Private-Key Encryption and Pseudorandomness (Part II)

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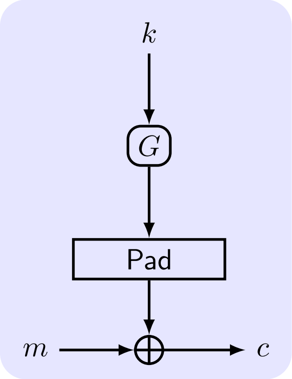
Outline

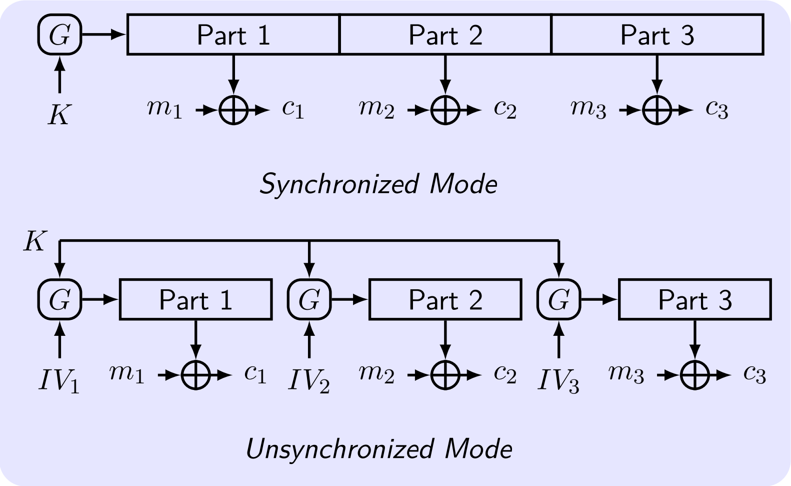
Content

Stream Ciphers



- **Idea**: Generalization of one-time pad
- **Stream cipher**: Enc. by XORing with pseudorandom stream (keystream)
- **Multiple messages**: Be concatenated into a single one and encrypted
- **Keystream**: Generated by a variable-length PRG
- **Strength**: Faster than block cipher
- **Weakness**: Difficult to be secure

Secure Multiple Encryptions Using a Stream Cipher

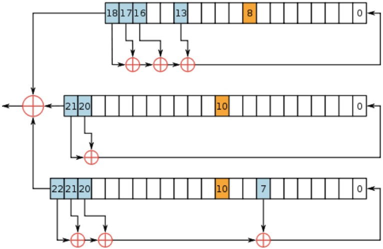


Initial vector IV is chosen *u.a.r* and public

Q: which mode is better in your opinion?

Questionable Security

- **State of the art**: No standardized and popular one. Security is questionable, e.g., RC4 in WEP protocol in 802.11, Linear Feedback Shift Registers (LFSRs) used in A5/1 for GSM.



WARNING

Don't use any stream cipher. If necessary, construct one from a block cipher.

- eStream project worked on secure stream ciphers. Salsa20/12 is a promising candidate.

Related Keys: Real World Cases

Keys (the IV-key pair) for multiple enc. must be independent

Attacks on 802.11b WEP

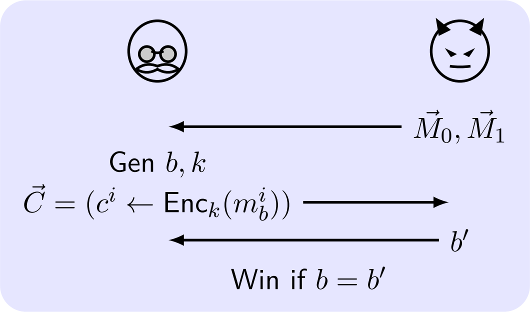
Unsynchronized mode: $\text{Enc}(m_i) := \langle IV_i, G(IV_i \| k) \oplus m_i \rangle$

- Length of IV is 24 bits, repeat IV after $2^{24} \approx$ 16M frames
- On some WiFi cards, IV resets to 0 after power cycle
- $IV_i = IV_{i-1} + 1$. For RC4, recover k after 40,000 frames

Security for Multiple Encryptions

The multiple-message eavesdropping experiment $\mathsf{PrivK}^{\mathsf{mult}}_{\mathcal{A},\Pi}(n)$:

1. $\mathcal{A}$ is given input 1^n, outputs $\vec{M}_0 = (m_0^1, \ldots, m_0^t)$, $\vec{M}_1 = (m_1^1, \ldots, m_1^t)$ with $\forall i, |m_0^i| = |m_1^i|$.
2. $k \leftarrow \mathsf{Gen}(1^n)$, a random bit $b \leftarrow \{0,1\}$ is chosen. Then $c^i \leftarrow \mathsf{Enc}_k(m_b^i)$ and $\vec{C} = (c^1, \ldots, c^t)$ is given to $\mathcal{A}$.
3. $\mathcal{A}$ outputs b'. If $b' = b$, $\mathsf{PrivK}^{\mathsf{mult}}_{\mathcal{A},\Pi} = 1$, otherwise 0.



Definition of Multi-Encryption Security

Definition 1

Π has **indistinguishable multiple encryptions in the presence of an eavesdropper** if $\forall$ PPT $\mathcal{A}$, $\exists$ negl such that

$$\Pr\left[\mathsf{PrivK}^{\mathsf{mult}}_{\mathcal{A},\Pi}(n) = 1\right] \le \frac{1}{2} + \mathsf{negl}(n).$$

Question:

Does any cipher we have learned so far have indistinguishable multiple encryptions in the presence of an eavesdropper?

Attack On Deterministic Multiple Encryptions

Question:

Generally, if Π's encryption function is **deterministic**, i.e., a plaintext will be always encrypted into the same ciphertext with the same key, is Π multiple-encryption-secure?

Attack:

For the deterministic encryption, the adversary may generate $m_0^1 = m_0^2$ and $m_1^1 \neq m_1^2$, and then outputs $b' = 0$ if $c^1 = c^2$, otherwise $b' = 1$.

Chosen-Plaintext Attacks (CPA)

CPA: the adversary has the ability to obtain the encryption of plaintexts of its choice

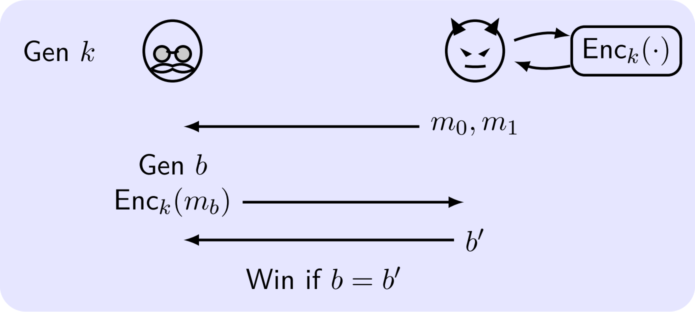
A story in WWII

- Navy cryptanalysts believe the ciphertext "AF" means "Midway island" in Japanese messages
- But the general did not believe that Midway island would be attacked
- Navy cryptanalysts sent a plaintext that the freshwater supplies at Midway island were low
- Japanese intercepted the plaintext and sent a ciphertext that "AF" was low in water
- The US forces dispatched three aircraft carriers and won

CPA Indistinguishability Experiment

The CPA indistinguishability experiment $\mathsf{PrivK}^{\mathsf{cpa}}_{\mathcal{A},\Pi}(n)$:

1. $k \leftarrow \mathsf{Gen}(1^n)$
2. $\mathcal{A}$ is given input 1^n and **oracle access** $\mathcal{A}^{\mathsf{Enc}_k(\cdot)}$ to $\mathsf{Enc}_k(\cdot)$, outputs m_0, m_1 of the same length
3. $b \leftarrow \{0,1\}$. Then $c \leftarrow \mathsf{Enc}_k(m_b)$ is given to $\mathcal{A}$
4. $\mathcal{A}$ **continues to have oracle access** to $\mathsf{Enc}_k(\cdot)$, outputs b'
5. If $b' = b$, $\mathcal{A}$ succeeded $\mathsf{PrivK}^{\mathsf{cpa}}_{\mathcal{A},\Pi} = 1$, otherwise 0



Definition of CPA Security

Definition 2

Π has **indistinguishable encryptions under a CPA (CPA-secure)** if $\forall$ PPT $\mathcal{A}$, $\exists$ negl such that

$$\Pr\left[\mathsf{PrivK}^{\mathsf{cpa}}_{\mathcal{A},\Pi}(n) = 1\right] \leq \frac{1}{2} + \mathsf{negl}(n).$$

- Q: Is any cipher we have learned so far CPA-secure? Why?

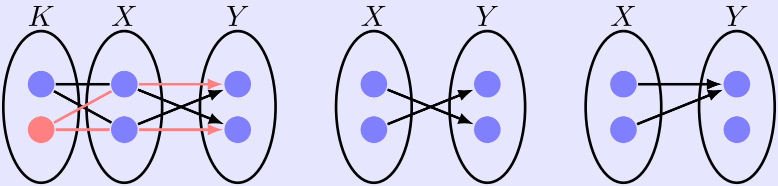
Proposition 3

Any private-key encryption scheme that is CPA-secure also is ***multiple-encryption-secure****.*

- Q: Does **multiple-encryption-security** mean CPA-security? (homework)

Content

Concepts on Pseudorandom Functions



$F : K \times X \to Y$ 2D function

$F_k(x) \stackrel{\text{def}}{=} F(k, x)$ Keyed function

$f : X \to Y$ Look-up table

- **Keyed function** $F : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$
  $F_k : \{0,1\}^* \to \{0,1\}^*$, $F_k(x) \stackrel{\text{def}}{=} F(k, x)$
- **Look-up table** f: $\{0,1\}^n \to \{0,1\}^n$ with size $= ?$ bits
- **Function family** Func_n: all functions $\{0,1\}^n \to \{0,1\}^n$.
  $|\mathsf{Func}_n| = 2^{n \cdot 2^n}$
- **Length Preserving**: $\ell_{key}(n) = \ell_{in}(n) = \ell_{out}(n)$

Definition of Pseudorandom Function

Intuition: A PRF F generates a function F_k that is indistinguishable from truly random selected function f (look-up table) in Func_n.
However, the function has **exponential length**. Give D the deterministic **oracle access** $D^{\mathcal{O}}$ to the functions $\mathcal{O}$.

Definition 4

An efficient length-preserving, keyed function F is a **pseudorandom function (PRF)** if $\forall$ PPT distinguishers D,

$$\left|\Pr[D^{F_k(\cdot)}(1^n) = 1] - \Pr[D^{f(\cdot)}(1^n) = 1]\right| \leq \mathsf{negl}(n),$$

where f is chosen *u.a.r* from Func_n.

Q: Is the fixed-length OTP a PRF?

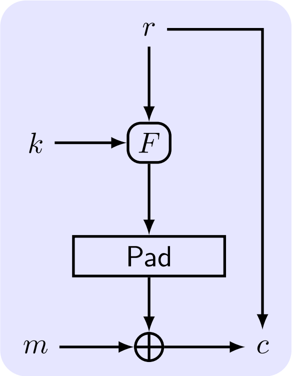
Questions

Q: Without knowing the key and the oracle access, could anyone learn something about the output from the input with a non-negligible probability?

Let $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^n$ be a PRF. Is G a PRF?

- $G((k_1, k_2), x) = F(k_1, x) \| F(k_2, x)$
- $G(k, x) = F(k, x \oplus 1^n)$
- $G(k, x) = \begin{cases} F(k, x) & \text{when } x \neq 0^n \\ 0^n & \text{otherwise} \end{cases}$
- $G(k, x) = \begin{cases} F(k, x) & \text{when } x \neq 0^n \\ k & \text{otherwise} \end{cases}$
- $G(k, x) = F(k, x) \bigoplus F(k, x \oplus 1^n)$

CPA-Security from Pseudorandom Function



Construction 5

- *Fresh random string r.*
- $F_k(r)$*:* $|k| = |m| = |r| = n$.
- Gen*:* $k \in \{0,1\}^n$.
- Enc*:* $s := F_k(r) \oplus m$,
  $c := \langle r, s \rangle$.
- Dec*:* $m := F_k(r) \oplus s$.

Theorem 6

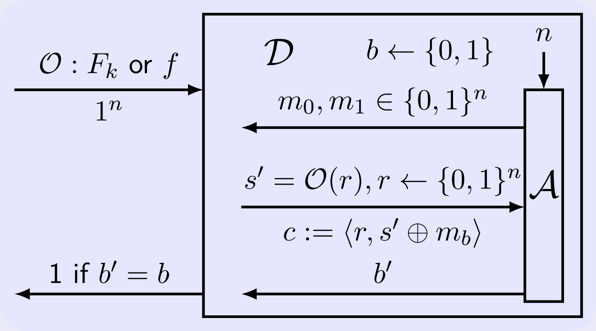
If F is a PRF, this fixed-length encryption scheme Π is CPA-secure.

Proof of CPA-Security from PRF

Idea: First, analyze the security in an idealized world where f is used in $\tilde{\Pi}$; next, claim that if Π is insecure when F_k was used then this would imply F_k is not PRF by reduction.

Proof.

Reduce D to $\mathcal{A}$:



□

Proof of CPA-Security from PRF (Cont.)

Proof.

Analyze $\Pr[\text{Break}]$, Break means $\text{PrivK}^{\text{cpa}}_{\mathcal{A},\tilde{\Pi}}(n) = 1$:
$\mathcal{A}$ collects $\{\langle r_i, f(r_i)\rangle\}$, $i = 1, \ldots, q(n)$ with $q(n)$ queries;
The challenge $c = \langle r_c, f(r_c) \oplus m_b\rangle$.

- Repeat: $r_c \in \{r_i\}$ with probability $\frac{q(n)}{2^n}$. $\mathcal{A}$ can know m_b.
- $\overline{\text{Repeat}}$: As OTP, $\Pr[\text{Break}] = \frac{1}{2}$

$$\begin{aligned}\Pr[\text{Break}] &= \Pr[\text{Break} \wedge \text{Repeat}] + \Pr[\text{Break} \wedge \overline{\text{Repeat}}] \\ &\le \Pr[\text{Repeat}] + \Pr[\text{Break} | \overline{\text{Repeat}}] \\ &\le \frac{q(n)}{2^n} + \frac{1}{2}.\end{aligned}$$

$\Pr[D^{F_k(\cdot)}(1^n) = 1] = \Pr[\text{PrivK}^{\text{cpa}}_{\mathcal{A},\Pi}(n) = 1] = \frac{1}{2} + \varepsilon(n)$.
$\Pr[D^{f(\cdot)}(1^n) = 1] = \Pr[\text{PrivK}^{\text{cpa}}_{\mathcal{A},\tilde{\Pi}}(n) = 1] = \Pr[\text{Break}] \le \frac{1}{2} + \frac{q(n)}{2^n}$.
$\Pr[D^{F_k(\cdot)}(1^n) = 1] - \Pr[D^{f(\cdot)}(1^n) = 1] \ge \varepsilon(n) - \frac{q(n)}{2^n}$. $\varepsilon(n)$ is negligible. □

CPA-Security from PRF for Arbitrary-Length

- For arbitrary-length messages, $m = m_1, \ldots, m_\ell$

$$c := \langle r_1, F_k(r_1) \oplus m_1, r_2, F_k(r_2) \oplus m_2, \ldots, r_\ell, F_k(r_\ell) \oplus m_\ell \rangle$$

Corollary 7

If F is a PRF, then Π is CPA-secure for arbitrary-length messages.

- **Efficiency**: $|c| = 2|m|$.

Pseudorandom Permutations

- **Bijection**: F is one-to-one and onto
- **Permutation**: A bijective function from a set to itself
- **Keyed permutation**: $\forall k, F_k(\cdot)$ is permutation
- F is a bijection $\iff$ F^{-1} is a bijection

Definition 8

An efficient, keyed permutation F is a **strong pseudorandom permutation (PRP)** if $\forall$ PPT distinguishers D,

$$\left|\Pr[D^{F_k(\cdot),F_k^{-1}(\cdot)}(1^n) = 1] - \Pr[D^{f(\cdot),f^{-1}(\cdot)}(1^n) = 1]\right| \leq \mathsf{negl}(n),$$

where f is chosen *u.a.r* from the set of permutations on n-bit strings.

If F is a pseudorandom permutation then is it a PRF?

Questions

Let $X = \{0, 1\}$ (1 bit), answer the following questions.

1. What are the functions in the permutation over X?
2. $K = \{0, 1\}$, what is the simplest permutation $F(k, x)$ over X?
3. Is your F a secure PRP?
4. Is your F a secure PRF?
5. What if $X = \{0, 1\}^{128}$ and $K = \{0, 1\}^{128}$?
6. Could you give a (or another) PRP over $X = \{0, 1\}^{128}$?

Proposition 9

IF F is a PRP and additionally $\ell_{in}(n) \geq n$, then F is also a PRF.

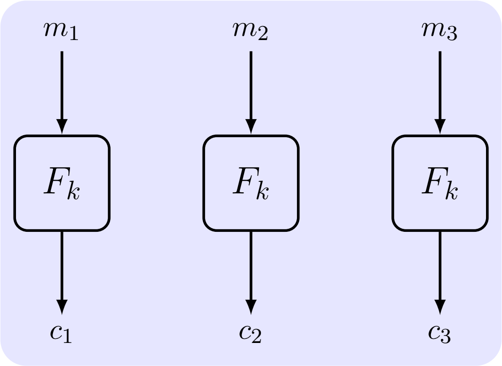
Content

PRF, PRP, PRG, and Modes of Operation

Modes of Operation:

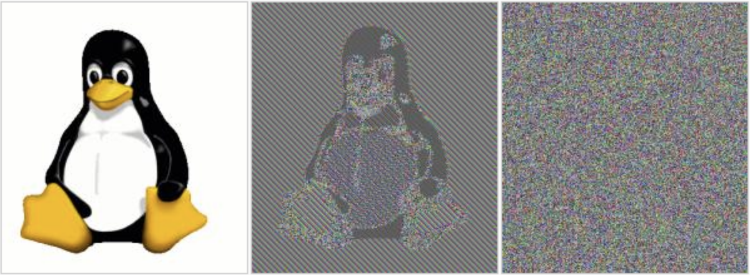
- A way of encrypting arbitrary-length messages using a PRP or PRF
- A way of constructing a PRG from a PRP or PRF

Electronic Code Book (ECB) Mode



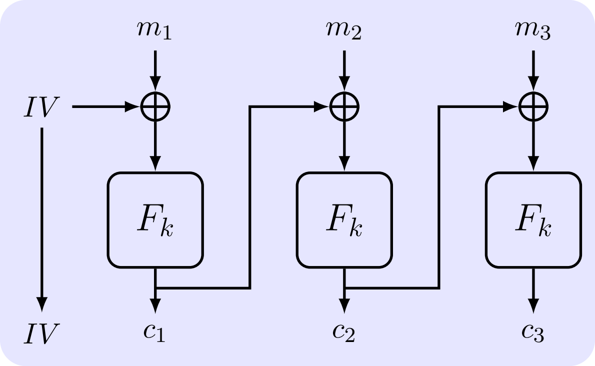
- Q: is it indistinguishable in the presence of an eavesdropper?
- Q: can F be any PRF?

Attack on ECB mode



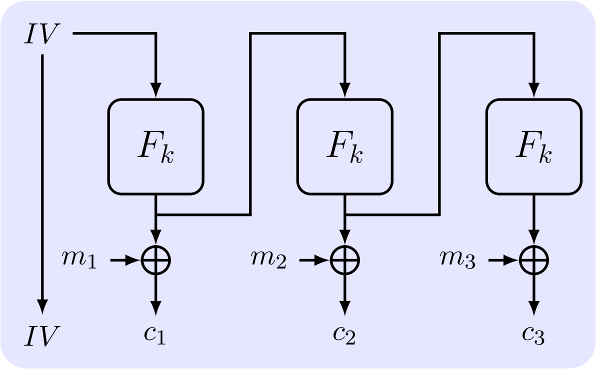
Original image | Encrypted using ECB mode | Modes other than ECB result in pseudo-randomness

Cipher Block Chaining (CBC) Mode



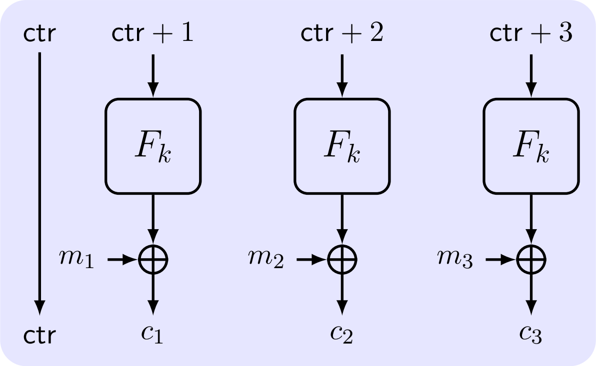
- IV: initial vector, a fresh random string.
- Q: is it CPA-secure? what if IV is always 0?
- Q: is the encryption parallelizable, i.e., outputting c_2 before getting c_1?
- Q: can F be any PRF?

Output Feedback (OFB) Mode



- Q: is it CPA-secure?
- Q: is the encryption parallelizable?
- Q: can F be any PRF?

Counter (CTR) Mode



- ctr is an IV
- Q: is it CPA-secure?
- Q: is the encryption parallelizable?
- Q: can F be any PRF?

CTR Mode Is CPA-secure

Theorem 10

If F is a PRF, then randomized CTR mode is CPA-secure.

Proof.

The message length and the number of query are $q(n)$.
Overlap: the sequence for the challenge overlaps the sequences for the queries from the adversary.
ctr^*: ctr in the challenge. ctr_i: ctr in the queries, $i = 1, \ldots, q(n)$.
Overlap: $\mathsf{ctr}_i - q(n) < \mathsf{ctr}^* < \mathsf{ctr}_i + q(n)$.

$$\Pr[\text{Overlap}] \leq \frac{2q(n) - 1}{2^n} \cdot q(n)$$

$\square$

Proof of CPA-secure CTR Mode (Cont.)

Proof.

See proof of theorem 6. (1) Analyze Break : $\mathsf{PrivK}^{\mathsf{cpa}}_{\mathcal{A},\tilde{\Pi}}(n) = 1$.

$$\begin{aligned}
\Pr[\mathsf{Break}] &= \Pr[\mathsf{Break} \wedge \mathsf{Overlap}] + \Pr[\mathsf{Break} \wedge \overline{\mathsf{Overlap}}] \\
&\le \Pr[\mathsf{Overlap}] + \Pr[\mathsf{Break} | \overline{\mathsf{Overlap}}] \\
&\le \frac{2q(n)^2}{2^n} + \frac{1}{2}.
\end{aligned}$$

(2) Reduce D to $\mathcal{A}$

$$\Pr[D^{f(\cdot)}(1^n) = 1] = \Pr[\mathsf{PrivK}^{\mathsf{cpa}}_{\mathcal{A},\tilde{\Pi}}(n) = 1] \le \frac{2q(n)^2}{2^n} + \frac{1}{2}$$

$$\Pr[D^{F_k(\cdot)}(1^n) = 1] = \Pr[\mathsf{PrivK}^{\mathsf{cpa}}_{\mathcal{A},\Pi}(n) = 1] \le \frac{1}{2} + \varepsilon(n)$$

If F is PRP, $\varepsilon(n)$ is negligible.

□

IV Should Not Be Predictable

If IV is predictable, then CBC/OFB/CTR mode is not CPA-secure.
Q: Why? (homework)

Bug in SSL/TLS 1.0

IV for record $\#i$ is last CT block of record $\#(i-1)$.

API in OpenSSL

```
void AES_cbc_encrypt (
    const unsigned char *in,
    unsigned char       *out,
    size_t              length,
    const AES_KEY       *key,
    unsigned char       *ivec,     User supplies IV
    AES_ENCRYPT or AES_DECRYPT);
```

Content

Security Against CCA

The CCA indistinguishability experiment $\mathsf{PrivK}^{\mathsf{cca}}_{\mathcal{A},\Pi}(n)$:

1. $k \leftarrow \mathsf{Gen}(1^n)$.
2. $\mathcal{A}$ is given input 1^n and oracle access $\mathcal{A}^{\mathsf{Enc}_k(\cdot)}$ and $\mathcal{A}^{\mathsf{Dec}_k(\cdot)}$, outputs m_0, m_1 of the same length.
3. $b \leftarrow \{0,1\}$. $c \leftarrow \mathsf{Enc}_k(m_b)$ is given to $\mathcal{A}$.
4. $\mathcal{A}$ continues to have oracle access **except for c**, outputs b'.
5. If $b' = b$, $\mathcal{A}$ succeeded $\mathsf{PrivK}^{\mathsf{cca}}_{\mathcal{A},\Pi} = 1$, otherwise 0.

Definition 11

Π has **indistinguishable encryptions under a CCA (CCA-secure)** if $\forall$ PPT $\mathcal{A}$, $\exists$ negl such that

$$\Pr\left[\mathsf{PrivK}^{\mathsf{cca}}_{\mathcal{A},\Pi}(n) = 1\right] \leq \frac{1}{2} + \mathsf{negl}(n).$$

Understanding CCA-security

- In real world, the adversary might conduct CCA by influencing what gets decrypted
  - If the communication is not authenticated, then an adversary may send certain ciphertexts on behalf of the honest party
- CCA-security implies "**non-malleability**"
- None of the above scheme is CCA-secure

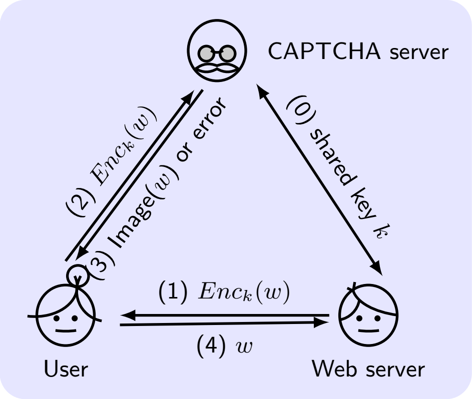
CCA against Construction 5

$\mathcal{A}$ gives m_0, m_1 and gets $c = \langle r, F_k(r) \oplus m_b \rangle$, and then queries c' which is the same with c except that a single bit is flipped. The $m' = c' \oplus F_k(r)$ should be the same with m_b except ____?

Q: Show that the above modes (CBC, OFB and CTR) are also not CCA-secure. (homework)

Padding-Oracle Attacks: Real-world Case

CAPTCHA server will return an error when deciphering the CT of a CAPTCHA text received from a user.

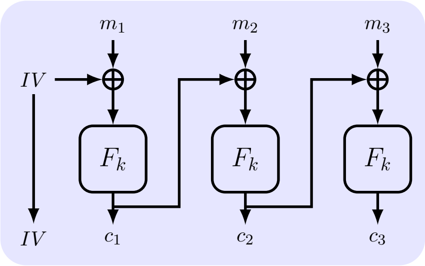


Padding-Oracle Attacks

PKCS #5 Padding: append b bytes of b to the message in order to make the total length a multiple of the block length (append a dummy block if needed). The decryption server will return a **Bad Padding Error** for incorrect padding.

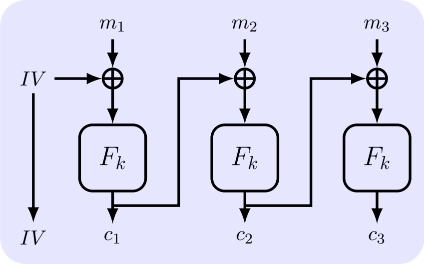
Padding-Oracle Attacks:

- In a one-block CBC, by modifying the 1st byte of IV, attacker can learn whether m is NULL. If yes, error will occur.



- append $\{b\}^b$ as a dummy block if m is NULL
- change the 1st byte of IV from x to y, get decrypted block $(x \oplus y \oplus b)\|\{b\}^{b-1}$, and trigger an error

Padding-Oracle Attacks (Cont.)



- If no error, then learn whether m is 1 byte by modifying the 2nd byte of IV and so on (changing the ciphertext)
- Once learn the length of m, learn the last byte of m (s) by modifying the one before the last block in the ciphertext
- $m_{last} = \cdots s\|\{b\}^b$, $c_{last-1} = \cdots t\|\{\cdot\}^b$
- modify c_{last-1} to $c'_{last-1} = \cdots u\|(\{\cdot\}^b \oplus \{b\}^b \oplus \{b+1\}^b)$
- Q: If no padding error, then $s =$?

Summary

- Asymptotic approach, proof of reduction, indistinguishable
- PRG, PRF, PRP, stream cipher, block cipher
- Security/construction against eavesdropping/CPA
- EBC, CBC, OFB, CTR
- CCA, padding-oracle attack