Let $M(\omega) = |H(e^{j\omega})|$ and $\theta'(\omega) = \frac{d\theta(\omega)}{d\omega}$. Also note that $M(\omega) = M(-\omega)$, $M'(\omega) = M'(-\omega)$ and $\theta'(\omega) = \theta'(-\omega)$. Therefore,

$$D = \frac{1}{2\pi}\int_0^{\pi} \left\{|M'(\omega) + M(\omega)\theta'(\omega)|^2 + |M'(\omega) - M(\omega)\theta'(\omega)|^2\right\} d\omega.$$

Now since the integrand is positive for all ω, it is sufficient to minimize the integrand to minimize D. Therefore,

$$\frac{d}{d\theta'(\omega)} \left\{|M'(\omega) + M(\omega)\theta'(\omega)|^2 + |M'(\omega) - M(\omega)\theta'(\omega)|^2\right\} = 0.$$

Simplifying this, we obtain

$$2M^2(\omega)\theta'(\omega) = 0 \quad \Rightarrow \quad \theta'(\omega) = 0.$$

However, since $\theta(\omega)$ is odd, the only function that satisfies $\theta'(\omega) = 0$ is $\theta(\omega) = 0$.

6.64. (a) From Table 5.1 we know that when a signal is real and even, then its Fourier transform is also real and even. Therefore, using duality, we may say that if the Fourier transform of a signal is real and even, then the signal is real and even. Therefore, $h_r[n] = h_r[-n]$.

By using the time shift property, we know that if $H(e^{j\omega}) = H_r(e^{j\omega})e^{-j\omega M}$, then

$$h[n] = h_r[n - M].$$

(b) We have

$$h[M + n] = h_r[M + n - M] = h_r[n].$$

Also,

$$h[M - n] = h_r[M - n - M] = h_r[-n].$$

Since $h_r[n] = h_r[-n]$,

$$h[M + n] = h[M - n].$$

(c) Since $h[n]$ is causal, $h[-k] = 0$ for $k > 0$. But due to the symmetry property,

$$h[-k] = h_r[-k - M] = h_r[k + M] = h[k + 2M].$$

Therefore,

$$h[k + 2M] = 0 \quad \text{for } k > 0.$$

It follows that

$$h[n] = 0 \quad \text{for } n > 2M.$$

6.65. (a) We have

$$|[B(e^{j\omega})|^2 = \frac{1}{1 + \tan^2(\omega/2)} = \frac{1}{\sec^2(\omega/2)} = \cos^2(\omega/2).$$

(b) If $B(e^{j\omega}) = a\cos(\omega/2)$, then

$$|[B(e^{j\omega})|^2 = aa^*\cos^2(\omega/2).$$

If we want this to be the same as part (a), then $aa^* = 1$. Therefore,

$$a = e^{j\theta(\omega)}.$$

(c) Taking the Fourier transform of the given difference equation we obtain

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \alpha + \beta e^{-j\omega\gamma} = e^{-j\omega\gamma/2}[\alpha e^{j\omega\gamma/2} + \beta e^{-j\omega\gamma/2}].$$

Comparing with

$$B(e^{j\omega}) = e^{-j\theta(\omega)}\left[\frac{1}{2}e^{j\omega/2} + \frac{1}{2}e^{-j\omega/2}\right],$$

we find that $H(e^{j\omega}) = B(e^{j\omega})$ when

$$\alpha = \beta = \frac{1}{2}, \qquad \gamma = 1.$$

6.66. (a) Since $h_k[n] = e^{j2\pi nk/N}h_0[n]$, we have

$$H_k(e^{j\omega}) = H_0(e^{j(\omega - 2\pi k/N)}).$$

Below are shown the sketches of $H_k(e^{j\omega})$ for $N = 16$ in Figure S6.66.

(b) Overall frequency response of the system is $H_{ov}(e^{j\omega}) = \sum_{k=0}^{N-1} H_k(e^{j\omega})$. For this to be an identity system, we require that $H_{ov}(e^{j\omega}) = 1$ for all ω. Therefore, we want the non-zero portions of the $H_k(e^{j\omega})$s to be non-overlapping and yet cover the region from $-\pi$ to π. We see that this is achieved by having $\omega_c = \pi/N$.

(c) Since $H_{ov}(e^{j\omega}) = \sum_{k=0}^{N-1} H_k(e^{j\omega})$, we have

$$h_{ov}[n] = \sum_{k=0}^{N-1} h_k[n] = \sum_{k=0}^{N-1} h_0[n]e^{j2\pi kn/N} = h_0[n]\sum_{k=0}^{N-1} e^{j2\pi kn/N}.$$

Therefore,

$$r[n] = \sum_{k=0}^{N-1} e^{j2\pi kn/N} = \begin{cases} N, & n = 0, \pm N, \pm 2N, \cdots \\ 0, & \text{otherwise} \end{cases}$$

Therefore, $r[n] = N\sum_{k=-\infty}^{\infty} \delta[n - kN]$ and is as sketched in Figure S6.66.

Figure S6.66

(d) In order for $h[n]$ to be the impulse response of an identity system, we require that $h[n] = \delta[n]$. From part (c), we know that

$$h[n] = h_0[n]\sum_{k=-\infty}^{\infty} \delta[n - kN].$$

Therefore, the necessary and sufficient condition for $h[n]$ to be $\delta[n]$ is

$$h_0[0] = \frac{1}{N} \quad \text{and} \quad h_0[kN] = 0 \quad \text{for } k = \pm 1, \pm 2, \cdots.$$

Chapter 7 Answers

7.1. From the Nyquist sampling theorem, we know that only if $X(j\omega) = 0$ for $|\omega| > \omega_s/2$ will be signal be recoverable from its samples. Therefore, $X(j\omega) = 0$ for $|\omega| > 5000\pi$.

7.2. From the Nyquist theorem, we know that the sampling frequency in this case must be at least $\omega_s = 2000\pi$. In other words, the sampling period should be at most $T = 2\pi/(\omega_s) = 1 \times 10^{-3}$. Clearly, only (a) and (c) satisfy this condition.

7.3. (a) We can easily show that $X(j\omega) = 0$ for $|\omega| > 4000\pi$. Therefore, the Nyquist rate for this signal is $\omega_N = 2(4000\pi) = 8000\pi$.

(b) From Table 4.2 we know that, $X(j\omega)$ is a rectangular pulse for which $X(j\omega) = 0$ for $|\omega| > 4000\pi$. Therefore, the Nyquist rate for this signal is $\omega_N = 2(4000\pi) = 8000\pi$.

(c) From Tables 4.1 and 4.2, we know that $X(j\omega)$ is the convolution of two rectangular pulses each of which is zero for $|\omega| > 4000\pi$. Therefore, $X(j\omega) = 0$ for $|\omega| > 8000\pi$ and the Nyquist rate for this signal is $\omega_N = 2(8000\pi) = 16000\pi$.

7.4. If the signal $x(t)$ has a Nyquist rate of ω_0, then its Fourier transform $X(j\omega) = 0$ for $|\omega| > \omega_0/2$.

(a) From chapter 4,

$$y(t) = x(t) + x(t-1) \overset{FT}{\longleftrightarrow} Y(j\omega) = X(j\omega) + e^{-j\omega t}X(j\omega).$$

Clearly, we can only guarantee that $Y(j\omega) = 0$ for $|\omega| > \omega_0/2$. Therefore, the Nyquist rate for $y(t)$ is also ω_0.

(b) From chapter 4,

$$y(t) = \frac{dx(t)}{dt} \overset{FT}{\longleftrightarrow} Y(j\omega) = j\omega X(j\omega).$$

Clearly, we can only guarantee that $Y(j\omega) = 0$ for $|\omega| > \omega_0/2$. Therefore, the Nyquist rate for $y(t)$ is also ω_0.

(c) From chapter 4,

$$y(t) = x^2(t) \overset{FT}{\longleftrightarrow} Y(j\omega) = (1/2\pi)[X(j\omega) * X(j\omega)].$$

Clearly, we can guarantee that $Y(j\omega) = 0$ for $|\omega| > \omega_0$. Therefore, the Nyquist rate for $y(t)$ is $2\omega_0$.

(d) From chapter 4,

$$y(t) = x(t)\cos(\omega_0 t) \overset{FT}{\longleftrightarrow} Y(j\omega) = (1/2)X(j(\omega - \omega_0)) + (1/2)X(j(\omega + \omega_0)).$$

Clearly, we can guarantee that $Y(j\omega) = 0$ for $|\omega| > \omega_0 + \omega_0/2$. Therefore, the Nyquist rate for $y(t)$ is $3\omega_0$.

Using Table 4.2

$$p(t) \xleftrightarrow{FT} \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k2\pi/T).$$

From Table 4.1,

$$p(t-1) \xleftrightarrow{FT} \frac{2\pi}{T}e^{-j\omega}\sum_{k=-\infty}^{\infty}\delta(\omega - k\frac{2\pi}{T}) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\frac{2\pi}{T})e^{-jk\frac{2\pi}{T}}.$$

Since $y(t) = x(t)p(t-1)$, we have

$$\begin{aligned} Y(j\omega) &= (1/2\pi)[X(j\omega) * \mathcal{FT}\{p(t-1)\}] \\ &= (1/T)\sum_{k=-\infty}^{\infty} X(j(\omega - k\frac{2\pi}{T}))e^{-jk\frac{2\pi}{T}} \end{aligned}$$

Therefore, $Y(j\omega)$ consists of replicas of $X(j\omega)$ shifted by $k2\pi/T$ and added to each other (see Figure S7.5). In order to recover $x(t)$ from $y(t)$, we need to be able to isolate one replica of $X(j\omega)$ from $Y(j\omega)$.

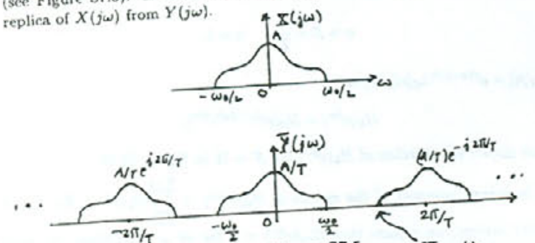


Figure S7.5

From the figure, it is clear that this is possible if we multiply $Y(j\omega)$ with

$$H(j\omega) = \begin{cases} T, & |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$

where $(\omega_0/2) < \omega_c < (2\pi/T) - (\omega_0/2)$.

7.6. Consider the signal $w(t) = x_1(t)x_2(t)$. The Fourier transform $W(j\omega)$ of $w(t)$ is given by

$$W(j\omega) = \frac{1}{2\pi}[X_1(j\omega) * X_2(j\omega)].$$

Since $X_1(j\omega) = 0$ for $|\omega| \ge \omega_1$ and $X_2(j\omega) = 0$ for $|\omega| \ge \omega_2$, we may conclude that $W(j\omega) = 0$ for $|\omega| \ge \omega_1 + \omega_2$. Consequently, the Nyquist rate for $w(t)$ is $\omega_s = 2(\omega_1 + \omega_2)$. Therefore, the *maximum* sampling period which would still allow $w(t)$ to be recovered is $T = 2\pi/(\omega_s) = \pi/(\omega_1 + \omega_2)$.

7.7. We note that

$$x_1(t) = h_1(t) * \left\{\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)\right\}.$$

From Figure 7.7 in the textbook, we know that the output of the zero-order hold may be written as

$$x_0(t) = h_0(t) * \left\{\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)\right\},$$

where $h_0(t)$ is as shown in Figure S7.7. By taking the Fourier transform of the two above equations, we have

$$\begin{aligned} X_1(j\omega) &= H_1(j\omega)X_p(j\omega) \\ X_0(j\omega) &= H_0(j\omega)X_p(j\omega) \end{aligned}$$

We now need to determine a frequency response $H_d(j\omega)$ for a filter which produces $x_1(t)$ at its output when $x_0(t)$ is its input. Therefore, we need

$$X_0(j\omega)H_d(j\omega) = X_1(j\omega).$$

The triangular function $h_1(t)$ may be obtained by convolving two rectangular pulses as shown in Figure S7.7.

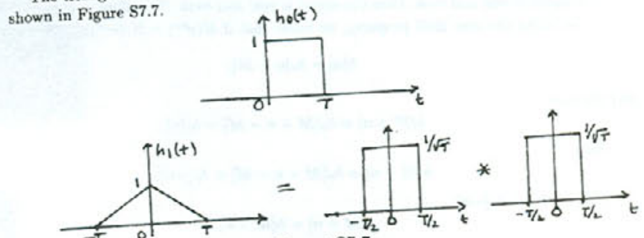


Figure S7.7

Therefore,

$$h_1(t) = \{(1/\sqrt{T})h_0(t+T/2)\} * \{(1/\sqrt{T})h_0(t+T/2)\}.$$

Taking the Fourier transform of both sides of the above equation,

$$H_1(j\omega) = \frac{1}{T}e^{j\omega T}H_0(j\omega)H_0(j\omega).$$

Therefore,

$$\begin{aligned} X_1(j\omega) &= H_1(j\omega)X_p(j\omega) \\ &= \frac{1}{T}e^{j\omega T}H_0(j\omega)H_0(j\omega)X_p(j\omega) \\ &= \frac{1}{T}e^{j\omega T}H_0(j\omega)X_0(j\omega) \end{aligned}$$

Therefore,

$$H_d(j\omega) = \frac{1}{T}e^{j\omega T}H_0(j\omega) = e^{j\omega T/2}\frac{2\sin(\omega T/2)}{\omega T}.$$

7.8. (a) Yes, aliasing does occur in this case. This may be easily shown by considering the sinusoidal term of $x(t)$ for $k = 5$. This term is a signal of the form $y(t) = (1/2)^5\sin(5\pi t)$. If $x(t)$ is sampled as $T = 0.2$, then we will always be sampling $y(t)$ at exactly its zero-crossings (This is similar to the idea presented in Figure 7.17 of your textbook). Therefore, the signal $y(t)$ appears to be identical to the signal $(1/2)^5\sin(0\pi t)$ for all time in the sampled signal. Therefore, the sinusoid $y(t)$ of frequency 5π is aliased into a sinusoid of frequency 0 in the sampled signal.

(b) The lowpass filter performs band limited interpolation on the signal $\hat{x}(t)$. But since aliasing has already resulted in the loss of the sinusoid $(1/2)^5\sin(5\pi t)$, the output will be of the form

$$x_r(t) = \sum_{k=0}^{4}\left(\frac{1}{2}\right)^k \sin(k\pi t).$$

The Fourier series representation of this signal is of the form

$$x_r(t) = \sum_{k=-4}^{4} a_k e^{-j(k\pi/t)}, \quad \text{where} \quad a_k = \begin{cases} 0, & k = 0 \\ -j(1/2)^{k+1}, & 1 \le k \le 4 \\ j(1/2)^{-k+1}, & -4 \le k \le -1 \end{cases}$$

7.9. The Fourier transform $X(j\omega)$ of $x(t)$ is as shown in Figure S7.9.

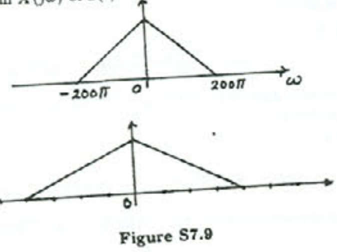


Figure S7.9

We know from the results on impulse-train sampling that

$$G(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)),$$

where $T = 2\pi/\omega_s = 1/75$. Therefore, $G(j\omega)$ is as shown in Figure S7.9. Clearly, $G(j\omega) = (1/T)X(j\omega) = 75X(j\omega)$ for $|\omega| \le 50\pi$.

7.10. (a) We know that $x(t)$ is not a band-limited signal. Therefore, it *cannot* undergo impulse-train sampling without aliasing.

(b) From the given $X(j\omega)$ it is clear that the signal $x(t)$ which is bandlimited. That is, $X(j\omega) = 0$ for $|\omega| > \omega_0$. Therefore, it must be possible to perform impulse-train sampling on this signal without experiencing aliasing. The minimum sampling rate required would be be $\omega_s = 2\omega_0$. This implies that the sampling period can at most be $T = 2\pi/\omega_s = \pi/\omega_0$.

(c) When $x(t)$ undergoes impulse train sampling with $T = 2\pi/\omega_0$, we would obtain the signal $g(t)$ with Fourier transform

$$G(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k2\pi/T)).$$

This is as shown in the Figure S7.10.

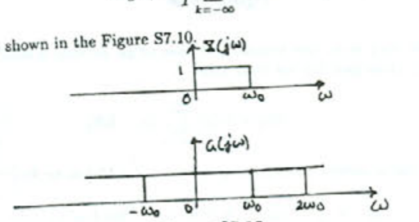


Figure S7.10

It is clear from the figure that no aliasing occurs, and that $X(j\omega)$ can be recover by using a filter with frequency response

$$H(j\omega) = \begin{cases} T, & 0 \le \omega \le \omega_0 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, the given statement is true.

7.11. We know from Section 7.4 that

$$X_d(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - 2\pi k)/T).$$

(a) Since $X_d(e^{j\omega})$ is just formed by shifting and summing replicas of $X(j\omega)$, we may as that if $X_d(e^{j\omega})$ is real, then $X(j\omega)$ must also be real.

(b) $X_d(e^{j\omega})$ consists of replicas of $X(j\omega)$ which are scaled by $1/T$. Therefore, if $X_d($ has a maximum of 1, then $X(j\omega)$ will have a maximum of $T = 0.5 \times 10^{-3}$.

(c) The region $3\pi/4 \le |\omega| \le \pi$ in the discrete-time domain corresponds to the re $3\pi/(4T) \le |\omega| \le \pi/T$ in the continuous-time domain. Therefore, if $X_d(e^{j\omega}) =$ $3\pi/4 \le |\omega| \le \pi$, then $X(j\omega) = 0$ for $1500\pi \le |\omega| \le 2000\pi$. But since we already $X(j\omega) = 0$ for $|\omega| \ge 2000\pi$, we have $X(j\omega) = 0$ for $|\omega| \ge 1500\pi$.

d) In this case, since π in discrete-time frequency domain corresponds to 2000π in the continuous-time frequency domain, this condition translates to $X(j\omega) = (j(\omega - 2000\pi))$.

From Section 7.4, we know that the discrete and continuous-time frequencies Ω and ω are related by $\Omega = \omega T$. Therefore, in this case for $\Omega = \frac{3\pi}{4}$, we find the corresponding value of ω to be $\omega = \frac{3\pi}{4}\frac{1}{T} = 3000\pi/4 = 750\pi$.

For this problem, we use an approach similar to the one used in Example 7.2. We assume that

$$x_c(t) = \frac{\sin(\pi t/T)}{\pi t}.$$

The overall output is

$$y_c(t) = x_c(t - 2T) = \frac{\sin[(\pi/T)(t - 2T)]}{\pi(t - 2T)}.$$

From $x_c(t)$, we obtain the corresponding discrete-time signal $x_d[n]$ to be

$$x_d[n] = x_c(nT) = \frac{1}{T}\delta[n].$$

Also, we obtain from $y_c(t)$, the corresponding discrete-time signal $y_d[n]$ to be

$$y_d[n] = y_c(nT) = \frac{\sin[\pi(n-2)]}{\pi T(n-2)}.$$

We note that the right-hand side of the above equation is always zero when $n \neq 2$. When $n = 2$, we may evaluate the value of the ratio using L' Hospital's rule to be $1/T$. Therefore,

$$y_d[n] = \frac{1}{T}\delta[n-2].$$

We conclude that the impulse response of the filter is

$$h_d[n] = \delta[n-2].$$

14. For this problem, we use an approach similar to the one used in Example 7.2. We assume that

$$x_c(t) = \frac{\sin(\pi t/T)}{\pi t}.$$

The overall output is

$$y_c(t) = \frac{d}{dt}x_c(t - \frac{T}{2}) = \frac{(\pi/T)\cos[(\pi/T)(t - T/2)]}{\pi(t - T/2)} - \frac{\pi\sin[(\pi/T)(t - T/2)]}{(\pi(t - T/2))^2}.$$

From $x_c(t)$, we obtain the corresponding discrete-time signal $x_d[n]$ to be

$$x_d[n] = x_c(nT) = \frac{1}{T}\delta[n].$$

Also, we obtain from $y_c(t)$, the corresponding discrete-time signal $y_d[n]$ to be

$$y_d[n] = y_c(nT) = \frac{(\pi/T)\cos[\pi(n - 1/2)]}{\pi T(n - 1/2)} - \frac{\sin[\pi(n - 1/2)]}{\pi T^2(n - 1/2)^2}.$$

The first term in the right-hand side of the above equation is always zero because $\cos[\pi(n - 1/2)] = 0$. Therefore,

$$y_d[n] = -\frac{\sin[\pi(n - 1/2)]}{\pi T^2(n - 1/2)^2}.$$

We conclude that the impulse response of the filter is

$$h_d[n] = -\frac{\sin[\pi(n - 1/2)]}{\pi T(n - 1/2)^2}.$$

7.15. In this problem we are interested in the lowest rate which $x[n]$ may be sampled without the possibility of aliasing. We use the approach used in Example 7.4 to solve this problem. To find the lowest rate at which $x[n]$ may be sampled while avoiding the possibility of aliasing, we must find an N such that

$$\frac{2\pi}{N} \geq 2\left(\frac{3\pi}{7}\right) \Rightarrow N \leq \frac{7}{3}.$$

Therefore, N can at most be 2.

7.16. Although the signal $x_1[n] = 2\sin(\pi n/2)/(\pi n)$ satisfies the first two conditions, it does not satisfy the third condition. This is because the Fourier transform $X_1(e^{j\omega})$ of this signal is a rectangular pulse which is zero for $\pi/2 < |\omega| < \pi/2$. We also note that the signal $x[n] = 4[\sin(\pi n/2)/(\pi n)]^2$ satisfies the first two conditions. From our numerous encounters with this signal, we know that its Fourier transform $X(e^{j\omega})$ is given by the periodic convolution of $X_1(e^{j\omega})$ with itself. Therefore, $X(e^{j\omega})$ will be a triangular function in the range $0 \leq |\omega| \leq \pi$. This obviously satisfies the third condition as well. Therefore, the desired signal is $x[n] = 4[\sin(\pi n/2)/(\pi n)]^2$.

7.17. In this problem, we wish to determine the effect of decimating the impulse response of the given filter by a factor of 2. As explained in Section 7.5.2, the process of decimation may be broken up into two steps. In the first step we perform impulse train sampling on $h[n]$ to obtain

$$h_p[n] = \sum_{k=-\infty}^{\infty} h[2k]\delta[n - 2k].$$

The decimated sequence is then obtained using

$$h_1[n] = h[2n] = h_p[2n].$$

Using eq. (7.37), we obtain the Fourier transform $H_p(e^{j\omega})$ of $h_p[n]$ to be

$$H_p(e^{j\omega}) = (1/2)H(e^{j\omega}) + (1/2)H(e^{j(\omega - \pi)}).$$

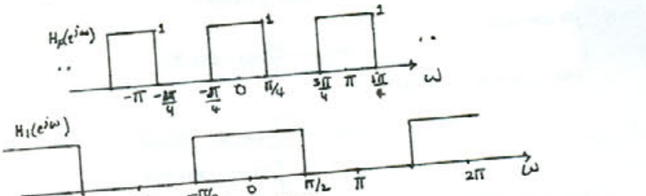


Figure S7.17

This is as shown in Figure S7.17.

From eq. (7.49) we know that the Fourier transform of the decimated impulse response is

$$H_1(e^{j\omega}) = H_p(e^{j\omega/2}).$$

In other words, $H_1(e^{j\omega})$ is $H_p(e^{j\omega})$ expanded by a factor of 2. This is as shown in the figure above. Therefore, $h_1[n] = h[2n]$ is the impulse response of an ideal lowpass filter with a passband gain of unity and a cutoff frequency of $\pi/2$.

7.18. From Figure 7.37, it is clear interpolation by a factor of 2 results in the frequency response getting compressed by a factor of 2. Interpolation also results in a magnitude scaling by a factor of 2. Therefore, in this problem, the interpolated impulse response will correspond to an ideal lowpass filter with cutoff frequency $\pi/$ and a passband gain of 2.

7.19. The Fourier transform of $x[n]$ is given by

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \omega_1 \\ 0, & \text{otherwise} \end{cases}.$$

This is as shown in Figure S7.19.

(a) When $\omega_1 \leq 3\pi/5$, the Fourier transform $X_1(e^{j\omega})$ of the output of the zero-insertion system is as shown in Figure S7.19. The output $W(e^{j\omega})$ of the lowpass filter is as shown in Figure S7.19. The Fourier transform of the output of the decimation system $Y(e^{j\omega})$ is an expanded or stretched out version of $W(e^{j\omega})$. This is as shown in Figure S7.19.

Therefore,

$$y[n] = \frac{1}{5}\frac{\sin(5\omega_1 n/3)}{\pi n}.$$

(b) When $\omega_1 > 3\pi/5$, the Fourier transform $X_1(e^{j\omega})$ of the output of the zero-insertion system is as shown in Figure S7.19. The output $W(e^{j\omega})$ of the lowpass filter is as shown in Figure S7.19.

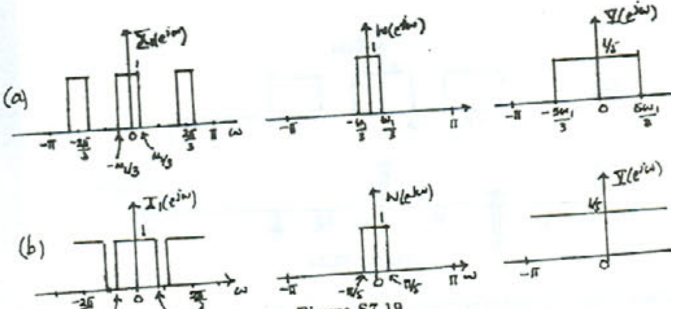


Figure S7.19

The Fourier transform of the output of the decimation system $Y(e^{j\omega})$ is an exp or stretched out version of $W(e^{j\omega})$. This is as shown in Figure S7.19. Therefore,

$$y[n] = \frac{1}{5}\delta[n].$$

7.20. (a) Suppose that $X(e^{j\omega})$ is as shown in Figure S7.20, then the Fourier transform X of the output of S_A, the Fourier transform $X_1(e^{j\omega})$ of the output of the lowpass and the Fourier transform $X_B(e^{j\omega})$ of the output of S_B are all shown in the below. Clearly this system accomplishes the filtering task.

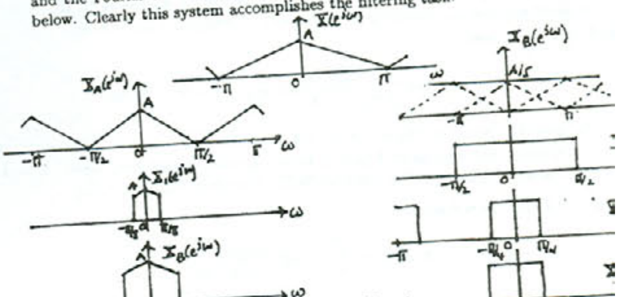


Figure S7.20

(b) Suppose that $X(e^{j\omega})$ is as shown in Figure S7.20, then the Fourier transform $X_1(e^{j\omega})$ of the output of the firs of the output of S_B, the Fourier transform $X_1(e^{j\omega})$ of the output of the filter, the Fourier transform $X_A(e^{j\omega})$ of the output of S_A, the Fourier transfor of the output of the first lowpass filter are all shown in the figures below. Clearly this system does not accomplish the filtering task.

7.21. (a) The Nyquist rate for the given signal is $2 \times 5000\pi = 10000\pi$. Therefore, in order to be able to recover $x(t)$ from $x_p(t)$, the sampling period must at most be $T_{max} = \frac{2\pi}{10000\pi} = 2 \times 10^{-4}$ sec. Since the sampling period used is $T = 10^{-4} < T_{max}$, $x(t)$ can be recovered from $x_p(t)$.

(b) The Nyquist rate for the given signal is $2 \times 15000\pi = 30000\pi$. Therefore, in order to be able to recover $x(t)$ from $x_p(t)$, the sampling period must at most be $T_{max} = \frac{2\pi}{30000\pi} = 0.66 \times 10^{-4}$ sec. Since the sampling period used is $T = 10^{-4} > T_{max}$, $x(t)$ cannot be recovered from $x_p(t)$.

(c) Here, $\mathcal{I}m\{X(j\omega)\}$ is not specified. Therefore, the Nyquist rate for the signal $x(t)$ is indeterminate. This implies that one cannot guarantee that $x(t)$ would be recoverable from $x_p(t)$.

(d) Since $x(t)$ is real, we may conclude that $X(j\omega) = 0$ for $|\omega| > 5000$. Therefore, the answer to this part is identical to that of part (a).

(e) Since $x(t)$ is real, $X(j\omega) = 0$ for $|\omega| > 15000\pi$. Therefore, the answer to this part is identical to that of part (b).

(f) If $X(j\omega) = 0$ for $|\omega| > \omega_1$, then $X(j\omega) * X(j\omega) = 0$ for $|\omega| > 2\omega_1$. Therefore, in this part, $X(j\omega) = 0$ for $|\omega| > 7500\pi$. The Nyquist rate for this signal is $2 \times 7500\pi = 15000\pi$. Therefore, in order to be able to recover $x(t)$ from $x_p(t)$, the sampling period must at most be $T_{max} = \frac{2\pi}{15000\pi} = 1.33 \times 10^{-4}$ sec. Since the sampling period used is $T = 10^{-4} < T_{max}$, $x(t)$ can be recovered from $x_p(t)$.

(g) If $|X(j\omega)| = 0$ for $\omega > 5000\pi$, then $X(j\omega) = 0$ for $\omega > 5000\pi$. Therefore, the answer to this part is identical to the answer of part (a).

7.22. Using the properties of the Fourier transform, we obtain

$$Y(j\omega) = X_1(j\omega)X_2(j\omega).$$

Therefore, $Y(j\omega) = 0$ for $|\omega| > 1000\pi$. This implies that the Nyquist rate for $y(t)$ is $2 \times 1000\pi = 2000\pi$. Therefore, the sampling period T can at most be $2\pi/(2000\pi) = 10^{-3}$ sec. Therefore we have to use $T < 10^{-3}$ sec in order to be able to recover $y(t)$ from $y_p(t)$.

7.23. (a) We may express $p(t)$ as

$$p(t) = p_1(t) - p_1(t - \Delta),$$

where $p_1(t) = \sum_{k=-\infty}^{\infty} \delta(t - k2\Delta)$. Now,

$$P_1(j\omega) = \frac{\pi}{\Delta} \sum_{k=-\infty}^{\infty} \delta(\omega - \pi/\Delta).$$

Therefore,

$$P(j\omega) = P_1(j\omega) - e^{-j\omega\Delta} P_1(j\omega)$$

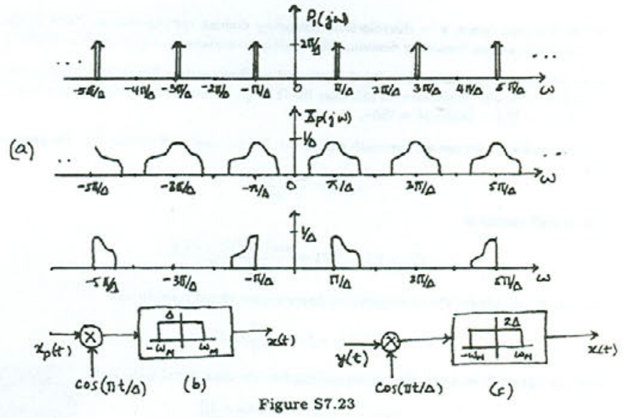


Figure S7.23

is as shown in Figure S7.23.

Now,

$$X_p(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * P(j\omega)\right].$$

Therefore, $X_p(j\omega)$ is as sketched below for $\Delta < \pi/(2\omega_M)$. The corresponding $Y(j\omega)$ is also sketched in Figure S7.23.

(b) The system which can be used to recover $x(t)$ from $x_p(t)$ is as shown in Figure S7.23.

(c) The system which can be used to recover $x(t)$ from $x(t)$ is as shown in Figure S7.23.

(d) We see from the figures sketched in part (a) that aliasing is avoided when $\omega_M \le \pi/\Delta$. Therefore, $\Delta_{max} = \pi/\omega_M$.

7.24. We may express $s(t)$ as $s(t) = \hat{s}(t) - 1$, where $\hat{s}(t)$ is as shown in Figure S7.24.

We may easily show that

$$\hat{S}(j\omega) = \sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k\Delta/T)}{k}\delta(\omega - k2\pi/T).$$

From this, we obtain

$$S(j\omega) = \hat{S}(j\omega) - 2\pi\delta(\omega) = \sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k\Delta/T)}{k}\delta(\omega - k2\pi/T) - 2\pi\delta(\omega).$$

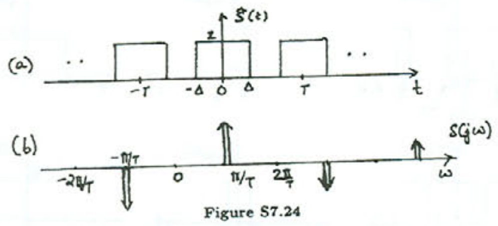


Figure S7.24

Clearly, $S(j\omega)$ consists of impulses spaced every $2\pi/T$.

(a) If $\Delta = T/3$, then

$$S(j\omega) = \sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k/3)}{k}\delta(\omega - k2\pi/T) - 2\pi\delta(\omega).$$

Now, since $w(t) = s(t)x(t)$,

$$W(j\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k/3)}{k}X(j(\omega - k2\pi/T)) - 2\pi X(j\omega).$$

Therefore, $W(j\omega)$ consists of replicas of $X(j\omega)$ which are spaced $2\pi/T$ apart. In order to avoid aliasing, ω_M should be less that π/T. Therefore, $T_{max} = \pi/\omega_M$.

(b) If $\Delta = T/3$, then

$$S(j\omega) = \sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k/4)}{k}\delta(\omega - k2\pi/T) - 2\pi\delta(\omega).$$

We note that $S(j\omega) = 0$ for $k = 0, \pm 2, \pm 4, \cdots$. This is as sketched in Figure S7.24

Therefore, the replicas of $X(j\omega)$ in $W(j\omega)$ are now spaced $4\pi/T$ apart. In order to avoid aliasing, ω_M should be less that $2\pi/T$. Therefore, $T_{max} = 2\pi/\omega_M$.

7.25. Here, $x_r(kT)$ can be written as

$$x_r(kT) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\sin[\pi(k-n)]}{\pi(k-n)}.$$

Note that when $n \ne k$,

$$\frac{\sin[\pi(k-n)]}{\pi(k-n)} = 0$$

and when $n = k$,

$$\frac{\sin[\pi(k-n)]}{\pi(k-n)} = 1.$$

Therefore,

$$x_r(kT) = x(kT).$$

7.26. We note that

$$P(j\omega) = \frac{2\pi}{T}\delta(\omega - k2\pi/T).$$

Also, since $x_p(t) = x(t)p(t)$,

$$X_p(j\omega) = \frac{1}{2\pi}\{X(j\omega) * P(j\omega)\} = \frac{1}{T}X(j(\omega - k2\pi/T)).$$

This is sketched in Figure S7.26.

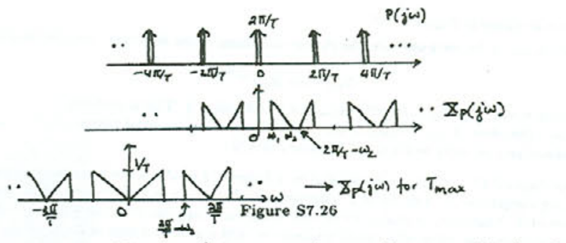


Figure S7.26

Note that as T increases, $\frac{2\pi}{T} - \omega_2$ approaches zero. Also, we note that there is aliasing when

$$2\omega_1 - \omega_2 < \frac{2\pi}{T} - \omega_2 < \omega_2.$$

If $2\omega_1 - \omega_2 \ge 0$ (as given) then it is easy to see that aliasing does not occur when

$$0 \le \frac{2\pi}{T} - \omega_2 \le 2\omega_1 - \omega_2.$$

For maximum T, we must choose the minimum allowable value for $\frac{2\pi}{T} - \omega_2$ (which is zero). This implies that $T_{max} = 2\pi/\omega_2$. We plot $X_p(j\omega)$ for this case in Figure S7.26. Therefore, $A = T$, $\omega_b = 2\pi/T$, and $\omega_a = \omega_b - \omega_1$.

7.27. (a) Let $X_1(j\omega)$ denote the Fourier transform of the signal $x_1(t)$ obtained by multiplying $x(t)$ with $e^{-j\omega_0 t}$ Let $X_2(j\omega)$ be the Fourier transform of the signal $x_2(t)$ obtained at the output of the lowpass filter. Then, $X_1(j\omega)$, $X_2(j\omega)$, and $X_p(j\omega)$ are as shown in Figure S7.27.

(b) The Nyquist rate for the signal $x_2(t)$ is $2 \times (\omega_2 - \omega_1)/2 = \omega_2 - \omega_1$. Therefore, the sampling period T must be at most $2\pi/(\omega_2 - \omega_1)$ in order to avoid aliasing.

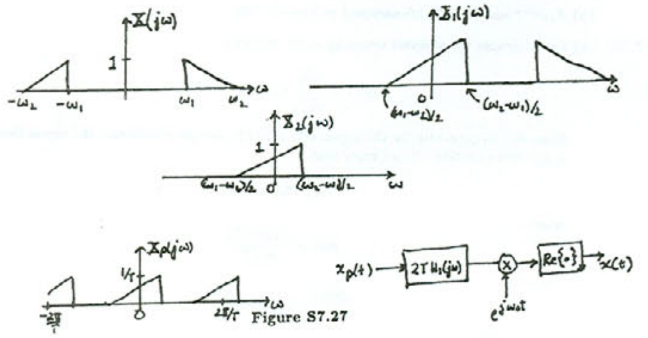


Figure S7.27

(c) A system that can be used to recover $x(t)$ from $x_p(t)$ is shown in Figure S7.27.

7.28. (a) The fundamental frequency of $x(t)$ is 20π rad/sec. From Chapter 4 we know that the Fourier transform of $x(t)$ is given by

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - 20\pi k).$$

This is as sketched below. The Fourier transform $X_c(j\omega)$ of the signal $x_c(t)$ is also sketched in Figure S7.28.

Note that

$$P(j\omega) = \frac{2\pi}{5 \times 10^{-3}} \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k/(5 \times 10^{-3}))$$

and

$$X_p(j\omega) = \frac{1}{2\pi}\left[X_c(j\omega) * P(j\omega)\right].$$

Therefore, $X_p(j\omega)$ is as shown in the Figure S7.28. Note that the impulses from adjacent replicas of $X_c(j\omega)$ add up at 200π. Now the Fourier transform $X(e^{j\Omega})$ of the sequence $x[n]$ is given by

$$X(e^{j\Omega}) = X_p(j\omega)|_{\omega=\Omega T}.$$

This is as shown in the Figure S7.28.

Since the impulses in $X(e^{j\omega})$ are located at multiples of a 0.1π, the signal $x[n]$ is periodic. The fundamental period is $2\pi/(0.1\pi) = 20$.

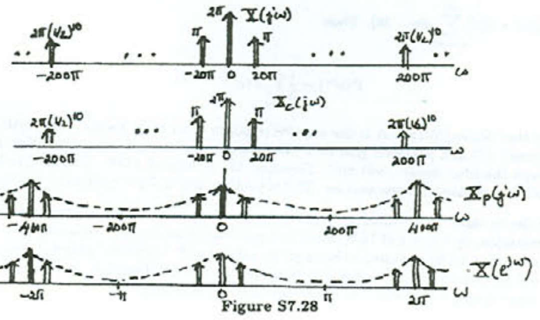


Figure S7.28

(b) The Fourier series coefficients of $x[n]$ are

$$a_k = \begin{cases} \frac{2\pi}{T}\left(\frac{1}{2}\right)^k, & k = 0, \pm1, \pm2, \cdots, \pm9 \\ \frac{4\pi}{T}\left(\frac{1}{2}\right)^{10}, & k = 10 \end{cases}.$$

7.29. From Section 7.1.1 we know that

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k2\pi/T)).$$

$X(e^{j\omega})$, $Y(e^{j\omega})$, $Y_p(j\omega)$, and $Y_c(j\omega)$ are as shown in Figure S7.29.

7.30. (a) Since $x_c(t) = \delta(t)$, we have

$$\frac{dy_c(t)}{dt} + y_c(t) = \delta(t).$$

Taking the Fourier transform we obtain

$$j\omega Y(j\omega) + Y(j\omega) = 1.$$

Therefore,

$$Y_c(j\omega) = \frac{1}{j\omega + 1}, \quad \text{and} \quad y_c(t) = e^{-t}u(t).$$

(b) Since $y_c(t) = e^{-t}u(t)$,

$$y[n] = y_c(nT) = e^{-nT}u[n].$$

Therefore,

$$Y(e^{j\omega}) = \frac{1}{1 - e^{-T}e^{-j\omega}}.$$

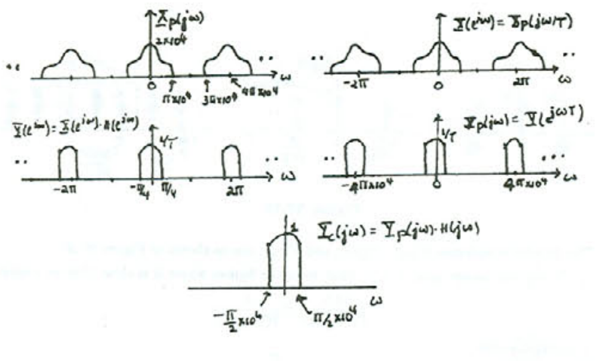


Figure S7.29

Also,

$$H(e^{j\omega}) = \frac{W(e^{j\omega})}{Y(e^{j\omega})} = \frac{1}{1/(1 - e^{-T}e^{-j\omega})} = 1 - e^{-T}e^{-j\omega}.$$

Therefore,

$$h[n] = \delta[n] - e^{-T}\delta[n-1].$$

7.31. In this problem for the sake of clarity we will use the variable Ω to denote discrete frequency.

Taking the Fourier transform of both sides of the given difference equation we obtain

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}.$$

Given that the sampling rate is greater than the Nyquist rate, we have

$$X(e^{j\Omega}) = \frac{1}{T}X_c(j\Omega/T), \qquad \text{for } -\pi \le \Omega \le \pi.$$

Therefore,

$$Y(e^{j\Omega}) = \frac{\frac{1}{T}X_c(j\Omega/T)}{1 - \frac{1}{2}e^{-j\Omega}}$$

for $-\pi \le \Omega \le \pi$. From this we get

$$\hat{Y}(j\omega) = Y(e^{j\omega T}) == \frac{\frac{1}{T}X_c(j\omega)}{1 - \frac{1}{2}e^{-j\omega T}}$$

for $-\pi/T \le \omega \le \pi/T$. In this range, $\hat{Y}(j\omega) = Y_c(j\omega)$. Therefore,

$$H_c(j\omega) = \frac{Y_c(j\omega)}{X_c(j\omega)} = \frac{1/T}{1 - \frac{1}{2}e^{-j\omega T}}.$$

7.32. Let $p[n] = \sum_{k=-\infty}^{\infty} \delta[n - 1 - 4k]$. Then from Chapter 5,

$$P(e^{j\omega}) = e^{-j\omega}\frac{2\pi}{4}\sum_{k=-\infty}^{\infty}\delta(\omega - 2\pi k/4) = \frac{\pi}{2}\sum_{k=-\infty}^{\infty} e^{-j2\pi k/4}\delta(\omega - 2\pi k/4).$$

Therefore,

$$\begin{aligned} G(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{j\theta})X(e^{j(\omega-\theta)})d\theta \\ &= \frac{1}{4}\sum_{k=0}^{3} e^{-j2\pi k/4}X(e^{j(\omega - 2\pi k/4)}) \end{aligned}$$

Since $X(e^{j\omega}) = 0$ for $\pi/4 \le |\omega| \le \pi$, $G(e^{j\omega})$ is as shown in Figure S7.32.

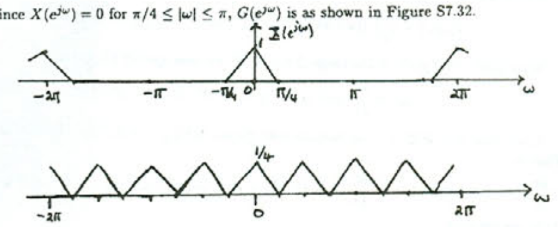


Figure S7.32

Clearly, in order to isolate just $X(e^{j\omega})$ we need to use an ideal lowpass filter with cutoff frequency $\pi/4$ and passband gain of 4. Therefore, in the range $|\omega| < \pi$,

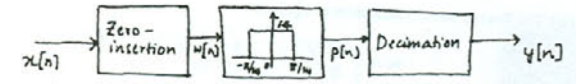
$$H(e^{j\omega}) = \begin{cases} 4, & |\omega| < \pi/4 \\ 0, & \pi/4 \le |\omega| \le \pi \end{cases}.$$

7.33. Let $y[n] = x[n]\sum_{k=-\infty}^{\infty}\delta[n-3k]$. Then

$$Y(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{3} X(e^{j(\omega-2\pi k/3)}).$$

Note that $\sin(\pi n/3)/(\pi n/3)$ is the impulse response of an ideal lowpass filter with cutoff frequency $\pi/3$ and passband gain of 3. Therefore, we now require that $y[n]$ when passed through this filter should yield $x[n]$. Therefore, the replicas of $X(e^{j\omega})$ contained in $Y(e^{j\omega})$ should not overlap with one another. This is possible only if $X(e^{j\omega}) = 0$ for $\pi/3 \le |\omega| \le \pi$.

7.34. In order to make $X(e^{j\omega})$ occupy the entire region from $-\pi$ to π, the signal $x[n]$ must be downsampled by a factor of 14/3. Since it is not possible to directly downsample by a non-integer factor, we first upsample the signal by a factor of 3. Therefore, after the upsampling we will need to reduce the sampling rate by $14/3 \times 3 = 14$. Therefore, the overall system for performing the sampling rate conversion is shown in Figure S7.34.



$w[n] = \begin{cases} x[\frac{n}{3}], & n = 0, \pm 3, \pm 6, \ldots \\ 0, & \text{otherwise} \end{cases}$ $\qquad y[n] = p[14n]$

Figure S7.34

7.35. (a) The signals $x_p[n]$ and $x_d[n]$ are sketched in Figure S7.35.

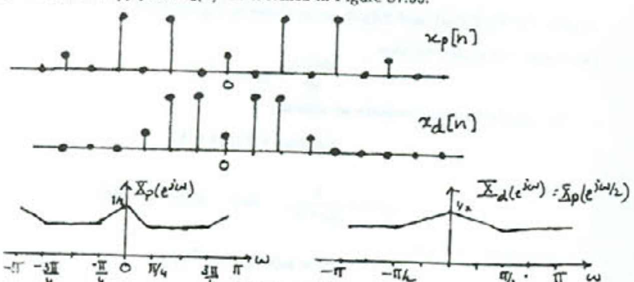


Figure S7.35

(b) $X_p(e^{j\omega})$ and $X_d(e^{j\omega})$ are sketched in Figure S7.35.

7.36. (a) Let us denote the sampled signal by $x_p(t)$. We have

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT).$$

Since the Nyquist rate for the signal $x(t)$ is $2\pi/T$, we can reconstruct the signal $x_p(t)$. From Section 7.2, we know that

$$x(t) = x_p(t) * h(t),$$

where

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

Therefore,

$$\frac{dx(t)}{dt} = x_p(t) * \frac{dh(t)}{dt}.$$

Denoting $\frac{dh(t)}{dt}$ by $g(t)$, we have

$$\frac{dx(t)}{dt} = x_p(t) * g(t) = \sum_{n=-\infty}^{\infty} x(nT)g(t-nT).$$

Therefore,

$$g(t) = \frac{dh(t)}{dt} = \frac{\cos(\pi t/T)}{t} - \frac{T\sin(\pi t/T)}{\pi t^2}.$$

(b) No.

7.37. We may write $p(t)$ as

$$p(t) = p_1(t) + p_1(t-\Delta),$$

where

$$p_1(t) = \sum_{k=-\infty}^{\infty}\delta(t - 2\pi k/W).$$

Therefore,

$$P(j\omega) = (1 + e^{-j\Delta\omega})P_1(j\omega),$$

where

$$P_1(j\omega) = W\sum_{k=-\infty}^{\infty}\delta(\omega - kW). \qquad \text{(S7.37}$$

Let us denote the product $p(t)f(t)$ by $g(t)$. Then,

$$g(t) = p(t)f(t) = p_1(t)f(t) + p_1(t-\Delta)f(t).$$

This may be written as

$$g(t) = ap_1(t) + bp_1(t-\Delta).$$

Therefore,

$$G(j\omega) = (a + be^{-j\omega\Delta})P_1(j\omega),$$

with $P_1(j\omega)$ is specified in eq. (S7.37-1). Therefore,

$$G(j\omega) = W\sum_{k=-\infty}^{\infty}[a + be^{-jk\Delta W}]\delta(\omega - kW).$$

We now have

$$y_1(t) = x(t)p(t)f(t).$$

Therefore,

$$Y_1(j\omega) = \frac{1}{2\pi}[G(j\omega) * X(j\omega)].$$

This gives us

$$Y_1(j\omega) = \frac{W}{2\pi}\sum_{k=-\infty}^{\infty}[a + be^{-jk\Delta W}]X(j(\omega - kW)).$$

In the range $0 < \omega < W$, we may specify $Y_1(j\omega)$ as

$$Y_1(j\omega) = \frac{W}{2\pi}\left[(a+b)X(j\omega) + (a + be^{-j\Delta W})X(j(\omega - W))\right].$$

Since $Y_2(j\omega) = Y_1(j\omega)H_1(j\omega)$, in the range $0 < \omega < W$ we may specify $Y_2(j\omega)$ as

$$Y_2(j\omega) = \frac{jW}{2\pi}\left[(a+b)X(j\omega) + (a + be^{-j\Delta W})X(j(\omega - W))\right].$$

Since $y_3(t) = x(t)p(t)$, in the range $0 < \omega < W$ we may specify $Y_3(j\omega)$ as

$$Y_3(j\omega) = \frac{W}{2\pi}\left[2X(j\omega) + (1 + e^{-j\Delta W})X(j(\omega - W))\right].$$

Given that $0 < W\Delta < \pi$, we require that $Y_2(j\omega) + Y_3(j\omega) = KX(j\omega)$ for $0 < \omega < W$. That is,

$$\frac{W}{2\pi}\left[(2 + ja + jb)X(j\omega)\right] + \frac{W}{2\pi}\left[(1 + e^{-j\Delta W} + ja + jbe^{-j\Delta W})X(j(\omega - W))\right] = KX(j\omega).$$

This implies that

$$1 + e^{-j\Delta W} + ja + jbe^{-j\Delta W} = 0.$$

Solving this we obtain

$$a = 1, \qquad b = -1,$$

when $W\Delta = \pi/2$. More generally, we get

$$a = \sin(W\Delta) + \frac{(1 + \cos(W\Delta))}{\tan(W\Delta)} \quad \text{and} \quad b = -\frac{1 + \cos(W\Delta)}{\sin(W\Delta)},$$

except when $W\Delta = \pi/2$. Finally, we also get $K = \frac{2\pi}{W}[1/(2 + ja + jb)]$.

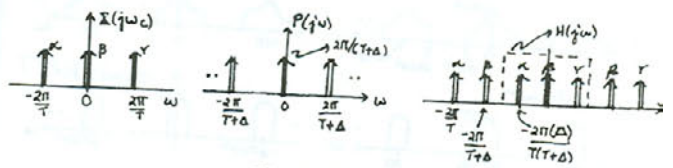


Figure S7.38

7.38. The Fourier transforms $X(j\omega)$, $P(j\omega)$, and $Y(j\omega)$ are as shown in Figure S7.38. Clearly, we cannot have $\Delta = 0$. Also, from the figures above it is clear that we require

$$\frac{2\pi\Delta}{T(T+\Delta)} \le \frac{1}{2(T+\Delta)}.$$

This implies that

$$\Delta \le \frac{T}{4\pi}.$$

Also from the figures, it is clear that

$$a = \frac{\frac{2\pi\Delta}{T(T+\Delta)}}{\frac{2\pi}{T}} = \frac{\Delta}{T+\Delta}.$$

7.39. (a) Using Trigonometric identities,

$$\cos\left(\frac{\omega_s}{2}t + \phi\right) = \cos\left(\frac{\omega_s}{2}t\right)\cos(\phi) - \sin\left(\frac{\omega_s}{2}t\right)\sin(\phi).$$

Therefore,

$$g(t) = -\sin\left(\frac{\omega_s}{2}t\right)\sin(\phi).$$

(b) By replacing ω_s with $2\pi/T$, and t by NT in the above equation, we get

$$\begin{aligned} g(nT) &= -\sin\left(\frac{2\pi}{2T}nT\right)\sin(\phi) \\ &= -\sin(n\pi)\sin(\phi). \end{aligned}$$

Clearly, the right-hand side of the above equation is zero for $n = 0, \pm 1, \pm 2, \cdots$.

(c) From parts (a) and (b), we get

$$\begin{aligned} x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) &= \sum_{n=-\infty}^{\infty}\delta(t-nT)\left\{\cos\left(\frac{\omega_s}{2}nT\right)\cos(\phi) + g(nT)\right\} \\ &= \sum_{n=-\infty}^{\infty}\delta(t-nT)\cos\left(\frac{\omega_s}{2}nT\right)\cos(\phi) \end{aligned}$$

When this signal is passed through a lowpass filter, we are in effect performing band-limited interpolation. This results in the signal

$$y(t) = \cos\left(\frac{\omega_s}{2}t\right)\cos(\phi).$$

7.40. (a) The Fourier transform $V(j\omega)$ is as shown in Figure S7.40.

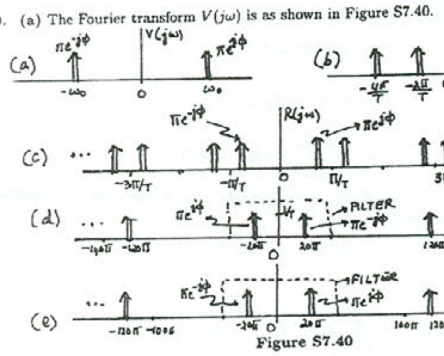


Figure S7.40

(b) The Fourier transform $I(j\omega)$ is

$$I(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - 2\pi k/T).$$

This is as shown in Figure S7.40.

(c) The Nyquist rate for $v(t)$ is $2\omega_0$. Therefore,

$$\frac{2\pi}{T_{max}} = 2\omega_0 \quad\Rightarrow\quad T_{max} = \frac{\pi}{\omega_0}.$$

The cutoff frequency of the lowpass filter has to be ω_0.

(d) Now,

$$R(j\omega) = \frac{1}{2\pi}\frac{1}{T}\sum_{k=-\infty}^{\infty}V(j(\omega - 2\pi k/T)).$$

Since $\omega_0 = 2\pi(60)$ rad/sec, we have $2\pi/T = 120\pi + 20\pi = 140\pi$. Therefore $R(j\omega)$ is as shown in Figure S7.40.

Therefore, $v_a(t)$ obtained by passing $r(t)$ through a lowpass filter with cutoff frequency $2\pi(20)$ rad/sec is

$$v_a(t) = \frac{1}{T}\cos(20\pi t - \phi).$$

Therefore,

$$\omega_a = 20\pi, \quad \phi_a = -\phi, \quad \text{and} \quad A_a = \frac{1}{T}.$$

(e) Here, $2\pi/T = 120\pi - 20\pi = 100\pi$. Therefore, $R(j\omega)$ is as shown in Figure S7.40. It follows that

$$v_a(t) = \frac{1}{T}\cos(20\pi t + \phi).$$

and

$$\omega_a = 20\pi, \quad \phi_a = \phi, \quad \text{and} \quad A_a = \frac{1}{T}.$$

7.41. In this problem, to avoid confusion we use the variable Ω to indicate discrete-time frequency.

(a) The Nyquist rate for the signal $x(t)$ is $2\omega_M$. Therefore, the sampling theorem states that $x(t)$ has to be sampled at least every π/ω_M. In this part, $T < \pi/\omega_M$. Therefore, $y_c(t)$ will be equal to $x(t)$ as long as $y[n] = x[n]$. Now,

$$\begin{aligned} s[n] &= x(nT_0) + \alpha x(nT_0 - T_0) \\ &= x[n] + \alpha x[n-1]. \end{aligned}$$

Therefore, if we require $y[n] = x[n]$ then,

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{S(e^{j\Omega})} = \frac{X(e^{j\Omega})}{X(e^{j\Omega}) + \alpha e^{-j\Omega}X(e^{j\Omega})} = \frac{1}{1 + \alpha e^{-j\Omega}}.$$

Therefore, the difference equation for the filter $h[n]$ is

$$y[n] + \alpha y[n-1] = s[n].$$

(b) From Figures P7.41(a) and (b), we have

$$H_{eq}(j\omega) = \frac{A}{T_0}H(e^{j\omega T_0}), \tag{S7.41-1}$$

where $H_{eq}(j\omega)$ is the system response of the overall continuous-time system. Since we require that $y_c(t) = x(t)$,

$$H_{eq}(j\omega) = \frac{Y_c(j\omega)}{S_c(j\omega)} = \frac{1}{1 + \alpha e^{-j\omega T_0}}. \tag{S7.41-2}$$

Comparing this with eq.(S7.41-1), we get $A = T_0$.

(c) We require a T which avoids aliasing. Therefore, $T < \pi/\omega_M$. We also require that

$$H_{eq}(j\omega) = \frac{1}{1 + \alpha e^{-j\omega T_0}}, \qquad -\omega_M \le \omega \le \omega_M.$$

But,

$$H_{eq}(j\omega) = \frac{A}{T}H(e^{j\omega T}), \qquad -\frac{\pi}{T} \le \omega \le \frac{\pi}{T}.$$

For these to be consistent, we need $A = T$ and

$$H(e^{j\Omega}) = \frac{1}{1 + \alpha e^{-j\Omega T/T_0}}$$

for $-\pi \le \Omega \le \pi$.

7.42. In this problem, to avoid confusion we use the variable Ω to indicate discrete-time frequency.

Using Parseval's theorem and the fact that $X_c(j\omega) = 0$ for $|\omega| \ge \omega_0$, we get

$$E_c = \int_{-\infty}^{\infty}|x_c(t)|^2 dt = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}|X_c(j\omega)|^2 d\omega.$$

Also, using Parseval's theorem we have

$$E_d = \sum_{n=-\infty}^{\infty}|x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\Omega})|^2 d\Omega.$$

But since $X(e^{j\Omega}) = \frac{1}{T}X_c(j\Omega/T)$ for $-\pi \le \Omega \le \pi$, we may write

$$E_d = \frac{1}{2\pi T^2}\int_{-\pi}^{\pi}|X_c(j\Omega/T)|^2 d\Omega.$$

Replacing Ω/T by ω, we get

$$E_d = \frac{1}{2\pi T}\int_{-\pi/T}^{\pi/T}|X_c(j\omega)|^2 d\omega.$$

Also, since $2\pi/T \ge 2\omega_0$, we may rewrite the above equation as

$$E_d = \frac{1}{2\pi T}\int_{-\omega_0}^{\omega_0}|X_c(j\omega)|^2 d\omega = \frac{E_c}{T}.$$

7.43. Throughout this problem, to avoid confusion we use the variable Ω to indicate discrete-time frequency.

Taking the Fourier transform of both sides of the given differential equation, we get

$$H(j\omega) = \frac{Y_c(j\omega)}{X_c(j\omega)} = \frac{1}{-\omega^2 + 4j\omega + 3}.$$

Taking the inverse Fourier transform of the partial fraction expansion of $H(j\omega)$, we obtain

$$h(t) = \frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^{-3t}u(t).$$

Now, $x_p(t) = \sum_{n=-\infty}^{\infty}x[n]\delta(t - nT)$. Therefore, $X_p(j\omega) = X(e^{j\omega T})$. Also,

$$X_c(j\omega) = TX_p(j\omega) = TX(e^{j\omega T}) \qquad \text{for} \quad -\pi/T \le \omega \le \pi/T$$

and 0 otherwise. From this we get

$$Y_c(j\omega) = H(j\omega)TX(e^{j\omega T}) \qquad \text{for} \quad -\pi/T \le \omega \le \pi/T$$

and 0 otherwise. Then, one period of $Y_p(j\omega)$ may be specified as

$$Y_p(j\omega) = \frac{1}{T}Y_c(j\omega) = H(j\omega)X(e^{j\omega T}) \qquad \text{for} \quad -\pi/T \le \omega \le \pi/T.$$

Therefore, one period of $Y(e^{j\Omega})$ is

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(j\Omega/T), \qquad \text{for} \quad -\pi \le \Omega \le \pi.$$

Denoting the frequency response of the equivalent system, by $H(e^{j\omega})$, we have

$$H(e^{j\omega}) = H(j\Omega/T), \qquad \text{for } -\pi \le \Omega \le \pi.$$

Note that $H(e^{j\omega})$ represents the Fourier transform of the sequence $h[n]$ obtained by lowpass filtering $h(t)$ (with a filter of cutoff frequency π/T) and sampling the result every T. Therefore,

$$h[n] = \left[h(t) * \frac{\sin(\pi t/T)}{\pi t/T}\right]_{t=nT} = \left[\frac{T}{2}\int_0^{\infty}[e^{-\tau} - e^{-3\tau}]\frac{\sin(\pi(t-\tau)/T)}{\pi(t-\tau)/T}d\tau\right]_{t=nT}.$$

7.44. (a) We have

$$y_p(t) = \sum_{k=-\infty}^{\infty}\cos\left(\frac{2\pi k}{N}\right)\delta(t - kT).$$

If $\omega_0 = 2\pi/NT$, then

$$\begin{aligned} y_p(t) &= \sum_{k=-\infty}^{\infty}\cos(\omega_0 kT)\delta(t - kT) \\ &= \sum_{k=-\infty}^{\infty}\cos(\omega_0 t)\delta(t - kT) \\ &= \cos(\omega_0 t)\sum_{k=-\infty}^{\infty}\delta(t - kT). \end{aligned}$$

Let the range of T be $T_{min} \le T \le T_{max}$. Then with T_{min}, we want to obtain the smallest frequency ω_1 and with T_{max}, we want to obtain the largest frequency ω_2. Therefore,

$$T_{min} = \frac{2\pi}{N\omega_2}, \quad \text{and} \quad T_{max} = \frac{2\pi}{N\omega_1}.$$

(b) Let $c(t) = \cos(\omega_0 t)$ and $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$. Then

$$Y_p(j\omega) = \frac{1}{2\pi}\left[C(j\omega) * P(j\omega)\right].$$

This is as shown in Figure S7.44.

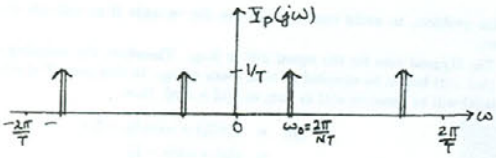


Figure S7.44

(c) To avoid aliasing in $Y(j\omega)$, we require that $2\omega_0 < 2\pi/T$. Therefore, $4\pi/NT < 2\pi/T$. This implies that $N > 2$. Therefore, the minimum value of N is 3. By inspection of $Y(j\omega)$, we obtain $\omega_2 < \omega_c < 4\pi/(3T)$. This keeps the sinusoid at frequency ω_2 while rejecting contributions from cosines centered around $2\pi/T$ and $-2\pi/T$.

(d) We have

$$G(j\omega) = \begin{cases} T, & -\omega_c \le \omega \le \omega_c \\ \text{arbitrary}, & \text{otherwise} \end{cases}.$$

7.45. (a) The Nyquist rate for the signal $x_c(t)$ is $4\pi \times 10^4$. Therefore, the maximum value of T that can be used to sample $x_c(t)$ is

$$T_{max} = \frac{2\pi}{4\pi \times 10^4} = 5 \times 10^{-5}.$$

(b) We have

$$y[n] = T \sum_{k=-\infty}^{n} x[k] = T \sum_{k=-\infty}^{\infty} x[k]u[n-k] = T\{x[n] * u[n]\}.$$

Therefore,

$$h[n] = Tu[n].$$

(c) We have

$$\lim_{n\to\infty} y[n] = \lim_{n\to\infty} T \sum_{k=-\infty}^{n} x[k] = TX(e^{j0}).$$

Also,

$$\lim_{t\to\infty} x_c(t) = X_c(j0).$$

Therefore, eq. (P7.45-2) requires that

$$TX(e^{j0}) = X_c(j0).$$

Now,

$$X(e^{j\omega}) = X_p(j\omega/T)$$

and

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - 2\pi k/T)).$$

To avoid aliasing at $\omega = 0$ in $X_p(\omega)$, we require that $(2\pi/T) > 2\pi \times 10^4$. This implies that $T < 10^{-4}$. With this condition,

$$X(e^{j0}) = (1/T)X_c(j0).$$

7.46. We have

$$\begin{aligned} x_r[mN] &= \sum_{k=-\infty}^{\infty} x[kN]\frac{N\omega_c}{2\pi}\frac{\sin[\omega_c(mN - kN)]}{\omega_c(mN - kN)} \\ &= \sum_{k=-\infty}^{\infty} x[kN]\frac{\sin 2\pi(m-k)}{2\pi(m-k)} \end{aligned}$$

Note that $[\sin[2\pi(m-k)]]/[2\pi(m-k)]$ is 1 when $m = k$, and zero otherwise. Therefore,

$$x_r[mN] = x[mN].$$

7.47. Let us define a signal

$$x_p[n] = x[n]\sum_{k=-\infty}^{\infty}\delta[n - 3k] = \sum_{k=-\infty}^{\infty} x[3k]\delta[n - 3k].$$

From Section 7.5.1, we know that the Fourier transform of $x_p[n]$ is

$$X_p(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} X(e^{j(\omega - 2\pi/3)}).$$

Since $X(e^{j\omega}) = 0$ for $\pi/3 \le |\omega| \le \pi$, there is no aliasing among the replicas of $X(e^{j\omega})$ in $X_p(e^{j\omega})$. This is shown in the Figure S7.47.

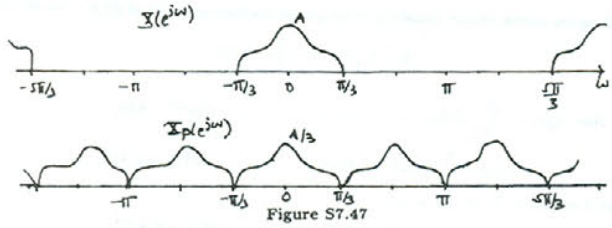


Figure S7.47

In order to be able to recover $x[n]$ from $x_p[n]$, it is clear that we need to pass $x_p[n]$ through a lowpass filter with cutoff frequency $\pi/3$ and passband gain 3. Therefore,

$$\begin{aligned} x[n] &= x_p[n] * \frac{3\sin(\pi n/3)}{\pi n} \\ &= \{\sum_{k=-\infty}^{\infty} x[3k]\delta[n-3k]\} * \frac{3\sin(\pi n/3)}{\pi n} \\ &= \sum_{k=-\infty}^{\infty} x[3k]\frac{\sin[\pi(n-3k)/3]}{\pi(n-3k)/3}. \end{aligned}$$

7.48. In Figure S7.49, we plot the signal $\cos(\pi n/4)$.

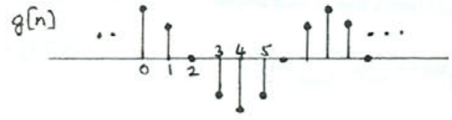


Figure S7.48

Note that the signal $g[n]$ contains every fourth sample of $x[n]$. If the signal $x[n]$ were $\cos[\pi(n+2)/4]$ (see Figure S7.48), then $g[n]$ would be zero for all n. Therefore, there would be no way of recovering $x[n]$ from $g[n]$. Therefore, ϕ_0 should never be $\pi/2$ in order for the given equation to be true.

7.49. (a) Let the signals $x_{d_1}[n]$ and $x_{d_2}[n]$ be inputs to system A. Let the corresponding outputs be $x_{p_1}[n]$ and $x_{p_2}[n]$. Now, consider an input of the form $x_{d_3}[n] = \alpha_1 x_{d_1}[n] + \alpha_2 x_{d_2}[n]$.

This gives an output which is

$$x_{p_3}[n] = \begin{cases} \alpha_1 x_{d_1}[n/N] + \alpha_2 x_{d_2}[n/N], & n = 0, \pm N, \pm 2N, \cdots \\ 0, & \text{otherwise} \end{cases}.$$

Therefore, $x_{p_3}[n] = \alpha_1 x_{p_1}[n] + \alpha_2 x_{p_2}[n]$. This implies that the system is linear.

(b) Let us consider a signal $x_d[n]$ as shown in Figure S7.49. The output of the system $x_p[n]$ is then as shown in the figure. Let us now define a new input $x_{d_1}[n] = x_d[n-1]$. The corresponding output $x_{p_1}[n]$ is shown in the Figure S7.49. Clearly, $x_{p_1}[n] \ne x_p[n]$. Therefore, the system in *not* time invariant.

(c) We have $X_p(e^{j\omega}) = X_d(e^{j\omega N})$. This is as shown in Figure S7.49.

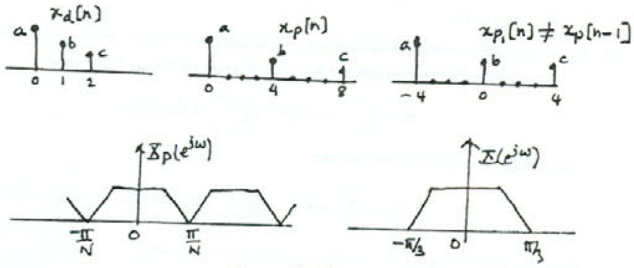


Figure S7.49

(d) $X(e^{j\omega})$ is as sketched in Figure S7.49.

7.50. (a) We have

$$h_0[n] = u[n] - u[n-N].$$

This is as shown in the Figure S7.50.

Figure S7.50

(b) We require that $H(e^{j\omega})H_0(e^{j\omega}) = N$ for $|\omega| < \omega_s/2$ and zero otherwise. Here, $\omega_s/2 = \pi/N$. But

$$H_0(e^{j\omega}) = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}.$$

Therefore,

$$H(e^{j\omega}) = \begin{cases} N\frac{1-e^{-j\omega}}{1-e^{-j\omega N}}, & |\omega| < \pi/N \\ 0, & (\pi/N) \le |\omega| \le \pi \end{cases}.$$

(c) We have

$$h_1[n] = \frac{1}{N}\left[h_0[n] * h_0[-n]\right].$$

(d) Again we have $H(e^{j\omega}) = N/H_1(e^{j\omega})$ for $|\omega| < \pi/N$ and zero otherwise. But from part (c),

$$H_1(e^{j\omega}) = (1/N^2)|H_0(e^{j\omega})|^2.$$

Therefore,

$$H(e^{j\omega}) = \begin{cases} N^2\left|\frac{1-e^{-j\omega}}{1-e^{-j\omega N}}\right|^2, & |\omega| < \pi/N \\ 0, & (\pi/N) \le |\omega| \le \pi \end{cases}$$

7.51. (a) This is possible only of $h[kL] = 0$ for $k = \pm1, \pm2, \cdots$ and $h[0] = 1$.

(b) N must be odd. In this case, α is an integer. If N is even, α is not an integer. If α were an integer, shifting $h[n]$ by α would make $h[n]$ an even sequence. This is impossible with N even.

(c) N can be odd or even. This time, α is allowed to be fractional. Thus, an even length filter can be designed which is a linear-phase causal symmetric FIR filter.

7.52. (a) Since,

$$\tilde{X}(j\omega) = X(j\omega)P(j\omega),$$

we have

$$\tilde{x}(t) = x(t) * p(t).$$

(b) Taking the inverse Fourier transform of $P(j\omega)$, we have

$$p(t) = \frac{1}{\omega_0}\sum_{k=-\infty}^{\infty}\delta\left(t - \frac{2\pi k}{\omega_0}\right).$$

From part (a), we have

$$\begin{aligned}\tilde{x}(t) &= p(t) * x(t) \\ &= \frac{1}{\omega_0}\sum_{k=-\infty}^{\infty} x\left(t - \frac{2\pi k}{\omega_0}\right)\end{aligned}$$

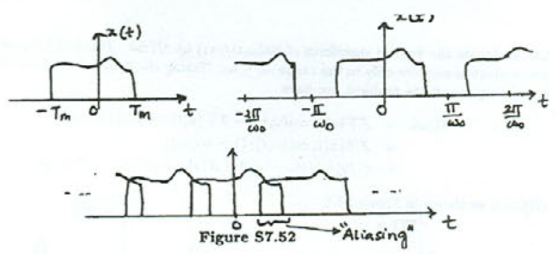


Figure S7.52

Noting that $x(t)$ is time-limited so that $x(t) = 0$ for $|t| > \pi/\omega_0$, we assume that $x(t)$ is as shown in Figure S7.52. Then, $\tilde{x}(t)$ is as shown in the figure below. Clearly, $x(t)$ can be recovered from $\tilde{x}(t)$ by multiplying it with the function

$$w(t) = \begin{cases} \omega_0, & |t| \le \pi/\omega_0 \\ 0, & \text{otherwise} \end{cases}.$$

(c) If $x(t)$ is not constrained to be zero for $|t| > \pi/\omega_0$, then $\tilde{x}(t)$ is as shown in Figure S7.52. Clearly, there is "time-domain aliasing" between the replicas of $x(t)$ in $\tilde{x}(t)$. Therefore, $x(t)$ cannot be recovered from $\tilde{x}(t)$.

Chapter 8 Answers

8.1. Using Table 4.1, take the inverse Fourier transform of $Y(j(\omega - \omega_c))$. This gives

$$y(t) = 2x(t)e^{j\omega_c t}.$$

Therefore,

$$m(t) = 2e^{j\omega_c t}.$$

8.2. (a) The Fourier transform $Y(j\omega)$ of $y(t)$ is given by

$$Y(j\omega) = X(j(\omega - \omega_c)).$$

Clearly $Y(j\omega)$ is just a shifted version of $X(j\omega)$. Therefore, $x(t)$ may be recovered from $y(t)$ simply by multiplying $y(t)$ by $e^{-j\omega_c t}$. There is no constraint that needs to be placed on ω_c to ensure that $x(t)$ is recoverable from $y(t)$.

(b) We know that

$$y_1(t) = \mathcal{R}e\{y(t)\} = x(t)\cos(\omega_c t).$$

The Fourier transform $Y_1(j\omega)$ of $y_1(t)$ is as shown in Figure S8.2

$$Y_1(j\omega) = \frac{1}{2}X(j(\omega - \omega_c)) + \frac{1}{2}X(j(\omega + \omega_c))$$

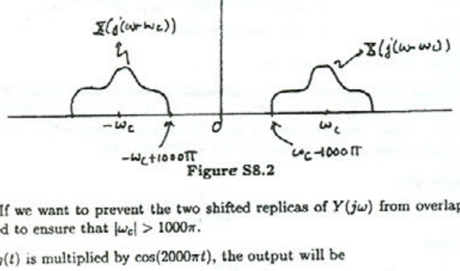


Figure S8.2

If we want to prevent the two shifted replicas of $Y(j\omega)$ from overlapping, then we need to ensure that $|\omega_c| > 1000\pi$.

8.3 When $g(t)$ is multiplied by $\cos(2000\pi t)$, the output will be

$$x_1(t) = g(t)\cos(2000\pi t) = x(t)\sin(2000\pi t)\cos(2000\pi t) = \frac{1}{2}x(t)\sin(4000\pi t).$$

The Fourier transform of this signal is

$$X_1(j\omega) = \frac{1}{4j}X(j(\omega - 4000\pi)) - \frac{1}{4j}X(j(\omega + 4000\pi)).$$

This implies that $X_1(j\omega)$ is zero for $|\omega| \le 2000\pi$. When $y(t)$ is passed through a lowpass filter with cutoff frequency 2000π, the output will clearly be zero. Therefore $y(t) = 0$.

8.4. Consider the signal

$$\begin{aligned}y(t) &= g(t)\sin(400\pi t) \\ &= \sin(200\pi t)\sin^2(400\pi t) + 2\sin^3(400\pi t) \\ &= \sin(200\pi t)[(1-\cos(800\pi t))/2] + 2\sin(400\pi t)[(1-\cos(800\pi t)/2] \\ &= (1/2)\sin(200\pi t) - (1/4)\{\sin(1000\pi t) - \sin(600\pi t)\} \\ &\quad + \sin(400\pi t) - (1/2)\{\sin(1200\pi t) - \sin(400\pi t)\}\end{aligned}$$

If this signal is passed through a lowpass filter with cutoff frequency 400π, then the output will be

$$y_1(t) = \sin(200\pi t).$$

8.5. The signal $x(t)$ is as shown in the Figure S8.5.

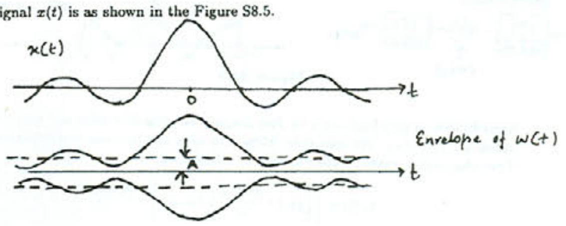


Figure S8.5

The envelope of the signal $w(t)$ is as shown in the Figure S8.5. Clearly is we want to use asynchronous demodulation to recover the signal $x(t)$, we need to ensure that A is greater than the height h of the highest sidelobe (see Figure S8.5). Let us now determine the height of the highest sidelobe. The first zero-crossing of the signal $x(t)$ occurs at time t_0 such that

$$1000\pi t_0 = \pi, \quad \Rightarrow \quad t_0 = 1/1000.$$

Similarly, the second zero-crossing happens at time t_1 such that

$$1000\pi t_1 = 2\pi, \quad \Rightarrow \quad t_1 = 2/1000.$$

The highest sidelobe occurs at time $(t_0 + t_1)/2$, that is, at time $t_2 = 3/2000$. At this time, the amplitude of the signal $x(t)$ is

$$x(t_2) = \frac{\sin(3\pi/2)}{\pi 3/2000} = -\frac{2000}{3\pi}.$$

Therefore, A should at least be $\frac{2000}{3\pi}$. The modulation index corresponding to the smallest permissible value of A is

$$m = \frac{\text{Max. value of } x(t)}{\text{Min. possible value of} A} = \frac{1000}{2000/3\pi} = \frac{3\pi}{2}.$$