EBU5375 Signals and Systems: Continuous-time systems in the frequency domain

Dr Jesús Requena Carrión

Queen Mary University of London

Agenda

Agenda

What is this week about?

The main topics covered by this course are organised as follows:

Week 1: CT and DT signals and systems in the time domain.

Week 2: CT signals and systems in the frequency domain.

Week 3: DT signals and systems in the frequency domain.

Week 4: Sampling theory and communication systems.

What have we learnt so far?

1. CT and DT **signals in the time domain**: basic signals, representation, properties, classification, manipulations in the time domain (shift, reflection, amplification) ...
2. CT and DT **systems in the time domain**: properties, LTI systems, impulse response, convolution ...
3. CT **signals in the frequency domain**: Fourier series and Fourier transform.

The notion of frequency: sinusoidal signals

Consider the signals

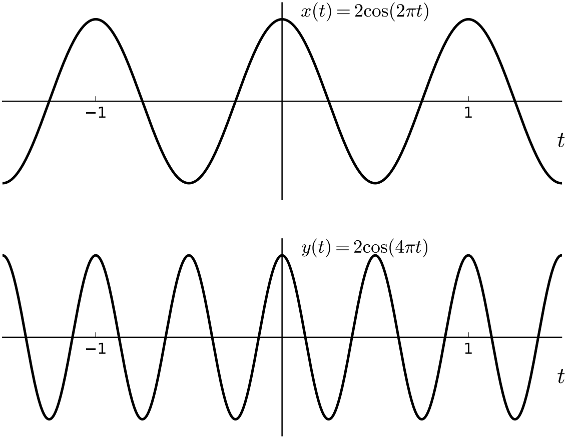
$$\begin{aligned} x(t) &= 2\cos(2\pi t) \\ y(t) &= 2\cos(4\pi t) \end{aligned}$$

(a) The frequency of $x(t)$ is **higher** than the frequency of $y(t)$.

(b) The frequency of $x(t)$ is **lower** than the frequency of $y(t)$.

(c) The frequencies of $x(t)$ and $y(t)$ are **equal**.

The notion of frequency: sinusoidal signals



The notion of frequency: complex exponentials

Consider the signals

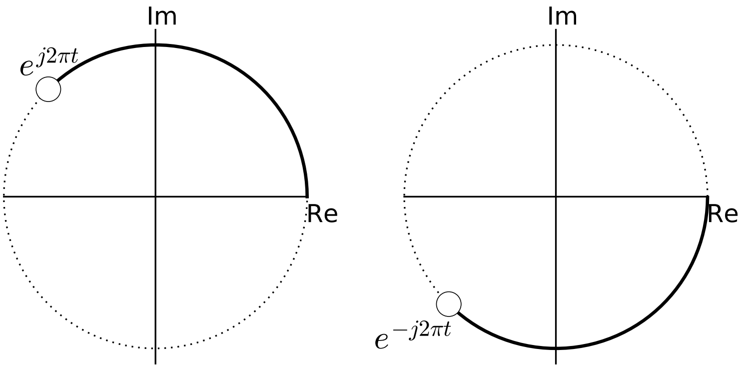
$$x(t) = e^{j2\pi t}$$
$$y(t) = e^{-j2\pi t}$$

(a) The period of $x(t)$ is $T_x = 1$ and the period of $y(t)$ is $T_y = -1$.

(b) The period of $x(t)$ is $T_x = -1$ and the period of $y(t)$ is $T_y = 1$.

(c) The periods of $x(t)$ and $y(t)$ are $T_x = T_y = 1$.

The notion of frequency: complex exponentials



Note that the sum of the complex exponentials $e^{j2\pi t}$ and $e^{-j2\pi t}$ results in a real signal! That's why the magnitude of the FT of real signals is always symmetric :)

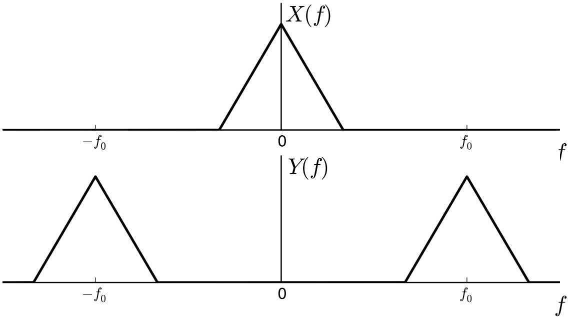
The frequency domain and the Fourier transform

$$x(t) \overset{FT}{\Longleftrightarrow} X(f), X(\omega)$$

The f-domain		**The ω-domain**
$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$	**Analysis**	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$	**Synthesis**	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$

Fourier transform of CT signals

Consider the Fourier transforms:



(a) $X(f)$ has low frequencies and $Y(f)$ high frequencies.

(b) $X(f)$ has high frequencies and $Y(f)$ low frequencies.

(c) $Y(f)$ has low and high frequencies, $X(f)$ neither of them.

Properties of FT: temporal displacement

Consider the FT pairs:

$$x(t) \overset{FT}{\Longleftrightarrow} X(f)$$
$$y(t) \overset{FT}{\Longleftrightarrow} Y(f)$$

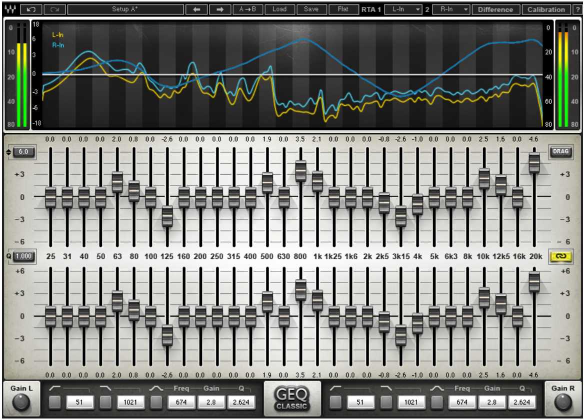
If $y(t) = x(t - 10)$, then

(a) $|Y(f)| = |X(f - 10)|$.
(b) $|Y(f)| = |X(f - 0.1)|$.
(c) $|Y(f)| = |X(f)|$.

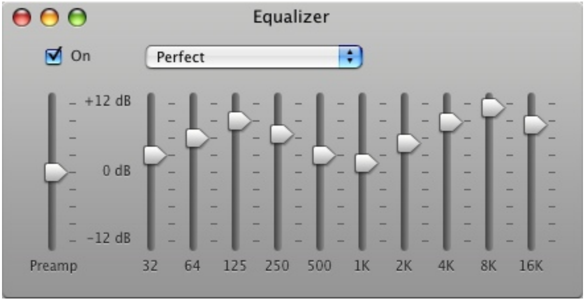
Agenda

Frequencies, what for?

Digital equalisers



Digital equalisers



Applications

- By looking at the **frequency domain we can extract useful information**. Why? Because many natural phenomena are **cyclic** (electromagnetic radiation, movement of planets, circadian rhythms. . .).
- It can be easier to **understand signal distortions caused by physical media in the frequency domain**. Why? Because media can often be described as linear and time-invariant.
- **Modulation techniques for transmitting data** can be best understood in the frequency domain.
- **Signal processing techniques** can be best understood in the frequency domain.

Internet of Things

IoT devices:

- **Measure and process physical signals**, and information might be more apparent in the frequency domain.
- **Digitise physical signals**, and the process of digitisation can be best understood in the frequency domain.
- **Transmit information** by using modulation techniques and they can be understood in the frequency domain.

Agenda

Introduction

The frequency domain: what for?

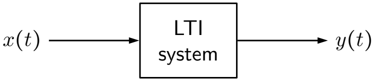
The convolution theorem

Introduction to filters

Linear Time-Invariant systems

LTI systems are defined by two basic properties:

1. **Linearity**: Combinations of inputs produce combinations of their outputs.
2. **Time invariance**: Delayed inputs produce delayed outputs.

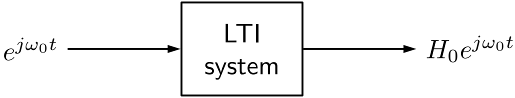


$$
\begin{aligned}
x_1(t) &\quad\rightarrow\quad y_1(t) \\
x_2(t) &\quad\rightarrow\quad y_2(t)
\end{aligned}
$$

$$
\begin{aligned}
A_1x_1(t) + A_2x_2(t) &\quad\rightarrow\quad A_1y_1(t) + A_2y_2(t) \\
x_1(t - t_0) &\quad\rightarrow\quad y_1(t - t_0)
\end{aligned}
$$

Linear Time-Invariant systems and complex exponentials

A pure frequency ω at the input produces the **same pure frequency** ω at the output (with different amplitude and phase):

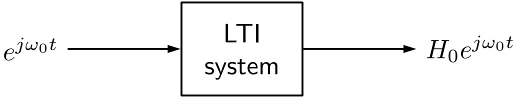


$$
\begin{array}{ccc}
e^{j\omega_0 t} & \longrightarrow & H_0 e^{j\omega_0 t} \\
e^{j\omega_1 t} & \longrightarrow & H_1 e^{j\omega_1 t} \\
 & \vdots & \\
e^{j\omega t} & \longrightarrow & H(\omega) e^{j\omega t}
\end{array}
$$

$$A_0 e^{j\omega_0 t} + A_1 e^{j\omega_1 t} \longrightarrow$$

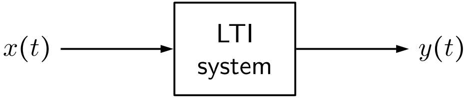
Linear Time-Invariant systems and general signals

A pure frequency ω at the input produces the **same pure frequency** ω at the output (with different amplitude and phase):



$$
\begin{array}{ccc}
e^{j\omega_0 t} & \longrightarrow & H_0 e^{j\omega_0 t} \\
e^{j\omega_1 t} & \longrightarrow & H_1 e^{j\omega_1 t} \\
 & \vdots & \\
e^{j\omega t} & \longrightarrow & H(\omega) e^{j\omega t}
\end{array}
\qquad
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \longrightarrow
$$

Linear Time-Invariant systems and general signals

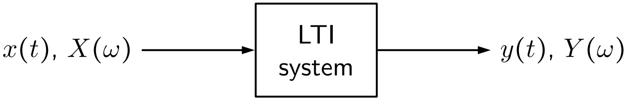


$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \longrightarrow y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j\omega t}d\omega$$

$$y(t) \overset{FT}{\Longleftrightarrow} Y(\omega) = X(\omega)H(\omega)$$

$H(\omega)$ is the **frequency response** or **transfer function** of the LTI system.

Linear Time-Invariant systems: Summary



$$y(t) = x(t) \star h(t) \overset{FT}{\Longleftrightarrow} Y(\omega) = X(\omega)H(\omega)$$

Is there any relationship between $h(t)$ and $H(\omega)$? Can you guess?

The Convolution Theorem

Question: What is the Fourier transform of a convolution?

Consider $y(t) = x(t) \star h(t)$. Let us calculate its Fourier transform.

$$
\begin{aligned}
Y(\omega) &= \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \right] e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h(t-\tau) e^{-j\omega(t-\tau)} e^{-j\omega\tau} d\tau dt \\
&= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega(t-\tau)} dt \right] e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) H(\omega) e^{-j\omega\tau} d\tau \\
&= X(\omega) H(\omega)
\end{aligned}
$$

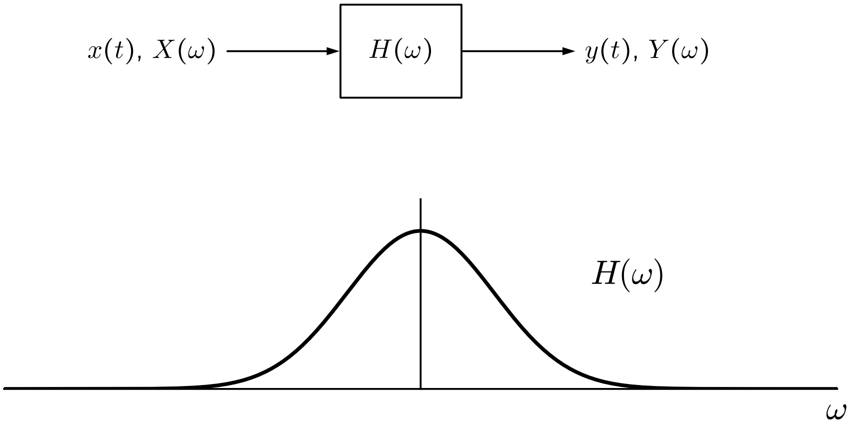
Linear Time-Invariant systems: Summary

$$x(t),\, X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow y(t),\, Y(\omega)$$

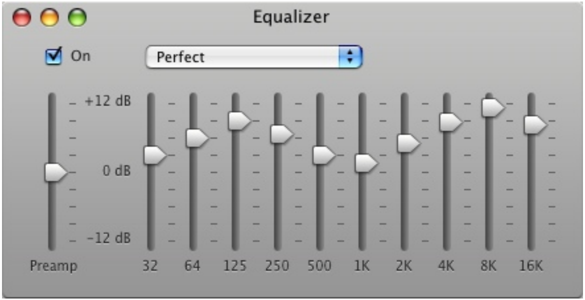
$$y(t) = x(t) \star h(t) \quad \overset{FT}{\Longleftrightarrow} \quad Y(\omega) = X(\omega)H(\omega)$$

$$\begin{aligned} x(t) &\quad \overset{FT}{\Longleftrightarrow} \quad X(\omega) \\ h(t) &\quad \overset{FT}{\Longleftrightarrow} \quad H(\omega) \\ y(t) &\quad \overset{FT}{\Longleftrightarrow} \quad Y(\omega) \end{aligned}$$

What does the frequency response tell us?



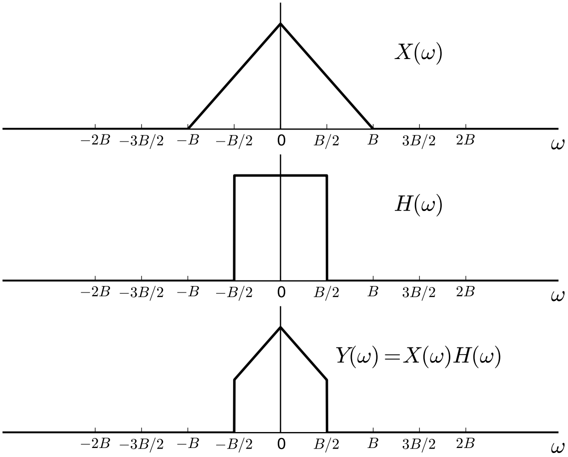
What does the frequency response tell us?



Agenda

LTI systems as filters

The frequency response $H(\omega)$ shows that LTI systems act as **frequency filters** since they allow certain frequencies at the input to pass whereas they stop other frequencies.



LTI systems as filters

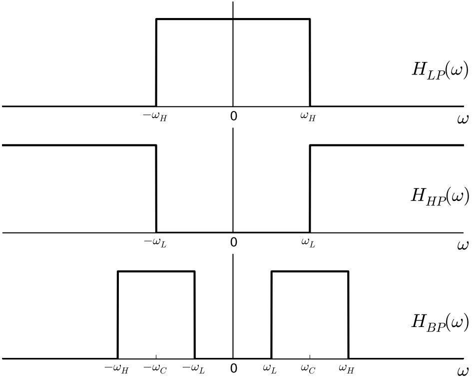
The frequency response of LTI systems are characterised by

- The **stopband**: interval of frequencies that are not allowed to pass.
- The **passband**: interval of frequencies that are allowed to pass.
- A **bandwidth**: width of the passband (ONLY POSITIVE FREQUENCIES ARE CONSIDERED).

There are three basic types of filters:

- **Lowpass** filters: Low frequencies pass.
- **Highpass** filters: High frequencies pass.
- **Bandpass** filters: Frequencies within an intermediate band pass.

Ideal filters



Lowpass filter:

$$H_{LP}(\omega)$$
$$B_{LP} = \omega_H - 0 = \omega_H$$

Highpass filter:

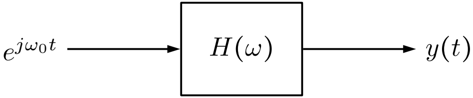
$$H_{HP}(\omega)$$
$$B_{HP} = \infty - \omega_L = \infty$$

Bandpass filter:

$$H_{BP}(\omega)$$
$$B_{BP} = \omega_H - \omega_L$$

ω_L: lower cutoff, ω_H: upper cutoff, ω_C: centre frequency

Example: Filtering complex exponentials



Example: Filtering sinusoidal signals

$$\cos\omega_0 t \longrightarrow \boxed{H(\omega)} \longrightarrow y(t)$$

Example: Filtering periodic signals

$$x_T(t) \longrightarrow \boxed{H(\omega)} \longrightarrow y(t)$$

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

The inverse filter

An LTI system with frequency response $H_2(\omega)$ is said to be the inverse filter of another LTI system with frequency response $H_1(\omega)$ if

$$H_2(\omega) = \frac{1}{H_1(\omega)}$$

$$x(t) \longrightarrow \boxed{H_1(\omega)} \xrightarrow{y(t)} \boxed{H_2(\omega)} \longrightarrow z(t)$$

In this example,

$$Z(\omega) = Y(\omega)H_2(\omega) = X(\omega)H_1(\omega)H_2(\omega) = X(\omega)H_1(\omega)\frac{1}{H_1(\omega)} = X(\omega)$$

as long as $H_1(\omega) \neq 0$.

Remember!

- In continuous-time, the higher the frequency of a signal, the faster its amplitude changes.
- A frequency-domain description of a signal tells us how fast the amplitude of the signal changes: it can be fast, slow or even both!
- LTI systems can also be described in the frequency domain: the frequency response tells us which frequencies the system attenuates and which frequencies it amplifies.

Never forget this:

$$y(t) = x(t) \star h(t) \overset{FT}{\Longleftrightarrow} Y(\omega) = X(\omega)H(\omega)$$