EBU6503
Control Theory

State Variables

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States

What do we mean by a system "state"?

- **So far, we have referred to how a system *output* will behave….its *state* could be its *position* or *speed* or *temperature*, etc. We have the input/output relationship only.**
- **But we have not considered the *state* of elements *inside* the system.**
- **We do not have a complete description of a system until we can describe the condition of a minimum number of system states.**

States

What is the minimum number of states needed to completely describe a system?

- **An *nth* order system requires a minimum of *n* states.**
- **These must be linearly independent.**

If we represent a system by a function of its states then we have a *time-domain* model of the system. So we can analyse the time response directly, without using the frequency response to predict it.

State Model

A State Model of a system reduces an *nth* order system to a set of *n* 1st order relationships.

We know that systems can be represented mathematically by differential equations.

For example we looked at the differential equation representing the simple electrical series LCR circuit (input applied voltage, output resulting current).

State Model

To introduce the idea of a state model, consider a system whose input is u(t) and whose output is y(t), e.g. the 3rd order system:

$$a\dddot{y}(t) + b\ddot{y}(t) + c\dot{y}(t) + dy(t) = Ku(t)$$

This 3rd order system needs 3 independent states. Label the states as *x* and make them the output and its derivatives.

State Model

Therefore:

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

$$x_3(t) = \ddot{y}(t)$$

State Model

Then:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = x_3(t)$$

$$\dot{x}_3(t) = \dddot{y}(t)$$

Some Simple Magic

State Model

Finally:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dfrac{-d}{a} & \dfrac{-c}{a} & \dfrac{-b}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{K}{a} \end{bmatrix} u(t)$$

$$y(t) = x_1(t)$$

State Model

In general, the *State Equations* can be written in the form:

$$[\dot{x}] = [A][x] + [B][u]$$

$$[y] = [C][x] + [D][u]$$

State Model

Where:

A is called the *plant* or *companion* matrix

B is called the *excitation* matrix

C is called the *output* matrix

D is called the *direct transmission* matrix

Solution of State Equations

Because the state equations are matrix equations they are suitable for computer solution and are therefore useful.

Also, they give us a complete picture of system behaviour (instead of telling us only the output) because they allow us to analyse what is happening inside the system.

Observability and Controllability

Observability **is whether or not changes in all the states can be observed at the output.**

Controllability **is whether or not all the states can be affected by a change at the input.**

Back to the Solution of the State Equations

The state equations are suitable for computer solution, but hand solution can be tedious and is limited to low order systems (at most 3rd order).

Solution of State Equations

The first equation must be solved for x, and then this allows us to solve the second equation (which is often just called the output equation). Because they are differential equations we use Laplace Transforms.

$$[\dot{x}] = [A][x] + [B][u]$$

$$[y] = [C][x] + [D][u]$$

Solution of State Equations

Let's ignore the matrix brackets (to be lazy ☺).

Using Laplace to solve the first equation:

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$[sI - A]X(s) = x(0) + BU(s)$$

$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}BU(s)$$

Solution of State equations

The *Resolvent Matrix* is: $\emptyset(s) = [sI - A]^{-1}$

So we get:

$$X(s) = \emptyset(s)x(0) + \emptyset(s)BU(s)$$

And then:

$$x(t) = L^{-1}\emptyset(s)x(0) + L^{-1}\emptyset(s)BU(s)$$

The *Transition Matrix* is:

$$\emptyset(t) = L^{-1}\emptyset(s) = L^{-1}[sI - A]^{-1} = e^{At}$$

Transition Matrix

The *Transition Matrix* is:

$$\emptyset(t) = L^{-1}\emptyset(s) = L^{-1}[sI - A]^{-1} = e^{At}$$

The label e^{At}

is used for the transition matrix because we know that the functions of time describing the states will be exponentials. This is because the system is described as a set of first order differential equations.

Solution of State Equations

- **If u(t)=0, we get the *unforced response*, ie, the response due only to any non-zero initial conditions.**
- **The *forced response* is the response to an input u(t) with no initial conditions.**
- **The *total response* is the sum of the forced and unforced responses.**
- **The system *modes* are described by the decay rates, amplitudes, of the states x(t).**