Theoretical Constructions of Pseudorandom Objects

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Outline

1. One-Way Functions

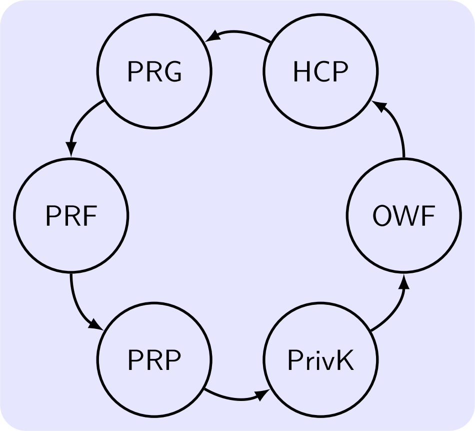
2. From OWF to PRP

Content

1 One-Way Functions

2 From OWF to PRP

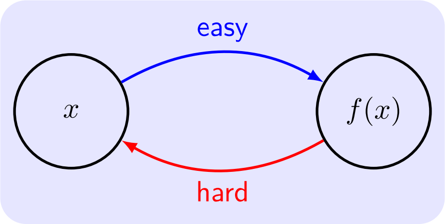
Overview



One of contributions of modern cryptography

The existence of one-way functions is equivalent to the existence of all (non-trivial) private-key cryptography.

One-Way Functions (OWF)



The inverting experiment $\mathsf{Invert}_{\mathcal{A},f}(n)$:

1. Choose input $x \leftarrow \{0,1\}^n$. Compute $y := f(x)$.
2. $\mathcal{A}$ is given 1^n and y as input, and outputs x'.
3. $\mathsf{Invert}_{\mathcal{A},f}(n) = 1$ if $f(x') = y$, otherwise 0.

Definitions of OWF/OWP [Yao]

For polynomial-time algorithm M_f and $\mathcal{A}$.

Definition 1

A function $f : \{0,1\}^* \to \{0,1\}^*$ is **one-way** if:

1. (Easy to compute): $\exists$ M_f: $\forall x, M_f(x) = f(x)$.
2. (Hard to invert): $\forall$ $\mathcal{A}$, $\exists$ negl such that

$$\Pr[\mathsf{Invert}_{\mathcal{A},f}(n) = 1] \leq \mathsf{negl}(n).$$

or

$$\Pr_{x \leftarrow \{0,1\}^n}[\mathcal{A}(f(x)) \in f^{-1}(f(x))] \leq \mathsf{negl}(n).$$

Definition 2

Let $f : \{0,1\}^* \to \{0,1\}^*$ be length-preserving, and f_n be the restriction of f to the domain $\{0,1\}^n$. A OWP f is a **one-way permutation** if $\forall n$, f_n is a bijection.

Candidate One-Way Function

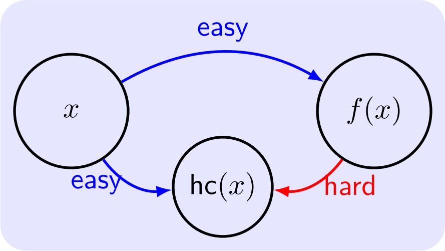
- **Multiplication and factoring**:
  $f_{\mathsf{mult}}(x, y) = (xy, \|x\|, \|y\|)$, x and y are equal-length primes.
- **Modular squaring and square roots**:
  $f_{\mathsf{square}}(x) = x^2 \bmod N$.
- **Discrete exponential and logarithm**:
  $f_{g,p}(x) = g^x \bmod p$.
- **Subset sum problem**:
  $f(x_1, \ldots, x_n, J) = (x_1, \ldots, x_n, \sum_{j \in J} x_j)$.
- **Cryptographically secure hash functions**:
  Practical solutions for one-way computation.

Examples

$f:\ \{0,1\}^{128} \rightarrow \{0,1\}^{128}$ is a OWF. Is f' OWF?

- $f'(x) = f(x) \| x$
- $f'(x \| x') = f(x) \| x'$
- $f'(x) = f(x) \oplus f(x)$
- $f'(x) = \begin{cases} f(x) & \text{if } x[0,1,2,3] \neq 1010 \\ x & \text{otherwise} \end{cases}$
- $f'(x) = \begin{cases} f(x) & \text{if } x \neq 1010 \| 0^{124} \\ x & \text{otherwise} \end{cases}$
- more examples in homework

Hard-Core Predicates (HCP) [Blum-Micali]



Definition 3

A function hc : $\{0,1\}^* \to \{0,1\}$ is a **hard-core predicate of a function** f if (1) hc can be computed in polynomial time, and (2) $\forall$ PPT $\mathcal{A}$, $\exists$ negl such that

$$\Pr_{x \leftarrow \{0,1\}^n}[\mathcal{A}(f(x)) = \mathsf{hc}(x)] \le \frac{1}{2} + \mathsf{negl}(n).$$

A HCP for Any OWF

Theorem 4

f is OWF. Then $\exists$ an OWF g along with an HCP gl *for g. If f is a permutation then so is g.*

Q: is $\mathsf{gl}(x) = \bigoplus_{i=1}^{n} x_i$ the HCP of any OWF?

Proof.

$g(x, r) \overset{\text{def}}{=} (f(x), r)$, for $|x| = |r|$, and define

$$\mathsf{gl}(x, r) \overset{\text{def}}{=} \bigoplus_{i=1}^{n} x_i \cdot r_i.$$

r is generated uniformly at random. [Goldreich and Levin] □

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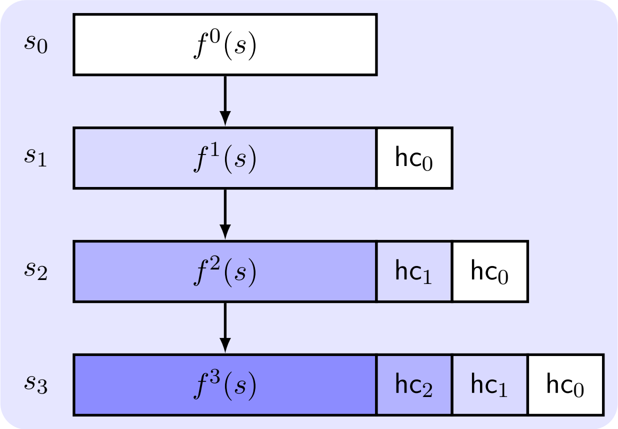
PRG from OWP: Blum-Micali Generator

Theorem 5

f is an OWP and hc *is an HCP of f. Then $G(s) \overset{def}{=} (f(s), \mathsf{hc}(s))$ constitutes a PRG with expansion factor $\ell(n) = n + 1$, then $\forall$ polynomial $p(n) > n$, $\exists$ a PRG with expansion factor $\ell(n) = p(n)$.*

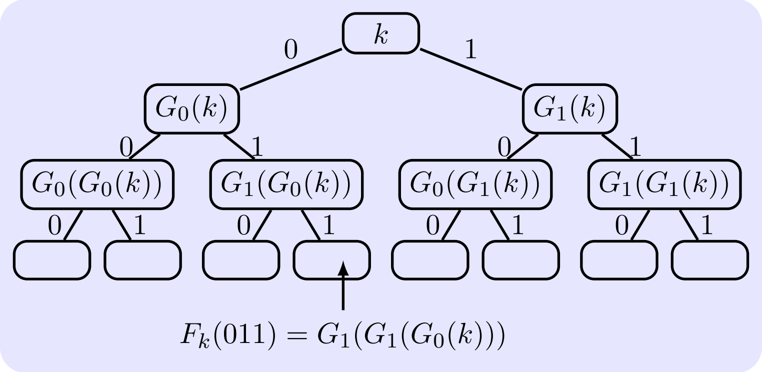


PRF from PRG [Goldreich, Goldwasser, Micali]

Theorem 6

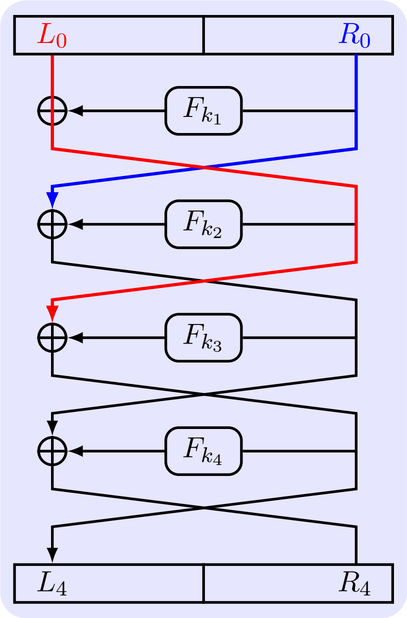
If ∃ a PRG with expansion factor $\ell(n) = 2n$, then ∃ a PRF.

$G(k) = G_0(k) \| G_1(k)$



$$F_k(x_1 x_2 \cdots x_n) = G_{x_n}(\cdots(G_{x_2}(G_{x_1}(k)))\cdots), G(s) = (G_0(s), G_1(s)).$$

PRP from PRF [Lucy, Rackoff]



$F^{(r)}$ is an r-round Feistel network with the mangler function F.

Theorem 7

If F is a length-preserving PRF, then $F^{(3)}$ is a PRP, and $F^{(4)}$ is a strong PRP, that maps $2n$-bit strings to $2n$-bit strings (and uses a key of length $3n$ and $4n$).

Show that 1- or 2-round Feistel network is not a PRF.

Necessary Assumptions

Theorem 8

Assume that $\exists$ OWP. Then $\exists$ PRG, PRF, strong PRP, and CCA-secure private-key encryption schemes.

Proposition 9

If $\exists$ a private-key encryption scheme that has indistinguishable encryptions in the presence of an eavesdropper, then $\exists$ an OWF.

Proof.

$f(k, m, r) \overset{\text{def}}{=} (\mathsf{Enc}_k(m, r), m)$, where $|k| = n, |m| = 2n, |r| = \ell(n)$. See the textbook for details. □

Summary

- OWF implies secure private-key encryption scheme
- Secure private-key encryption scheme implies OWF