To authenticate a voice stream, which approach do you prefer?

MAC Padding

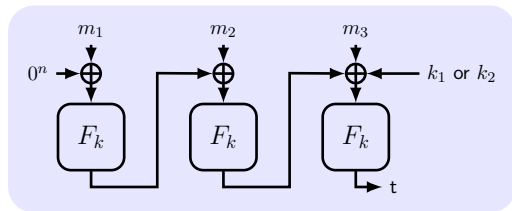
Padding must be invertible!

$$m_0 \neq m_1 \Rightarrow \text{pad}(m_0) \neq \text{pad}(m_1).$$

ISO: pad with “100...00”. Add dummy block if needed.

Q: What if no dummy block?

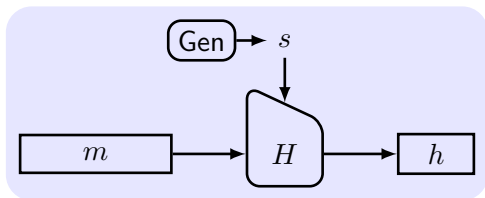
CMAC (Cipher-based MAC from NIST): $\text{key} = (k, k_1, k_2)$.



- No final encryption: extension attack thwarted by keyed XOR.
- No dummy block: ambiguity resolved by use of k_1 or k_2 .

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Defining Hash Function



Definition 7

A **hash function (compression function)** is a pair of PPT algorithms (Gen, H) satisfying:

- a key $s \leftarrow \text{Gen}(1^n)$, s is **not kept secret**.
- $H^s(x) \in \{0, 1\}^{\ell(n)}$, where $x \in \{0, 1\}^*$ and ℓ is polynomial.

If H^s is defined only for $x \in \{0, 1\}^{\ell'(n)}$ and $\ell'(n) > \ell(n)$, then (Gen, H) is a **fixed-length** hash function.

Defining Collision Resistance

- **Collision** in H : $x \neq x'$ and $H(x) = H(x')$.
- **Collision Resistance**: infeasible for any PPT alg. to find.

The collision-finding experiment $\text{Hashcoll}_{\mathcal{A}, \Pi}(n)$:

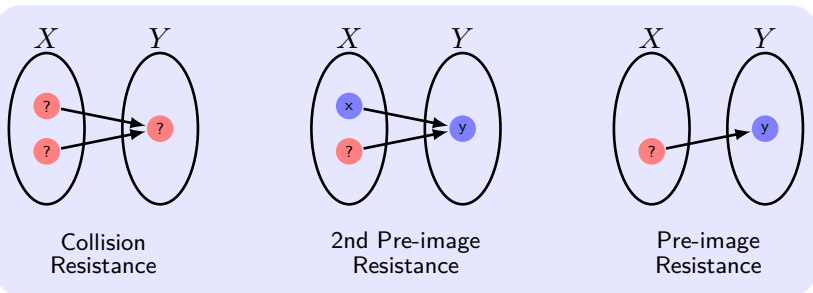
- 1 $s \leftarrow \text{Gen}(1^n)$.
- 2 \mathcal{A} is given s and outputs x, x' .
- 3 $\text{Hashcoll}_{\mathcal{A}, \Pi}(n) = 1 \iff x \neq x' \wedge H^s(x) = H^s(x')$.

Definition 8

$\Pi (\text{Gen}, H^s)$ is **collision resistant** if \forall PPT \mathcal{A} , $\exists \text{negl}$ such that

$$\Pr[\text{Hashcoll}_{\mathcal{A}, \Pi}(n) = 1] \leq \text{negl}(n).$$

Weaker Notions of Security for Hash Functions



- **Collision resistance:** It is hard to find $(x, x'), x' \neq x$ such that $H(x) = H(x')$.
- **Second pre-image resistance:** Given s and x , it is hard to find $x' \neq x$ such that $H^s(x') = H^s(x)$.
- **Pre-image resistance:** Given s and $y = H^s(x)$, it is hard to find x' such that $H^s(x') = y$.

H is CRHF. Is H' CRHF?

- $H'(m) = H(m) \oplus H(m \oplus 1^{|m|})$
- $H'(m) = H(m) \| H(0)$
- $H'(m) = H(m) \| H(m)$
- $H'(m) = H(m) \oplus H(m)$
- $H'(m) = H(m[0, \dots, |m| - 2])$
- $H'(m) = H(m \| 0)$
- $H'(m) = H(m)[0, \dots, |m| - 1]$

Applications of Hash Functions

- **Fingerprinting and Deduplication:** $H(\text{alargefile})$ for virus fingerprinting, deduplication, P2P file sharing
- **Merkle Trees:**
 $H(H(H(\text{file1}), H(\text{file2})), H(H(\text{file3}), H(\text{file4})))$
fingerprinting multiple files / parts of a file
- **Password Hashing:** $(\text{salt}, H(\text{salt}, \text{pw}))$ mitigating the risk of leaking password stored in the clear
- **Key Derivation:** $H(\text{secret})$ deriving a key from a high-entropy (but not necessarily uniform) shared secret
- **Commitment Schemes:** $H(\text{info})$ hiding the committed info; binding the commitment to a info

The “Birthday” Problem

The “Birthday” Problem

Q: “What size group of people do we need to take such that with probability $1/2$ some pair of people share a birthday?” **A:** 23.

Lemma 9

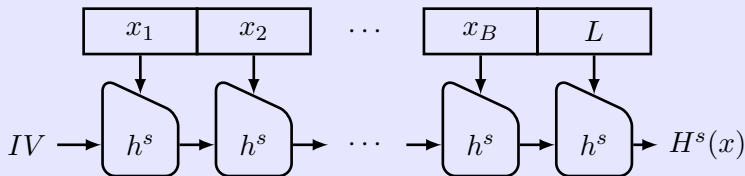
Choose q elements u.a.r from a set of size N , the probability that $\exists i \neq j$ with $y_i = y_j$ is $\text{coll}(q, N)$, then $\text{coll}(q, N) \leq \frac{q^2}{2N}$.

How many different meaningful sentences are below?

It is **hard/difficult/challenging/impossible** to **imagine/believe** that we will **find/locate/hire** another **employee/person** having similar **abilities/skills/character** as Alice. She has done a **great/super** job.

A principle: The length of hash value should be long enough.

The Merkle-Damgård Transform



Construction 10

Construct **variable-length** CRHF (Gen, H) from fixed-length (Gen, h) (2ℓ bits $\rightarrow \ell$ bits, $\ell = \ell(n)$):

- Gen : remains unchanged
- H : key s and string $x \in \{0, 1\}^*$, $L = |x| < 2^\ell$:
 - $B := \lceil \frac{L}{\ell} \rceil$ (# blocks). **Pad x with 0s.** ℓ -bit blocks x_1, \dots, x_B . $x_{B+1} := L$, L is encoded using ℓ bits
 - $z_0 := IV = 0^\ell$. For $i = 1, \dots, B + 1$, compute $z_i := h^s(z_{i-1} || x_i)$

Security of the Merkle-Damgård Transform

Theorem 11

If (Gen, h) is a fixed-length CRHF, then (Gen, H) is a CRHF.

Proof.

Idea: a collision in H^s yields a collision in h^s .

Two messages $x \neq x'$ of respective lengths L and L' such that $H^s(x) = H^s(x')$. # blocks are B and B' .

$x_{B+1} := L$ is necessary since **Padding with 0s** will lead to the same input with different messages.

1 $L \neq L': z_B \| L \neq z_{B'} \| L'$

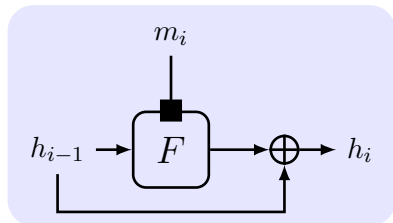
2 $L = L': z_{i^*-1} \| x_{i^*} \neq z'_{i^*-1} \| x'_{i^*}$

So there must be $x \neq x'$ such that $h^s(x) = h^s(x')$. □

Security on MD transform variations in Homework.

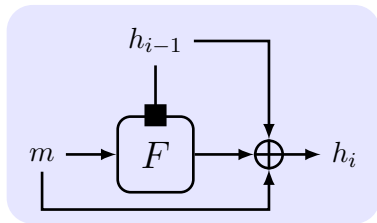
CRHF from Block Cipher

Davies-Meyer (SHA-1/2, MD5)



$$h_i = F_{m_i}(h_{i-1}) \oplus h_{i-1}$$

Miyaguchi-Preneel (Whirlpool)



$$h_i = F_{h_{i-1}}(m_i) \oplus h_{i-1} \oplus m$$

Theorem 12

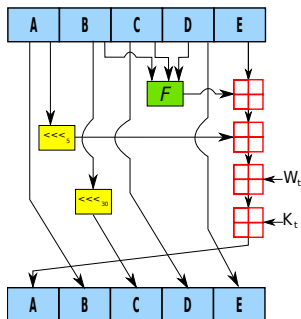
If F is modeled as an ideal cipher, then Davies-Meyer construction yields a CRHF.

Q: what if $h_i = F_{m_i}(h_{i-1})$ without XOR with h_{i-1} ?

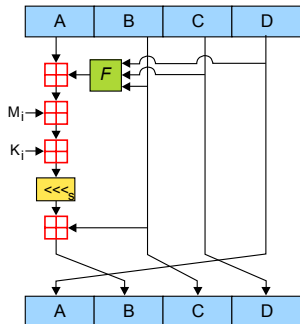
Q: what if F is not ideal such that $\exists x, F_k(x) = x$?

Cryptographic Hash Functions: SHA-1 and MD5

SHA-1:



MD5:



A, B, C, D and E are 32-bit words of the state; F is a nonlinear function that varies; $\lll n$ denotes a left bit rotation by n places; W_t/M_t is the expanded message word of round t ; K_t is the round constant of round t ; \boxplus denotes addition modulo 2^{32} .

- Finding a collision in 128-bit MD5 requires time $2^{20.96}$
- Finding a collision in 160-bit SHA-1 requires time 2^{51}

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Construction 13

$(\widetilde{\text{Gen}}, H)$ is a CRHF. $(\text{Gen}, \text{Mac}, \text{Vrfy})$ is a fixed-length MAC.

- $\text{Gen}'(1^n): (k, s). s \leftarrow \widetilde{\text{Gen}}, k \leftarrow \text{Gen}.$
- $\text{Mac}'_{s,k}(m): t := \text{Mac}_k(H^s(m)).$
- $\text{Vrfy}'_{s,k}(m, t): 1 \iff \text{Vrfy}_k(H^s(m), t) = 1.$

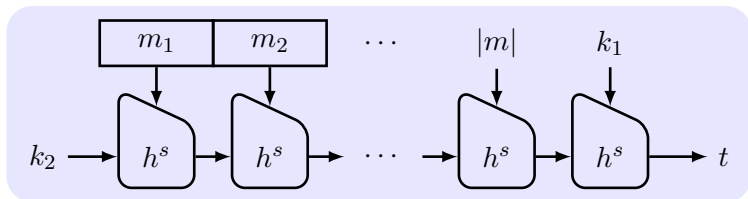
Theorem 14

The construction is a secure MAC for arbitrary-length messages.

Idea of proof: if the adversary has forged a tag on the “new message” m^* , then

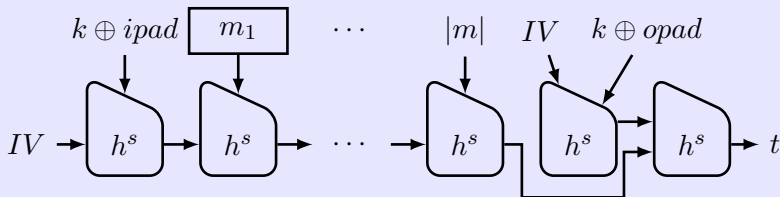
- Case 1: If there is a queried messages m such that $H^s(m) = H^s(m^*)$, then there is a collision in H^s .
- Case 2: If there is no queried messages m such that $H^s(m) = H^s(m^*)$, then the adversary has forged a valid tag on the “new message” $H^s(m^*)$ for MAC.

Nested MAC (NMAC)



- NMAC is a MAC using CRHF (MD transform) without using PRF.
- k_2 is not needed once h is CRHF, while it is needed if h is **weak collision resistance**: It is hard to find $(x, x'), x' \neq x$ such that $H_{k_2}^s(x) = H_{k_2}^s(x')$ **without knowing** k_2 .
- **Disadvantage**: IV of H must be modified.

Hash-based MAC (HMAC)



Construction 15

$(\widetilde{\text{Gen}}, h)$ is a fixed-length CRHF. $(\widetilde{\text{Gen}}, H)$ is the Merkle-Damgård transform. IV , $opad$ ($0x36$), $ipad$ ($0x5C$) are constants.

- $\text{Gen}(1^n)$: Output (s, k) . $s \leftarrow \widetilde{\text{Gen}}, k \leftarrow \{0, 1\}^n$ u.a.r
- $\text{Mac}_{s,k}(m)$: $t := H_{IV}^s((k \oplus opad) \| H_{IV}^s((k \oplus ipad) \| m))$
- $\text{Vrfy}_{s,k}(m, t)$: $1 \iff t \stackrel{?}{=} \text{Mac}_{s,k}(m)$

(In)Security of (Before-)HMAC ⁴

We investigate HMAC's security by showing the insecurity of some before-HMAC designs.

- **Prepend the key** $H^s(k\|x)$: Vulnerable to length extension attack. Given $H^s(k\|x)$ and the length of x , get the valid tag $H^s(k\|x\|x')$ for a new message $x\|x'$.
- **Append the key** $H^s(x\|k)$: A collision in the **weak** CRHF has a collision in the MAC. Recall that there is an appended key for **weak** CRHF in NMAC.
- **Envelope** $H^s(k\|x\|k)$: Some known vulnerabilities with this approach, even when two different keys are used. This needs reasonable pseudorandomness assumptions on h .
- **Two-key nest** $H^s(k\|H^s(k\|x))$: NMAC with 2 keys, HMAC with 1 key

⁴from BonehShoup v0.5 p.303

Remarks on HMAC

- HMAC is based on NMAC which was first published in a paper “Keying Hash Functions for Message Authentication” by Mihir Bellare, Ran Canetti, and Hugo Krawczyk in 1996.
- HMAC became an industry standard (RFC2104) in 1997
- HMAC is faster than CBC-MAC

Verification timing attacks

Keyczar crypto library (Python):

```
def Verify(key, msg, sig_bytes):  
    return HMAC(key, msg) == sig_bytes
```

The problem: implemented as a byte-by-byte comparison

In Xbox 360, a difference of 2.2 milliseconds between rejection times of i vs. $i + 1$ bytes.

Don't implement it yourself

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Definition of Information-Theoretic MAC Security

It is impossible to achieve "perfect" MAC, as the adversary can output a valid tag with probability $1/2^{|t|}$ at least.

The one-time MAC experiment $\text{Macforge}_{\mathcal{A}, \Pi}^{1\text{-time}}$:

- 1 $k \leftarrow \text{Gen}$.
- 2 \mathcal{A} outputs a message m' , and is given a tag $t' \leftarrow \text{Mac}_k(m')$, and outputs (m, t) .
- 3 $\text{Macforge}_{\mathcal{A}, \Pi}^{1\text{-time}} = 1 \iff \text{Vrfy}_k(m, t) = 1 \wedge m \neq m'$.

Definition 16

A MAC Π is **one-time ε -secure** if \forall PPT \mathcal{A} :

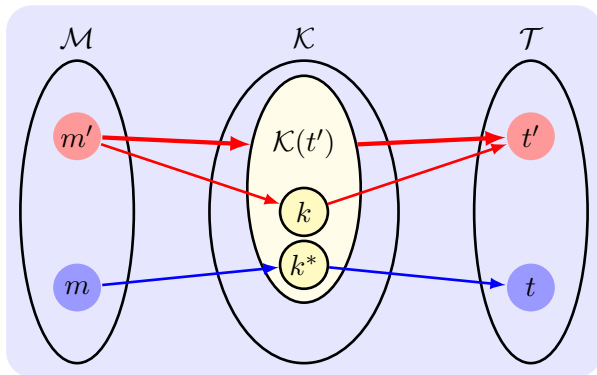
$$\Pr[\text{Macforge}_{\mathcal{A}, \Pi}^{1\text{-time}} = 1] \leq \varepsilon.$$

Understanding Information-Theoretic MACs

An adversary will forge the tag in the following steps.

- 1 Obtain a tag t' from a MAC query for a fixed message m'
- 2 Obtain $\mathcal{K}(t') \stackrel{\text{def}}{=} \{k \mid \text{Vrfy}_k(m', t') = 1\}$ by using his unlimited computing power
- 3 Output (m, t) using a key k^* from $\mathcal{K}(t')$

Question: What if $\mathcal{K}(t')$ is too large or too small?



Definition 17

A function $h: \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{T}$ is a **Strongly Universal (or pairwise-independent) Function (SUF)** if for all distinct $m, m' \in \mathcal{M}$ and all $t, t' \in \mathcal{T}$, it holds that:

$$\Pr[h_k(m) = t \wedge h_k(m') = t'] = 1/|\mathcal{T}|^2.$$

where the probability is taken over uniform choice of $k \in \mathcal{K}$.

Construction 18

- Let $h: \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{T}$ be an SUF.
- Gen: $k \leftarrow \{0, 1\}^n$ u.a.r.
- Mac $_k(m)$: $t := h_k(m)$.
- Vrfy $_k(m, t)$: $1 \iff t \stackrel{?}{=} h_k(m)$. (If $m \notin \mathcal{M}$, then output 0.)

Construction of An SUF

Theorem 19

For any prime P , the function h is an SUF:

$$h_{a,b}(m) \stackrel{\text{def}}{=} [a \cdot m + b \mod p]$$

Proof.

$h_{a,b}(m) = t$ and $h_{a,b}(m') = t'$, only if $a \cdot m + b = t \mod p$ and $a \cdot m' + b = t' \mod p$. We have $a = [(t - t') \cdot (m - m')^{-1} \mod p]$ and $b = [t - a \cdot m \mod p]$, which means there is a unique key (a, b) . Since there are $|\mathcal{T}|^2$ keys,

$$\Pr[h_k(m) = t \wedge h_k(m') = t'] = \frac{1}{|\mathcal{T}|^2}.$$



Security of Construction from An SUF

Theorem 20

If h is an SUF, Construction is a $1/|\mathcal{T}|$ -secure MAC.

Proof.

Assume that \mathcal{A} is deterministic and receives tag t' for the message m' , where m' is fixed. The pair (m, t) that \mathcal{A} outputs is a deterministic function of (m', t') .

$$\begin{aligned}\Pr[\text{Macforge}_{\mathcal{A}, \Pi}^{1-\text{time}} = 1] &= \sum_{t' \in \mathcal{T}} \Pr[\text{Macforge}_{\mathcal{A}, \Pi}^{1-\text{time}} = 1 \wedge h_k(m') = t'] \\ &= \sum_{t' \in \mathcal{T}} \Pr[h_k(m) = t \wedge h_k(m') = t'] \\ &= \sum_{t' \in \mathcal{T}} \frac{1}{|\mathcal{T}^2|} = \frac{1}{|\mathcal{T}|}\end{aligned}$$



Limitations on Information-Theoretic MACs

Limitations: Any ℓ -time 2^{-n} -secure MAC requires keys of length at least $(\ell + 1) \cdot n$.

Theorem 21

Let Π be a 1-time 2^{-n} -secure MAC where all keys are the same length. Then the keys must have length at least $2n$.

Proof.

Let $\mathcal{K}(t') \stackrel{\text{def}}{=} \{k \mid \text{Vrfy}_k(m', t') = 1\}$. For any t' , $|\mathcal{K}(t')| \leq 2^{-n} \cdot |\mathcal{K}|$. Otherwise, (m', t') would be a valid forgery with probability at least $|\mathcal{K}(t')|/|\mathcal{K}| > 2^{-n}$. The probability that \mathcal{A} outputs a valid forgery by guessing k from $|\mathcal{K}(t')|$ is at least

$$\sum_{t'} \Pr[\text{Mac}_k(m') = t'] \cdot \frac{1}{|\mathcal{K}(t')|} \geq \sum_{t'} \Pr[\text{Mac}_k(m') = t'] \cdot \frac{2^n}{|\mathcal{K}|} = \frac{2^n}{|\mathcal{K}|}$$

As the probability is at most 2^{-n} , $|\mathcal{K}| \geq 2^{2n}$. Since all keys have the same length, each key must have length at least $2n$. \square

- Authentication means existential unforgeability.
- Secure MAC is constructed by using PRF.
- Secure MAC is constructed by using keyed CRHF.
- Information-theoretic MAC security requires very, very long key.