EBU6503
Control Theory

Laplace Transform

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The definition of Laplace Transform

The Laplace transform $X(s)$ of the time-domain signal $x(t)$ is defined by the integral

$$X(s) = \int_{0}^{\infty} x(t) e^{-st} dt$$

The integral differs from the Fourier transform in two main ways:

1. The integration is from $t = 0$ to $t = \infty$. (It is causal).
1. The Laplace integral contains the factor e^{-st} (rather than $e^{-j\omega t}$ as in the Fourier transform).

The Laplace variable s

The Laplace variable s is a complex number, where $s = \sigma + j\omega$.

Hence, $e^{-st} = e^{-(\sigma + j\omega)t} = e^{-\sigma t} e^{-j\omega t}$

The term $e^{-j\omega t}$ is identical to the term in the Fourier integral.

Depending on the value and sign of σ , the factor represents a growing or decaying exponential.

The product $e^{-\sigma t} e^{-j\omega t}$, therefore represents an exponentially growing or decaying complex frequency component.

$$e^{-st} = e^{-(\sigma + j\omega)t} = e^{-\sigma t} e^{-j\omega t} = e^{-\sigma t}(\cos \omega t - j \sin \omega t)$$

The Laplace variable $s = \sigma + j\omega$ is sometimes called the complex frequency. The s plane is called the complex frequency domain.

The Laplace transform pairs

As with Fourier transforms, the relationship between the time-domain model of a signal and its Laplace transform is unique.

A signal and its Laplace transform form a transform pair, which is denoted by the double-headed arrow

$$x(t) \leftrightarrow X(s)$$

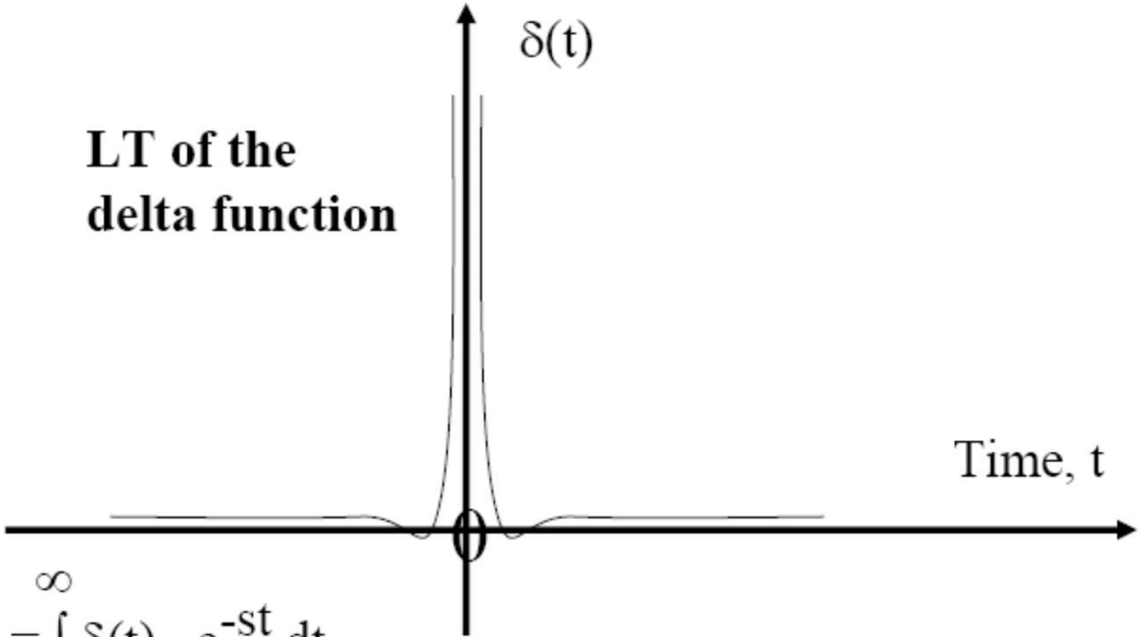
Example:

Work out the LT of the signal $x(t) = Ae^{-\alpha t}$.

The resulting transform pair: $Ae^{-\alpha t} \xleftrightarrow{LT} \dfrac{A}{s+\alpha}$

This provides a starting point from which the transforms of many other functions can be worked out.

The Laplace transform pairs



$$X(s) = \int_0^\infty \delta(t).\ e^{-st}\, dt$$

$$X(s) = \int_0^0 \delta(t).e^{-s0}\, dt = \int_0^0 \delta(t).1\, dt = 1$$

$$\boxed{\boldsymbol{\delta(t) \longleftrightarrow 1}}$$

Application of the LT to sinusoids

Try the signal is a cosine $\quad x(t) = A\cos(\omega t)$

$$X(s) = \int_0^\infty A\cos(\omega t).\ e^{-st}\, dt$$

$$X(s) = A\int_0^\infty [e^{j\omega t} + e^{-j\omega t}]/2\ .\ e^{-st}\, dt$$

$$X(s) = (A/2)\int_0^\infty [e^{j\omega t}\, e^{-st} + e^{-j\omega t}\, e^{-st}]\, dt$$

but these ----------- -----------

are standard results for the exponential, and allow solutions for sinusoids (via the LT) that are rigorous enough to satisfy everyone.

Application of the LT to sinusoids

$A.e^{-\alpha t} \longleftrightarrow A/(\alpha + s)$, and $A.e^{+\alpha t} \longleftrightarrow A/(s - \alpha)$

apply to:

$$X(s) = (A/2)\int_0^{\infty} [e^{j\omega t} e^{-st} + e^{-j\omega t} e^{-st}]\, dt$$

$$X(s) = (A/2)\,[1/(s - j\omega) + 1/(s + j\omega)]$$

$$X(s) = (A/2)\,\frac{[s + j\omega + s - j\omega]}{[(s - j\omega).(s + j\omega)]} = (A/2)\,\frac{2s}{[s^2 - j\omega s - j^2\omega^2 + j\omega s]}$$

$$X(s) = (\cancel{2}As/\cancel{2}).[1/(s^2 + \omega^2)] = As/(s^2 + \omega^2)$$

Class exercise: Try the signal is a sine $x(t) = A\sin(\omega t)$

Using standard LT results

If we have e.g.

$X(t) = (2/3)u(t) - e^{-2t} + (1/3)e^{-3t}$

we can take the LT directly:

$X(s) = (2/3) \int u(t).e^{-st}\, dt \;-\; \int e^{-2t}.e^{-st}\, dt \;+\; (1/3) \int e^{-3t}\, e^{-st}\, dt$

but also note these are standard results:

$X(s) = (2/3)1/s \;-\; 1/(s + 2) + (1/3).(1 / (s + 3))$

and this is the quicker way to the answer.

Properties of the Laplace transform

- Linearity
- Right shift in time
- Time Scaling
- Multiplication by a Power of t
- Multiplication by an Exponential
- Multiplication by a Sinusoid
- Differentiation in the Time Domain
- Integration
- Convolution

Linearity

The Laplace transform is a linear operation; that is, if $x(t) \leftrightarrow X(s)$ and $v(t) \leftrightarrow V(s)$

, then for any real or complex scalars a, b,

$$ax(t) + bv(t) \leftrightarrow aX(s) + bV(s)$$

Right Shift in Time

If $x(t) \leftrightarrow X(s)$, then for any positive or negative real number c,

$$x(t-c)u(t-c) \leftrightarrow e^{-cs}X(s)$$

The equation above is equal to the c-second right shift of $x(t)u(t)$

Why multiply $x(t)$ by $u(t)$?

A c-second right shift in the time domain corresponds to multiplication by in the Laplace transform domain (or s-domain).

Differentiation in the Time Domain

If $x(t) \leftrightarrow X(s)$, then

$$\dot{x}(t) \leftrightarrow sX(s) - X(0)$$

Where $\dot{x}(t) = dx(t)/dt$. Thus, differentiation in the time domain corresponds to multiplication by s in the Laplace transform domain [plus subtraction of the initial value $x(0)$].

By computing the transform of the derivative of x(t) to prove this property

$$\int_0^\infty \dot{x}(t)\, e^{-st}\, dt$$

The integral can be computed "by parts" as follows: with $v = e^{-st}$ and $\omega = x(t)$

$dv = -se^{-st}$ and $d\omega = [dx(t)/dt] = \dot{x}(t)$ Then……..

Differentiation in the Time Domain

$$\int_0^\infty \dot{x}(t)\,e^{-st}dt = vw\Big|_{t=0}^{t=\infty} - \int_0^\infty w\,dv$$

$$= e^{-st}x(t)\Big|_{t=0}^{t=\infty} - \int_0^\infty x(t)(-s)e^{-st}dt$$

$$= \lim_{t\to\infty}[e^{-st}x(t)] - x(0) + sX(s)$$

$$\lim_{t\to\infty}[e^{-st}x(t)] \rightarrow 0$$

Thus, $\int_0^\infty \dot{x}(t)\,e^{-st}dt = sX(s) - x(0)$

It should be pointed out that if x(t) is discontinuous at t=0, it is necessary to take the initial time in above equation to be at 0^- (an infinitesimally small negative number). In other words, the transform pair becomes $\dot{x}(t) = sX(s) - x(0^-)$

Note that if x(t)=0 for t<0, then $x(0^-)=0$ and $\dot{x}(t) = sX(s)$

Integration

If $x(t) \leftrightarrow X(s)$, then $\int_0^t x(\lambda)d\lambda \leftrightarrow \frac{1}{s}X(s)$

By above property, the Laplace transform of the integral of x(t) is equal to X(s) divided by s. This transform pair follows directly for the derivative property. To see this, let v(t) denote the integral of x(t) given by

$$v(t) = \begin{cases} \int_0^t x(\lambda)d\lambda, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Then $x(t) = \dot{v}(t)$ for t>0, and since v(t)=0 for t<0, by the derivative transform pair $X(s) = sV(s)$.

Therefore, $V(s) = \frac{1}{s}X(s)$

The inverse Laplace transform

Like the Fourier transform the Laplace transform has an inverse which is formally defined by an integral

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\,ds$$

The integral in above equation is evaluated along the path $s = c + j\omega$ in the complex plane from $c - j\infty$ to $c + j\infty$.

The evaluation of this integral requires a knowledge of contour integration and complex variable theory and is seldom if ever used in routine Laplace transform work.

A frequently-used method is the partial-fraction technique.

The inverse Laplace transform

The method of partial fractions relies on the fact that finding the Laplace transform or its inverse is a linear operation.

1. Split up a transform into a sum of simpler transforms

2. Find the inverse of the overall transform by finding the inverse of each simpler function separately.

3. Add them together

Extensive tables of Laplace transform pairs have been prepared by various authors to remove the need to work out a transform or its inverse from first principles.

A short table of Laplace transform pairs

A short table of some of the more commonly used transform pairs is listed:

Time function $x(t)$	**Laplace transform** $X(s)$
$\delta(t)$	1
$u(t)$	$1/s$
t	$1/s^2$
t^n	$n!/s^{n+1}$
$e^{-\alpha t}$	$1/(s+\alpha)$
$te^{-\alpha t}$	$1/(s+\alpha)^2$
$te^{-\alpha t}/n!$	$1/(s+\alpha)^{n+1}$
$\sin \omega t$	$\omega/(s^2+\omega^2)$
$\cos \omega t$	$s/(s^2+\omega^2)$
$e^{-\alpha t}\sin \omega t$	$\omega/[(s+\alpha)^2+\omega^2]$
$e^{-\alpha t}\cos \omega t$	$(s+\alpha)/[(s+\alpha)^2+\omega^2]$

Method of partial fractions - example 1

If $X(s) = (s + 1) / (s (s + 2))$

find the original signal, $x(t)$.

$RHS = (s + 1) / (s (s + 2)) = A/s + B/(s + 2)$

where we need to find A and B.

$(s + 1) / (s (s + 2)) = A/s + B/(s + 2)$

now multiply by $s(s + 2)$ gives:

$(s + 1) = A(s + 2) + Bs$

$(s + 1) = A(s + 2) + Bs$

let $s = 0$

$1 = 2A$ therefore $A = 1/2$

let $s = -2$

$-1 = -2B$ therefore $B = 1/2$

therefore:

$x(t) = (1/2).u(t) + (1/2).e^{-2t}$

Method of using standard results

If $X(s) = 1 / (s + 4)^3$

find the original signal, $x(t)$.
In the table of standard results we have that:

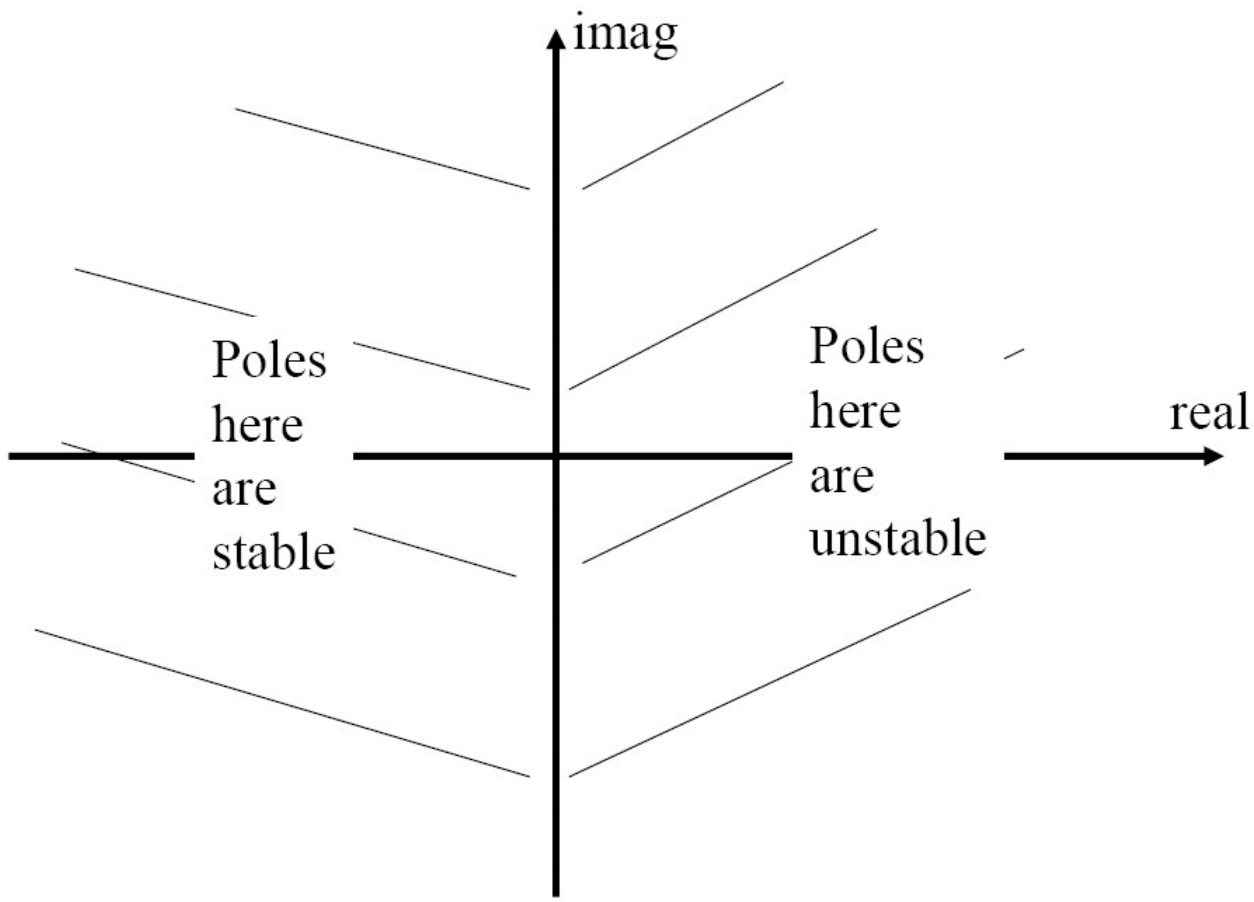
$$1 / (s + a)^{n+1} \longleftrightarrow t^n \, . \, e^{-at} / n!$$

$$LT^{-1}\{1 / (s + 4)^3\} = t^2 \, . \, e^{-4t} / 2!$$

$$= t^2/2 \, . \, e^{-4t}$$

There are many such standard results that can be used to invert the LT. Not quite as easy as the Z Transform though …

Pole zero stability rule for LT



Example on Pole zero stability rule for LT

Is our previous example a ‘stable’ signal?

$$X(s) = \frac{s+4}{(s.(s+2).(s+3))}$$

poles are at s = 0, s = -2, s = -3

so it has one pole on the edge, which is consistent with the fact that the signal contains a term, u(t), which does not die away with time.

The transfer function for CTS

We know that

$$y(t) = x(t) * h(t)$$

the output signal

input signal

Unit impulse response

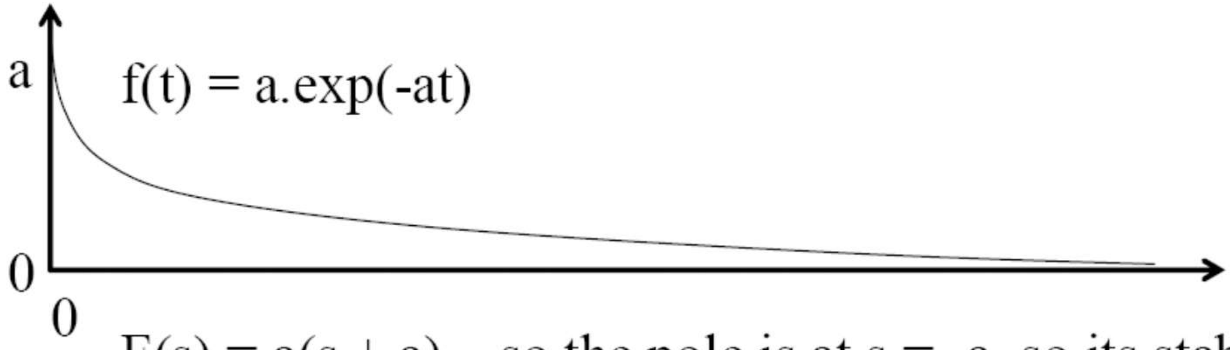
The convolution in the 'continuous time domain' can be replaced by the multiplication in s-transform domain.

$$Y(s) = X(s) \cdot H(s) \qquad H(s) = \frac{Y(s)}{X(s)}$$

$H(s)$ is defined as transfer function.

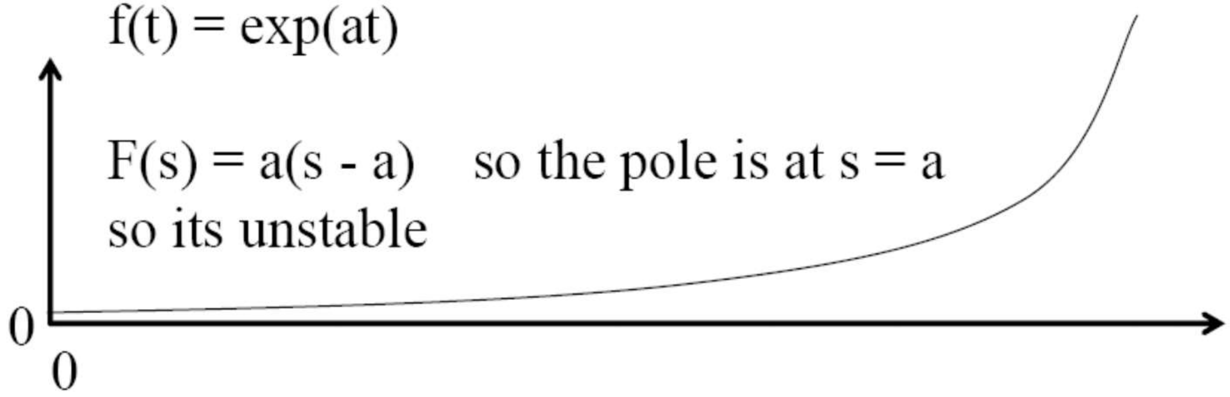
$$H(s) \leftrightarrow h(t)$$

Consider certain signals:

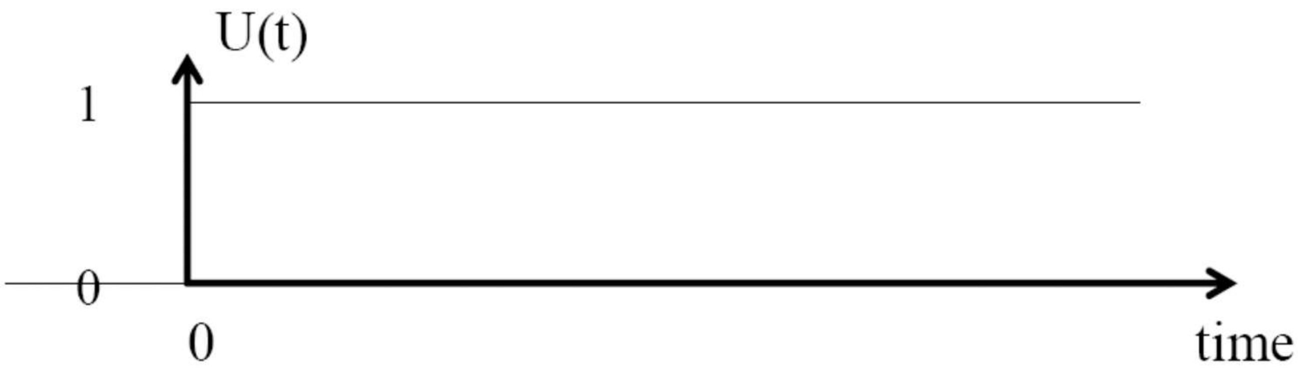


F(s) = a(s + a)   so the pole is at s = -a so its stable!

f(t) = exp(at)



Consider certain signals:

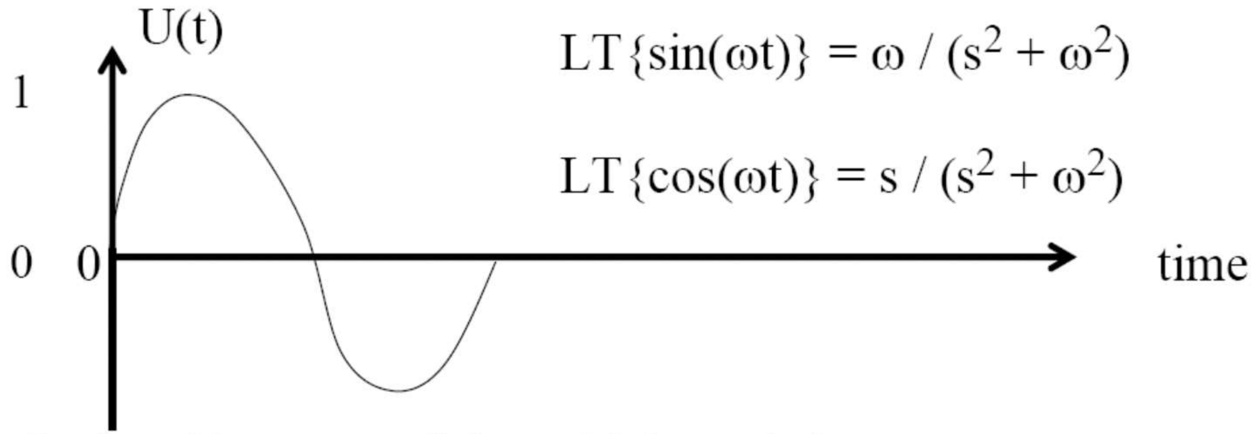


If $x(t) = u(t)$, as above, then $X(s) = 1/s$

so the single pole is at $s = 0$

this is right on the border between stable and unstable and this is what you would expect since $u(t)$ neither grows or decays with increasing time.

Consider certain signals:



$LT\{\sin(\omega t)\} = \omega / (s^2 + \omega^2)$

$LT\{\cos(\omega t)\} = s / (s^2 + \omega^2)$

So for either type of sinusoid the pole is at:

$(s^2 + \omega^2) = 0$, which can again be solved by using the quadratic formula:

$s = [\ -0 \ +/-\ \sqrt{(0 - 4\omega^2)}\ ] / 2 = +/-\ j\omega$ this is again

right on the border between stability and instability.

Examples on evaluating stability via transfer function

1) H(s) = (s2+ 1) / (s + 1)2

2) H(s) = (s2-s + 1) / (s2+ s + 1)

3) H(s) = (s2+ 2) / (s + 1)(s2+ 0.2s + 1)