Poles, Zeroes, Stability & Solving Differential Equations

Introduction

- The inverse Laplace transform is usually calculated directly by using tables, as solving the integral is a rather difficult task.
- The procedure is first to convert a function $F(s)$ into a sum of partial fractions and then find the inverse Laplace transform of the fractions through the given tables.
- The expansion into partial fractions is a very useful method for systems analysis and design, as the influence of every characteristic root or eigenvalue is visualized.
- In the usual case the Laplace transform of a function is expressed as a rational function of s, that is, it is given as a ratio of two polynomials of s.

Transfer function

- The transfer function provides a basis for determining important system response characteristics without solving the complete differential equation. As defined, the transfer function is a rational function in the complex variable $s = \sigma + j\omega,$
- Consider the following complex polynomial equation:

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

Transfer function

- It is often convenient to factor the polynomials in the numerator and denominator, and to write the transfer function in terms of those factors:

$$H(s) = \frac{N(s)}{D(s)} = K\frac{(s-z_1)(s-z_2)\dots(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_{n-1})(s-p_n)},$$

- where the numerator and denominator polynomials, $N(s)$ and $D(s)$, have real coefficients defined by the system's differential equation and $K = b_m/a_n$.

Transfer function and poles/zeros

- the z_i's are the roots of the equation $N(s) = 0$ and are defined to be the system ***zeros***.
- the p_i's are the roots of the equation $D(s) = 0,$ and are defined to be the system ***poles.***
- In the polynomial equation (previous slide), the factors in the numerator and denominator are written so that when $s = z_i$,the numerator $N(s) = 0$ and the transfer function vanishes, i.e. $\lim_{s \to z_i} H(s) = 0.$
- and similarly when $s = p_i$, the denominator polynomial $D(s) = 0,$ and the value of the transfer unction becomes unbounded, $\lim_{s \to p_i} H(s) = \infty$

Poles and Zeros

- All of the coefficients of polynomials *N*(*s*) and *D*(*s*) are real, therefore the poles and zeros **must** be either <u>purely real</u>, or appear in <u>complex conjugate pairs</u>. In general for the poles, either *pi* = *σi*, or else

  $p_i, p_{i+1} = \sigma_i \pm j\omega_i.$

- The existence of a single complex pole without a corresponding conjugate pole would generate complex coefficients in the polynomial *D*(*s*). Similarly, the system zeros are either real or appear in complex conjugate pairs.

Example of poles and zeros

- The poles and zeros are properties of the transfer function, and therefore of the differential equation describing the input-output system dynamics.
- Together with the gain constant *K* they completely characterize the differential equation, and provide a complete description of the system.
- Problem: A linear system is described by the differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 2\frac{du}{dt} + 1.$$

Find the system poles and zeros.

Example of Poles and Zeros

Solution: From the differential equation the transfer function is

$$H(s) = \frac{2s+1}{s^2+5s+6}. \tag{5}$$

which may be written in factored form

$$\begin{aligned} H(s) &= \frac{1}{2}\frac{s+1/2}{(s+3)(s+2)} \\ &= \frac{1}{2}\frac{s-(-1/2)}{(s-(-3))(s-(-2))}. \end{aligned} \tag{6}$$

The system therefore has a single real zero at $s = -1/2$, and a pair of real poles at $s = -3$ and $s = -2$.

Problem to solve

A system has a pair of complex conjugate poles $p_1, p_2 = -1 \pm j2$, a single real zero $z_1 = -4$, and a gain factor $K = 3$. Find the differential equation representing the system.

Solution

Solution: The transfer function is

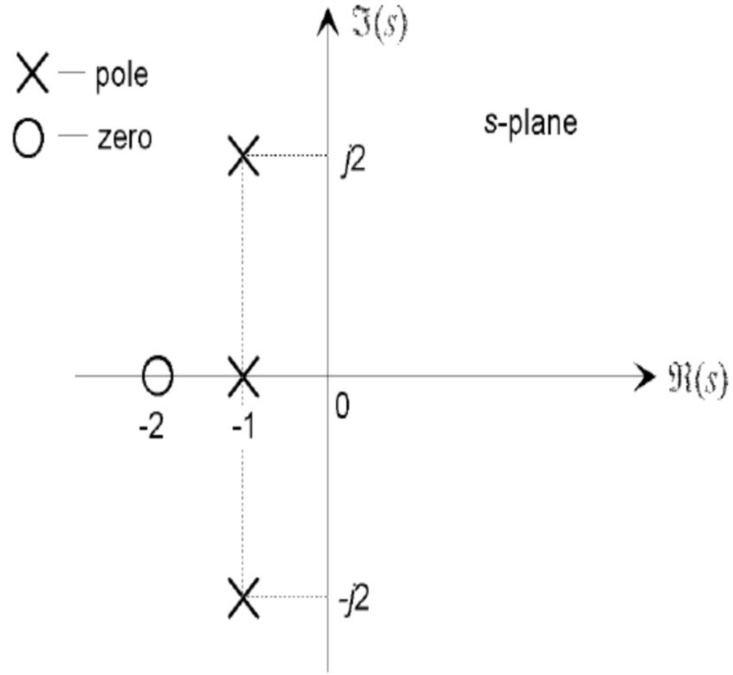
$$\begin{aligned} H(s) &= K\frac{s-z}{(s-p_1)(s-p_2)} \\ &= 3\frac{s-(-4)}{(s-(-1+j2))(s-(-1-j2))} \\ &= 3\frac{(s+4)}{s^2+2s+5} \end{aligned}$$

and the differential equation is

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=3\frac{du}{dt}+12u$$

Pole Zero plot

- A system is characterized by its poles and zeros in the sense that they allow reconstruction of the input/output differential equation
- In general, the poles and zeros of a transfer function may be complex, and the system dynamics may be represented graphically by plotting their locations on the complex *s*-plane, whose axes represent the real and imaginary parts of the complex variable *s*.
- Such plots are known as *pole-zero plots*.



Pole zero plot.

- It is usual to mark a zero location by a circle (◦) and a pole location a cross (×).
- The location of the poles and zeros provide qualitative insights into the response characteristics of a system.

$$H(s) = \frac{(3s+6)}{(s^3+3s^2+7s+5)} = 3\frac{(s-(-2))}{(s-(-1))(s-(-1-2j))(s-(-1+2j))}$$

Analysis of the plot

1. A real pole $p_i = -\sigma$ in the left-half of the s-plane defines an exponentially decaying component , $Ce^{-\sigma t}$, in the homogeneous response. The rate of the decay is determined by the pole location; poles far from the origin in the left-half plane correspond to components that decay rapidly, while poles near the origin correspond to slowly decaying components.

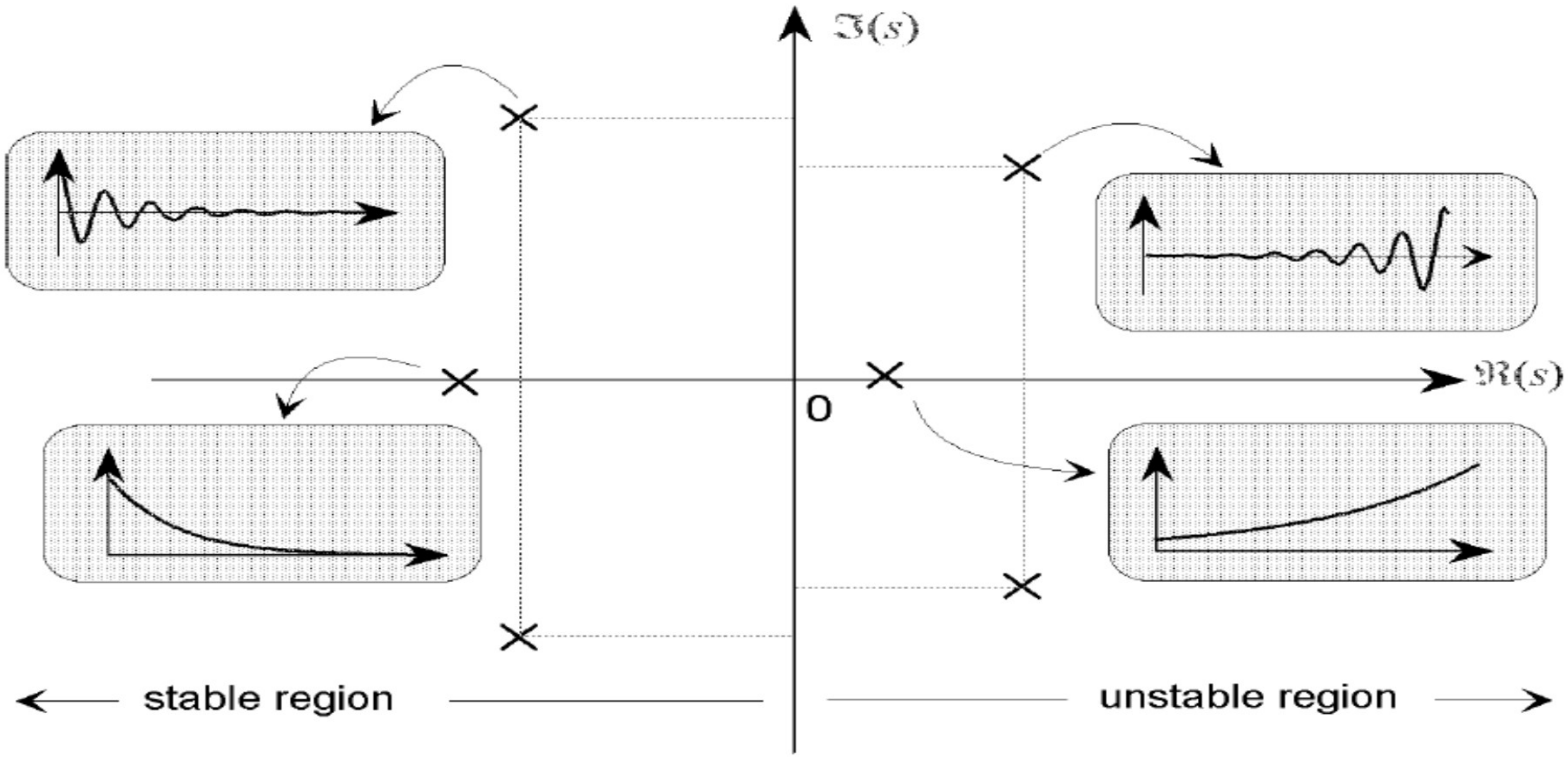
2. A pole at the origin $p_i = 0$ defines a component that is constant in amplitude and defined by the initial conditions.

3. A real pole in the right-half plane corresponds to an exponentially increasing component $Ce^{\sigma t}$ in the homogeneous response; thus defining the system to be unstable.

4. A complex conjugate pole pair $\sigma \pm j\omega$ in the left-half of the s-plane combine to generate a response component that is a decaying sinusoid of the form $Ae^{-\sigma t}\sin(\omega t + \phi)$ where A and ϕ are determined by the initial conditions. The rate of decay is specified by σ; the frequency of oscillation is determined by ω.

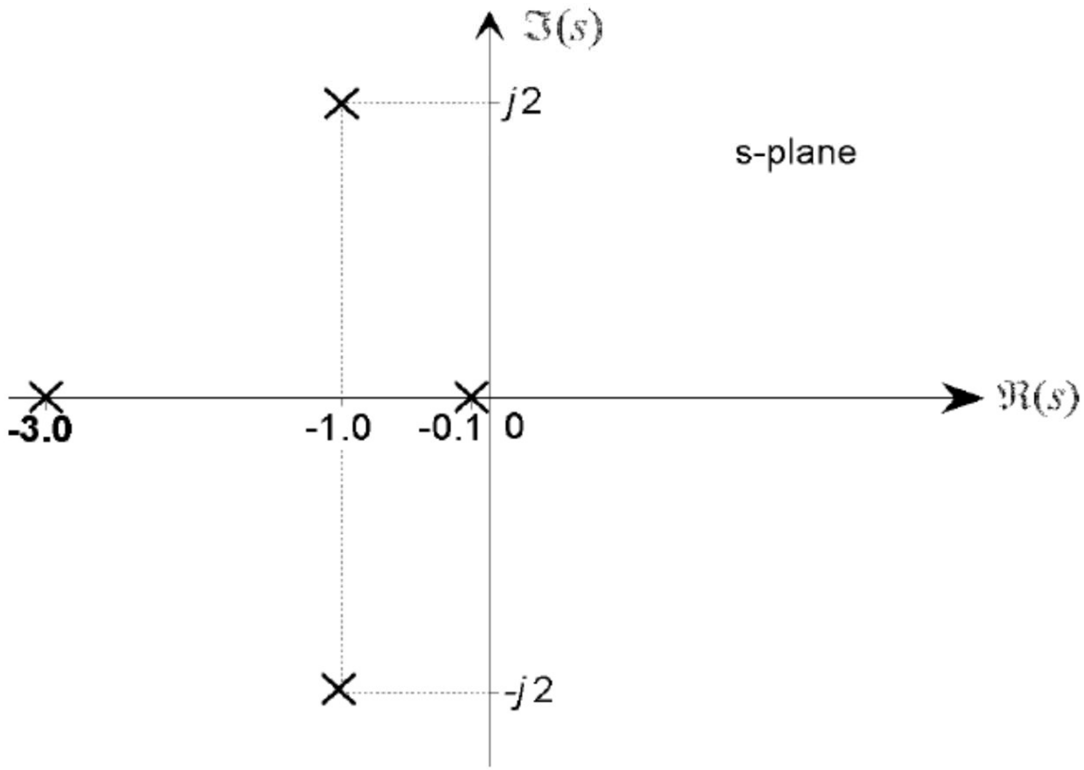
5. An imaginary pole pair, that is a pole pair lying on the imaginary axis, $\pm j\omega$ generates an oscillatory component with a constant amplitude determined by the initial conditions.

Analysis of the plot



Problem to solve

Comment on the expected form of the response of a system with a pole-zero plot shown in Fig. 3 to an arbitrary set of initial conditions.



Solution

Solution: The system has four poles and no zeros. The two real poles correspond to decaying exponential terms C_1e^{-3t} and $C_2e^{-0.1t}$, and the complex conjugate pole pair introduce an oscillatory component $Ae^{-t}\sin(2t+\phi)$, so that the total homogeneous response is

$$y_h(t) = C_1e^{-3t} + C_2e^{-0.1t} + Ae^{-t}\sin(2t+\phi) \tag{12}$$

Although the relative strengths of these components in any given situation is determined by the set of initial conditions, the following general observations may be made:

1. The term e^{-3t}, with a time-constant τ of 0.33 seconds, decays rapidly and is significant only for approximately 4τ or 1.33seconds.
2. The response has an oscillatory component $Ae^{-t}\sin(2t+\phi)$ defined by the complex conjugate pair, and exhibits some overshoot. The oscillation will decay in approximately four seconds because of the e^{-t} damping term.
3. The term $e^{-0.1t}$, with a time-constant $\tau = 10$ seconds, persists for approximately 40 seconds. It is therefore the *dominant* long term response component in the overall homogeneous response.