EBU5375 Signals and systems: The Fourier transform in discrete time

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Quick review

The Fourier transform of discrete-time signals

Some important properties

Discrete-time filters in the frequency domain

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Complex exponentials in DT

The discrete-time complex exponential $e^{j\Omega n}$

(a) Is always periodic.
(b) Is periodic for $\Omega = 2\pi k/N$, where k, N integers.
(c) Is periodic for $\Omega = 2\pi k/N$, where k, N integers and N is the period.

Complex exponentials in DT

Given a discrete-time interval of duration N

(a) An infinite number of complex exponentials with period N exist.

(b) There are N complex exponential with period N.

(c) There is no periodic complex exponential with period N.

Complex exponentials in DT

Given two complex exponentials $x_A[n] = e^{j\Omega_A n}$ and $x_B[n] = e^{j\Omega_B n}$, where Ω_A and Ω_B,

(a) $x_A[n]$ is a high-frequency signal and $x_B[n]$ a low-frequency signal.

(b) $x_A[n]$ is a low-frequency signal and $x_B[n]$ a high-frequency signal.

(c) With the above information, it is not possible to determine it.

Complex exponentials

CT complex exponentials	DT complex exponentials
Always periodic	Only periodic for $\Omega = 2\pi k/N$, k, N integers
Different frequencies produce different signals	Frequencies within an interval of size 2π produce different signals
There exist infinite complex exponentials with period T, namely those of frequencies $\frac{2\pi}{T}$, $2\frac{2\pi}{T}$,$3\frac{2\pi}{T}$,...	There only exist N complex exponentials with period N, namely those of frequencies $\frac{2\pi}{N}$, $2\frac{2\pi}{N}$,...,$N\frac{2\pi}{N}$

Low and high frequencies in discrete-time

Since a signal with frequency Ω_1 is the same as a signal with frequency $\Omega_1 + 2\pi$, we will in general only consider an interval of frequencies of size 2π, usually the interval $[-\pi, \pi]$. In this interval:

- Low frequencies are close to $\Omega = 0$.
- High frequencies are close to $\Omega = -\pi$ and $\Omega = \pi$.

(In the interval, $[\pi, 3\pi]$ low frequencies are around $\Omega = 2\pi$, and high frequencies around $\Omega = \pi$ and $\Omega = 3\pi$; in the interval $[3\pi, 5\pi]$ low frequencies are around $4\Omega = \pi$, and high frequencies around $\Omega = 3\pi$ and $\Omega = 5\pi$, and so on)

Fourier series representation of DT periodic signals

We have shown that a periodic signal $x_N[n]$ with period N can be expressed as a sum of N complex exponentials:

$$x_N[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x_N[n] e^{-jk\Omega_0 n}$$

The frequencies of the complex exponentials are multiples of the fundamental frequency $\Omega_0 = \frac{2\pi}{N}$ and they are distributed within an interval of size 2π, for instance for the interval $[0, 2\pi]$ they are:

$$0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, \ldots, k\frac{2\pi}{N}, \ldots, (N-1)\frac{2\pi}{N}$$

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The Fourier transform of discrete-time signals

Non-periodic discrete-time signals can also be expressed as a linear combination of complex exponentials with different frequencies. In other words, DT signals also have a **Fourier transform**.

If we compare the **Fourier transform of DT signals** with the **Fourier series of DT signals** we note that:

- The Fourier series of DT signals consists of N harmonic frequencies in an interval of size 2π.
- The Fourier transform of DT signals uses every frequency within an interval of size 2π.

If we compare the **Fourier transform of DT signals** and the **Fourier transform of CT signals**, we note that:

- The Fourier transform of a CT signal uses all the frequencies within the interval $[-\infty, \infty]$.
- The Fourier transform of a DT uses all the frequencies within an interval of size 2π.

The Fourier transform of discrete-time signals

Given a signal $x[n]$, we denote by $X(\Omega)$ its Fourier transform:

$$x[n] \overset{FT}{\Longleftrightarrow} X(\Omega)$$

The equations for the Fourier transform of DT signals are:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\Omega) e^{j\Omega n} d\Omega$$

Example

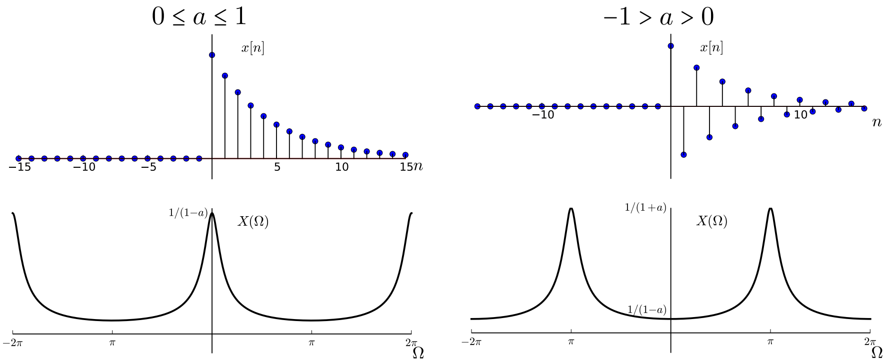
Obtain the Fourier transform of the signal $x[n] = \delta[n]$ and sketch it.

Example

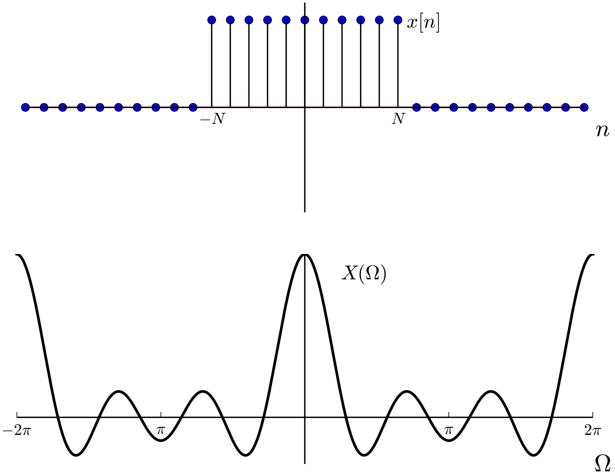
Obtain the Fourier transform of the signal $x[n] = \delta[n - n_0]$ and sketch it.

Example

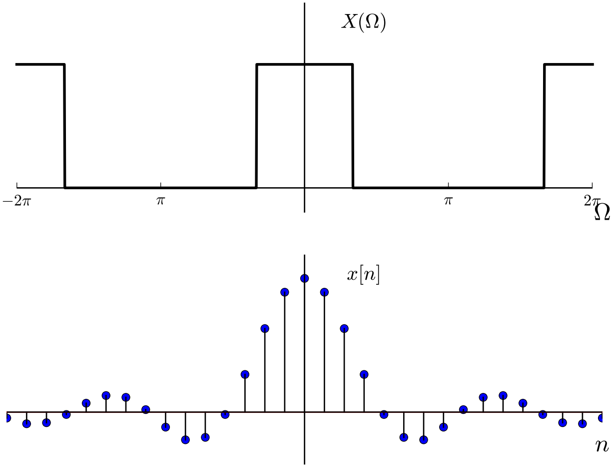
Draw the signal $x[n] = a^n u[n]$ and its Fourier transform $X(\Omega)$ for $0 \leq a \leq 1$ and $-1 > a > 0$.



Example



Example



Remember

- The Fourier transform describes a signal in the frequency domain, i.e. describes its frequency components.
- Because of the nature of DT complex exponentials, we only need an interval of frequencies of size 2π.
- If we want to consider all the frequencies, we only need to replicate the Fourier transform in the original interval. Consequently, the Fourier transform of DT signals can be seen as a periodic function.
- Finally, low frequency components are located around the frequencies $\Omega = 0, \pm 2\pi, \pm, 4\pi, \ldots, \pm k2\pi, \ldots$, whereas high frequencies are around $\Omega = \pi, \pm 3\pi, \ldots, \pm(2k+1)\pi, \ldots$.

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Properties I

Consider

$$x[n] \overset{FT}{\Longleftrightarrow} X(\Omega)$$
$$x_1[n] \overset{FT}{\Longleftrightarrow} X_1(\Omega)$$
$$x_2[n] \overset{FT}{\Longleftrightarrow} X_2(\Omega)$$

- Periodicity

$$X(\Omega + 2\pi) = X(\Omega)$$

- Leariry

$$Ax_1[n] + Bx_2[n] \overset{FT}{\Longleftrightarrow} AX_1(\Omega) + BX_2(\Omega)$$

- Time shift

$$x[n - n_0] \overset{FT}{\Longleftrightarrow} e^{j\Omega n_0} X(\Omega)$$

Properties II

- Frequency shift

$$e^{j\Omega_0 n}x[n] \overset{FT}{\Longleftrightarrow} X(\Omega - \Omega_0)$$

- Reflection

$$x[-n] \overset{FT}{\Longleftrightarrow} X(-\Omega_0)$$

- Real signals

$$\begin{aligned} x[n]\text{real} \quad &\Longrightarrow \quad X(\Omega_0) = X^*(-\Omega_0) \\ &\Longrightarrow \quad |X(\Omega_0)| = |X(-\Omega_0)| \\ &\Longrightarrow \quad \angle X(\Omega_0) = -\angle X(-\Omega_0) \end{aligned}$$

Properties III

- Convolution

$$x_1[n] * x_2[n] \overset{FT}{\Longleftrightarrow} X_1(\Omega)X_2(\Omega)$$

- Parseval's relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} |X(\Omega)|^2 d\Omega$$

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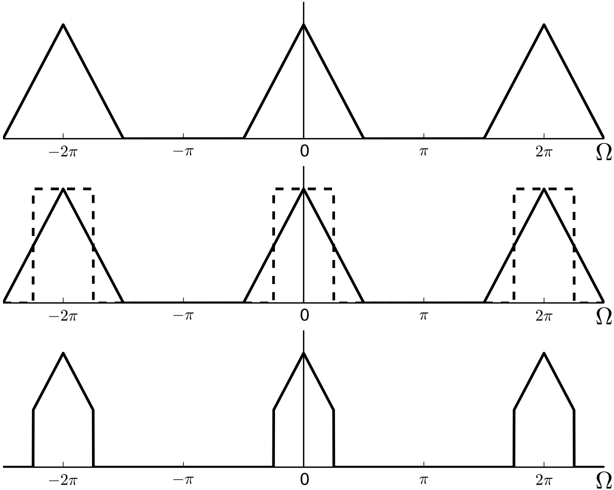
Linear-time invariant systems and the Fourier transform

$$x[n],\ X(\Omega) \longrightarrow \boxed{H(\Omega)} \longrightarrow y[n],\ Y(\Omega)$$

$$y[n] = x[n] \star h[n] \quad \overset{FT}{\Longleftrightarrow} \quad Y(\Omega) = X(\Omega)H(\Omega)$$

$$\begin{aligned} x[n] &\overset{FT}{\Longleftrightarrow} X(\Omega) \\ h[n] &\overset{FT}{\Longleftrightarrow} H(\Omega) \\ y[n] &\overset{FT}{\Longleftrightarrow} Y(\Omega) \end{aligned}$$

Linear-time invariant systems and the Fourier transform



Lowpass, highpass and bandpass filters