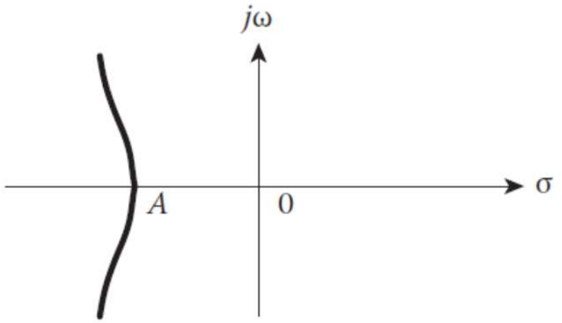
Solution to the Problems of Root Locus

Problems to Solve (1)

1. The following figure depicts with a bold line a segment of a root locus of the characteristic equation of a system with open-loop transfer function

$$GH(s) = \frac{K\left(s+(10/3)\right)}{s(s+3)(s+6)}, \quad K>0$$



$$GH(s) = \frac{K\left(s+(10/3)\right)}{s(s+3)(s+6)}, \quad K>0$$

   a) Plot the rest of the straight-line segments of the locus, which are on the real axis.
   b) Mark the direction of the locus for every segment.
   c) Find the abscissa of point A.
   d) What is the value of K at the point A?
   e) Find the asymptotes of the locus.
   f) Discuss the stability of the system.

Solution to Problem (1)

The open-loop transfer function is

$$GH(s) = \frac{K\left(s+(10/3)\right)}{s(s+3)(s+6)} \tag{P6.1.1}$$

1. The poles are $p_1 = 0, p_2 = -3, p_3 = -6$ ($n = 3$).
2. The zeros are $z_1 = -10/3$ ($m = 1$).
3. The number of separate branches of the locus is $\max(3, 1) = 3$.
4. The intersection of the asymptotes is

$$\sigma_\alpha = \frac{\sum_{i=1}^{3} p_i - \sum_{j=1}^{1} z_j}{n-m} = \frac{-9-(-10/3)}{3-1} = -\frac{17}{6} = -2.83 \tag{P6.1.2}$$

Thus, the abscissa of point A is –2.83.

Solution to Problem (1)

5. The angle of the asymptotes is

$$\left.\begin{array}{l} \sphericalangle\varphi_\alpha = \dfrac{(2\rho+1)\pi}{n-m} \\ \rho = 0,1 \end{array}\right\} \Rightarrow \left.\begin{array}{ll} \hat{\varphi}_{\alpha_1} = 90^\circ & (\rho=0) \\ \hat{\varphi}_{\alpha_2} = 270^\circ & (\rho=1) \end{array}\right\} \tag{P6.1.3}$$

6. The segments of the real axis, which can be segments of the RL for $K > 0$, are the segment from 0 to –3 and the segment from –10/3 to –6.
7. The breakaway points of the RL are the roots of Equation P6.1.5. The characteristic equation of the system is

$$\begin{aligned} &1+\frac{K\left(s+(10/3)\right)}{s(s+3)(s+6)} = 0 \Rightarrow s(s+3)(s+6)+K\left(s+(10/3)\right) = 0 \Rightarrow \\ &K = -\frac{s(s+3)(s+6)}{s+(10/3)} = -\frac{s^3+9s^2+185}{s+(10/3)} \end{aligned} \tag{P6.1.4}$$

Solution to Problem (1)

and

$$\frac{dK}{ds}=0 \overset{(P6.1.4)}{\Rightarrow} 2s^3+19s^2+60s+60=0 \qquad (P6.1.5)$$

The roots of Equation P6.1.5 are $s_1 = -2$, $s_{2,3} = -3.75 \pm j0.9682$. Apparently the root $s_1 = -2$ is also a breakaway point of the RL, since from relationship (P6.1.4) it follows that $K = 6 \in R$. Hence,

$$s_b = -2 \qquad (P6.1.6)$$

Solution to Problem (1)

8. We continue with Routh's tabulation in order to find the intersections of RL with the imaginary axis.

$$\text{C.E.:} \quad s^3 + 9s^2 + s(K+18) + \frac{10}{3}K = 0 \tag{P6.1.7}$$

Routh's tabulation is

$$\begin{array}{c|cc} s^3 & 1 & K+18 \\ s^2 & 9 & \frac{10}{3}K \\ s^1 & b & \\ s^0 & \frac{10}{3}K & \end{array}$$

where

$$b = \frac{9(K+18) - (10/3)K}{9} = 8.63K + 18 \tag{P6.1.8}$$

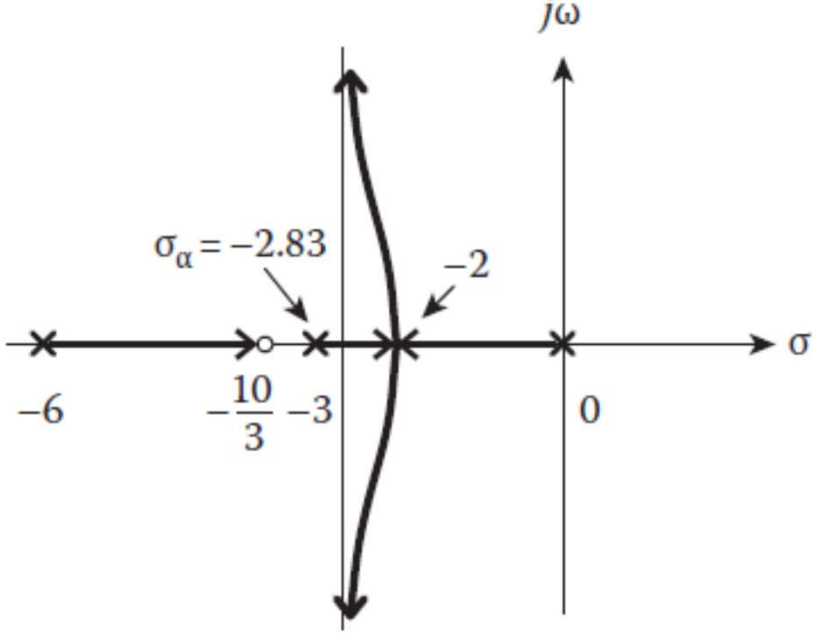
In order to find the critical value of K that ensures the stability of the system from row s^1, we have

$$b = 0 \overset{(8)}{\Rightarrow} K = -\frac{18}{8.63} = -2.08 < 0$$

The critical value of K is negative; thus, there are no intersections of the RL with the imaginary axis.

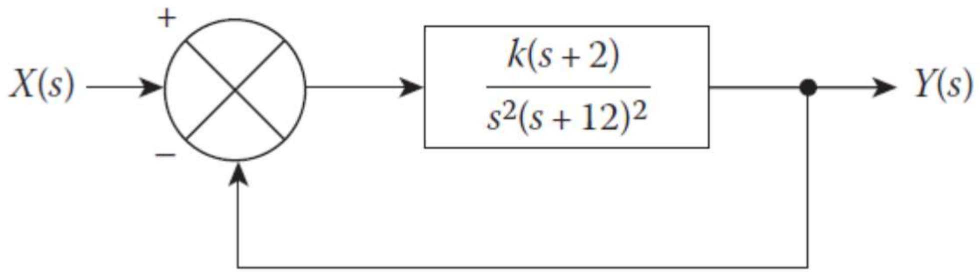
Solution to Problem (1)

9. We now plot the RL for $K > 0$.



Problems to Solve (2)

2. For the system shown in the following figure, design the RL diagram of the characteristic equation for K > 0, and find out the values of K for which the system is stable.



Solution to Problem (2)

The open-loop transfer function is

$$GH(s) = \frac{K(s+2)}{s^2(s+12)^2} \tag{P6.2.1}$$

1. The poles are $p_1 = p_2 = 0$, $p_3 = p_4 = -2$ ($n = 4$).
2. The zeros are $z_1 = -2$ ($m = 1$).
3. The number of separate branches of RL is $\max(4, 1) = 4$.
4. The intersection of the asymptotes is

$$\sigma_\alpha = \frac{\sum_{i=1}^{4} p_i - \sum_{j=1}^{1} z_j}{n-m} = \frac{-24-(-2)}{4-1} = -\frac{22}{3} = -7.33 \tag{P6.2.2}$$

5. The angles of the asymptotes are

$$\left.\begin{aligned} \sphericalangle\varphi_\alpha &= \frac{(2\rho+1)\pi}{n-m} \\ \rho &= 0,1,2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \hat{\varphi}_{\alpha_1} &= \frac{180^\circ}{3} = 60^\circ && \text{for } (\rho = 0) \\ \hat{\varphi}_{\alpha_2} &= \frac{3\cdot 180^\circ}{3} = 180^\circ && \text{for } (\rho = 1) \\ \hat{\varphi}_{\alpha_3} &= \frac{5\cdot 180^\circ}{3} = 300^\circ && \text{for } (\rho = 2) \end{aligned}\right\} \tag{P6.2.3}$$

Solution to Problem (2)

6. The segments of the real axis, which can also be segments of RL for $K > 0$, are between (–2, –12] and [–12, –∞).
7. The breakaway points of the RL are roots of Equation P6.2.6. The characteristic equation of the system is

$$1+\frac{K(s+2)}{s^2(s+12)^2}=0 \Rightarrow s^2(s+12)^2+K(s+2)=0 \Rightarrow \tag{P6.2.4}$$

$$s^4+24s^3+144s^2+Ks+2K=0$$

Solving for K, we get

$$K=-\frac{s^2(s+12)^2}{s+2} \tag{P6.2.5}$$

$$\frac{dK}{ds}=0 \overset{(\text{P6.2.5})}{\Rightarrow} 3s^4+56s^3+288s^2-576s=0 \tag{P6.2.6}$$

An apparent root of Equation P6.2.6 is $s = 0$, which is acceptable as a departure point of the branches from the real axis; as for $s = 0$, from the relationship (P6.2.5), we get $K = 0$.

Solution to Problem (2)

8. The intersections of RL with the imaginary axis can be found as follows:
   We proceed with Routh's tabulation for the characteristic equation of relationship (P6.2.7):

$$s^4 + 24s^3 + 144s^2 + Ks + 2K = 0 \tag{P6.2.7}$$

Routh's tabulation is

$$\begin{array}{c|ccc}
s^4 & 1 & 144 & 2K \\
s^3 & 24 & K & \\
s^2 & \dfrac{3456-K}{24} & 2K & \\
s^1 & \dfrac{K(3456-K)-1152K}{3456-K} & & \\
s^0 & 2K & &
\end{array}$$

Solution to Problem (2)

From row s^1, we get

$$\frac{K(3456-K)-1152K}{3456-K}=0 \Rightarrow 2304K-K^2=0 \Rightarrow K(2304-K)=0 \qquad \text{(P6.2.8)}$$

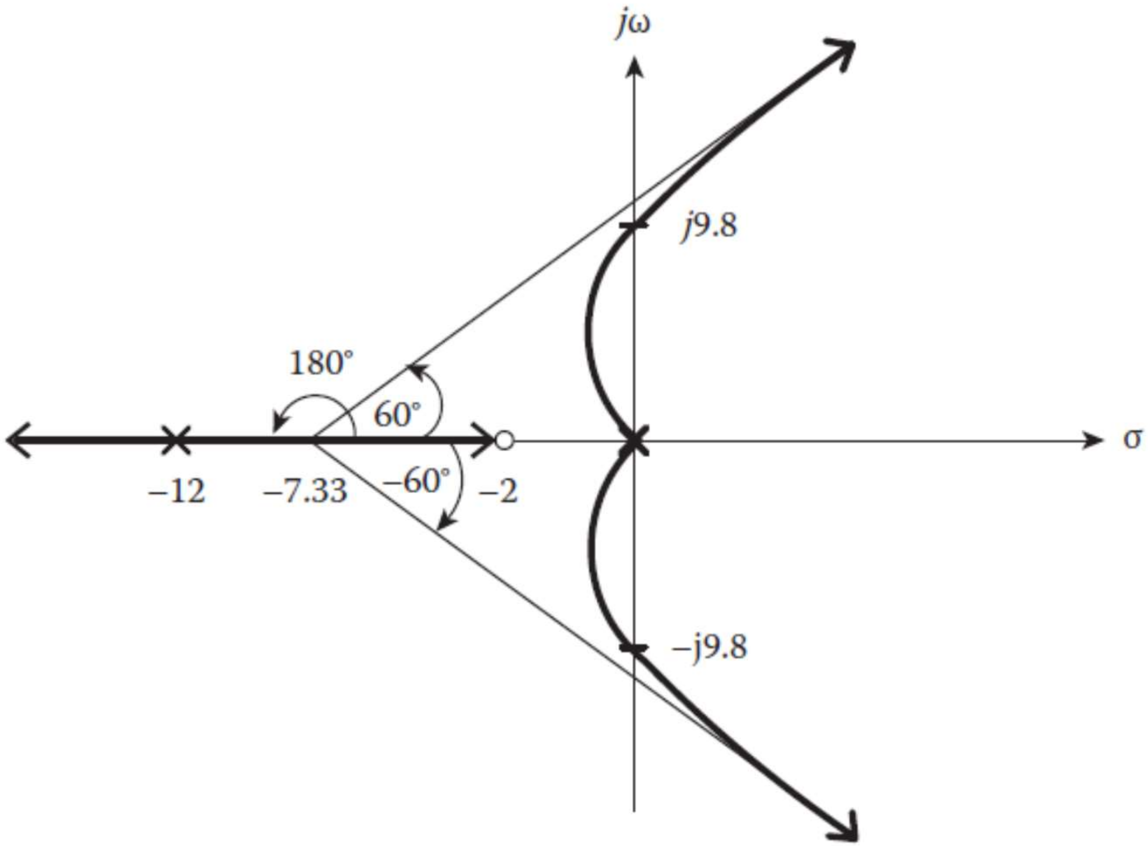
$$\Rightarrow K_c = 2304$$

From the auxiliary equation of row s^2, we have

$$\frac{3456-K_c}{24}s^2+2K_c \Rightarrow \omega_c = 9.8\,\text{rad/s} \qquad \text{(P6.2.9)}$$

Solution to Problem (2)

9. By using the previous findings, we now plot the RL of the system for $K > 0$.



We observe that for $0 < K < 2304$ the system is stable.

Problem

3. The loop transfer function of a system is

$$G(s)H(s) = \frac{K}{s(s^2 + 4s + 8)}, \quad K > 0$$

   a) Find the asymptotes and the angles of departure of the RL.
   b) Compute the critical value of K so that the closed-loop system is stable, and find the intersections of RL with the imaginary axis.
   c) Plot the RL of the characteristic equation of the system.

Solution to Problem (3)

a. The loop transfer function is

$$G(s)H(s) = \frac{K}{s(s^2 + 4s + 8)} \tag{P6.3.1}$$

The intersection of the asymptotes is

$$\sigma_\alpha = \frac{\sum_{i=1}^{3} p_i - \sum_{j=0} z_j}{n-m} = \frac{(-2-2j)+(-2+2j)+0}{3} = -\frac{4}{3} \tag{P6.3.2}$$

The angles of the asymptotes for $K > 0$ are

$$\left.\begin{array}{l} \sphericalangle\varphi_\alpha = \dfrac{(2\rho+1)\pi}{n-m} \\ \rho = 0,1,2 \end{array}\right\} \Rightarrow \left.\begin{array}{ll} \hat{\varphi}_{\alpha_1} = \dfrac{180^\circ}{3} = 60^\circ & \text{for } (\rho = 0) \\ \hat{\varphi}_{\alpha_2} = \dfrac{3 \cdot 180^\circ}{3} = 180^\circ & \text{for } (\rho = 1) \\ \hat{\varphi}_{\alpha_3} = \dfrac{5 \cdot 180^\circ}{3} = 300^\circ & \text{for } (\rho = 2) \end{array}\right\} \tag{P6.3.3}$$

Solution to Problem (3)

The angle of departure from the complex pole $(-2 + 2j)$ is given by:

$$\sphericalangle \varphi_d = 180^\circ + \sphericalangle GH'(s) \tag{P6.3.4}$$

where

$$\sphericalangle(GH'(s)) = \sphericalangle\left(\left.\frac{K}{s(s+2+2j)}\right|_{s=-2+2j}\right) = \sphericalangle\left(\frac{K}{4j(-2+2j)}\right) \Rightarrow$$

$$\sphericalangle(GH'(s)) = 0^\circ - 90^\circ - \tan^{-1}\left(\frac{2}{-2}\right) = 0^\circ - 90^\circ - 135^\circ = -225^\circ \tag{P6.3.5}$$

Thus,

$$\sphericalangle \varphi_d = 180^\circ - 225^\circ = -45^\circ \tag{P6.3.6}$$

Solution to Problem (3)

b. The characteristic equation is

$$1+G(s)H(s)=0 \overset{(P6.3.1)}{\Rightarrow} s(s^2+4s+8)+K=0 \Rightarrow \\ s^3+4s^2+8s+K=0 \qquad \text{(P6.3.11)}$$

Routh's tabulation is

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 4 & K \\ s^1 & 8-\dfrac{K}{4} & \\ s^0 & K & \end{array}$$

The closed-loop system is stable if

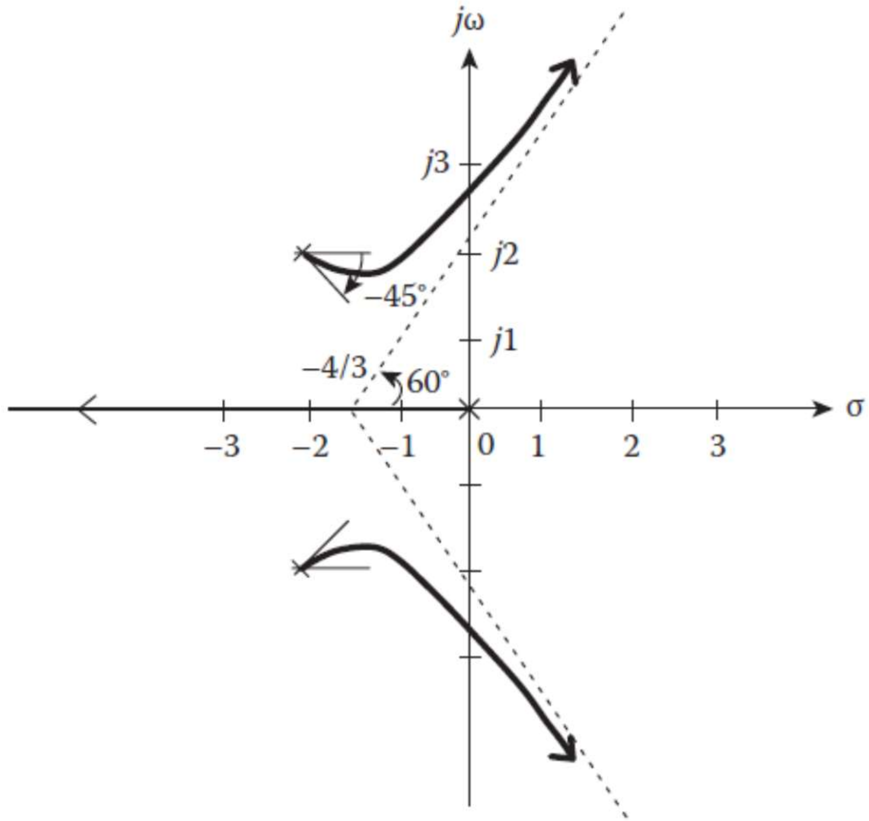
$$\left.\begin{array}{l} 8-\dfrac{K}{4}>0 \\ K>0 \end{array}\right\} \Rightarrow 0<K<32 \qquad \text{(P6.3.12)}$$

Given that $K > 0$, for $K = K_c = 32$, the RL intersects the imaginary axis. We find the intersection $\pm j\omega_c$ from the auxiliary equation of row s^2:

$$4s^2+K_c=0 \Rightarrow s^2=-\frac{32}{4} \Rightarrow s=\pm j2\sqrt{2} \Rightarrow \\ \omega_c=2\sqrt{2}\ \text{rad/s} \qquad \text{(P6.3.13)}$$

Solution to Problem (3)

c. Based on the previous computations, we plot the root-locus diagram of the system.



Problem to Solve

4. Given the following loop transfer function sketch the RL of the characteristic equation for K > 0 and for K < 0.

$$GH(s) = \frac{K}{s(s+1)(s+3)(s+4)}$$

Solution to Problem (4)

The open-loop transfer function is

$$GH(s) = \frac{K}{s(s+1)(s+3)(s+4)} \tag{P6.4.1}$$

1. The poles are $p_1 = 0$, $p_2 = -1$, $p_3 = -3$, $p_4 = -4$ ($n = 4$).
2. There are no zeros.
3. There are $4 = \max(4, 0)$ separate loci.
4. The intersection of the asymptotes is

$$\sigma_\alpha = \frac{\sum_{i=1}^{4} p_i}{n-m} = \frac{0 + (-1) + (-3) + (-4)}{3} = -\frac{8}{4} = -2 \tag{P6.4.2}$$

Solution to Problem (4)

5. The angles of the asymptotes are

$$\sphericalangle\varphi_\alpha = \begin{cases} \dfrac{(2\rho+1)\pi}{n-m} & \text{for } K>0 \qquad \text{(P6.4.3)} \\ \dfrac{(2\rho)\pi}{n-m} & \text{for } K<0 \qquad \text{(P6.4.4)} \end{cases}$$

$$\rho = 0,1,2,\ldots,n-m-1 \Rightarrow \rho = 0,1,2,3$$

Hence, for $K>0$, we have

$$\left.\begin{aligned} \hat{\varphi}_{a_1} &= \frac{180^\circ}{4} = 45^\circ \\ \hat{\varphi}_{a_2} &= \frac{3\cdot 180^\circ}{4} = 135^\circ \\ \hat{\varphi}_{a_3} &= \frac{5\cdot 180^\circ}{4} = 225^\circ \\ \hat{\varphi}_{a_4} &= \frac{7\cdot 180^\circ}{4} = 315^\circ \end{aligned}\right\} \qquad \text{(P6.4.5)}$$

Solution to Problem (4)

While for $K < 0$, we have

$$\left.\begin{aligned}
\hat{\varphi}_{a_1} &= \frac{0 \cdot 180^\circ}{4} = 0^\circ \\
\hat{\varphi}_{a_2} &= \frac{2 \cdot 180^\circ}{4} = 90^\circ \\
\hat{\varphi}_{a_3} &= \frac{4 \cdot 180^\circ}{4} = 180^\circ \\
\hat{\varphi}_{a_4} &= \frac{6 \cdot 180^\circ}{4} = 270^\circ
\end{aligned}\right\} \tag{P6.4.6}$$

6. The segments of the real axis that belong to the RL for $K > 0$ are those between 0 and –1 and between –3 and –4.

   For $K < 0$, the RL includes the segment that begins from 0 and tends to $+\infty$, the segment between –1 to –3, and the segment from –4 that tends to $-\infty$.

Solution to Problem (4)

7. In order to find the breakaway points, we have

$$\text{C.E.:} \quad 1+GH(s)=0 \overset{(\text{P6.4.1})}{\Rightarrow} K=-s(s+1)(s+3)(s+4) \tag{P6.4.7}$$

$$\frac{dK}{ds}=0 \overset{(\text{P6.4.7})}{\Rightarrow} 2s^3+12s^2+19s+6=0 \tag{P6.4.8}$$

By solving (8), we get the roots

$$s_1=-2, \quad s_2 \simeq -0.42, \quad \text{and} \quad s_3 \simeq -3.58.$$

The root $s_{b_1}=-2$ is the breakaway point for $K<0$, and the roots $s_{b_2}=-0.42$ and $s_{b_3}=-3.58$ are the breakaway points for $K>0$.

Solution to Problem (4)

8. In order to find the intersections of RL with the imaginary axis, we proceed with Routh's tabulation.

   The characteristic equation is

$$s^4 + 8s^3 + 19s^2 + 12s + K = 0 \tag{P6.4.9}$$

   Routh's tabulation is

$$\begin{array}{c|ccc} s^4 & 1 & 19 & K \\ s^3 & 8 & 12 & \\ s^2 & 17.5 & K & \\ s^1 & \dfrac{210-8K}{17.5} & & \\ s^0 & K & & \end{array}$$

From row s^1, it follows that

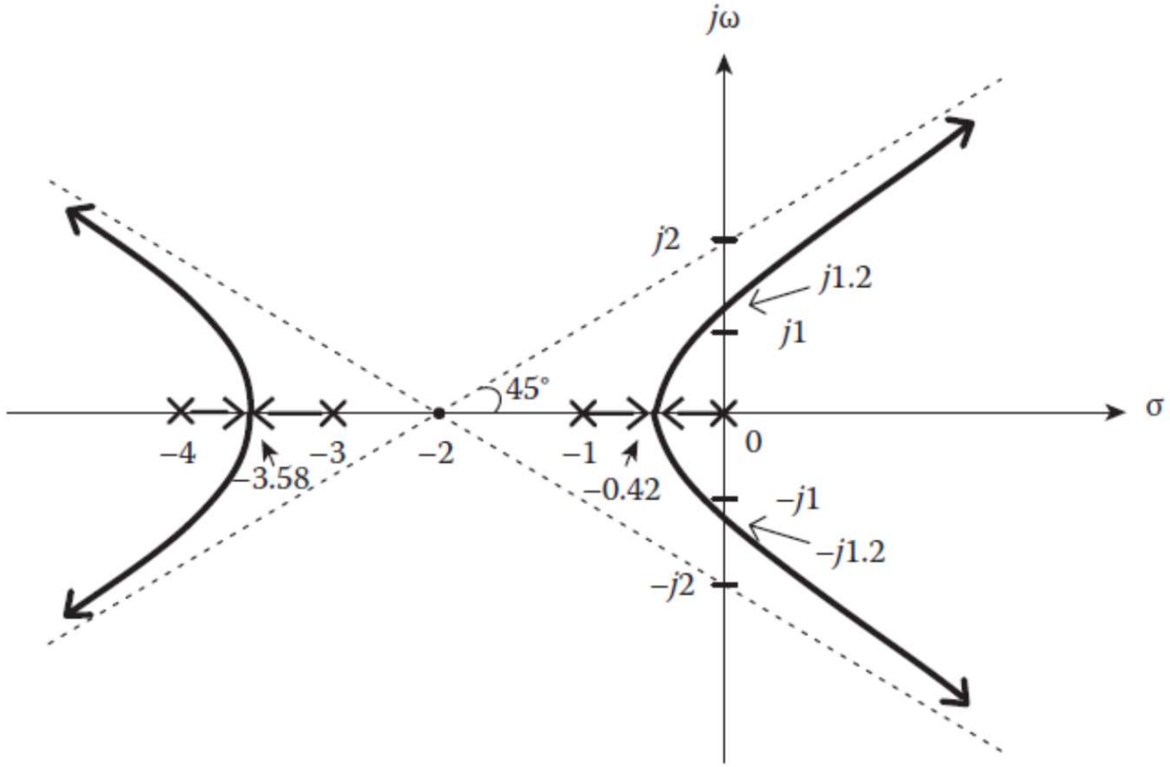
$$\frac{210-8K}{17.5} = 0 \Rightarrow K_c = \frac{210}{8} = 26.25 \tag{P6.4.10}$$

The intersections are determined with the use of the auxiliary equation of row s^2:

$$\begin{aligned} &17.5s^2 + K_c = 0 \Rightarrow 17.5s^2 + 26.25 = 0 \Rightarrow \\ &s = \pm 1.2j \end{aligned} \tag{P6.4.11}$$

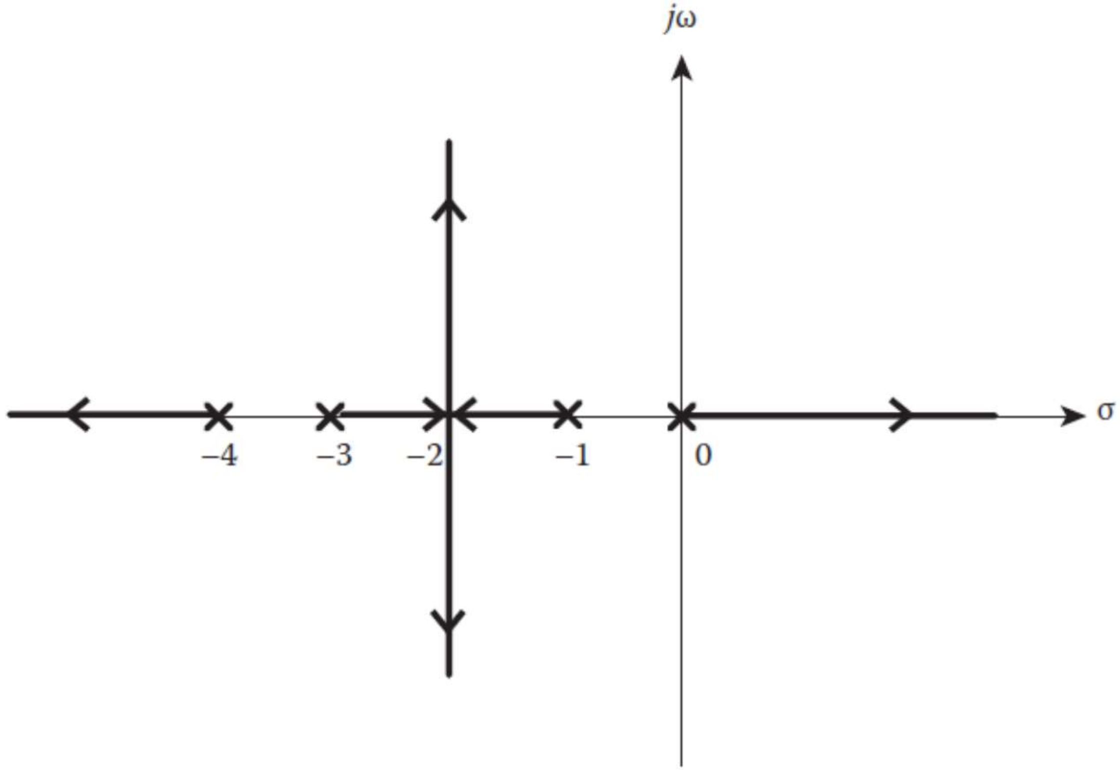
Solution to Problem (4)

9. Based on the previous, we now plot the RL diagram.
   For $K > 0$, the RL is:



Solution to Problem (4)

For $K < 0$, the RL is shown in:



Problem to Solve

5. Plot the RL of the characteristic equation of the system when the loop transfer function is

$$GH(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+16)}, \quad K > 0$$

Solution to Problem (5)

The open-loop transfer function is

$$G(s)H(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+16)} \tag{P6.5.1}$$

1. The poles are $p_1 = 0$, $p_2 = 1$, $p_3 = -2 + j3.465$, $p_4 = -2 - j3.465$ and $(n = 4)$.
2. The zeros are $z_1 = -1$ $(m = 1)$.
3. There are $4 = \max(4,1)$ separate loci.
4. The intersection of the asymptotes is

$$\sigma_\alpha = \frac{1+(-2-j3.465)+(-2+j3.465)-(-1)}{3} = -\frac{2}{3} \tag{P6.5.2}$$

5. The angles of the asymptotes are

$$\left.\begin{array}{l} \sphericalangle\varphi_\alpha = \dfrac{(2\rho+1)\pi}{n-m} \\ \rho = 0,1,2 \end{array}\right\} \Rightarrow \left.\begin{array}{ll} \hat{\varphi}_{\alpha_1} = \dfrac{180^\circ}{3} = 60^\circ & (\rho = 0) \\ \hat{\varphi}_{\alpha_2} = \dfrac{3\cdot 180^\circ}{3} = 180^\circ & (\rho = 1) \\ \hat{\varphi}_{\alpha_3} = \dfrac{5\cdot 180^\circ}{3} = 300^\circ & (\rho = 2) \end{array}\right\} \tag{P6.5.3}$$

Solution to Problem (5)

6. In order to find the angles of departure from the complex poles, we proceed as follows:
   For the complex pole $(-2 + j3.465)$, we have

$$\sphericalangle\varphi_d = 180^\circ - (\Sigma\varphi_p - \Sigma\varphi_z) =$$

$$180^\circ - \left(\tan^{-1}\frac{3.465}{2} + 180^\circ - \tan^{-1}\frac{3.465}{3} + 90^\circ - 180^\circ + \tan^{-1}\frac{3.465}{1}\right) \Rightarrow$$

$$\varphi_d = 180^\circ - (180^\circ - 60^\circ + 180^\circ - 49.1^\circ + 90^\circ - 180^\circ + 73.9^\circ) \simeq -54.8^\circ$$

Hence,

$$\hat{\varphi}_{d(-2+j3.465)} = -54.8^\circ \tag{P6.5.4}$$

The angle of departure from the complex pole $(-2 - j3.465)$, due to symmetry, is

$$\hat{\varphi}_{d(-2-j3.465)} = 54.8^\circ \tag{P6.5.5}$$

Solution to Problem (5)

7. In order to find the breakaway points, we have

$$\text{C.E.}: \quad 1+G(s)H(s)=0 \overset{\text{(P6.5.1)}}{\Rightarrow} K=-\frac{s(s-1)(s^2+4s+16)}{(s+1)} \tag{P6.5.6}$$

$$\frac{dK}{ds}=0 \overset{\text{(P6.5.6)}}{\Rightarrow} 3s^4+10s^3+21s^2+24s-16=0 \tag{P6.5.7}$$

By solving (P6.5.7), we get the roots

$$\left.\begin{aligned} s_{1,2} &= -0.7595 \pm j2.1637 \\ s_3 &= -2.2627 \\ s_4 &= 0.4483 \end{aligned}\right\} \tag{P6.5.8}$$

The complex roots $s_{1,2}$ are rejected. The roots s_3 and s_4 are accepted as breakaway points, because they give real values of K, that is, 35.48 and 2.048, respectively.

Solution to Problem (5)

8. In order to find the intersections of RL with the imaginary axis, we proceed with Routh's tabulation.

   The characteristic equation is

$$s^4 + 3s^3 + 12s^2 + (K-16)s + K = 0 \tag{P6.5.9}$$

Routh's tabulation is

$$\begin{array}{c|ccc} s^4 & 1 & 12 & K \\ s^3 & 3 & K-16 & \\ s^2 & \dfrac{52-K}{3} & K & \\ s^1 & b & & \\ s^0 & K & & \end{array}$$

where

$$b = \frac{\left((52-K)/3\right)(K-16) - 3K}{(52-K)/3} \tag{P6.5.10}$$

Solution to Problem (5)

We now compute the critical values of K for stability.

From row s^1 it suffices that $b = 0$.

$$(\text{P6.5.10}) \Rightarrow K^2 - 59K + 832 = 0 \Rightarrow \begin{cases} \rightarrow K_1 = 35.7 \\ \rightarrow K_2 = 23.3 \end{cases}$$

Thus, we get two values for critical stability:

$$K_{c_1} = 35.7 \quad \text{and} \quad K_{c_2} = 23.3 \tag{P6.5.11}$$

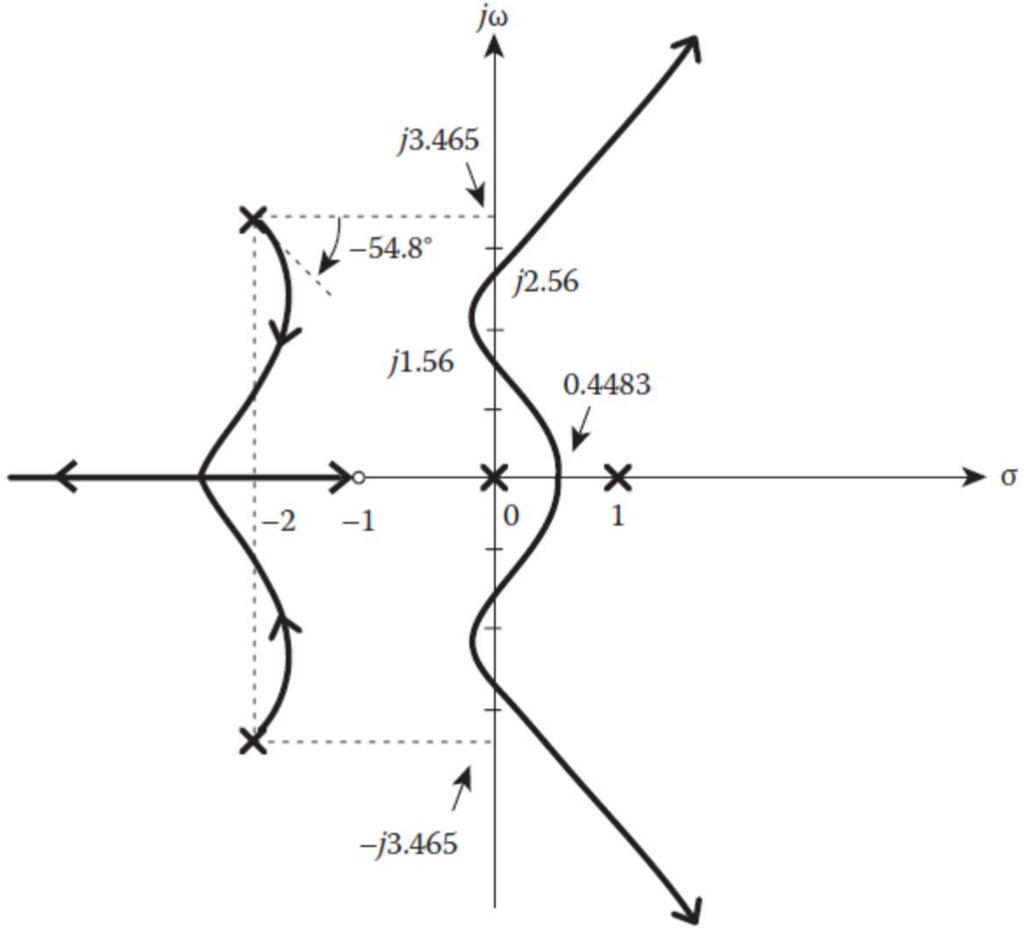
The intersections of RL with the imaginary axis are found with the use of the auxiliary equation of row s^2:

$$\frac{52 - K_{cr}}{3} s^2 + K_{cr} = 0 \tag{P6.5.12}$$

From (P6.5.12) for $K_{cr_1} = 35.7$, we get $j\omega_{cr_1} = j2.56$, and for $K_{cr_2} = 23.3$, we have $j\omega_{cr_2} = j1.56$.

Solution to Problem (5)

9. Based on the previous queries, we plot the RL of the characteristic equation of the system.



Problem to Solve

6. The loop transfer function of a system is given by

$$G(s)H(s) = \frac{K}{s(s+1)(s^2+4s+8)}, \quad K > 0$$

a) Find the asymptotes and the angles of departure of the RL.
b) Find the breakaway points (if there are any) of the RL. Take into account that one root of the equation $4s^3 + 15s^2 + 24s + 8 = 0$ is $s = -1.6549 + j1.3432$.
c) Find the critical value of K so that the system is stable.
d) Plot the RL.

Solution to Problem (6)

The open-loop transfer function is

$$GH(s) = \frac{K}{s(s+1)(s^2+4s+8)}, \quad K > 0 \tag{P6.6.1}$$

a. The intersection of the asymptotes is

$$\sigma_\alpha = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n-m} = \frac{0+(-1)+(-2+j2)+(-2-j2)}{4-0} = -\frac{5}{4} \tag{P6.6.2}$$

The angles of the asymptotes for $K > 0$ are

$$\left.\begin{aligned} \sphericalangle\varphi_\alpha &= \frac{(2\rho+1)\pi}{n-m} \\ \rho &= 0,1,2,\ldots,n-m-1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \hat{\varphi}_{\alpha_1} &= \frac{180^\circ}{4} = 45^\circ & (\rho = 0) \\ \hat{\varphi}_{\alpha_2} &= \frac{3\cdot 180^\circ}{4} = 135^\circ & (\rho = 1) \\ \hat{\varphi}_{\alpha_3} &= \frac{5\cdot 180^\circ}{4} = 225^\circ & (\rho = 2) \\ \hat{\varphi}_{\alpha_4} &= \frac{7\cdot 180^\circ}{4} = 315^\circ & (\rho = 3) \end{aligned}\right\} \tag{P6.6.3}$$

Solution to Problem (6)

The angle of departure from the complex pole $(-2 + j2)$ is

$$\sphericalangle\varphi_d = 180^\circ + \sphericalangle GH'(s) \tag{P6.6.4}$$

where

$$GH'(s) = \sphericalangle\frac{K}{s(s+1)(s+2+j2)} = 0^\circ - 90^\circ - \tan^{-1}\left(\frac{2}{-2}\right) - \tan^{-1}\left(\frac{2}{-1}\right)$$

$$= 0^\circ - 90^\circ - 135^\circ - 116.37^\circ \simeq -341.57^\circ$$

Hence,

$$\sphericalangle\varphi_d \overset{(P6.6.4)}{=} 180^\circ - 341.57^\circ = -161.57^\circ \tag{P6.6.5}$$

The angle of departure from the complex conjugate pole $(-2 - j2)$ is, due to symmetry of RL, 161.57°. The angle of departure from the pole to 0 is 180° and from the pole to –1 is 0°.

Solution to Problem (6)

b. The breakaway points satisfy the following relationship:

$$\sum_{i=1}^{n} \frac{1}{(s_b + p_i)} = \sum_{j=1}^{m} \frac{1}{(s_b + z_j)} \tag{P6.6.6}$$

$$(\text{P6.6.6}) \Rightarrow \frac{1}{s_b} + \frac{1}{s_b + 1} + \frac{1}{s_b + 2 - j2} + \frac{1}{s_b + 2 + j2} = 0 \Rightarrow$$

$$4{s_b}^3 + 15{s_b}^2 + 24s_b + 8 = 0 \tag{P6.6.7}$$

The roots of Equation P6.6.7 are $s_1 = -0.4402$ and $s_{2,3} = -1.6549 \pm j1.3432$. The breakaway point is the root $s_1 = s_b = -0.4402$ for which we get a real value of K.

c. The characteristic equation is

$$s^4 + 5s^3 + 12s^2 + 8s + K = 0 \tag{P6.6.8}$$

Solution to Problem (6)

Routh's tabulation is

$$
\begin{array}{c|ccc}
s^4 & 1 & 12 & K \\
s^3 & 5 & 8 & \\
s^2 & 10.4 & K & \\
s^1 & b & & \\
s^0 & K & &
\end{array}
$$

where

$$b = \frac{10.4 \cdot 8 - 5K}{10.4} \tag{P6.6.9}$$

The closed-loop system is stable for $b > 0$ and $K > 0$:

$$b > 0 \Rightarrow 8 - \frac{5K}{10.4} > 0 \Rightarrow K < 16.64 \tag{P6.6.10}$$

Thus,

$$0 < K < 16.64 \tag{P6.6.11}$$

For $K_c = 16.64$, the roots are on the imaginary axis.

Solution to Problem (6)

By substituting the intersections with the imaginary axis in the auxiliary equation of row s^2, we get

$$10.4s^2 + K_c = 0 \Rightarrow \pm j\omega_c = \pm j1.2649 \tag{P6.6.12}$$

d. The number of separate loci is $\max(n, m) = \max(4, 0) = 4$. These depart from the poles of the open-loop transfer function, $p_1 = 0$, $p_2 = -1$, and $p_{3,4} = -2 \pm j2$. The number of branches that approach infinity is $n - m = 4$, as $K \rightarrow \infty$.

Solution to Problem (6)

Based on the previous queries we now plot the RL of the characteristic equation of the system: