EBU4375 Signals and Systems Theory

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Fourier Transform

Fourier Transform

- The Fourier Series (FS) can only be applied to periodic signals

- Non-periodic signals cannot be analysed using Fourier series, the Fourier Transform (FT) is required

The Fourier Transform (FT) is defined as:

In this course, the Fourier Transform will always be denoted by an uppercase letter or symbol, whereas time signals will be denoted by lowercase letters or symbols.

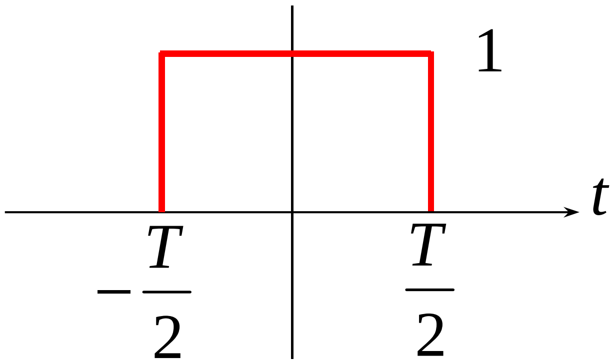
$$X(f) = \int_{t=-\infty}^{t=\infty} e^{-j\omega t} \cdot x(t)\,dt$$

X(*f*) is the frequency signal

x(*t*) is the time signal

Fourier Transform – Rectangular Pulse (also known as Rect Function)

$$x(t) = \begin{cases} 1, & -\dfrac{T}{2} \le t \le \dfrac{T}{2} \\ 0, & all\ other\ t \end{cases}$$



Fourier Transform – Rectangular Pulse (also known as Rect Function)

x(t) is an even signal

Method 1:

even odd

$$X(f) = \int_{t=-\infty}^{t=\infty} e^{-j\omega t} \cdot x(t)dt = \int_{t=-\infty}^{t=\infty} (\cos(\omega t) - j\sin(\omega t)) \cdot x(t)dt$$

$$X(f) = 2\int_{0}^{T/2} (1)\cos(\omega t)dt = \frac{2}{\omega}\left[\sin(\omega t)\Big|_{t=0}^{t=T/2}\right] = \frac{2}{\omega}\sin\left(\frac{\omega T}{2}\right) = \frac{1}{\pi f}\sin(\pi f T)$$

Let's recall the sinc function $\text{sinc}(\theta) = \dfrac{\sin(\pi\theta)}{\pi\theta}$. After some mathematical Manipulation, we have $X(f) = T\dfrac{\sin(\pi f T)}{\pi f T}$. Therefore,

$$\boxed{X(f) = T\,\text{sinc}(f\,T) = T\,\text{sinc}\left(\frac{T\omega}{2\pi}\right)}$$

Fourier Transform – Rectangular Pulse (also known as Rect Function)

<u>Method 2:</u>

$$X(f)=\int_{t=-\infty}^{t=\infty} e^{-j\omega t}\cdot x(t)dt=\int_{t=-T/2}^{t=T/2} e^{-j\omega t}\cdot 1\,dt=\left[\frac{e^{-j\omega t}}{-j\omega}\Bigg|\begin{matrix}t=T/2\\ t=-T/2\end{matrix}\right]$$

Therefore

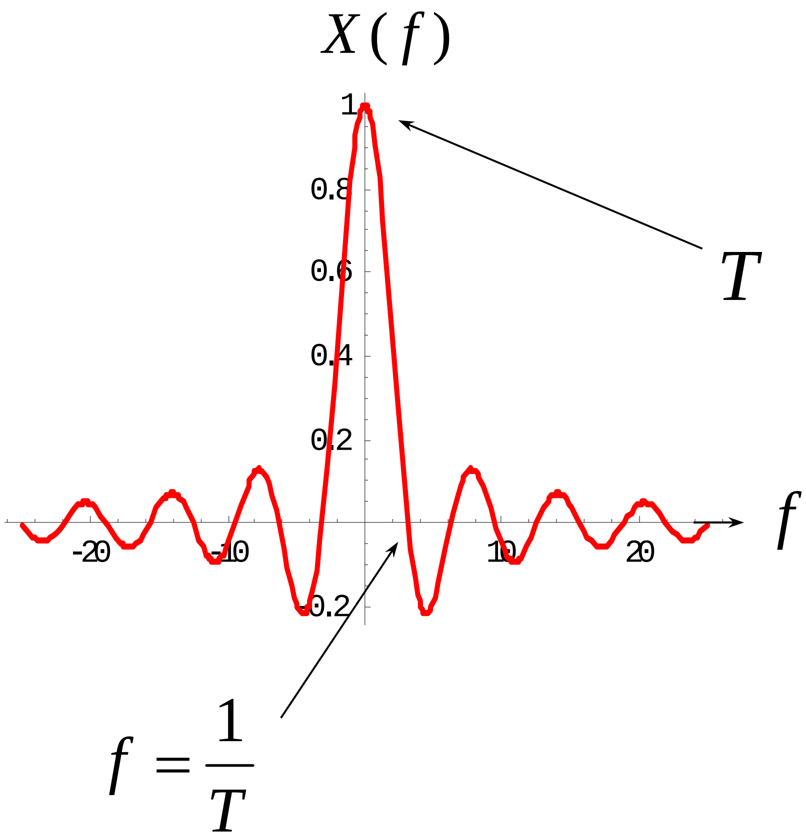
$$X(f)=\frac{1}{-j\omega}\left[e^{-j\omega T/2}-e^{+j\omega T/2}\right]=\frac{1}{j\omega}\left[e^{+j\omega T/2}-e^{-j\omega T/2}\right]=\frac{1}{j2\pi f}\left[e^{+j\pi f T}-e^{-j\pi f T}\right]$$

Using identity $\sin\theta=\frac{1}{2j}\left[e^{+j\theta}-e^{-j\theta}\right]$, we get $X(f)=\frac{1}{j2\pi f}\left[2j\sin(\pi f T)\right]$.

And after some mathematical manipulation, we get $X(f)=T\left[\frac{\sin(\pi f T)}{\pi f T}\right]$.

Using identity $\mathrm{sinc}(\theta)=\left[\frac{\sin(\pi\theta)}{\pi\theta}\right]$, we get $\boxed{X(f)=T\,\mathrm{sinc}(f\,T)}$.

Fourier Transform – Rectangular Pulse (also known as Rect Function)

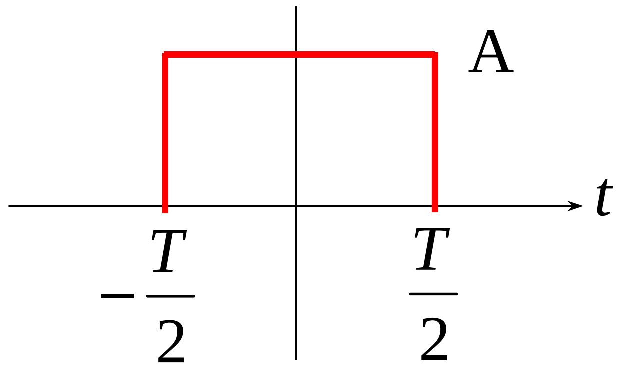


$$X(f) = 0, \qquad f = \frac{1}{T}, \frac{2}{T}, \frac{3}{T}, \cdots$$

Fourier Transform – Rectangular Pulse (also known as Rect Function – General Formula)

The Fourier transform of a rectangular pulse is given as

$$\mathrm{A}\,\mathrm{rect}\left(\frac{t}{T}\right) \quad \leftrightarrow \quad \mathrm{A}T\,\mathrm{sinc}\left(f\;T\right) = \mathrm{A}T\,\mathrm{sinc}\left(\frac{\omega\,T}{2\pi}\right)$$



Inverse Fourier Transform

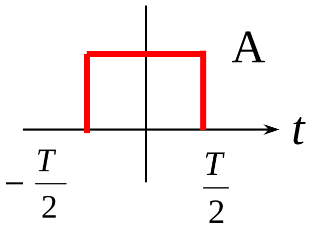
Give a signal x(t) with Fourier transform X(f), x(t) can be recomputed from X(f) by application of the inverse Fourier transform give by

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{j\omega t} df$$

To denote the fact that X(f) is the Fourier transform of x(t), or that X(f) is the inverse Fourier transform of x(t), the transform pair notation

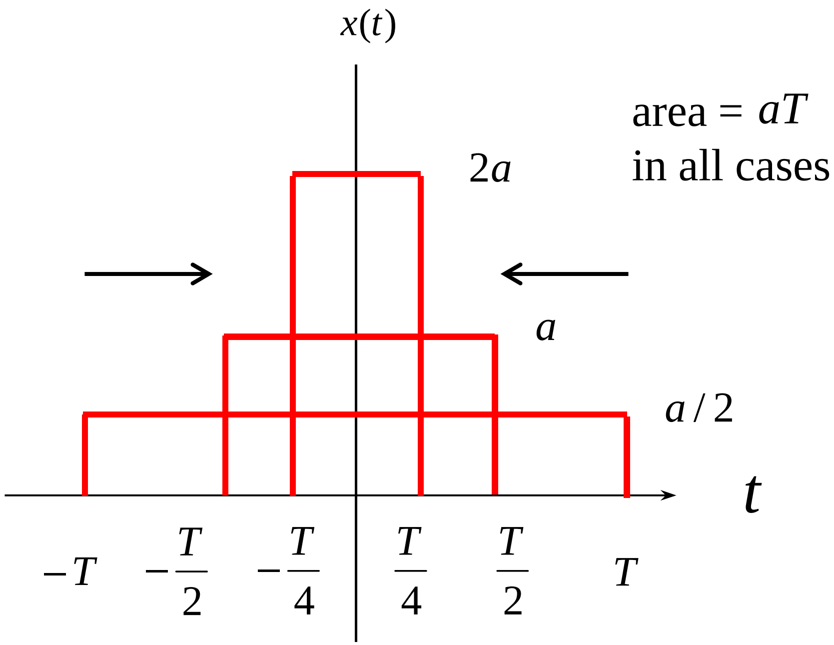
$$x(t) \leftrightarrow X(f)$$

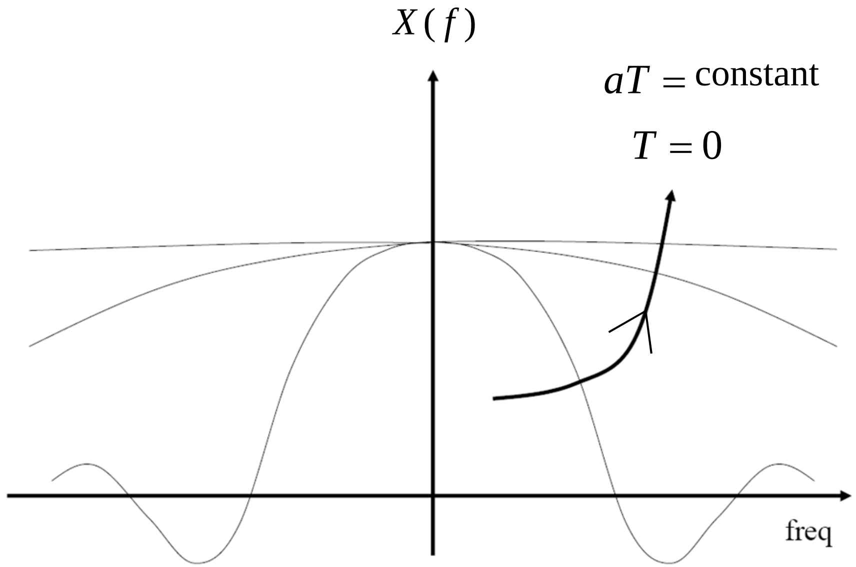
will sometimes be used. One of most fundamental transform pairs in the Fourier theory is the pair



$$\leftrightarrow \quad AT\,\mathrm{sinc}\left(\frac{T\omega}{2\pi}\right) = AT\,\mathrm{sinc}(T\ f)$$

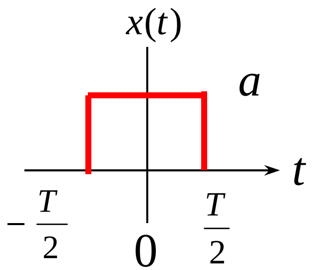
Fourier Transform – Delta Function





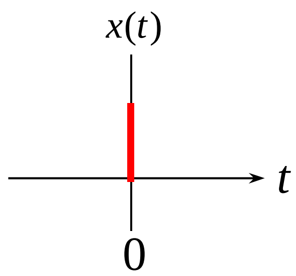
Fourier Transform – Delta Function

We know that



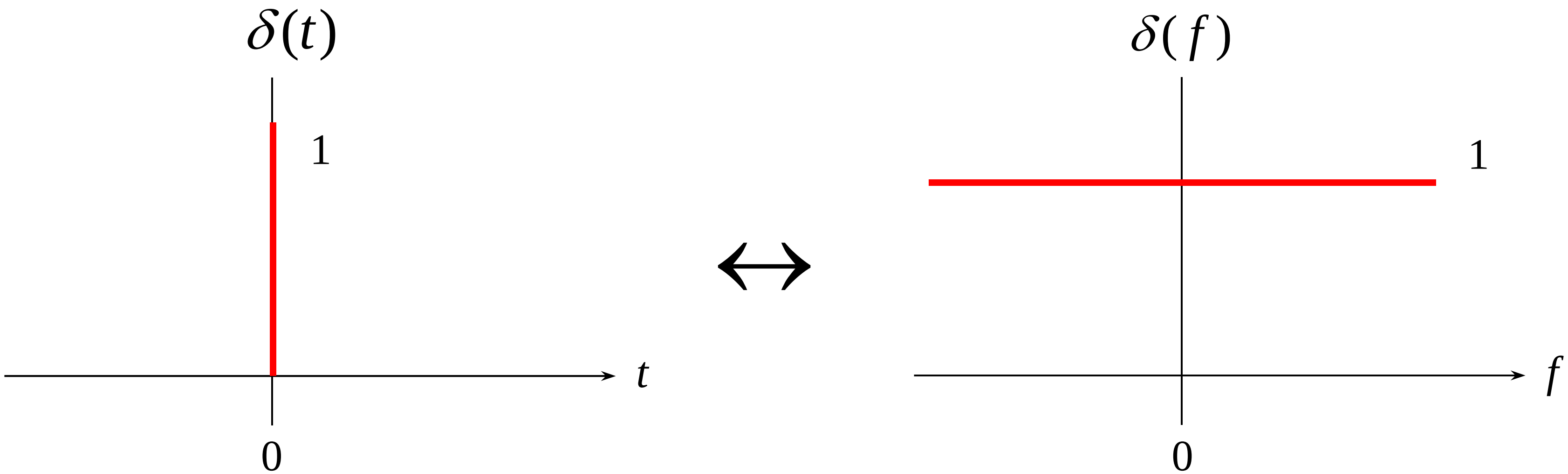
$$\leftrightarrow \quad aT\,\mathrm{sinc}(f\,T)$$

Note that sinc(0) = 1. We also know that the area of a delta function is $aT = 1$, therefore



$$\leftrightarrow \quad 1$$

Fourier Transform – Delta Function



Fourier Transform Table

Fourier-Transform Pairs

Time Function	*Fourier Transform*
$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\ \mathrm{sinc}\ (fT)$
$\mathrm{sinc}\ (2Wt)$	$\frac{1}{2W}\mathrm{rect}\left(\frac{f}{2W}\right)$
$\exp(-at)u(t), \qquad a > 0$	$\frac{1}{a + j2\pi f}$
$\exp(-a\lvert t\rvert), \qquad a > 0$	$\frac{2a}{a^2 + (2\pi f)^2}$
$\exp(-\pi t^2)$	$\exp(-\pi f^2)$

Fourier Transform Table

Fourier-Transform Pairs

Time Function	*Fourier Transform*
$\begin{cases} 1 - \dfrac{\lvert t \rvert}{T}, & \lvert t \rvert < T \\ 0, & \lvert t \rvert \geq T \end{cases}$	$T \operatorname{sinc}^2(fT)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$
$\exp(j2\pi f_c t)$	$\delta(f - f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$
$\sin(2\pi f_c t)$	$\dfrac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$

Fourier Transform Table

Fourier-Transform Pairs

Time Function	*Fourier Transform*
$\mathrm{sgn}(t)$	$\dfrac{1}{j\pi f}$
$\dfrac{1}{\pi t}$	$-j\,\mathrm{sgn}(f)$
$u(t)$	$\dfrac{1}{2}\delta(f) + \dfrac{1}{j2\pi f}$
$\displaystyle\sum_{i=-\infty}^{\infty} \delta(t - iT_0)$	$\displaystyle\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right)$

Trigonometric Identities

Some useful trigonometric identities:

$$\exp(\pm j\theta) = \cos\theta \pm j\sin\theta$$
$$\cos\theta = \tfrac{1}{2}[\exp(j\theta) + \exp(-j\theta)]$$
$$\sin\theta = \frac{1}{2j}[\exp(j\theta) - \exp(-j\theta)]$$
$$\sin^2\theta + \cos^2\theta = 1$$
$$\cos^2\theta - \sin^2\theta = \cos(2\theta)$$
$$\cos^2\theta = \tfrac{1}{2}[1 + \cos(2\theta)]$$
$$\sin^2\theta = \tfrac{1}{2}[1 - \cos(2\theta)]$$
$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$$

Fourier Transform Properties

Part I:

- Linearity
- Time Shift
- Time Scaling
- Multiplication by a Complex Exponential (Frequency Shift)
- Tutorial

Part II:

- Multiplication by a Sinusoid
- Differentiation in the Time Domain
- Integration in the Time Domain
- Convolution in the Time Domain
- Multiplication in the Time Domain
- Tutorial

Fourier Transform Properties – Linearity

The Fourier transform is a linear operation; that is, if $x(t) \leftrightarrow X(f)$ and $v(t) \leftrightarrow V(f)$, then for any real or complex scalars a, b

$$\boxed{ax(t) + bv(t) \leftrightarrow aX(f) + bV(f)}$$

The properties of linearity can be proved by computing the Fourier transform of $ax(t) + bv(t)$: By definition of the Fourier transform,

$$ax(t) + bv(t) \leftrightarrow \int_{-\infty}^{\infty} [ax(t) + bv(t)] e^{-j\omega t} dt$$

By linearity of integration,

$$\int_{-\infty}^{\infty} [ax(t) + bv(t)] e^{-j\omega t} dt = a \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} v(t) e^{-j\omega t} dt$$

and thus $\quad ax(t) + bv(t) \leftrightarrow aX(f) + bV(f)$

Fourier Transform Properties – Time Shift

If $x(t) \leftrightarrow X(f)$, then for any positive or negative real number c,

$$\boxed{x(t-c) \leftrightarrow X(f)e^{-j\omega c}}$$

Note that if $c > 0$, then $x(t-c)$ is c-second right sift of $x(t)$;

if $c < 0$, then $x(t-c)$ is c-second left shift of $x(t)$.

Thus the above transform pair is valid for both left and right shifts of $x(t)$.

To verify this prosperity, first apply the definition of the Fourier transform

$$x(t-c) \leftrightarrow \int_{-\infty}^{\infty} x(t-c)e^{-j\omega t}dt$$

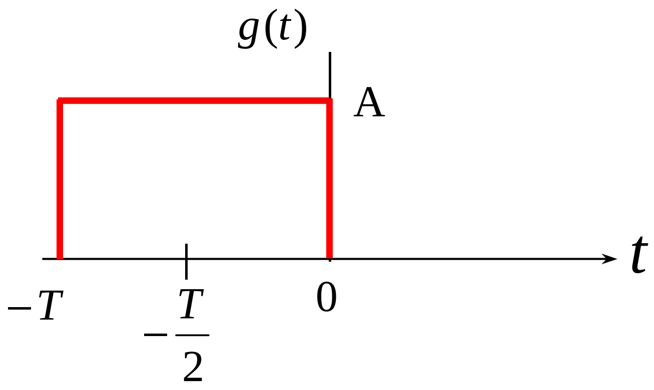
Let $\bar{t} = t - c$, then $t = \bar{t} + c$ and $dt = d\,\bar{t}$

$$x(t-c) \leftrightarrow \int_{-\infty}^{\infty} x\left(\bar{t}\right)e^{-j\omega\left(\bar{t}+c\right)}d\,\bar{t}$$

$$\leftrightarrow \left[\int_{-\infty}^{\infty} x\left(\bar{t}\right)e^{-j\omega\bar{t}}d\,\bar{t}\right]e^{-j\omega c}$$

$$\leftrightarrow X(f)e^{-j\omega c}$$

Fourier Transform Properties – Time Shift



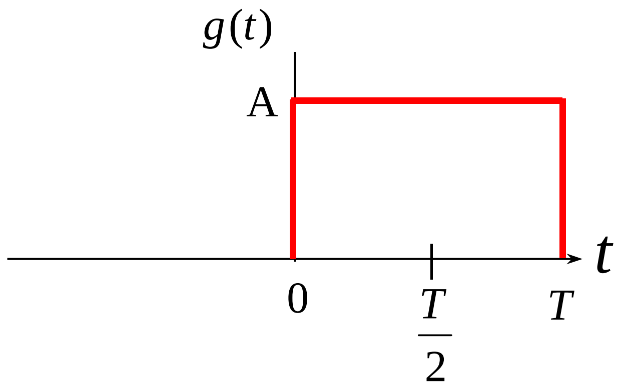
$$g(t) = \mathrm{A}\,\mathrm{rect}\left(\frac{t+T/2}{T}\right)$$

$$\boxed{\mathrm{A}\,\mathrm{rect}\left(\frac{t}{T}\right) \quad \leftrightarrow \quad \mathrm{A}T\,\mathrm{sinc}\left(f\ T\right)}$$

$$\boxed{\begin{array}{c} x(t) \leftrightarrow X(f) \\ x(t-t_0) \leftrightarrow X(f)e^{-j\omega t_0} \end{array}}$$

Therefore,

$$G(f) = \mathrm{A}T\,\mathrm{sinc}\left(f\ T\right)e^{j2\pi\ f\frac{T}{2}}$$



$$g(t) = \mathrm{A}\,\mathrm{rect}\left(\frac{t-T/2}{T}\right)$$

$$\boxed{\mathrm{A}\,\mathrm{rect}\left(\frac{t}{T}\right) \quad \leftrightarrow \quad \mathrm{A}T\,\mathrm{sinc}\left(f\ T\right)}$$

$$\boxed{\begin{array}{c} x(t) \leftrightarrow X(f) \\ x(t-t_0) \leftrightarrow X(f)e^{-j\omega t_0} \end{array}}$$

Therefore,

$$G(f) = \mathrm{A}T\,\mathrm{sinc}\left(f\ T\right)e^{-j2\pi\ f\frac{T}{2}}$$

Fourier Transform Properties – Time Scaling

If $x(t) \leftrightarrow X(f)$, for any positive real number a, $\boxed{x(at) \leftrightarrow \frac{1}{a} X(\frac{f}{a})}$

To verify this prosperity, first apply the definition of the Fourier transform

$$x(at) \leftrightarrow \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

Let $\bar{t} = at$, then $t = \bar{t}/a$ and $d\bar{t} = adt$,

$$x(at) \leftrightarrow \int_{-\infty}^{\infty} x\left(\bar{t}\right) e^{\left(-j\frac{\omega}{a}\bar{t}\right)} \frac{1}{a} d\bar{t}$$

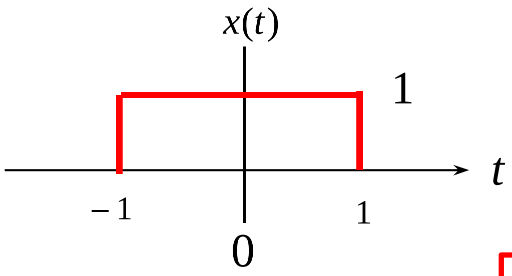
$$\leftrightarrow \frac{1}{a} \int_{-\infty}^{\infty} x\left(\bar{t}\right) e^{\left(-j\frac{\omega}{a}\bar{t}\right)} d\bar{t}$$

$$\leftrightarrow \frac{1}{a} X\left(\frac{f}{a}\right)$$

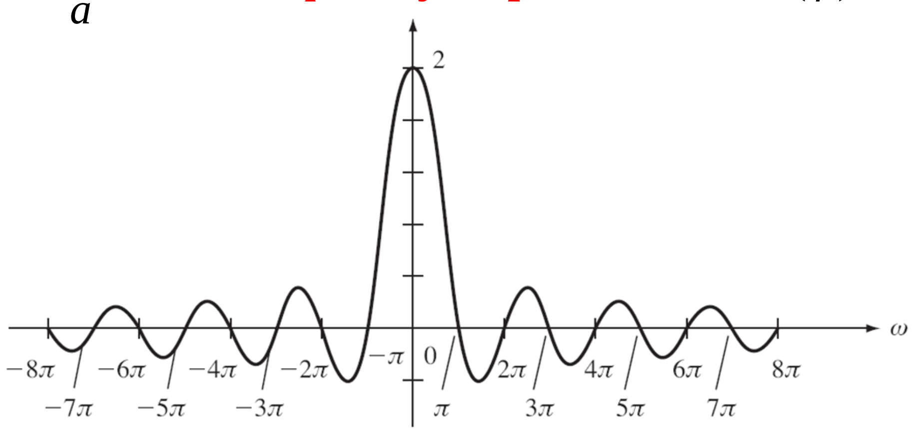
Fourier Transform Properties – Time Scaling

If $0<a<1$, $x(at)$ is a time expansion of $x(t)$ and $X(\frac{f}{a})$ is a frequency compression of $X(f)$

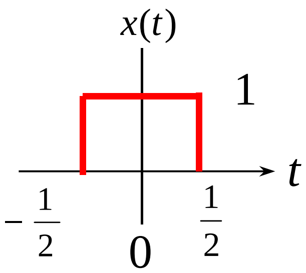
If $a>1$, $x(at)$ is a time compression of $x(t)$ and $X(\frac{f}{a})$ is a frequency expansion of $X(f)$



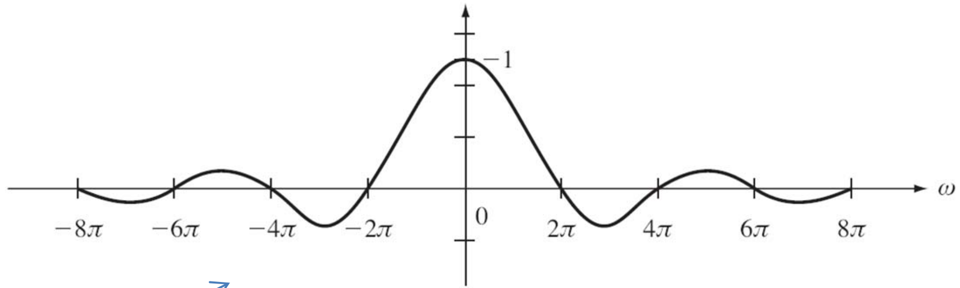
$x(t) \leftrightarrow 2\mathrm{sinc}(2f)$



$$\boxed{x(at) \leftrightarrow \frac{1}{a} X(\frac{f}{a})}$$



$x(2t) \leftrightarrow \mathrm{sinc}(f)$



time compression

frequency expansion

Fourier Transform Properties – Multiplication by a Complex Exponential (Frequency Shift)

If $x(t) \leftrightarrow X(f)$, then

$$\boxed{x(t)e^{j\omega_0 t} \leftrightarrow X(f - f_0)}$$

So, multiplication a complex exponential in the time domain corresponds to a frequency shift in the frequency domain.

The proof of this properties follows directly from the definition of the Fourier transform. You can do the verification after the lecture.

Fourier Transform Properties – Tutorial

1) Find the Fourier transform of

   i) $g(t) = e^{-t} \sin(2\pi f_o\, t)\, u(t)$

   ii) $g(t) = 8\,\mathrm{rect}(t/4)\cos(2\pi 10^6\, t)$

2) Find the inverse Fourier transform of

   $G(f) = 12\,\mathrm{sinc}(4f)\sin(4\pi f)$

Fourier Transform Properties

Part I:

- Linearity
- Time Shift
- Time Scaling
- Multiplication by a Complex Exponential (Frequency Shift)
- Tutorial

Part II:

- Multiplication by a Sinusoid
- Differentiation in the Time Domain
- Integration in the Time Domain
- Convolution in the Time Domain
- Multiplication in the Time Domain
- Tutorial

Fourier Transform Properties – Multiplication by a Sinusoid

If $x(t) \leftrightarrow X(f)$, for any real number f_0, where $\omega_0 = 2\pi f_0$

$$\boxed{\begin{aligned} x(t)\sin(\omega_0 t) &\leftrightarrow \frac{1}{2j}\left[X(f - f_0) - X(f + f_0)\right] \\ x(t)\cos(\omega_0 t) &\leftrightarrow \frac{1}{2}\left[X(f - f_0) + X(f + f_0)\right] \end{aligned}}$$

The proof of this property follows directly from the definition of the Fourier transform and Euler's identity.

The signals $x(t)\sin(\omega_0 t)$ and $x(t)\cos(\omega_0 t)$ can be viewed as amplitude-modulated signals. More precisely, they are called the modulation theorems of the Fourier transform.

The above relationships show that modulation of a carrier by a signal $x(t)$ results in the frequency translations $X(f + f_0)$, $X(f - f_0)$ of Fourier transform $X(f)$.

Fourier Transform Properties – Differentiation in the Time Domain

If $x(t) \leftrightarrow X(f)$, then

$$\boxed{\frac{d}{dt}x(t) \leftrightarrow j\omega X(f)}$$

It follows from the above equation that differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain. To prove this property, observe that the Fourier transform of $dx(t)/dt$ is

$$\int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t} dt$$

The integral can be computed "by parts" as follows: with $v = e^{-j\omega t}$ and $\omega = x(t)$ $dv = -j\omega e^{-j\omega t}$ and $d\omega = [dx(t)/dt]$. Then,

$$\int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t} dt = v\omega \Big|_{t=-\infty}^{t=\infty} - \int_{-\infty}^{\infty} \omega dv$$

Then, $x(t) \to 0$ as $t \to \infty$,

$$= e^{-j\omega t} x(t) \Big|_{t=-\infty}^{t=\infty} - \int_{-\infty}^{\infty} x(t)(-j\omega)e^{-j\omega t} dt \;=\; j\omega \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = j\omega X(f)$$

Fourier Transform Properties – Integration in the Time Domain

Suppose that $x(t)$ has the Fourier transform $X(f)$. Then the integration of a time function $x(t)$ results in the following generalized transform in the frequency

$$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}X(f)+\pi X(0)\delta(f)$$

where $\delta(f)$ is the impulse function in the frequency domain.

Note that if the signal $x(t)$ has no dc component (e.g. $X(0)=0$), then the above equation reduces to

$$\boxed{\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}X(f)}$$

This means the integration of a time function has the effect of dividing its Fourier transform by the factor of $j\omega$.

Fourier Transform Properties – Convolution in the Time Domain

Given two signals $x(t)$ and $v(t)$ with Fourier transforms $X(f)$ and $V(f)$, the Fourier transform of the convolution $x(t)*v(t)$ is equal to the product $X(f)V(f)$ which results in the transform pair

$$\boxed{x(t)*v(t) \leftrightarrow X(f)V(f)}$$

This means that convolution in the time domain corresponds to multiplication in the frequency domain.

To prove it, first recall that by definition of convolution, $x(t)*v(t) = \int_{-\infty}^{\infty} x(\lambda)v(t-\lambda)d\lambda$

Hence, the Fourier transform of $x(t)*v(t)$ is given by

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\lambda)v(t-\lambda)d\lambda\right]e^{-j\omega t}dt \quad \text{rewritten} \quad \int_{-\infty}^{\infty} x(\lambda)\left[\int_{-\infty}^{\infty} v(t-\lambda)e^{-j\omega t}dt\right]d\lambda$$

Let $\bar{t} = t-\lambda$ in the second integral,

$$\int_{-\infty}^{\infty} x(\lambda)\left[\int_{-\infty}^{\infty} v(\bar{t})e^{-j\omega(\bar{t}+\lambda)}d\bar{t}\right]d\lambda \quad \text{rewritten} \quad \left[\int_{-\infty}^{\infty} x(\lambda)e^{-j\omega\lambda}d\lambda\right]\left[\int_{-\infty}^{\infty} v(\bar{t})e^{-j\omega\bar{t}}d\bar{t}\right]$$

Fourier Transform Properties – Multiplication in the Time Domain

If $x(t) \leftrightarrow X(f)$ and $v(t) \leftrightarrow V(f)$, then

$$\boxed{x(t)v(t) \leftrightarrow \left[X(f) * V(f)\right] = \int_{-\infty}^{\infty} X(\lambda)V(f-\lambda)d\lambda}$$

It is seen that multiplication in the time domain corresponds to convolution in the Fourier transform domain.

The proof of this property follows from the definition of the Fourier transform and the manipulation of integrals.

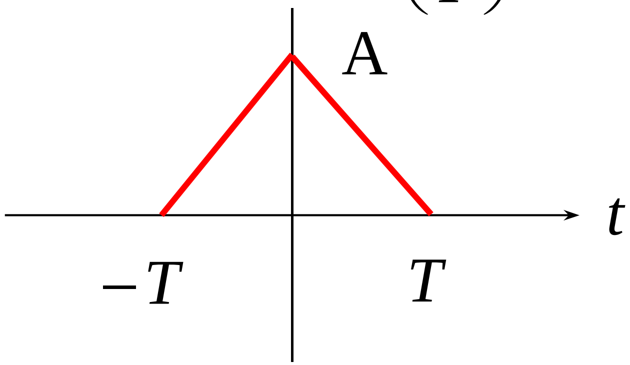
Fourier Transform Properties – Tutorial

1) Find the Fourier transform of a triangular pulse

$$g(t) = \begin{cases} A\left[1-\dfrac{|t|}{T}\right], & |t| < T \\ 0, & |t| \geq T \end{cases}$$

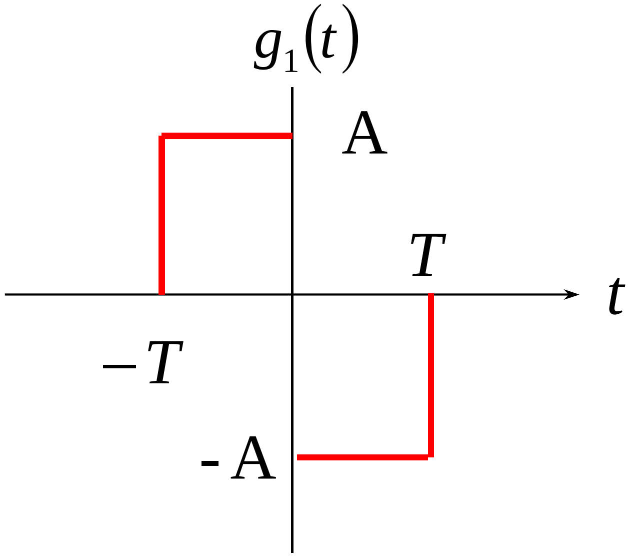
Let’s draw it

$$g(t) = \mathrm{A}\,\mathrm{tri}\left(\frac{t}{T}\right)$$



Fourier Transform Properties – Tutorial

Now let's by defining a *doublet pulse* which looks like this



It is the superposition of 2 shifted rectangular functions, i.e.,

$$g_1(t) = \mathrm{Arect}\left(\frac{t+T/2}{T}\right) - \mathrm{Arect}\left(\frac{t-T/2}{T}\right)$$

Fourier Transform Properties – Tutorial

$$g_1(t) = \mathrm{A}\,\mathrm{rect}\left(\frac{t+T/2}{T}\right) - \mathrm{A}\,\mathrm{rect}\left(\frac{t-T/2}{T}\right)$$

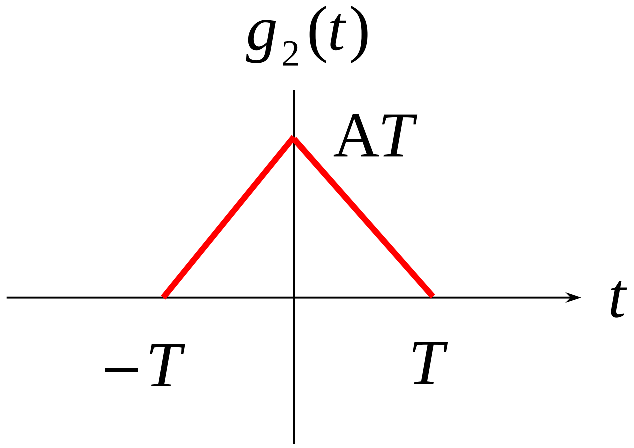
What is the Fourier transform of this doublet pulse?

$$G_1(f) = \mathrm{A}T\,\mathrm{sinc}(f\,T)\left\{e^{j\pi f T} - e^{-j\pi f T}\right\}$$

$$= j2\mathrm{A}T\,\mathrm{sinc}(f\,T)\sin(\pi\, f\,T)$$

Fourier Transform Properties – Tutorial

Integrating $g_1(t)$, we get



Therefore, using the integration property, we get the Fourier Transform of $g_2(t)$

$$G_2(f) = \frac{1}{j\omega} G_1(f)$$

$$= \mathrm{A}T\mathrm{sinc}(f\ T)\frac{\sin(\pi\ f\ T)}{\pi\ f}$$

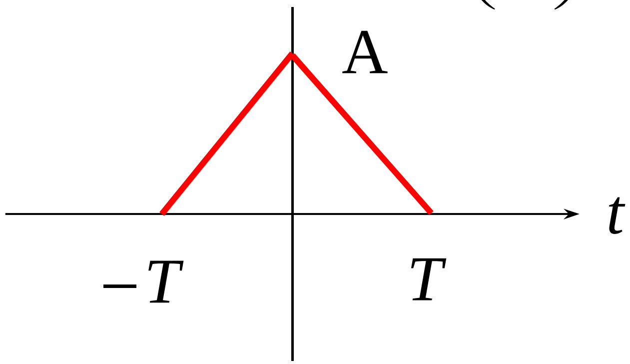
$$= \mathrm{A}T^2\mathrm{sinc}^2(f\ T)$$

Fourier Transform Properties – Tutorial

Keep in mind that $g(t) = g_2(t)/T$, and therefore $G(f) = G_2(f)/T$.

The Fourier transform of a triangular pulse is given as

$$\text{Atri}\left(\frac{t}{T}\right) \leftrightarrow \text{A}T\ \text{sinc}^2(f\,T)$$



Fourier Transform Properties – Tutorial

2) Find the inverse Fourier transform of

$$G(f) = 16\text{sinc}^2(4(f-10^6)) + 16\text{sinc}^2(4(f+10^6))$$

3) Find the Fourier transform of

i) $g(t) = 10\ \text{tri}(2t - 1/2)$

ii) $g(t) = 8\ \text{tri}(t/2)\cos(2\pi 10^6\, t)$