EBU4375 Signals and Systems Theory

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Orthogonal Signal Space and Fourier Series

Orthogonal Signal Space and Fourier Series

1) **Periodic continuous-time signals** can be expressed as a weighted sum of sinusoids (or a weighted sum of complex exponential functions). In this case, the frequency spectrum can be generated by computing the ***Fourier series***

2) The resulting representations are referred to as the **continuous-time Fourier series (CTFS)**

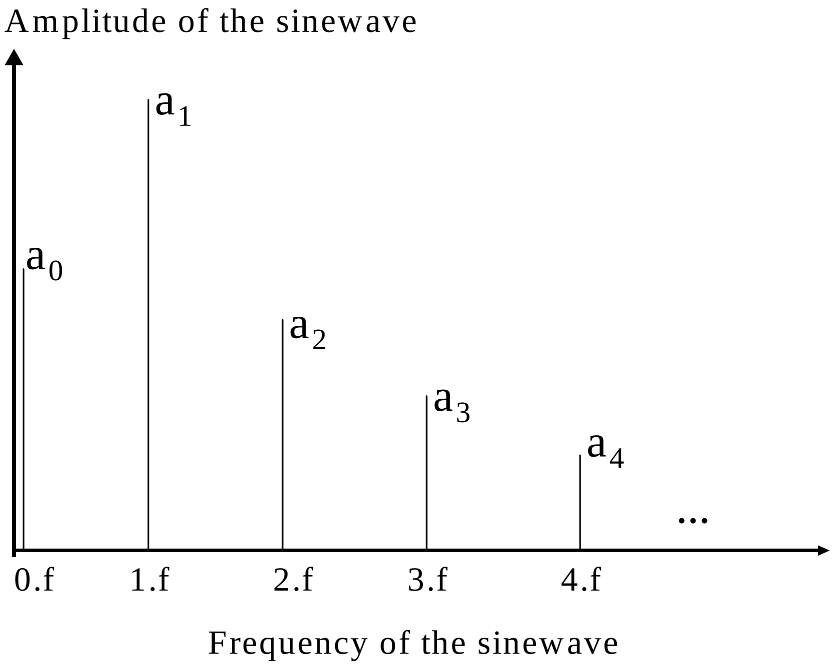
3) The Fourier series is named after the French physicist **Jean Baptiste Fourier** (1768-1830), who was the first one to propose that periodic waveforms could be represented by **a sum of sinusoids (or complex exponentials)**

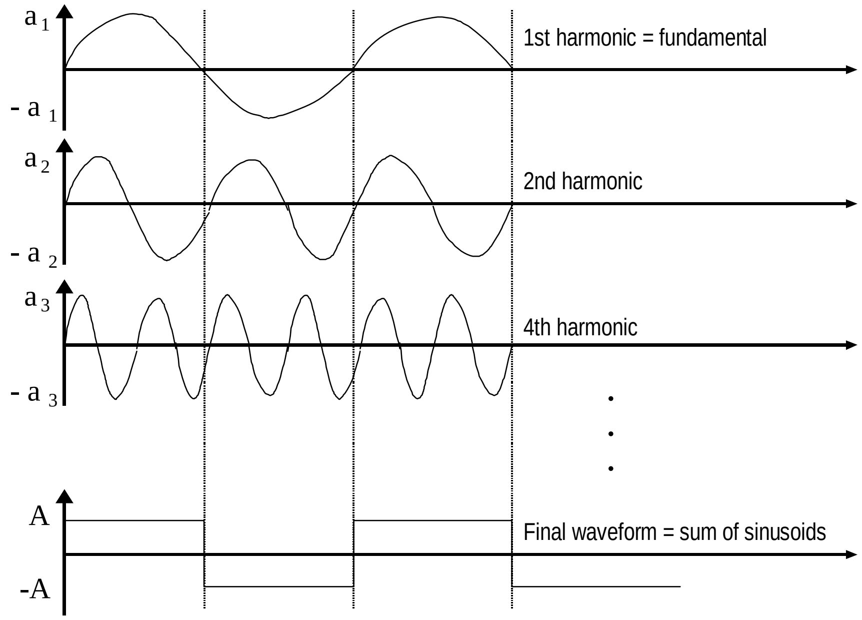
Orthogonal Signal Space and Fourier Series

1) **Periodic continuous-time** signals $\longrightarrow$ We use **Continuous-Time Fourier series (CTFS)** to decompose such signals into their frequency components

2) **Aperiodic continuous-time** signals $\longrightarrow$ We use **Continuous-Time Fourier Transform (CTFT)** to decompose such signals into their frequency components

Orthogonal Signal Space and Fourier Series

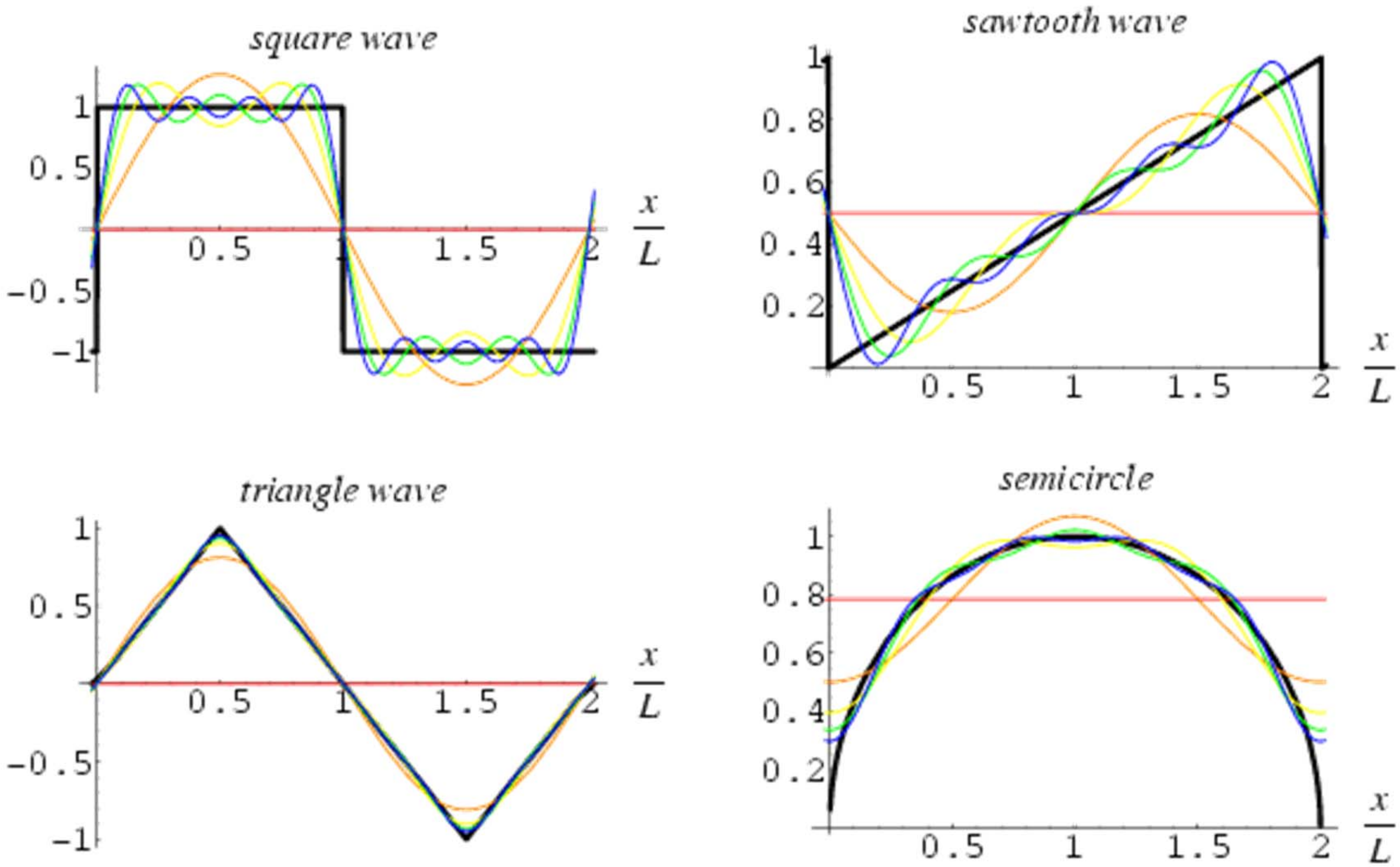




This diagram represents the frequency domain

This diagram represents the time domain. NOTE: the bottom line is not the sum of the first 3 lines.

Orthogonal Signal Space and Fourier Series



Orthogonal Signal Space

Definition *Two non-zero signals $p(t)$ and $q(t)$ are said to be orthogonal over interval $t = [t_1, \ t_2]$ if*

$$\int_{t_1}^{t_2} p(t)q^*(t)\mathrm{d}t = \int_{t_1}^{t_2} p^*(t)q(t)\mathrm{d}t = 0,$$

where the superscript $*$ denotes the complex conjugation operator. In addition if both signals $p(t)$ and $q(t)$ also satisfy the unit magnitude property:

$$\int_{t_1}^{t_2} p(t)p^*(t)\mathrm{d}t = \int_{t_1}^{t_2} q(t)q^*(t)\mathrm{d}t = 1,$$

they are said to be orthonormal to each other over the interval $t = [t_1, \ t_2]$.

Orthogonal Signal Space

Example

Show that

(i) functions $\cos(2\pi t)$ and $\cos(3\pi t)$ are orthogonal over interval $t = [0,\ 1]$;

(ii) functions $\exp(\mathrm{j}2t)$ and $\exp(\mathrm{j}4t)$ are orthogonal over interval $t = [0, \pi]$;

(iii) functions $\cos(t)$ and t are orthogonal over interval $t = [-1, 1]$.

Solution

$$\int_0^1 \cos(2\pi t)\cos(3\pi t)\mathrm{d}t = \frac{1}{2}\int_0^1 [\cos(\pi t) + \cos(5\pi t)]\mathrm{d}t$$

$$= \frac{1}{2}\left[\frac{1}{\pi}\sin(\pi t) + \frac{1}{5\pi}\sin(5\pi t)\right]_0^1 = 0.$$

Therefore, the functions $\cos(2\pi t)$ and $\cos(3\pi t)$ are orthogonal over interval $t = [0, 1]$.

Orthogonal Signal Space

(ii)

$$\int_0^{\pi} e^{j2t}e^{-j4t}\,dt = \int_0^{\pi} e^{-j2t}\,dt = \frac{1}{-2j}[e^{-j2t}]_0^{\pi} = -\frac{1}{2j}[e^{-j2\pi} - 1]_0^{\pi} = 0,$$

implying that the functions $\exp(j2t)$ and $\exp(j4t)$ are orthogonal over interval $t = [0, \pi]$.

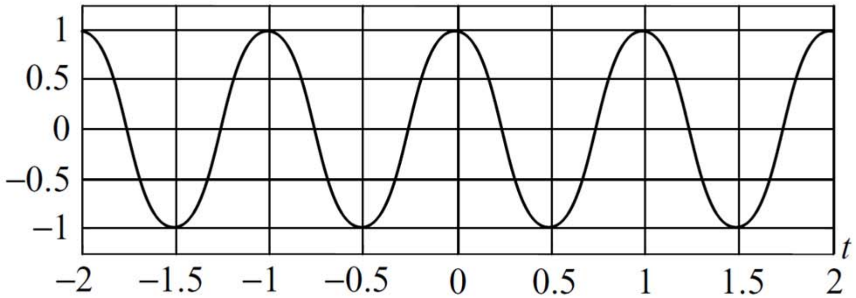
(iii)

$$\int_{-1}^{1} t\cos(t)dt = [t\sin(t) + \cos(t)]_{-1}^{1} = [1 \cdot \sin(1) + \cos(1)]$$

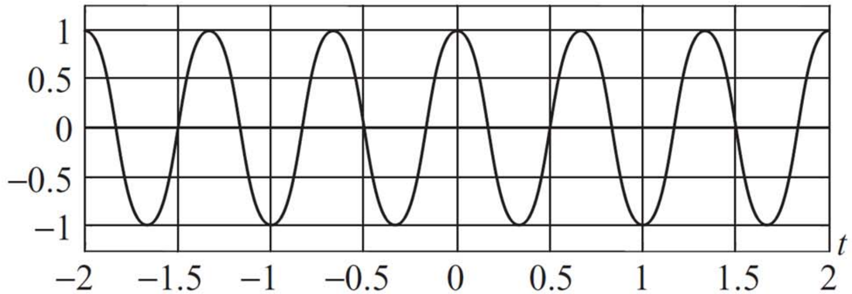
$$- [(-1) \cdot \sin(-1) + \cos(-1)] = 0,$$

implying that the functions $\cos(t)$ and t are orthogonal over interval $t = [-1, 1]$.

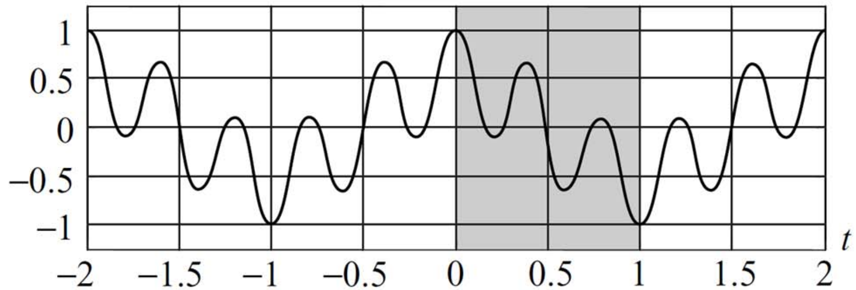
Orthogonal Signal Space



(a)



(b)



Graphical illustration of the orthogonality condition for the functions $\cos(2\pi t)$ and $\cos(3\pi t)$

(a) Waveform for $\cos(2\pi t)$.

(b) Waveform for $\cos(3\pi t)$.

(c) Waveform for $\cos(2\pi t) \times \cos(3\pi t)$.

Orthogonal Signal Space

Example

Show that the set $\{1, \cos(\omega_0 t), \cos(2\omega_0 t), \cos(3\omega_0 t), \ldots, \sin(\omega_0 t), \sin(2\omega_0 t), \sin(3\omega_0 t), \ldots\}$, consisting of all possible harmonics of sine and cosine waves with fundamental frequency of ω_0, is an orthogonal set over any interval $t = [t_0, t_0 + T_0]$, with duration $T_0 = 2\pi/\omega_0$.

Solution

It may be noted that the set $\{1, \cos(\omega_0 t), \cos(2\omega_0 t), \cos(3\omega_0 t), \ldots, \sin(\omega_0 t), \sin(2\omega_0 t), \sin(3\omega_0 t), \ldots\}$ contains three types of functions: 1, $\{\cos(m\omega_0 t)\}$, and $\{\sin(n\omega_0 t)\}$ for arbitrary integers $m, n \in Z^+$, where Z^+ is the set of positive integers. We will consider all possible combinations of these functions.

Orthogonal Signal Space

Case 1 The following proof shows that functions $\{\cos(m\omega_0 t),\ m \in Z^+\}$ are orthogonal to each other over interval $t = [t_0,\ t_0 + T_0]$ with $T_0 = 2\pi/\omega_0$.

$$\int_{\langle T_0 \rangle} \cos(m\omega_0 t)\cos(n\omega_0 t)\mathrm{d}t = \int_{t_0}^{t_0+T_0} \cos(m\omega_0 t)\cos(n\omega_0 t)\mathrm{d}t \quad \text{for any arbitrary } t_0.$$

Using the trigonometric identity $\cos(m\omega_0 t)\cos(n\omega_0 t) = (1/2)[\cos((m - n)\omega_0 t) + \cos((m + n)\omega_0 t)]$, the above integral reduces as follows:

$$\int_{\langle T_0 \rangle} \cos(m\omega_0 t)\cos(n\omega_0 t)\mathrm{d}t = \begin{cases} \left[\dfrac{\sin(m-n)\omega_0 t}{2(m-n)\omega_0} + \dfrac{\sin(m+n)\omega_0 t}{2(m+n)\omega_0}\right]_{t_0}^{t_0+T_0} & m \neq n \\ \left[\dfrac{t}{2} + \dfrac{\sin 2m\omega_0 t}{4m\omega_0}\right]_{t_0}^{t_0+T_0} & m = n, \end{cases}$$

Orthogonal Signal Space

or

$$\int_{\langle T_0 \rangle} \cos(m\omega_0 t)\cos(n\omega_0 t)\mathrm{d}t = \begin{cases} 0 & m \neq n \\ \dfrac{T_0}{2} & m = n, \end{cases}$$

for $m, n \in Z^+$.

Case 2 By following the procedure outlined in case 1, it is straightforward to show that

$$\int_{\langle T_0 \rangle} \sin(m\omega_0 t)\sin(n\omega_0 t)\mathrm{d}t = \begin{cases} 0 & m \neq n \\ \dfrac{T_0}{2} & m = n, \end{cases}$$

for $m, n \in Z^+$.

Orthogonal Signal Space

Case 3 To verify that functions $\{\cos(m\omega_0 t)\}$ and $\{\sin(n\omega_0 t)\}$ are mutually orthogonal, consider the following:

$$\int_{\langle T_0\rangle} \cos(m\omega_0 t)\sin(n\omega_0 t)\mathrm{d}t = \int_{t_0}^{t_0+T_0} \cos(m\omega_0 t)\sin(n\omega_0 t)\mathrm{d}t$$

$$= \begin{cases} \dfrac{1}{2}\displaystyle\int_{t_0}^{t_0+T_0} [\sin((m+n)\omega_0 t) - \sin((m-n)\omega_0 t)]\mathrm{d}t & m \neq n \\ \dfrac{1}{2}\displaystyle\int_{t_0}^{t_0+T_0} [\sin(2m\omega_0 t)\mathrm{d}t & m = n \end{cases}$$

$$= \begin{cases} -\dfrac{1}{2}\left[\dfrac{\cos((m+n)\omega_0 t)}{(m+n)\omega_0}\right]_{t_0}^{t_0+T_0} + \dfrac{1}{2}\left[\dfrac{\cos((m-n)\omega_0 t)}{(m-n)\omega_0}\right]_{t_0}^{t_0+T_0} & m \neq n \\ -\dfrac{1}{2}\left[\dfrac{\cos(2n\omega_0 t)}{2m\omega_0}\right]_{t_0}^{t_0+T_0} & m = n \end{cases}$$

$$= \begin{cases} 0 & m \neq n \\ 0 & m = n, \end{cases}$$

for $m, n \in Z^+$, which proves that $\{\cos(m\omega_0 t)\}$ and $\{\sin(n\omega_0 t)\}$ are orthogonal over interval $t = [t_0, t_0 + T_0]$ with $T_0 = 2\pi/\omega_0$.

Orthogonal Signal Space

Case 4 The following proof demonstrates that the function "1" is orthogonal to $\cos(m\omega_0 t)\}$ and $\{\sin(n\omega_0 t)\}$:

$$\int_{\langle T_0 \rangle} 1 \cdot \cos(m\omega_0 t)\mathrm{d}t = \left[\frac{\sin(m\omega_0 t)}{m\omega_0}\right]_{t_0}^{t_0+T_0}$$

$$= \left[\frac{\sin(m\omega_0 t_0 + 2m\pi) - \sin(m\omega_0 t_0)}{m\omega_0}\right] = 0$$

and

$$\int_{\langle T_0 \rangle} 1 \cdot \sin(m\omega_0 t)\mathrm{d}t = \left[-\frac{\cos(m\omega_0 t)}{m\omega_0}\right]_{t_0}^{t_0+T_0}$$

$$= -\left[\frac{\cos(m\omega_0 t_0 + 2m\pi) - \cos(m\omega_0 t_0)}{m\omega_0}\right] = 0$$

Trigonometric CTFS

Definition *An arbitrary periodic function $x(t)$ with fundamental period T_0 can be expressed as follows:*

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)),$$

where $\omega_0 = 2\pi/T_0$ is the fundamental frequency of $x(t)$ and coefficients a_0, a_n, and b_n are referred to as the trigonometric CTFS coefficients. The coefficients are calculated as follows:

$$a_0 = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) \mathrm{d}t,$$

$$a_n = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \cos(n\omega_0 t) \mathrm{d}t,$$

and

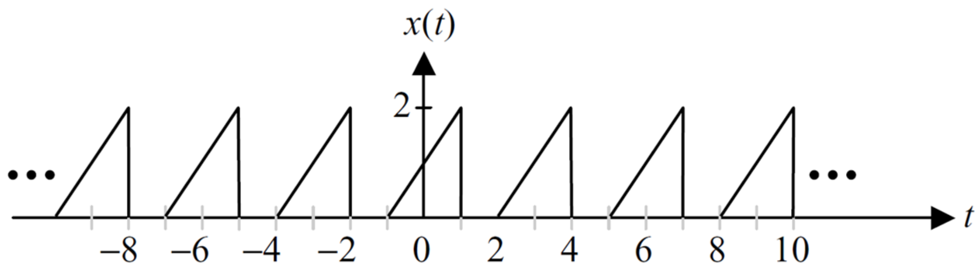
$$b_n = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \sin(n\omega_0 t) \mathrm{d}t.$$

Trigonometric CTFS

Example

Calculate the trigonometric CTFS coefficients of the periodic signal $x(t)$ defined over one period $T_0 = 3$ as follows:

$$x(t) = \begin{cases} t+1 & -1 \le t \le 1 \\ 0 & 1 < t < 2. \end{cases}$$



$$a_0 = \frac{1}{T_0}\int_{\langle T_0\rangle} x(t)\mathrm{d}t = \frac{1}{3}\int_{-1}^{1}(t+1)\mathrm{d}t = \frac{1}{3}\left[\frac{1}{2}t^2 + t\right]_{-1}^{1} = \frac{2}{3}$$

Trigonometric CTFS

The CTFS coefficients a_n are given by

$$a_n = \frac{2}{T_0} \int\limits_{\langle T_0 \rangle} x(t)\cos(n\omega_0 t)\mathrm{d}t = \frac{2}{3} \int\limits_{-1}^{1} (t+1)\cos(n\omega_0 t)\mathrm{d}t$$

$$= \frac{2}{3} \int\limits_{-1}^{1} \underbrace{t\cos(n\omega_0 t)}_{\text{odd function}}\,\mathrm{d}t + \frac{2}{3} \int\limits_{-1}^{1} \underbrace{\cos(n\omega_0 t)}_{\text{even function}}\,\mathrm{d}t.$$

Since the integral of odd functions within the limit $[-t_0, t_0]$ is zero,

$$\int\limits_{-1}^{1} t\cos(n\omega_0 t)\mathrm{d}t = 0,$$

and the value of a_n is given by

$$a_n = \frac{2}{3} \int\limits_{-1}^{1} \cos(n\omega_0 t)\mathrm{d}t = \frac{4}{3} \int\limits_{0}^{1} \cos(n\omega_0 t)\mathrm{d}t = \frac{4}{3} \left[\frac{\sin(n\omega_0 t)}{n\omega_0} \right]_0^1 = \frac{4\sin(n\omega_0)}{3n\omega_0}$$

Trigonometric CTFS

Substituting $\omega_0 = 2\pi/3$, we obtain

$$a_n = \begin{cases} 0 & n = 3k \\ \dfrac{\sqrt{3}}{n\pi} & n = 3k+1 \\ -\dfrac{\sqrt{3}}{n\pi} & n = 3k+2, \end{cases}$$

for $k \in Z$. Similarly, the CTFS coefficients b_n are given by

$$b_n = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \sin(n\omega_0 t) \mathrm{d}t = \frac{2}{3} \int_{-1}^{1} (t+1) \sin(n\omega_0 t) \mathrm{d}t$$

$$= \frac{2}{3} \int_{-1}^{1} \underbrace{t \sin(n\omega_0 t)}_{\text{even function}} \mathrm{d}t + \frac{2}{3} \int_{-1}^{1} \underbrace{\sin(n\omega_0 t)}_{\text{odd function}} \mathrm{d}t.$$

Trigonometric CTFS

Since the integral of odd functions within the limits $[-t_0, t_0]$ is zero,

$$\int_{-1}^{1} \sin(n\omega_0 t)\mathrm{d}t = 0,$$

and the value of b_n is given by

$$b_n = \frac{2}{3}\int_{-1}^{1} t\sin(n\omega_0 t)\mathrm{d}t = \frac{4}{3}\int_{0}^{1} t\sin(n\omega_0 t)\mathrm{d}t$$

$$= \frac{4}{3}\left[-t\frac{\cos(n\omega_0 t)}{n\omega_0} + \frac{\sin(n\omega_0 t)}{(n\omega_0)^2}\right]_0^1 = -\frac{4\cos(n\omega_0)}{3n\omega_0} + \frac{4\sin(n\omega_0)}{3(n\omega_0)^2}.$$

Substituting $\omega_0 = 2\pi/3$, we obtain

$$b_n = \begin{cases} -\dfrac{2}{n\pi} & n = 3k \\ \dfrac{1}{n\pi} + \dfrac{3\sqrt{3}}{2(n\pi)^2} & n = 3k+1 \\ \dfrac{1}{n\pi} - \dfrac{3\sqrt{3}}{2(n\pi)^2} & n = 3k+2, \end{cases}$$

Trigonometric CTFS

for $k \in Z$. The periodic signal $x(t)$ is therefore expressed as follows:

$$x(t) = \underbrace{\frac{2}{3}}_{x_{\mathrm{av}}(t)} + \underbrace{\sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{3}t\right)}_{\mathrm{Ev}\{x(t)-a_0\}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi}{3}t\right)}_{\mathrm{Odd}\{x(t)-a_0\}},$$

Coefficient a_0 represents the average value of signal $x(t)$, referred to as $x_{av}(t)$. The cosine terms collectively represent the zero-mean even component of signal $x(t)$, denoted by $\mathrm{Ev}\{x(t) - a_0\}$, while the sine terms collectively represent the zero-mean odd component of $x(t)$, denoted by $\mathrm{Odd}\{x(t) - a_0\}$.

Exponential CTFS

Definition *An arbitrary periodic function $x(t)$ with a fundamental period T_0 can be expressed as follows:*

$$x(t) = \sum_{n=0}^{\infty} D_n e^{jn\omega_0 t},$$

where the exponential CTFS coefficients D_n are calculated as

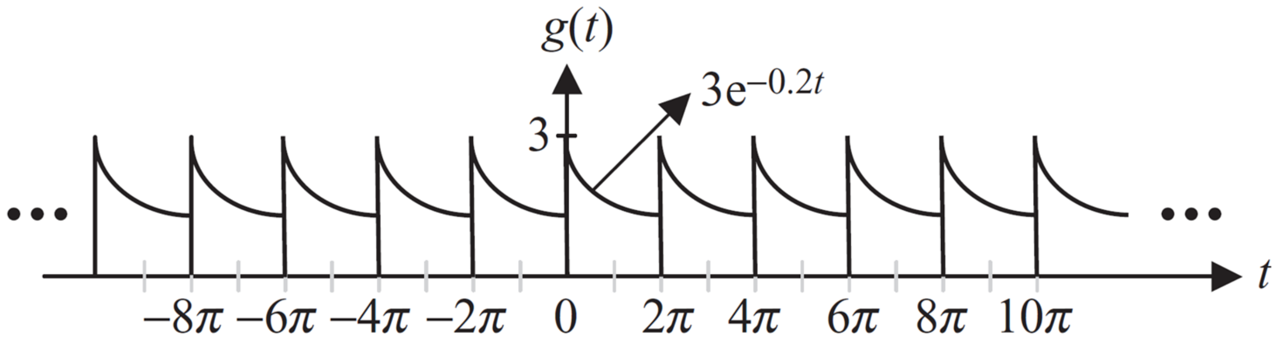
$$D_n = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) e^{-jn\omega_0 t} dt,$$

ω_0 being the fundamental frequency given by $\omega_0 = 2\pi / T_0$.

Exponential CTFS

Example

Calculate the exponential CTFS coefficients for the periodic function $g(t)$



Solution

By inspection, the fundamental period $T_0 = 2\pi$, which gives the fundamental frequency $\omega_0 = 2\pi/2\pi = 1$. The exponential CTFS coefficients D_n are given by

$$D_n = \frac{1}{T_0}\int_{\langle T_0\rangle} g(t)\mathrm{e}^{-\mathrm{j}n\omega_0 t}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} 3\mathrm{e}^{-0.2t}\mathrm{e}^{-\mathrm{j}n\omega_0 t}\,\mathrm{d}t = \frac{3}{2\pi}\int_0^{2\pi} \mathrm{e}^{-(0.2+\mathrm{j}n\omega_0)\,t}\,\mathrm{d}t$$

Exponential CTFS

or

$$D_n = -\frac{3}{2\pi}\left[\frac{e^{-(0.2+jn\omega_0)t}}{(0.2+jn\omega_0)}\right]_0^{2\pi} = \frac{3}{2\pi}\frac{1}{(0.2+jn\omega_0)}\left[1-e^{-(0.2+jn\omega_0)2\pi}\right].$$

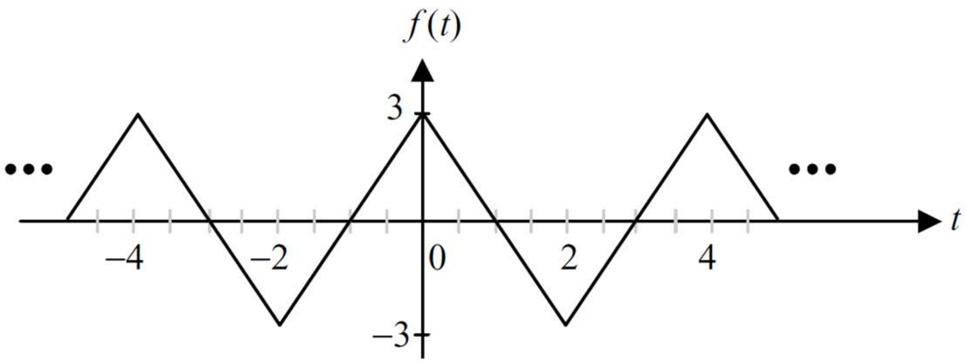
Substituting $\omega_0 = 1$, we obtain the following expression for the exponential CTFS coefficients:

$$\begin{aligned} D_n &= \frac{3}{2\pi(0.2+jn)}\left[1-e^{-(0.2+jn)2\pi}\right] \\ &= \frac{3}{2\pi(0.2+jn)}\left[1-e^{-0.4\pi}\right] \approx \frac{0.3416}{(0.2+jn)}. \end{aligned}$$

Exponential CTFS

Example

Calculate the exponential CTFS coefficients for $f(t)$ as shown



Solution

Since the fundamental period $T_0 = 4$, the angular frequency $\omega_0 = 2\pi/4 = \pi/2$. The exponential CTFS coefficients D_n are calculated directly from the definition as follows:

$$D_n = \frac{1}{T_0}\int_{\langle T_0\rangle} f(t)\mathrm{e}^{-\mathrm{j}n\omega_0 t}\,\mathrm{d}t = \frac{1}{4}\int_{-2}^{2} f(t)\mathrm{e}^{-\mathrm{j}n\omega_0 t}\,\mathrm{d}t$$

$$= \frac{1}{4}\int_{-2}^{2} \underbrace{f(t)\cos(n\omega_0 t)}_{\text{even function}}\mathrm{d}t - \mathrm{j}\frac{1}{4}\int_{-2}^{2} \underbrace{f(t)\sin(n\omega_0 t)}_{\text{odd function}}\mathrm{d}t.$$

Exponential CTFS

Since the integration of an odd function within the limits $[t_0, -t_0]$ is zero,

$$D_n = \frac{1}{4}\int_{-2}^{2} f(t)\cos(n\omega_0 t)\mathrm{d}t = \frac{1}{2}\int_{0}^{2}(3-3t)\cos(n\omega_0 t)\mathrm{d}t,$$

which simplifies to

$$\begin{aligned} D_n &= \frac{1}{2}\left[(3-3t)\frac{\sin(n\omega_0 t)}{n\omega_0} - 3\frac{\cos(n\omega_0 t)}{(n\omega_0)^2}\right]_0^2 \\ &= \frac{3}{2}\left[-\frac{\sin(2n\omega_0)}{n\omega_0} - \frac{\cos(2n\omega_0)}{(n\omega_0)^2} + \frac{1}{(n\omega_0)^2}\right]. \end{aligned}$$

Substituting $\omega_0 = \pi/2$, we obtain

$$D_n = \frac{3}{2}\left[-\frac{\sin(n\pi_0)}{0.5n\pi} - \frac{\cos(n\pi)}{(0.5n\pi)^2} + \frac{1}{(0.5n\pi)^2}\right] = \frac{6}{(n\pi)^2}[1-(-1)^n]$$

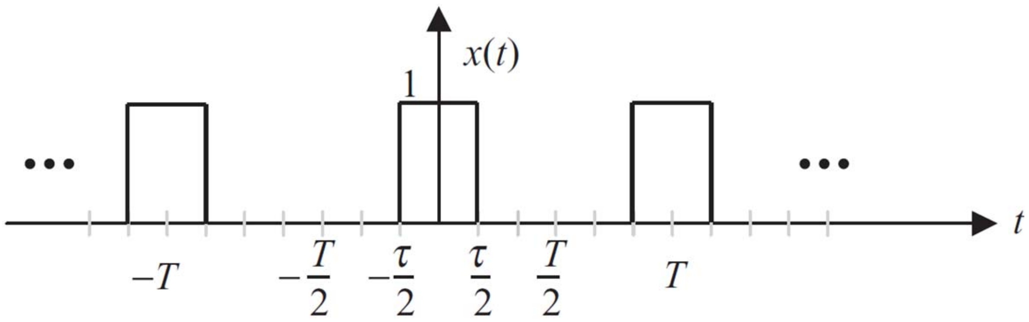
or

$$D_n = \begin{cases} 0 & n \text{ is even} \\ \dfrac{12}{(n\pi)^2} & n \text{ is odd.} \end{cases}$$

Exponential CTFS

Example

Calculate the exponential Fourier series of the signal $x(t)$



Case I For $n = 0$, the exponential CTFS coefficients are given by

$$D_n = \frac{1}{T}[t]_{-\tau/2}^{\tau/2} = \frac{\tau}{T}.$$

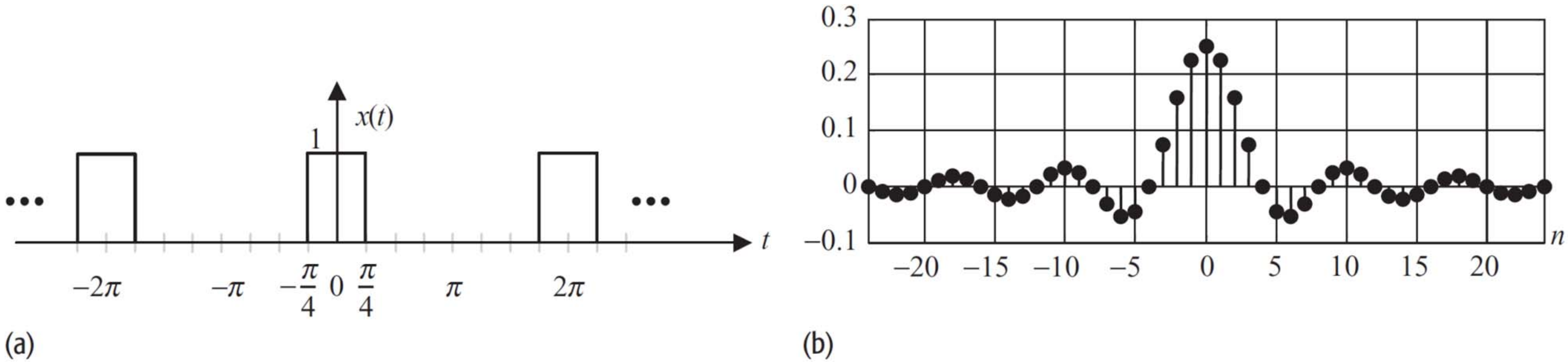
Case II For $n \neq 0$, the exponential CTFS coefficients are given by

$$D_n = -\frac{1}{\mathrm{j}n\omega_0 T}[\mathrm{e}^{-\mathrm{j}n\omega_0 t}]_{-\tau/2}^{\tau/2} = \frac{1}{n\pi}\sin\left(\frac{n\pi\tau}{T}\right)$$

Exponential CTFS

or

$$D_n = \frac{\tau}{T}\frac{\sin\left(\pi\frac{n\tau}{T}\right)}{\left(\pi\frac{n\tau}{T}\right)} = \frac{\tau}{T}\mathrm{sinc}\left(\frac{n\tau}{T}\right)$$



Exponential CTFS coefficients for the signal $x(t)$ with $\tau = \pi/2$ and $T = 2\pi$.
(a) Waveform for $x(t)$. (b) Exponential CTFS coefficients.

$$\longrightarrow \quad D_n = \frac{1}{4}\mathrm{sinc}\left(\frac{n}{4}\right)$$

More Examples on Trigonometric CTFS

Any periodic signal[1] $x(t)$ can be expressed as a weighted sum of trigonometric functions:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n}{T} t + \sum_{m=1}^{\infty} b_m \sin \frac{2\pi m}{T} t \tag{1}$$

where the period is T seconds. The hard part is finding the weights. You calculate these by

$$a_0 = \frac{1}{T} \int_0^T x(t) dt \tag{2}$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos \left(\frac{2\pi n}{T} t \right) dt \tag{3}$$

$$b_m = \frac{2}{T} \int_0^T x(t) \sin \left(\frac{2\pi m}{T} t \right) dt. \tag{4}$$

Remember, cosine is an even function and sine is an odd function:

$$\cos(-t) = \cos(t), \qquad \sin(-t) = -\sin(t). \tag{5}$$

Also, for any integer n

$$\cos(t + 2\pi n) = \cos(t), \qquad \sin(t + 2\pi n) = \sin(t). \tag{6}$$

We can convert one into the other too:

$$\cos(t) = \sin(t + \pi/2), \qquad \sin(t) = \cos(t - \pi/2). \tag{7}$$

And when we integrate and differentiate them:

$$\int \cos(t) dt = \sin(t) \Longrightarrow \frac{d}{dt} \sin(t) = \cos(t) \tag{8}$$

$$\int \sin(t) dt = -\cos(t) \Longrightarrow \frac{d}{dt} \cos(t) = -\sin(t). \tag{9}$$

For the problems below, it is quite helpful to know how to integrate by parts. Given two functions of t, $U(t)$ and $v(t)$, the integral of their product is given simply by[2]

$$\int U(t)v(t)dt = V(t)U(t) - \int V(t)u(t)dt \tag{10}$$

where

$$u(t) := \frac{d}{dt} U(t) \tag{11}$$

$$V(t) := \int v(t)dt. \tag{12}$$

[1]Satisfying the Dirichlet conditions, that is.

[2]Derived by the product rule of derivatives.

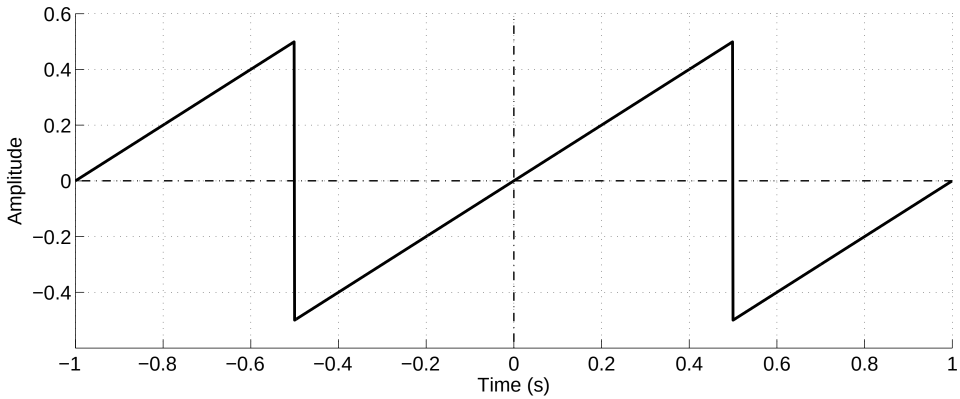


Figure 1: A sawtooth wave of period 1 s and amplitude $A = 0.5$. Just look at those sharp edges! Will it shiv your ear drums?

1 Sawtooth Wave

Figure 1 shows two periods of a sawtooth wave. This signal is defined over one period as

$$x(t) := \frac{2A}{T}t, \; -T/2 \leq t < T/2. \tag{13}$$

Since $x(t) = -x(-t)$, this is an odd signal, and so all weights on the cosine terms $\{a_n\}$ will be zero. Also, since this signal spends as much time above as below zero, $a_0 = 0$. So, we calculate $\{b_m\}$. First, notice the following:

$$b_m = \frac{2}{T}\int_0^T x(t)\sin\left(\frac{2\pi n}{T}t\right)dt = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin\left(\frac{2\pi n}{T}t\right)dt. \tag{14}$$

Since $x(t)$ is periodic, we can take any period; and so for convenience we have shifted the domain of integration to make it easier.

Now, substituting in for $x(t)$, we see

$$b_m = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{2A}{T}t\sin\left(\frac{2\pi n}{T}t\right)dt = \frac{4}{T}\int_0^{\frac{T}{2}} \frac{2A}{T}t\sin\left(\frac{2\pi n}{T}t\right)dt \tag{15}$$

since $t\sin t$ is an even function. To use integration by parts, we first define

$$U(t) := t \Longrightarrow u(t) = \frac{d}{dt}U(t) = 1 \tag{16}$$

$$v(t)dt = \sin\left(\frac{2\pi n}{T}t\right)dt \Longrightarrow V(t) = \int \sin\left(\frac{2\pi n}{T}t\right)dt = -\frac{T}{2\pi n}\cos\left(\frac{2\pi n}{T}t\right) \tag{17}$$

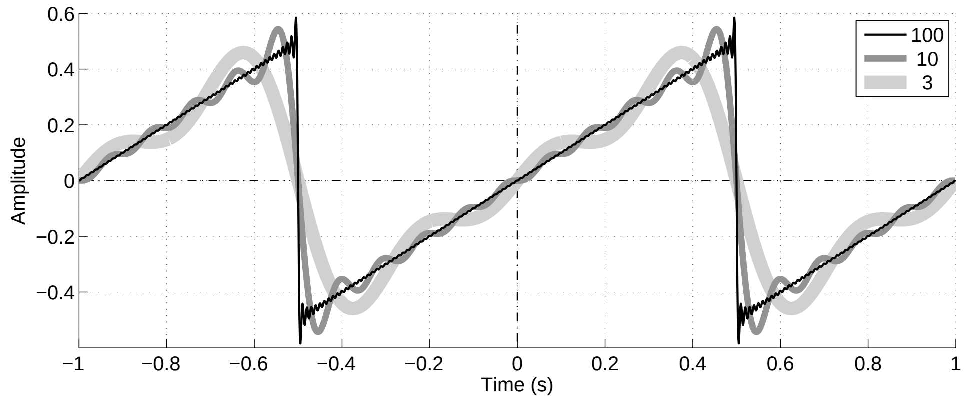


Figure 2: Fourier series reconstructions of the sawtooth wave in Fig. 1 for three numbers of components.

and so by integration by parts, we see

$$
\begin{aligned}
b_m = \frac{8A}{T^2}\int_0^{\frac{T}{2}} t\sin\left(\frac{2\pi m}{T}t\right)dt &= V(t)U(t)\Big|_0^{\frac{T}{2}} - \int_0^{\frac{T}{2}} V(t)u(t)dt \\
&= \frac{8A}{T^2}\left[\cancelto{-\frac{T^2}{4\pi m}\cos\pi m}{-t\frac{T}{2\pi m}\cos\left(\frac{2\pi m}{T}t\right)\Big|_0^{\frac{T}{2}}} + \cancelto{0}{\int_0^{\frac{T}{2}}\frac{T}{2\pi m}\cos\left(\frac{2\pi m}{T}t\right)dt}\right] \\
&= -\frac{2A}{\pi m}\cos\pi m. \qquad (18)
\end{aligned}
$$

Thus, to reconstruct the sawtooth wave with period T with sine waves, we only need to do

$$
x(t) = \sum_{m=1}^{\infty} -\frac{2A\cos\pi m}{\pi m}\sin\frac{2\pi m}{T}t. \qquad (19)
$$

Figure 2 shows the results for three different numbers of sine waves.

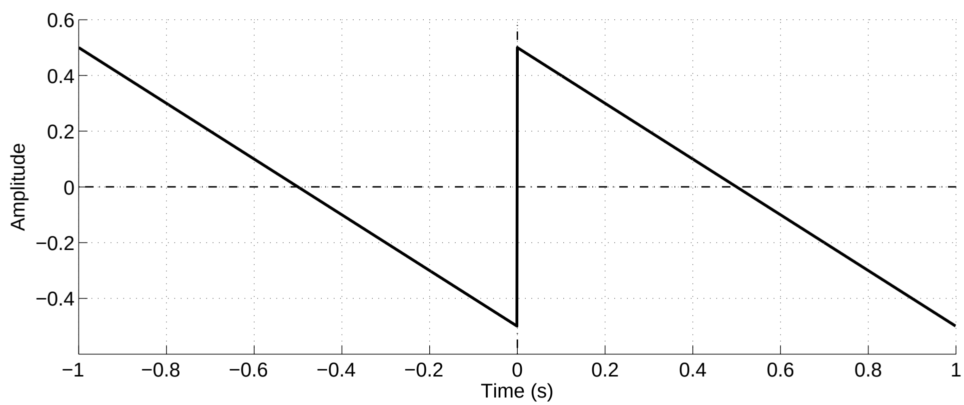


Figure 3: A reverse sawtooth wave of period 1 s and a half period delay with amplitude $A = 0.5$.

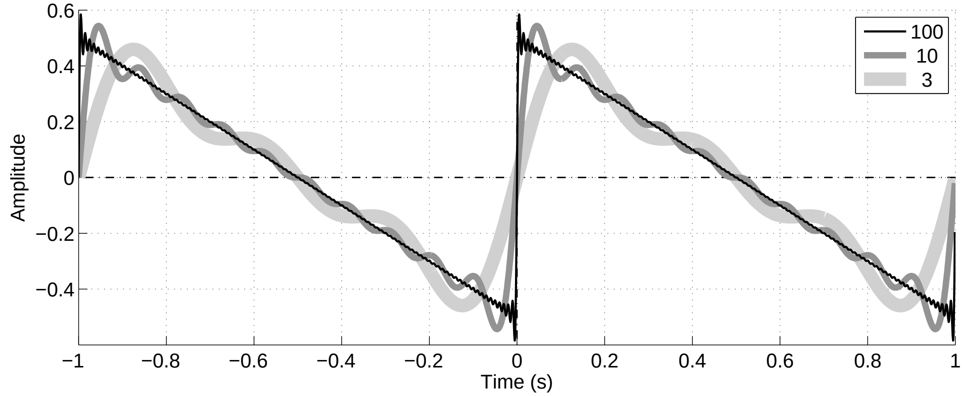


Figure 4: Fourier series reconstructions of the reverse sawtooth wave in Fig. 3.

2 Reverse Sawtooth Wave with Half-period Delay

Figure 3 shows a reverse sawtooth wave with a delay half its period. This signal is defined over one period as

$$x(t) := \frac{2A}{T}\left(\frac{T}{2} - t\right),\ 0 \le t < T. \tag{20}$$

Unlike the sawtooth wave we consider above, this is an even signal, and so all $\{b_m\}$ will be zero. Also, $a_0 = 0$ since it spends as much time above as below zero. But since this one is just a flipped and delayed version of the sawtooth above, and since we did all that work already, and since we are really hungry, do we really have to go through all those calculations again?[3]

We can be completely green by realizing that all we need to do is multiply the Fourier series of the sawtooth wave in (19) by -1, and add a half-period delay to the trigonometric functions! Thus, to reproduce the reverse sawtooth wave with period T, and a delay of $T/2$, we see that we just need to add up a bunch of sine waves as follows:

$$x(t) = \sum_{m=1}^{\infty} \frac{2A\cos \pi m}{\pi m} \sin \frac{2\pi m}{T}(t - T/2) \tag{21}$$

Figure 4 shows the results!

3 Triangle Wave

Figure 5 shows two periods of a triangle wave, which is defined over one period as

$$x(t) := \begin{cases} -\frac{4A}{T}\left(t + \frac{T}{2}\right), & -\frac{T}{2} \le t < -\frac{T}{4} \\ \frac{4A}{T}t, & -\frac{T}{4} \le t < \frac{T}{4} \\ -\frac{4A}{T}\left(t - \frac{T}{2}\right), & \frac{T}{4} \le t < \frac{T}{2}. \end{cases} \tag{22}$$

Obviously, $x(-t) = -x(t)$, and so we are left with an odd signal, which means we only need to find $\{b_m\}$, since $\{a_n = 0\}$. Finally, since it spends as much time above as below zero, $a_0 = 0$.

[3]No!

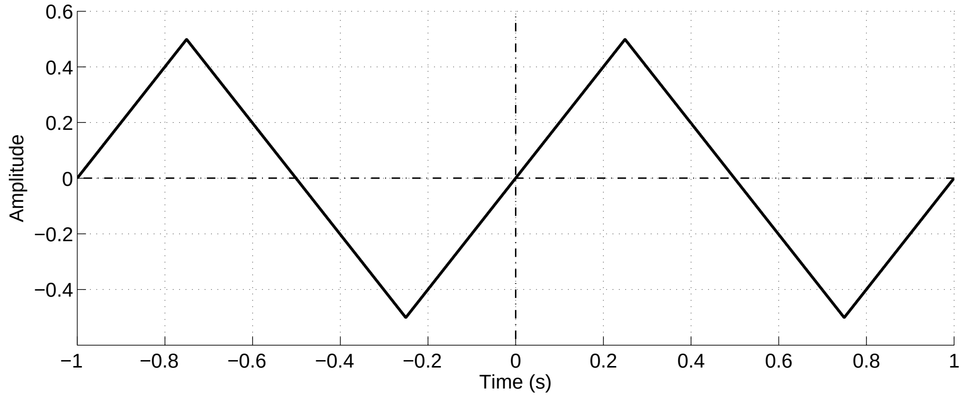


Figure 5: A triangle wave of period 1 s and amplitude $A = 0.5$.

Now, substituting in for $x(t)$, and expanding everything to hell, we see

$$\begin{aligned}
b_m = \frac{2}{T}\int_{-\frac{T}{2}}^{-\frac{T}{4}} -\frac{4A}{T}\left(t+\frac{T}{2}\right)\sin\left(\frac{2\pi m}{T}t\right)dt + \frac{2}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \frac{4A}{T}t\sin\left(\frac{2\pi m}{T}t\right)dt \\
+ \frac{2}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} -\frac{4A}{T}\left(t-\frac{T}{2}\right)\sin\left(\frac{2\pi m}{T}t\right)dt \\
= \frac{2}{T}\int_{-\frac{T}{2}}^{-\frac{T}{4}} -\frac{4A}{T}t\sin\left(\frac{2\pi m}{T}t\right)dt + \frac{2}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \frac{4A}{T}t\sin\left(\frac{2\pi m}{T}t\right)dt + \frac{2}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} -\frac{4A}{T}t\sin\left(\frac{2\pi m}{T}t\right)dt \\
+ \frac{2A}{\pi m}\cos\left(\frac{2\pi m}{T}t\right)\Bigg|_{-\frac{T}{2}}^{-\frac{T}{4}} - \frac{2A}{\pi m}\cos\left(\frac{2\pi m}{T}t\right)\Bigg|_{\frac{T}{4}}^{\frac{T}{2}} \\
= \frac{16A}{T^2}\int_0^{\frac{T}{4}} t\sin\left(\frac{2\pi m}{T}t\right)dt - \frac{16A}{T^2}\int_{\frac{T}{4}}^{\frac{T}{2}} t\sin\left(\frac{2\pi m}{T}t\right)dt + \frac{4A}{\pi m}\left[\cos\frac{\pi m}{2} - \cos\pi m\right] \qquad (23)
\end{aligned}$$

since $t\sin t$ is an even function. We have solved these integrals in the sawtooth case above, whereby we see

$$\begin{aligned}
\int_0^{\frac{T}{4}} t\sin\left(\frac{2\pi m}{T}t\right)dt = \cancelto{-\frac{T^2}{8\pi m}\cos\frac{\pi m}{2}}{-t\frac{T}{2\pi m}\cos\left(\frac{2\pi m}{T}t\right)\Bigg|_0^{\frac{T}{4}}} + \int_0^{\frac{T}{4}} \frac{T}{2\pi m}\cos\left(\frac{2\pi m}{T}t\right)dt \\
= -\frac{T^2}{8\pi m}\cos\frac{\pi m}{2} + \left(\frac{T}{2\pi m}\right)^2\sin\left(\frac{2\pi m}{T}t\right)\Bigg|_0^{\frac{T}{4}} \\
= -\frac{T^2}{8\pi m}\cos\frac{\pi m}{2} + \left(\frac{T}{2\pi m}\right)^2\sin\frac{\pi m}{2} \qquad (24)
\end{aligned}$$

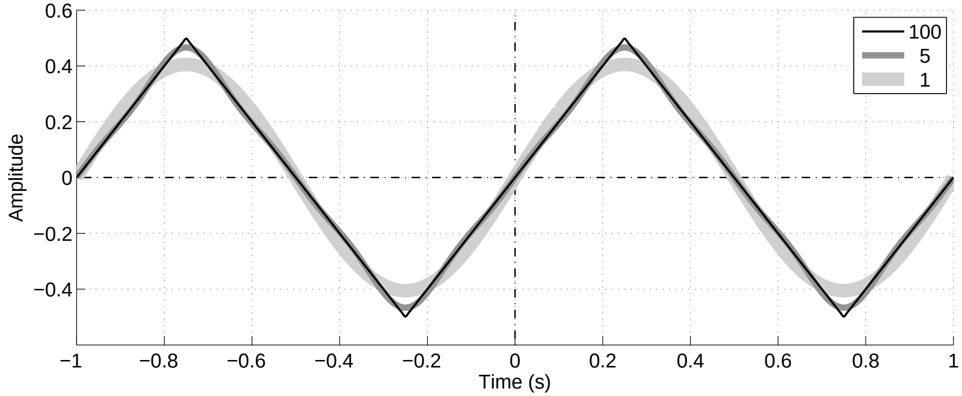


Figure 6: Fourier series reconstructions of the triangle wave in Fig. 5.

$$\int_{\frac{T}{4}}^{\frac{T}{2}} t \sin\left(\frac{2\pi m}{T} t\right) dt = -t \frac{T}{2\pi m} \cos\left(\frac{2\pi m}{T} t\right)\Big|_{\frac{T}{4}}^{\frac{T}{2}} + \int_{\frac{T}{4}}^{\frac{T}{2}} \frac{T}{2\pi m} \cos\left(\frac{2\pi m}{T} t\right) dt$$
$$= -\frac{T^2}{4\pi m} \cos \pi m + \frac{T^2}{8\pi m} \cos \frac{\pi m}{2} + \left(\frac{T}{2\pi m}\right)^2 \sin\left(\frac{2\pi m}{T} t\right)\Big|_{\frac{T}{4}}^{\frac{T}{2}}$$
$$= -\frac{T^2}{4\pi m} \cos \pi m + \frac{T^2}{8\pi m} \cos \frac{\pi m}{2} - \left(\frac{T}{2\pi m}\right)^2 \sin \frac{\pi m}{2}. \quad (25)$$

Putting it all together, we see

$$b_m = \frac{4A}{\pi m}\left[\cos\frac{\pi m}{2} - \cos \pi m\right] + \frac{16A}{T^2}\left[-\frac{T^2}{8\pi m} \cos\frac{\pi m}{2} + \left(\frac{T}{2\pi m}\right)^2 \sin\frac{\pi m}{2}\right]$$
$$- \frac{16A}{T^2}\left[-\frac{T^2}{4\pi m} \cos \pi m + \frac{T^2}{8\pi m} \cos\frac{\pi m}{2} - \left(\frac{T}{2\pi m}\right)^2 \sin\frac{\pi m}{2}\right] \quad (26)$$

and after nearly everything cancels, we finally get the simple rather stupid expression

$$b_m = \frac{8A}{\pi^2 m^2} \sin\frac{\pi m}{2}. \quad (27)$$

With all that, we see that we just need to add up a bunch of sine waves as follows:

$$x(t) = \sum_{m=1}^{\infty} \frac{8A \sin \pi m/2}{\pi^2 m^2} \sin\frac{2\pi m}{T} t. \quad (28)$$

Fourier reconstructions of the triangle wave are shown in Fig. 6.

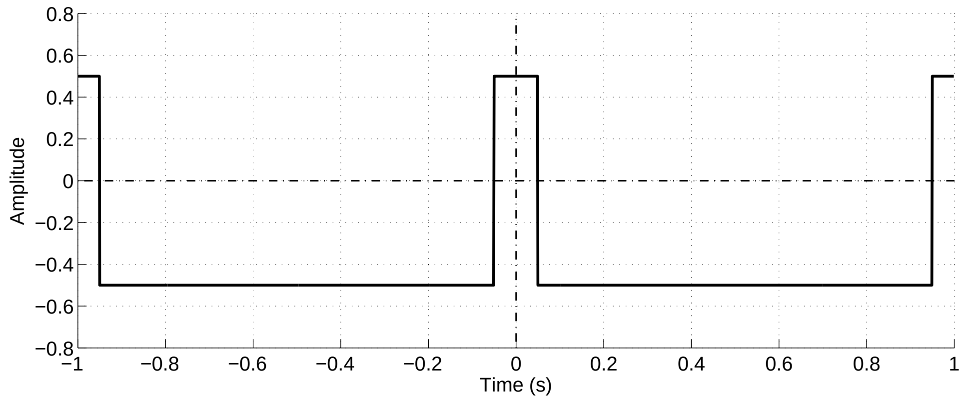


Figure 7: A square pulse of period 1 s, amplitude $A = 0.5$, and a pulsewidth of $\tau = 0.1$ s.

4 Even τ-pulse

Figure 7 shows two periods of a square pulse wave with a pulse width τ, and amplitude A. We define this signal generally over one period as

$$x(t) := \begin{cases} A, & 0 \leq t < \tau/2 \\ -A, & \tau/2 \leq t < T - \tau/2 \\ A, & T - \tau/2 \leq t < T. \end{cases} \tag{29}$$

Since $x(t) = x(-t)$, this is an even signal, and so all weights on the sine terms $\{b_m\}$ will be zero. But, does this signal spend as much time above as below zero?

$$\begin{aligned} a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\int_0^{\tau/2} Adt + \frac{1}{T}\int_{\tau/2}^{T-\tau/2} (-A)dt + \frac{1}{T}\int_{T-\tau/2}^{T} Adt \\ = \frac{1}{T}A\tau/2 - \frac{1}{T}A(T-\tau) + \frac{1}{T}A\tau/2 = A\left(\frac{2\tau}{T} - 1\right). \end{aligned} \tag{30}$$

Note what happens as $\tau \to T/2$, which is where the signal spends as much time above as below zero. The value of $a_0 \to 0$, which means we have a sensible answer!

So, now we bite our tounge and calculate $\{a_n\}$:

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T x(t)\cos\left(\frac{2\pi n}{T}t\right)dt \\ &= \frac{2}{T}\int_0^{\tau/2} A\cos\left(\frac{2\pi n}{T}t\right)dt + \frac{2}{T}\int_{\tau/2}^{T-\tau/2} (-A)\cos\left(\frac{2\pi n}{T}t\right)dt + \frac{2}{T}\int_{T-\tau/2}^{T} A\cos\left(\frac{2\pi n}{T}t\right)dt. \end{aligned} \tag{31}$$

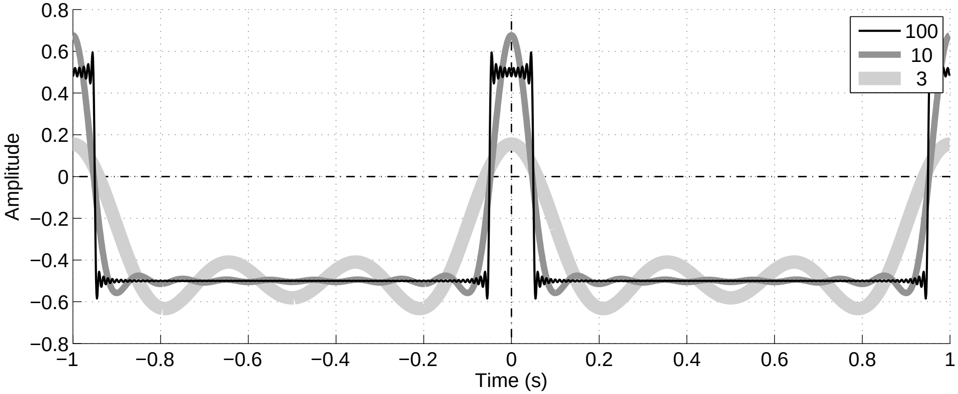


Figure 8: Fourier series reconstructions of the square pulse wave in Fig. 7.

Since there is no need to use integration by parts here, we see

$$a_n = \cancel{\frac{2}{T}\frac{T}{2\pi n}A\sin\left(\frac{2\pi n}{T}t\right)\Bigg|_0^{\frac{\tau}{2}}} \rightarrow \frac{A}{\pi n}\sin\left(\frac{\pi n\tau}{T}\right) + \frac{2}{T}\frac{T}{2\pi n}(-A)\sin\left(\frac{2\pi n}{T}t\right)\Bigg|_{\frac{\tau}{2}}^{T-\frac{\tau}{2}}$$

$$+ \cancel{\frac{2}{T}\frac{T}{2\pi n}A\sin\left(\frac{2\pi n}{T}t\right)\Bigg|_{T-\frac{\tau}{2}}^{T}} \rightarrow \frac{A}{\pi n}\sin\left(\frac{\pi n\tau}{T}\right). \tag{32}$$

Now we see

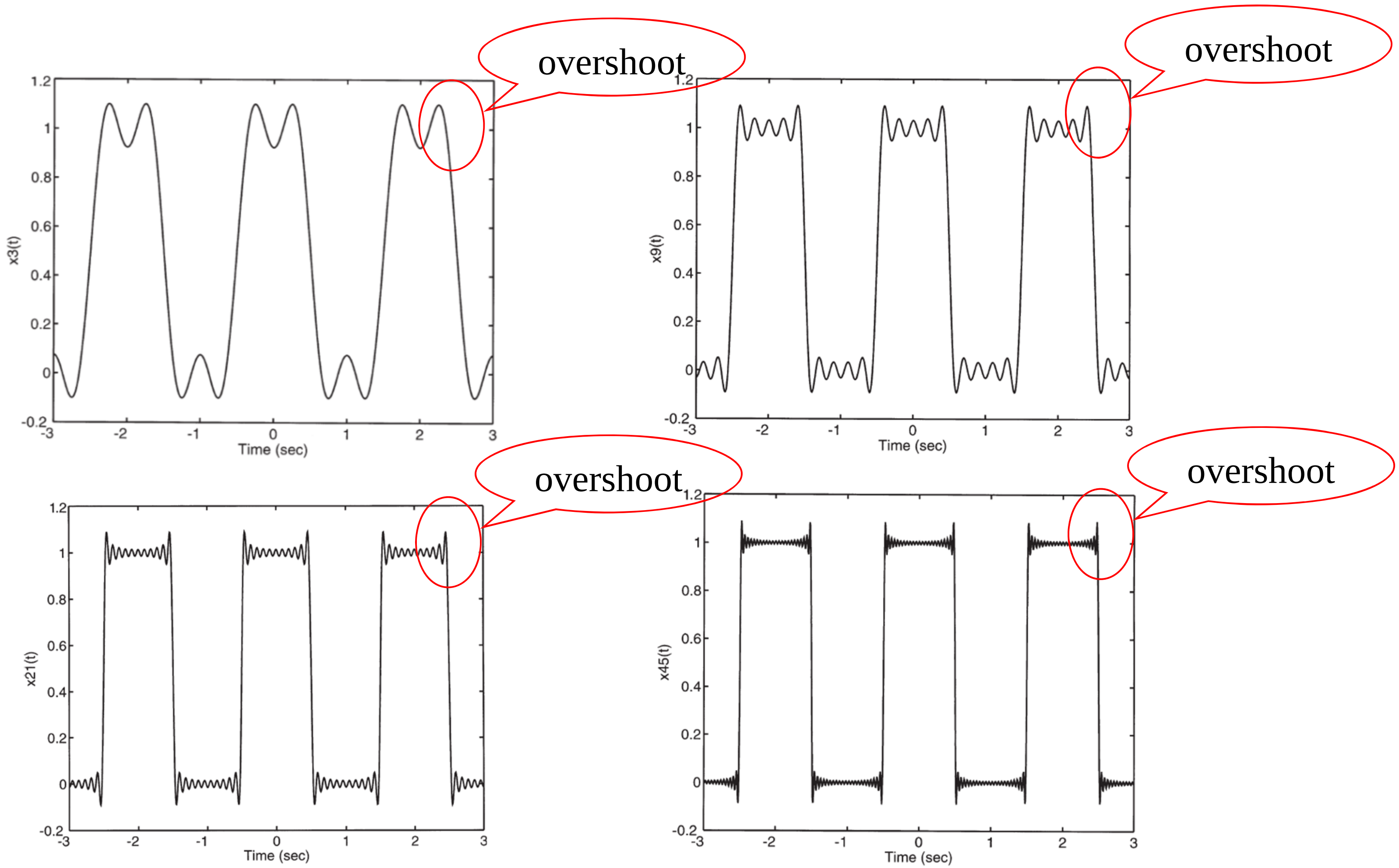
$$\begin{aligned} a_n &= 2\frac{A}{\pi n}\sin\left(\frac{\pi n\tau}{T}\right) - \frac{A}{\pi n}\left[\sin\left(\frac{2\pi n(T-\tau/2)}{T}\right) - \sin\left(\frac{\pi n\tau}{T}\right)\right] \\ &= 2\frac{A}{\pi n}\sin\left(\frac{\pi n\tau}{T}\right) - \frac{A}{\pi n}\left[\sin\left(2\pi n - \frac{\pi n\tau}{T}\right) - \sin\left(\frac{\pi n\tau}{T}\right)\right] = \frac{4A}{\pi n}\sin\left(\frac{\pi n\tau}{T}\right) \end{aligned} \tag{33}$$

Finally, to reconstruct the square pulse wave with period T and pulse width τ, we only need

$$x(t) = A\left(\frac{2\tau}{T} - 1\right) + \sum_{n=1}^{\infty}\frac{4A}{\pi n}\sin\left(\frac{\pi n\tau}{T}\right)\cos\frac{2\pi n}{T}t. \tag{34}$$

Figure 8 shows the results for three different numbers of cosine waves.

Gibbs Phenomenon



Gibbs Phenomenon

- **The overshoot at the corners is still present even in the limit as N approaches to infinity. This characteristic was first discovered by Josiah Willard Gibbs (1893-1903), and this overshoot is referred to as the *Gibbs phenomenon***

- **Now let x(t) be an arbitrary periodic signal. As a consequence of the Gibbs phenomenon, the Fourier series representation of x(t) is not actually equal to the true value of x(t) at any points where x(t) is discontinuous**

- **If x(t) is discontinuous at $t = t_1$, the Fourier series representation is off by approximately 9% at t_1^+ and t_1^-**