

EBU4375 Signals and Systems Theory

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Convolution (CT Systems)

Convolution (CT Systems) – Recap

The impulse-response (IR) $h(t)$ of a continuous-time LTI system is defined to be the response following excitation by the signal $\delta(t)$ i.e.

$$h(t) = T\{\delta(t)\}$$

The diagram illustrates the definition of the impulse response $h(t)$ as the system response to an impulse excitation $\delta(t)$ via the transfer function T . The equation $h(t) = T\{\delta(t)\}$ is centered at the top. Three arrows point from descriptive text below to the components of the equation: an arrow from the left points to $h(t)$ with the label "system response to the impulse"; an arrow from the bottom points to T with the label "Transfer Function"; and an arrow from the right points to $\delta(t)$ with the label "“impulse” excitation".

system response to the impulse

Transfer Function

“impulse” excitation

Convolution (CT Systems) – Recap

Recall the earlier result that a general signal could be expressed by

$$x(t) = \int_{-\infty}^{\infty} d\tau x(\tau) \delta(t - \tau)$$

Since the system is **linear**, the response $y(t)$ to an excitation $x(t)$ can be written as

$$\begin{aligned} y(t) = T\{x(t)\} &= T\left\{\int_{-\infty}^{\infty} d\tau x(\tau) \delta(t - \tau)\right\} \\ &= \int_{-\infty}^{\infty} d\tau x(\tau) T\{\delta(t - \tau)\} \end{aligned} \quad (1)$$

Time-invariance implies

$$h(t - \tau) = T\{\delta(t - \tau)\} \quad (2)$$

$$(2) \rightarrow (1) \Rightarrow y(t) = \int_{-\infty}^{\infty} d\tau x(\tau) h(t - \tau) \quad (3)$$

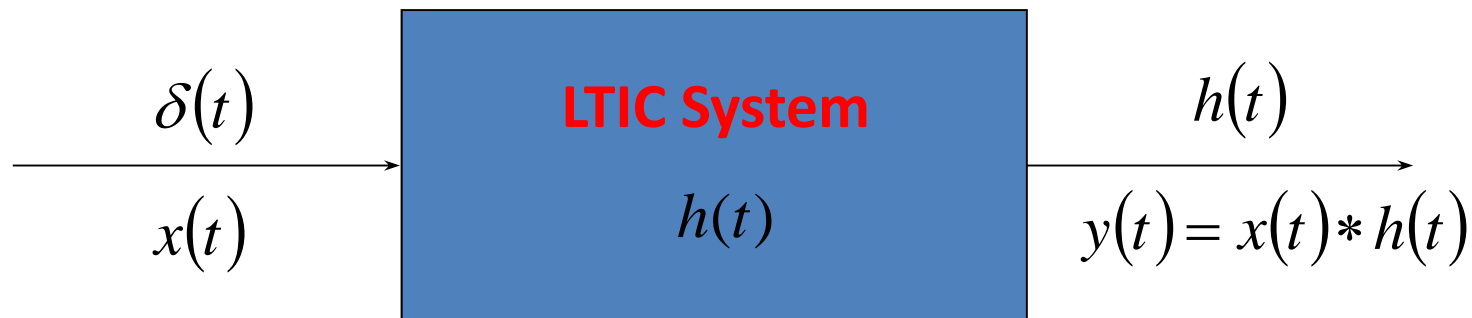
(3) says that the continuous-time response of an LTI system is entirely characterised by its impulse response $h(t)$.

Convolution (CT Systems) – Recap

Equation (3) defines the convolution operation, i.e.

$$y(t) = x(t) * h(t) \equiv \int_{-\infty}^{\infty} d\tau x(\tau) h(t - \tau) \quad (4)$$

so that (4) is the *convolution integral*.



Convolution (CT Systems) – General Formula

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

Convolution (CT Systems) – Example 1

To find the output of the system with impulse response $h(t) = e^{-2t}u(t)$

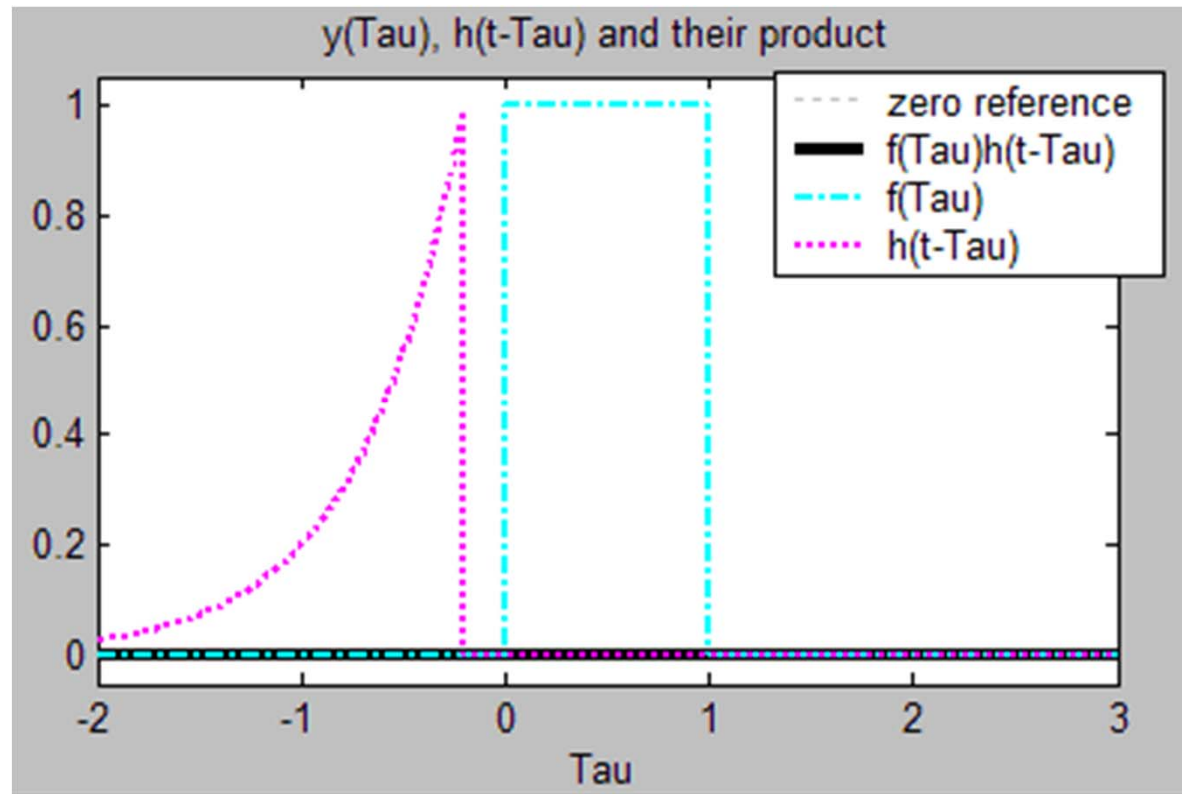
to the input $f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$

we will use the convolution integral $y(t) = \int_{-\infty}^{\infty} f(\tau)h(t - \tau)d\tau$

Because the input function has three distinct regions $t < 0$, $0 < t < 1$ and $t > 1$, we will need to split up the integral into three parts.

Convolution (CT Systems) – Example 1

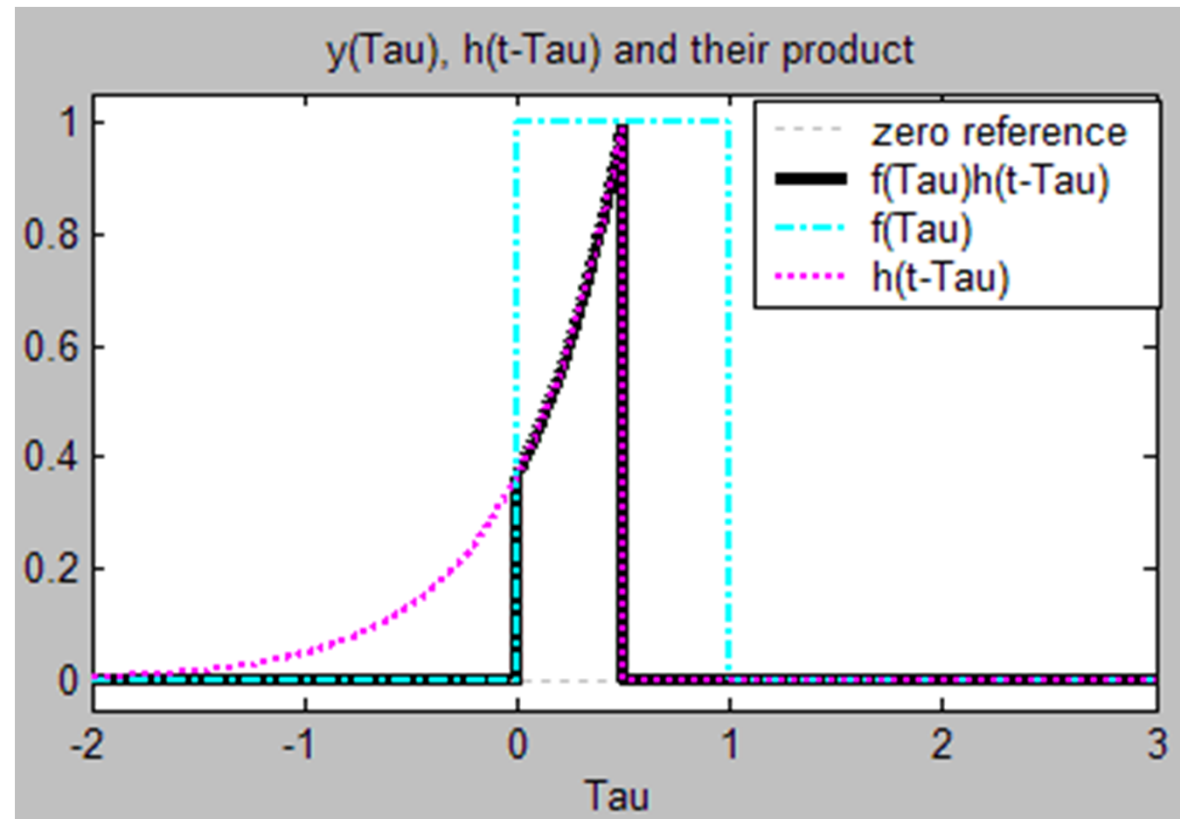
Part I: $t < 0$



Therefore, the result for the first part of our solution is $y(t) = 0 \quad t < 0$

Convolution (CT Systems) – Example 1

Part II: $0 < t < 1$



Convolution (CT Systems) – Example 1

Part II: $0 < t < 1$

We can now evaluate the integral of the solid black line:

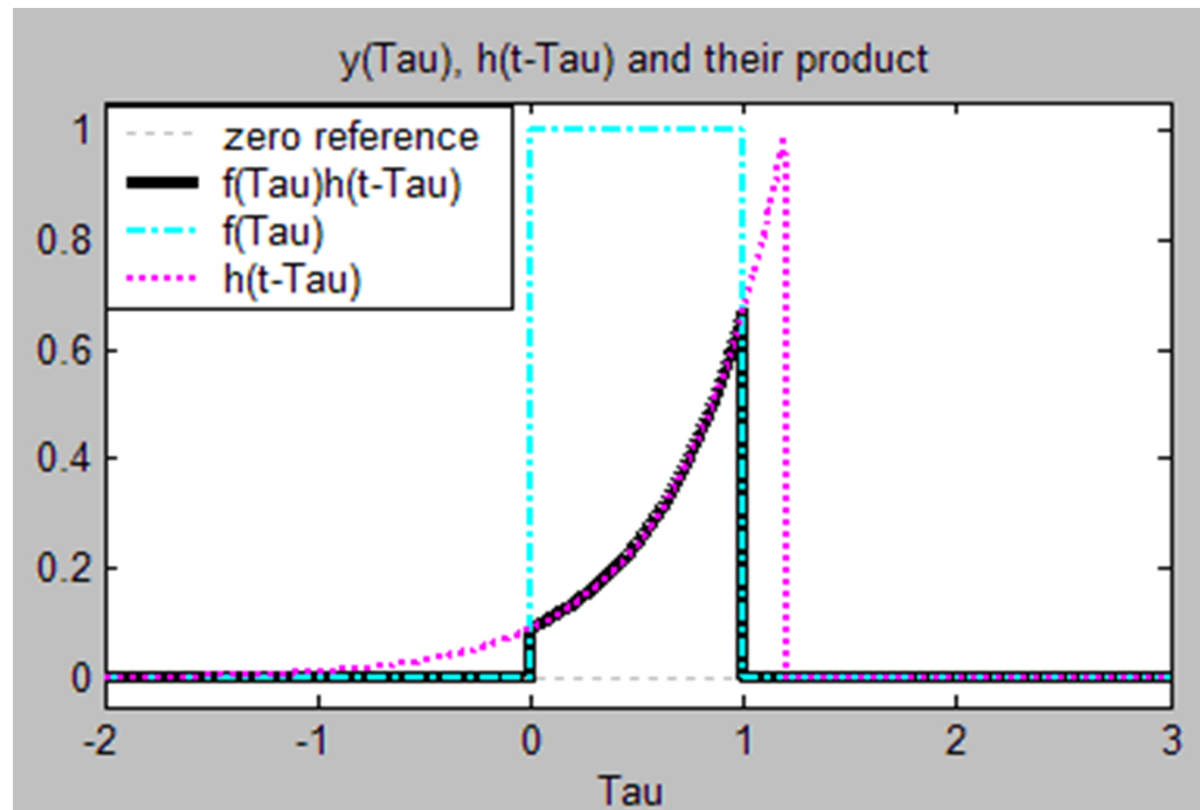
$$\begin{aligned} y(t) &= \int_0^{\infty} f(\tau)h(t-\tau)d\tau = \int_0^t f(\tau)h(t-\tau)d\tau \\ &= \int_0^t (1)(e^{-2(t-\tau)})d\tau = e^{-2t} \int_0^t e^{2\tau} d\tau \\ &= e^{-2t} \left(\frac{1}{2} e^{2\tau} \Big|_0^t \right) = \frac{1}{2} (1 - e^{-2t}) \end{aligned}$$

As such, the result for the second part of our solution is

$$y(t) = \frac{1}{2} (1 - e^{-2t}) \quad 0 < t < 1$$

Convolution (CT Systems) – Example 1

Part III: $t > 1$



Convolution (CT Systems) – Example 1

Part III: $t > 1$

We can now evaluate the integral:

$$\begin{aligned} y(t) &= \int_0^{\infty} f(\tau)h(t-\tau)d\tau = \int_0^1 f(\tau)h(t-\tau)d\tau \\ &= \int_0^1 (1)(e^{-2(t-\tau)})d\tau = e^{-2t} \int_0^1 e^{2\tau} d\tau \\ &= e^{-2t} \left(\frac{1}{2} e^{2\tau} \Big|_0^1 \right) = e^{-2(t-1)} \frac{1}{2} (1 - e^{-2}) \end{aligned}$$

Thus, the result for the third part of our solution is

$$y(t) = e^{-2(t-1)} \frac{1}{2} (1 - e^{-2}) \quad t > 1$$

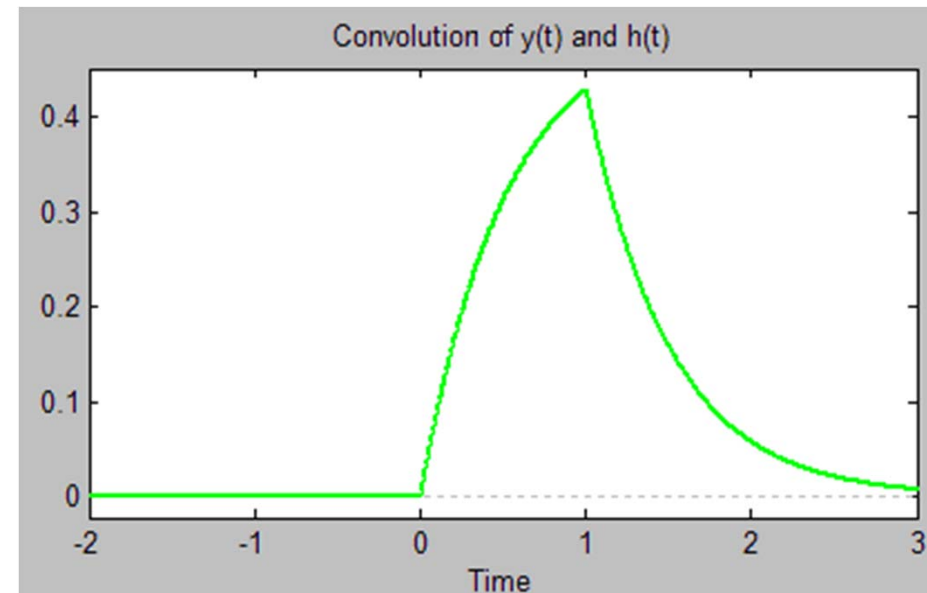
Convolution (CT Systems) – Example 1

The complete answer:

$$y(t) = 0 \quad t < 0$$

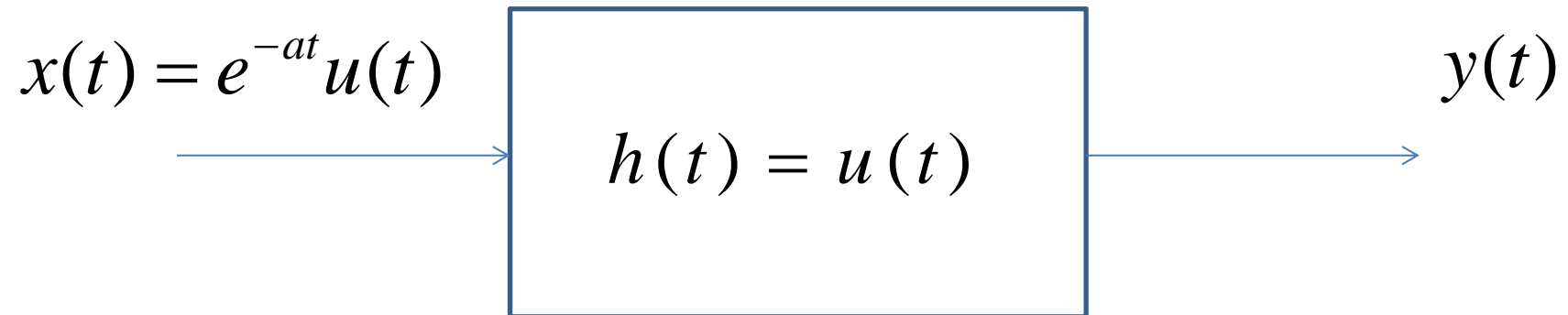
$$y(t) = \frac{1}{2}(1 - e^{-2t}) \quad 0 < t < 1$$

$$y(t) = e^{-2(t-1)} \frac{1}{2}(1 - e^{-2}) \quad t > 1$$



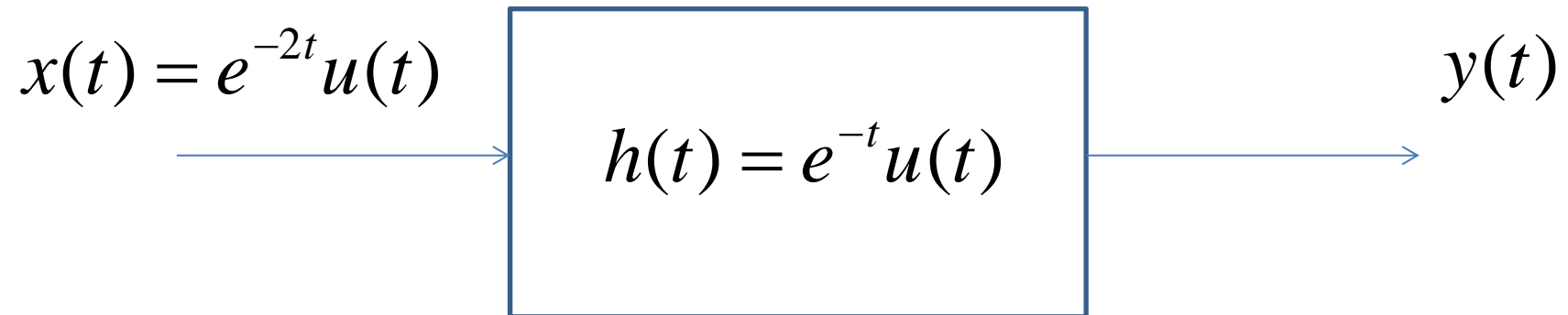
Convolution (CT Systems) – Example 2

Find the output of the following system:



Convolution (CT Systems) – Example 3

Find the output of the following system:



Convolution (CT Systems) – Example 4

Find the output of the following system:

