APPENDIX G

Matrices, Determinants, and Systems of Equations

To Accompany

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By

Norman S. Nise

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APPENDIX G

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G.1 Matrix Definitions and Notations

Matrix

An $m \times n$ *matrix* is a rectangular or square array of elements with m rows and n columns. An example of a matrix is shown in Eq. (G.1).

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{G.1}$$

For each subscript, a_{ij}, i = the row, and j = the column. If $m = n$, the matrix is said to be a *square matrix*.

Vector

If a matrix has just one row, it is called a *row vector*. An example of a row vector follows:

$$\mathbf{B} = [b_{11} \quad b_{12} \quad \cdots \quad b_{1n}] \tag{G.2}$$

If a matrix has just one column, it is called a *column vector*. An example of a column vector follows:

$$\mathbf{C} = \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix} \tag{G.3}$$

Partitioned Matrix

A matrix can be partitioned into component matrices or vectors. For example, let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \left[\begin{array}{cc:cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ \hdashline a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tag{G.4}$$

where

$$\mathbf{A}_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}; \quad \mathbf{A}_{12} = \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \\ a_{33} & a_{34} \end{bmatrix}$$

$$\mathbf{A}_{21} = [a_{41} \quad a_{42}]; \quad \mathbf{A}_{22} = [a_{43} \quad a_{44}]$$

Null Matrix

A matrix with all elements equal to zero is called the *null matrix*; that is, $a_{ij} = 0$ for all i and j. An example of a null matrix follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & & \cdots & & 0 \\ 0 & 0 & 0 & & \cdots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{G.5}$$

Diagonal Matrix

A square matrix with all elements off of the diagonal equal to zero is said to be a *diagonal matrix*; that is, $a_{ij} = 0$ for $i \neq j$. An example of a diagonal matrix follows:

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \tag{G.6}$$

Identity Matrix

A diagonal matrix with all diagonal elements equal to unity is called an *identity matrix* and is denoted by $\mathbf{I}$; that is, $a_{ij} = 1$ for $i = j$, and $a_{ij} = 0$ for $i \neq j$. An example of an identity matrix follows:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{G.7}$$

Symmetric Matrix

A square matrix for which $a_{ij} = a_{ji}$ is called a *symmetric matrix*. An example of a symmetric matrix follows:

$$\mathbf{A} = \begin{bmatrix} 3 & 8 & 7 \\ 8 & 9 & 2 \\ 7 & 2 & 4 \end{bmatrix} \tag{G.8}$$

Matrix Transpose

The *transpose* of matrix $\mathbf{A}$, designated $\mathbf{A}^T$, is formed by interchanging the rows and columns of $\mathbf{A}$. Thus, if $\mathbf{A}$ is an $m \times n$ matrix with elements a_{ij}, the transpose is an $n \times m$ matrix with elements a_{ji}. An example follows. Given

$$\mathbf{A} = \begin{bmatrix} 1 & 7 & 9 \\ 2 & 6 & -3 \\ 4 & 8 & 5 \\ -1 & 3 & -2 \end{bmatrix} \tag{G.9}$$

then

$$\mathbf{A}^T = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 7 & 6 & 8 & 3 \\ 9 & -3 & 5 & -2 \end{bmatrix} \tag{G.10}$$

Determinant of a Square Matrix

The *determinant* of a square matrix is denoted by det **A**, or

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix} \tag{G.11}$$

The determinant of a 2×2 matrix,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{G.12}$$

is evaluated as

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \tag{G.13}$$

Minor of an Element

The *minor*, M_{ij}, of element a_{ij} of det **A** is the determinant formed by removing the ith row and the jth column from det **A**. As an example, consider the following determinant:

$$\det \mathbf{A} = \begin{vmatrix} 3 & 8 & 7 \\ 6 & 9 & 2 \\ 5 & 1 & 4 \end{vmatrix} \tag{G.14}$$

The minor M_{32} is the determinant formed by removing the third row and the second column from det **A**. Thus,

$$M_{32} = \begin{vmatrix} 3 & 7 \\ 6 & 2 \end{vmatrix} = -36 \tag{G.15}$$

Cofactor of an Element

The *cofactor*, C_{ij}, of element a_{ij} of det **A** is defined to be

$$C_{ij} = (-1)^{(i+j)} M_{ij} \tag{G.16}$$

For example, given the determinant of Eq. (G.14),

$$C_{21} = (-1)^{(2+1)} M_{21} = (-1)^3 \begin{vmatrix} 8 & 7 \\ 1 & 4 \end{vmatrix} = -25 \tag{G.17}$$

Evaluating the Determinant of a Square Matrix

The determinant of a square matrix can be evaluated by expanding minors along any row or column. Expanding along any row, we find

$$\det \mathbf{A} = \sum_{k=1}^{n} a_{ik}C_{ik} \tag{G.18}$$

where n = number of columns of $\mathbf{A}$; i is the ith row selected to expand by minors; and C_{ik} is the cofactor of a_{ik}. Expanding along any column, we find

$$\det \mathbf{A} = \sum_{k=1}^{m} a_{kj}C_{kj} \tag{G.19}$$

where m = number of rows of $\mathbf{A}$; j is the jth column selected to expand by minors; and C_{kj} is the cofactor of a_{kj}. For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ -5 & 6 & -7 \\ 8 & 5 & 4 \end{bmatrix} \tag{G.20}$$

then, expanding by minors on the third column, we find

$$\det \mathbf{A} = 2\begin{vmatrix} -5 & 6 \\ 8 & 5 \end{vmatrix} - (-7)\begin{vmatrix} 1 & 3 \\ 8 & 5 \end{vmatrix} + 4\begin{vmatrix} 1 & 3 \\ -5 & 6 \end{vmatrix} = -195 \tag{G.21}$$

Expanding by minors on the second row, we find

$$\det \mathbf{A} = -(-5)\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} + 6\begin{vmatrix} 1 & 2 \\ 8 & 4 \end{vmatrix} - (-7)\begin{vmatrix} 1 & 3 \\ 8 & 5 \end{vmatrix} = -195 \tag{G.22}$$

Singular Matrix

A matrix is *singular* if its determinant equals zero.

Nonsingular Matrix

A matrix is *nonsingular* if its determinant does not equal zero.

Adjoint of a Matrix

The *adjoint* of a square matrix, $\mathbf{A}$, written adj $\mathbf{A}$, is the matrix formed from the transpose of the matrix $\mathbf{A}$ after all elements have been replaced by their cofactors. Thus,

$$\operatorname{adj} \mathbf{A} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T \tag{G.23}$$

For example, consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 5 \\ 6 & 8 & 7 \end{bmatrix} \tag{G.24}$$

Hence,

$$\operatorname{adj}\mathbf{A} = \begin{bmatrix} \begin{vmatrix} 4 & 5 \\ 8 & 7 \end{vmatrix} & -\begin{vmatrix} -1 & 5 \\ 6 & 7 \end{vmatrix} & \begin{vmatrix} -1 & 4 \\ 6 & 8 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 8 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 6 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 6 & 8 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} -12 & 10 & -2 \\ 37 & -11 & -8 \\ -32 & 4 & 6 \end{bmatrix} \tag{G.25}$$

Rank of a Matrix

The *rank* of a matrix, $\mathbf{A}$, equals the number of linearly independent rows or columns. The rank can be found by finding the highest-order square submatrix that is nonsingular. For example, consider the following:

$$\mathbf{A} = \begin{bmatrix} 1 & -5 & 2 \\ 4 & 7 & -5 \\ -3 & 15 & -6 \end{bmatrix} \tag{G.26}$$

The determinant of $\mathbf{A} = 0$. Since the determinant is zero, the 3×3 matrix is singular. Choosing the submatrix

$$\mathbf{A} = \begin{bmatrix} 1 & -5 \\ 4 & 7 \end{bmatrix} \tag{G.27}$$

whose determinant equals 27, we conclude that $\mathbf{A}$ is of rank 2.

G.2 Matrix Operations

Addition

The sum of two matrices, written $\mathbf{A} + \mathbf{B} = \mathbf{C}$, is defined by $a_{ij} + b_{ij} = c_{ij}$. For example,

$$\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ -1 & 8 \end{bmatrix} \tag{G.28}$$

Subtraction

The difference between two matrices, written $\mathbf{A} - \mathbf{B} = \mathbf{C}$, is defined by $a_{ij} - b_{ij} = c_{ij}$. For example,

$$\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 7 & 2 \end{bmatrix} \tag{G.29}$$

Multiplication

The product of two matrices, written $\mathbf{AB} = \mathbf{C}$, is defined by $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. For example, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{G.30}$$

then

$$C = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}) & (a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}) & (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}) & (a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}) \end{bmatrix} \tag{G.31}$$

Notice that multiplication is defined only if the number of columns of **A** equals the number of rows of **B**.

Multiplication by a Constant

A matrix can be multiplied by a constant by multiplying every element of the matrix by that constant. For example, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{G.32}$$

then

$$k\mathbf{A} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix} \tag{G.33}$$

Inverse

An $n \times n$ square matrix, **A**, has an inverse, denoted by $\mathbf{A}^{-1}$, which is defined by

$$\mathbf{AA}^{-1} = \mathbf{I} \tag{G.34}$$

where **I** is an $n \times n$ identity matrix. The inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{\operatorname{adj}\mathbf{A}}{\det\mathbf{A}} \tag{G.35}$$

For example, find the inverse of **A** in Eq. (G.24) . The adjoint was calculated in Eq. (G.25) . The determinant of **A** is

$$\det\mathbf{A} = 1\begin{vmatrix} 4 & 5 \\ 8 & 7 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 3 \\ 8 & 7 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -34 \tag{G.36}$$

Hence,

$$\mathbf{A}^{-1} = \frac{\begin{bmatrix} -12 & 10 & -2 \\ 37 & -11 & -8 \\ -32 & 4 & 6 \end{bmatrix}}{-34} = \begin{bmatrix} 0.353 & -0.294 & 0.059 \\ -1.088 & 0.324 & 0.235 \\ 0.941 & -0.118 & -0.176 \end{bmatrix} \tag{G.37}$$

G.3 Matrix and Determinant Identities

The following are identities that apply to matrices and determinants.

Matrix Identities

Commutative Law

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{G.38}$$

$$\mathbf{AB} \neq \mathbf{BA} \tag{G.39}$$

Associative Law

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \tag{G.40}$$

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} \tag{G.41}$$

Transpose of Sum

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \tag{G.42}$$

Transpose of Product

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \tag{G.43}$$

Determinant Identities

Multiplication of a Single Row or Single Column of a Matrix, A, by a Constant
If a single row or single column of a matrix, $\mathbf{A}$, is multiplied by a constant, k, forming the matrix, $\tilde{\mathbf{A}}$, then

$$\det \tilde{\mathbf{A}} = k \det \mathbf{A} \tag{G.44}$$

Multiplication of All Elements of an $n \times n$ Matrix, A, by a Constant

$$\det (k\mathbf{A}) = k^n \det \mathbf{A} \tag{G.45}$$

Transpose

$$\det \mathbf{A}^T = \det \mathbf{A} \tag{G.46}$$

Determinant of the Product of Square Matrices

$$\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B} \tag{G.47}$$

$$\det \mathbf{AB} = \det \mathbf{BA} \tag{G.48}$$

G.4 Systems of Equations

Representation

Assume the following system of n linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n} &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n} &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn} &= b_n \end{aligned} \tag{G.49}$$

This system of equations can be represented in vector-matrix form as

$$\mathbf{Ax} = \mathbf{B} \tag{G.50}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

For example, the following system of equations,

$$5x_1 + 7x_2 = 3 \tag{G.51a}$$

$$-8x_1 + 4x_2 = -9 \tag{G.51b}$$

can be represented in vector-matrix form as $\mathbf{Ax} = \mathbf{B}$, or

$$\begin{bmatrix} 5 & 7 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \end{bmatrix} \tag{G.52}$$

Solution via Matrix Inverse

If $\mathbf{A}$ is nonsingular, we can premultiply Eq. (G.50) by $\mathbf{A}^{-1}$, yielding the solution $\mathbf{x}$. Thus,

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{B} \tag{G.53}$$

For example, premultiplying both sides of Eq. (G.52) by $\mathbf{A}^{-1}$, where

$$\mathbf{A}^{-1} = \begin{bmatrix} 5 & 7 \\ -8 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 0.0526 & -0.0921 \\ 0.1053 & 0.0658 \end{bmatrix} \tag{G.54}$$

we solve for $\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$ as follows:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.0526 & -0.0921 \\ 0.1053 & 0.0658 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 0.987 \\ -0.276 \end{bmatrix} \tag{G.55}$$

Solution via Cramer's Rule

Equation (G.53) allows us to solve for all unknowns, x_i, where $i = 1$ to n. If we are interested in a single unknown, x_k, then Cramer's rule can be used. Given Eq. (G.50), Cramer's rule states that

$$x_k = \frac{\det \mathbf{A}_k}{\det \mathbf{A}} \tag{G.56}$$

where $\mathbf{A}_k$ is a matrix formed by replacing the kth column of $\mathbf{A}$ by $\mathbf{B}$. For example, solve Eq. (G.52). Using Eq. (G.56) with

$$\mathbf{A} = \begin{bmatrix} 5 & 7 \\ -8 & 4 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$$

we find

$$x_1 = \frac{\begin{vmatrix} 3 & 7 \\ -9 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ -8 & 4 \end{vmatrix}} = \frac{75}{76} = 0.987 \tag{G.57}$$

and

$$x_2 = \frac{\begin{vmatrix} 5 & 3 \\ -8 & -9 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ -8 & 4 \end{vmatrix}} = \frac{-21}{76} = -0.276 \qquad \text{(G.58)}$$

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