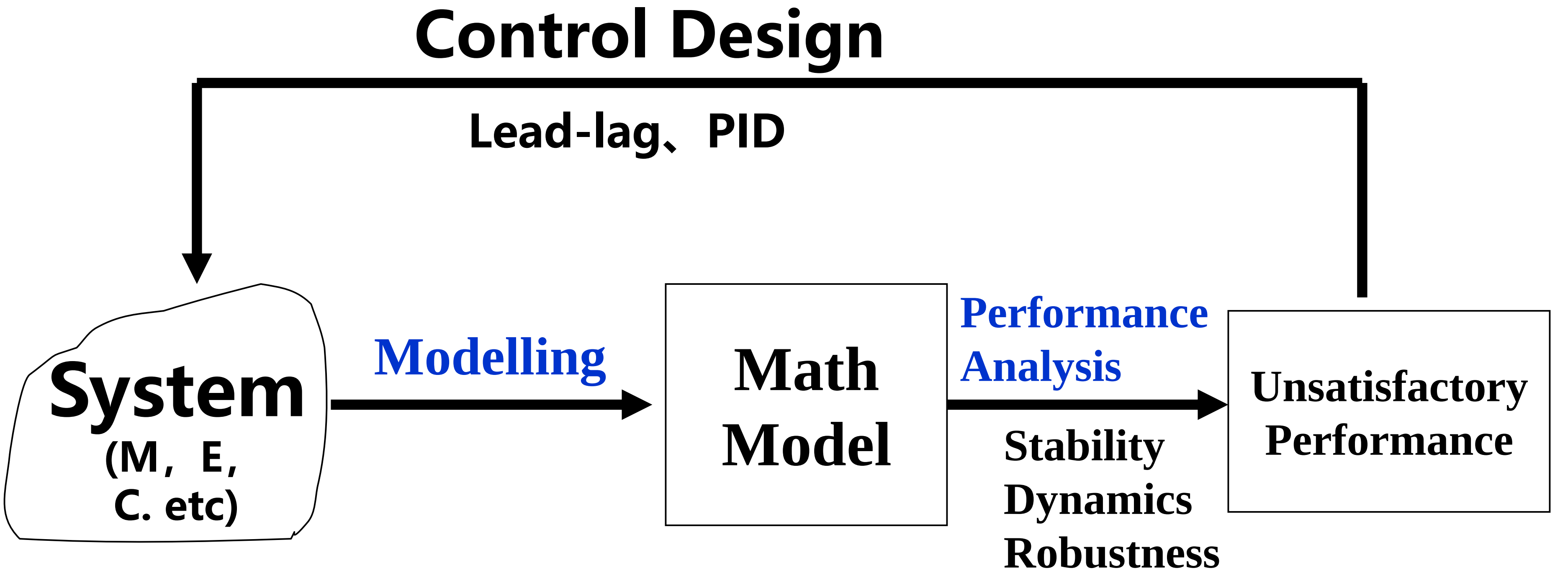
State Space Analysis



State Space Analysis

Introduction

Why?

A state space model describes the relationship between internal states and input-output of a system, hence， analysis and synthesis based on a state space model can reveal the dynamic feature of the system in depth in order to realize optimal control.

What?

State space analysis and synthesis, including state space modelling， controllability and observability as the basis of state feedback, Lyapunov stability， state feedback control, and linear quadratic optimal control design.

State Space Analysis

1 State space model

2 Controllability and observability

3 Lyapunov stability criterion

4 State space equation solution

5 State feedback control

6 Linear quadratic optimal control

1 State space model

1.1 State and state space

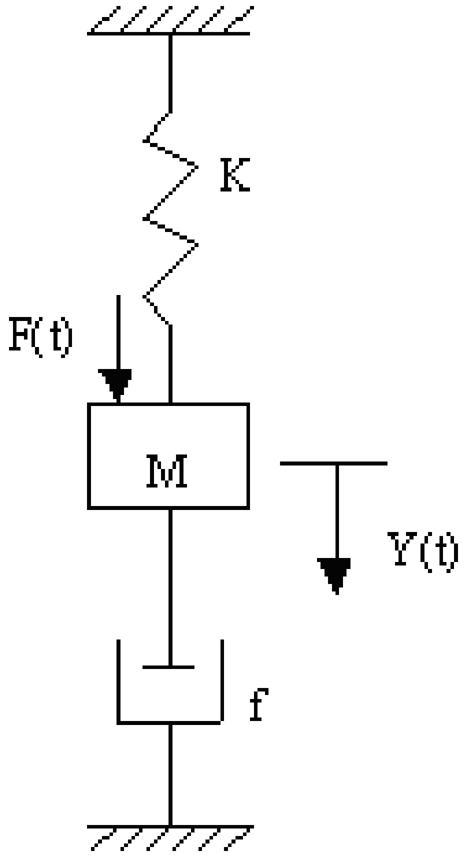
Example：Spring-damper system

Given F(t)，if current position and velocity are known, future response is determined。

If only know position and velocity，future response is undetermined。

Position, velocity and acceleration are independent, any two of them can determine the another. Thus, one of them is redundant.

Position and velocity can be chosen as the state at time t.



弹簧-阻尼器系统

1.1 State and state space

•**State** is a set of internal information, which are sufficient and necessary to determine future behavior of a system when external input is given.

•**State variables** is a minimal set of variables to determine the state of the system at a time instance.

•**State vector** is a vector whose elements are the state variables.

•**State space is spanned by** n state variables.

❑**State trajectory: from $X(t)=X(t_0)$, as t changing, in the state space, $X(t)$ movement is a trajectory.**

1.2 State space description

Relationship between state and input variables can be described by a set of n simultaneous, first-order differential equation with n variables, called state equation.

Output variables can be related to state and input variables by output equation。

State and output equations form the state space model, or dynamic equations。

State space model describes the relationship between the internal state and external input and output, hence called internal description model. It reveals system's dynamics better than input-output models.

1.2 State space description

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

for $t \geq t_0$ and initial conditions, $\mathbf{x}(t_0)$, where

$\mathbf{x}$ = state vector

$\dot{\mathbf{x}}$ = derivative of the state vector with respect to time

$\mathbf{y}$ = output vector

$\mathbf{u}$ = input or control vector

$\mathbf{A}$ = system matrix

$\mathbf{B}$ = input matrix

$\mathbf{C}$ = output matrix

$\mathbf{D}$ = feedforward matrix

1.2 State space description

Select state variable has to satisfy its definition, firstly to check independence, none can be derived by others; then to check whether sufficiently determine system’s state.

Number of state variables should equal to number of energy storage units.

Three approaches to select state variables (not limited to):

（1）Select output of storage units as state variables;

（2）Select output and their derivatives as state variables (as the number of independent energy storage units)

（3）Select variables to convert state equation to a certain canonic form.

Example: state space model for a mass-spring-damper system

Select state variable $x_1 = y(t),\ x_2 = \dot{y}(t)$

Newton's law

$$M\frac{d^2y}{dt^2} = F - F_k - F_f$$

$$F_k(t) = Ky(t)$$

$$F_f(t) = f\frac{dy(t)}{dt}$$

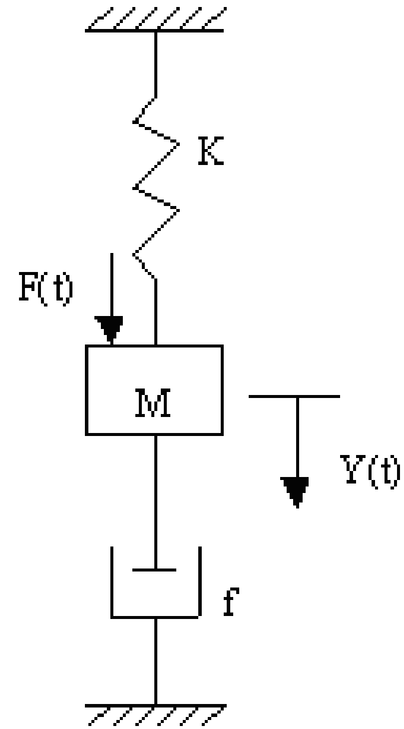
State equation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{K}{M}x_1 - \dfrac{f}{M}x_2 + \dfrac{1}{M}F \end{cases}$$

Output equation $y = x_1$

State space representation

$$\begin{cases} \dot{x} = Ax + BF \\ y = Cx \end{cases}$$



弹簧-阻尼器系统

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ -\dfrac{K}{M} & -\dfrac{f}{M} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

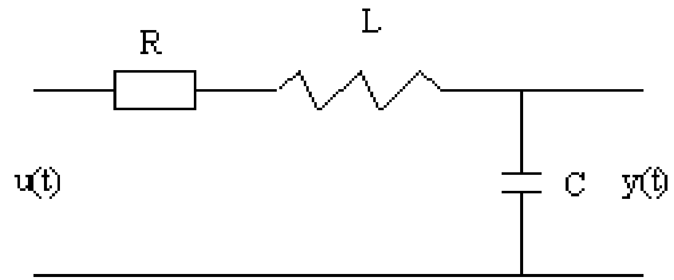
Example 1 state space model of a RLC circuit

Select state variable

$$x_1 = q, \; x_2 = i$$

Electrical circuit principles

$$\begin{cases} \dfrac{dq}{dt} = i \\ iR + L\dfrac{di}{dt} + \dfrac{1}{C}q = u \end{cases}$$



RLC网络

q:electric charge

i: current

L: inductance

C:capacitor

$$\begin{cases} \dfrac{dq}{dt} = i \\ \dfrac{di}{dt} = -\dfrac{1}{LC}q - \dfrac{R}{L}i + \dfrac{1}{L}u \end{cases}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{1}{LC}x_1 - \dfrac{R}{L}x_2 + \dfrac{1}{L}u \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{LC} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} u$$

$$y = \frac{q}{C} = \frac{x_1}{C} = \begin{bmatrix} \dfrac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Summary from examples：

（1） selection of state variable is nonunique, hence, state equation is nonunique（but unique in the similarity sense）；

（2）number of state variables is determined;

（3）state variable can have physical meaning or not. State variable can be measurable or unmeasurable.

Different physical systems can have a similar form of mathematical model.

1.3 Linear transformation of state equations

Although both state variables and equations are nonunique, different state equations can be obtained through linear transformation. Thus, state equations are unique in the similarity sense.

Through linear transformation, a general model can be converted into a canonic form to simplify system analysis and design.

Linear transformation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

$$x(t) = P\bar{x}(t)$$

$$\bar{x}(t) = P^{-1}x(t)$$

$$\begin{cases} P\dot{\bar{x}} = AP\bar{x} + Bu \\ y = CP\bar{x} \end{cases}$$

$$\begin{cases} \dot{\bar{x}} = P^{-1}AP\bar{x} + P^{-1}Bu \\ y = CP\bar{x} \end{cases}$$

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t)$$

$$y = \bar{C}\bar{x}$$

$$\bar{A} = P^{-1}AP \qquad \bar{B} = P^{-1}B \qquad \bar{C} = CP$$

$$A = P\bar{A}P^{-1} \qquad B = P\bar{B} \qquad C = \bar{C}P^{-1}$$

$$|\lambda I - \bar{A}| = |\lambda I - P^{-1}AP| = |\lambda P^{-1}P - P^{-1}AP|$$

$$= |P^{-1}\lambda IP - P^{-1}AP| = |P^{-1}(\lambda I - A)P|$$

$$= |P^{-1}||\lambda I - A||P| = |P^{-1}||P||\lambda I - A|$$

$$= |P^{-1}P||\lambda I - A| = |\lambda I - A|$$

A nonsingular transformation changes state variables and equation parameters, but eigenvalues of the system are unchangeable.

Example 2 given state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Linear transformation (composed by eigenvectors)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}$$

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} -1 & -2 & -3 \\ 1 & 4 & 9 \\ -1 & -8 & -27 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\bar{B} = P^{-1}B = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1 \\ 0.5 \end{bmatrix}$$

converted state equation

$$\dot{\bar{x}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \bar{x} + \begin{bmatrix} 0.5 \\ -1 \\ 0.5 \end{bmatrix} u$$

New state matrix A is diagonal, no coupling between state variables. Such a form is useful for control analysis and design.

1.4 Converting a transfer function to state space

Consider the differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u$$

Choose the state variables x_i , differentiating both sides yields:

$$\begin{aligned} x_1 &= y \\ x_2 &= \frac{dy}{dt} \\ x_3 &= \frac{d^2 y}{dt^2} \\ &\vdots \\ x_n &= \frac{d^{n-1} y}{dt^{n-1}} \end{aligned}$$

$$\begin{aligned} \dot{x}_1 &= \frac{dy}{dt} \\ \dot{x}_2 &= \frac{d^2 y}{dt^2} \\ \dot{x}_3 &= \frac{d^3 y}{dt^3} \\ &\vdots \\ \dot{x}_n &= \frac{d^n y}{dt^n} \end{aligned}$$

1.4 Converting a transfer function to state space

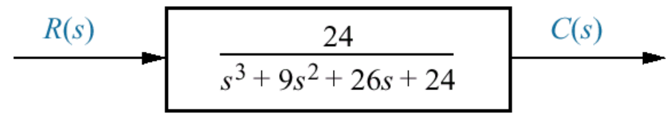
Substituting the definitions into differential equation

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -a_0 x_1 - a_1 x_2 \cdots - a_{n-1} x_n + b_0 u
\end{aligned}
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}
$$

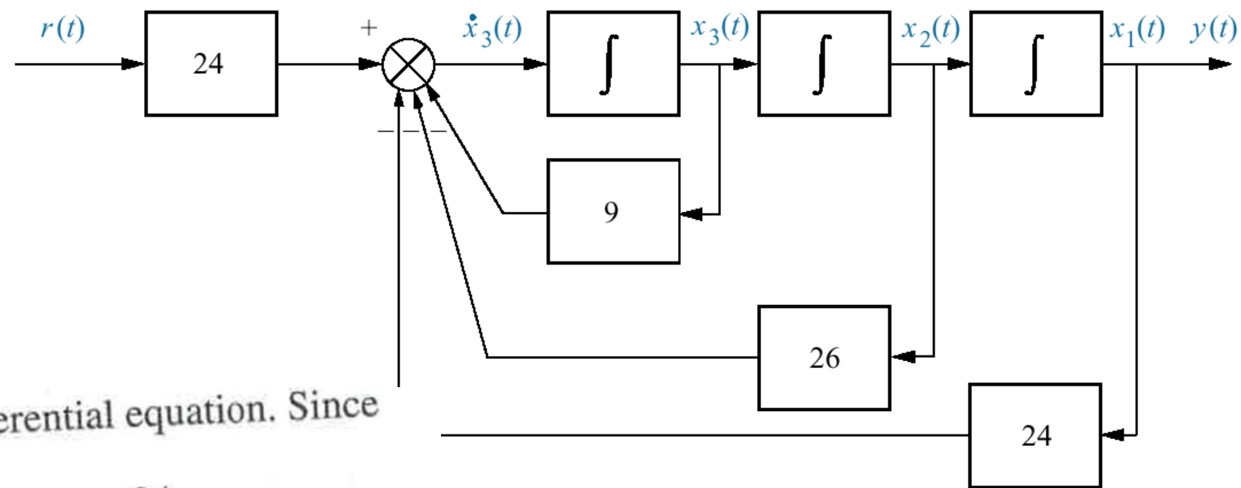
$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u \qquad (3.52)
$$

A transfer function with constant term in numerator



(a)

Figure 3.10
a. Transfer function;
b. equivalent block diagram showing phase-variables.
Note: *y(t)* = *c(t)*



(b)

Step 1 Find the associated differential equation. Since

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$

cross-multiplying yields

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

Assume zero initial conditions:

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

Step 2 Select the state variables.

Choosing the state variables as successive derivatives, we get

$$x_1 = c$$

$$x_2 = \dot{c}$$

$$x_3 = \ddot{c}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$

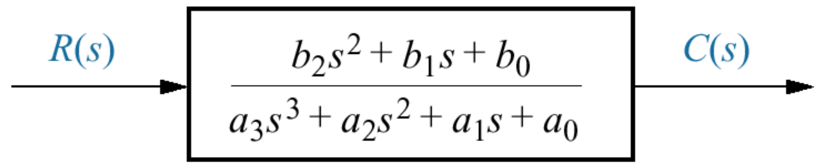
In vector-matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

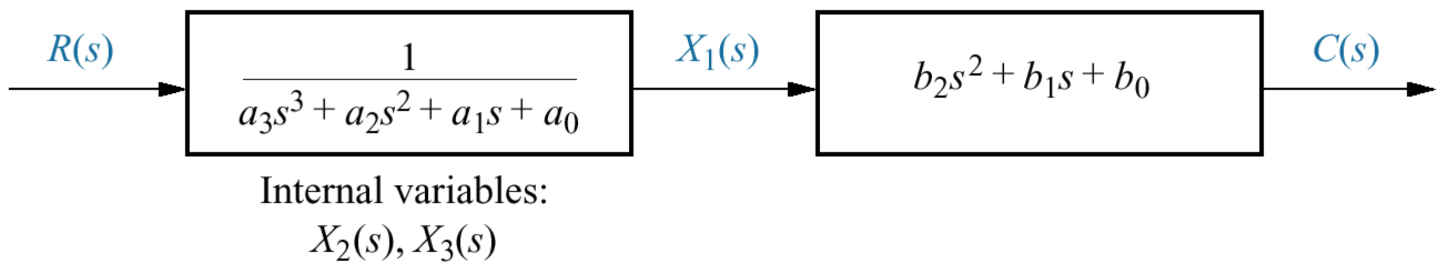
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A transfer function with polynomial in numerator

Figure 3.11
Decomposing a transfer function



(*a*)



(*b*)

. The second transfer function with just the numerator yields

$$Y(s) = C(s) = (b_2s^2 + b_1s + b_0)\,X_1(s)$$

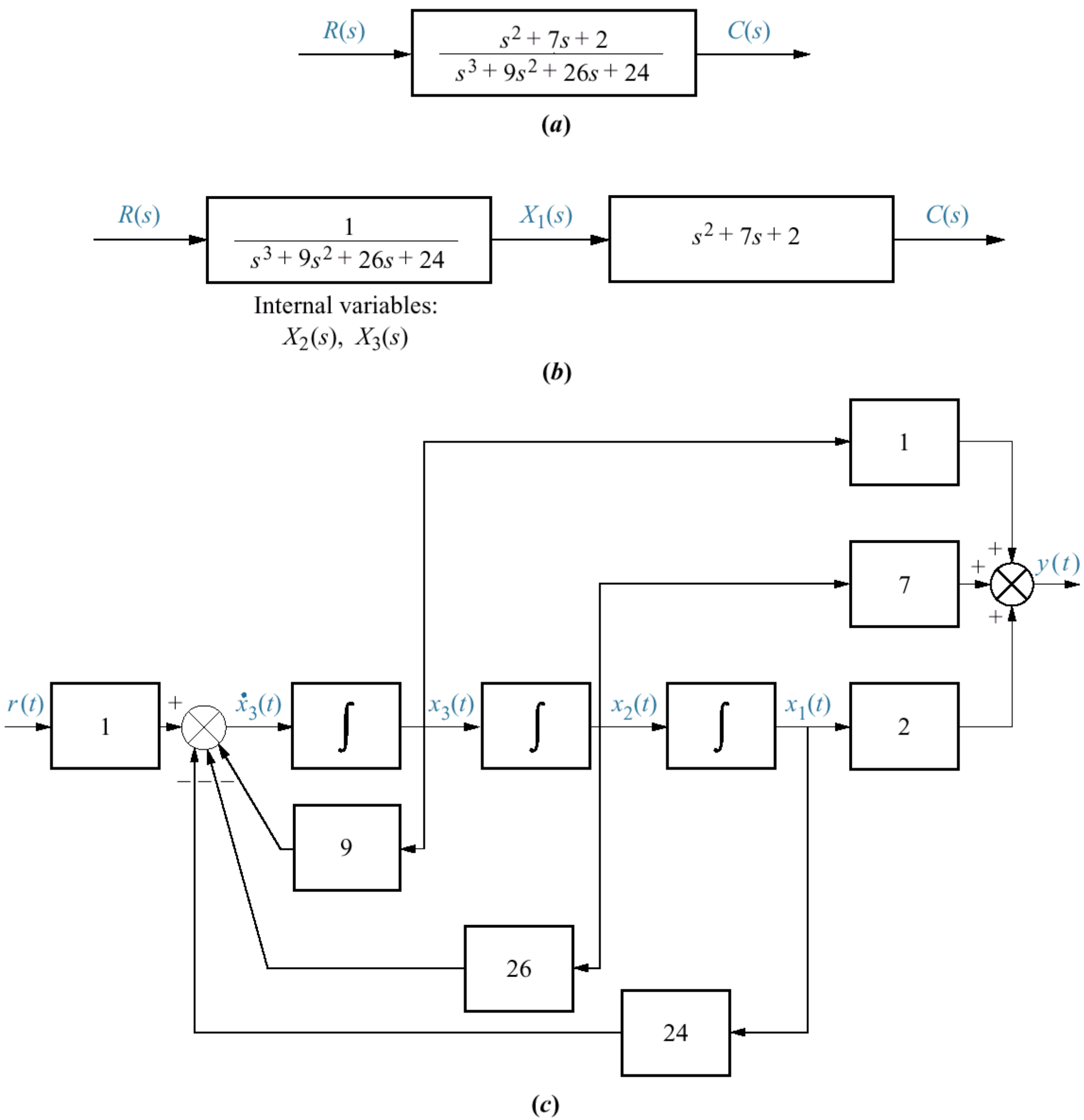
after taking the inverse Laplace transform with zero initial conditions

$$y(t) = b_2\frac{d^2x_1}{dt^2} + b_1\frac{dx_1}{dt} + b_0x_1 \qquad ($$

$$y(t) = b_0x_1 + b_1x_2 + b_2x_3$$

Example

Figure 3.12
a. Transfer function;
b. decomposed transfer function;
c. equivalent block diagram. Note: *y(t) = c(t)*



Step 1: separate the system into two cascaded block

Step 2: Find the state equation for the block containing the denominator

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

Step 3: Introduce the effect of the block with the numerator

$$C(s) = (b_2 s^2 + b_1 s + b_0)X_1(s) = (s^2 + 7s + 2)X_1(s)$$
$$c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1$$

but

$$x_1 = x_1$$
$$\dot{x}_1 = x_2$$
$$\ddot{x}_1 = x_3$$

$$y = c(t) = b_2 x_3 + b_1 x_2 + b_0 x_1 = x_3 + 7x_2 + 2x_1$$

$$y = [b_0 \quad b_1 \quad b_2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [2 \quad 7 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State Space Analysis

1 State space model

2 Controllability and observability

3 Lyapunov stability criterion

4 State equation solution

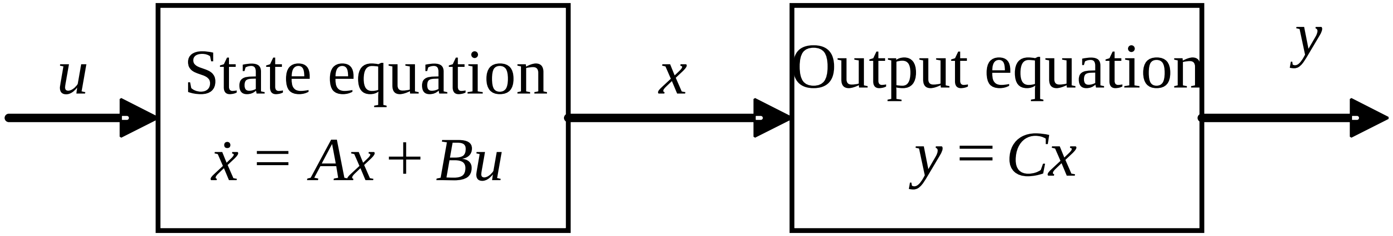
5 State feedback control

6 Linear quadratic optimal control

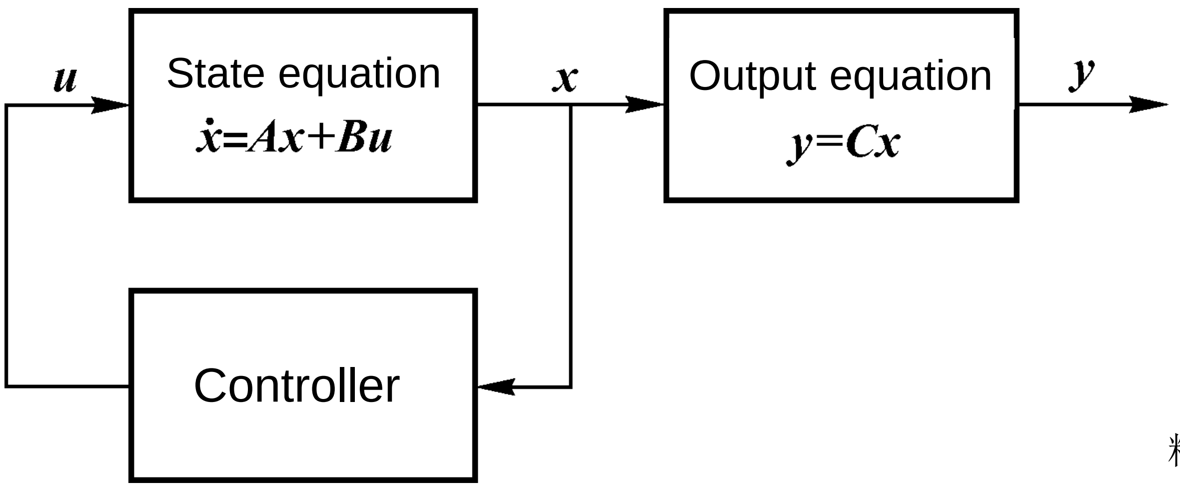
2.1 Controllability and observability

Controllability and observability was proposed by Kalman in 1960.

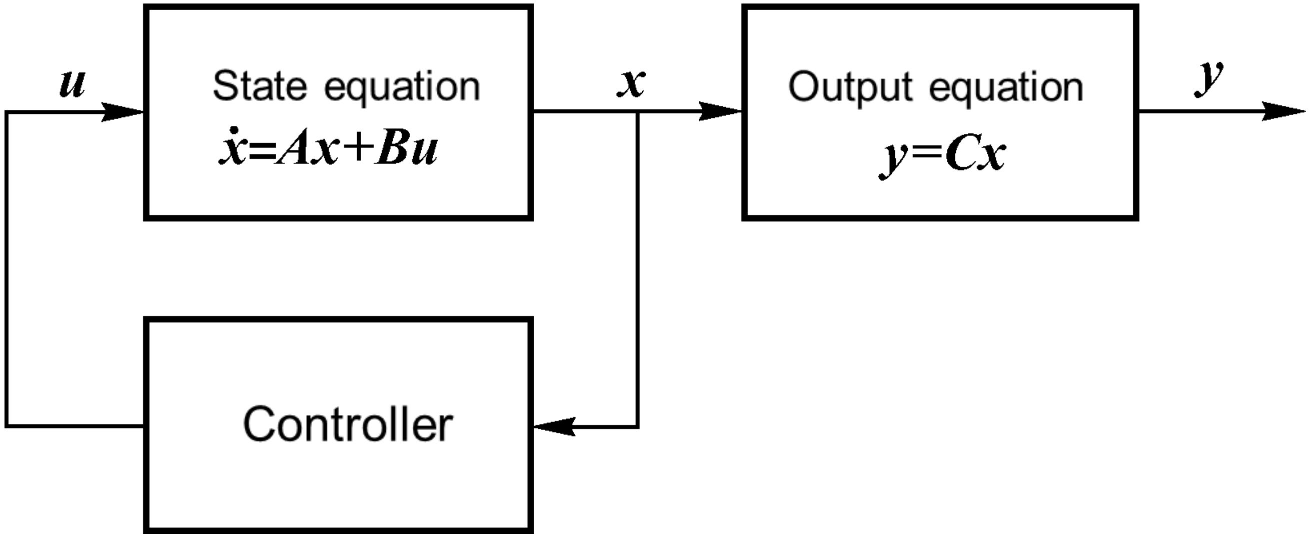
State space description shown in Figure.



State feedback can achieve optimal control，as shown in Figure



2.1 Controllability and observability

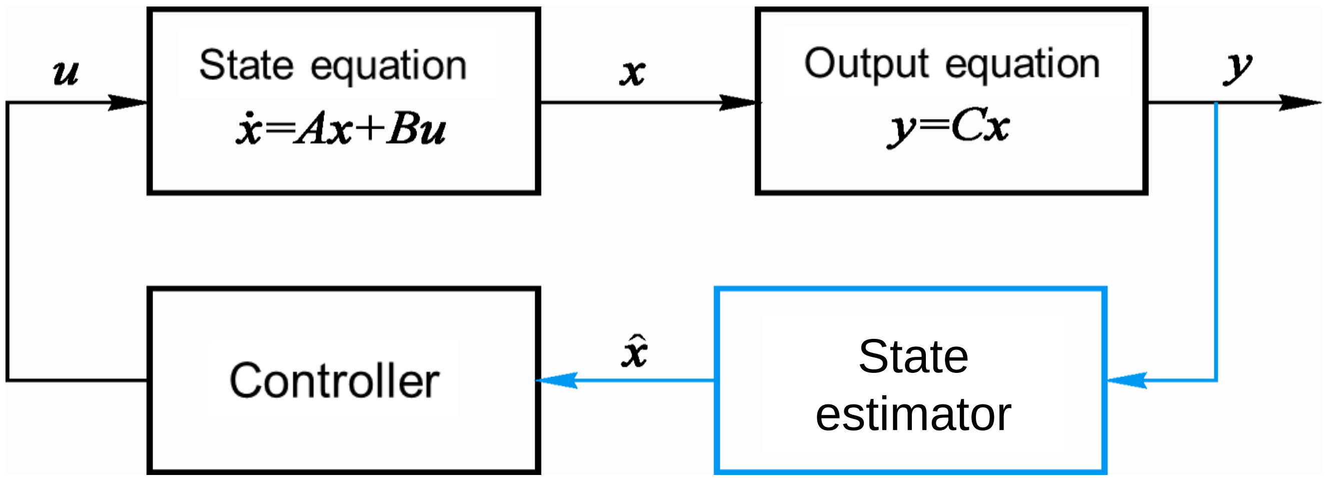


•Control system design seeks control actions so that the system can reach desired state.

•Primary question: Can the state be controlled?

—— state controllability

2.1 Controllability and observability

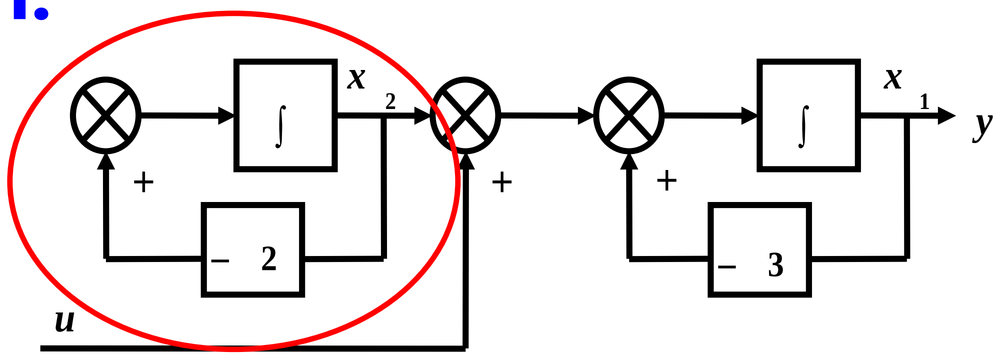


- **In order to realize state feedback, state has to be measured, but may not be measurable, hence, need to be estimated from output.**
- **State estimation is to design state estimator to estimate state from output to realize state feedback.**
- **Primary question： can state be estimated from output? —— state observability.**

2.1 Controllability and observability

Controllability: **ability of *u*(t) to inference *x*(t) and *y*(t), answers whether *u*(t) can move *x*(t) arbitrarily**

Consider system:



Obviously， *u* can only control x_1, but not x_2. We say x_1 is controllable, x_2 is not controllable.

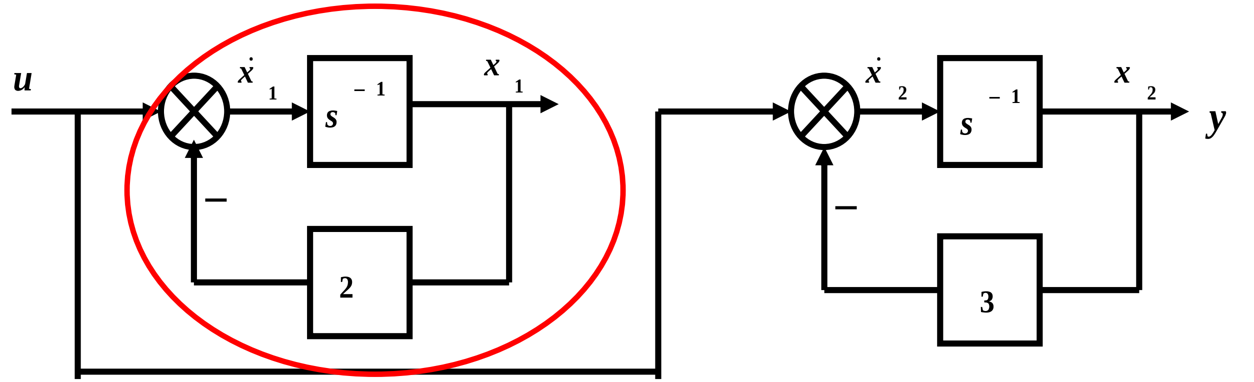
If all states are controllable， then the system is state controllable. If there is any state uncontrollable, then the system is state uncontrollable.

2.1 Controllability and observability

Observability: **ability using output $y(t)$ to identify $x(t)$, it answers whether states can be inferred from outputs.**

state not inferable by y(t) is not observable.

Consider system:

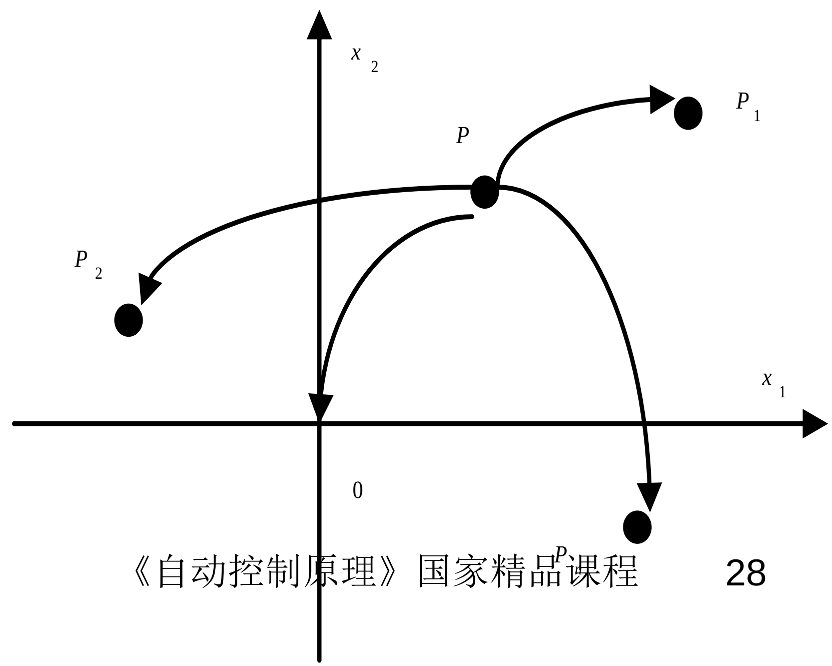


x_2 is observable, x_1 is not.

2.2 Controllability of LTI systems

1. Controllability definition

If all states are controllable, the system is called complete state controllable, in short, controllable.



(2)Controllability of discrete systems

Definition within a limited sample period, [0， n]， if there is a unconstrained control sequence $u(0),\cdots,u(n-1)$ able to transfer any initial state $x(0)$ to any final state $x(n)$, then the system is called complete state observable, in short, observable.

In above definition, both initial and final states can be any non-zero finite point. Although it is generic, but not convenient for mathematical development.

Without losing generality, either the final or initial state can be the origin of the state space. This is normally called reachability.

For LTI (continuous or discrete) systems, controllability and reachability are equivalent.

2. Controllability criterion

Controllability criterion: the sufficient and necessary condition for a LTI (continuous or discrete) system to be completely controllable is that the controllability matrix consisting of A and B is full rank, *i.e.*

$$rank S_c = rank \begin{bmatrix} B & AB & A^2 B & \cdots & A^{n-1} B \end{bmatrix} = n$$

Example 3 determine the controllability of the following system

$$\dot{x} = \begin{bmatrix} -1 & 2 & 2 \\ 0 & -2 & 0 \\ 1 & 3 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Answer:

$$S_c = \begin{bmatrix} B & AB & A^2 B \end{bmatrix} = \begin{bmatrix} 0 & 2 & -8 \\ 0 & 0 & 0 \\ 1 & -3 & 11 \end{bmatrix} \qquad rank S_c = 2 < n = 3$$

So, the system is not controllable.

Example 4 determine the controllability of the following system.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Answer:

$$S_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix}$$

$$rankS_c = 3 = n$$

So, the system is controllable.

Example 5 determine the controllability of the following system.

$$\dot{x} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} x + \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} u$$

Answer:

$$S_c = \begin{bmatrix} 2 & 1 & 3 & 2 & 5 & 4 \\ 1 & 1 & 2 & 2 & 4 & 4 \\ -1 & -1 & -2 & -2 & -4 & -4 \end{bmatrix}$$

2nd and 3rd rows are proportional， $rankS_c = 2 < 3$

So, the system is not controllable.

Example 6 Determine the controllability of the following system.

$$x(k+1)=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & -2\\ -1 & 1 & 0\end{bmatrix}x(k)+\begin{bmatrix}1\\0\\1\end{bmatrix}u(k)$$

Answer: because

$$rankS_c=\begin{bmatrix}B & AB & A^2B\end{bmatrix}=rank\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & -2\\ 1 & -1 & -3\end{bmatrix}$$

$$=rank\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & -2\\ 0 & -2 & -4\end{bmatrix}=rank\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & -2\\ 0 & 0 & -2\end{bmatrix}=3=n$$

So, the system is controllable.

Example 7 determine the controllability of the following system.

$$x(k+1)=\begin{bmatrix} -2 & 2 & -1 \\ 0 & -2 & 0 \\ 1 & -4 & 0 \end{bmatrix} x(k)+\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(k)$$

Answer:

$$S_c=\begin{bmatrix} 0 & 0 & -1 & 2 & 2 & -4 \\ 0 & 1 & 0 & -2 & 0 & 4 \\ 1 & 0 & 0 & -4 & -1 & 10 \end{bmatrix}$$

Since the determinant of the first 3 columns of S_c is not zero,

$$rankS_c = 3$$

Thus, the system is fully controllable.

3. 2nd controllability criterion

Theorem： controllability of a LTI system (continuous or discrete) is invariant under any non-singular linear transformation.

Proof： for a LTI continuous system, $\boldsymbol{S}$: $x = A\dot{x} + Bu$

Apply a non-singular transform, $x = P\bar{x}$, $\bar{\boldsymbol{S}}$: $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$

For $\boldsymbol{S}$:

$$rankS_c = rank\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

$$rankS_c = rank\begin{bmatrix} P\bar{B} & P\bar{A}P^{-1}P\bar{B} & P\bar{A}^2P^{-1}P\bar{B} & \cdots & P\bar{A}^{n-1}P^{-1}P\bar{B} \end{bmatrix}$$

$$= rank\begin{bmatrix} P\bar{B} & P\bar{A}\bar{B} & P\bar{A}^2\bar{B} & \cdots & P\bar{A}^{n-1}\bar{B} \end{bmatrix}$$

$$= rank\left\{P\begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \bar{A}^2\bar{B} & \cdots & \bar{A}^{n-1}\bar{B} \end{bmatrix}\right\}$$

P is invertible, therefore

$$rankS_c = rank\begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \bar{A}^2\bar{B} & \cdots & \bar{A}^{n-1}\bar{B} \end{bmatrix} = rank\bar{S}_c$$

Similar proof for discrete systems applies.

2nd controllability criterion——individual eigenvalue

A LTI system $\dot{x} = Ax + Bu$

with individual eigenvalues is controllable if and only if its diagonal standard form after a non-singular transform:

$$\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & \cdots & \cdots & 0 \\ \vdots & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix} \bar{x} + \bar{B} u$$

$\bar{B}$ does not have any row whose elements are all zero.

This criterion applies to discrete systems as well.

It is simple to determine controllability of diagonal systems.

E.g. for the following 4 systems

1) $\dot{x} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} u$ **controllable**

2) $\dot{x} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 5 \\ 7 \end{bmatrix} u$ **uncontrollable**

3) $\dot{x} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 4 & 0 \\ 7 & 5 \end{bmatrix} u$ **controllable**

4) $\dot{x} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 4 & 0 \\ 7 & 5 \end{bmatrix} u$ **uncontrollable**

2^{nd} controllability criterion——repeated eigenvalues

A LTI system with repeated eigenvalues and every repeated eigenvalue corresponding to an eigenvector is controllable if and only if in its Jorden normal form after a non-singular transform

$$\dot{\bar{x}} = \begin{bmatrix} J_1 & & \cdots & 0 \\ & J_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & J_k \end{bmatrix} \bar{x} + \bar{B} u$$

$$J_i \quad (i = 1, 2, \cdots, k)$$

elements of $\bar{B}$ corresponding to the last row of each Jorden block, J_i, $i = 1, 2, \ldots, k$ are not all zeros.

This criterion applies to both continuous and discrete systems.

E.g. the following 4 systems

(1) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-4 & 1\\ 0 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 2\end{bmatrix}u$$

controllable

(2) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-4 & 1\\ 0 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}2\\ 0\end{bmatrix}u$$

uncontrollable

(3) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}-4 & 1 & 0 & 0\\ 0 & -4 & 0 & 0\\ 0 & 0 & -3 & 1\\ 0 & 0 & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1\\ 0 & 0\\ 2 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

controllable

(4) $$\dot{x}=\begin{bmatrix}-4 & 1 & 0 & 0\\ 0 & -4 & 0 & 0\\ 0 & 0 & -3 & 1\\ 0 & 0 & 0 & -3\end{bmatrix}x+\begin{bmatrix}0 & 1\\ 0 & 0\\ 2 & 0\\ 0 & 2\end{bmatrix}u$$

uncontrollable

2.3 LTI system's observability

1. Observability definition

Definition: Consider a LTI system, if for a given input, $u(t)$, there is a limited observing time, $t_1 > t_0$, such that the initial state $x(t_0)$ can be uniquely determined by the output, $y(t)$, within the period $t \in [t_0, t_1]$ then the state $x(t_0)$ is observable.

If every state is observable, then the system is complete state observable, in short observable.

Observability of discrete systems is defined similarly.

2. Observability criterion

A LTI (continuous, discrete) system $\{A, B, C\}$is observable if and only if the observability matrix S_o has a full rank, i.e.

$$\operatorname{rank} S_o = \operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

Example 8 Determine observability of the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Answer:

$$rankS_o = rank \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 2 & -1 \\ -2 & 1 \end{bmatrix} = rank \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} = 2$$

Thus，the system is observable.

Example 9 Determine the observability of the following system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \qquad y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

Answer

$$rankS_o = rank\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 1 < n = 2$$

Hence，the system is unobservable.

Example 10 Determine observability of the given system

$$x(k+1)=\begin{bmatrix}2 & 0 & 3\\ -1 & -2 & 0\\ 0 & 1 & 2\end{bmatrix}x(k)$$

$$y(k)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}x(k)$$

Answer: according to the criterion

$$rankS_o = rank\begin{bmatrix}C\\ CA\\ CA^2\end{bmatrix} = rank\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & 3\\ -1 & -2 & 0\\ 4 & 3 & 12\\ 0 & 4 & -3\end{bmatrix} = 3 = n$$

Thus, the system is observable。

Example 11 given system as follows

$$x(k+1)=\begin{bmatrix}1 & 0 & -1\\ 0 & -2 & 1\\ 3 & 0 & 2\end{bmatrix}x(k)+\begin{bmatrix}2\\ -1\\ 1\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}x(k)$$

determine its observability.

Answer:

$$rankS_o = rank\begin{bmatrix}C\\ CA\\ CA^2\end{bmatrix} = rank\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 3 & 0 & 2\\ 1 & 0 & -1\\ 9 & 0 & 1\\ -2 & 0 & -3\end{bmatrix} = 2 < 3 = n$$

Therefore, the system is not observable.

3. 2nd observability criterion

Theorem： observability of a LTI system (continuous or discrete) is invariant under any non-singular linear transformation.

2nd observability criterion： for a LTI system, $\dot{x} = Ax + Bu, y = Cx$ with repeated eigenvalues， and every repeated eigenvalue corresponding to only one eigenvector, the system is observable if and only if the Jorden normal form of the system

$$\begin{aligned} \dot{\bar{x}} &= \begin{bmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_k \end{bmatrix} \bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

has non-zero elements of $\bar{C}$ corresponding to the 1st columns of all Jorden blocks, $J_i, i = 1, \ldots, k$.

$$\dot{x} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} x \qquad y = \begin{bmatrix} 6 & 4 & 5 \end{bmatrix} x$$

observable

$$\dot{x} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} x \qquad y = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix} x$$

unobservable

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} x \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

observable

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} x \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

unobservable

$$\dot{x} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \qquad y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x$$

observable

$$\dot{x} = \begin{bmatrix} -3 & 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} x \qquad y = \begin{bmatrix} 8 & -1 & 0 & 2 & 4 \end{bmatrix} x$$

unobservable

2.4 Controllable and observable canonic forms

Selecting different state variables results different state equations. Some of them are canonic forms.

State space canonic forms are convenient for system analysis.

Diagonal forms are convenient for state transition calculation, controllability and observability analysis.

Observable form is convenient for observer design.

This section explains how to transform state space models into controllable and observable canonic forms.

For discussion， a single input single output continuous or discrete LTI system is denoted as $\{A, b, c\}$, after a linear transformation, it is denoted as $\{\bar{A}, \bar{b}, \bar{c}\}$

1. Controllable canonic form

Definition: following state equation is called controllable or Brunovsky canonic form

$$\bar{A} = \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Theorem： System having controllable canonic form is controllable. A controllable state space model can be converted into the controllable canonic form through the linear transformation, $\bar{x} = Tx$, where

$$T = \begin{bmatrix} T_1 \\ T_1 A \\ \vdots \\ T_1 A^{n-1} \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} S_c^{-1}, \quad S_c = \begin{bmatrix} b & Ab & \cdots & A^{n-1} b \end{bmatrix}$$

Example 12 Convert the system to controllable canonic form.

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \qquad S_c = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}, \quad rankS_c = 2$$

Answer: check $S_c = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}$, rank$S_c = 2$, controllable

Characteristic equation, $\det(sI - A) = s^2 - 3s + 2$

1st controllable canonic form $\bar{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Because

$$S_c^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} S_c^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Therefore

$$T = \begin{bmatrix} T_1 \\ T_1 A \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Example 13 Convert system to controllable canonic form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad 1]\, x$$

$$P = Q_c = \begin{bmatrix} 2 & 4 & 16 \\ 1 & 6 & 8 \\ 1 & 2 & 12 \end{bmatrix}, \quad P^{-1} = Q_c^{-1} = \begin{bmatrix} 1.75 & -0.5 & -2 \\ -0.125 & 0.25 & 0 \\ -0.125 & 0 & 0.25 \end{bmatrix}$$

Check

$$\bar{A} = P^{-1} A P = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 9 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{b} = P^{-1} b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c} = cP = \begin{bmatrix} 1 & 2 & 12 \end{bmatrix}$$

2. Observable canonic form

Definition： following state equation is called observable canonic form

$$\bar{A} = \begin{bmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \vdots & \vdots \\ \vdots & \ddots & \ddots & -a_{n-2} \\ 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}, \qquad \bar{c} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$$

Theorem： dynamic equation with observable canonic form is observable. A observable dynamic equation can be transformed to observable canonic form with transformation matrix as follows.

$$P = \begin{bmatrix} p_1 & Ap_1 & A^2 p_1 & \cdots & A^{n-1} p_1 \end{bmatrix}$$

$$p_1 = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = S_0^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Example 14 transform dynamic equation to observable form.

$$\dot{x}=\begin{bmatrix}1 & 2 & 0\\ 3 & -1 & 1\\ 0 & 2 & 0\end{bmatrix}x+\begin{bmatrix}2\\ 1\\ 1\end{bmatrix}u, \qquad y=\begin{pmatrix}0 & 0 & 1\end{pmatrix}x$$

Answer: 1） check observability: since

$$S_0=\begin{bmatrix}c\\ cA\\ cA^2\end{bmatrix}=\begin{bmatrix}0 & 0 & 1\\ 0 & 2 & 0\\ 6 & -2 & 2\end{bmatrix}, \qquad rankS_0=3$$

Observable, hence can transform it to observable canonic form

Characteristic equation $\left|sI-A\right|=s^3-9s+2=0$

Canonic form

$$\bar{A}=\begin{bmatrix}0 & 0 & -2\\ 1 & 0 & 9\\ 0 & 1 & 0\end{bmatrix}, \qquad \bar{c}=\begin{pmatrix}0 & 0 & 1\end{pmatrix}$$

2）transformation matrix

$$S_o^{-1}=\begin{bmatrix} -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$p_1=S_0^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}\frac{1}{6}\\0\\0\end{bmatrix},\quad Ap_1=\begin{bmatrix}\frac{1}{6}\\\frac{1}{2}\\0\end{bmatrix},\quad A^2p_1=\begin{bmatrix}\frac{7}{6}\\0\\1\end{bmatrix}$$

$$P=\begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{7}{6} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.5 dual principle

Definition: LTI systems $S_1\{A,B,C\}$ and $S_2\{A^*,B^*,C^*\}$ satisfy $A^*=A^T$, $B^*=B^T$ and $C^*=C^T$, then are called dual systems.

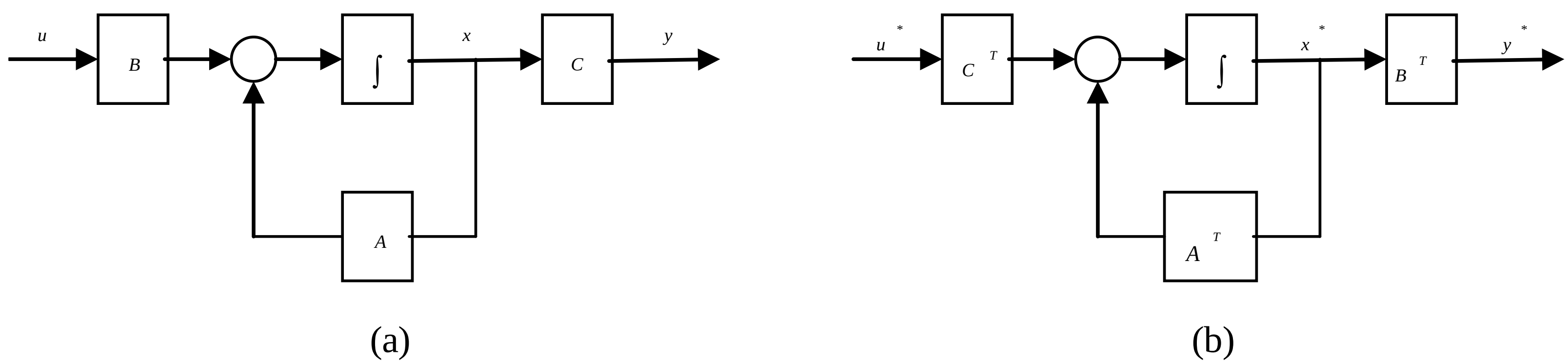


Figure 5.7 dual systems

Dual principle: Controllability of S_1 is equivalent to observability of S_2. Observability of S_1 is equivalent to controllability of S_2.

Theorem Dual systems have the same characteristic equation, their transfer function matrices are mutual transpositions. i.e.

$$\det(sI-A)=\det(sI-A^*) \qquad G(s)=\left[G^*(s)\right]^T$$

State Space Analysis

1 State space model

2 Controllability and observability

3 Lyapunov stability criterion

4 State equation solution

5 State feedback control

6 Linear quadratic optimal control

3 Lyapunov stability criterion

- A system subject to a disturbance will be away from the equilibrium state. Its response is then one of three cases:

(1) Bounded response;

(2) Unbounded response;

(3) bounded also back to the original equilibrium state.

- Lyapunov classified the three cases as stable, unstable and asymptotically stable.
- For unstable system, its response is unbounded, or oscillated. Thus, the primary condition for a system to work is stable.
- Lyapunov used norm as a measure for the state space dimension. The concept of norm will be introduced firstly as a preliminary.

3.1 Lyapunov stability definitions

1. Positive definite function

(1) Scalar case

1) When $x = 0, V(x) = 0$; when $x \neq 0$, $V(x) > 0$, then $V(x)$ is positive definite；

2) If $V(x)$=0 only at origin and a few states, elsewhere, $V(x) > 0$, then, $V(x)$ is semi-positive definite；

3) If $-V(x)$ is positive definite, then, $V(x)$ is negative definite；

4) If $-V(x)$ is semi-positive definite, then, $V(x)$ is semi-negative definite；

5) If $V(x) > 0$ for some states, but $V(x) < 0$ for other states， then $V(x)$ is indefinite.

Easy to check positive definiteness：

(1) Positive definite $V(x)=x_1^2+2x_2^2$

(2) Semi-positive definite $V(x)=(x_1+x_2)^2$

(3) Negative definite $V(x)=-(x_1^2+2x_2^2)$

(4) Semi-negative definite $V(x)=-(x_1+x_2)^2$

(5) Indefinite $V(x)=x_1x_2+x_2^2$

(2) Scalar quadratic function positive definiteness

$$V(x) = x^T P x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$V(x)$ is a scalar quadratic function. P is a real symmetric matrix

Sylvester's criterion: denote principal minors of P

$$\Delta_1 = P_{11} \qquad \Delta_2 = \begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} \qquad \Delta_n = \begin{vmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{vmatrix}$$

V(x) positive definite iff $\Delta_1 > 0 \quad \Delta_2 > 0 \quad \Delta_n > 0$

V(x) negative definite iff $\Delta_i \begin{cases} > 0, & \text{even } i \\ < 0, & \text{odd } i \end{cases}$

2. Norm

- Among many different definitions, Euclid norm is most common, is an extension of length in 2D and 3D space.
- Vector norm in n-D vector space

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

- Matrix norm:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \qquad \|A\| = \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}^2}$$

3. Equilibrium

• A system without input action is in free movement. If the free movement finally reaches to a state unmovable, then such a state is an equilibrium of the system.

• For a continuous system, $\dot{x} = f(x)$, the equilibrium satisfies $f(x_e) = 0$. For a discrete system, $x(k+1) = f(x(k))$, the equilibrium satisfies $x_e = f(x_e, k)$ for all k. x_e

•For non-singular A， a LTI system has only one equilibrium, $x_e = 0$.

•For singular A, a LTI system has an infinity number of equilibriums.

•Nonlinear systems may have multiple equilibriums all determined by equilibrium equations.

Example determine the equilibrium of the following system

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = x_1 + x_2 - x_2^3 \end{cases}$$

Answer from equilibrium definition，it should satisfy

$$x_{1e} = 0 \qquad x_{2e} - x_{2e}^3 = 0$$

$$x_{1e} + x_{2e} - x_{2e}^3 = 0 \qquad x_{2e}(1 + x_{2e})(1 - x_{2e}) = 0$$

Hence，the system has three equilibriums:

$$x_{e1} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \qquad x_{e2} = \begin{bmatrix} 0 & -1 \end{bmatrix}^T \qquad x_{e3} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$$

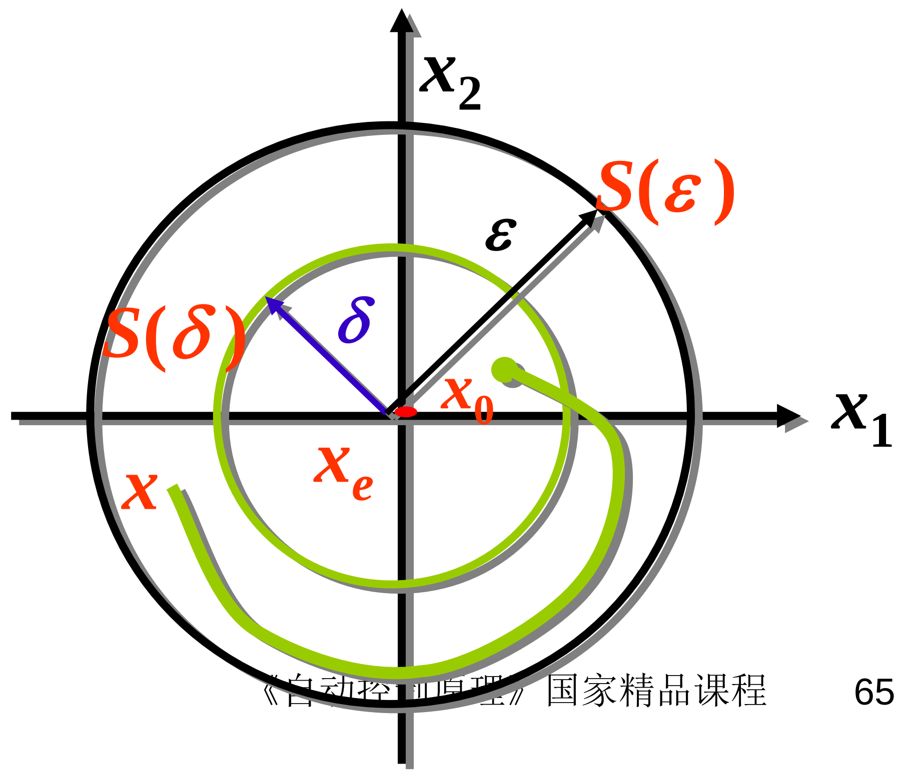
4. Lyapunov stability definition

In 1892, Lyapunov gave a generic stability definition.

(1) Stable: if for a real number, $\varepsilon > 0$, there exists a real number, $\delta(\varepsilon, t_0) > 0$, such that from any initial state satisfying $\|x_0 - x_e\| \le \delta(\varepsilon, t_0)$, the response at anytime satisfies $\|x - x_e\| \le \varepsilon$, then the equilibrium of the system is stable. If selections of δ and t_0 are independent each other, then the equilibrium is uniformly stable.

Lyapunov stability——

system's response is bounded

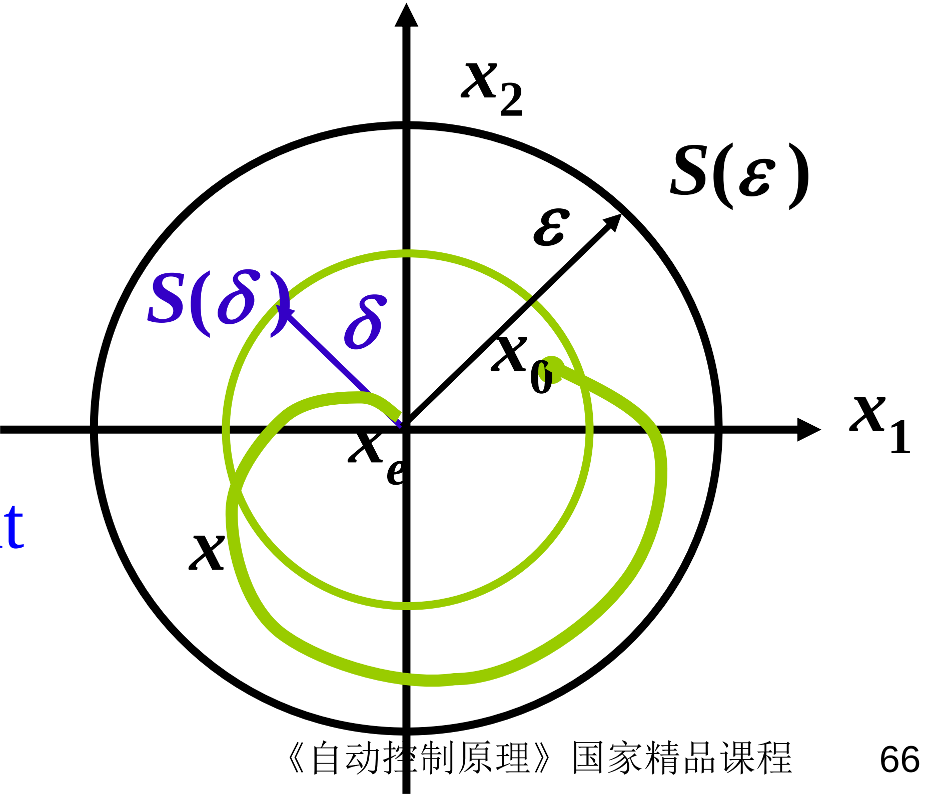


(2) Asymptoticly stable: if the equilibrium is Lyapunov stable, and as $t \to \infty$, $x(t) \to x_e$, i.e. $\lim_{t\to\infty}\|x(t) - x_e\| = 0$, then the equilibrium is asymptoticly stable.

（3）globally (asymptoticly) stable：if for any large δ, the system is always stable, then the system is globally (asymptoticly) stable. If the system is always asymptoticly stable, then it is globally asymptoticly stable.

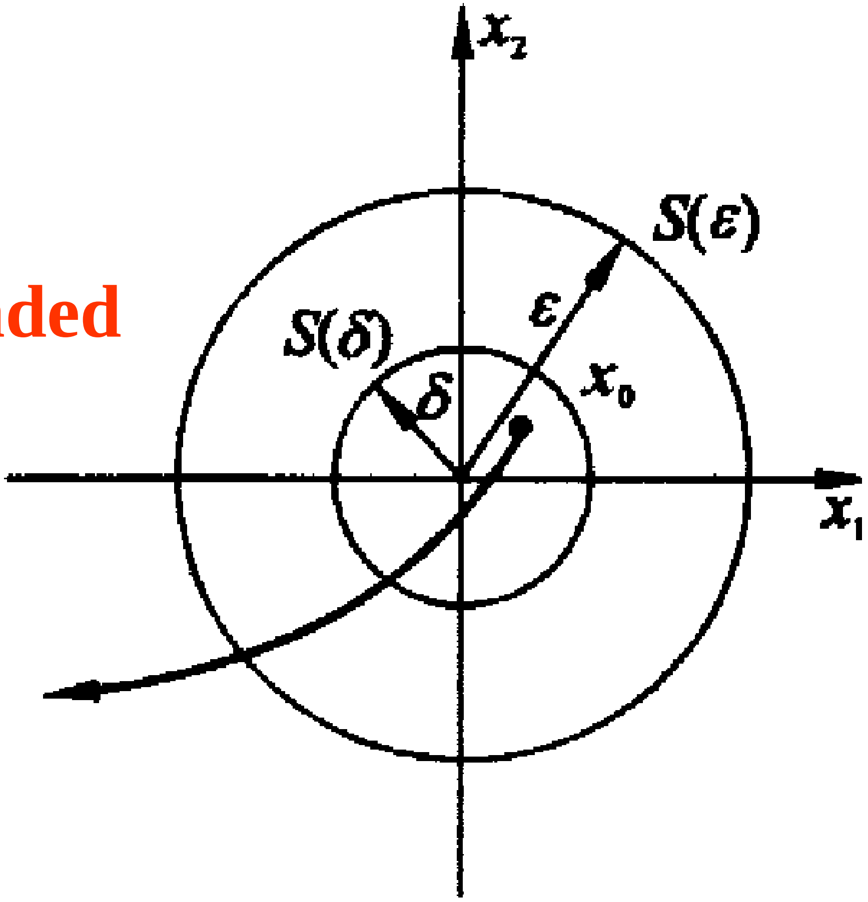
Bounded response converges to equilibrium

Stability in classical control theory is equivalent Lyapunov asymptotic stability



(4) unstable: for a real number $\varepsilon > 0$, choosing whatever small δ, if always there is a trajectory starting from $s(\delta)$ going unbounded, then the system is unstable.

system response unbounded



3.2 Lyapunov stability criterion

Lyapunov stability criterion was proposed in 1892, giving necessary and sufficient conditions for continuous (nonlinear and linear) systems to be asymptotic stable. It was extended to discrete systems in 1958.

Many mechanical systems consume energy and its total energy decreases with time, until the minimum energy state. Thus, energy can be used to measure mechanical system stability. However, kinetic and potential energies do not explicitly exist in general systems.

Lyapunov abstracted the concept of "energy" to construct a positive definite function, similar to "energy", called Lyapunov function. By analyzing whether the positive definite function representing "energy", whether it decreases with time, i.e. to analyze whether the derivative of Lyapunov function is a negative definite function, the system stability can be determined.

3.2 Lyapunov stability criterion

Consider a dynamic (linear/nonlinear) system as $\dot{x} = f(x, t)$ with equilibrium as $x_e = 0$

Continuous system Lyapunov stability criterion: If there exists a scalar function, $V(x)$, which, for all states, $x(t)$, has continuous 1st order partial derivatives, and is positive definite, and if

- $\dot{V}(x)$ is negative definite, then the equilibrium is asymptoticly stable.
- $\dot{V}(x)$ is negative definite, and $\|x\| \to \infty$, $V(x) \to \infty$, then the equilibrium is globally asymptoticly stable.
- $\dot{V}(x)$ is negative semi-definite, then equilibrium is Lyapunov stable.
- $\dot{V}(x)$ is negative semi-definite, and $\dot{V}(x)$ is not always zero, then equilibrium is asymptoticly stable.
- $\dot{V}(x)$ is positive definite, then the equilibrium is unstable.

Example 16 Analyze nonlinear continuous system equilibrium stability

$$\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2) \end{cases}$$

Answer: determine equilibrium

$$\begin{cases} \dot{x}_{e1} = x_{e2} - x_{e1}(x_{e1}^2 + x_{e2}^2) = 0 \\ \dot{x}_{e2} = -x_{e1} - x_{e2}(x_{e1}^2 + x_{e2}^2) = 0 \end{cases}$$

$$\begin{cases} x_{e2} = x_{e1}(x_{e1}^2 + x_{e2}^2) \\ x_{e1} + x_{e2}(x_{e1}^2 + x_{e2}^2) = 0 \end{cases}$$

$$x_{e1} + x_{e1}(x_{e1}^2 + x_{e2}^2)^2 = 0$$

$$x_{e1}[1 + (x_{e1}^2 + x_{e2}^2)^2] = 0$$

$x_{e1} = 0 \quad x_{e2} = 0$ equilibrium is $x_e = [x_{e1} \quad x_{e2}]^T = [0 \quad 0]^T$

Take Lyapunov function as $V(x) = x_1^2 + x_2^2$

$$\dot{V}(x) = \frac{dV(x)}{dt} = \frac{\partial V}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial V}{\partial x_2} \cdot \frac{dx_2}{dt} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$\dot{V}(x) = 2x_1[x_2 - x_1(x_1^2 + x_2^2)] + 2x_2[-x_1 - x_2(x_1^2 + x_2^2)] = -2(x_1^2 + x_2^2)^2$$

$\dot{V}(x)$ negative definite, equilibrium is asymptoticly stable, when $\|x\| \to \infty$, $V(x) \to \infty$, hence is globally asymptotic stable.

Example 17 check linear system stability

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 \end{cases} \qquad \dot{x}_2 \neq 0$$

Answer (0,0) is the only equilibrium. Take Lyapunov function as $V(x) = x_1^2 + x_2^2$

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1x_2 + 2x_2(-x_1 - x_2) = -2x_2^2$$

$\dot{V}(x)$ is negative semi-definite. Equilibrium is Lyapunov stable.
If $x_1 \neq 0$ and $x_2 = 0$, then $\dot{x}_2 \neq 0$, hence,
if x is not always 0, $\dot{V}(x)$ is not always 0.

Therefore, equilibrium is global asymptotic stable.

Lyapunov function is not unique. We can also take

$$V(x) = \frac{1}{2}[(x_1 + x_2)^2 + 2x_1^2 + x_2^2]$$

Example 18 check linear system stability

$$\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = -x_1 + x_2 \end{cases}$$

Answer (0,0) is the only equilibrium. Take Lyapunov function as

$$V(x) = x_1^2 + x_2^2$$

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1(x_1 + x_2) + 2x_2(-x_1 + x_2) = 2(x_1^2 + x_2^2)$$

$\dot{V}(x)$ is positive definite. The system equilibrium is unstable.

Lyapunov stability is most general, suitable for linear and nonlinear, time variant and invariant systems.

Main problem: difficult to fine a Lyapunov function.

Lyapunov stability theory itself does not provide a general method to construct Lyapunov function.

For nonlinear systems, select a positive definite first, then take its derivative. If the derivative is positive or negative definite, then the selected function is Lyapunov function. Otherwise, a reselection is necessary.

For linear systems, quadratic form can be used to construct Lyapunov function.

Methods to construct Lyapunov function and stability criterion for linear systems are discussed as follows.

3.3 Linear continuous system Lyapunov stability

Linear continuous time variant system $\dot{x}(t) = A(t)x(t)$

Positive definite quadratic function can be Lyapunov function:

$$V(x,t) = x^T(t)P(t)x(t)$$

$$\dot{V}(x,t) = \dot{x}^T(t)P(t)x(t) + x^T(t)\dot{P}(t)x(t) + x^T(t)P(t)\dot{x}(t)$$

$$= x^T(t)A^T(t)P(t)x(t) + x^T(t)\dot{P}(t)x(t) + x^T(t)P(t)A(t)x(t)$$

$$= x^T(t)[A^T(t)P(t) + \dot{P}(t) + P(t)A(t)]x(t)$$

Let $A^T(t)P(t) + \dot{P}(t) + P(t)A(t) = -Q(t)$ Lyapunov matrix differential equation

$$\dot{V}(x,t) = -x^T(t)Q(t)x(t)$$

If Q is positive definite, then, $\dot{V}(x)$ is negative definite. Thus, $V(x,t) = x^T(t)P(t)x(t)$ constructed by a positive real symmetric matrix, P, is Lyapunov function.

Linear continuous system Lyapunov stability criterion: Linear continuous (time variant or invariant) system is stable if and only if for a given positive real symmetric matrix, $Q(t)$, there exists a

$$A^T(t)P(t)+\dot{P}(t)+P(t)A(t)=-Q(t)$$

For LTI systems, the equation is Lyapunov matrix algebraic equation.

$$A^T P+PA=-Q$$

Check LTI continuous system stability procedure is: firstly, take a positive real symmetric matrix, Q, then from $A^T P+PA=-Q$, solve P, finally check whether P is positive definite to determine system stability.

Since Q can be freely chosen, and the result is independent from Q, for simplicity, normally take $Q=I$.

Example 19 Analyse linear continuous system stability

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x$$

Answer: insert $Q = I$ into $A^T P + PA = -Q$

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -P_{12} & -P_{22} \\ P_{11} - P_{12} & P_{12} - P_{22} \end{bmatrix} + \begin{bmatrix} -P_{12} & P_{11} - P_{12} \\ -P_{22} & P_{12} - P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2P_{12} & P_{11} - P_{12} - P_{22} \\ P_{11} - P_{12} - P_{22} & 2P_{12} - 2P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Matrix equality gives following equations

$$\begin{cases} -2P_{12} = -1 \\ P_{11} - P_{12} - P_{22} = 0 \\ 2P_{12} - 2P_{22} = -1 \end{cases}$$

leads to $P = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{bmatrix}$

Check positive definiteness: since

$$P_{11} = \frac{3}{2} > 0$$

$$\begin{vmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{vmatrix} = \begin{vmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{vmatrix} = \frac{5}{4} > 0$$

P is positive definite. Therefore, the system is (global) asymptotic stable, Lyapunov function is

$$V(x) = x^T Px = \frac{3}{2} x_1^2 + x_1 x_2 + x_2^2$$

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4 Solving state equations

4.1 Solving continuous LTI state equation

- 1. Solving homogeneous state equation
- Let LTI homogeneous state equation $\dot{x} = Ax$

matrix exponent approach

$$x(t) = (I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots)x(0)$$

From matrix theory

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots$$

$$x(t) = e^{At}x(0)$$

State transfer matrix, the same as A, an $n \times n$ square matrix

State transfer matrix

$$\Phi(t) = e^{At}$$

State transfer matrix properties

$$\Phi(t)|_{t=0} = \Phi(0) = I$$

$$\Phi^{-1}(t) = \Phi(-t)$$

$$\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2) = \Phi(t_2)\Phi(t_1)$$

$$\Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Phi(t_2 - t_0) = \Phi(t_1 - t_0)\Phi(t_2 - t_1)$$

$$[\Phi(t)]^n = \Phi(nt)$$

Calculate state transfer matrix

(1) Solving through its definition

$$e^{At} = I + At + \frac{A^2}{2!}t^2 + \cdots + \frac{A^k}{k!}t^k + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}t^k$$

Suitable for computer algorithm, not an analytic solution

e^{At} dimension

(2) Solving through Laplace transformation

$$e^{At} = L^{-1}[(sI - A)^{-1}]$$

Key point is to solve the inverse of $(SI - A)$ i.e. resolvent matrix, then apply the inverse Laplace transform.

(2) Solving through Laplace transform

Homogeneous equation: $\dot{x} = Ax$, Initial state: $x(0) = x_0$

Laplace transform: $sX(s) - x(0) = AX(s)$

Tidy up: $X(s) = (sI - A)^{-1}x(0)$

Inverse Laplace transform $x(t) = L^{-1}[(sI - A)^{-1}]x(0)$

Since: $x(t) = e^{At}x(0)$

Uniqueness leads to: $e^{At} = L^{-1}[\boxed{(sI - A)^{-1}}]$

resolvent matrix

Hence $e^{At} = I + At + \frac{A^2}{2!}t^2 + \cdots = L^{-1}\left[(sI - A)^{-1}\right]$

$L\left[e^{At}\right] = (sI - A)^{-1}$

Key task：？？

Example using Laplace transform solving the following system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Answer:

1

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

2

$$(sI - A)^{-1} = \frac{adj(sI - A)}{|sI - A|} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

③

$$(sI-A)^{-1}=\begin{bmatrix}\dfrac{2}{s+1}-\dfrac{1}{s+2} & \dfrac{1}{s+1}-\dfrac{1}{s+2}\\ \dfrac{-2}{s+1}+\dfrac{2}{s+2} & \dfrac{-1}{s+1}+\dfrac{2}{s+2}\end{bmatrix}$$

$$\Downarrow$$

$$e^{At}=\Phi(t)=L^{-1}[(sI-A)^{-1}]=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$$

④

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\Phi(t)\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}$$

2. Solving nonhomogeneous state equations

Laplace transform approach

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$(sI - A)X(s) = x(0) + BU(s)$$

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

$$x(t) = L^{-1}\left[(sI - A)^{-1}\right]x(0) + L^{-1}\left[(sI - A)^{-1}BU(s)\right]$$

$$x(t) = \Phi(t)x(0) + L^{-1}\left[(sI - A)^{-1}BU(s)\right]$$

Example 22 using Laplace transform to solve example 21。

$$(sI-A)^{-1}BU(s)=\frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}=\begin{bmatrix} \dfrac{1}{s(s+1)(s+2)} \\ \dfrac{1}{(s+1)(s+2)} \end{bmatrix}$$

$$L^{-1}\left[(sI-A)^{-1}BU(s)\right]=L^{-1}\begin{bmatrix} \dfrac{1}{s(s+1)(s+2)} \\ \dfrac{1}{(s+1)(s+2)} \end{bmatrix}=L^{-1}\begin{bmatrix} \dfrac{1/2}{s}-\dfrac{1}{s+1}+\dfrac{1/2}{s+2} \\ \dfrac{1}{s+1}-\dfrac{1}{s+2} \end{bmatrix}$$

$$=\begin{bmatrix} \dfrac{1}{2}-e^{-t}+\dfrac{1}{2}e^{-2t} \\ e^{-t}-e^{-2t} \end{bmatrix}$$

The same result

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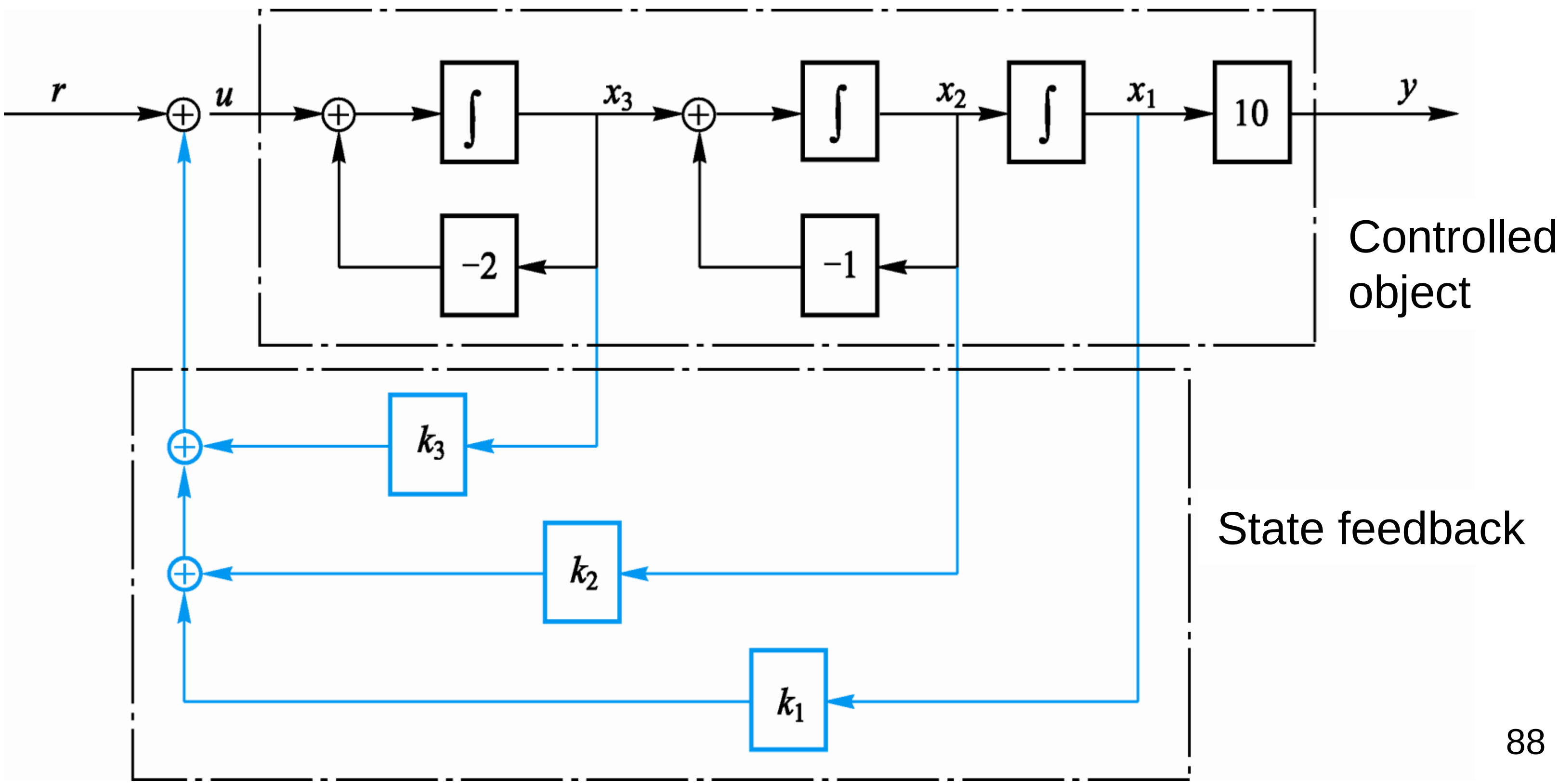
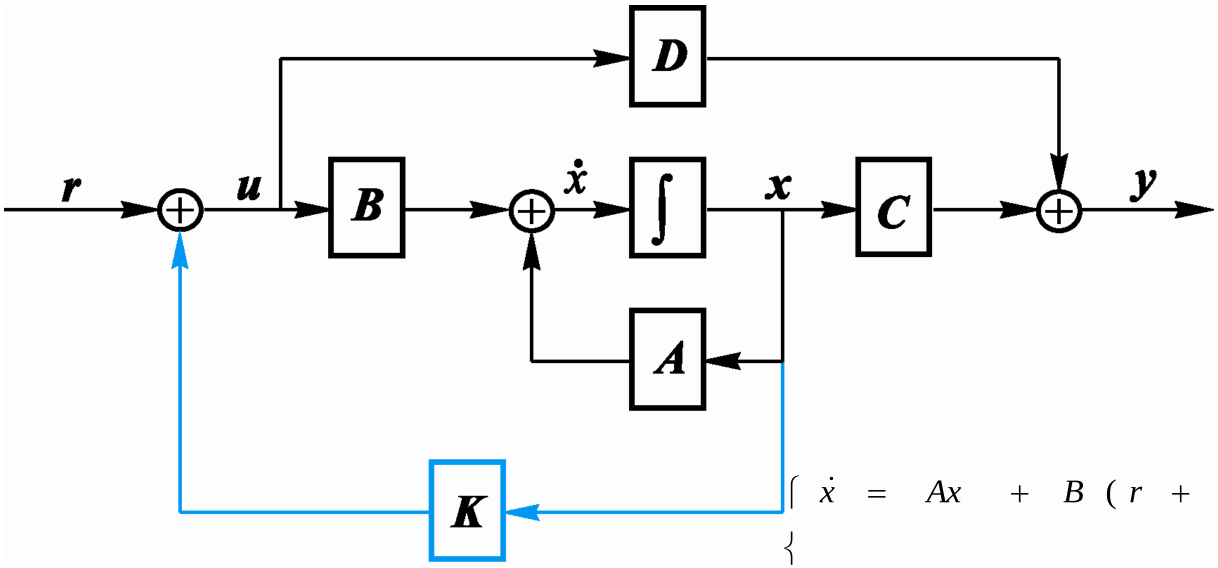
5 state feedback control and observer design

- 5.1 state feedback

•State feedback: each state multiplying corresponding coefficients feedback to input, then adding reference, the result is control signal of the system under control.



•MIMO state feadback system, signal lines are signal vectors.



$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$u = r + Kx$$

$$\begin{cases} \dot{x} = Ax + B(r + Kx) = (A + BK)x + Br \\ y = Cx + D(r + Kx) = (C + DK)x + Dr \end{cases}$$

System without direct-transit (D matrix)

$$\begin{cases} \dot{x} = (A + BK)x + Br \\ y = Cx \end{cases}$$

State feedback closed-loop transfer function matrix

$$G_K(s) = C(sI - A - BK)^{-1}B$$

5.2 State feedback design approach

Pole assignment theorem: Closed-loop poles of a linear (continuous or discrete) multivariable system can be assigned freely if and only if the system is controllable.

If unstable pole is uncontrollable, the system is not stabilizable through state feedback.

If all of uncontrollable poles are stable, state feedback can allocate controllable poles to desired locations so that the closed-loop system is stable. Such a system is called stabilizable.

Theorem: A linear (continuous or discrete) system is stabilizable if and only if its uncontrollable poles are stable.

$\det[\lambda I - (A + bK)]$

For a linear (continuous or discrete) single input system $\{A, b, c\}$, the basic design approach of state feedback matrix to achieve desired pole assignment is to design K such that $\det[\lambda I - (A + bK)] = f^*(\lambda)$, where $f^*(\lambda)$ is desired characteristic polynomial.

Example 6.3 calculate state feedback gain matrix

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Answer: （1）for continuous systems

$$u = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k_1 x_1 + k_2 x_2$$

$$\det[\lambda I - A - BK] = \det \begin{bmatrix} \lambda & -1 \\ -k_1 & \lambda - k_2 \end{bmatrix} = \lambda^2 - k_2 \lambda - k_1$$

Desired characteristic polynomial $f^*(\lambda) = \lambda^2 + a_1\lambda + a_0$

State feedback gain matrix $K = [k_1 \quad k_2] = [-a_0 \quad -a_1]$

（2） for discrete system

discretization

$$\begin{bmatrix} x_1[(k+1)T] \\ x_2[(k+1)T] \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u(kT)$$

State feedback

$$u(kT) = [k_1 \quad k_2] \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix} = k_1 x_1(kT) + k_2 x_2(kT)$$

Closed-loop system equation

$$x[(k+1)T] = \begin{bmatrix} 1 + \frac{1}{2}k_1 T^2 & T + \frac{1}{2}k_2 T^2 \\ k_1 T & 1 + k_2 T \end{bmatrix} x(kT)$$

Closed-loop characteristic polynomial

$$\left|\lambda I - A - bK\right| = \lambda^2 - (\frac{1}{2}k_1T^2 + k_2T + 2)\lambda - \frac{1}{2}k_1T^2 + Tk_2 + 1$$

Desired characteristic polynomial $f^*(\lambda) = \lambda^2 + a_1\lambda + a_0$

Equalize coefficients of these two polynomials leading to two equations

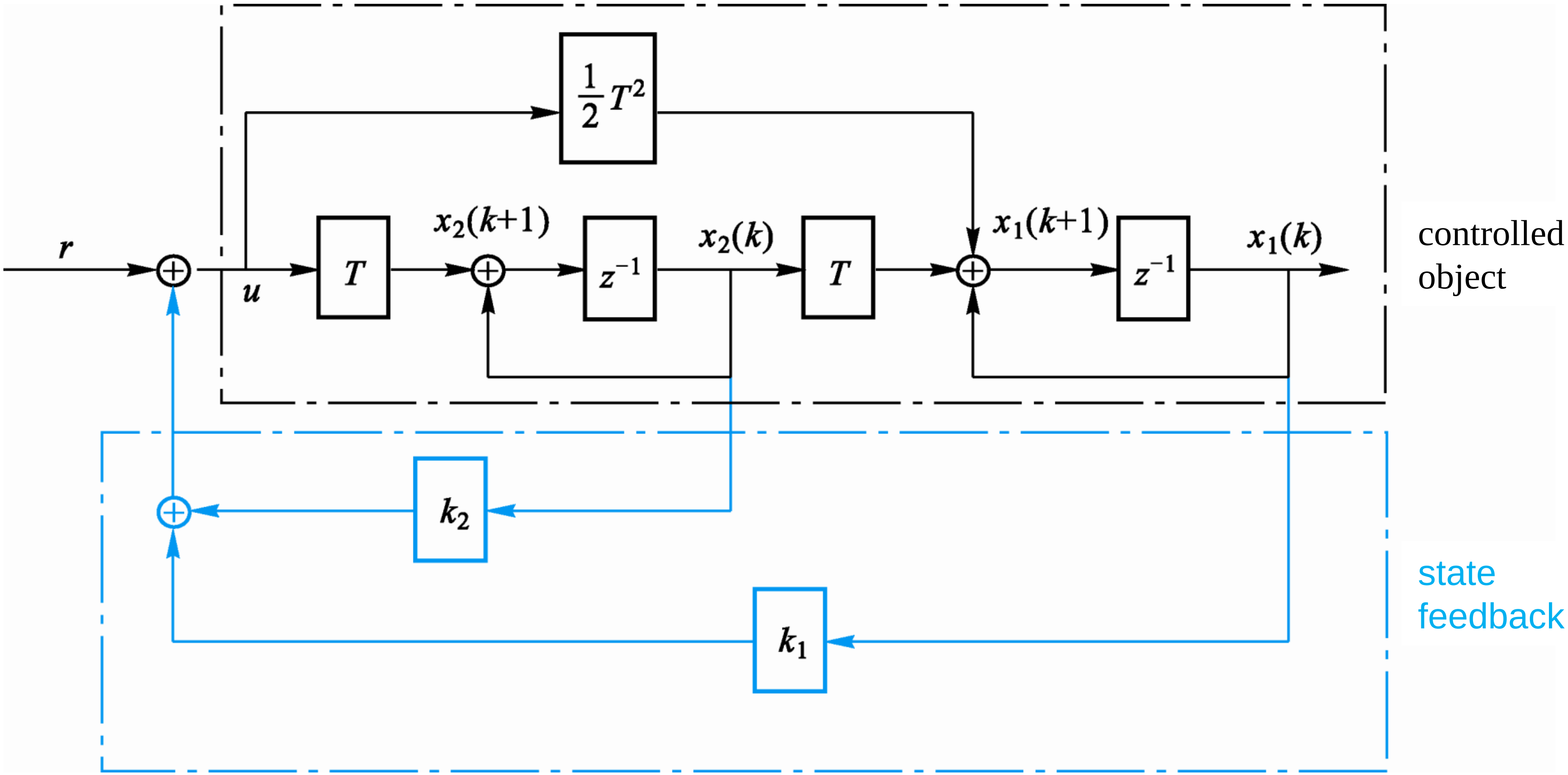
$$-(Tk_2 + \frac{1}{2}k_1T^2 + 2) = a_1$$

$$-\frac{1}{2}k_1T^2 + Tk_2 + 1 = a_0$$

Solution gives state feedback coefficients

$$k_1 = -\frac{1}{T^2}(1 + a_1 + a_0) \qquad k_2 = -\frac{1}{2T}(3 + a_1 - a_0)$$

Discrete time closed-loop system structure shown as follows



Procedure of pole assignment to design state feedback matrix:

1) Obtain controllable canonic form：transformation matrix T

$$S_C = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} S_C^{-1}$$

$$T = \begin{bmatrix} T_1 \\ T_1 A \\ \vdots \\ T_1 A^{n-1} \end{bmatrix}$$

2）Obtain characteristic polynomial

$$f(\lambda) = \det[\lambda I - A] = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

3）Desired characteristic polynomial based on pole locations

$$f^*(\lambda) = \lambda^n + a^*_{n-1}\lambda^{n-1} + \cdots + a^*_1\lambda + a^*_0$$

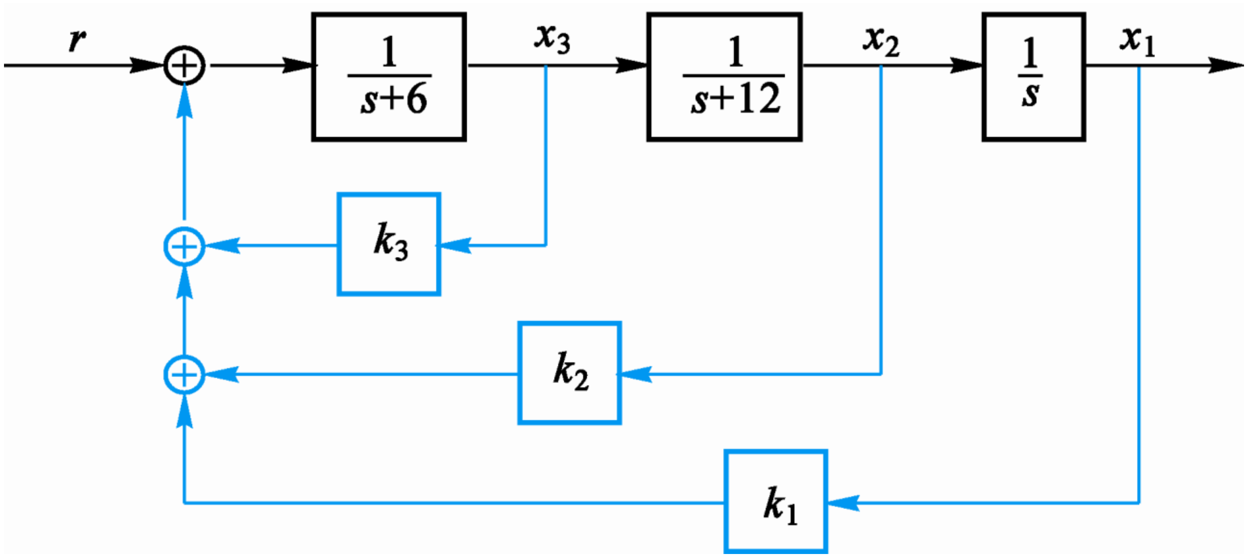
4) State feedback gain matrix in controllable canonic form

$$\overline{K} = \begin{bmatrix} a_0 - a^*_0 & a_1 - a^*_1 & \cdots & a_{n-1} - a^*_{n-1} \end{bmatrix}$$

5）Convert to state feedback gain for original state $K = \overline{K}T$

6）State feedback law $u = Kx + r$

Example 25 Design state feedback gain matrix so that closed-loop eigenvalues are $\lambda_{1,2} = -7.07 \pm j7.07, \lambda_3 = -100$



Answer: Derive state equation of the controlled system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -12 & 1 \\ 0 & 0 & -6 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Since

$$rankS_C = rank\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -18 \\ 1 & -6 & 36 \end{bmatrix} = 3$$

System is controllable, hence pole can be freely assigned.

$$S_c^{-1} = \begin{bmatrix} 72 & 6 & 1 \\ 18 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} T_1 \\ T_1 A \\ T_1 A^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -12 & 1 \end{bmatrix}$$

$$T_1 = [0 \quad 0 \quad 1] S_C^{-1} = [1 \quad 0 \quad 0]$$

system characteristic polynomial

$$f(\lambda) = \det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda + 12 & -1 \\ 0 & 0 & \lambda + 6 \end{bmatrix}$$

$$= \lambda(\lambda + 12)(\lambda + 6) = \lambda^3 + 18\lambda^2 + 72\lambda$$

$a_2 = 18 \qquad a_1 = 72 \qquad a_0 = 0$

Desired characteristic polynomial

$$f^*(\lambda)=(\lambda+7.07-j7.07)(\lambda+7.07+j7.07)(\lambda+100)$$

$$=\lambda^3+114.14\lambda^2+1514\lambda+9997$$

$$a_2^*=114.14 \qquad a_1^*=1514 \qquad a_0^*=9997$$

hence

$$\bar{K}=\begin{bmatrix}-9997 & 72-1514 & 18-114.14\end{bmatrix}=\begin{bmatrix}-9997 & -1442 & -96.14\end{bmatrix}$$

$$K=\bar{K}T=\begin{bmatrix}-9997 & -288.32 & -96.14\end{bmatrix}$$

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6 Linear quadratic optimal control

Among optimal control problems, if the system is linear and performance index is quadratic, then it is a linear quadratic regulator (LQR) problem.

Since quadratic index has clear physical meaning, represents performance requirements for many practical engineering problems, mathematically tractable, resulting in a linear state feedback law easily implementable, it has wide applications in engineering systems.

6.1 Linear quadratic optimal control problem

- Assume linear system is fully controllable

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

Performance index of optimal control is a quadratic function of state and control vectors

$$J = \frac{1}{2}[x(t_f) - x_d(t_f)]^T P [x(t_f) - x_d(t_f)]$$

$$+ \frac{1}{2}\int_{t_0}^{t_f} \{[x(t) - x_d(t)]^T Q(t)[x(t) - x_d(t)] + u^T(t)R(t)u(t)\} dt$$

The above is called linear as quadratic optimal control problem.

Linear quadratic regulator problem has universal meaning

6.2 Linear system finite time state regulartor

Theorem: Assume linear time variant system state equation is

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

Quadratic performance index is

$$J = \frac{1}{2}x(t_f)^T Px(t_f) + \frac{1}{2}\int_{t_0}^{t_f}[x(t)^T Q(t)x(t) + u^T(t)R(t)u(t)]\,dt$$

Then, optimal control uniquely exists as follows

$$u^*(t) = -R^{-1}(t)B^T(t)K(t)x(t)$$

$$\dot{K}(t) = -K(t)A(t) - A^T(t)K(t) + K(t)B(t)R^{-1}(t)B^T(t)K(t) - Q(t)$$

$$K(t_f) = P$$

$$\dot{x}(t) = [A(t) - B(t)R^{-1}(t)B^T(t)K(t)]\,x(t)$$

$$J^* = \frac{1}{2}x^T(t_0)K(t_0)x(t_0)$$

Example 27: Assume system is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Performance index is

$$J = \frac{1}{2}\int_{t_0}^{t_f} [x^T(t)Qx(t) + u^T(t)Ru(t)]\,dt \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = 1$$

Determine the optimal control law.

Answer: assume positive matrix, satisfies matrix Riccati equation $K(t) = \begin{bmatrix} k_{11}(t) & k_{12}(t) \\ k_{12}(t) & k_{22}(t) \end{bmatrix}$

$$\dot{K}(t) = -K(t)A(t) - A^T(t)K(t) + K(t)B(t)R^{-1}(t)B^T(t)K(t) - Q(t)$$

$$\begin{bmatrix} \dot{k}_{11} & \dot{k}_{12} \\ \dot{k}_{12} & \dot{k}_{22} \end{bmatrix} = -\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{k}_{11} & \dot{k}_{12} \\ \dot{k}_{12} & \dot{k}_{22} \end{bmatrix} = \begin{bmatrix} k_{12}^2 - 1 & -k_{11} + k_{12}k_{22} \\ -k_{11} + k_{12}k_{22} & -2k_{12} + k_{22}^2 \end{bmatrix}$$

Obtain differential equation as follows

$$\dot{k}_{11} = k_{12}^2 - 1$$

$$\dot{k}_{12} = -k_{11} + k_{12}k_{22}$$

$$\dot{k}_{22} = -2k_{12} + k_{22}^2$$

Numerical solution gives $K(t)$ values from $t = 0$ to $t = t_f$ so that optimal state feedback control is

$$u^*(t) = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} k_{11}(t) & k_{12}(t) \\ k_{12}(t) & k_{22}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -k_{12}(t)x_1(t) - k_{22}(t)x_2(t)$$

Since state feedback coefficient is time variant, when design optimal control, it is necessary to calculate k_{12} and k_{22} firstly, then save on a computer to be accessed in online control.

Summary

- State is an information set, which, with known external input, is sufficient and necessary to determine system future behavior.
- A set of first order differential equations describing relationship between state and input variables is called state equation. Equation describing output variable dependence on state and input variables is called output equation.
- State variable can be: physical variables of energy storage units, outputs and their derivatives, variables to form certain canonic forms
- State variables are not unique, but number of states is determined. State variables can have or have no clear physical meanings; can be measurable or unmeasurable. State equation is not unique, but unique in the sense of similarity.
- Eliminating state variables leads to system differential equation.
- Linear transformations convert state equation from one to another.

Lyapunov stability definition:

(1) Stable: if for a given $\varepsilon > 0$, exists another $\delta(\varepsilon, t_0) > 0$, such that any system response with initial state satisfying $\|x_0 - x_e\| \leq \delta(\varepsilon, t_0)$ at anytime satisfying $\|x - x_e\| \leq \varepsilon$, then equilibrium x_e is stable. If δ is independent from t_0, then equilibrium is uniformly stable.

(2) Asymptotic stable: If equilibrium is Lyapunov stable and when $t \to \infty$, $x(t) \to x_e$, i.e. $\lim_{t\to\infty}\|x(t) - x_e\| = 0$, then it is asymptotic stable.

(3) Globally asymptotic stable: If for any large δ, system is always stable, then it is globally stable. If the system is always asymptotic stable, then it is globally asymptotic stable.

(4) unstable：If for some $\varepsilon > 0$, whatever how small δ is, among trajectories from $s(\delta)$, there exists at least one going out, then the equilibrium is unstable.

Lyapunov stability criterion:

Continuous systems (linear or nonlinear)

Nonlinear continuous system Lyapunov stability criterion:

If there is a scalar function, $V(x)$, for all $x(t)$ having continuous first order partial derivatives, and $\dot{V}(x)$ is positive definite, then

If $\dot{V}(x)$ is negative definite, equilibrium is asymptotic stable;

If $\dot{V}(x)$ is negative definite and $\|x\| \to \infty$, $V(x) \to \infty$, then equilibrium is globally asymptotic stable;

If $\dot{V}(x)$ is negative semi-definite, equilibrium is Lyapunov stable;

If $\dot{V}(x)$ is negative semi-definite and $\dot{V}(x)$ is not always 0, equilibrium is globally asymptotic stable;

If $\dot{V}(x)$ is positive definite, equilibrium is unstable.

Lyapunov stability criterion:

Linear continuous system Lyapunov stability criterion: Linear continuous (time variant / invariant) system is stable if and only if for a given positive definite real symmetric matrix, $Q(t)$, there is a

$$A^T(t)P(t)+\dot{P}(t)+P(t)A(t)=-Q(t)$$

For LTI systems， differential equation becomes algebraic equation

$$A^TP+PA=-Q$$

Linear continuous time invariant system

1． LTI homogeneous state equation solution $x(t) = e^{At} x(0)$

2． State transition matrix $\Phi(t) = e^{At}$

3. State transition matrix properties：

$\Phi(t)|_{t=0} = \Phi(0) = I$ $\quad \Phi^{-1}(t) = \Phi(-t)$ $\quad \Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2) = \Phi(t_2)\Phi(t_1)$

$[\Phi(t)]^n = \Phi(nt)$ $\quad \Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Phi(t_2 - t_0) = \Phi(t_1 - t_0)\Phi(t_2 - t_1)$

4. Laplace transform approach $\boldsymbol{e^{At} = L^{-1}[(sI - A)^{-1}]}$

5. LTI non-homogenous state equation solution

$$x(t) = \Phi(t - t_0)x(t_0) + \int_{t_0}^{t} \Phi(t - \tau)Bu(\tau)d\tau$$

$$x(t) = L^{-1}\left[(sI - A)^{-1}\right]x(0) + L^{-1}\left[(sI - A)^{-1}BU(s)\right]$$

Linear quadratic optimal control: concept and linear system finite time state regulator design approach.

PPT adopted from 《自动控制原理》国家精品课程网站

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