EBU6503
Control Theory

Frequency Response and Root Locus

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Stability

If G(s) is the forward path transfer function and H(s) is the feedback path transfer function, the open loop transfer function is G(s)H(s).

Closed-loop stability is achieved when

(magnitude of GH is ≤ 1

when the phase of GH is -180°)

Frequency Response

If the variable "s" in the open loop transfer function $G(s)H(s)$ is replaced by "$j\omega$" we then have the system's open loop frequency response $GH(j\omega)$.

A plot of $GH(j\omega)$ in magnitude and phase versus frequency can be analysed to discuss the system's stability and the system's transient response.

Frequency Response

If GH(jω) is plotted on polar graph paper the result is called a Nyquist Diagram.

If it is plotted on a Nichol’s Chart it is called a Nichol’s Diagram, and useful information can be obtained more easily from this.

(Other plots such as Bode Plot can be used)

Absolute and Relative Stability

- A system is Absolutely Stable if it is stable on closed loop.
- A measure of its Relative Stability are the values of Gain Margin and Phase Margin (both of the figures must be known).
- Good relative stability is a compromise between too much overshoot (lightly damped) and too slow (heavily damped)

Absolute and Relative Stability

Measures of good relative stability are empirical figures (i.e. not mathematically calculated) and can be given approximately as:

- **GM of between 10 and 12 dB**
- ***and* PM of between 45 and 50°**

System Response

A system's response to an input signal (i.e. a change in its set-point value) comprises two parts:

- **Transient Response.** This is the output change with time until the output settles down.
- **Steady-state Response.** This is the output value (or function) after the transients have died away.

Transient Response

A system's closed-loop transient response can be estimated from its open-loop frequency response plot.

The necessary information can be more easily obtained using a Nichol's diagram.

Transient Response

From the Nichol’s open-loop plot we can obtain two closed-loop values:

Maximum closed loop magnitude M_{max}

and resonant frequency ω_r

These can be used to calculate ζ and ω_n

If the system is second order then these two values are accurate, otherwise they are approximate.

Transient Response

ζ and ω_n can be used to calculate the closed-loop system's:

- Peak overshoot
- Time to peak overshoot
- Settling time
- Rise time.

If the system is not second order, then the transient response calculated is its second order approximation.

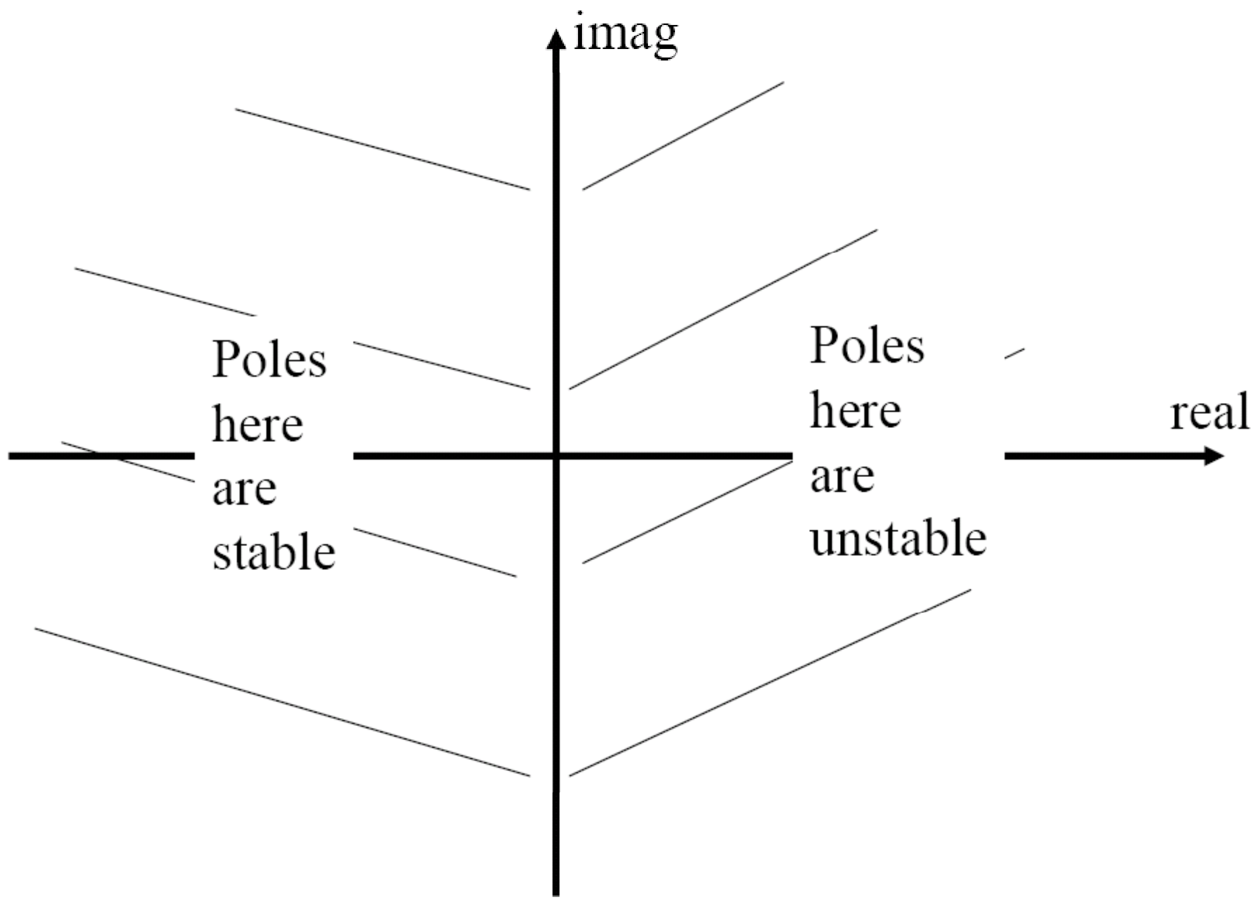
Steady-State Response

The system's steady-state response can be found using the Laplace final value theorem:

Limit $f(t)_{t \to \infty}$=limit $sF(s)_{s \to 0}$

where F(s) is the system's closed loop transfer function.

S-Plane and Closed-Loop Response



CL poles and pole dominance

If a closed loop system is stable:

- **All its poles are in the LH half of the s-plane.**
- **The nearer a pole is to the origin, the longer its time-constant**
- **Poles nearer the origin dominate the response**
- **If a pair of poles is nearer the origin than all others, then the system can be approximated by a second order system**

Root Locus

- The term “Root Locus” refers to an s-plane diagram of the locus of a system’s closed loop poles as some parameter (usually a loop gain constant K) varies.
- The locus can then be used to analyse the system’s CL behaviour for different values of K.
- The CL locus can be derived from the open loop poles

Root Locus

Example:

Consider the unity feedback system whose forward path transfer function is

G(s)= $$\frac{K}{s(s+4)}$$

Example

The closed loop transfer function F(s):

F(s)= $$\frac{K}{s^2+4s+K}$$

Example

The poles of F(s) are:

$$s = -2 \pm \sqrt{4 - K}$$

Plot the poles on an s-plane for:

- **K=0**
- **Critical damping**
- **K$\rightarrow\infty$**

Root Locus

This is a simple example to illustrate the technique.

The root locus can be used to calculate K that would give a particular closed loop response, e.g. a given value of ζ (and hence peak overshoot).

Root Locus

Relationship between closed loop pole position and ζ and ω:

Root Locus

There are rules to let us construct the root locus for any system:

1. Plot the open loop poles and zeroes
2. He number of loci is equal to the order of the characteristic equation
3. Each locus starts at an open looppole when K=0 and finishes either at an open loop zero or infinity when K=∞

Root Locus

4. **Loci either move along the real axis or occur as complex conjugate pairs of loci**
5. **A point on the real axis is part of the locus if the number of poles and zeroes to the right of the point concerned is odd**
6. **When the locus is far enough from the open loop poles and zeroes, it becomes asymptotic to lines making angles to the real axis given by:**

P=number of OL poles

Z=number of OL zeroes

n=odd integer

$$\frac{\pm n\pi}{P - Z}$$

Root Locus

7. The asymptotes intersect the real axis at a point x given by

$$x = \frac{\Sigma poles - \Sigma zeroes}{P - Z}$$

8. The break-away point between two poles or break-in point between two zeroes can be found by calculating the roots of the derivative of the characteristic equation

Root Locus

9. **The angle of departure from a complex pole or zero is given by**

$$Angle = 180 - \phi_P + \phi_Z$$

   where φ_P is the sum of all the angles subtended by the other poles and φ_Z is the sum of the angles of any zeroes.

10. **The limiting value of K for stability may be found using the Routh Array on the characteristic equation. This allows the value of the loci at the intersection of the imaginary axis to be determined.**

Routh Array

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

Root Locus Example

Example:

A unity feedback system has forward path transfer function:

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

Sketch the root locus plot and then determine the value of K to give a damping ratio of approximately 0.5.