CHAPTER 2

FOURIER REPRESENTATION OF SIGNALS AND SYSTEMS

In mathematical terms, a signal is ordinarily described as a *function of time*, which is how we usually see the signal when its waveform is displayed on an oscilloscope. However, as pointed out in Chapter 1, from the perspective of a communication system it is important that we know the *frequency content* of the signal in question. The mathematical tool that relates the frequency-domain description of the signal to its time-domain description is the *Fourier transform*. There are in fact several versions of the Fourier transform available. In this chapter, we confine the discussion primarily to two specific versions:

- The *continuous Fourier transform*, or the Fourier transform (FT) for short, which works with continuous functions in both the time and frequency domains.
- The *discrete Fourier transform*, or DFT for short, which works with discrete data in both the time and frequency domains.

Much of the material presented in this chapter focuses on the Fourier transform, since the primary motivation of the chapter is to determine the frequency content of a continuous-time signal or to evaluate what happens to this frequency content when the signal is passed through a *linear time-invariant (LTI) system*. In contrast, the discrete Fourier transform, discussed toward the end of the chapter, comes into its own when the requirement is to evaluate the frequency content of the signal on a digital computer or to evaluate what happens to the signal when it is processed by a digital device as in digital communications.

The extensive material presented in this chapter teaches the following lessons:

- *Lesson 1: The Fourier transform of a signal specifies the complex amplitudes of the components that constitute the frequency-domain description or spectral content of the signal. The inverse Fourier transform uniquely recovers the signal, given its frequency-domain description.*

- *Lesson 2: The Fourier transform is endowed with several important properties, which, individually and collectively, provide invaluable insight into the relationship between a signal defined in the time domain and its frequency domain description.*

- *Lesson 3: A signal can only be strictly limited in the time domain or the frequency domain, but not both.*

- *Lesson 4: Bandwidth is an important parameter in describing the spectral content of a signal and the frequency response of a linear time-invariant filter.*

► *Lesson 5: A widely used algorithm called the fast Fourier transform algorithm provides a powerful tool for computing the discrete Fourier transform; it is the mathematical tool for digital computations involving Fourier transformation.*

2.1 *The Fourier Transform*[1]

DEFINITIONS

Let $g(t)$ denote a *nonperiodic deterministic signal*, expressed as some function of *time t*. By definition, the *Fourier transform* of the signal $g(t)$ is given by the integral

$$G(f) = \int_{-\infty}^{\infty} g(t) \exp(-j2\pi ft)\, dt \tag{2.1}$$

where $j = \sqrt{-1}$, and the variable f denotes *frequency*; the exponential function $\exp(-j2\pi ft)$ is referred to as the *kernel* of the formula defining the Fourier transform. Given the Fourier transform $G(f)$, the original signal $g(t)$ is recovered exactly using the formula for the *inverse Fourier transform:*

$$g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi ft)\, df \tag{2.2}$$

where the exponential $\exp(j2\pi ft)$ is the *kernel* of the formula defining the inverse Fourier transform. The two kernels of Eqs. (2.1) and (2.2) are therefore the complex conjugate of each other.

Note also that in Eqs. (2.1) and (2.2) we have used a lowercase letter to denote the time function and an uppercase letter to denote the corresponding frequency function. The functions $g(t)$ and $G(f)$ are said to constitute a *Fourier-transform pair*. In Appendix 2, we derive the definitions of the Fourier transform and its inverse, starting from the Fourier series of a periodic waveform.

We refer to Eq. (2.1) as the *analysis equation*. Given the time-domain behavior of a system, we are enabled to analyze the frequency-domain behavior of a system. The basic advantage of transforming the time-domain behavior into the frequency domain is that *resolution into eternal sinusoids presents the behavior as the superposition of steady-state effects*. For systems whose time-domain behavior is described by linear differential equations, the separate steady-state solutions are usually simple to understand in theoretical as well as experimental terms.

Conversely, we refer to Eq. (2.2) as the *synthesis equation*. Given the superposition of steady-state effects in the frequency-domain, we can *reconstruct the original time-domain behavior of the system without any loss of information*. The analysis and synthesis equations, working side by side as depicted in Fig. 2.1, enrich the representation of signals and

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[1]Joseph Fourier studied the flow of heat in the early 19th century. Understanding heat flow was a problem of both practical and scientific significance at that time and required solving a partial-differential equation called the heat equation. Fourier developed a technique for solving partial-differential equations that was based on the assumption that the solution was a weighted sum of harmonically related sinusoids with unknown coefficients, which we now term the *Fourier series*. Fourier's initial work on heat conduction was submitted as a paper to the Academy of Sciences of Paris in 1807 and rejected after review by Lagrange, Laplace, and Legendre. Fourier persisted in developing his ideas in spite of being criticized for a lack of rigor by his contemporaries. Eventually, in 1822, he published a book containing much of his work, *Theorie analytique de la chaleur*, which is now regarded as one of the classics of mathematics.

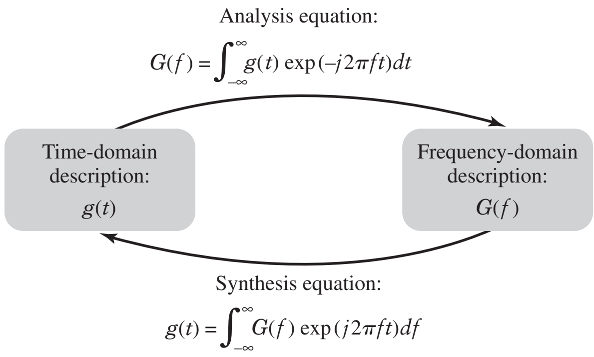


FIGURE 2.1 Sketch of the interplay between the synthesis and analysis equations embodied in Fourier transformation.

systems by making it possible to view the representation in two interactive domains: the time domain and the frequency domain.

For the Fourier transform of a signal $g(t)$ to exist, it is sufficient, but not necessary, that $g(t)$ satisfies three conditions known collectively as *Dirichlet's conditions:*

1. The function $g(t)$ is single-valued, with a finite number of maxima and minima in any finite time interval.
2. The function $g(t)$ has a finite number of discontinuities in any finite time interval.
3. The function $g(t)$ is absolutely integrable—that is,

$$\int_{-\infty}^{\infty} |g(t)|\, dt < \infty$$

We may safely ignore the question of the existence of the Fourier transform of a time function $g(t)$ when it is an accurately specified description of a physically realizable signal (e.g., voice signal, video signal). In other words, physical realizability is a sufficient condition for the existence of a Fourier transform. For physical realizability of a signal $g(t)$, the energy of the signal defined by $\int_{-\infty}^{\infty} |g(t)|^2\, dt$ must satisfy the condition

$$\int_{-\infty}^{\infty} |g(t)|^2\, dt < \infty$$

Such a signal is referred to as an energy-like signal or simply an *energy signal*. What we are therefore saying is that *all energy signals are Fourier transformable*.

■ NOTATIONS

The formulas for the Fourier transform and the inverse Fourier transform presented in Eqs. (2.1) and (2.2) are written in terms of two variables: *time* t measured in *seconds* (s) and frequency f measured in *hertz* (Hz). The frequency f is related to the *angular frequency* ω as

$$\omega = 2\pi f$$

which is measured in *radians per second* (rad/s). We may simplify the expressions for the exponents in the integrands of Eqs. (2.1) and (2.2) by using ω instead of f. However, the use of f is preferred over ω for two reasons. First, the use of frequency results in mathematical *symmetry* of Eqs. (2.1) and (2.2) with respect to each other in a natural way. Second, the spectral contents of communication signals (i.e., voice and video signals) are usually expressed in hertz.

A convenient *shorthand* notation for the transform relations of Eqs. (2.1) and (2.2) is to write

$$G(f) = \mathbf{F}[g(t)] \tag{2.3}$$

and

$$g(t) = \mathbf{F}^{-1}[G(f)] \tag{2.4}$$

where $\mathbf{F}[\,]$ and $\mathbf{F}^{-1}[\,]$ play the roles of *linear operators*. Another convenient shorthand notation for the *Fourier-transform pair*, represented by $g(t)$ and $G(f)$, is

$$g(t) \rightleftharpoons G(f) \tag{2.5}$$

The shorthand notations described in Eqs. (2.3) through (2.5) are used in the text where appropriate.

Continuous Spectrum

By using the Fourier transform operation, a pulse signal $g(t)$ of finite energy is expressed as a continuous sum of exponential functions with frequencies in the interval $-\infty$ to ∞. The amplitude of a component of frequency f is proportional to $G(f)$, where $G(f)$ is the Fourier transform of $g(t)$. Specifically, at any frequency f, the exponential function $\exp(j2\pi ft)$ is weighted by the factor $G(f)\,df$, which is the contribution of $G(f)$ in an infinitesimal interval df centered on the frequency f. Thus we may express the function $g(t)$ in terms of the continuous sum of such infinitesimal components, as shown by the integral

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)\,df$$

Restating what was mentioned previously, the Fourier transformation provides us with a tool to resolve a given signal $g(t)$ into its complex exponential components occupying the entire frequency interval from $-\infty$ to ∞. In particular, the Fourier transform $G(f)$ of the signal defines the frequency-domain representation of the signal in that it specifies complex amplitudes of the various frequency components of the signal. We may equivalently define the signal in terms of its time-domain representation by specifying the function $g(t)$ at each instant of *time* t. The signal is uniquely defined by either representation.

In general, the Fourier transform $G(f)$ is a complex function of frequency f, so that we may express it in the form

$$G(f) = |G(f)|\exp[j\theta(f)] \tag{2.6}$$

where $|G(f)|$ is called the *continuous amplitude spectrum* of $g(t)$, and $\theta(f)$ is called the *continuous phase spectrum* of $g(t)$. Here, the spectrum is referred to as a *continuous spectrum* because both the amplitude and phase of $G(f)$ are uniquely defined for all frequencies.

For the special case of a real-valued function $g(t)$, we have

$$G(-f) = G^*(f)$$

where the asterisk denotes complex conjugation. Therefore, it follows that if $g(t)$ is a *real-valued function of time* t, then

$$|G(-f)| = |G(f)|$$

and

$$\theta(-f) = -\theta(f)$$

Accordingly, we may make the following statements on the spectrum of a real-valued signal:

1. The amplitude spectrum of the signal is an even function of the frequency; that is, the amplitude spectrum is *symmetric* with respect to the origin $f = 0$.
2. The phase spectrum of the signal is an odd function of the frequency; that is, the phase spectrum is *antisymmetric* with respect to the origin $f = 0$.

These two statements are summed up by saying that the spectrum of a real-valued signal exhibits *conjugate symmetry*.

EXAMPLE 2.1 Rectangular Pulse

Consider a box function or *rectangular pulse* of duration T and amplitude A, as shown in Fig. 2.2(*a*). To define this pulse mathematically in a convenient form, we use the notation

$$\operatorname{rect}(t) = \begin{cases} 1, & -\dfrac{1}{2} \le t \le \dfrac{1}{2} \\ 0, & t < -\dfrac{1}{2} \text{ or } t > \dfrac{1}{2} \end{cases} \tag{2.7}$$

which stands for a *rectangular function* of unit amplitude and unit duration centered at $t = 0$. Then, in terms of this "standard" function, we may express the rectangular pulse of Fig. 2.2(*a*) simply as

$$g(t) = A \operatorname{rect}\left(\frac{t}{T}\right)$$

The Fourier transform of the rectangular pulse $g(t)$ is given by

$$\begin{aligned} G(f) &= \int_{-T/2}^{T/2} A \exp(-j2\pi f t)\, dt \\ &= AT\left(\frac{\sin(\pi f T)}{\pi f T}\right) \end{aligned} \tag{2.8}$$

To simplify the notation in the preceding and subsequent results, we introduce another standard function—namely, the *sinc function*—defined by

$$\operatorname{sinc}(\lambda) = \frac{\sin(\pi\lambda)}{\pi\lambda} \tag{2.9}$$

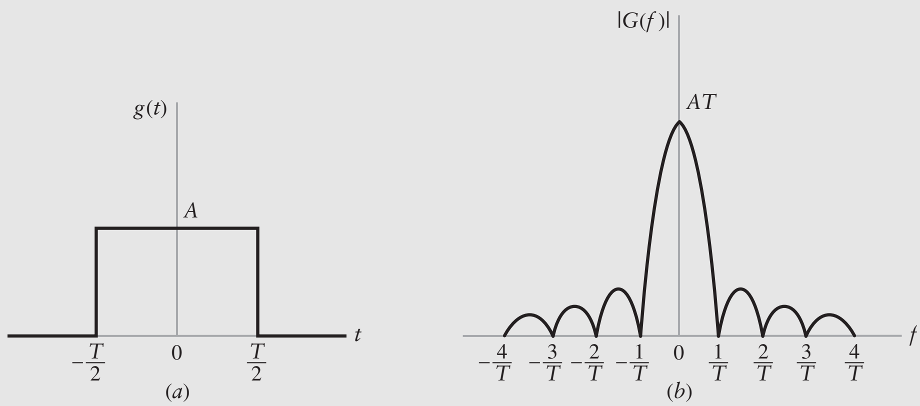


FIGURE 2.2 (*a*) Rectangular pulse. (*b*) Amplitude spectrum.

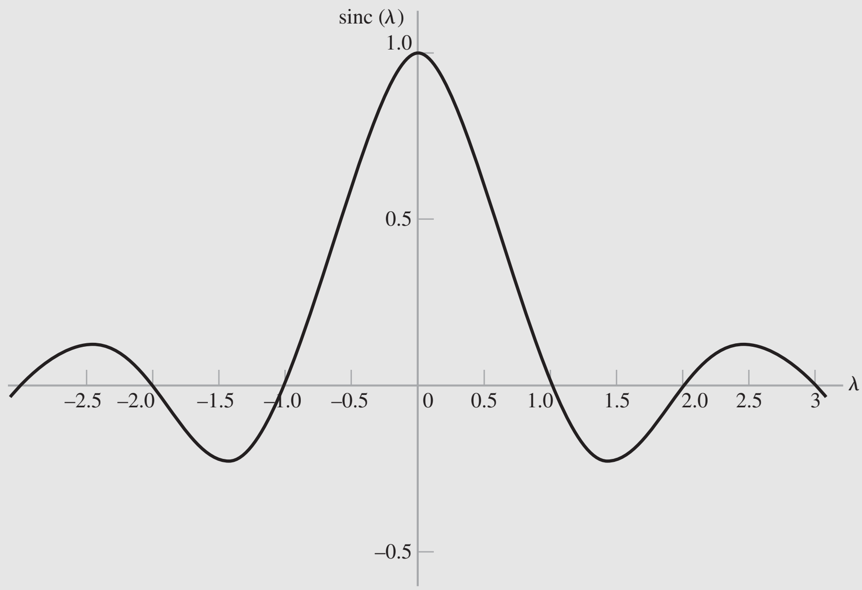


FIGURE 2.3 The sinc function.

where λ is the independent variable. The sinc function plays an important role in communication theory. As shown in Fig. 2.3, it has its maximum value of unity at $\lambda = 0$, and approaches zero as λ approaches infinity, oscillating through positive and negative values. It goes through zero at $\lambda = \pm 1, \pm 2, \ldots$, and so on.

Thus, in terms of the sinc function, we may rewrite Eq. (2.8) as

$$A \operatorname{rect}\left(\frac{t}{T}\right) \rightleftharpoons AT \operatorname{sinc}(fT) \tag{2.10}$$

The amplitude spectrum $|G(f)|$ is shown plotted in Fig. 2.2(*b*). The first zero-crossing of the spectrum occurs at $f = \pm 1/T$. As the pulse duration T is decreased, this first zero-crossing moves up in frequency. Conversely, as the pulse duration T is increased, the first zero-crossing moves toward the origin.

This example shows that the relationship between the time-domain and frequency-domain descriptions of a signal is an *inverse* one. That is, a pulse narrow in time has a significant frequency description over a wide range of frequencies, and vice versa. We shall have more to say on the inverse relationship between time and frequency in Section 2.3.

Note also that in this example, the Fourier transform $G(f)$ is a real-valued and symmetric function of frequency f. This is a direct consequence of the fact that the rectangular pulse $g(t)$ shown in Fig. 2.2(a) is a symmetric function of *time t*.

EXAMPLE 2.2 **Exponential Pulse**

A *truncated decaying exponential pulse* is shown in Fig. 2.4(*a*). We define this pulse mathematically in a convenient form using the unit step function:

$$u(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases} \tag{2.11}$$

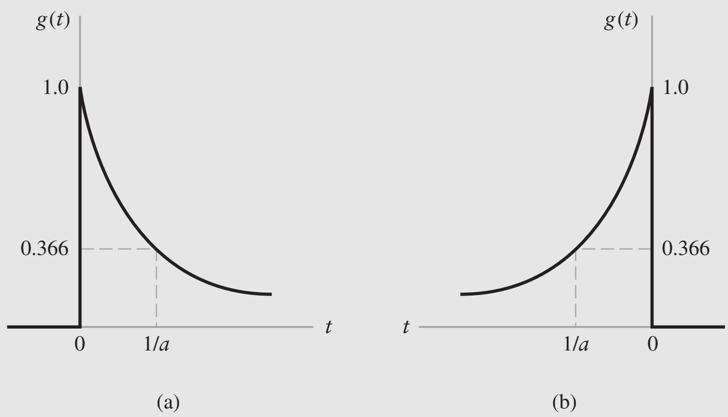


FIGURE 2.4 (*a*) Decaying exponential pulse. (*b*) Rising exponential pulse.

We may then express the decaying exponential pulse of Fig. 2.4(*a*) as

$$g(t) = \exp(-at)u(t)$$

Recognizing that $g(t)$ is zero for $t < 0$, the Fourier transform of this pulse is

$$\begin{aligned} G(f) &= \int_0^{\infty} \exp(-at) \exp(-j2\pi ft)\, dt \\ &= \int_0^{\infty} \exp[-t(a + j2\pi f)]\, dt \\ &= \frac{1}{a + j2\pi f} \end{aligned}$$

The Fourier-transform pair for the decaying exponential pulse of Fig. 2.4(*a*) is therefore

$$\exp(-at)u(t) \rightleftharpoons \frac{1}{a + j2\pi f} \tag{2.12}$$

A *truncated rising exponential pulse* is shown in Fig. 2.4(*b*), which is defined by

$$g(t) = \exp(at)u(-t)$$

Note that $u(-t)$ is equal to unity for $t < 0$, one-half at $t = 0$, and zero for $t > 0$. With $g(t)$ equal to zero for $t > 0$, the Fourier transform of this pulse is

$$G(f) = \int_{-\infty}^{0} \exp(at) \exp(-j2\pi ft)\, dt$$

Replacing t with $-t$, we may next write

$$\begin{aligned} G(f) &= \int_0^{\infty} \exp[-t(a - j2\pi f)]\, dt \\ &= \frac{1}{a - j2\pi f} \end{aligned}$$

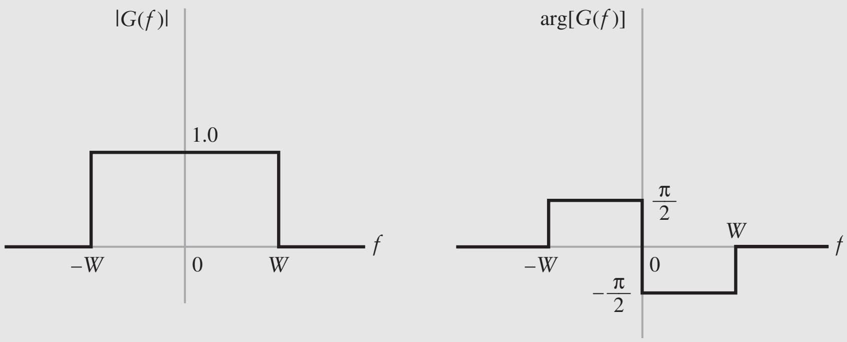


FIGURE 2.5 Frequency function $G(f)$ for Problem 2.2.

The Fourier-transform pair for the rising exponential pulse of Fig. 2.4(b) is therefore

$$\exp(-at)u(-t) \rightleftharpoons \frac{1}{a - j2\pi f} \tag{2.13}$$

The decaying and rising exponential pulses of Fig. 2.4 are both asymmetric functions of time t. Their Fourier transforms are therefore complex valued, as shown in Eqs. (2.12) and (2.13). Moreover, from these Fourier-transform pairs, we readily see that truncated decaying and rising exponential pulses have the same amplitude spectrum, but the phase spectrum of the one is the negative of the phase spectrum of the other.

► **Drill Problem 2.1** Evaluate the Fourier transform of the damped sinusoidal wave $g(t) = \exp(-t)\sin(2\pi f_c t)u(t)$, where $u(t)$ is the unit step function. ◄

► **Drill Problem 2.2** Determine the inverse Fourier transform of the frequency function $G(f)$ defined by the amplitude and phase spectra shown in Fig. 2.5. ◄

2.2 *Properties of the Fourier Transform*

It is useful to have insight into the relationship between a time function $g(t)$ and its Fourier transform $G(f)$, and also into the effects that various operations on the function $g(t)$ have on the transform $G(f)$. This may be achieved by examining certain properties of the Fourier transform. In this section, we describe fourteen properties, which we will prove, one by one. These properties are summarized in Table A8.1 of Appendix 8 at the end of the book.

PROPERTY 1 Linearity (Superposition) *Let* $g_1(t) \rightleftharpoons G_1(f)$ *and* $g_2(t) \rightleftharpoons G_2(f)$. *Then for all constants* c_1 *and* c_2, *we have*

$$c_1 g_1(t) + c_2 g_2(t) \rightleftharpoons c_1 G_1(f) + c_2 G_2(f) \tag{2.14}$$

The proof of this property follows simply from the linearity of the integrals defining $G(f)$ and $g(t)$.

Property 1 permits us to find the Fourier transform $G(f)$ of a function $g(t)$ that is a linear combination of two other functions $g_1(t)$ and $g_2(t)$ whose Fourier transforms $G_1(f)$ and $G_2(f)$ are known, as illustrated in the following example.

EXAMPLE 2.3 Combinations of Exponential Pulses

Consider a *double exponential pulse* (defined by (see Fig. 2.6(*a*))

$$\begin{aligned} g(t) &= \begin{cases} \exp(-at), & t > 0 \\ 1, & t = 0 \\ \exp(at), & t < 0 \end{cases} \\ &= \exp(-a|t|) \end{aligned} \tag{2.15}$$

This pulse may be viewed as the sum of a truncated decaying exponential pulse and a truncated rising exponential pulse. Therefore, using the linearity property and the Fourier-transform pairs of Eqs. (2.12) and (2.13), we find that the Fourier transform of the double exponential pulse of Fig. 2.6(*a*) is

$$\begin{aligned} G(f) &= \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} \\ &= \frac{2a}{a^2 + (2\pi f)^2} \end{aligned}$$

We thus have the following Fourier-transform pair for the double exponential pulse of Fig. 2.6(*a*):

$$\exp(-a|t|) \rightleftharpoons \frac{2a}{a^2 + (2\pi f)^2} \tag{2.16}$$

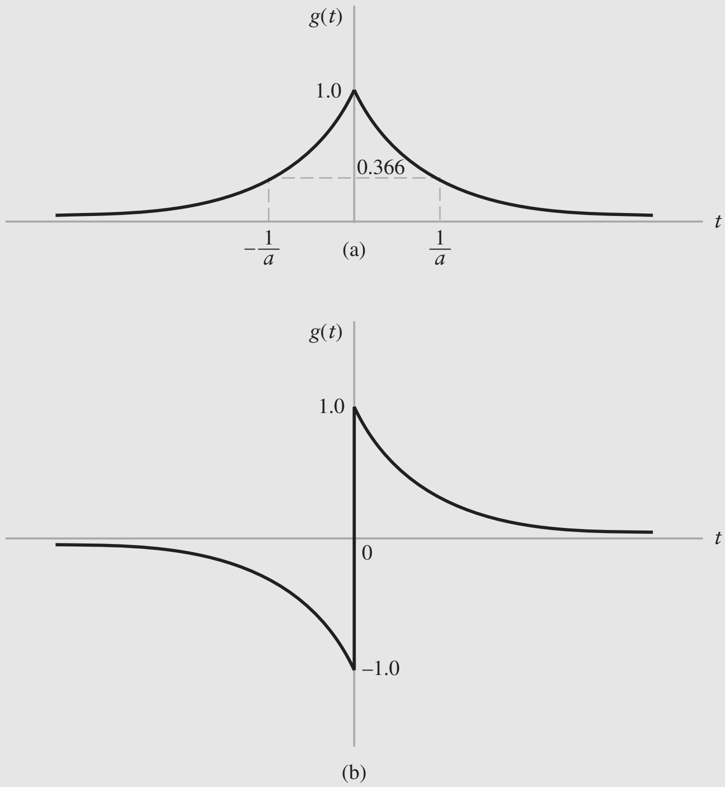


FIGURE 2.6 (*a*) Double-exponential pulse (symmetric). (*b*) Another double-exponential pulse (odd-symmetric).

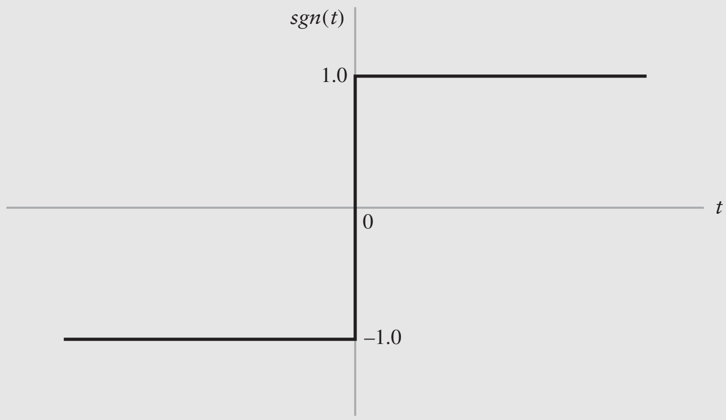


FIGURE 2.7 Signum function.

Note that because of the symmetry in the time domain, as in Fig. 2.6(a), the spectrum is real and symmetric; this is a general property of such Fourier-transform pairs.

Another interesting combination is the difference between a truncated decaying exponential pulse and a truncated rising exponential pulse, as shown in Fig. 2.6(b). Here we have

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 0, & t = 0 \\ -\exp(at), & t < 0 \end{cases} \tag{2.17}$$

We may formulate a compact notation for this composite signal by using the *signum function* that equals $+1$ for positive time and -1 for negative time, as shown by

$$\mathrm{sgn}(t) = \begin{cases} +1, & t > 0 \\ \ \ 0, & t = 0 \\ -1, & t < 0 \end{cases} \tag{2.18}$$

The signum function is shown in Fig. 2.7. Accordingly, we may reformulate the composite signal $g(t)$ defined in Eq. (2.17) simply as

$$g(t) = \exp(-a|t|)\,\mathrm{sgn}(t)$$

Hence, applying the linearity property of the Fourier transform, we readily find that in light of Eqs. (2.12) and (2.13), the Fourier transform of the signal $g(t)$ is given by

$$\begin{aligned} \mathbf{F}[\exp(-a|t|)\,\mathrm{sgn}(t)] &= \frac{1}{a + j2\pi f} - \frac{1}{a - j2\pi f} \\ &= \frac{-j4\pi f}{a^2 + (2\pi f)^2} \end{aligned}$$

We thus have the Fourier-transform pair

$$\exp(-a|t|)\,\mathrm{sgn}(t) \rightleftharpoons \frac{-j4\pi f}{a^2 + (2\pi f)^2} \tag{2.19}$$

In contrast to the Fourier-transform pair of Eq. (2.16), the Fourier transform in Eq. (2.19) is odd and purely imaginary. It is a general property of Fourier-transform pairs that apply to an *odd-symmetric* time function, which satisfies the condition $g(-t) = -g(t)$, as in Fig. 2.6(b); such a time function has an odd and purely imaginary function as its Fourier transform.

PROPERTY 2 Dilation *Let $g(t) \rightleftharpoons G(f)$. Then, the dilation property or similarity property states that*

$$g(at) \rightleftharpoons \frac{1}{|a|} G\left(\frac{f}{a}\right) \tag{2.20}$$

where the dilation factor—namely, a—is a real number.

To prove this property, we note that

$$\mathbf{F}[g(at)] = \int_{-\infty}^{\infty} g(at) \exp(-j2\pi ft)\, dt$$

Set $\tau = at$. There are two cases that can arise, depending on whether the dilation factor a is positive or negative. If $a > 0$, we get

$$\begin{aligned} \mathbf{F}[g(at)] &= \frac{1}{a} \int_{-\infty}^{\infty} g(\tau) \exp\left[-j2\pi\left(\frac{f}{a}\right)\tau\right] d\tau \\ &= \frac{1}{a} G\left(\frac{f}{a}\right) \end{aligned}$$

On the other hand, if $a < 0$, the limits of integration are interchanged so that we have the multiplying factor $-(1/a)$ or, equivalently, $1/|a|$. This completes the proof of Eq. (2.20).

Note that the dilation factors a and $1/a$ used in the time and frequency functions in Eq. (2.20) are reciprocals. In particular, the function $g(at)$ represents $g(t)$ *compressed* in time by the factor a, whereas the function $G(f/a)$ represents $G(f)$ expanded in frequency by the same factor a, assuming that $0 < a < 1$. Thus, the dilation rule states that the compression of a function $g(t)$ in the time domain is equivalent to the expansion of its Fourier transform $G(f)$ in the frequency domain by the same factor, or vice versa.

For the special case when $a = -1$, the dilation rule of Eq. (2.20) reduces to the *reflection property*, which states that *if $g(t) \rightleftharpoons G(f)$, then*

$$g(-t) \rightleftharpoons G(-f) \tag{2.21}$$

Referring to Fig. 2.4, we see that the rising exponential pulse shown in part (b) of the figure is the *reflection* of the decaying exponential pulse shown in part (a) with respect to the vertical axis. Hence, applying the reflection rule to Eq. (2.12) that pertains to the decaying exponential pulse, we readily see that the Fourier transform of the rising exponential pulse is $1/(a - j2\pi f)$, which is exactly what we have in Eq. (2.13).

PROPERTY 3 Conjugation Rule *Let $g(t) \rightleftharpoons G(f)$. Then for a complex-valued time function $g(t)$, we have*

$$g^*(t) \rightleftharpoons G^*(-f) \tag{2.22}$$

where the asterisk denotes the complex-conjugate operation.

To prove this property, we know from the inverse Fourier transform that

$$g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi ft)\, df$$

Taking the complex conjugates of both sides yields

$$g^*(t) = \int_{-\infty}^{\infty} G^*(f)\exp(-j2\pi ft)\,df$$

Next, replacing f with $-f$ gives

$$\begin{aligned} g^*(t) &= -\int_{\infty}^{-\infty} G^*(-f)\exp(j2\pi ft)\,df \\ &= \int_{-\infty}^{\infty} G^*(-f)\exp(j2\pi ft)\,df \end{aligned}$$

That is, $g^*(t)$ is the inverse Fourier transform of $G^*(-f)$, which is the desired result.

As a corollary to the conjugation rule of Eq. (2.22), we may state that if $g(t) \rightleftharpoons G(f)$, then

$$g^*(-t) \rightleftharpoons G^*(f) \tag{2.23}$$

This result follows directly from Eq. (2.22) by applying the reflection rule described in Eq. (2.21).

PROPERTY 4 Duality *If* $g(t) \rightleftharpoons G(f)$, *then*

$$G(t) \rightleftharpoons g(-f) \tag{2.24}$$

This property follows from the relation defining the inverse Fourier transform of Eq. (2.21) by first replacing t with $-t$, thereby writing it in the form

$$g(-t) = \int_{-\infty}^{\infty} G(f)\exp(-j2\pi ft)\,df$$

Finally, interchanging t and f (i.e., replacing t with f in the left-hand side of the equation and f with t in the right-hand side), we get

$$g(-f) = \int_{-\infty}^{\infty} G(t)\exp(-j2\pi ft)\,dt$$

which is the expanded part of Eq. (2.24) in going from the time domain to the frequency domain.

EXAMPLE 2.4 Sinc Pulse

Consider a signal $g(t)$ in the form of a sinc function, as shown by

$$g(t) = A\,\mathrm{sinc}(2Wt)$$

To evaluate the Fourier transform of this function, we apply the duality and dilation properties to the Fourier-transform pair of Eq. (2.10). Then, recognizing that the rectangular function is an even function of time, we obtain the result

$$A\,\mathrm{sinc}(2Wt) \rightleftharpoons \frac{A}{2W}\,\mathrm{rect}\left(\frac{f}{2W}\right) \tag{2.25}$$

which is illustrated in Fig. 2.8. We thus see that the Fourier transform of a sinc pulse is zero for $|f| > W$. Note also that the sinc pulse itself is only asymptotically limited in time in the sense that it approaches zero as time t approaches infinity; it is this asymptotic characteristic that makes the sinc function into an energy signal and therefore Fourier transformable.

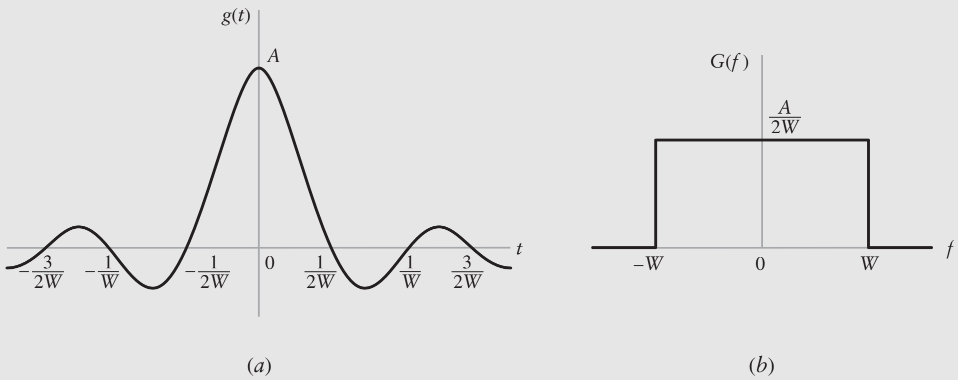


FIGURE 2.8 (*a*) Sinc pulse $g(t)$. (*b*) Fourier transform $G(f)$.

PROPERTY 5 Time Shifting *If* $g(t) \rightleftharpoons G(f)$, *then*

$$g(t - t_0) \rightleftharpoons G(f)\exp(-j2\pi f t_0) \tag{2.26}$$

where t_0 *is a real constant time shift.*

To prove this property, we take the Fourier transform of $g(t - t_0)$ and then set $\tau = (t - t_0)$ or, equivalently, $t = \tau + t_0$. We thus obtain

$$\begin{aligned} \mathbf{F}[g(t - t_0)] &= \exp(-j2\pi f t_0)\int_{-\infty}^{\infty} g(\tau)\exp(-j2\pi\tau)\,d\tau \\ &= \exp(-j2\pi f t_0)G(f) \end{aligned}$$

The time-shifting property states that if a function $g(t)$ is shifted along the time axis by an amount t_0, the effect is equivalent to multiplying its Fourier transform $G(f)$ by the factor $\exp(-j2\pi f t_0)$. This means that the amplitude of $G(f)$ is unaffected by the time shift, but its phase is changed by the linear factor $-2\pi f t_0$, which varies linearly with frequency f.

PROPERTY 6 Frequency Shifting *If* $g(t) \rightleftharpoons G(f)$, *then*

$$\exp(j2\pi f_c t)g(t) \rightleftharpoons G(f - f_c) \tag{2.27}$$

where f_c *is a real constant frequency.*

This property follows from the fact that

$$\begin{aligned} \mathbf{F}[\exp(j2\pi f_c t)g(t)] &= \int_{-\infty}^{\infty} g(t)\exp[-j2\pi t(f - f_c)]\,dt \\ &= G(f - f_c) \end{aligned}$$

That is, multiplication of a function $g(t)$ by the factor $\exp(j2\pi f_c t)$ is equivalent to shifting its Fourier transform $G(f)$ along the frequency axis by the amount f_c. This property is a special case of the *modulation theorem* discussed later under Property 11; basically, a shift of the range of frequencies in a signal is accomplished by using the process of modulation. Note the duality between the time-shifting and frequency-shifting operations described in Eqs. (2.26) and (2.27).

EXAMPLE 2.5 Radio Frequency (RF) Pulse

Consider the pulse signal $g(t)$ shown in Fig. 2.9(a), which consists of a sinusoidal wave of unit amplitude and frequency f_c, extending in duration from $t = -T/2$ to $t = T/2$. This signal is sometimes referred to as an *RF pulse* when the frequency f_c falls in the radio-frequency band. The signal $g(t)$ of Fig. 2.9(a) may be expressed mathematically as follows:

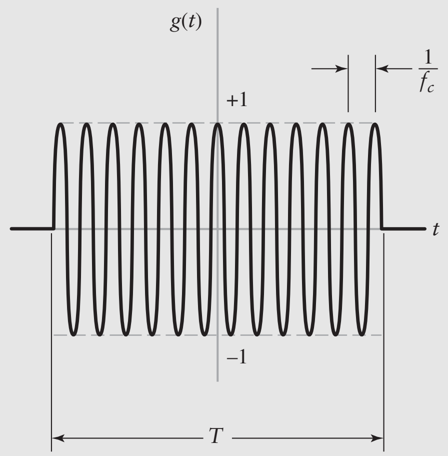
$$g(t) = \text{rect}\left(\frac{t}{T}\right)\cos(2\pi f_c t) \tag{2.28}$$

To find the Fourier transform of the RF signal, we first use *Euler's formula* to write

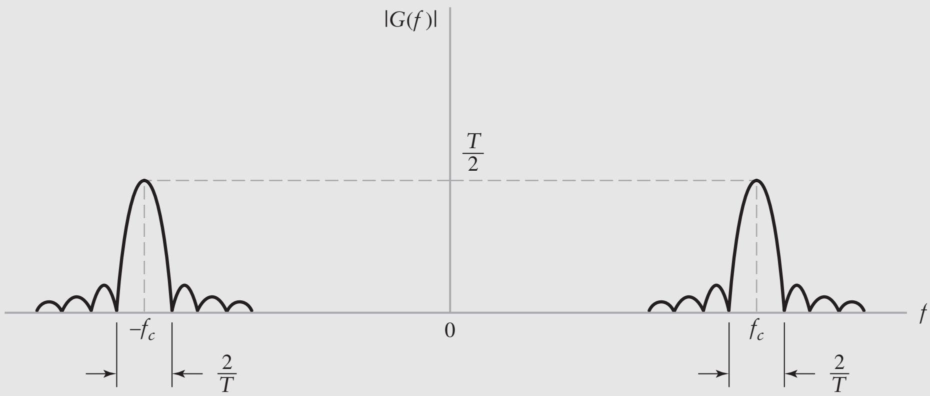
$$\cos(2\pi f_c t) = \frac{1}{2}[\exp(j2\pi f_c t) + \exp(-j2\pi f_c t)]$$

Therefore, applying the frequency-shifting property to the Fourier-transform pair of Eq. (2.10), and then invoking the linearity property of the Fourier transform, we get the desired result

$$\text{rect}\left(\frac{t}{T}\right)\cos(2\pi f_c t) \rightleftharpoons \frac{T}{2}\{\text{sinc}[T(f - f_c)] + \text{sinc}[T(f + f_c)]\} \tag{2.29}$$



(a)



(b)

FIGURE 2.9 (a) RF pulse of unit amplitude and duration T. (b) Amplitude spectrum.

In the special case of $f_c T \gg 1$—that is, the frequency f_c is large compared to the reciprocal of the pulse duration T—we may use the approximate result

$$G(f) \approx \begin{cases} \dfrac{T}{2}\operatorname{sinc}[T(f - f_c)], & f > 0 \\ 0, & f = 0 \\ \dfrac{T}{2}\operatorname{sinc}[T(f + f_c)], & f < 0 \end{cases} \tag{2.30}$$

Under the condition $f_c T \gg 1$, the amplitude spectrum of the RF pulse is shown in Fig. 2.9(*b*). This diagram, in relation to Fig. 2.2(*b*), clearly illustrates the frequency-shifting property of the Fourier transform.

PROPERTY 7 Area Under $g(t)$ *If* $g(t) \rightleftharpoons G(f)$, *then*

$$\int_{-\infty}^{\infty} g(t)\,dt = G(0) \tag{2.31}$$

That is, the area under a function $g(t)$ is equal to the value of its Fourier transform $G(f)$ at $f = 0$.

This result is obtained simply by putting $f = 0$ in Eq. (2.1) defining the Fourier transform of the function $g(t)$.

► **Drill Problem 2.3** Suppose $g(t)$ is real valued with a complex-valued Fourier transform $G(f)$. Explain how the rule of Eq. (2.31) can be satisfied by such a signal. ◄

PROPERTY 8 Area Under $G(f)$ *If* $g(t) \rightleftharpoons G(f)$, *then*

$$g(0) = \int_{-\infty}^{\infty} G(f)\,df \tag{2.32}$$

That is, the value of a function $g(t)$ at $t = 0$ is equal to the area under its Fourier transform $G(f)$.

The result is obtained simply by putting $t = 0$ in Eq. (2.2) defining the inverse Fourier transform of $G(f)$.

► **Drill Problem 2.4** Continuing with Problem 2.3, explain how the rule of Eq. (2.32) can be satisfied by the signal $g(t)$ described therein. ◄

PROPERTY 9 Differentiation in the Time Domain *Let* $g(t) \rightleftharpoons G(f)$ *and assume that the first derivative of $g(t)$ with respect to time t is Fourier transformable. Then*

$$\frac{d}{dt}g(t) \rightleftharpoons j2\pi f G(f) \tag{2.33}$$

That is, differentiation of a time function $g(t)$ has the effect of multiplying its Fourier transform $G(f)$ by the purely imaginary factor $j2\pi f$.

This result is obtained simply in two steps. In step 1, we take the first derivative of both sides of the integral in Eq. (2.2) defining the inverse Fourier transform of $G(f)$. In step 2, we interchange the operations of integration and differentiation.

We may generalize Eq. (2.33) for higher order derivatives of the time function $g(t)$ as follows:

$$\frac{d^n}{dt^n}g(t) \rightleftharpoons (j2\pi f)^n G(f) \tag{2.34}$$

which includes Eq. (2.33) as a special case. Equation (2.34) assumes that the Fourier transform of the higher order derivative of $g(t)$ exists.

EXAMPLE 2.6 Unit Gaussian Pulse

Typically, a pulse signal $g(t)$ and its Fourier transform $G(f)$ have different mathematical forms. This observation is illustrated by the Fourier-transform pairs studied in Examples 2.1 through 2.5. In this example, we consider an exception to this observation. In particular, we use the differentiation property of the Fourier transform to derive the particular form of a *pulse signal that has the same mathematical form as its own Fourier transform.*

Let $g(t)$ denote the pulse signal expressed as a function of time t, and $G(f)$ denote its Fourier transform. Differentiating the Fourier transform formula of Eq. (2.1) with respect to frequency f, we may write

$$-j2\pi t g(t) \rightleftharpoons \frac{d}{df}G(f)$$

or, equivalently,

$$2\pi t g(t) \rightleftharpoons j\frac{d}{df}G(f) \tag{2.35}$$

Suppose we now impose the following condition on the left-hand sides of Eqs. (2.33) and (2.35):

$$\frac{d}{dt}g(t) = -2\pi t g(t) \tag{2.36}$$

Then in a corresponding way, it follows that the right-hand sides of these two equations must (after cancelling the common multiplying factor j) satisfy the condition

$$\frac{d}{df}G(f) = -2\pi f G(f) \tag{2.37}$$

Equations (2.36) and (2.37) show that the pulse signal $g(t)$ and its Fourier transform $G(f)$ have exactly the same mathematical form. In other words, provided that the pulse signal $g(t)$ satisfies the differential equation (2.36), then $G(f) = g(f)$, where $g(f)$ is obtained from $g(t)$ by substituting f for t. Solving Eq. (2.36) for $g(t)$, we obtain

$$g(t) = \exp(-\pi t^2) \tag{2.38}$$

The pulse defined by Eq. (2.38) is called a *Gaussian pulse*, the name being derived from the similarity of the function to the Gaussian probability density function of probability theory (see Chapter 8). It is shown plotted in Fig. 2.10. By applying Eq. (2.31), we find that the area under

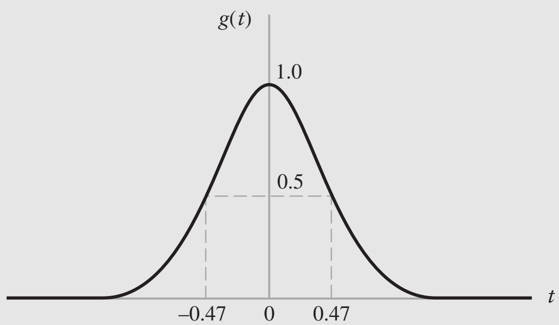


FIGURE 2.10 Gaussian pulse.

this Gaussian pulse is unity, as shown by

$$\int_{-\infty}^{\infty} \exp(-\pi t^2)\, dt = 1 \tag{2.39}$$

When the central ordinate and the area under the curve of a pulse are both unity, as in Eqs. (2.38) and (2.39), we say that the Gaussian pulse is a *unit pulse*. We conclude therefore that the unit Gaussian pulse is its own Fourier transform, as shown by

$$\exp(-\pi t^2) \rightleftharpoons \exp(-\pi f^2) \tag{2.40}$$

PROPERTY 10 Integration in the Time Domain *Let $g(t) \rightleftharpoons G(f)$. Then provided that $G(0) = 0$, we have*

$$\int_{-\infty}^{t} g(\tau)\, d\tau \rightleftharpoons \frac{1}{j2\pi f} G(f) \tag{2.41}$$

That is, integration of a time function $g(t)$ has the effect of dividing its Fourier transform $G(f)$ by the factor $j2\pi f$, provided that $G(0)$ is zero.

This property is verified by expressing $g(t)$ as

$$g(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} g(\tau)\, d\tau\right]$$

and then applying the time-differentiation property of the Fourier transform to obtain

$$G(f) = (j2\pi f)\left\{\mathbf{F}\left[\int_{-\infty}^{t} g(\tau)\, d\tau\right]\right\}$$

from which Eq. (2.41) follows immediately.

It is a straightforward matter to generalize Eq. (2.41) to multiple integration; however, the notation becomes rather cumbersome.

Equation (2.41) assumes that $G(0)$—that is, the area under $g(t)$—is zero. The more general case pertaining to $G(0) \neq 0$ is deferred to Section 2.4.

EXAMPLE 2.7 Triangular Pulse

Consider the *doublet pulse* $g_1(t)$ shown in Fig. 2.11(*a*). By integrating this pulse with respect to time, we obtain the *triangular pulse* $g_2(t)$ shown in Fig. 2.11(*b*). We note that the doublet pulse $g_1(t)$ consists of two rectangular pulses: one of amplitude A, defined for the interval $-T \leq t \leq 0$; and the other of amplitude $-A$, defined for the interval $0 \leq t \leq T$. Applying the time-shifting property of the Fourier transform to Eq. (2.10), we find that the Fourier transforms of these two rectangular pulses are equal to $AT\,\text{sinc}(fT)\exp(j\pi fT)$ and $-AT\,\text{sinc}(fT)\exp(-j\pi fT)$, respectively. Hence, invoking the linearity property of the Fourier transform, we find that the Fourier transform $G_1(f)$ of the doublet pulse $g_1(t)$ of Fig. 2.11(*a*) is given by

$$\begin{aligned} G_1(f) &= AT\,\text{sinc}(fT)[\exp(j\pi fT) - \exp(-j\pi fT)] \\ &= 2jAT\,\text{sinc}(fT)\sin(\pi fT) \end{aligned} \tag{2.42}$$

We further note from Eq. (2.42) that $G_1(0)$ is zero. Hence, using Eqs. (2.41) and (2.42), we find that the Fourier transform $G_2(f)$ of the triangular pulse $g_2(t)$ of Fig. 2.11(*b*) is given by

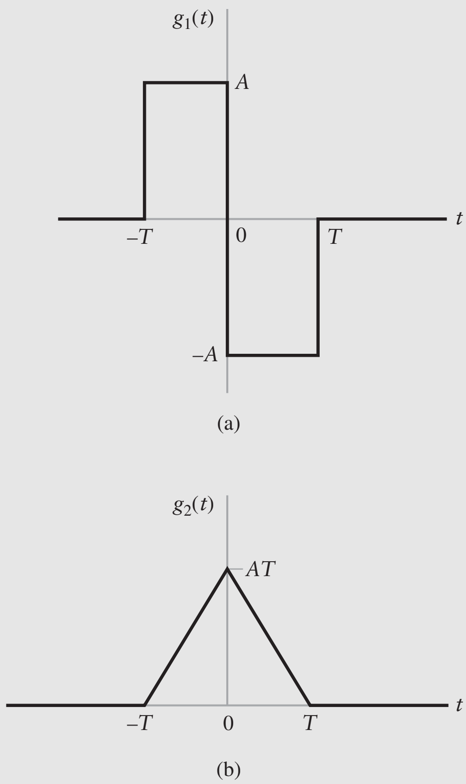


FIGURE 2.11 (*a*) Doublet pulse $g_1(t)$. (*b*) Triangular pulse $g_2(t)$ obtained by integrating $g_1(t)$ with respect to time t.

$$\begin{aligned} G_2(f) &= \frac{1}{j2\pi f} G_1(f) \\ &= AT \frac{\sin(\pi f T)}{\pi f} \operatorname{sinc}(fT) \\ &= AT^2 \operatorname{sinc}^2(fT) \end{aligned} \tag{2.43}$$

Note that the doublet pulse of Fig. 2.11(*a*) is real and odd-symmetric and its Fourier transform is therefore odd and purely imaginary, whereas the triangular pulse of Fig. 2.11(*b*) is real and symmetric and its Fourier transform is therefore symmetric and purely real.

EXAMPLE 2.8 Real and Imaginary Parts of a Time Function

Thus far in the chapter, we have discussed the Fourier representation of various signals, some being purely real, others being purely imaginary, yet others being complex valued with real and imaginary parts. It is therefore apropos that at this stage in the Fourier analysis of signals, we use this example to develop a number of general formulas pertaining to complex signals and their spectra.

Expressing a complex-valued function $g(t)$ in terms of its real and imaginary parts, we may write

$$g(t) = \mathbf{Re}[g(t)] + j\mathbf{Im}[g(t)] \tag{2.44}$$

where **Re** denotes "the real part of" and **Im** denotes the "imaginary part of." The complex conjugate of $g(t)$ is defined by

$$g^*(t) = \mathbf{Re}[g(t)] - j\mathbf{Im}[g(t)] \tag{2.45}$$

Adding Eqs. (2.44) and (2.45) gives

$$\mathbf{Re}[g(t)] = \frac{1}{2}[g(t) + g^*(t)] \tag{2.46}$$

and subtracting them yields

$$\mathbf{Im}[g(t)] = \frac{1}{2}[g(t) - g^*(t)] \tag{2.47}$$

Therefore, applying the conjugation rule of Eq. (2.22), we obtain the following two Fourier-transform pairs:

$$\left.\begin{aligned} \mathbf{Re}[g(t)] &\rightleftharpoons \frac{1}{2}[G(f) + G^*(-f)] \\ \mathbf{Im}[g(t) &\rightleftharpoons \frac{1}{2}[G(f) - G^*(-f)] \end{aligned}\right\} \tag{2.48}$$

From the second line of Eq. (2.48), it is apparent that in the case of a real-valued time function $g(t)$, we have $G(f) = G^*(-f)$; that is, the Fourier transform $G(f)$ exhibits conjugate symmetry, confirming a result that we stated previously in Section 2.2.

PROPERTY 11 Modulation Theorem *Let* $g_1(t) \rightleftharpoons G_1(f)$ *and* $g_2(t) \rightleftharpoons G_2(f)$. *Then*

$$g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2(f - \lambda)\, d\lambda \tag{2.49}$$

To prove this property, we first denote the Fourier transform of the product $g_1(t)g_2(t)$ by $G_{12}(f)$, so that we may write

$$g_1(t)g_2(t) \rightleftharpoons G_{12}(f)$$

where

$$G_{12}(f) = \int_{-\infty}^{\infty} g_1(t)g_2(t) \exp(-j2\pi ft)\, dt$$

For $g_2(t)$, we next substitute the inverse Fourier transform

$$g_2(t) = \int_{-\infty}^{\infty} G_2(f') \exp(j2\pi f't)\, df'$$

in the integral defining $G_{12}(f)$ to obtain

$$G_{12}(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(t)G_2(f') \exp[-j2\pi(f - f')t]\, df'dt$$

Define $\lambda = f - f'$. Then, eliminating the variable f' and interchanging the order of integration, we obtain (after rearranging terms)

$$G_{12}(f) = \int_{-\infty}^{\infty} G_2(f - \lambda)\left[\int_{-\infty}^{\infty} g_1(t) \exp(-j2\pi\lambda t)\, dt\right] d\lambda$$

assuming that f is fixed. The inner integral (inside the square brackets) is recognized simply as $G_1(\lambda)$; we may therefore write

$$G_{12}(f) = \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)\,d\lambda$$

which is the desired result. This integral is known as the *convolution integral* expressed in the frequency domain, and the function $G_{12}(f)$ is referred to as the *convolution* of $G_1(f)$ and $G_2(f)$. We conclude that *the multiplication of two signals in the time domain is transformed into the convolution of their individual Fourier transforms in the frequency domain*. This property is also known as the *modulation theorem*. We have more to say on the practical implications of this property in subsequent chapters.

In a discussion of convolution, the following shorthand notation is frequently used:

$$G_{12}(f) = G_1(f) \star G_2(f)$$

Accordingly, we may reformulate Eq. (2.49) in the following symbolic form:

$$g_1(t)g_2(t) \rightleftharpoons G_1(f) \star G_2(f) \tag{2.50}$$

where the symbol $\star$ denotes convolution. Note that convolution is *commutative*; that is,

$$G_1(f) \star G_2(f) = G_2(f) \star G_1(f)$$

which follows directly from Eq. (2.50).

Property 12 Convolution Theorem *Let $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$. Then*

$$\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)\,d\tau \rightleftharpoons G_1(f)G_2(f) \tag{2.51}$$

Equation (2.51) follows directly by combining Property 4 (duality) and Property 11 (modulation). We may thus state that the *convolution of two signals in the time domain is transformed into the multiplication of their individual Fourier transforms in the frequency domain*. This property is known as the *convolution theorem*. Its use permits us to exchange a convolution operation in the time domain for a multiplication of two Fourier transforms, an operation that is ordinarily easier to manipulate. We have more to say on convolution later in the chapter when the issue of filtering is discussed.

Using the shorthand notation for convolution, we may rewrite Eq. (2.51) in the simple form

$$g_1(t) \star g_2(t) \rightleftharpoons G_1(f)G_2(f) \tag{2.52}$$

Note that Properties 11 and 12, described by Eqs. (2.49) and (2.51), respectively, are the dual of each other.

► **Drill Problem 2.5** Develop the detailed steps that show that the modulation and convolution theorems are indeed the dual of each other. ◄

Property 13 Correlation Theorem *Let $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$. Then, assuming that $g_1(t)$ and $g_2(t)$ are complex valued,*

$$\int_{-\infty}^{\infty} g_1(t)g_2^*(t-\tau)\,dt \rightleftharpoons G_1(f)G_2^*(f) \tag{2.53}$$

where $G_2^*(f)$ is the complex conjugate of $G_2(f)$, and τ is the time variable involved in defining the inverse Fourier transform of the product $G_1(f)G_2^*(f)$.

To prove Eq. (2.53), we begin by reformulating the convolution integral with the roles of the time variables t and τ interchanged, in which case we may simply rewrite Eq. (2.51) as

$$\int_{-\infty}^{\infty} g_1(t)g_2(\tau - t)\,dt \rightleftharpoons G_1(f)G_2(f) \tag{2.54}$$

As already pointed out in the statement of Property 13, the inverse Fourier transform of the product term $G_1(f)G_2(f)$ has τ as its time variable; that is, $\exp(j2\pi f\tau)$ is its kernel. With the formula of Eq. (2.54) at hand, Eq. (2.53) follows directly by combining reflection rule (special case of the dilation property) and conjugation rule.

The integral on the left-hand side of Eq. (2.53) defines a measure of the *similarity* that may exist between a pair of complex-valued signals. This measure is called *correlation*, on which we have more to say later in the chapter.

► **Drill Problem 2.6** Develop the detailed steps involved in deriving Eq. (2.53), starting from Eq. (2.51). ◄

► **Drill Problem 2.7** Prove the following properties of the convolution process:

(a) The commutative property: $g_1(t) \star g_2(t) = g_2(t) \star g_1(t)$

(b) The associative property: $g_1(t) \star [g_2(t) \star g_3(t)] = [g_1(t) \star g_2(t)] \star g_3(t)$

(c) The distributive property: $g_1(t) \star [g_2(t) + g_3(t)] = g_1(t) \star g_2(t) + g_1(t) \star g_3(t)$ ◄

PROPERTY 14 Rayleigh's Energy Theorem *Let* $g(t) \rightleftharpoons G(f)$. *Then*

$$\int_{-\infty}^{\infty} |g(t)|^2\,dt = \int_{-\infty}^{\infty} |G(f)|^2\,df \tag{2.55}$$

To prove Eq. (2.55), we set $g_1(t) = g_2(t) = g(t)$ in Eq. (2.53), in which case the correlation theorem reduces to

$$\int_{-\infty}^{\infty} g(t)g^*(t - \tau)\,dt \rightleftharpoons G(f)G^*(f) = |G(f)|^2$$

In expanded form, we may write

$$\int_{-\infty}^{\infty} g(t)g^*(t - \tau)\,dt = \int_{-\infty}^{\infty} |G(f)|^2 \exp(j2\pi f\tau)\,df \tag{2.56}$$

Finally, putting $\tau = 0$ in Eq. (2.56) and recognizing that $g(t)g^*(t) = |g(t)|^2$, we get the desired result.

Equation (2.55), known as *Rayleigh's energy theorem*, states that the total energy of a Fourier-transformable signal equals the total area under the curve of squared amplitude spectrum of this signal. Determination of the energy is often simplified by invoking the Rayleigh energy theorem, as illustrated in the following example.

EXAMPLE 2.9 Sinc Pulse (continued)

Consider again the sinc pulse $A\ \mathrm{sinc}\,(2Wt)$. The energy of this pulse equals

$$E = A^2 \int_{-\infty}^{\infty} \mathrm{sinc}^2(2Wt)\,dt$$

The integral in the right-hand side of this equation is rather difficult to evaluate. However, we note from Example 2.4 that the Fourier transform of the sinc pulse $A\ \text{sinc}(2Wt)$ is equal to $(A/2W)\ \text{rect}(f/2W)$; hence, applying Rayleigh's energy theorem to the problem at hand, we readily obtain the desired result:

$$\begin{aligned} E &= \left(\frac{A}{2W}\right)^2 \int_{-\infty}^{\infty} \text{rect}^2\left(\frac{f}{2W}\right) df \\ &= \left(\frac{A}{2W}\right)^2 \int_{-W}^{W} df \\ &= \frac{A^2}{2W} \end{aligned} \tag{2.57}$$

This example clearly illustrates the usefulness of Rayleigh's energy theorem.

▶ **Drill Problem 2.8** Considering the pulse function $\text{sinc}(t)$, show that

$$\int_{-\infty}^{\infty} \text{sinc}^2(t)\, dt = 1.$$

◀

2.3 *The Inverse Relationship Between Time and Frequency*

The properties of the Fourier transform discussed in Section 2.2 show that the time-domain and frequency-domain descriptions of a signal are *inversely* related to each other. In particular, we may make two important statements:

1. If the time-domain description of a signal is changed, the frequency-domain description of the signal is changed in an *inverse* manner, and vice versa. This inverse relationship prevents arbitrary specifications of a signal in both domains. In other words, *we may specify an arbitrary function of time or an arbitrary spectrum, but we cannot specify both of them together*.
2. If a signal is strictly limited in frequency, the time-domain description of the signal will trail on indefinitely, even though its amplitude may assume a progressively smaller value. We say a signal is *strictly limited in frequency* or *strictly band limited* if its Fourier transform is exactly zero outside a finite band of frequencies. The sinc pulse is an example of a strictly band-limited signal, as illustrated in Fig. 2.8. This figure also shows that the sinc pulse is only *asymptotically limited in time*. In an inverse manner, if a signal is *strictly limited in time* (i.e., the signal is exactly zero outside a finite time interval), then the spectrum of the signal is infinite in extent, even though the amplitude spectrum may assume a progressively smaller value. This behavior is exemplified by both the rectangular pulse (described in Fig. 2.2) and the triangular pulse (described in Fig. 2.11(b)). Accordingly, we may state that *a signal cannot be strictly limited in both time and frequency*.

■ Bandwidth

The *bandwidth* of a signal provides a measure of the *extent of the significant spectral content of the signal for positive frequencies*. When the signal is strictly band limited, the bandwidth is well defined. For example, the sinc pulse described in Fig. 2.8(a) has a bandwidth equal to W. However, when the signal is not strictly band limited, which is generally the case, we encounter difficulty in defining the bandwidth of the signal. The difficulty arises because the meaning of the word "significant" attached to the spectral content of the signal is mathematically imprecise. Consequently, there is no universally accepted definition of bandwidth.

Nevertheless, there are some commonly used definitions for bandwidth. In this section, we consider three such definitions; the formulation of each definition depends on whether the signal is low-pass or band-pass. A signal is said to be *low-pass* if its significant spectral content is centered around the origin $f = 0$. A signal is said be *band-pass* if its significant spectral content is centered around $\pm f_c$, where f_c is a constant frequency.

When the spectrum of a signal is symmetric with a *main lobe* bounded by well-defined *nulls* (i.e., frequencies at which the spectrum is zero), we may use the main lobe as the basis for defining the bandwidth of the signal. The rationale for doing so is that the main spectral lobe contains the significant portion of the signal energy. If the signal is low-pass, the bandwidth is defined as one half the total width of the main spectral lobe, since only one half of this lobe lies inside the positive frequency region. For example, a rectangular pulse of duration T seconds has a main spectral lobe of total width $(2/T)$ hertz centered at the origin, as depicted in Fig. 2.2(b). Accordingly, we may define the bandwidth of this rectangular pulse as $(1/T)$ hertz. If, on the other hand, the signal is band-pass with main spectral lobes centered around $\pm f_c$, where f_c is large, the bandwidth is defined as the width of the main lobe for positive frequencies. This definition of bandwidth is called the *null-to-null bandwidth*. For example, an RF pulse of duration T seconds and frequency f_c has main spectral lobes of width $(2/T)$ hertz centered around $\pm f_c$, as depicted in Fig. 2.9(b). Hence, we may define the null-to-null bandwidth of this RF pulse as $(2/T)$ hertz. On the basis of the definitions presented here, we may state that shifting the spectral content of a low-pass signal by a sufficiently large frequency has the effect of doubling the bandwidth of the signal. Such a frequency translation is attained by using the process of modulation, which is discussed in detail in Chapter 3.

Another popular definition of bandwidth is the *3-dB bandwidth*. Specifically, if the signal is low-pass, the 3-dB bandwidth is defined as the separation between zero frequency, where the amplitude spectrum attains its peak value, and the *positive frequency* at which the amplitude spectrum drops to $1/\sqrt{2}$ of its peak value. For example, the decaying exponential and rising exponential pulses defined in Fig. 2.4 have a 3-dB bandwidth of $(a/2\pi)$ hertz. If, on the other hand, the signal is band-pass, centered at $\pm f_c$, the 3-dB bandwidth is defined as the separation (along the positive frequency axis) between the two frequencies at which the amplitude spectrum of the signal drops to $1/\sqrt{2}$ of the peak value at f_c. The 3-dB bandwidth has an advantage in that it can be read directly from a plot of the amplitude spectrum. However, it has a disadvantage in that it may be misleading if the amplitude spectrum has slowly decreasing tails.

Yet another measure for the bandwidth of a signal is the *root mean-square (rms) bandwidth*, defined as the square root of the second moment of a properly normalized form of the squared amplitude spectrum of the signal about a suitably chosen point. We assume that the signal is low-pass, so that the second moment may be taken about the origin. As for the normalized form of the squared amplitude spectrum, we use the nonnegative function $|G(f)|^2 \Big/ \int_{-\infty}^{\infty} |G(f)|^2\, df$, in which the denominator applies the correct normalization in the sense that the integrated value of this ratio over the entire frequency axis is unity. We may thus formally define the rms bandwidth of a low-pass signal $g(t)$ with Fourier transform $G(f)$ as follows:

$$W_{\text{rms}} = \left(\frac{\displaystyle\int_{-\infty}^{\infty} f^2 |G(f)|^2 \, df}{\displaystyle\int_{-\infty}^{\infty} |G(f)|^2 \, df} \right)^{1/2} \tag{2.58}$$

An attractive feature of the rms bandwidth W_{rms} is that it lends itself more readily to mathematical evaluation than the other two definitions of bandwidth, although it is not as easily measured in the laboratory.

■ Time-Bandwidth Product

For any family of pulse signals that differ by a time-scaling factor, the product of the signal's duration and its bandwidth is always a constant, as shown by

$$(\text{duration}) \times (\text{bandwidth}) = \text{constant}$$

The product is called the *time-bandwidth product* or *bandwidth-duration product*. The constancy of the time-bandwidth product is another manifestation of the inverse relationship that exists between the time-domain and frequency-domain descriptions of a signal. In particular, if the duration of a pulse signal is decreased by compressing the time scale by a factor a, say, the frequency scale of the signal's spectrum, and therefore the bandwidth of the signal, is expanded by the same factor a, by virtue of Property 2 (dilation), and the time-bandwidth product of the signal is thereby maintained constant. For example, a rectangular pulse of duration T seconds has a bandwidth (defined on the basis of the positive-frequency part of the main lobe) equal to $(1/T)$ hertz, making the time-bandwidth product of the pulse equal unity. The important point to note here is that whatever definition we use for the bandwidth of a signal, the time-bandwidth product remains constant over certain classes of pulse signals. The choice of a particular definition for bandwidth merely changes the value of the constant.

To be more specific, consider the rms bandwidth defined in Eq. (2.58). The corresponding definition for the *rms duration* of the signal $g(t)$ is

$$T_{\text{rms}} = \left(\frac{\displaystyle\int_{-\infty}^{\infty} t^2 |g(t)|^2 \, dt}{\displaystyle\int_{-\infty}^{\infty} |g(t)|^2 \, dt} \right)^{1/2} \tag{2.59}$$

where it is assumed that the signal $g(t)$ is centered around the origin. It may be shown that using the rms definitions of Eqs. (2.58) and (2.59), the time-bandwidth product has the following form:

$$T_{\text{rms}} W_{\text{rms}} \geq \frac{1}{4\pi} \tag{2.60}$$

where the constant is $(1/4\pi)$. It may also be shown that the Gaussian pulse satisfies this condition with the equality sign. For the details of these calculations, the reader is referred to Problem 2.51.

2.4 *Dirac Delta Function*

Strictly speaking, the theory of the Fourier transform, as described in Sections 2.2 and 2.3, is applicable only to time functions that satisfy the Dirichlet conditions. Such functions include energy signals—that is, signals for which the condition

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$$

holds. However, it would be highly desirable to extend the theory in two ways:

1. To combine the theory of Fourier series and Fourier transform into a unified framework, so that the Fourier series may be treated as a special case of the Fourier transform. (A review of the Fourier series is presented in Appendix 2.)
2. To expand applicability of the Fourier transform to include power signals—that is, signals for which the condition

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 \, dt < \infty$$

holds.

It turns out that both of these objectives are met through the "proper use" of the *Dirac delta function* or *unit impulse*.

The Dirac delta function, denoted by $\delta(t)$, is defined as having zero amplitude everywhere except at $t = 0$, where it is infinitely large in such a way that it contains unit area under its curve. Specifically, $\delta(t)$ satisfies the pair of relations

$$\delta(t) = 0, \qquad t \neq 0 \tag{2.61}$$

and

$$\int_{-\infty}^{\infty} \delta(t) \, dt = 1 \tag{2.62}$$

An implication of this pair of relations is that the delta function $\delta(t)$ must be an even function of time t.

For the delta function to have meaning, however, it has to appear as a factor in the integrand of an integral with respect to time and then, strictly speaking, only when the other factor in the integrand is a continuous function of time. Let $g(t)$ be such a function, and consider the product of $g(t)$ and the time-shifted delta function $\delta(t - t_0)$. In light of the two defining equations (2.61) and (2.62), we may express the integral of the product $g(t)\delta(t - t_0)$ with respect to time t as follows:

$$\int_{-\infty}^{\infty} g(t)\delta(t - t_0) \, dt = g(t_0) \tag{2.63}$$

The operation indicated on the left-hand side of this equation sifts out the value $g(t_0)$ of the function $g(t)$ at time $t = t_0$, where $-\infty < t < \infty$. Accordingly, Eq. (2.63) is referred to as the *sifting property* of the delta function. This property is sometimes used as the defining equation of a delta function; in effect, it incorporates Eqs. (2.61) and (2.62) into a single relation.

Noting that the delta function $\delta(t)$ is an even function of t, we may rewrite Eq. (2.63) in a way that emphasizes its resemblance to the convolution integral, as shown by

$$\int_{-\infty}^{\infty} g(\tau)\delta(t - \tau) \, d\tau = g(t) \tag{2.64}$$

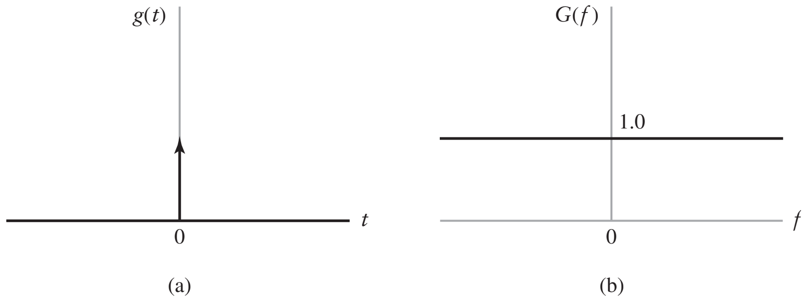


FIGURE 2.12 (*a*) The Dirac delta function $\delta(t)$. (*b*) Spectrum of $\delta(t)$.

or, using the notation for convolution:

$$g(t) \star \delta(t) = g(t)$$

In words, the convolution of any time function $g(t)$ with the delta function $\delta(t)$ leaves that function completely unchanged. We refer to this statement as the *replication property* of the delta function.

By definition, the Fourier transform of the delta function is given by

$$\mathbf{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) \exp(-j2\pi ft)\, dt$$

Hence, using the sifting property of the delta function and noting that $\exp(-j2\pi ft)$ is equal to unity at $t = 0$, we obtain

$$\mathbf{F}[\delta(t)] = 1$$

We thus have the Fourier-transform pair for the Dirac delta function:

$$\delta(t) \rightleftharpoons 1 \tag{2.65}$$

This relation states that the spectrum of the delta function $\delta(t)$ extends uniformly over the entire frequency interval, as shown in Fig. 2.12.

It is important to realize that the Fourier-transform pair of Eq. (2.65) exists only in a limiting sense. The point is that no function in the ordinary sense has the two properties of Eqs. (2.61) and (2.62) or the equivalent sifting property of Eq. (2.63). However, we can imagine a sequence of functions that have progressively taller and thinner peaks at $t = 0$, with the area under the curve remaining equal to unity, whereas the value of the function tends to zero at every point except $t = 0$, where it tends to infinity. That is, we may view the delta function as *the limiting form of a pulse of unit area as the duration of the pulse approaches zero*. It is immaterial what sort of pulse shape is used.

In a rigorous sense, the Dirac delta function belongs to a special class of functions known as *generalized functions* or *distributions*. Indeed, in some situations its use requires that we exercise considerable care. Nevertheless, one beautiful aspect of the Dirac delta function lies precisely in the fact that a rather intuitive treatment of the function along the lines described herein often gives the correct answer.

EXAMPLE 2.10 The Delta Function as a Limiting Form of the Gaussian Pulse

Consider a Gaussian pulse of unit area, defined by

$$g(t) = \frac{1}{\tau} \exp\left(-\frac{\pi t^2}{\tau^2}\right) \tag{2.66}$$

where τ is a variable parameter. The Gaussian function $g(t)$ has two useful properties: (1) its derivatives are all continuous, and (2) it dies away more rapidly than any power of t. The delta function $\delta(t)$ is obtained by taking the limit $\tau \to 0$. The Gaussian pulse then becomes infinitely narrow in duration and infinitely large in amplitude, yet its area remains finite and fixed at unity. Figure 2.13(*a*) illustrates the sequence of such pulses as the parameter τ is permitted to decrease.

The Gaussian pulse $g(t)$, defined here, is the same as the unit Gaussian pulse $\exp(-\pi t^2)$ derived in Example 2.6, except for the fact that it is now scaled in time by the factor τ and scaled in amplitude by the factor $1/\tau$. Therefore, applying the linearity and dilation properties of the Fourier transform to the Fourier transform pair of Eq. (2.40), we find that the Fourier transform of the Gaussian pulse $g(t)$ defined in Eq. (2.66) is also Gaussian, as shown by

$$G(f) = \exp(-\pi\tau^2 f^2)$$

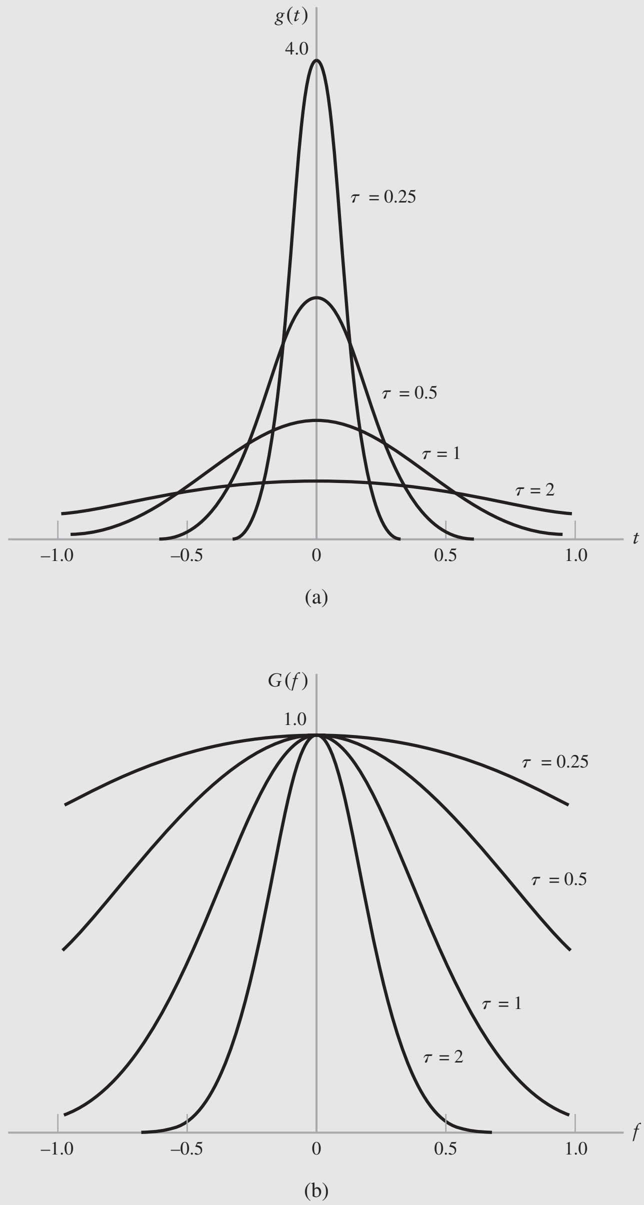


FIGURE 2.13
(*a*) Gaussian pulses of varying duration.
(*b*) Corresponding spectra.

Figure 2.13(b) illustrates the effect of varying the parameter τ on the spectrum of the Gaussian pulse $g(t)$. Thus putting $\tau = 0$, we find, as expected, that the Fourier transform of the delta function is unity.

■ Applications of the Delta Function

1. *dc Signal.*
By applying the duality property to the Fourier-transform pair of Eq. (2.65) and noting that the delta function is an even function, we obtain

$$1 \rightleftharpoons \delta(f) \tag{2.67}$$

Equation (2.67) states that a *dc signal* is transformed in the frequency domain into a delta function $\delta(t)$ occurring at zero frequency, as shown in Fig. 2.14. Of course, this result is intuitively satisfying.

Invoking the definition of Fourier transform, we readily deduce from Eq. (2.67) the useful relation

$$\int_{-\infty}^{\infty} \exp(-j2\pi ft)\, dt = \delta(f)$$

Recognizing that the delta function $\delta(f)$ is real valued, we may simplify this relation as follows:

$$\int_{-\infty}^{\infty} \cos(2\pi ft)\, dt = \delta(f) \tag{2.68}$$

which provides yet another definition for the delta function, albeit in the frequency domain.

2. *Complex Exponential Function.*
Next, by applying the frequency-shifting property to Eq. (2.67), we obtain the Fourier-transform pair

$$\exp(j2\pi f_c t) \rightleftharpoons \delta(f - f_c) \tag{2.69}$$

for a complex exponential function of frequency f_c. Equation (2.69) states that the complex exponential function $\exp(j2\pi f_c t)$ is transformed in the frequency domain into a delta function $\delta(f - f_c)$ occurring at $f = f_c$.

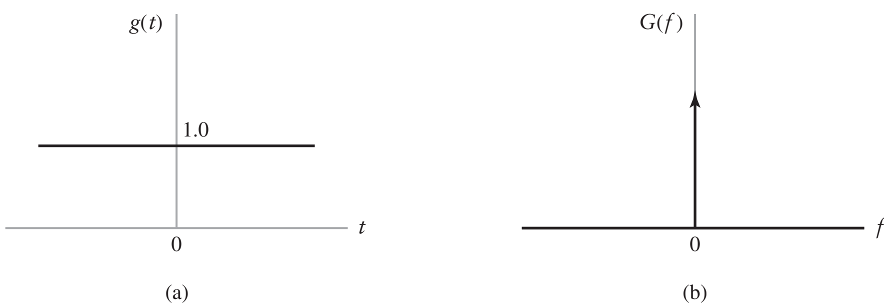


Figure 2.14 (a) dc signal. (b) Spectrum.

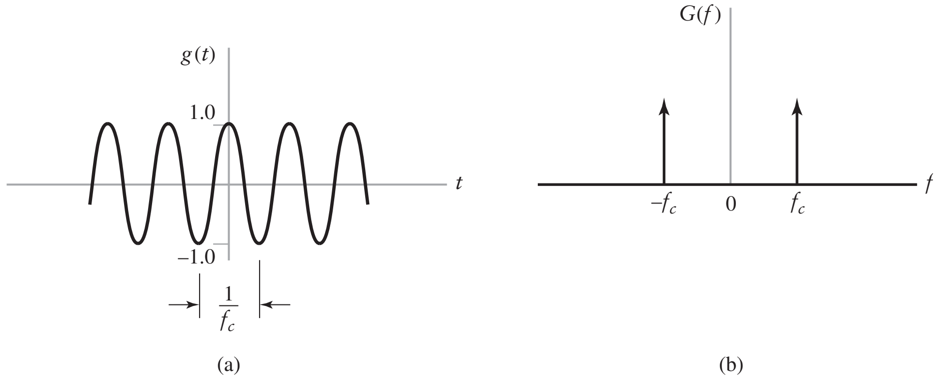


FIGURE 2.15 (*a*) Cosine function. (*b*) Spectrum.

3. *Sinusoidal Functions.*
Consider next the problem of evaluating the Fourier transform of the cosine function $\cos(2\pi f_c t)$. We first use *Euler's formula* to write

$$\cos(2\pi f_c t) = \frac{1}{2}[\exp(j2\pi f_c t) + \exp(-j2\pi f_c t)] \tag{2.70}$$

Therefore, using Eq. (2.69), we find that the cosine function $\cos(2\pi f_c t)$ is represented by the Fourier-transform pair

$$\cos(2\pi f_c t) \rightleftharpoons \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)] \tag{2.71}$$

In other words, the spectrum of the cosine function $\cos(2\pi f_c t)$ consists of a pair of delta functions occurring at $f = \pm f_c$, each of which is weighted by the factor 1/2, as shown in Fig. 2.15.

Similarly, we may show that the sine function $\sin(2\pi f_c t)$ is represented by the Fourier-transform pair

$$\sin(2\pi f_c t) \rightleftharpoons \frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)] \tag{2.72}$$

which is illustrated in Fig. 2.16.

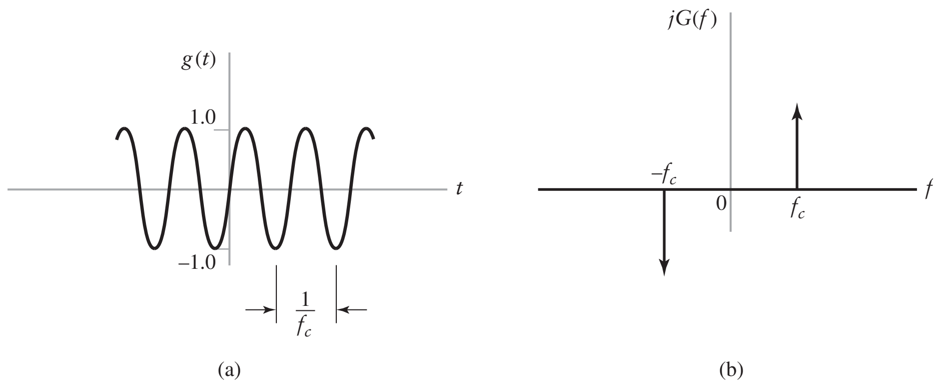


FIGURE 2.16 (*a*) Sine function. (*b*) Spectrum.

► **Drill Problem 2.9** Determine the Fourier transform of the squared sinusoidal signals:

(i) $g(t) = \cos^2(2\pi f_c t)$

(ii) $g(t) = \sin^2(2\pi f_c t)$ ◄

4. *Signum Function.*

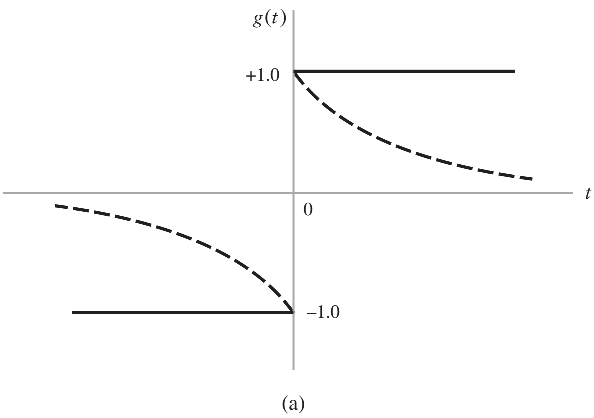
The *signum function* $\text{sgn}(t)$ equals $+1$ for positive time and -1 for negative time, as shown by the solid curve in Fig. 2.17(a). The signum function was defined previously in Eq. (2.18); this definition is reproduced here for convenience of presentation:

$$\text{sgn}(t) = \begin{cases} +1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

The signum function does not satisfy the Dirichlet conditions and therefore, strictly speaking, it does not have a Fourier transform. However, we may define a Fourier transform for the signum function by viewing it as the limiting form of the odd-symmetric double-exponential pulse

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 0, & t = 0 \\ -\exp(at), & t < 0 \end{cases} \tag{2.73}$$

as the parameter a approaches zero. The signal $g(t)$, shown as the dashed curve in Fig. 2.17(a), does satisfy the Dirichlet conditions. Its Fourier transform was derived in



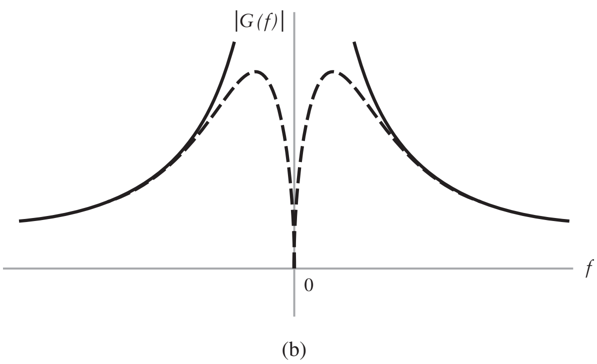


FIGURE 2.17 (a) Signum function (continuous curve), and double-exponential pulse (dashed curve). (b) Amplitude spectrum of signum function (continuous curve), and that of double-exponential pulse (dashed curve).

Example 2.3; the result is given by [see Eq. (2.19)]:

$$G(f) = \frac{-j4\pi f}{a^2 + (2\pi f)^2}$$

The amplitude spectrum $|G(f)|$ is shown as the dashed curve in Fig. 2.17(b). In the limit as a approaches zero, we have

$$\begin{aligned} \mathbf{F}[\text{sgn}(t)] &= \lim_{a \to 0} \frac{-4j\pi f}{a^2 + (2\pi f)^2} \\ &= \frac{1}{j\pi f} \end{aligned}$$

That is,

$$\text{sgn}(t) \rightleftharpoons \frac{1}{j\pi f} \tag{2.74}$$

The amplitude spectrum of the signum function is shown as the continuous curve in Fig. 2.17(b). Here we see that for small a, the approximation is very good except near the origin on the frequency axis. At the origin, the spectrum of the approximating function $g(t)$ is zero for $a > 0$, whereas the spectrum of the signum function goes to infinity.

5. *Unit Step Function.*
The *unit step function* $u(t)$ equals $+1$ for positive time and zero for negative time. Previously defined in Eq. (2.11), it is reproduced here for convenience:

$$u(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases}$$

The waveform of the unit step function is shown in Fig. 2.18(a). From this defining equation and that of the signum function, or from the waveforms of Figs. 2.17(a) and 2.18(a), we see that the unit step function and signum function are related by

$$u(t) = \frac{1}{2}[\text{sgn}((t) + 1)] \tag{2.75}$$

Hence, using the linearity property of the Fourier transform and the Fourier-transform pairs of Eqs. (2.67) and (2.75), we find that the unit step function is represented by the Fourier-transform pair

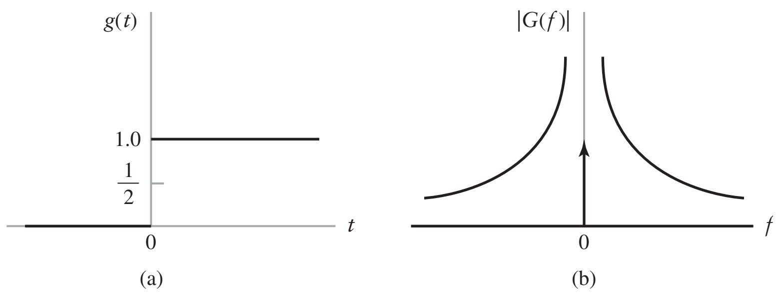


FIGURE 2.18 (a) Unit step function. (b) Amplitude spectrum.

$$u(t) \rightleftharpoons \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) \tag{2.76}$$

This means that the spectrum of the unit step function contains a delta function weighted by a factor of 1/2 and occurring at zero frequency, as shown in Fig. 2.18(b).

6. *Integration in the Time Domain (Revisited).*
The relation of Eq. (2.41) describes the effect of integration on the Fourier transform of a signal $g(t)$, assuming that $G(0)$ is zero. We now consider the more general case, with no such assumption made.

Let

$$y(t) = \int_{-\infty}^{t} g(\tau)\, d\tau \tag{2.77}$$

The integrated signal $y(t)$ can be viewed as the convolution of the original signal $g(t)$ and the unit step function $u(t)$, as shown by

$$y(t) = \int_{-\infty}^{\infty} g(\tau)u(t-\tau)\, d\tau$$

where the time-shifted unit step function $u(t-\tau)$ is itself defined by

$$u(t-\tau) = \begin{cases} 1, & \tau < t \\ \dfrac{1}{2}, & \tau = t \\ 0, & \tau > t \end{cases}$$

Recognizing that convolution in the time domain is transformed into multiplication in the frequency domain in accordance with Property 12, and using the Fourier-transform pair of Eq. (2.76) for the unit step function $u(t)$, we find that the Fourier transform of $y(t)$ is

$$Y(f) = G(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] \tag{2.78}$$

where $G(f)$ is the Fourier transform of $g(t)$. According to the sifting property of a delta function formulated in the frequency domain, we have

$$G(f)\delta(f) = G(0)\delta(f)$$

Hence, we may rewrite Eq. (2.78) in the equivalent form:

$$Y(f) = \frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\delta(f)$$

In general, the effect of integrating the signal $g(t)$ is therefore described by the Fourier-transform pair

$$\int_{-\infty}^{t} g(\tau)\, d\tau \rightleftharpoons \frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\delta(f) \tag{2.79}$$

This is the desired result, which includes Eq. (2.41) as a special case (i.e., $G(0) = 0$).

► **Drill Problem 2.10** Consider the function

$$g(t) = \delta\left(t + \frac{1}{2}\right) - \delta\left(t - \frac{1}{2}\right)$$

which consists of the difference between two delta functions at $t = \pm\frac{1}{2}$. The integration of $g(t)$ with respect to time t yields the unit rectangular function $\text{rect}(t)$. Using Eq. (2.79), show that

$$\text{rect}(t) \rightleftharpoons \text{sinc}(f)$$

which is a special form of Eq. (2.10). ◀

2.5 *Fourier Transforms of Periodic Signals*

It is well known that by using the Fourier series, a periodic signal can be represented as a sum of complex exponentials. (Appendix 2 presents a review of the Fourier series.) Also, in a limiting sense, Fourier transforms can be defined for complex exponentials, as demonstrated in Eqs. (2.69), (2.71), and (2.72). Therefore, it seems reasonable to represent a periodic signal in terms of a Fourier transform, provided that this transform is permitted to include delta functions.

Consider then a periodic signal $g_{T_0}(t)$, where the subscript T_0 denotes the *period* of the signal. We know that $g_{T_0}(t)$ can be represented in terms of the *complex exponential Fourier series* as (see Appendix 2)

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_0 t) \tag{2.80}$$

where c_n is the *complex Fourier coefficient*, defined by

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_{T_0}(t) \exp(-j2\pi n f_0 t)\, dt \tag{2.81}$$

and f_0 is the *fundamental frequency* defined as the reciprocal of the period T_0; that is,

$$f_0 = \frac{1}{T_0} \tag{2.82}$$

Let $g(t)$ be a pulselike function, which equals $g_{T_0}(t)$ over one period and is zero elsewhere; that is,

$$g(t) = \begin{cases} g_{T_0}(t), & -\frac{T_0}{2} \le t \le \frac{T_0}{2} \\ 0, & \text{elsewhere} \end{cases} \tag{2.83}$$

The periodic signal $g_{T_0}(t)$ may now be expressed in terms of the function $g(t)$ as the infinite summation

$$g_{T_0}(t) = \sum_{m=-\infty}^{\infty} g(t - mT_0) \tag{2.84}$$

Based on this representation, we may view $g(t)$ as a *generating function*, in that it generates the periodic signal $g_{T_0}(t)$. Being pulselike with some finite energy, the function $g(t)$ is Fourier transformable. Accordingly, in light of Eqs. (2.82) and (2.83), we may rewrite the formula for the complex Fourier coefficient c_n as follows:

$$\begin{aligned} c_n &= f_0 \int_{-\infty}^{\infty} g(t) \exp(-j2\pi n f_0 t)\, dt \\ &= f_0 G(n f_0) \end{aligned} \tag{2.85}$$

where $G(nf_0)$ is the Fourier transform of $g(t)$, evaluated at the frequency $f = nf_0$. We may thus rewrite the formula of Eq. (2.80) for the reconstruction of the periodic signal $g_{T_0}(t)$ as

$$g_{T_0}(t) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0) \exp(j2\pi nf_0 t) \tag{2.86}$$

Therefore, eliminating $g_{T_0}(t)$ between Eqs. (2.84) and (2.86), we may now write

$$\sum_{m=-\infty}^{\infty} g(t - mT_0) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0) \exp(j2\pi nf_0 t) \tag{2.87}$$

which defines one form of *Poisson's sum formula*.

Finally, using Eq. (2.69), which defines the Fourier transform of a complex exponential function, in Eq. (2.87), we deduce the Fourier-transform pair:

$$\sum_{m=-\infty}^{\infty} g(t - mT_0) \rightleftharpoons f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0) \tag{2.88}$$

for the periodic signal $g_{T_0}(t)$ whose fundamental frequency $f_0 = (1/T_0)$. Equation (2.88) simply states that the Fourier transform of a periodic signal consists of delta functions occurring at integer multiples of the fundamental frequency f_0, including the origin, and that each delta function is weighted by a factor equal to the corresponding value of $G(nf_0)$. Indeed, this relation merely provides a method to display the frequency content of the periodic signal $g_{T_0}(t)$.

It is of interest to observe that the pulselike function $g(t)$, constituting one period of the periodic signal $g_{T_0}(t)$, has a *continuous spectrum* defined by $G(f)$. On the other hand, the periodic signal $g_{T_0}(t)$ itself has a *discrete spectrum*. In words, we may therefore sum up the transformation embodied in Eq. (2.88) as follows:

> Periodicity in the time domain has the effect of changing the spectrum of a pulselike signal into a discrete form defined at integer multiples of the fundamental frequency, and vice versa.

EXAMPLE 2.11 Ideal Sampling Function

An *ideal sampling function*, or *Dirac comb*, consists of an infinite sequence of uniformly spaced delta functions, as shown in Fig. 2.19(*a*). We denote this waveform by

$$\delta_{T_0}(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0) \tag{2.89}$$

We observe that the generating function $g(t)$ for the ideal sampling function $\delta_{T_0}(t)$ consists simply of the delta function $\delta(t)$. We therefore have $G(f) = 1$, and

$$G(nf_0) = 1 \qquad \text{for all } n$$

Thus, the use of Eq. (2.88) yields the new result

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) \rightleftharpoons f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0) \tag{2.90}$$

Equation (2.90) states that the Fourier transform of a periodic train of delta functions, spaced T_0 seconds apart, consists of another set of delta functions weighted by the factor $f_0 = (1/T_0)$ and regularly spaced f_0 Hz apart along the frequency axis as in Fig. 2.19(*b*). In the special case of $T_0 = 1$, a periodic train of delta functions is, like a Gaussian pulse, its own Fourier transform.

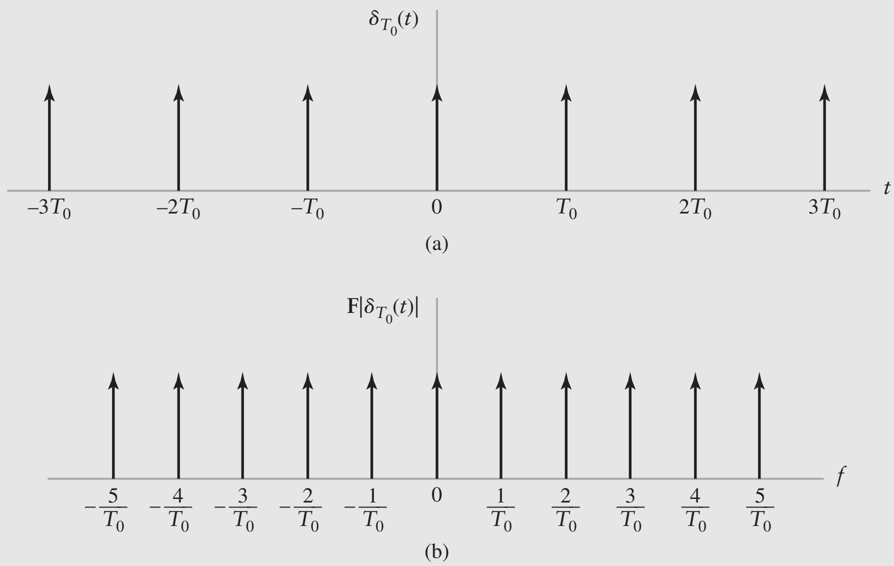


FIGURE 2.19 (*a*) Dirac comb. (*b*) Spectrum.

Applying the inverse Fourier transform to the right-hand side of Eq. (2.90), we get the relationship

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) = f_0 \sum_{n=-\infty}^{\infty} \exp(j2\pi n f_0 t) \tag{2.91}$$

On the other hand, applying the Fourier transform to the left-hand side of Eq. (2.90), we get the *dual* relationship:

$$T_0 \sum_{m=-\infty}^{\infty} \exp(j2\pi m f T_0) = \sum_{n=-\infty}^{\infty} \delta(f - n f_0) \tag{2.92}$$

where we have used the relation of Eq. (2.82) rewritten in the form $T_0 = 1/f_0$. Equations (2.91) and (2.92) are the *dual* of each other, in that in the delta functions show up in the time domain in Eq. (2.91) whereas in Eq. (2.92) the delta functions show up in the frequency domain.

► **Drill Problem 2.11** Using the Euler formula $\cos x = \frac{1}{2}[\exp(jx) + \exp(-jx)]$, reformulate Eqs. (2.91) and (2.92) in terms of cosinusoidal functions. ◄

2.6 *Transmission of Signals Through Linear Systems: Convolution Revisited*

With the Fourier transform theory presented in the previous sections at our disposal, we are now ready to turn our attention to the study of a special class of systems known to be *linear*. A *system* refers to any physical device or phenomenon that produces an output signal in response to an input signal. It is customary to refer to the input signal as the *excitation* and to the output signal as the *response*. In a *linear* system, the *principle of superposition* holds; that is, *the response of a linear system to a number of excitations applied simultaneously is equal to the sum of the responses of the system when each excitation is applied individually*. Important examples of linear systems include *filters* and *communication channels* operating in their linear region. A filter refers to a frequency-selective device that is used to limit the spectrum of a signal to some band of frequencies. A channel refers to a physical medium that connects the transmitter and receiver of a communication system. We wish to evaluate the effects of transmitting signals through linear filters and communication channels. This evaluation may be carried out in two ways, depending on the description adopted for the filter or channel. That is, we may use time-domain or frequency-domain ideas, as described below.

TIME RESPONSE

In the time domain, a linear system is described in terms of its *impulse response*, which is defined as *the response of the system (with zero initial conditions) to a unit impulse or delta function $\delta(t)$ applied to the input of the system*. If the system is *time invariant*, then this property implies that a time-shifted unit impulse at the input of the system produces an impulse response at the output, shifted by exactly the same amount. In other words, the shape of the impulse response of a linear time-invariant system is the same no matter when the unit impulse is applied to the system. Thus, assuming that the unit impulse or delta function is applied at time $t = 0$, we may denote the impulse response of a linear time-invariant system by $h(t)$. Let this system be subjected to an arbitrary excitation $x(t)$, as in Fig. 2.20(*a*). To determine the response $y(t)$ of the system, we begin by first approximating $x(t)$ by a staircase function composed of narrow rectangular pulses, each of duration $\Delta\tau$, as shown in Fig. 2.20(*b*). Clearly the approximation becomes better for smaller $\Delta\tau$. As $\Delta\tau$ approaches zero, each pulse approaches, in the limit, a delta function weighted by a factor equal to the height of the pulse times $\Delta\tau$. Consider a typical pulse, shown shaded in Fig. 2.20(*b*), which occurs at $t = \tau$. This pulse has an area equal to $x(\tau)\Delta\tau$. By definition, the response of the system to a unit impulse or delta function $\delta(t)$, occurring at $t = 0$, is $h(t)$. It follows therefore that the response of the system to a delta function, weighted by the factor $x(\tau)\Delta\tau$ and occurring at $t = \tau$, must be $x(\tau)h(t - \tau)\Delta\tau$. To find the response $y(t)$ at some time t, we

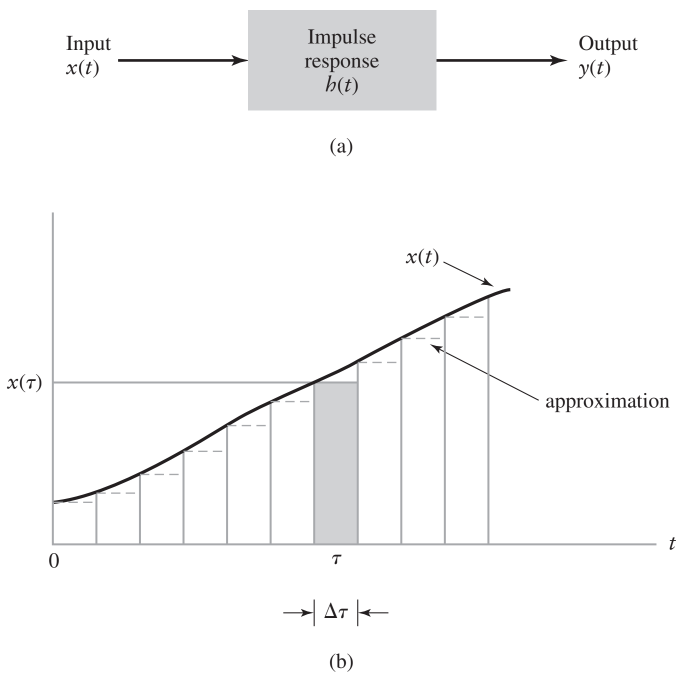


FIGURE 2.20 (*a*) Linear system with input $x(t)$ and output $y(t)$. (*b*) Staircase approximation of input $x(t)$.

apply the principle of superposition. Thus, summing the various infinitesimal responses due to the various input pulses, we obtain in the limit, as $\Delta\tau$ approaches zero,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \tag{2.93}$$

This relation is called the *convolution integral*.

In Eq. (2.93), three different time scales are involved: *excitation time* τ, *response time* t, and *system-memory time* $(t-\tau)$. This relation is the basis of time-domain analysis of linear time-invariant systems. It states that *the present value of the response of a linear time-invariant system is a weighted integral over the past history of the input signal, weighted according to the impulse response of the system*. Thus, the impulse response acts as a *memory function* for the system.

In Eq. (2.93), the excitation $x(t)$ is convolved with the impulse response $h(t)$ to produce the response $y(t)$. Since convolution is commutative, it follows that we may also write

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \tag{2.94}$$

where $h(t)$ is convolved with $x(t)$.

EXAMPLE 2.12 Tapped-Delay-Line Filter

Consider a linear time-invariant filter with impulse response $h(t)$. We make two assumptions:

1. *Causality*, which means that the impulse response $h(t)$ is zero for $t < 0$.
2. *Finite support*, which means that the impulse response of the filter is of some finite duration T_f, so that we may write $h(t) = 0$ for $t \geq T_f$.

Under these two assumptions, we may express the filter output $y(t)$ produced in response to the input $x(t)$ as

$$y(t) = \int_{0}^{T_f} h(\tau)x(t-\tau)\,d\tau \tag{2.95}$$

Let the input $x(t)$, impulse response $h(t)$, and output $y(t)$ be *uniformly sampled* at the rate $(1/\Delta\tau)$ samples per second, so that we may put

$$t = n\,\Delta\tau$$

and

$$\tau = k\,\Delta\tau$$

where k and n are integers, and $\Delta\tau$ is the *sampling period*. Assuming that $\Delta\tau$ is small enough for the product $h(\tau)x(t-\tau)$ to remain essentially constant for $k\,\Delta\tau \leq \tau \leq (k+1)\,\Delta\tau$ for all values of k and τ, we may approximate Eq. (2.95) by a *convolution sum* as shown by

$$y(n\,\Delta\tau) = \sum_{k=0}^{N-1} h(k\,\Delta\tau)x(n\,\Delta\tau - k\,\Delta\tau)\,\Delta\tau$$

where $N\,\Delta\tau = T_f$. Define the *weight*

$$w_k = h(k\,\Delta\tau)\,\Delta\tau, \qquad k = 0, 1, \ldots, N-1 \tag{2.96}$$

We may then rewrite the formula for $y(n\,\Delta\tau)$ as

$$y(n\,\Delta\tau) = \sum_{k=0}^{N-1} w_k x(n\,\Delta\tau - k\,\Delta\tau) \tag{2.97}$$

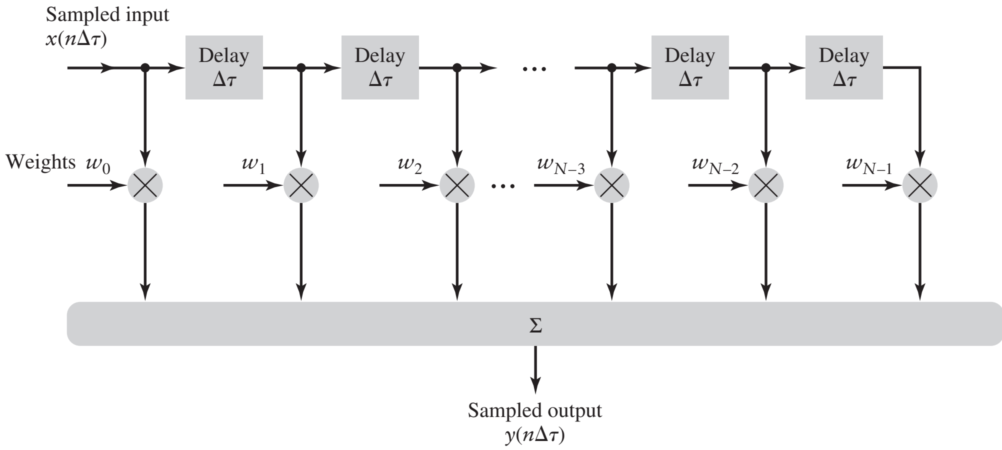


FIGURE 2.21 Tapped-delay-line filter.

Equation (2.97) may be realized using the structure shown in Fig. 2.21, which consists of a set of *delay elements* (each producing a delay of $\Delta\tau$ seconds), a set of *multipliers* connected to the *delay-line taps*, a corresponding set of *weights* supplied to the multipliers, and a *summer* for adding the multiplier outputs. This structure is known as a *tapped-delay-line filter* or *transversal filter*. Note that in Fig. 2.21 the tap-spacing or basic increment of delay is equal to the sampling period of the input sequence $\{x(n\,\Delta\tau)\}$.

Causality and Stability

A system is said to be *causal* if it does not respond before the excitation is applied. For a linear time-invariant system to be causal, it is clear that the impulse response $h(t)$ must vanish for negative time, as stated in Example 2.12. That is, we may formally state that the necessary and sufficient condition for a linear time-invariant system to be causal is

$$h(t) = 0, \qquad t < 0 \tag{2.98}$$

Clearly, for a system operating in *real time* to be physically realizable, it must be causal. However, there are many applications in which the signal to be processed is only available in *stored* form; in these situations, the system can be noncausal and yet physically realizable.

The system is said to be *stable* if the output signal is bounded for all bounded input signals. We refer to this requirement as the *bounded input–bounded output* (BIBO) *stability criterion*, which is well suited for the analysis of linear time-invariant systems. Let the input signal $x(t)$ be *bounded*, as shown by

$$|x(t)| < M \qquad \text{for all } t$$

where M is a positive real finite number. Taking the absolute values of both sides of Eq. (2.94), we have

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \right| \tag{2.99}$$

Next, we recognize that the absolute value of an integral is *bounded* by the integral of the absolute value of the integrand, as shown by

$$\left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau \right| \leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|\, d\tau$$
$$= M \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

Hence, substituting this inequality into Eq. (2.99) yields the important result

$$|y(t)| \leq M \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

It follows therefore that for a linear time-invariant system to be stable, the impulse response $h(t)$ must be absolutely integrable. That is, *the necessary and sufficient condition for BIBO stability of a linear time-invariant system* is described by

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty \tag{2.100}$$

where $h(t)$ is the impulse response of the system.

■ FREQUENCY RESPONSE

Consider next a linear time-invariant system of impulse response $h(t)$, which is driven by a complex exponential input of unit amplitude and frequency f; that is,

$$x(t) = \exp(j2\pi ft) \tag{2.101}$$

Using Eqs. (2.101) in (2.94), the response of the system is obtained as

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \exp[j2\pi f(t-\tau)]\, d\tau$$
$$= \exp(j2\pi ft) \int_{-\infty}^{\infty} h(\tau) \exp(-j2\pi f\tau)\, d\tau \tag{2.102}$$

Define the *transfer function* or *frequency response* of the system as the Fourier transform of its impulse response, as shown by

$$H(f) = \int_{-\infty}^{\infty} h(t) \exp(-j2\pi ft)\, dt \tag{2.103}$$

The terms transfer function and frequency response are used interchangably. The integral in the last line of Eq. (2.102) is the same as that of Eq. (2.103), except for the fact that τ is used in place of t. Hence, we may rewrite Eq. (2.102) in the form

$$y(t) = H(f) \exp(j2\pi ft) \tag{2.104}$$

Equation (2.104) states that the response of a linear time-invariant system to a complex exponential function of frequency f is the same complex exponential function multiplied by a constant coefficient $H(f)$.

Equation (2.103) is one definition of the transfer function $H(f)$. An alternative definition of the transfer function may be deduced by dividing Eq. (2.104) by (2.101) to obtain

$$H(f) = \left.\frac{y(t)}{x(t)}\right|_{x(t)=\exp(j2\pi ft)} \tag{2.105}$$

Consider next an arbitrary signal $x(t)$ applied to the system. The signal $x(t)$ may be expressed in terms of its inverse Fourier transform as

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)\,df \tag{2.106}$$

Equivalently, we may express $x(t)$ in the limiting form

$$x(t) = \lim_{\substack{\Delta f\to 0\\ f=k\,\Delta f}} \sum_{k=-\infty}^{\infty} X(f)\exp(j2\pi ft)\,\Delta f \tag{2.107}$$

That is, the input signal $x(t)$ may be viewed as a superposition of complex exponentials of incremental amplitude. Because the system is linear, the response to this superposition of complex exponential inputs is given by

$$\begin{aligned} y(t) &= \lim_{\substack{\Delta f\to 0\\ f=k\,\Delta f}} \sum_{k=-\infty}^{\infty} H(f)X(f)\exp(j2\pi ft)\,\Delta f \\ &= \int_{-\infty}^{\infty} H(f)X(f)\exp(j2\pi ft)\,df \end{aligned} \tag{2.108}$$

The Fourier transform of the output signal $y(t)$ is therefore readily obtained as

$$Y(f) = H(f)X(f) \tag{2.109}$$

According to Eq. (2.109), a linear time-invariant system may thus be described quite simply in the frequency domain by noting that *the Fourier transform of the output is equal to the product of the frequency response of the system and the Fourier transform of the input*.

Of course, we could have deduced the result of Eq. (2.109) directly by recognizing two facts:

1. The response $y(t)$ of a linear time-invariant system of impulse response $h(t)$ to an arbitrary input $x(t)$ is obtained by convolving $x(t)$ with $h(t)$, in accordance with Eq. (2.93).
2. The convolution of a pair of time functions is transformed into the multiplication of their Fourier transforms.

The alternative derivation of Eq. (2.109) above is presented primarily to develop an understanding of why the Fourier representation of a time function as a superposition of complex exponentials is so useful in analyzing the behavior of linear time-invariant systems.

The frequency response $H(f)$ is a characteristic property of a linear time-invariant system. It is, in general, a complex quantity, so that we may express it in the form

$$H(f) = |H(f)|\exp[j\beta(f)] \tag{2.110}$$

where $|H(f)|$ is called the *amplitude response* or *magnitude response*, and $\beta(f)$ the *phase* or *phase response*. In the special case of a linear system with real-valued impulse response $h(t)$, the frequency response $H(f)$ exhibits conjugate symmetry, which means that

$$|H(f)| = |H(-f)|$$

and

$$\beta(f) = -\beta(-f)$$

That is, the amplitude response $|H(f)|$ of a linear system with real-valued impulse response is an even function of frequency, whereas the phase $\beta(f)$ is an odd function of frequency.

In some applications it is preferable to work with the logarithm of $H(f)$, expressed in polar form, rather than with $H(f)$ itself. Define the natural logarithm

$$\ln H(f) = \alpha(f) + j\beta(f) \tag{2.111}$$

where

$$\alpha(f) = \ln|H(f)| \tag{2.112}$$

The function $\alpha(f)$ is one definition of the *gain* of the system. It is measured in *nepers*, whereas the phase $\beta(f)$ is measured in radians. Equation (2.111) indicates that the gain $\alpha(f)$ and phase $\beta(f)$ are the real and imaginary parts of the natural logarithm of the frequency response $H(f)$, respectively. The gain may also be expressed in *decibels* (dB) by using the definition

$$\alpha'(f) = 20 \log_{10}|H(f)| \tag{2.113}$$

The two gain functions $\alpha(f)$ and $\alpha'(f)$ are related by

$$\alpha'(f) = 8.69\alpha(f) \tag{2.114}$$

That is, 1 neper is equal to 8.69 dB.

From the discussion presented Section 2.3, we note that the *bandwidth* of a system is specified by the constancy of its amplitude response. The bandwidth of a low-pass system is thus defined as the frequency at which the amplitude response $|H(f)|$ is $1/\sqrt{2}$ times its value of zero frequency or, equivalently, the frequency at which the gain $\alpha'(f)$ drops by 3 dB below its value at zero frequency, as illustrated in Fig. 2.22(*a*). Correspondingly, the bandwidth of a band-pass system is defined as the range of frequencies over which the amplitude response $|H(f)|$ remains within $1/\sqrt{2}$ times its value at the mid-band frequency, as illustrated in Fig. 2.22(*b*).

■ PALEY–WIENER CRITERION

A necessary and sufficient condition for a function $\alpha(f)$ to be the gain of a causal filter is the convergence of the integral.

$$\int_{-\infty}^{\infty} \left(\frac{|\alpha(f)|}{1 + f^2} \right) df < \infty \tag{2.115}$$

This condition is known as the *Paley–Wiener criterion*. It states that, provided the gain $\alpha(f)$ satisfies the condition of Eq. (2.115), then we may associate with this gain a suitable phase $\beta(f)$ such that the resulting filter has a causal impulse response that is zero for negative time. In other words, the Paley–Wiener criterion is the frequency-domain equivalent of the causality requirement. A system with a realizable gain characteristic may have infinite attenuation [i.e., $\alpha(f) = -\infty$] for a discrete set of frequencies, but it cannot have infinite attenuation over a band of frequencies; otherwise, the Paley–Wiener criterion is violated.

► **Drill Problem 2.12** Discuss the following two issues, citing examples for your answers:

(**a**) Is it possible for a linear time-invariant system to be causal but unstable?

(**b**) Is it possible for such a system to be noncausal but stable? ◄

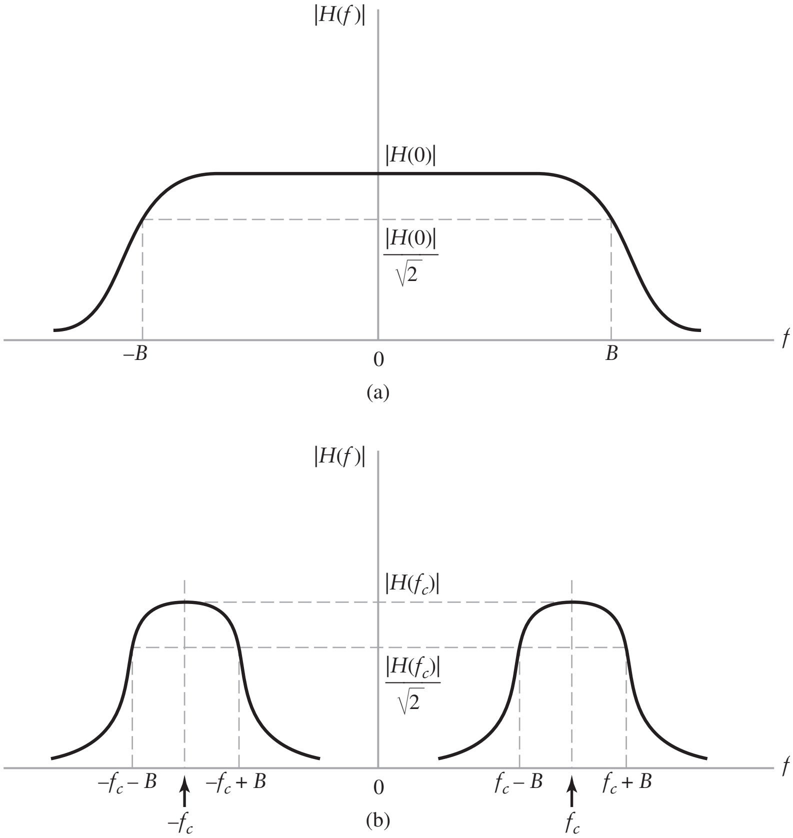


FIGURE 2.22 Illustration of the definition of system bandwidth. (*a*) Low-pass system. (*b*) Band-pass system.

► **Drill Problem 2.13** The impulse response of a linear system is defined by the Gaussian function

$$h(t) = \exp\left(-\frac{t^2}{2\tau^2}\right)$$

where τ is an adjustable parameter that defines pulse duration. Determine the frequency response of the system. ◄

► **Drill Problem 2.14** A tapped-delay-line filter consists of N weights, where N is odd. It is symmetric with respect to the center tap; that is, the weights satisfy the condition

$$w_n = w_{N-1-n}, \qquad 0 \le n \le N-1$$

(a) Find the amplitude response of the filter.

(b) Show that this filter has a linear phase response. What is the implication of this property?

(c) What is the time delay produced by the filter? ◄

2.7 *Ideal Low-Pass Filters*

As previously mentioned, a *filter* is a frequency-selective system that is used to limit the spectrum of a signal to some specified band of frequencies. Its frequency response is characterized by a *passband* and a *stopband*. The frequencies inside the passband are transmitted with little or no distortion, whereas those in the stopband are rejected. The filter may be of the *low-pass, high-pass, band-pass*, or *band-stop* type, depending on whether it transmits low, high, intermediate, or all but intermediate frequencies, respectively. We have already encountered examples of low-pass and band-pass systems in Fig. 2.22.

Filters, in one form or another, represent an important functional block in building communication systems. In this book, we will be concerned with the use of high-pass, low-pass, and band-pass filters.

In this section, we study the time response of the *ideal low-pass filter*, which transmits, without any distortion, all frequencies inside the passband and completely rejects all frequencies inside the stopband, as illustrated in Fig. 2.23. According to this figure, the frequency response of an ideal low-pass filter satisfies two necessary conditions:

1. *The amplitude response of the filter is a constant inside the passband* $-B \le f \le B$. (The constant in Fig. 2.23 is set equal to unity for convenience of presentation.)
2. *The phase response varies linearly with frequency inside the passband of the filter*. (Outside the passband, the phase response may assume arbitrary values.)

In mathematical terms, the transfer function of an ideal low-pass filter is therefore defined by

$$H(f) = \begin{cases} \exp(-j2\pi f t_0), & -B \le f \le B \\ 0, & |f| > B \end{cases} \tag{2.116}$$

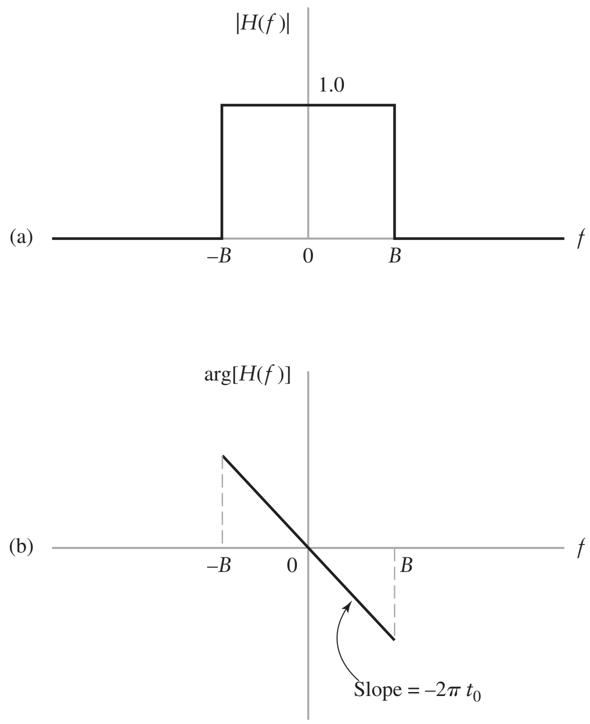


FIGURE 2.23 Frequency response of ideal low-pass filter. (*a*) Amplitude response. (*b*) Phase response; outside the band $-B \le f \le B$, the phase response assumes an arbitrary form (not shown in the figure).

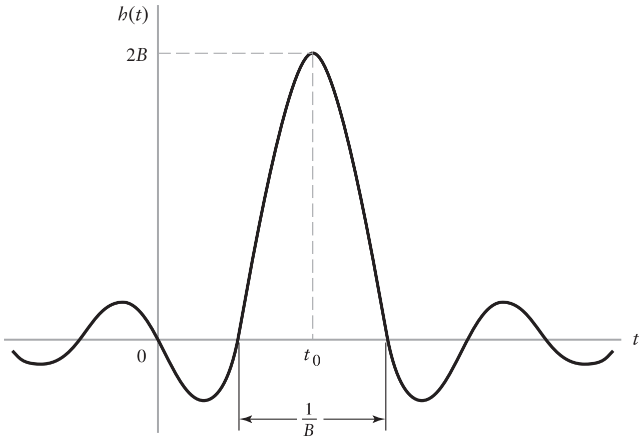


FIGURE 2.24 Impulse response of ideal low-pass filter.

The parameter B defines the *bandwidth* of the filter. The ideal low-pass filter is, of course, noncausal because it violates the Paley–Wiener criterion. This observation may also be confirmed by examining the impulse response $h(t)$. Thus, by evaluating the inverse Fourier transform of the transfer function of Eq. (2.116), we get

$$h(t) = \int_{-B}^{B} \exp[j2\pi f(t - t_0)]\, df \tag{2.117}$$

where the limits of integration have been reduced to the frequency band inside which $H(f)$ does not vanish. Equation (2.117) is readily integrated, yielding

$$\begin{aligned} h(t) &= \frac{\sin[2\pi B(t - t_0)]}{\pi(t - t_0)} \\ &= 2B \operatorname{sinc}[2B(t - t_0)] \end{aligned} \tag{2.118}$$

The impulse response has a peak amplitude of $2B$ centered on time t_0, as shown in Fig. 2.24 for $t_0 = 1/B$. The duration of the main lobe of the impulse response is $1/B$, and the build-up time from the zero at the beginning of the main lobe to the peak value is $1/2B$. We see from Fig. 2.24 that, for any finite value of t_0, there is some response from the filter before the time $t = 0$ at which the unit impulse is applied to the input; this observation confirms that the ideal low-pass filter is noncausal. Note, however, that we can always make the delay t_0 large enough for the condition

$$\left|\operatorname{sinc}[2B(t - t_0)]\right| \ll 1, \qquad \text{for } t < 0$$

to be satisfied. By so doing, we are able to build a causal filter that approximates an ideal low-pass filter, with the approximation improving with increasing delay t_0.

■ PULSE RESPONSE OF IDEAL LOW-PASS FILTERS

Consider a rectangular pulse $x(t)$ of unit amplitude and duration T, which is applied to an ideal low-pass filter of bandwidth B. The problem is to determine the response $y(t)$ of the filter.

The impulse response $h(t)$ of the filter is defined by Eq. (2.118). Clearly, the delay t_0 has no effect on the shape of the filter response $y(t)$. Without loss of generality, we may therefore simplify the exposition by setting $t_0 = 0$, in which case the impulse response of Eq. (2.118) reduces to

$$h(t) = 2B\,\text{sinc}(2Bt) \tag{2.119}$$

With the input $x(t) = 1$ for $-(T/2) \le t \le (T/2)$, the resulting response of the filter is given by the convolution integral

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\ &= 2B\int_{-T/2}^{T/2} \text{sinc}[2B(t-\tau)]\,d\tau \\ &= 2B\int_{-T/2}^{T/2} \left(\frac{\sin[2\pi B(t-\tau)]}{2\pi B(t-\tau)}\right) d\tau \end{aligned} \tag{2.120}$$

Define a new dimensionless variable

$$\lambda = 2\pi B(t-\tau)$$

Then, changing the integration variable from τ to λ, we may rewrite Eq. (2.120) as

$$\begin{aligned} y(t) &= \frac{1}{\pi}\int_{2\pi B(t-T/2)}^{2\pi B(t+T/2)} \left(\frac{\sin\lambda}{\lambda}\right) d\lambda \\ &= \frac{1}{\pi}\left[\int_0^{2\pi B(t+T/2)} \left(\frac{\sin\lambda}{\lambda}\right) d\lambda - \int_0^{2\pi B(t-T/2)} \left(\frac{\sin\lambda}{\lambda}\right) d\lambda\right] \\ &= \frac{1}{\pi}\{\text{Si}[2\pi B(t+T/2)] - \text{Si}[2\pi B(t-T/2)]\} \end{aligned} \tag{2.121}$$

In Eq. (2.121), we have introduced a new expression called the *sine integral*, which is defined by

$$\text{Si}(u) = \int_0^u \frac{\sin x}{x}\,dx \tag{2.122}$$

Unfortunately, the sine integral $\text{Si}(u)$ cannot be evaluated in closed form in terms of elementary functions. However, it can be integrated in a power series, which, in turn, leads to the graph plotted in Fig. 2.25. From this figure we make three observations:

1. The sine integral $\text{Si}(u)$ is an oscillatory function of u, having odd symmetry about the origin $u = 0$.
2. It has its maxima and minima at multiples of π.
3. It approaches the limiting value $(\pi/2)$ for large positive values of u.

In Fig. 2.25, we see that the sine integral $\text{Si}(u)$ oscillates at a frequency of $1/2\pi$. Correspondingly, the filter response $y(t)$ will also oscillate at a frequency equal to the cutoff frequency (i.e., bandwidth) B of the low-pass filter, as indicated in Fig. 2.26. The maximum value of $\text{Si}(u)$ occurs at $u_{\max} = \pi$ and is equal to

$$1.8519 = (1.179) \times \left(\frac{\pi}{2}\right)$$

We may show that the filter response $y(t)$ has maxima and minima at

$$t_{\max} = \pm\frac{T}{2} \pm \frac{1}{2B}$$

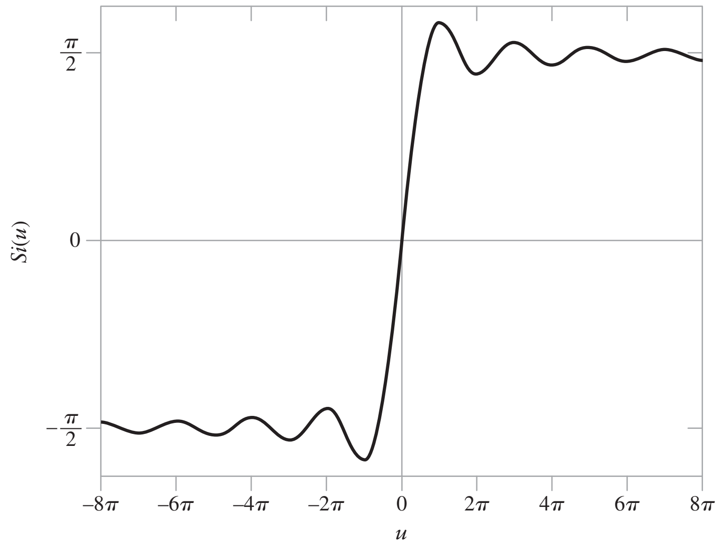


FIGURE 2.25 The sine integral $\mathrm{Si}(u)$.

with

$$y(t_{\max}) = \frac{1}{\pi}[\mathrm{Si}(\pi) - \mathrm{Si}(\pi - 2\pi BT)]$$
$$= \frac{1}{\pi}[\mathrm{Si}(\pi) + \mathrm{Si}(2\pi BT - \pi)]$$

where, in the second line, we have used the odd symmetric property of the sine integral. Let

$$\mathrm{Si}(2\pi BT - \pi) = \frac{\pi}{2}(1 \pm \Delta)$$

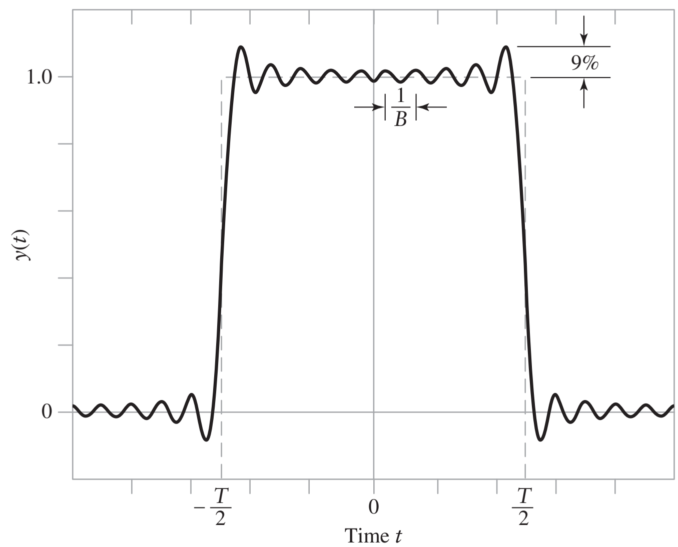


FIGURE 2.26 Ideal low-pass filter response for a square pulse.

where Δ is the absolute value of the deviation in the value of $\text{Si}(2\pi BT - \pi)$ expressed as a fraction of the final value $+\pi/2$. Thus, recognizing that

$$\text{Si}(\pi) = (1.179)(\pi/2)$$

we may redefine $y(t_{\max})$ as

$$\begin{aligned} y(t_{\max}) &= \frac{1}{2}(1.179 + 1 \pm \Delta) \\ &\approx 1.09 \pm \frac{1}{2}\Delta \end{aligned} \tag{2.123}$$

For a time-bandwidth product $BT \gg 1$, the fractional deviation Δ has a very small value, in which case we may make two important observations from Eq. (2.123):

1. The percentage overshoot in the filter response is approximately 9 percent.
2. The overshoot is practically independent of the filter bandwidth B.

The basic phenomenon underlying these two observations is called the *Gibbs phenomenon*. Figure 2.26 shows the oscillatory nature of the filter response and the 9 percent overshoot characterizing the response, assuming that $BT \gg 1$.

Figure 2.27, occupying pages 65 and 66, shows the filter response for four time-bandwidth products: $BT = 5$, 10, 20, and 100, assuming that the pulse duration T is 1 second. Table 2.1 shows the corresponding frequencies of oscillations and percentage overshoots for these time-bandwidth products, confirming observations 1 and 2.

TABLE 2.1 ***Oscillation Frequency and Percentage Overshoot for Varying Time-Bandwidth Product***

BT	Oscillation Frequency	Percentage Overshoot
5	5 Hz	9.11
10	10 Hz	8.98
20	20 Hz	8.99
100	100 Hz	9.63

Figure 2.28, occupying pages 67 and 68, shows the filter response for periodic square-wave inputs of different fundamental frequencies: $f_0 = 0.1$, 0.25, 0.5, and 1 Hz, and with the bandwidth of the low-pass filter being fixed at $B = 1$ Hz. From Fig. 2.28 we may make the following observations:

- For $f_0 = 0.1$ Hz, corresponding to a time-bandwidth product $BT = 5$, the filter somewhat distorts the input square pulse, but the shape of the input is still evident at the filter output. Unlike the input, the filter output has nonzero rise and fall times that are inversely proportional to the filter bandwidth. Also, the output exhibits oscillations (ringing) at both the leading and trailing edges.
- As the fundamental frequency f_0 of the input square wave increases, the low-pass filter cuts off more of the higher frequency components of the input. Thus, when $f_0 = 0.25$ Hz, corresponding to $BT = 2$, only the fundamental frequency and the first harmonic component pass through the filter; the rise and fall times of the output are now significant compared with the input pulse duration T. When $f_0 = 0.5$ Hz, corresponding to $BT = 1$, only the fundamental frequency component of the input square wave is preserved by the filter, resulting in an output that is essentially sinusoidal.

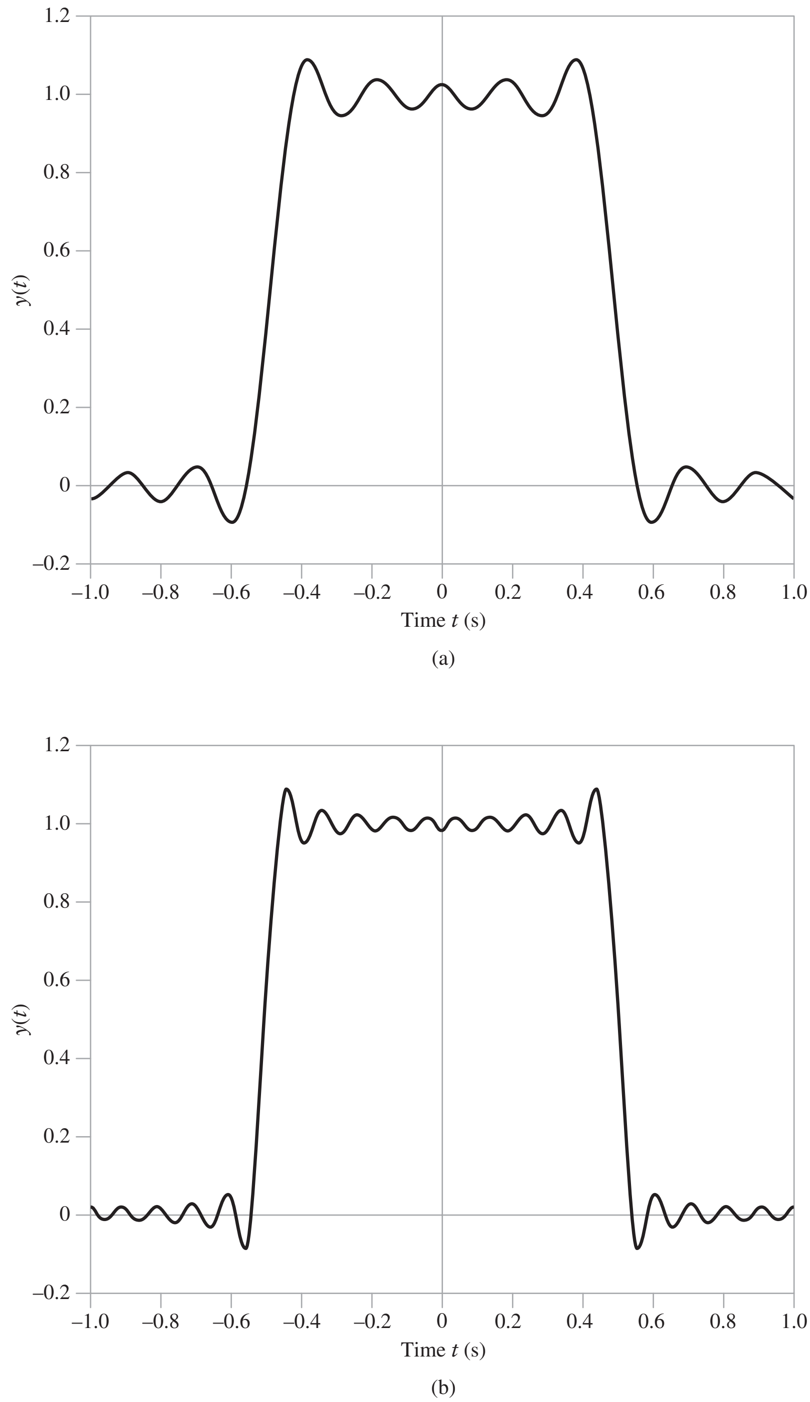
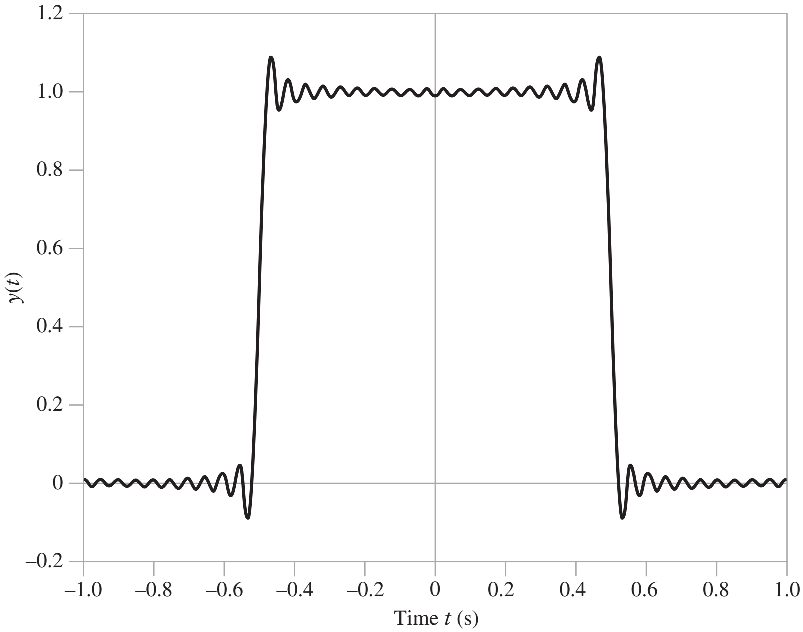
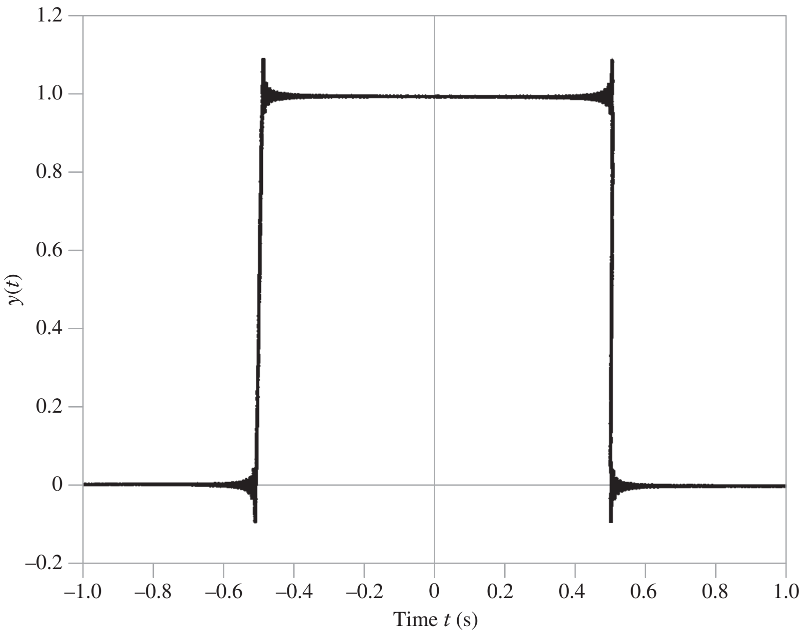


FIGURE 2.27 Pulse response of ideal low-pass filter for pulse duration $T = 1\text{s}$ and varying time-bandwidth (BT) product. (*a*) $BT = 5$. (*b*) $BT = 10$.

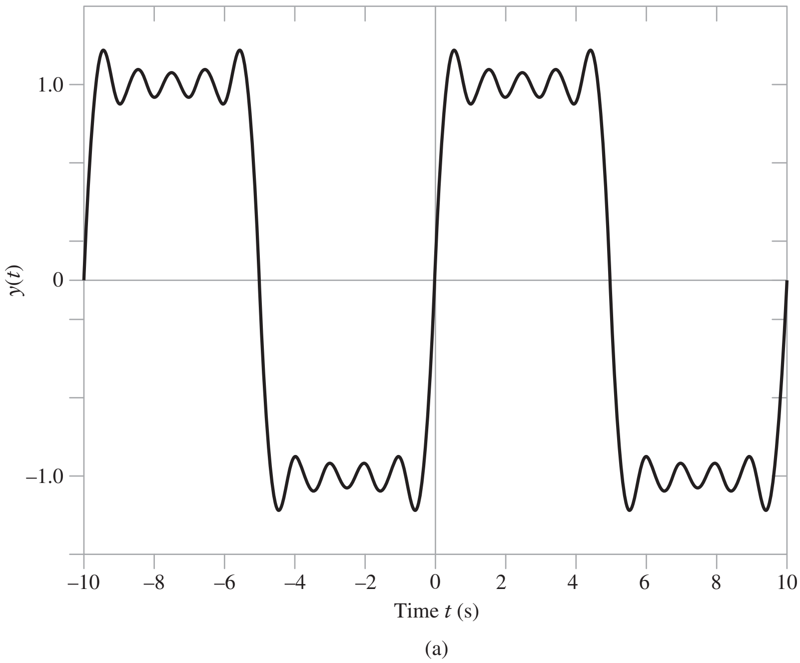


(c)



(d)

FIGURE 2.27 (*continued*) (*c*) $BT = 20$. (*d*) $BT = 100$.



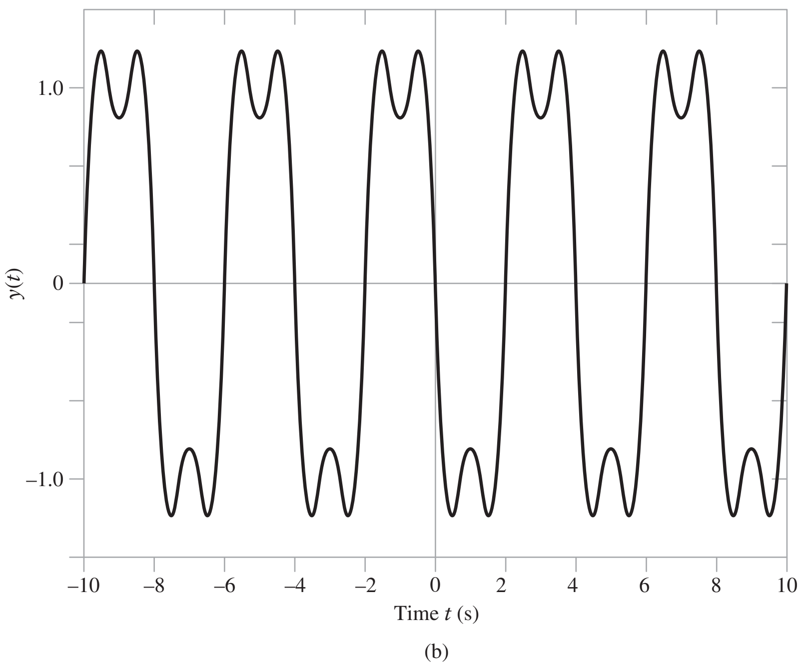


FIGURE 2.28 Response of ideal low-pass filter to a square wave of varying frequency f_0. (*a*) $f_0 = 0.1$ Hz. (*b*) $f_0 = 0.25$ Hz.

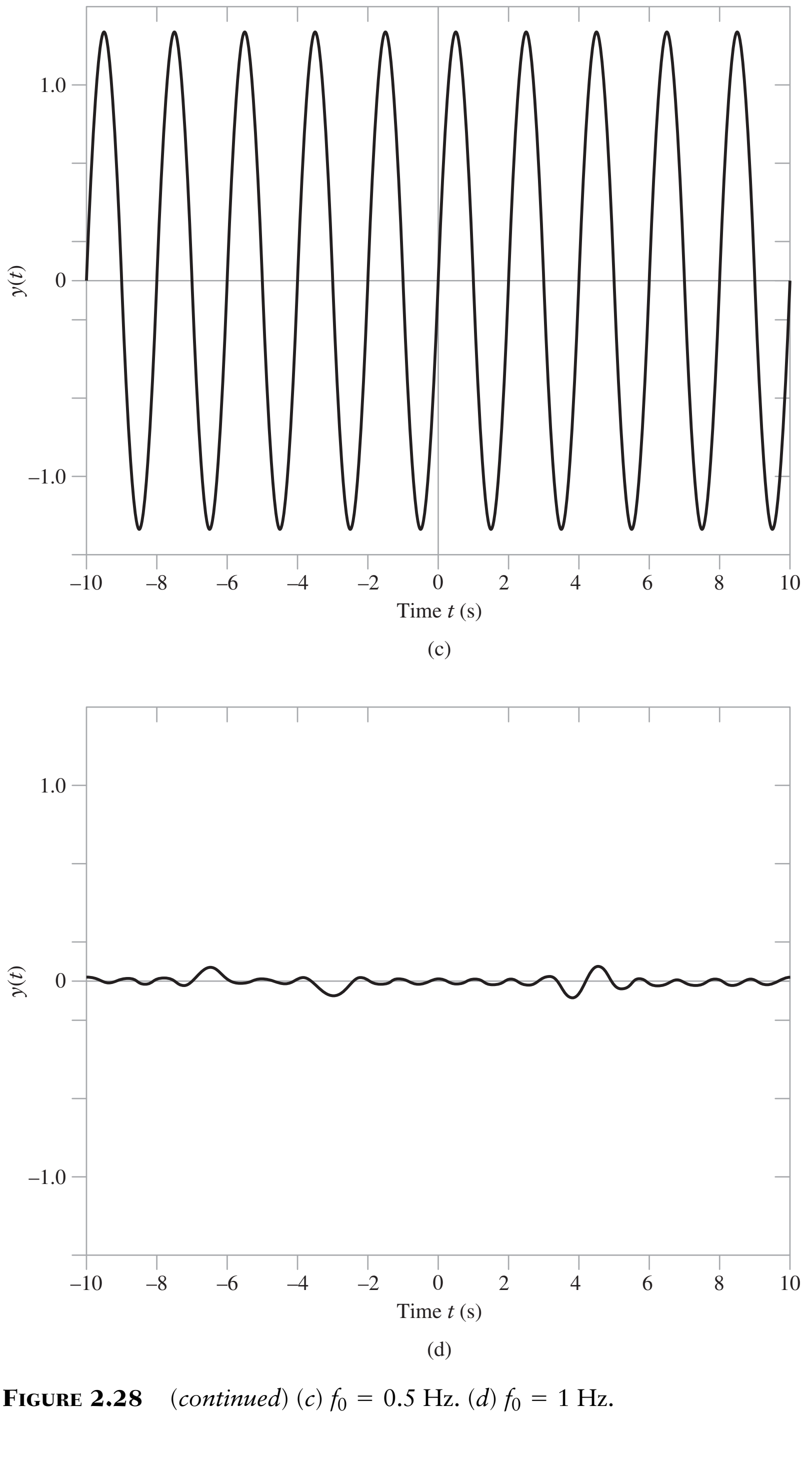


FIGURE 2.28 (*continued*) (*c*) $f_0 = 0.5$ Hz. (*d*) $f_0 = 1$ Hz.

- When the fundamental frequency of the input square wave is increased further to the high value $f_0 = 1$Hz, which corresponds to a time-bandwidth product $BT = 0.5$, the dc component becomes the dominant output, and the shape of the input square wave is completely destroyed by the filter.

From these results, we draw an important conclusion: *When using an ideal low-pass filter, we must use a time-bandwidth product $BT \geq 1$ to ensure that the waveform of the filter input is recognizable from the resulting output*. A value of BT greater than unity tends to reduce the rise time as well as decay time of the filter pulse response.

Approximation of Ideal Low-pass Filters

A filter may be characterized by specifying its impulse response $h(t)$ or, equivalently, its transfer function $H(f)$. However, the application of a filter usually involves the separation of signals on the basis of their spectra (i.e., frequency contents). This, in turn, means that the design of filters is usually carried out in the frequency domain. There are two basic steps involved in the design of a filter:

1. The *approximation* of a prescribed frequency response (i.e., amplitude response, phase response, or both) by a realizable transfer function.
2. The *realization* of the approximating transfer function by a physical device.

For an approximating transfer function $H(f)$ to be physically realizable, it must represent a *stable* system. Stability is defined here on the basis of the bounded input-bounded output criterion described in Eq. (2.100) that involves the impulse response $h(t)$. To specify the corresponding condition for stability in terms of the transfer function, the traditional approach is to replace $j2\pi f$ with s and recast the transfer function in terms of s. The new variable s is permitted to have a real part as well as an imaginary part. Accordingly, we refer to s as the *complex frequency*. Let $H'(s)$ denote the transfer function of the system, defined in the manner described herein. Ordinarily, the approximating transfer function $H'(s)$ is a rational function, which may therefore be expressed in the *factored* form

$$\begin{aligned} H'(s) &= H(f)\big|_{j2\pi f = s} \\ &= K\frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)} \end{aligned}$$

where K is a scaling factor; $z_1, z_2, \ldots, z_m$ are called the *zeros* of the transfer function, and $p_1, p_2, \ldots, p_n$ are called its *poles*. For a low-pass transfer function, the number of zeros, m, is less than the number of poles, n. If the system is causal, then the bounded input–bounded output condition for stability of the system is satisfied by restricting all the poles of the transfer function $H'(s)$ to be inside the left half of the s-plane; that is to say,

$$\mathbf{Re}([p_i]) < 0, \qquad \text{for all } i$$

Note that the condition for stability involves only the poles of the transfer function $H'(s)$; the zeros may indeed lie anywhere in the s-plane. Two types of systems may be distinguished, depending on locations of the m *zeros* in the s-plane:

- *Minimum-phase systems*, characterized by a transfer function whose poles and zeros are all restricted to lie inside the left hand of the s-plane.
- *Nonminimum-phase systems*, whose transfer functions are permitted to have zeros on the imaginary axis as well as the right half of the s-plane.

Minimum-phase systems distinguish themselves by the property that the phase response of this class of linear time-invariant systems is uniquely related to the gain response.

In the case of low-pass filters where the principal requirement is to approximate the ideal amplitude response shown in Fig. 2.23, we may mention two popular families of filters: *Butterworth filters* and *Chebyshev filters*, both of which have all their zeros at $s = \infty$. In a Butterworth filter, the poles of the transfer function $H'(s)$ lie on a circle with origin as the center and $2\pi B$ as the radius, where B is the 3-dB bandwidth of the filter. In a Chebyshev filter, on the other hand, the poles lie on an ellipse. In both cases, of course, the poles are confined to the left half of the s-plane.

Turning next to the issue of physical realization of the filter, we see that there are two basic options to do this realization, one analog and the other digital:

- *Analog filters*, built using (a) inductors and capacitors, or (b) capacitors, resistors, and operational amplifiers. The advantage of analog filters is the simplicity of implementation.
- *Digital filters*, for which the signals are sampled in time and their amplitude is also quantized. These filters are built using digital hardware; hence the name. An important feature of a digital filter is that it is *programmable*, thereby offering a high degree of flexibility in design. In effect, complexity is traded off for flexibility.

2.8 *Correlation and Spectral Density: Energy Signals*

In this section, we continue the characterization of signals and systems by considering the class of energy signals and therefore focusing on the notion of *energy*. (The characterization of signals and systems is completed in Section 2.9, where we consider the other class of signals, power signals.) In particular, we introduce a new parameter called *spectral density*, which is defined as the squared amplitude spectrum of the signal of interest. It turns out that the spectral density is the Fourier transform of the correlation function, which was first introduced under Property 13 in Section 2.2.

■ AUTOCORRELATION FUNCTION

Consider an energy signal $x(t)$ that, for the purpose of generality, is assumed to be complex valued. Following the material presented under the correlation theorem (Property 13) in Section 2.2, we formally define the *autocorrelation function* of the energy signal $x(t)$ for a *lag* τ as

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t - \tau)\, d\tau \tag{2.124}$$

According to this formula, the autocorrelation function $R_x(\tau)$ provides a measure of the similarity between the signal $x(t)$ and its delayed version $x(t - \tau)$. As such, it can be measured using the arrangement shown in Fig. 2.29. The time lag τ plays the role of a *scanning* or *searching variable*. Note that $R_x(\tau)$ is complex valued if $x(t)$ is complex valued.

From Eq. (2.124) we readily see that the value of the autocorrelation function $R_x(\tau)$ for $\tau = 0$ is equal to the energy of the signal $x(t)$; that is,

$$R_x(0) = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

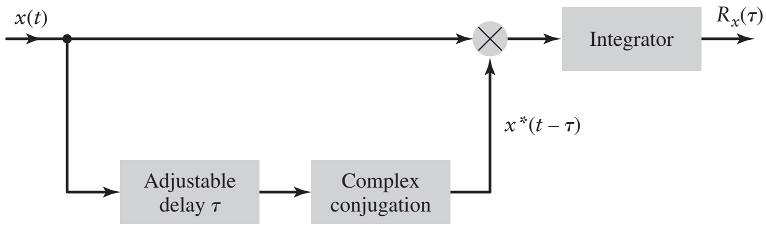


FIGURE 2.29 Scheme for measuring the autocorrelation function $R_x(\tau)$ of an energy signal $x(t)$ for lag τ.

■ ENERGY SPECTRAL DENSITY

The Rayleigh energy theorem, discussed under Property 14 in Section 2.2, is important because it not only provides a useful method for evaluating the energy of a pulse signal, but also it highlights the squared amplitude spectrum as the distribution of the energy of the signal measured in the frequency domain. It is in light of this theorem that we formally define the *energy spectral density* or *energy density spectrum* of an energy signal $x(t)$ as

$$\psi_x(f) = |X(f)|^2 \tag{2.125}$$

where $|X(f)|$ is the amplitude spectrum of $x(t)$. Clearly, the energy spectral density $\psi_x(f)$ is a nonnegative real-valued quantity for all f, even though the signal $x(t)$ may itself be complex valued.

■ WIENER–KHITCHINE RELATIONS FOR ENERGY SIGNALS

Referring to the correlation theorem described in Eq. (2.53), let $g_1(t) = g_2(t) = x(t)$, where $x(t)$ is an energy signal and therefore Fourier transformable. Under this condition, the resulting left-hand side of Eq. (2.53) defines the autocorrelation function $R_x(\tau)$ of the signal $x(t)$. Correspondingly, in the frequency domain, we have $G_1(f) = G_2(f) = X(f)$, in which case the right-hand side of Eq. (2.53) defines the energy spectral density $\psi_x(f)$. On this basis, we may therefore state that *given an energy signal $x(t)$, the autocorrelation function $R_x(\tau)$ and energy spectral density $\psi_x(f)$ form a Fourier-transform pair*. Specifically, we have the pair of relations:

$$\psi_x(f) = \int_{-\infty}^{\infty} R_x(\tau) \exp(-j2\pi f\tau)\, d\tau \tag{2.126}$$

and

$$R_x(\tau) = \int_{-\infty}^{\infty} \psi_x(f) \exp(j2\pi f\tau)\, df \tag{2.127}$$

Note, however, that the Fourier transformation in Eq. (2.126) is performed with respect to the adjustable lag τ. The pair of equations (2.126) and (2.127) constitutes the *Wiener–Khitchine relations for energy signals*.

From Eqs. (2.126) and (2.127) we readily deduce the following two properties:

1. By setting $f = 0$ in Eq. (2.126), we have

$$\int_{-\infty}^{\infty} R_x(\tau)\, d\tau = \psi_x(0)$$

which states that *the total area under the curve of the complex-valued autocorrelation function of a complex-valued energy signal is equal to the real-valued energy spectral $\psi_x(0)$ at zero frequency*.

2. By setting $\tau = 0$ in Eq. (2.127), we have

$$\int_{-\infty}^{\infty} \psi_x(f)\, df = R_x(0)$$

which states that *the total area under the curve of the real-valued energy spectral density of an energy signal is equal to the total energy of the signal*. This second result is merely another way of starting the Rayleigh energy theorem.

EXAMPLE 2.13 Autocorrelation Function of Sinc Pulse

From Example 2.4, the Fourier transform of the sinc pulse

$$x(t) = A\,\mathrm{sinc}(2Wt)$$

is given by

$$X(f) = \frac{A}{2W}\,\mathrm{rect}\left(\frac{f}{2W}\right)$$

Since the rectangular function rect $(f/2W)$ is unaffected by squaring, the energy spectral density of $x(t)$ is therefore

$$\psi_x(f) = \left(\frac{A}{2W}\right)^2 \mathrm{rect}\left(\frac{f}{2W}\right)$$

Taking the inverse Fourier transform of $\psi_x(f)$, we find that the autocorrelation function of the sinc pulse A sinc$(2Wt)$ is given by

$$R_x(\tau) = \frac{A^2}{2W}\,\mathrm{sinc}(2W\tau) \tag{2.128}$$

which has a similar waveform, plotted as a function of the lag τ, as the sinc pulse itself.

This example teaches us that sometimes it is easier to use an indirect procedure based on the energy spectral density to determine the autocorrelation function of an energy signal rather than using the formula for the autocorrelation function.

■ EFFECT OF FILTERING ON ENERGY SPECTRAL DENSITY

Suppose now the energy signal $x(t)$ is passed through a linear time-invariant system of transfer function $H(f)$, yielding the output signal $y(t)$ as illustrated in Fig. 2.20(a). Then, according to Eq. (2.109), the Fourier transform of the output $y(t)$ is related to the Fourier transform of the input $x(t)$ as follows:

$$Y(f) = H(f)X(f)$$

Taking the squared amplitude of both sides of this equation, we readily get

$$\psi_y(f) = |H(f)|^2 \psi_x(f) \tag{2.129}$$

where, by definition, $\psi_x(f) = |X(f)|^2$ and $\psi_y(f) = |Y(f)|^2$. Equation (2.129) states that *when an energy signal is transmitted through a linear time-invariant filter, the energy spectral density of the resulting output equals the energy spectral density of the input multiplied* *by the squared amplitude response of the filter*. The simplicity of this statement emphasizes the importance of spectral density as a parameter for characterizing the distribution of the energy of a Fourier transformable signal in the frequency domain.

Moreover, on the basis of the Wiener–Khintchine equations (2.126) and (2.127) and the relationship of Eq. (2.129), we may describe an *indirect method* for evaluating the effect of linear time-invariant filtering on the autocorrelation function of an energy signal:

1. Determine the Fourier transforms of $x(t)$ and $h(t)$, obtaining $X(f)$ and $H(f)$, respectively.
2. Use Eq. (2.129) to determine the energy spectral density $\psi_y(f)$ of the output $y(t)$.
3. Determine $R_y(\tau)$ by applying the inverse Fourier transform to $\psi_y(f)$ obtained under point 2.

EXAMPLE 2.14 Energy of Low-pass Filtered Version of Rectangular Pulse

A rectangular pulse of unit amplitude and unit duration is passed through an ideal low-pass filter of bandwidth B, as illustrated in Fig. 2.30(*a*). Part (*b*) of the figure depicts the waveform of the rectangular pulse. The amplitude response of the filter is defined by (see Fig. 2.30(*c*))

$$|H(f)| = \begin{cases} 1, & -B \le f \le B \\ 0, & \text{otherwise} \end{cases}$$

The rectangular pulse constituting the filter input has unit energy. We wish to evaluate the effect of varying the bandwidth B on the energy of the filter output.

We start with the Fourier transform pair:

$$\text{rect}(t) \rightleftharpoons \text{sinc}(f)$$

which represents the normalized version of the Fourier-transform pair given in Eq. (2.10). Hence, with the filter input defined by

$$x(t) = \text{rect}(t)$$

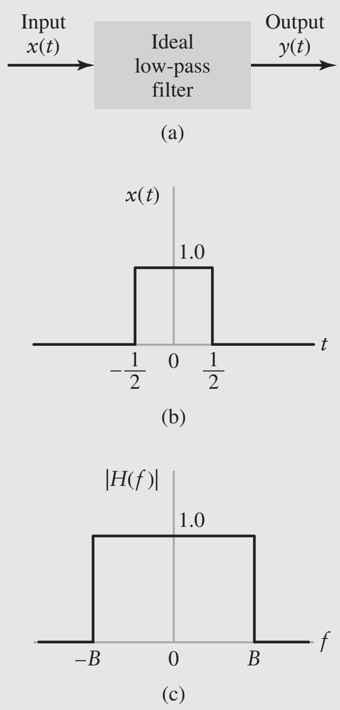


FIGURE 2.30 (*a*) Ideal low-pass filtering. (*b*) Filter input. (*c*) Amplitude response of the filter.

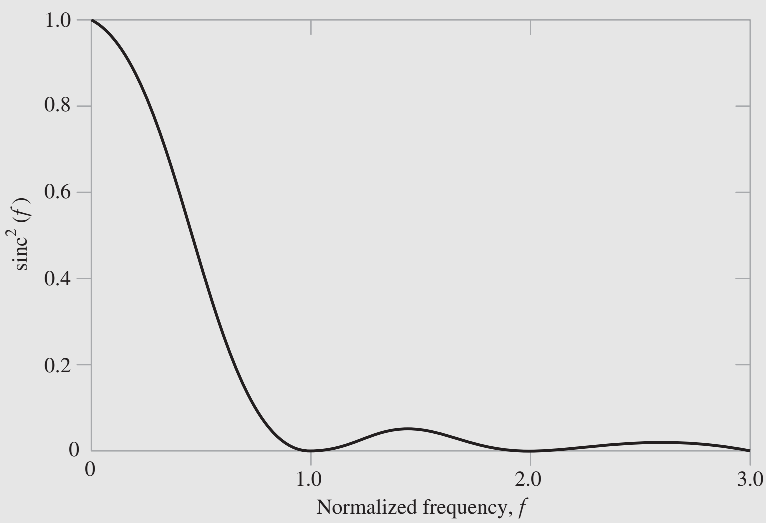


FIGURE 2.31 Energy spectral density of the filter input $x(t)$; only the values for positive frequencies are shown in the figure.

its Fourier transform equals

$$X(f) = \text{sinc}(f)$$

The energy spectral density of the filter input therefore equals

$$\begin{aligned}\psi_x(f) &= |X(f)|^2 \\ &= \text{sinc}^2(f)\end{aligned} \tag{2.130}$$

This normalized energy spectral density is plotted in Fig. 2.31.

To evaluate the energy spectral density $\psi_y(f)$ of the filter output $y(t)$, we use Eq. (2.129), obtaining

$$\begin{aligned}\psi_y(f) &= |H(f)|^2\psi_x(f) \\ &= \begin{cases}\psi_x(f), & -B \le f \le B \\ 0, & \text{otherwise}\end{cases}\end{aligned} \tag{2.131}$$

The energy of the filter output is therefore

$$\begin{aligned}E_y &= \int_{-\infty}^{\infty} \psi_y(f)\, df \\ &= \int_{-B}^{B} \psi_x(f)\, df \\ &= 2\int_0^B \psi_x(f)\, df \\ &= 2\int_0^B \text{sinc}^2(f)\, df\end{aligned} \tag{2.132}$$

Since the filter input has unit energy, we may also view the result given in Eq. (2.132) as *the ratio of the energy of the filter output to that of the filter input* for the general case of a rectangular pulse of arbitrary amplitude and arbitrary duration, processed by an ideal low-pass

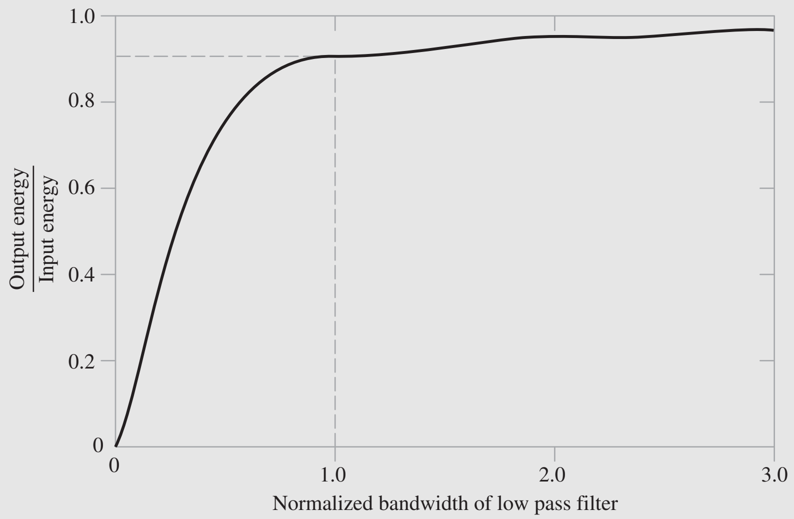


FIGURE 2.32 Output energy-to-input energy ratio versus normalized bandwidth.

filter of bandwidth B. Accordingly, we may in general write

$$\begin{aligned}\rho &= \frac{\text{Energy of filter output}}{\text{Energy of filter input}} \\ &= 2\int_0^B \operatorname{sinc}^2(f)\, df \qquad (2.133)\end{aligned}$$

According to Fig. 2.30(b), the rectangular pulse applied to the filter input has unit duration; hence, the variable f in Eq. (2.133) represents a *normalized frequency*. Equation (2.133) is plotted in Fig. 2.32. This figure shows that just over 90 percent of the total energy of a rectangular pulse lies inside the main spectral lobe of this pulse.

■ INTERPRETATION OF THE ENERGY SPECTRAL DENSITY

Equation (2.129) is important because it not only relates the output energy spectral density of a linear time-invariant system to the input energy spectral density, but it also provides a basis for the physical interpretation of the concept of energy spectral density itself. To be specific, consider the arrangement shown in Fig. 2.33(a), where an energy signal $x(t)$ is passed through a narrow-band filter followed by an *energy meter*. Figure 2.33(b) shows the idealized amplitude response of the filter. That is, the filter is a band-pass filter whose amplitude response is defined by

$$|H(f)| = \begin{cases} 1, & f_c - \dfrac{\Delta f}{2} \le |f| \le f_c + \dfrac{\Delta f}{2} \\ 0, & \text{otherwise} \end{cases} \qquad (2.134)$$

We assume that the *filter bandwidth* Δf is small enough for the amplitude response of the input signal $x(t)$ to be essentially flat over the frequency interval covered by the passband

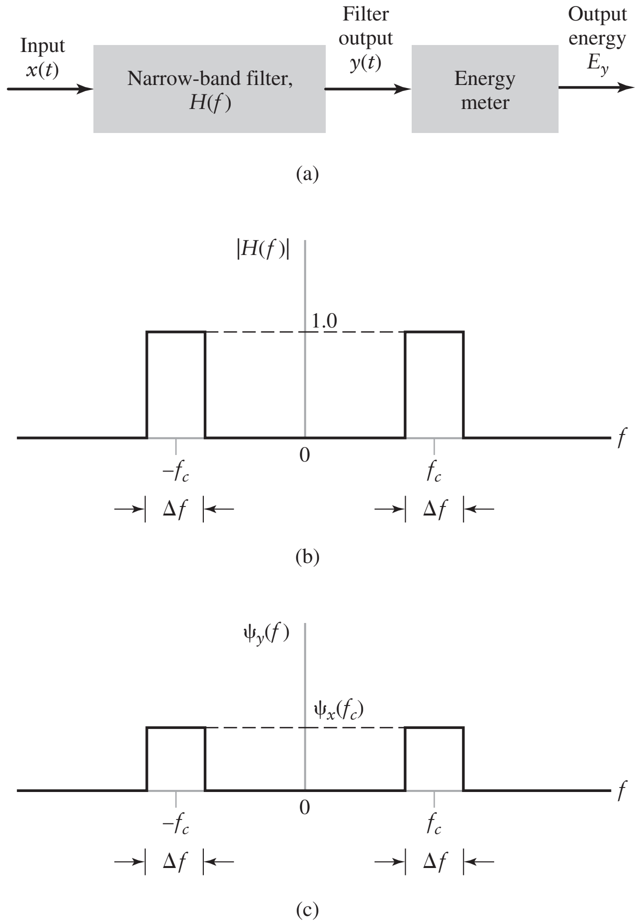


FIGURE 2.33 (*a*) Block diagram of system for measuring energy spectral density. (*b*) Idealized amplitude response of the filter. (*c*) Energy spectral density of the filter output.

of the filter. Accordingly, we may express the amplitude spectrum of the filter output by the approximate formula

$$\begin{aligned} |Y(f)| &= |H(f)||X(f)| \\ &\approx \begin{cases} |X(f_c)|, & f_c - \dfrac{\Delta f}{2} \le |f| \le f_c + \dfrac{\Delta f}{2} \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{2.135}$$

Correspondingly, the energy spectral density $\psi_y(f)$ of the filter output $y(t)$ is approximately related to the energy spectral density $\psi_x(f)$ of the filter input $x(t)$ as follows:

$$\psi_y(f) \approx \begin{cases} \psi_x(f_c), & f_c - \dfrac{\Delta f}{2} \le |f| \le f_c + \dfrac{\Delta f}{2} \\ 0, & \text{otherwise} \end{cases} \tag{2.136}$$

This relation is illustrated in Fig. 2.33(*c*), which shows that only the frequency components of the signal $x(t)$ that lie inside the narrow passband of the ideal band-pass filter reach the output. From Rayleigh's energy theorem, the energy of the filter output $y(t)$ is given by

$$E_y = \int_{-\infty}^{\infty} \psi_y(f)\, df$$

$$= 2\int_{0}^{\infty} \psi_y(f)\, df$$

In light of Eq. (2.136), we may approximate E_y as

$$E_y \approx 2\psi_x(f_c)\, \Delta f \tag{2.137}$$

The multiplying factor 2 accounts for the contributions of negative as well as positive frequency components. We may rewrite Eq. (2.137) in the form

$$\psi_x(f_c) \approx \frac{E_y}{2\Delta f} \tag{2.138}$$

Equation (2.138) states that the energy spectral density of the filter input at some frequency f_c equals the energy of the filter output divided by $2\Delta f$, where Δf is the filter bandwidth centered on f_c. We may therefore interpret the energy spectral density of an energy signal for any frequency f *as the energy per unit bandwidth, which is contributed by frequency components of the signal around the frequency* f.

The arrangement shown in the block diagram of Fig. 2.33(*a*) thus provides the basis for measuring the energy spectral density of an energy signal. Specifically, by using a *variable* band-pass filter to scan the frequency band of interest and determining the energy of the filter output for each midband frequency setting of the filter, a plot of the energy spectral density versus frequency is obtained. Note, however, for the formula of Eq. (2.138) to hold and therefore for the arrangement of Fig. 2.33(*a*) to work, the bandwidth Δf must remain fixed for varying f_c.

■ Cross-correlation of Energy Signals

The autocorrelation function provides a measure of the similarity between a signal and its own time-delayed version. In a similar way, we may use the *cross-correlation function* as a measure of the similarity between one signal and the time-delayed version of a second signal. Let $x(t)$ and $y(t)$ denote a pair of complex-valued energy signals. The cross-correlation function of this pair of signals is defined by

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y^*(t-\tau)\, dt \tag{2.139}$$

We see that if the two signals $x(t)$ and $y(t)$ are somewhat similar, then the cross-correlation function $R_{xy}(\tau)$ will be finite over some range of τ, thereby providing a quantitative measure of the similarity, or coherence, between them. The energy signals $x(t)$ and $y(t)$ are said to be *orthogonal* over the entire time interval if $R_{xy}(0)$ is zero — that is, if

$$\int_{-\infty}^{\infty} x(t)y^*(t)\, dt = 0 \tag{2.140}$$

Equation (2.139) defines one possible value for the cross-correlation function for a specified value of the delay variable τ. We may define a second cross-correlation function for the energy signals $x(t)$ and $y(t)$ as

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t)x^*(t - \tau)\, dt \tag{2.141}$$

From the definitions of the cross-correlation functions $R_{xy}(\tau)$ and $R_{yx}(\tau)$ just given, we obtain the fundamental relationship

$$R_{xy}(\tau) = R_{yx}^*(-\tau) \tag{2.142}$$

Equation (2.142) indicates that unlike convolution, correlation is not in general commutative; that is, $R_{xy}(\tau) \neq R_{yx}(\tau)$.

To characterize the cross-correlation behavior of energy signals in the frequency domain, we introduce the notion of *cross-spectral density*. Specifically, given a pair of complex-valued energy signals $x(t)$ and $y(t)$, we define their cross-spectral densities, denoted by $\psi_{xy}(f)$ and $\psi_{yx}(f)$, as the respective Fourier transforms of the cross-correlation functions $R_{xy}(\tau)$ and $R_{yx}(\tau)$, as shown by

$$\psi_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) \exp(-j2\pi f\tau)\, d\tau \tag{2.143}$$

and

$$\psi_{yx}(f) = \int_{-\infty}^{\infty} R_{yx}(\tau) \exp(-j2\pi f\tau)\, d\tau \tag{2.144}$$

In accordance with the correlation theorem (i.e., Property 13 of Section 2.2), we thus have

$$\psi_{xy}(f) = X(f)Y^*(f) \tag{2.145}$$

and

$$\psi_{yx}(f) = Y(f)X^*(f) \tag{2.146}$$

From this pair of relations, we readily see two properties of the cross-spectral density.

1. Unlike the energy spectral density, cross-spectral density is complex valued in general.
2. $\psi_{xy}(f) = \psi_{yx}^*(f)$ from which it follows that, in general, $\psi_{xy}(f) \neq \psi_{yx}(f)$.

► **Drill Problem 2.15** Derive the relationship of Eq. (2.142) between the two cross-correlation functions $R_{xy}(t)$ and $R_{yx}(t)$. ◄

► **Drill Problem 2.16** Consider the decaying exponential pulse

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 1, & t = 0 \\ 0, & t < 0 \end{cases}$$

Determine the energy spectral density of the pulse $g(t)$. ◄

► **Drill Problem 2.17** Repeat Problem 2.16 for the double exponential pulse

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 1, & t = 0 \\ \exp(at), & t < 0 \end{cases}$$ ◄

2.9 Power Spectral Density

In this section, we expand the important notion of spectral density to include the class of power signals. The *average power* of a signal $x(t)$ is defined by

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \tag{2.147}$$

The signal $x(t)$ is said to be a *power signal* if the condition

$$P < \infty$$

holds. Examples of power signals include periodic signals and noise. We consider periodic signals in this section. (Noise is considered in Chapter 8.)

To develop a frequency-domain distribution of power, we need to know the Fourier transform of the signal $x(t)$. However, this may pose a problem, because power signals have infinite energy and may therefore not be Fourier transformable. To overcome the problem, we consider a *truncated* version of the signal $x(t)$. In particular, we define

$$\begin{aligned} x_T(t) &= x(t)\,\mathrm{rect}\left(\frac{t}{2T}\right) \\ &= \begin{cases} x(t), & -T \le t \le T \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{2.148}$$

As long as the duration T is finite, the truncated signal $x_T(t)$ has finite energy; hence $x_T(t)$ is Fourier transformable. Let $X_T(f)$ denote the Fourier transform of $x_T(t)$; that is,

$$x_T(t) \rightleftharpoons X_T(f)$$

Using the truncated signal $x_T(t)$, we may rewrite Eq. (2.147) for the average power P in terms of $x_T(t)$ as

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |x_T(t)|^2 \, dt \tag{2.149}$$

Since $x_T(t)$ has finite energy, we may use the Rayleigh energy theorem to express the energy of $x_T(t)$ in terms of its Fourier transform $X_T(f)$ as

$$\int_{-\infty}^{\infty} |x_T(t)|^2 \, dt = \int_{-\infty}^{\infty} |X_T(f)|^2 \, df$$

where $|X_T(f)|$ is the amplitude spectrum of $x_T(t)$. Accordingly, we may rewrite Eq. (2.149) in the equivalent form

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |X_T(f)|^2 \, df \tag{2.150}$$

As the duration T increases, the energy of $x_T(t)$ increases. Correspondingly, the energy spectral density $|X_T(f)|^2$ increases with T. Indeed as T approaches infinity, so will $|X_T(f)|^2$. However, for the average power P to be finite, $|X_T(f)|^2$ must approach infinity at the same rate as T. This requirement ensures the *convergence* of the integral on the right-hand side of Eq. (2.150) in the limit as T approaches infinity. The convergence, in turn, permits us to *interchange the order in which the limiting operation and integration in Eq. (2.150) are performed*. We may then rewrite this equation as

$$P = \int_{-\infty}^{\infty} \left(\lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2 \right) df \tag{2.151}$$

Let the integrand in Eq. (2.151) be denoted by

$$S_x(f) = \lim_{T\to\infty} \frac{1}{2T}|X_T(f)|^2 \tag{2.152}$$

The frequency-dependent function $S_x(f)$ is called the *power spectral density* or *power spectrum* of the power signal $x(t)$, and the quantity $(|X_T(f)|^2/2T)$ is called the *periodogram* of the signal.

From Eq. (2.152), we readily see that the power spectral density is a nonnegative real-valued quantity for all frequencies. Moreover, from Eq. (2.152) we readily see that

$$P = \int_{-\infty}^{\infty} S_x(f)\, df \tag{2.153}$$

Equation (2.153) states: *the total area under the curve of the power spectral density of a power signal is equal to the average power of that signal*. The power spectral density of a power signal therefore plays a role similar to the energy spectral density of an energy signal.

▶ **Drill Problem 2.18** In an implicit sense, Eq. (2.153) embodies *Parseval's power theorem*, which states that for a *periodic signal* $x(t)$ we have

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |X(nf_0)|^2$$

where T is the period of the signal, f_0 is the fundamental frequency, and $X(nf_0)$ is the Fourier transform of $x(t)$ evaluated at the frequency nf_0. Prove this theorem. ◀

EXAMPLE 2.15 Modulated Wave

Consider the *modulated wave*

$$x(t) = g(t)\cos(2\pi f_c t) \tag{2.154}$$

where $g(t)$ is a power signal that is band-limited to B hertz. We refer to $x(t)$ as a "modulated wave" in the sense that the amplitude of the sinusoidal "carrier" of frequency f_c is varied linearly with the signal $g(t)$. (The subject of modulation is covered in detail in Chapter 3.) We wish to find the power spectral density of $x(t)$ in terms of that of $g(t)$, given that the frequency f_c is larger than the bandwidth B.

Let $g_T(t)$ denote the truncated version of $g(t)$, defined in a manner similar to that described in Eq. (2.148). Correspondingly, we may express the truncated version of $x(t)$ as

$$x_T(t) = g_T(t)\cos(2\pi f_c t) \tag{2.155}$$

Since

$$\cos(2\pi f_c t) = \frac{1}{2}[\exp(j2\pi f_c t) + \exp(-j2\pi f_c t)], \tag{2.156}$$

it follows from the frequency-shifting property (i. e., Property 6) of the Fourier transform that

$$X_T(f) = \frac{1}{2}[G_T(f - f_c) + G_T(f + f_c)] \tag{2.157}$$

where $G_T(f)$ is the Fourier transform of $g_T(t)$.

Given that $f_c > B$, we find that $G_T(f - f_c)$ and $G_T(f + f_c)$ represent nonoverlapping spectra; their product is therefore zero. Accordingly, using Eq. (2.157) to evaluate the squared amplitude of $X_T(f)$, we get

$$|X_T(f)|^2 = \frac{1}{4}[|G_T(f - f_c)|^2 + |G_T(f + f_c)|^2] \tag{2.158}$$

Finally, applying the definition of Eq. (2.152) for the power spectral density of the power signal $g(t)$ to Eq (2.158), we get the desired result:

$$S_x(f) = \frac{1}{4}[S_g(f - f_c) + S_g(f + f_c)] \tag{2.159}$$

Except for the scaling factor 1/4, *the power spectral density of the modulated wave $x(t)$ is equal to the sum of the power spectral density $S_g(f)$ shifted to the right by f_c and the $S_g(f)$ shifted to the left by the same amount f_c.*

2.10 *Numerical Computation of the Fourier Transform*

The material presented in this chapter clearly testifies to the importance of the Fourier transform as a theoretical tool for the representation of deterministic signals and linear time-invariant systems. The importance of the Fourier transform is further enhanced by the fact that there exists a class of algorithms called fast Fourier transform algorithms for the numerical computation of the Fourier transform in a highly efficient manner.

The fast Fourier transform algorithm is itself derived from the *discrete Fourier transform* in which, as the name implies, both time and frequency are represented in discrete form. The discrete Fourier transform provides an *approximation* to the Fourier transform. In order to properly represent the information content of the original signal, we have to take special care in performing the sampling operations involved in defining the discrete Fourier transform. A detailed treatment of the sampling process will be presented in Chapter 5. For the present, it suffices to say that given a band-limited signal, the sampling rate should be greater than twice the highest frequency component of the input signal. Moreover, if the samples are uniformly spaced by T_s seconds, the spectrum of the signal becomes periodic, repeating every $f_s = (1/T_s)$ Hz. Let N denote the number of frequency samples contained in an interval f_s. Hence, the *frequency resolution* involved in the numerical computation of the Fourier transform is defined by

$$\Delta f = \frac{f_s}{N} = \frac{1}{NT_s} = \frac{1}{T} \tag{2.160}$$

where $T = NT_s$ is the total duration of the signal.

Consider then a *finite data sequence* $\{g_0, g_1, \dots, g_{N-1}\}$. For brevity, we refer to this sequence as g_n, in which the subscript is the *time index* $n = 0, 1, \dots, N - 1$. Such a sequence may represent the result of sampling an analog signal $g(t)$ at times $t = 0, T_s, \dots, (N - 1)T_s$, where T_s is the sampling interval. The ordering of the data sequence defines the sample time in that $g_0, g_1, \dots, g_{N-1}$ denote samples of $g(t)$ taken at times $0, T_s, \dots, (N - 1)T_s$, respectively. Thus we have

$$g_n = g(nT_s) \tag{2.161}$$

We formally define the *discrete Fourier transform* (DFT) of the sequence g_n as

$$G_k = \sum_{n=0}^{N-1} g_n \exp\left(-\frac{j2\pi}{N}kn\right), \qquad k = 0, 1, \ldots, N-1 \tag{2.162}$$

The sequence $\{G_0, G_1, \ldots, G_{N-1}\}$ is called the *transform sequence*. For brevity, we refer to this new sequence as G_k, in which the subscript is the *frequency index* $k = 0, 1, \ldots, N-1$. Correspondingly, we define the *inverse discrete Fourier transform* (IDFT) of G_k as

$$g_n = \frac{1}{N}\sum_{k=0}^{N-1} G_k \exp\left(\frac{j2\pi}{N}kn\right), \qquad n = 0, 1, \ldots, N-1 \tag{2.163}$$

The DFT and the IDFT form a transform pair. Specifically, given the data sequence g_n, we may use the DFT to compute the transform sequence G_k; and given the transform sequence G_k, we may use the IDFT to recover the original data sequence g_n. A distinctive feature of the DFT is that for the finite summations defined in Eqs. (2.162) and (2.163), there is no question of convergence.

When discussing the DFT (and algorithms for its computation), the words "sample" and "point" are used interchangeably to refer to a sequence value. Also, it is common practice to refer to a sequence of length N as an *N-point sequence*, and refer to the DFT of a data sequence of length N as an *N-point DFT*.

■ INTERPRETATIONS OF THE DFT AND THE IDFT

We may visualize the DFT process, described in Eq. (2.162), as a collection of N *complex heterodyning* and *averaging* operations, as shown in Fig. 2.34(*a*); in the picture depicted herein, heterodyning refers to the multiplication of data sequence g_n by a complex exponential. We say that the heterodyning is complex in that samples of the data sequence are multiplied by *complex exponential sequences*. There are a total of N complex exponential sequences to be considered, corresponding to the frequency index $k = 0, 1, \ldots, N-1$. Their periods have been selected in such a way that each complex exponential sequence has precisely an integer number of cycles in the total interval 0 to $N-1$. The zero-frequency response, corresponding to $k = 0$, is the only exception.

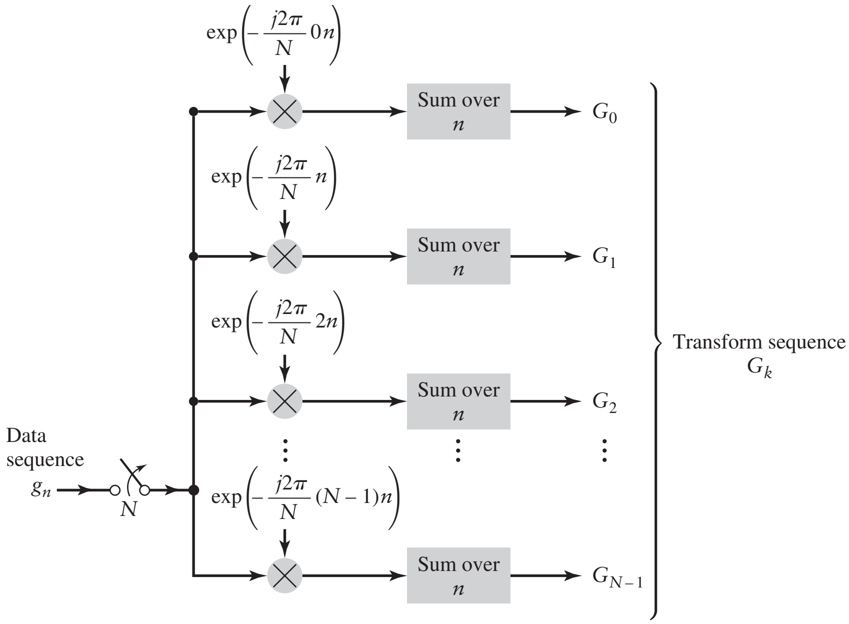
For the interpretation of the IDFT process, described in Eq. (2.163), we may use the scheme shown in Fig. 2.34(*b*). Here we have a collection of N *complex signal generators*, each of which produces the *complex exponential sequence*

$$\begin{aligned}\exp\left(\frac{j2\pi}{N}kn\right) &= \cos\left(\frac{2\pi}{N}kn\right) + j\sin\left(\frac{2\pi}{N}kn\right)\\ &= \left\{\cos\left(\frac{2\pi}{N}kn\right), \sin\left(\frac{2\pi}{N}kn\right)\right\}, \qquad k = 0, 1, \ldots, N-1\end{aligned} \tag{2.164}$$

Thus, in reality, each complex signal generator consists of a pair of generators that output a cosinusoidal and a sinusoidal sequence of k cycles per observation interval. The output of each complex signal generator is weighted by the complex Fourier coefficient G_k. At each time index n, an output is formed by summing the weighted complex generator outputs.

It is noteworthy that although the DFT and the IDFT are similar in their mathematical formulations, as described in Eqs. (2.162) and (2.163), their interpretations, as depicted in Figs. 2.34(*a*) and 2.34(*b*), are so completely different.

Also, the addition of harmonically related periodic signals, as in Figs. 2.34(*a*) and 2.34(*b*), suggests that the sequences G_k and g_n must be both periodic. Moreover, the proces-



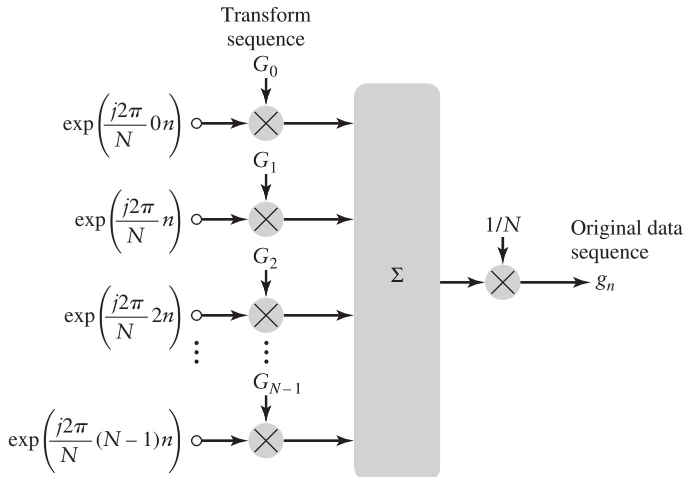


FIGURE 2.34 Interpretations of (*a*) the DFT as an analyzer of the data sequence g_n, and (*b*) the IDFT as a synthesizer of g_n.

sors shown in Figs. 2.34(*a*) and 2.34(*b*) must be linear, suggesting that the DFT and IDFT are both linear operations. This important property is also obvious from the defining equations (2.162) and (2.163).

■ FAST FOURIER TRANSFORM ALGORITHMS

In the discrete Fourier transform (DFT), both the input and the output consist of sequences of numbers defined at uniformly spaced points in time and frequency, respectively. This feature makes the DFT ideally suited for direct numerical evaluation on a digital computer.

Moreover, the computation can be implemented most efficiently using a class of algorithms called *fast Fourier transform (FFT) algorithms*.[2] An algorithm refers to a "recipe" that can be written in the form of a computer program.

FFT algorithms are computationally efficient because they use a greatly reduced number of arithmetic operations as compared to the brute force computation of the DFT. Basically, an FFT algorithm attains its computational efficiency by following a *divide-and-conquer strategy*, whereby the original DFT computation is decomposed successively into smaller DFT computations. In this section, we describe one version of a popular FFT algorithm, the development of which is based on such a strategy.

To proceed with the development, we first rewrite Eq. (2.162), defining the DFT of g_n, in the simplified form

$$G_k = \sum_{n=0}^{N-1} g_n W^{nk}, \qquad k = 0, 1, \ldots, N-1 \tag{2.165}$$

where the new coefficient W is defined by

$$W = \exp\left(-\frac{j2\pi}{N}\right) \tag{2.166}$$

From this definition, we see that

$$\begin{aligned} W^N &= \exp(-j2\pi) = 1 \\ W^{N/2} &= \exp(-j\pi) = -1 \\ W^{(k+lN)(n+mN)} &= W^{kn}, \quad \text{for} \quad m, l = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

That is, W^{kn} is periodic with period N. The *periodicity* of W^{kn} is a key feature in the development of FFT algorithms.

Let N, the number of points in the data sequence, be *an integer power of two*, as shown by

$$N = 2^L$$

where L is an integer. Since N is an even integer, $N/2$ is an integer, and so we may divide the data sequence into the first half and the last half of the points. Thus, we may rewrite Eq. (2.165) in the equivalent form

$$\begin{aligned} G_k &= \sum_{n=0}^{(N/2)-1} g_n W^{nk} + \sum_{n=N/2}^{N-1} g_n W^{nk} \\ &= \sum_{n=0}^{(N/2)-1} g_n W^{nk} + \sum_{n=0}^{(N/2)-1} g_{n+N/2} W^{k(n+N/2)} \\ &= \sum_{n=0}^{(N/2)-1} (g_n + g_{n+N/2} W^{kN/2}) W^{kn}, \qquad k = 0, 1, \ldots, N-1 \end{aligned} \tag{2.167}$$

Note that in the second line of Eq. (2.167), we changed the index of the second summation term so that both summation terms cover the same range. Since $W^{N/2} = -1$, we have

$$W^{kN/2} = (-1)^k$$

[2]The fast Fourier transform (FFT) algorithm has a long history. Its modern discovery (or rediscovery to be more precise) is attributed to Cooley and Tukey in 1965; see the paper by Cooley (1992) for details.

For the evaluation of Eq. (2.167), we proceed by considering two cases, one corresponding to even values of k and the other corresponding to odd values of k. For the case of even k, let $k = 2l$, where $l = 0, 1, \dots, (N/2)$. Hence, we define

$$x_n = g_n + g_{n+N/2} \tag{2.168}$$

Then, for even k we may put Eq. (2.167) into the new form

$$G_{2l} = \sum_{n=0}^{(N/2)-1} x_n (W^2)^{ln}, \qquad l = 0, 1, \dots, \frac{N}{2} - 1 \tag{2.169}$$

From the definition of W given in Eq. (2.166), we readily see that

$$\begin{aligned} W^2 &= \exp\left(-\frac{j4\pi}{N}\right) \\ &= \exp\left(-\frac{j2\pi}{N/2}\right) \end{aligned}$$

Hence, we recognize the sum on the right-hand side of Eq. (2.169) as the $(N/2)$-point DFT of the sequence x_n.

Consider next the remaining case of odd k, and let

$$k = 2l + 1, \qquad l = 0, 1, \dots, \frac{N}{2} - 1$$

Then, recognizing that for odd k, $W^{kN/2} = -1$, we may define

$$y_n = g_n - g_{n+N/2} \tag{2.170}$$

Hence, for the case of odd k, we may put Eq. (2.167) into the corresponding form

$$\begin{aligned} G_{2l+1} &= \sum_{n=0}^{(N/2)-1} y_n W^{(2l+1)n} \\ &= \sum_{n=0}^{(N/2)-1} [y_n W^n](W^2)^{ln}, \qquad l = 0, 1, \dots, \frac{N}{2} - 1 \end{aligned} \tag{2.171}$$

We recognize the sum on the right-hand side of Eq. (2.171) as the $(N/2)$-point DFT of the modified sequence $y_n W^n$. The coefficient W^n multiplying y_n is called a *twiddle factor*.

Equations (2.169) and (2.171) show that the even- and odd-valued samples of the transform sequence G_k can be obtained from the $(N/2)$-point DFTs of the sequences x_n and $y_n W^n$, respectively. The sequences x_n and y_n are themselves related to the original data sequence g_n by Eqs. (2.168) and (2.170), respectively. Thus, the problem of computing an N-point DFT is reduced to that of computing two $(N/2)$-point DFTs. This procedure is repeated a second time, whereby an $(N/2)$-point is decomposed into two $(N/4)$-point DFTs. The decomposition (or, more precisely, the divide-and-conquer procedure) is continued in this fashion until (after $L = \log_2 N$ stages), we reach the trivial case of N single-point DFTs.

Figure 2.35 illustrates the computations involved in applying the formulas of Eqs. (2.169) and (2.171) to an 8-point data sequence; that is, $N = 8$. In constructing the left-hand portions of the figure, we have used signal-flow graph notation. A *signal-flow graph* consists of an interconnection of *nodes* and *branches*. The *direction* of signal transmission along a branch is indicated by an arrow. A branch multiplies the variable at a node (to which it is connected) by the branch *transmittance*. A node sums the outputs of all incoming branches. The convention used for branch transmittances in Fig. 2.35 is as follows.

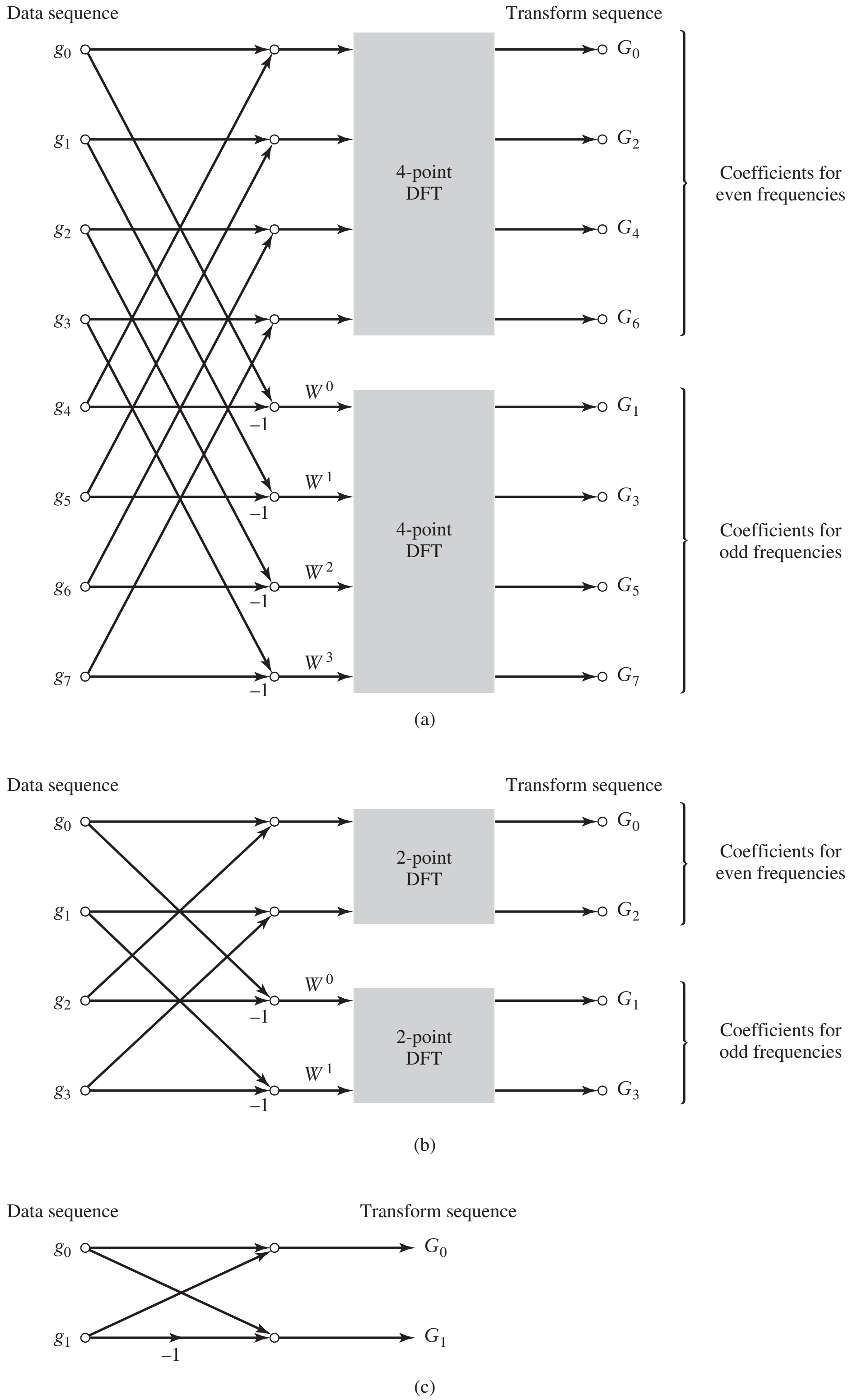


FIGURE 2.35 (*a*) Reduction of 8-point DFT into two 4-point DFTs. (*b*) Reduction of 4-point DFT into two 2-point DFTs. (*c*) Trivial case of 2-point DFT.

When no coefficient is indicated on a branch, the transmittance of that branch is assumed to be unity. For other branches, the transmittance of a branch is indicated by -1 or an integer power of W, placed alongside the arrow on the branch.

Thus, in Fig. 2.35(a), the computation of an 8-point DFT is reduced to that of two 4-point DFTs. The procedure for the 8-point DFT is mimicked to simplify the computation of the 4-point DFT. This is illustrated in Fig. 2.35(b), where the computation of a 4-point DFT is reduced to that of two 2-point DFTs. Finally, the computation of a 2-point DFT is shown in Fig. 2.35(c).

Combining the ideas described in Fig. 2.35, we obtain the complete signal-flow graph of Fig. 2.36 for the computation of the 8-point DFT. A repetitive structure, called a *butterfly*, can be discerned in the FFT algorithm of Fig. 2.36; a butterfly has two inputs and two outputs. Examples of butterflies (for the three stages of the algorithm) are illustrated by the bold-faced lines in Fig. 2.36.

For the general case of $N = 2^L$, the algorithm requires $L = \log_2 N$ stages of computation. Each stage requires $(N/2)$ butterflies. Each butterfly involves one complex multiplication and two complex additions (to be precise, one addition and one subtraction). Accordingly, the FFT structure described here requires $(N/2)\log_2 N$ complex multiplications

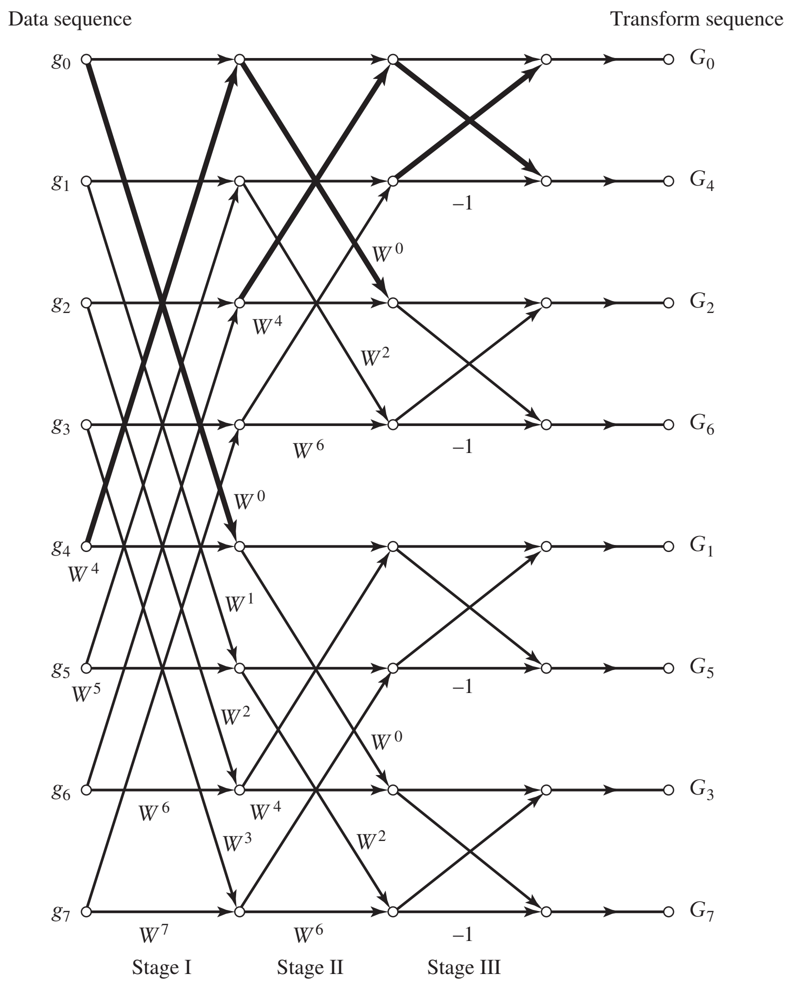


Figure 2.36 Decimation-in-frequency FFT algorithm.

and $N \log_2 N$ complex additions. (Actually, the number of multiplications quoted is pessimistic , because we may omit all twiddle factors $W^0 = 1$ and $W^{N/2} = -1$, $W^{N/4} = -j$, $W^{3N/4} = j$.) This computational complexity is significantly smaller than that of the N^2 complex multiplications and $N(N - 1)$ complex additions required for the *direct* computation of the DFT. The computational savings made possible by the FFT algorithm become more substantial as we increase the data length N.

We may establish two other important features of the FFT algorithm by carefully examining the signal-flow graph shown in Fig. 2.36:

1. At each stage of the computation, the new set of N complex numbers resulting from the computation can be stored in the same memory locations used to store the previous set. This kind of computation is referred to as *in-place computation*.
2. The samples of the transform sequence G_k are stored in a *bit-reversed order*. To illustrate the meaning of this latter terminology, consider Table 2.2 constructed for the case of $N = 8$. At the left of the table, we show the eight possible values of the frequency index k (in their natural order) and their 3-bit binary representations. At the right of the table, we show the corresponding bit-reversed binary representations and indices. We observe that the bit-reversed indices in the right-most column of Table 2.2 appear in the same order as the indices at the output of the FFT algorithm in Fig. 2.36.

TABLE 2.2 ***Illustrating Bit Reversal***

Frequency Index, k	*Binary Representation*	*Bit-Reversed Binary Representation*	*Bit-Reversed Index*
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

The FFT algorithm depicted in Fig. 2.36 is referred to as a *decimation-in-frequency algorithm*, because the transform (frequency) sequence G_k is divided successively into smaller subsequences. In another popular FFT algorithm, called a *decimation-in-time* algorithm, the data (time) sequence g_n is divided successively into smaller subsequences. Both algorithms have the same computational complexity. They differ from each other in two respects. First, for decimation-in-frequency, the input is in natural order, whereas the output is in bit-reversed order. The reverse is true for decimation-in-time. Second, the butterfly for decimation-in-time is slightly different from that for decimation-in-frequency. The reader is invited to derive the details of the decimation-in-time algorithm using the divide-and-conquer strategy that led to the development of the algorithm described in Fig. 2.36; See Problem 2.50.

■ COMPUTATION OF THE IDFT

The IDFT of the transform sequence G_k is defined by Eq. (2.163). We may rewrite this equation in terms of the complex parameter W as

$$g_n = \frac{1}{N}\sum_{k=0}^{N-1} G_k W^{-kn}, \qquad n = 0, 1, \dots, N - 1 \tag{2.172}$$

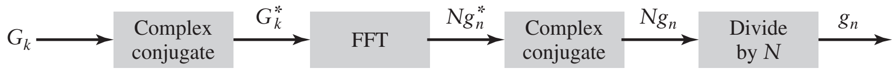


FIGURE 2.37 Use of the FFT algorithm for computing the IDFT.

Taking the complex conjugate of Eq. (2.172), multiplying by N, and recognizing from the definition of Eq. (2.166) that $W^* = W^{-1}$, we get

$$Ng_n^* = \sum_{k=0}^{N-1} G_k^* W^{kn}, \qquad 0, 1, \ldots, N-1 \tag{2.173}$$

The right-hand side of Eq. (2.173) is recognized as the N-point DFT of the complex-conjugated sequence G_k^*. Accordingly, Eq. (2.173) suggests that we may compute the desired sequence g_n using the scheme shown in Fig. 2.37, based on an N-point FFT algorithm. Thus, the same FFT algorithm can be used essentially to handle the computation of both the IDFT and the DFT.

2.11 *Theme Example: Twisted Pairs for Telephony*

The fundamental transmission medium for connecting homes to telephone central switching offices is the *twisted pair*. A twisted pair is usually a pair of solid copper wires with polyethylene sheathing. If the copper strand has a diameter of 0.4 mm, this cable size is referred to as #26 on the American Wire Gauge, or simply 26 AWG. A twisted pair is an example of a *transmission line*.

A transmission line consists of two conductors, each of which has its own inherent resistance and inductance. Since the two conductors are often in close proximity, there is also a capacitive effect between the two as well as potential conductance through the material that is used to insulate the two wires. A transmission line so constructed is often represented by the *lumped circuit* shown in Fig. 2.38. Although the impedances are shown as discrete elements in Fig. 2.38, it is more correct to consider them distributed through the length of the transmission line.

Depending upon the circuit element values in Fig. 2.38, it is clear that a transmission line will have a distorting effect on the transmitted signal. Furthermore, since the total impedance increases with the length of the line, so will the frequency response of the transmission line.

In Fig. 2.39, we show the typical response of a twisted pair with lengths of 2 to 8 kilometers. There are several observations to be made from the figure:

- Twisted pairs run directly from the central office to the home with one pair dedicated to each telephone line. Consequently, the transmission lines can be quite long.
- The results in Fig. 2.39 assume a continuous cable. In practice, there may be several splices in the cable, different gauge cables along different parts of the path, and so on. These discontinuities in the transmission medium will further affect the frequency response of the cable.

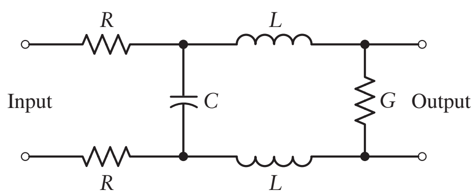


FIGURE 2.38 Lumped circuit model of a transmission line.

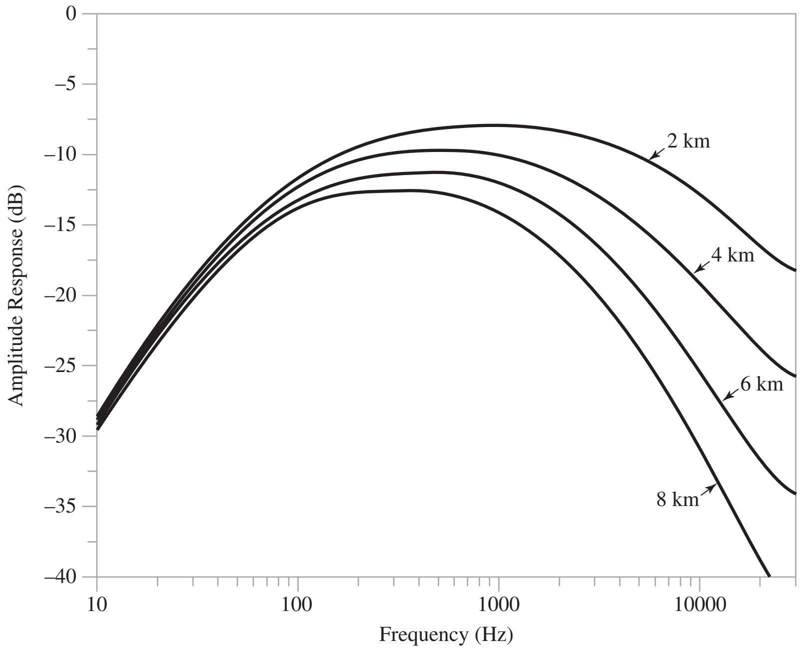


FIGURE 2.39 Typical frequency response of a 26-AWG twisted-pair transmission line of different lengths with $(600\ \Omega + 2\ \mu F)$ source and load impedances.

- We see that, for a 2-km cable, the frequency response is quite flat over the voice band from 300 to 3100 Hz for telephonic communication. However, for the 8-km cable, the frequency response starts to fall just above 1 kHz.
- The frequency response falls off at zero frequency because there is a capacitive connection at the load and the source. This capacitive connection is put to pratical use by enabling dc power to be transported along the cable to power the remote telephone handset.

Analysis of the frequency response of longer cables indicates that it can be improved by adding some reactive loading. For this reason, we often hear of *loaded lines* that include lumped inductors at regular intervals (typically 66 milli-henry (mH) approximately every two kilometers). The loading improves the frequency response of the circuit in the range corresponding to voice signals without requiring additional power. The disadvantage of loaded lines, however, is their degraded performance at high frequency. Services such as digital subscriber line (DSL) (discussed later in Chapter 7), which rely on the high-frequency response of the twisted pairs, do not work well over loaded telephone lines.

In most of what follows, in the rest of the book, we will usually assume that the medium does not affect the transmission, except possibly through the addition of noise to the signal. In practice, the medium may affect the signal in a variety of ways as illustrated in the theme example just described.

2.12 *Summary and Discussion*

In this chapter, we have described the Fourier transform as a fundamental tool for relating the time-domain and frequency-domain descriptions of a deterministic signal. The signal of interest may be an energy signal or a power signal. The Fourier transform includes the exponential Fourier series as a special case, provided that we permit the use of the Dirac delta function.

An inverse relationship exists between the time-domain and frequency-domain descriptions of a signal. Whenever an operation is performed on the waveform of a signal in the time domain, a corresponding modification is applied to the spectrum of the signal in the frequency domain. An important consequence of this inverse relationship is the fact that the time-bandwidth product of an energy signal is a constant; the definitions of signal duration and bandwidth merely affect the value of the constant.

An important signal processing operation frequently encountered in communication systems is that of linear filtering. This operation involves the convolution of the input signal with the impulse response of the filter or, equivalently, the multiplication of the Fourier transform of the input signal by the transfer function (i.e., Fourier transform of the impulse response) of the filter. Note, however, that the material presented in the chapter on linear filtering assumes that the filter is *time-invariant* (i.e., the shape of the impulse response of the filter is invariant with respect to when the unit impulse or delta function is applied to the filter).

Another important signal processing operation encountered in communication systems is that of correlation. This operation may provide a measure of similarity between a signal and a delayed version of itself, in which case we speak of the autocorrelation function. When the measure of similarity involves a pair of different signals, however, we speak of the cross-correlation function. The Fourier transform of the autocorrelation function is called the spectral density. The Fourier transform of the cross-correlation function is called the cross-spectral density. Discussions of correlation and spectral density presented in the chapter were confined to energy signals and power signals exemplified by pulse-like signals and periodic signals respectively; the treatment of noise (another realization of power signal) is deferred to Chapter 8.

The final part of the chapter was concerned with the discrete Fourier transform and its numerical computation. Basically, the discrete Fourier transform is obtained from the standard Fourier transform by uniformly sampling both the input signal and the output spectrum. The fast Fourier transform algorithm provides a practical means for the efficient implementation of the discrete Fourier transform on a digital computer. This makes the fast Fourier transform algorithm a powerful computational tool for spectral analysis and linear filtering.

Additional Problems

2.19 (**a**) Find the Fourier transform of the half-cosine pulse shown in Fig. 2.40(*a*).

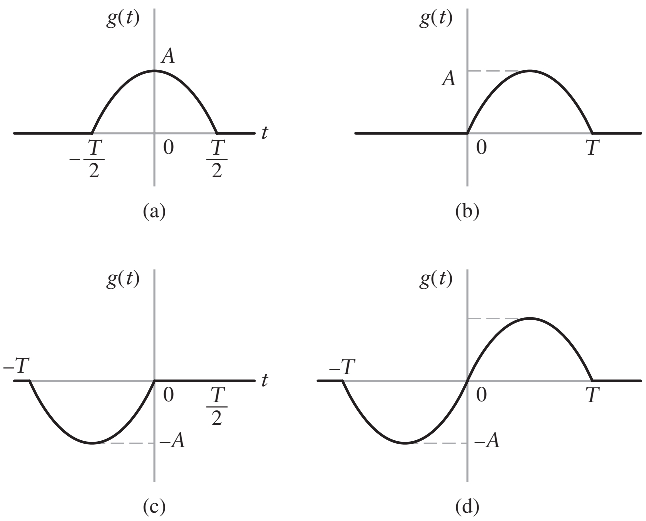


Figure 2.40
Problem 2.19

(b) Apply the time-shifting property to the result obtained in part (a) to evaluate the spectrum of the half-sine pulse shown in Fig. 2.40(*b*).

(c) What is the spectrum of a half-sine pulse having a duration equal to aT?

(d) What is the spectrum of the negative half-sine pulse shown in Fig. 2.40(*c*)?

(e) Find the spectrum of the single sine pulse shown in Fig. 2.40(*d*).

2.20 Any function $g(t)$ can be split umambiguously into an *even part* and an *odd part*, as shown by

$$g(t) = g_e(t) + g_o(t)$$

The even part is defined by

$$g_e(t) = \frac{1}{2}[g(t) + g(-t)]$$

and the odd part is defined by

$$g_o(t) = \frac{1}{2}[g(t) - g(-t)]$$

(a) Evaluate the even and odd parts of a rectangular pulse defined by

$$g(t) = A \operatorname{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$$

(b) What are the Fourier transforms of these two parts of the pulse?

2.21 The following expression may be viewed as an approximate representation of a pulse with finite rise time:

$$g(t) = \frac{1}{\tau}\int_{t-T}^{t+T} \exp\left(-\frac{\pi u^2}{\tau^2}\right) du$$

where it is assumed that $T \gg \tau$. Determine the Fourier transform of $g(t)$. What happens to this transform when we allow τ to become zero?

2.22 The Fourier transform of a signal $g(t)$ is denoted by $G(f)$. Prove the following properties of the Fourier transform:

(a) If a real signal $g(t)$ is an even function of time t, the Fourier transform $G(f)$ is purely real. If the real signal $g(t)$ is an odd function of time t, the Fourier transform $G(f)$ is purely imaginary.

(b) $t^n g(t) \rightleftharpoons \left(\frac{j}{2\pi}\right)^n G^{(n)}(f)$, where $G^{(n)}(f)$ is the nth derivative of $G(f)$ with respect to f.

(c) $\displaystyle\int_{-\infty}^{\infty} t^n g(t)\, dt = \left(\frac{j}{2\pi}\right)^n G^{(n)}(0)$

(d) $\displaystyle\int_{-\infty}^{\infty} g_1(t) g_2^*(t)\, dt = \int_{-\infty}^{\infty} G_1(f) G_2^*(f)\, df$

2.23 The Fourier transform $G(f)$ of a signal $g(t)$ is bounded by the following three inequalities:

(a) $\displaystyle |G(f)| \le \int_{-\infty}^{\infty} |g(t)|\, dt$

(b) $\displaystyle |j2\pi f G(f)| \le \int_{-\infty}^{\infty} \left|\frac{dg(t)}{dt}\right| dt$

(c) $\displaystyle |(j2\pi f)^2 G(f)| \le \int_{-\infty}^{\infty} \left|\frac{d^2 g(t)}{dt^2}\right| dt$

It is assumed that the first and second derivatives of $g(t)$ exist.

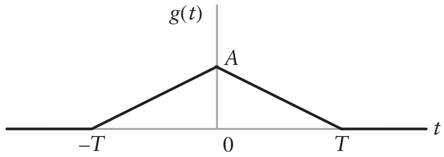


FIGURE 2.41 Problem 2.23

Construct these three bounds for the triangular pulse shown in Fig. 2.41 and compare your results with the actual amplitude spectrum of the pulse.

2.24 Consider the convolution of two signals $g_1(t)$ and $g_2(t)$. Show that

(a) $\frac{d}{dt}[g_1(t) \star g_2(t)] = \left[\frac{d}{dt}g_1(t)\right] \star g_2(t)$

(b) $\int_{-\infty}^{t} [g_1(\tau) \star g_2(\tau)]\, d\tau = \left[\int_{-\infty}^{t} g_1(\tau)\, d\tau\right] \star g_2(t)$

2.25 A signal $x(t)$ of finite energy is applied to a square-law device whose output $y(t)$ is defined by

$$y(t) = x^2(t)$$

The spectrum of $x(t)$ is limited to the frequency interval $-W \le f \le W$. Hence, show that the spectrum of $y(t)$ is limited to $-2W \le f \le 2W$. *Hint:* Express $y(t)$ as $x(t)$ multiplied by itself.

2.26 Evaluate the Fourier transform of the delta function by considering it as the limiting form of (a) rectangular pulse of unit area, and (b) sinc pulse of unit area.

2.27 The Fourier transform $G(f)$ of a signal $g(t)$ is defined by

$$G(f) = \begin{cases} 1, & f > 0 \\ \frac{1}{2}, & f = 0 \\ 0, & f < 0 \end{cases}$$

Determine the signal $g(t)$.

2.28 Consider a pulselike function $g(t)$ that consists of a small number of straight-line segments. Suppose that this function is differentiated with respect to time t twice so as to generate a sequence of weighted delta functions, as shown by

$$\frac{d^2g(t)}{dt^2} = \sum_i k_i \delta(t - t_i)$$

where the k_i are related to the slopes of the straight-line segments.

(a) Given the values of the k_i and t_i, show that the Fourier transform of $g(t)$ is given by

$$G(f) = -\frac{1}{4\pi^2 f^2} \sum_i k_i \exp(-j2\pi f t_i)$$

(b) Using this procedure, show that the Fourier transform of the trapezoidal pulse shown in Fig. 2.42 is given by

$$G(f) = \frac{A}{\pi^2 f^2 (t_b - t_a)} \sin[\pi f(t_b - t_a)] \sin[\pi f(t_b + t_a)]$$

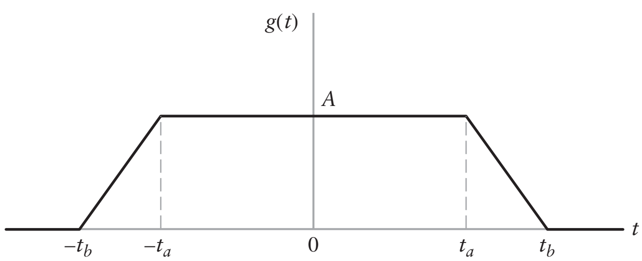


FIGURE 2.42 Problem 2.28

2.29 A rectangular pulse of amplitude A and duration $2t_a$ may be viewed as the limiting case of the trapezoidal pulse shown in Fig. 2.42 as t_b approaches t_a.

(**a**) Starting with the result given in part (b) of Problem 2.28, show that as t_b approaches t_a, this result approaches a sinc function.

(**b**) Reconcile the result derived in part (a) with the Fourier-transform pair of Eq. (2.10).

2.30 Let $x(t)$ and $y(t)$ be the input and output signals of a linear time-invariant filter. Using Rayleigh's energy theorem, show that if the filter is stable and the input signal $x(t)$ has finite energy, then the output signal $y(t)$ also has finite energy. That is, if

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$$

then

$$\int_{-\infty}^{\infty} |y(t)|^2 \, dt < \infty$$

2.31 (**a**) Determine the overall amplitude response of the cascade connection shown in Fig. 2.43 consisting of N identical stages, each with a time constant RC equal to τ_0.

(**b**) Show that as N approaches infinity, the amplitude response of the cascade connection approaches the Gaussian function $\exp\left(-\frac{1}{2}f^2T^2\right)$, where for each value of N, the time constant τ_0 is selected so that the condition

$$\tau_0^2 = \frac{T^2}{4\pi^2 N}$$

is satisfied.

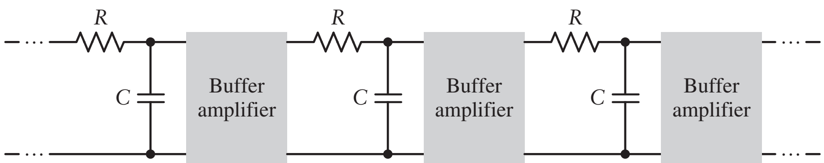


FIGURE 2.43 Problem 2.31

2.32 Suppose that, for a given signal $x(t)$, the integrated value of the signal over an interval T is required, as shown by

$$y(t) = \int_{t-T}^{t} x(\tau) \, d\tau$$

(**a**) Show that $y(t)$ can be obtained by transmitting the signal $x(t)$ through a filter with its transfer function given by

$$H(f) = T \operatorname{sinc}(fT) \exp(-j\pi fT)$$

(**b**) An adequate approximation to this transfer function is obtained by using a low-pass filter with a bandwidth equal to $1/T$, passband amplitude response T, and delay $T/2$. Assuming this low-pass filter to be ideal, determine the filter output at time $t = T$ due to a unit step function applied to the filter at $t = 0$, and compare the result with the corresponding output of the ideal integrator. Note that $\mathrm{Si}(\pi) = 1.85$ and $\mathrm{Si}(\infty) = \pi/2$.

2.33 Show that the two different pulses defined in parts (*a*) and (*b*) of Fig. 2.44 have the same energy spectral density:

$$\Psi_g(f) = \frac{4A^2T^2 \cos^2(\pi T f)}{\pi^2(4T^2f^2 - 1)^2}$$

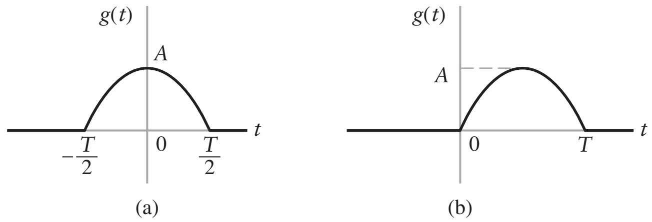


FIGURE 2.44 Problem 2.33

2.34 Determine and sketch the autocorrelation functions of the following exponential pulses:

(a) $g(t) = \exp(-at)u(t)$

(b) $g(t) = \exp(-a|t|)$

(c) $g(t) = \exp(-at)u(t) - \exp(at)u(-t)$

where $u(t)$ is the unit step function, and $u(-t)$ is its time-reversed version.

2.35 Determine and sketch the autocorrelation function of a Gaussian pulse defined by

$$g(t) = \frac{1}{t_0}\exp\left(-\frac{\pi t^2}{t_0^2}\right)$$

2.36 The Fourier transform of a signal is defined by $\operatorname{sinc}(f)$. Show that the autocorrelation function of this signal is triangular in form.

2.37 Specify two different pulse signals that have exactly the same autocorrelation function.

2.38 Consider a sinusoidal signal $g(t)$ defined by

$$g(t) = A_0 + A_1\cos(2\pi f_1 t + \theta_1) + A_2\cos(2\pi f_2 t + \theta_2)$$

(a) Determine the autocorrelation function $R_g(\tau)$ of this signal.

(b) What is the value of $R_g(0)$?

(c) Has any information about $g(t)$ been lost in obtaining the autocorrelation function? Explain.

2.39 Determine the autocorrelation function of the triplet pulse shown in Fig. 2.45.

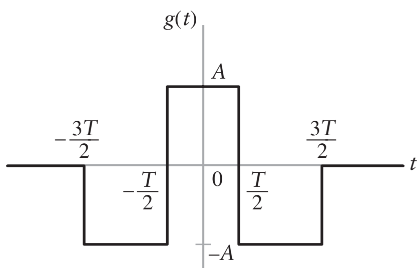


FIGURE 2.45
Problem 2.39

2.40 Let $G(f)$ denote the Fourier transform of a real-valued energy signal $g(t)$, and $R_g(\tau)$ denote its autocorrelation function. Show that

$$\int_{-\infty}^{\infty}\left[\frac{dR_g(\tau)}{d\tau}\right]d\tau = 4\pi^2\int_{-\infty}^{\infty} f^2|G(f)|^4\,df$$

2.41 Determine the cross-correlation function $R_{12}(\tau)$ of the rectangular pulse $g_1(t)$ and triplet pulse $g_2(t)$ shown in Fig. 2.46, and sketch it. What is $R_{21}(\tau)$? Are these two signals orthogonal to each other? Why?

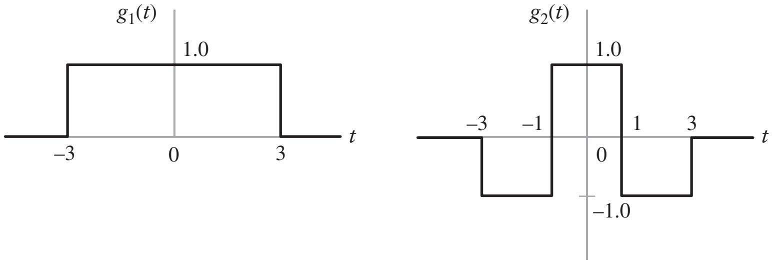


FIGURE 2.46 Problem 2.41

2.42 Consider two energy signals $g_1(t)$ and $g_2(t)$. These two signals are delayed by amounts equal to t_1 and t_2 seconds, respectively. Show that the time delays are additive in convolving the pair of delayed signals, whereas they are subtractive in cross-correlating them.

2.43 (a) An energy signal $x(t)$, its Fourier transform $X(f)$, autocorrelation function $R_x(\tau)$, and energy spectral density $\Psi_x(f)$ are all related, directly or indirectly. Construct a flow-graph that portrays all the possible direct relationships between them.

(b) If you are given the frequency-domain description $X(f)$, the autocorrelation function $R_x(\tau)$ can be calculated from $X(f)$. Outline two ways in which this calculation can be performed.

2.44 Find the autocorrelation function of a power signal $g(t)$ whose power spectral density is depicted in Fig. 2.47. What is the value of this autocorrelation function at the origin?

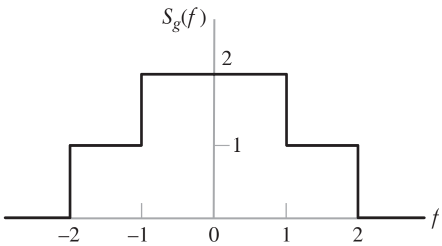


FIGURE 2.47 Problem 2.44

2.45 Consider the square wave $g(t)$ shown in Fig. 2.48. Find the power spectral density, average power, and autocorrelation function of this square wave. Does the wave have dc power? Explain your answer.

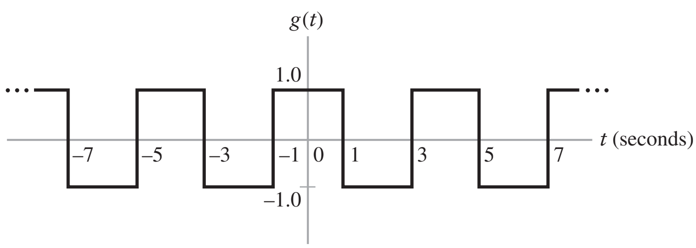


FIGURE 2.48 Problem 2.45

2.46 Consider two periodic signals $g_{p1}(t)$ and $g_{p2}(t)$ that have the following complex Fourier series representations:

$$g_{p1}(t) = \sum_{n=-\infty}^{\infty} c_{1,n} \exp\left(\frac{j2\pi nt}{T_0}\right)$$

and

$$g_{p2}(t) = \sum_{n=-\infty}^{\infty} c_{2,n} \exp\left(\frac{j2\pi nt}{T_0}\right)$$

The two signals have a common period equal to T_0.

Using the following definition of cross-correlation for a pair of periodic signals,

$$R_{12}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_{p1}(t) g_{p2}^*(t - \tau)\, dt$$

show that the prescribed pair of periodic signals satisfies the Fourier-transform pair

$$R_{12}(\tau) \rightleftharpoons \sum_{n=-\infty}^{\infty} c_{1,n} c_{2,n}^* \,\delta\left(f - \frac{n}{T_0}\right)$$

2.47 A periodic signal $g_p(t)$ of period T_0 is represented by the complex Fourier series

$$g_p(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi nt/T_0)$$

where the c_n are the complex Fourier coefficients. The autocorrelation function of $g_p(t)$ is defined by

$$R_{g_p}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_p(t) g_p^*(t - \tau)\, dt$$

(**a**) Consider the sinusoidal wave

$$g_p(t) = A \cos(2\pi f_c t + \theta)$$

Determine the autocorrelation function $R_{g_p}(\tau)$ and plot its waveform.

(**b**) Show that $R_{g_p}(0) = A^2/2$.

2.48 Repeat parts (a) and (b) of Problem 2.47 for the square wave:

$$g_p(t) = \begin{cases} A, & -\dfrac{T_0}{4} \le t \le \dfrac{T_0}{4} \\ 0, & \text{for the remainder of period } T_0 \end{cases}$$

2.49 Determine the power spectral density of (a) the sinusoidal wave of Problem 2.47, and (b) the square wave of Problem 2.48.

2.50 Following a procedure similar to that described in Section 2.10 that led to the flow graph of Fig. 2.36 for the 8-point FFT algorithm based on decimation-in-frequency, do the following:

(**a**) Develop the corresponding flow graph for the 8-point FFT algorithm based on decimation-in-time.

(**b**) Compare the flow graph obtained in part (a) with that described in Fig. 2.36, stressing the similarities and differences between these two basic methods for deriving the FFT algorithm.

ADVANCED PROBLEMS

2.51 (**a**) The *root mean-square* (*rms*) *bandwidth* of a low-pass signal $g(t)$ of finite energy is defined by

$$W_{\mathrm{rms}} = \left[\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 \, df}{\int_{-\infty}^{\infty} |G(f)|^2 \, df} \right]^{1/2}$$

where $|G(f)|^2$ is the energy spectral density of the signal. Correspondingly, *the root mean-square (rms) duration* of the signal is defined by

$$T_{\mathrm{rms}} = \left[\frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 \, dt}{\int_{-\infty}^{\infty} |g(t)|^2 \, dt} \right]^{1/2}$$

Using these definitions, show that

$$T_{\mathrm{rms}} W_{\mathrm{rms}} \geq \frac{1}{4\pi}$$

Assume that $|g(t)| \to 0$ faster than $1/\sqrt{|t|}$ as $|t| \to \infty$.

(**b**) Consider a Gaussian pulse defined by

$$g(t) = \exp(-\pi t^2)$$

Show that, for this signal, the equality

$$T_{\mathrm{rms}} W_{\mathrm{rms}} = \frac{1}{4\pi}$$

can be reached.

Hint: Use Schwarz's inequality (see *Appendix 5*).

$$\left\{ \int_{-\infty}^{\infty} [g_1^*(t) g_2(t) + g_1(t) g_2^*(t)] \, dt \right\}^2 \leq 4 \int_{-\infty}^{\infty} |g_1(t)|^2 \int_{-\infty}^{\infty} |g_2(t)|^2 \, dt$$

in which we set

$$g_1(t) = t g(t)$$

and

$$g_2(t) = \frac{dg(t)}{dt}$$

2.52 The *Hilbert transform* of a Fourier transformable signal $g(t)$ is defined by

$$\hat{g}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\tau)}{t - \tau} \, d\tau$$

Correspondingly, the *inverse Hilbert transform* is defined by

$$g(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{g}(\tau)}{t - \tau} \, d\tau$$

Using these two formulas, derive the following set of Hilbert-transform pairs:

$g(t)$	$\hat{g}(t)$
$\dfrac{\sin t}{t}$	$\dfrac{1 - \cos t}{t}$
$\text{rect}(t)$	$-\dfrac{1}{\pi} \ln\left\lvert \left(t - \dfrac{1}{2}\right) \bigg/ \left(t + \dfrac{1}{2}\right) \right\rvert$
$\delta(t)$	$\dfrac{1}{\pi t}$
$\dfrac{1}{1 + t^2}$	$\dfrac{t}{1 + t^2}$

2.53 Evaluate the inverse Fourier transform $g(t)$ of the one-sided frequency function

$$G(f) = \begin{cases} \exp(-f), & f > 0 \\ \dfrac{1}{2}, & f = 0 \\ 0, & f < 0 \end{cases}$$

Hence, show that $g(t)$ is complex, and that its real and imaginary parts constitute a Hilbert-transform pair.

2.54 A Hilbert transformer may be viewed as a device whose transfer function exhibits the following characteristics:

(**a**) The amplitude response is unity for all positive and negative frequencies.

(**b**) The phase response is $+90°$ for negative frequencies and $-90°$ for positive frequencies. Starting with the definition of the Hilbert transform given in Problem 2.52, demonstrate the frequency-domain description embodied in parts (a) and (b).

(**c**) Is the Hilbert transformer physically realizable? Justify your answer.