Digital Computer Control

1 Signal sampling and zero-order hold

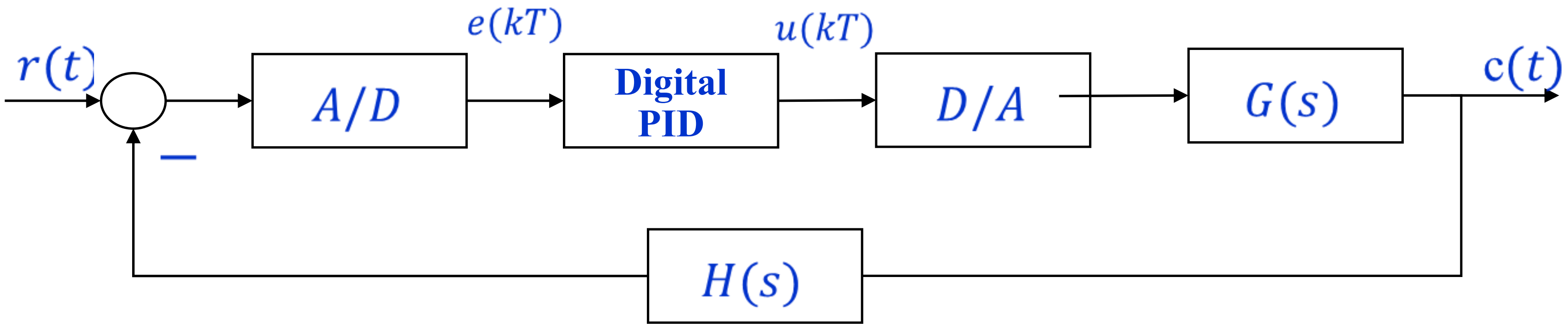
2 Differential equation

3 Digital PID control

4 Discrete system controllability and observability

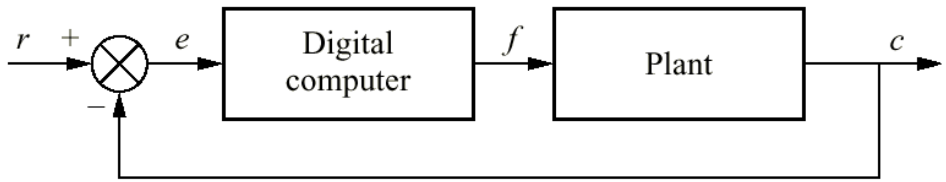
1.1 Computer Control System

Computer control system consists of a digital computer and a process. Digital computer works at discrete domain, process is in continuous domain. Output and input signals of process are continuous signals, Output and input signals of computer are discrete signal. Reference and feedback signals are continuous.

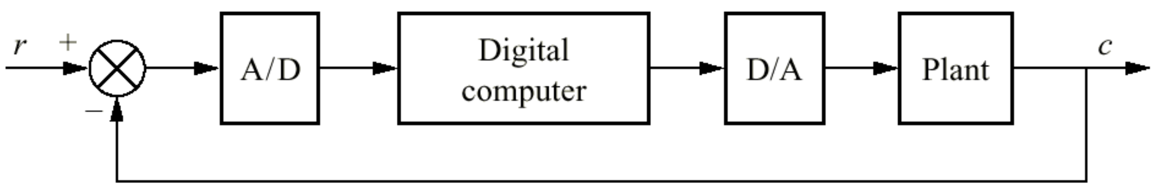


1.2 Digital Computer and A/D D/A Converter

Digital computer only can deal with binary data, the input and output data cannot be continuous (analog) signal. The input (error) signal must be converted into digital signal via (A/D), the output (control) signal must be converted back into analog signal.



(*a*)



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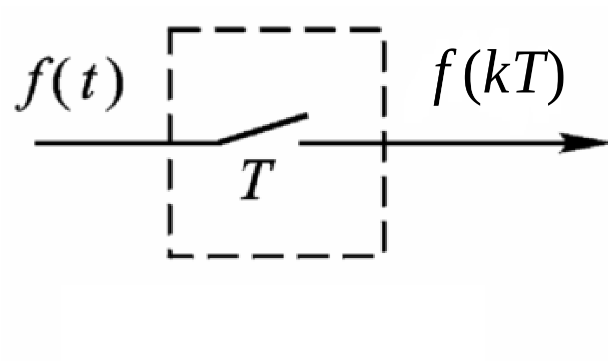
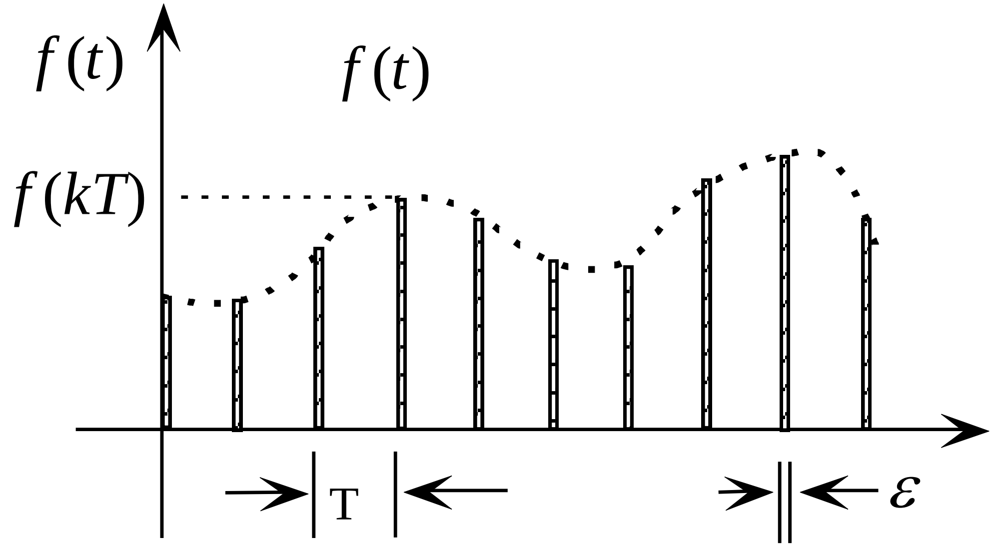
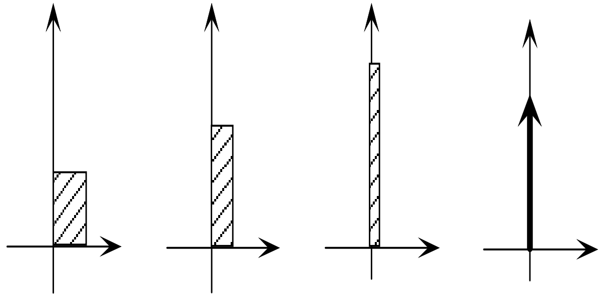
1.3 Sampling: From continuous signal to discrete signal

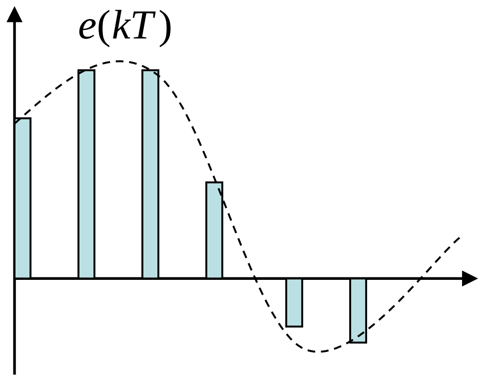
Sampling process

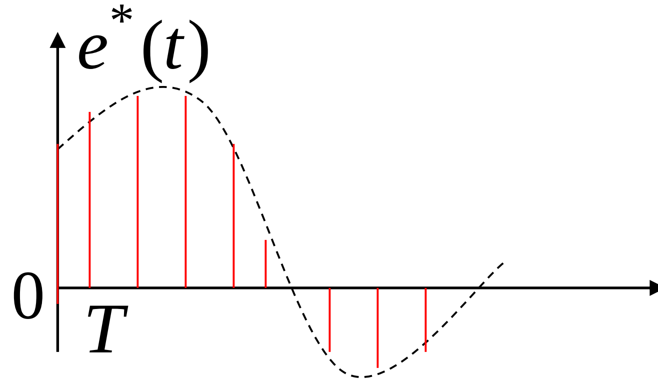




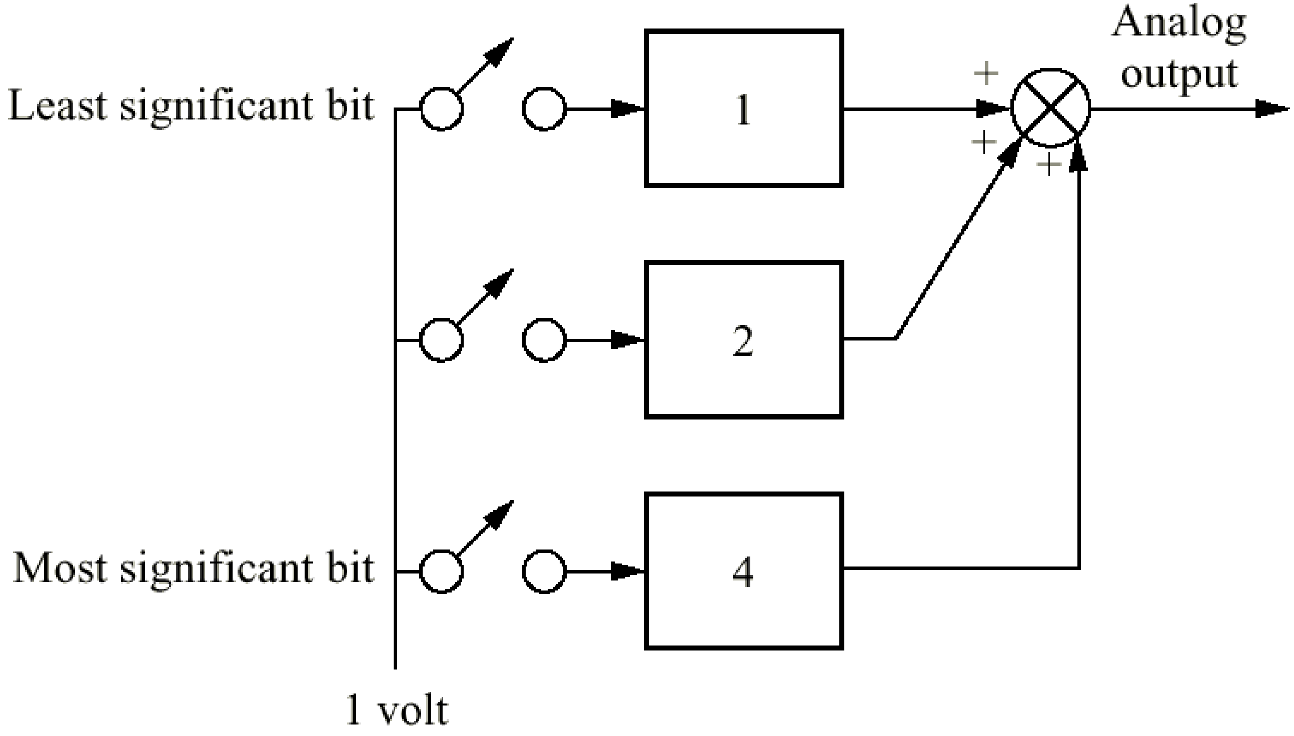
ε<<T **$u^*(kT)=u(kT)$**



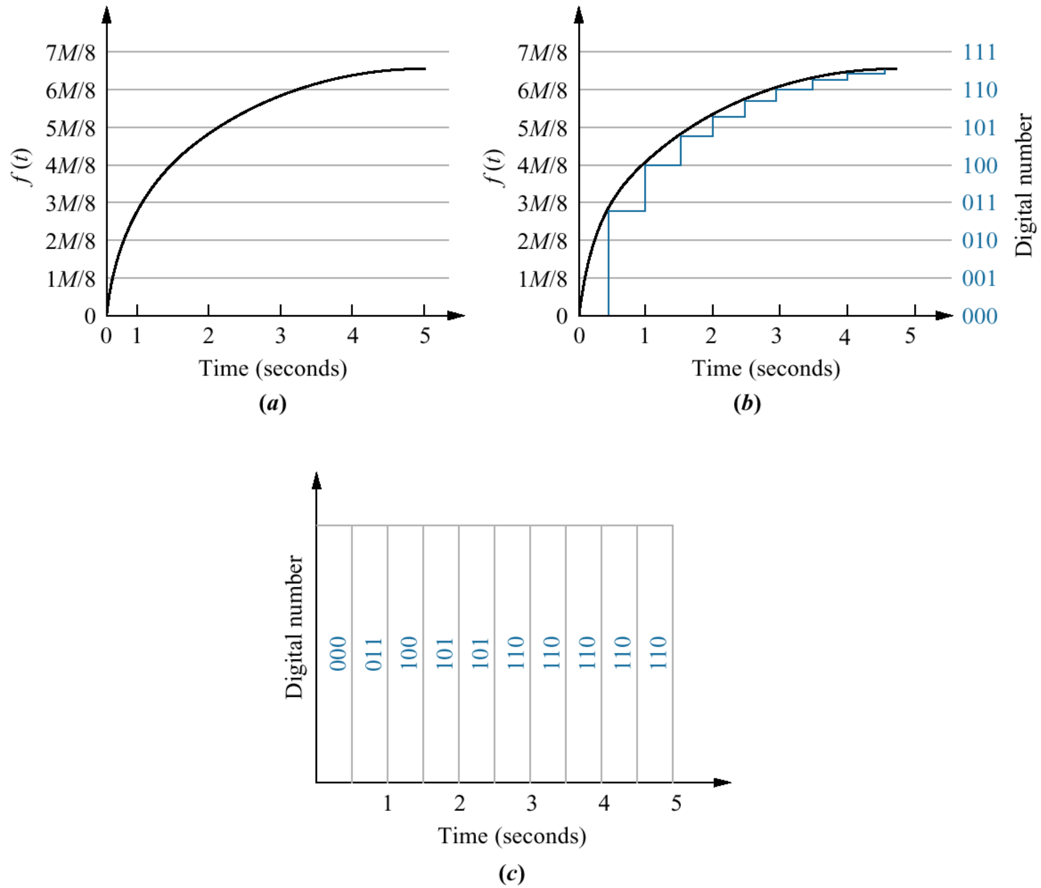




Digital-to-analog converter



Analog-to-Digital converter



Steps in analog-to-digital conversion:

a. analog signal;
b. analog signal after sample-and-hold;
c. conversion of samples to digital numbers

1.4 Signal Sampling

Sampling Categories：

1. **Equivalent interval sampling**
2. **Various interval sampling**
3. **Random interval sampling**

If there are multiple sampling switches in a system：

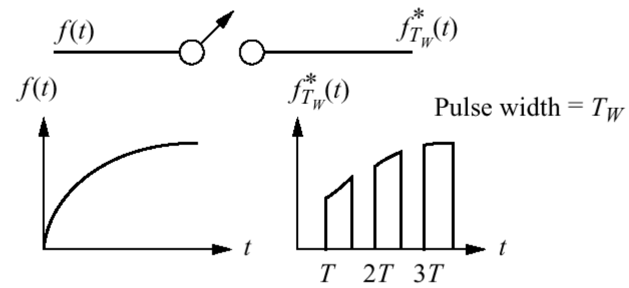
A single sampling interval is taken in the system, it is called synchronous sampling。

Multiple equivalent sampling intervals are used, but sampling intervals are not the same, it is called multiple speeds sampling.

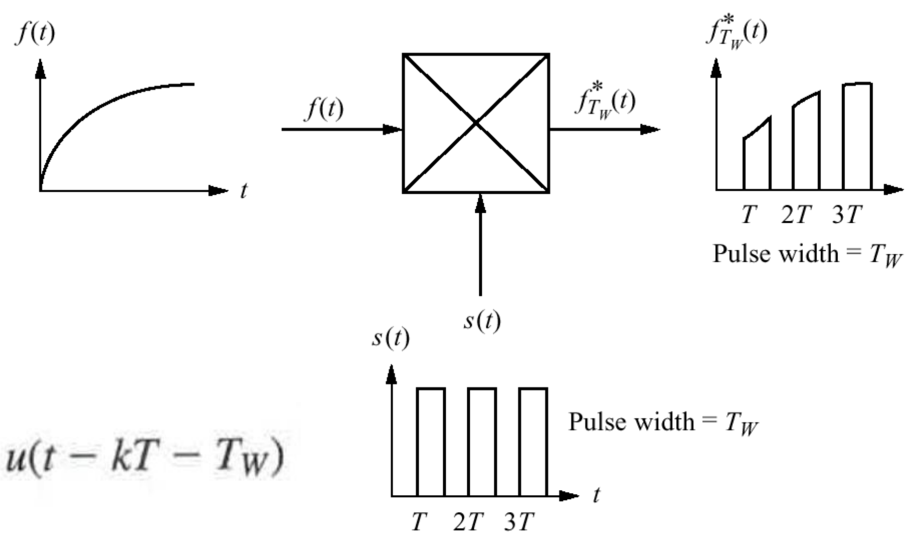
Two views of uniform-rate sampling:

a. switch opening and closing;

b. product of time waveform and sampling waveform



(a)



(b)

$$f^*_{T_W}(t) = f(t)s(t) = f(t)\sum_{k=-\infty}^{\infty} u(t-kT) - u(t-kT-T_W)$$

Assumption 1: the pulse width T_w is much smaller than the sampling period T.
Assumption 2: f(t) is constant during the sampling interval, i.e. *f(t)=f(kT)*

$$f_{T_W}^*(t) = \sum_{k=-\infty}^{\infty} f(kT)[u(t-kT) - u(t-kT-T_W)]$$

$$F_{T_W}^*(s) = \sum_{k=-\infty}^{\infty} f(kT)\left[\frac{e^{-kTs}}{s} - \frac{e^{-kTs-T_Ws}}{s}\right] = \sum_{k=-\infty}^{\infty} f(kT)\left[\frac{1-e^{-T_Ws}}{s}\right]e^{-kTs}$$

Replacing e^{-T_Ws} with its series expansion, we obtain

$$F_{T_W}^*(s) = \sum_{k=-\infty}^{\infty} f(kT)\left[\frac{1-\left\{1-T_Ws+\frac{(T_W s)^2}{2!}-\cdots\right\}}{s}\right]e^{-kTs}$$

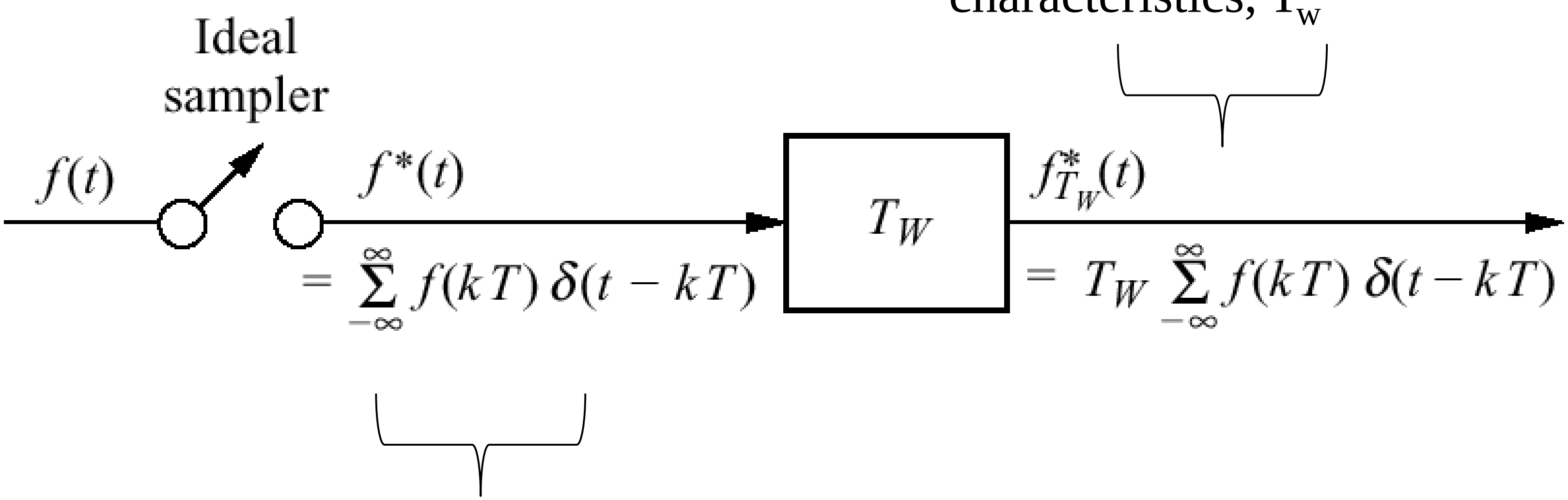
$$F_{T_W}^*(s) = \sum_{k=-\infty}^{\infty} f(kT)\left[\frac{T_Ws}{s}\right]e^{-kTs} = \sum_{k=-\infty}^{\infty} f(kT)T_We^{-kTs}$$

Finally, converting back to the time domain, we have

$$f_{T_W}^*(t) = T_W \sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT)$$

Model of sampling with a uniform rectangular pulse train

The portion depending upon the sampling waveform’s characteristics, T_w



The portion of an ideal sampler

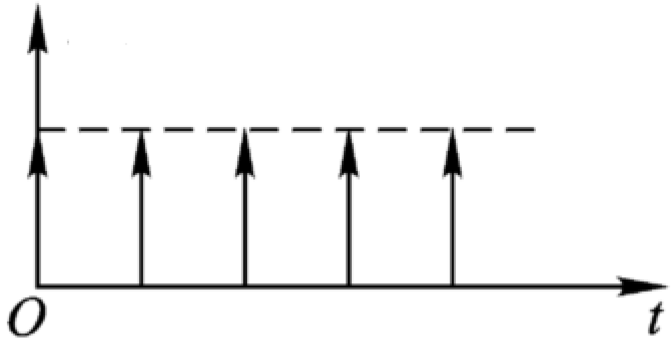
Mathematical description 1 of signal sampling

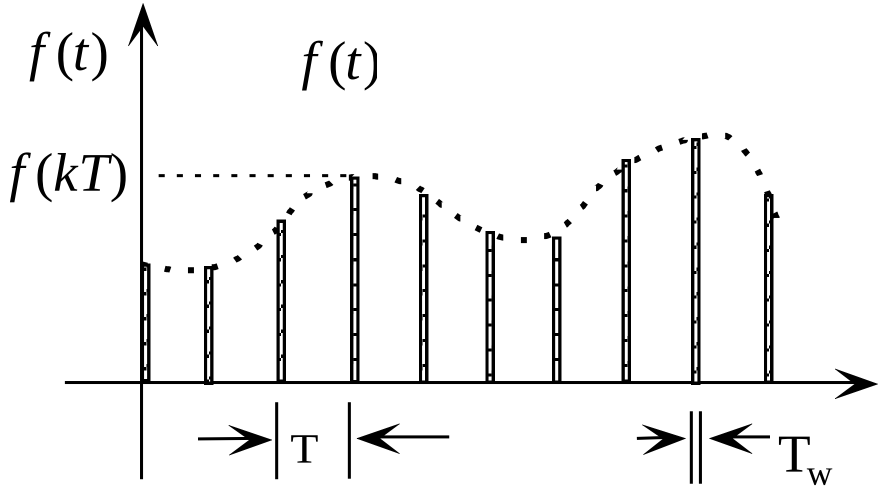
$$f^*(t)=\sum_{k=0}^{+\infty} f(kT)\delta(t-kT)$$

$$f^*(t)= f(t)\sum_{k=0}^{+\infty}\delta(t-kT)$$

$$f^*(t) = f(t)\delta_T(t)$$

$$\delta_T(t)=\sum_{k=0}^{+\infty}\delta(t-kT)$$





$$f^*(t)=\sum_{k=0}^{+\infty} f(kT)\delta(t-kT)$$

$$F^*(s)=\sum_{k=0}^{+\infty} f(kT)e^{-kTs}$$

Mathematical description 2 of signal sampling

$$f^*(t) = \sum_{k=0}^{+\infty} f(kT)\delta(t-kT)$$

$$f^*(t) = f(t)\sum_{k=0}^{+\infty} \delta(t-kT)$$

$$f^*(t) = f(t)\delta_T(t)$$

$$\delta_T(t) = \sum_{k=0}^{+\infty} \delta(t-kT)$$

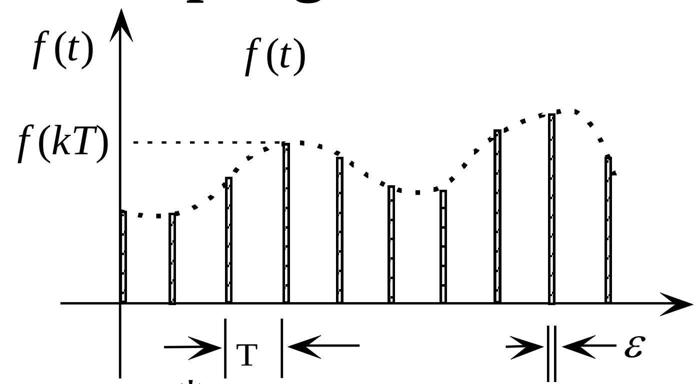
$\delta_T(t)$ is a periodic function, Fourie series is:

$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_s t}$$

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta_T(t) e^{-jk\omega_s t} dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T}$$

$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} e^{jk\omega_s t}$$



$$f^*(t) = f(t)\delta_T(t)$$

$$f^*(t) = f(t)\sum_{k=-\infty}^{+\infty} \frac{1}{T} e^{jk\omega_s t} = \frac{1}{T}\sum_{k=-\infty}^{+\infty} f(t) e^{jk\omega_s t}$$

$$F^*(s) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} L\left[f(t)e^{jk\omega_s t}\right]$$

$$\boxed{F^*(s) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} F(s+jk\omega_s)}$$

1.5 Recovering continuous signal from discrete signal

Question 1: Can the original continuous signal $f(t)$ be recovered from the discrete signal $f^*(t)$?

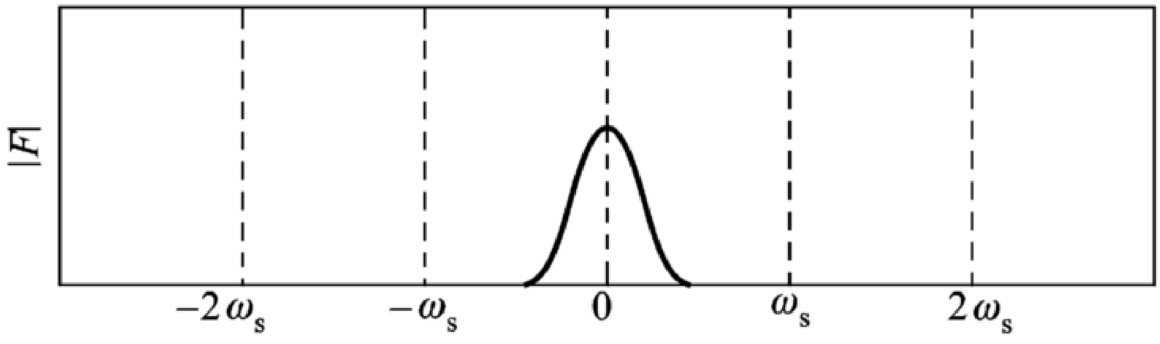
Question 2: What type of hold should be used?

Sampling Theorem

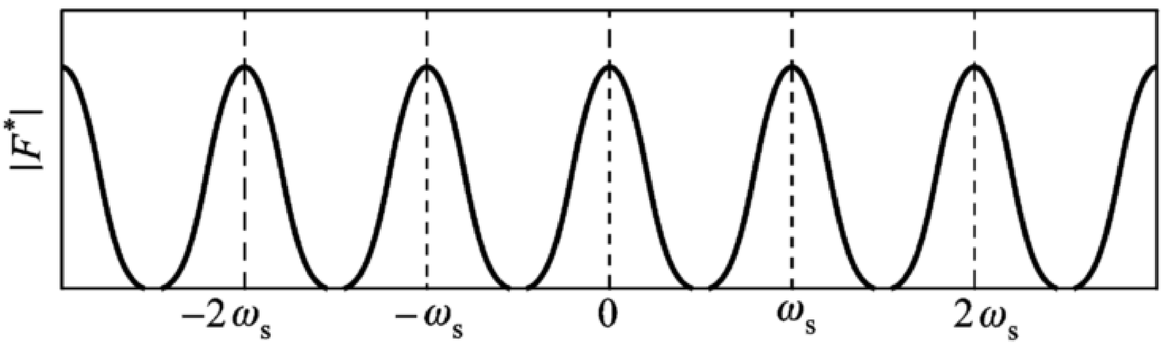
Sampling signal spectrum, and the relationship with the continuous signal spectrum (ω_s is the sampling rate**)**

$$F^*(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} F(j\omega + jk\omega_s)$$

$$F^*(j\omega) = \cdots + \frac{1}{T}F(j\omega - j2\omega_s) + \frac{1}{T}F(j\omega - j\omega_s) + \frac{1}{T}F(j\omega) + \frac{1}{T}F(j\omega + j\omega_s) + \cdots$$

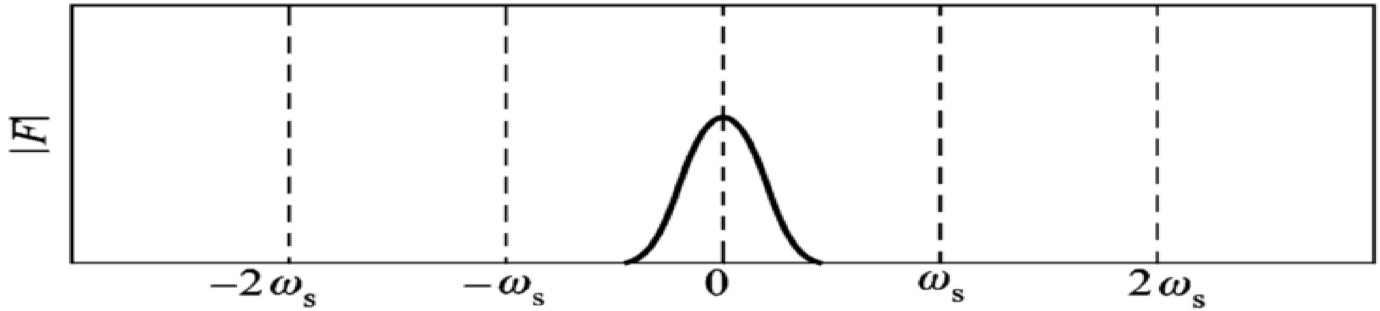


Continuous signal spectrum

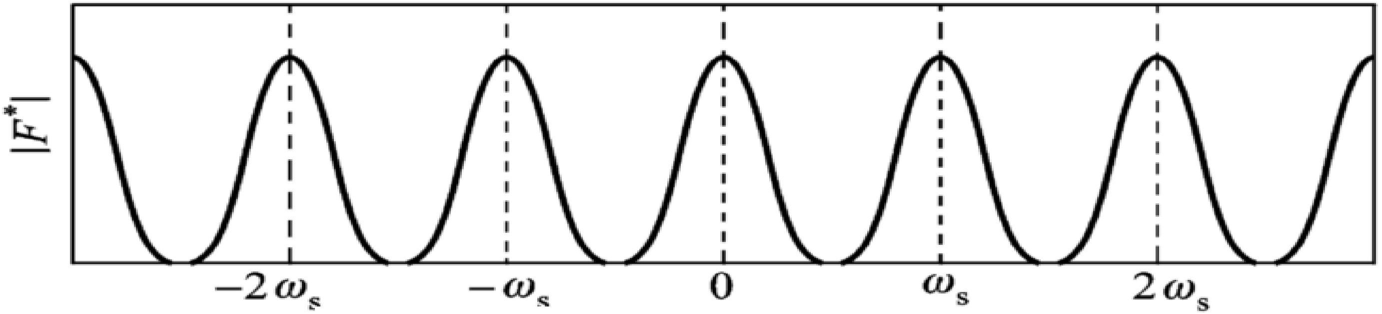


Sampled signal spectrum

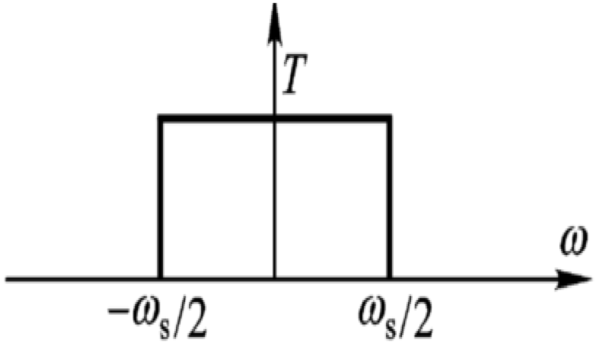
Recover the continuous signal from the sampling signal without any loss



Continuous signal spectrum

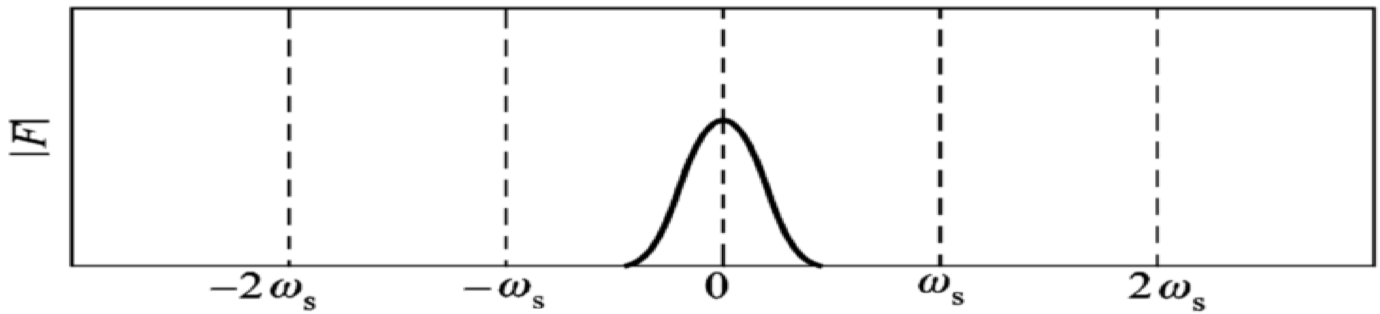


Sampled signal spectrum

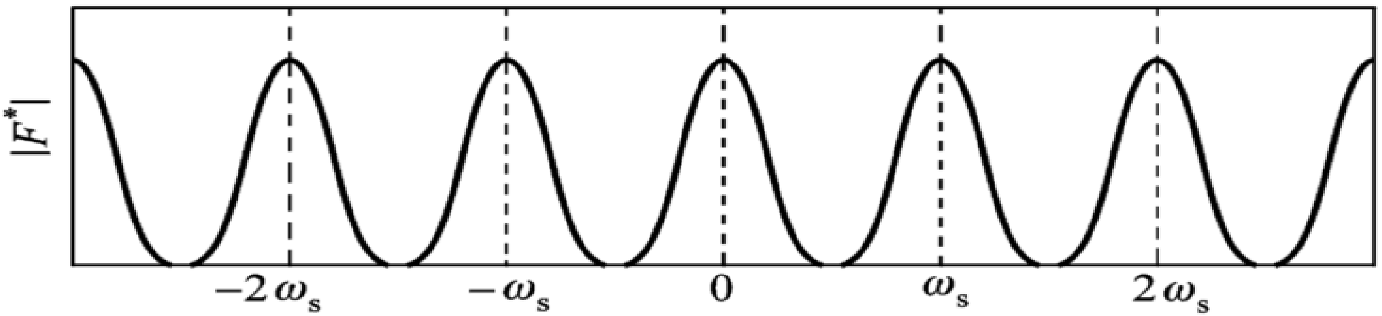


Ideal low pass filter

Recover the continuous signal from the sampling signal without any loss



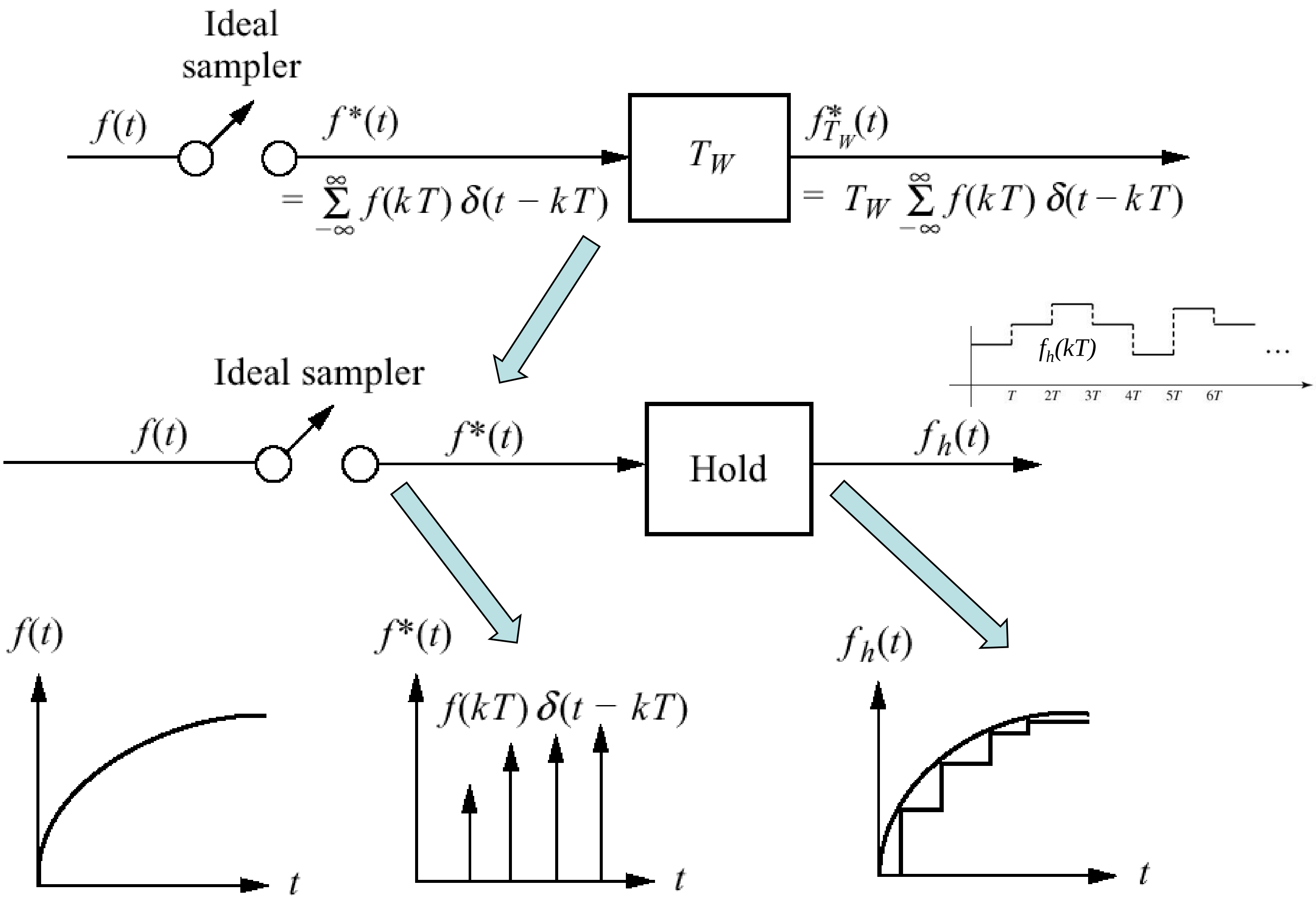
Continuous signal spectrum



Sampled signal spectrum

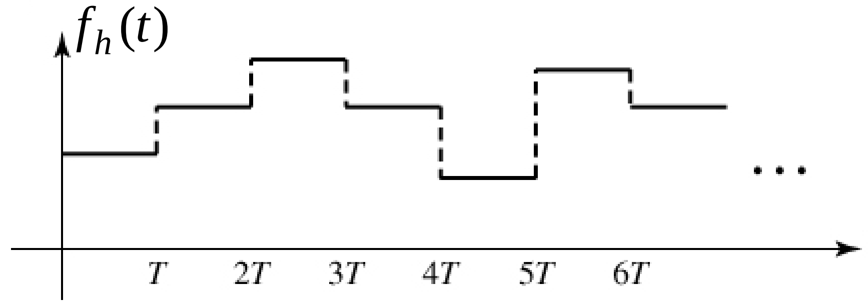
Shannon sampling theorem： The sampling rate w_s must be at least twice the bandwidth of the signal $w_{\max}$， i.e. $w_s >= 2w_{\max}$， in order to recover completely $f(t)$ from the sampling signal $f^*(t)$.

1.6 The zero-order hold



Transfer function of zero-order hold

$$f_h(t) = \sum_{k=0}^{+\infty} f(kT)\left[1(t-kT) - 1(t-kT-T)\right]$$



$$F_h(s) = \sum_{k=0}^{+\infty} f(kT)\{L\left[1(t-kT) - 1(t-kT-T)\right]\}$$

$$= \sum_{k=0}^{+\infty} f(kT)\left[\frac{1}{s}e^{-kTs} - \frac{1}{s}e^{-(k+1)Ts}\right]$$

$$= \sum_{k=0}^{+\infty} f(kT)e^{-kTs}\,\frac{1-e^{-Ts}}{s}$$

$$= \frac{1-e^{-Ts}}{s}\sum_{k=0}^{+\infty} f(kT)e^{-kTs}$$

$$= \frac{1-e^{-Ts}}{s}\quad F^*(s)$$

$$\boxed{G_{oh}(s) = \frac{F_h(s)}{F^*(s)} = \frac{1-e^{-Ts}}{s}}$$

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Differential equation

$$y(k)-7y(k-1)+16y(k-2)-12y(k-3)=x(k)$$

$$3k^2y(k+2)+\frac{2}{k+1}y(k)=x(k)$$

$$3y(k+2)-2y(k+1)y(k)=x(k)$$

$$y(k+3)-y^2(k)=x(k)$$

1. **Order of differential equation**
2. **forward（increment）、backward（decrement）**
3. **Linear, unlinear**
4. **Constant coefficient、various coefficient**

Approximation solution

Find out the approximation $$\tau \frac{dy(t)}{dt} + y(t) = Kx(t)$$

$$\frac{dy(t)}{dt}\Big|_{t=Kt} = \frac{1}{T}\{y[(k+1)T] - y(kT)\}$$

$$\frac{d^2y(t)}{dt^2}\Big|_{t=Kt} = \frac{1}{T^2}\{y[(k+2)T] - 2y[(k+1)T] + y(kT)\}$$

$$\frac{d^3y(t)}{dt^3}\Big|_{t=Kt} = \frac{1}{T^3}\{y[(k+3)T] - 3y[(k+2)T] + 3y[(k+1)T] - y(kT)\}$$

$$\frac{\tau}{T}\{y[(k+1)T] - y(kT)\} + y(kT) = Kx(kT)$$

$$y[(k+1)T] - (1 - \frac{T}{\tau})y(kT) = \frac{T}{\tau}Kx(kT)$$

Iterative solution

$$y(k)+2y(k-1)=u(k)-u(k-1) \qquad u(k)=\begin{cases} k^2 & k\geq 0 \\ 0 & k<0 \end{cases} \qquad y(0)=1$$

$$y(k)+2y(k-1)=k^2-(k-1)^2=2k-1$$

$$y(k)=-2y(k-1)+2k-1$$

$$y(1)=-2y(0)+2\times 1-1=-1$$

$$y(2)=-2y(1)+2\times 2-1=5$$

$$y(3)=-2y(2)+2\times 3-1=-5$$

$$y(4)=-2y(3)+2\times 4-1=17$$

$$\vdots$$

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Continuous PID Control

$$G_c(s) = \frac{U(s)}{E(s)} = K_P(1 + \frac{1}{T_I s} + T_D s) = K_P + K_I \frac{1}{s} + K_D s$$

$$u(t) = K_P\left[e(t) + \frac{1}{T_I}\int e(t)dt + T_D \frac{de(t)}{dt}\right]$$

$$= K_P e(t) + K_I \int e(t)dt + K_D \frac{de(t)}{dt}$$

Digital PID Control

$$\boxed{u(t)=K_P\left[e(t)+\frac{1}{T_I}\int e(t)dt+T_D\frac{de(t)}{dt}\right]}$$

$$u(k)=K_P\left\{e(k)+\sum_{m=0}^{k}\frac{T}{T_I}e(m)+\frac{T_D}{T}\left[e(k)-e(k-1)\right]\right\}$$

$$=K_Pe(k)+K_I'\sum_{m=0}^{k}e(m)+K_D'\left[e(k)-e(k-1)\right]$$

Positional PID algorithm

The output is determined by the current state and all the past states of the system. The disadvantage of the positional algorithm is that the use of all the past states could cumulate the errors.

Incremental PID algorithm is required.

Incremental PID algorithm

$$\Delta u(k) = u(k) - u(k-1)$$
$$= K_P\left[e(k) - e(k-1)\right] + K_I' e(k) + K_D'\left[e(k) - 2e(k-1) + e(k-2)\right]$$

$$\Delta u(k) = d_0 e(k) + d_1 e(k-1) + d_2 e(k-2)$$

$$d_0 = K_P + K_I' + K_D' = K_P(1 + \frac{T}{T_I} + \frac{T_D}{T})$$

$$d_1 = -K_P - 2K_D' = -K_P(1 + \frac{2T_D}{T})$$

$$d_2 = K_D' = K_P \cdot \frac{T_D}{T}$$

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4.1 Controllability of discrete systems

Definition: within a limited sample period [0, n], if there is a unconstrained control sequence u(0), … u(n-1) able to transfer any initial state x(0) to any final state x(n), then the system is called complete state controllable, in short, controllable

In above definition, both initial and final states can be any non-zero finite point. Although it is generic, but not convenient for mathematical development.

Without losing generality, either the final or initial state can be the origin of the state space. This is normally called reachability.

For LTI systems, controllability and reachability are equivalent.

4.2 Controllability criterion

Controllability criterion: the sufficient and necessary condition for a LTI system to be completely controllable is that the controllability matrix consisting of A and B is full rank, *i.e.*

$$rank S_c = rank\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n$$

Example: **Determine the controllability of the following system.**

$$x(k+1)=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & -2\\ -1 & 1 & 0\end{bmatrix}x(k)+\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}u(k)$$

Answer: because

$$rankS_c=\begin{bmatrix}B & AB & A^2B\end{bmatrix}=rank\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & -2\\ 1 & -1 & -3\end{bmatrix}$$

$$=rank\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & -2\\ 0 & -2 & -4\end{bmatrix}=rank\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & -2\\ 0 & 0 & -2\end{bmatrix}=3=n$$

So, the system is controllable.

Example: **determine the controllability of the following system.**

$$x(k+1)=\begin{bmatrix}-2 & 2 & -1\\ 0 & -2 & 0\\ 1 & -4 & 0\end{bmatrix}x(k)+\begin{bmatrix}0 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}u(k)$$

Answer:

$$S_c=\begin{bmatrix}0 & 0 & -1 & 2 & 2 & -4\\ 0 & 1 & 0 & -2 & 0 & 4\\ 1 & 0 & 0 & -4 & -1 & 10\end{bmatrix}$$

Since the determinant of the first 3 columns of S_c is not zero,

$$rank S_c = 3$$

Thus, the system is fully controllable.

4.3 Discrete system observability

1. Observability definition

Definition: Consider a LTI system, for a given control sequence *u(0), u(1), …, u(n-1)*, there is a limited period sample period [0, n], such that the initial state *x(0)* can be uniquely determined by the output sequence *y(0), y(1),…y(n)*, then the state *x(0)* is observable.

2. Observability criterion

A LTI discrete system {A,B,C} is observable if and only if the observability matric So has a full rank, i.e.

$$rank S_o = rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

Example: Determine observability of the given system

$$x(k+1)=\begin{bmatrix}2 & 0 & 3\\ -1 & -2 & 0\\ 0 & 1 & 2\end{bmatrix}x(k)$$

$$y(k)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}x(k)$$

Answer: according to the criterion

$$rankS_o=rank\begin{bmatrix}C\\ CA\\ CA^2\end{bmatrix}=rank\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & 3\\ -1 & -2 & 0\\ 4 & 3 & 12\\ 0 & 4 & -3\end{bmatrix}=3=n$$

Thus, the system is observable。

Example: **given system as follows**

$$x(k+1) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 3 & 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x(k)$$

determine its observability.

Answer:

$$rank S_o = rank \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = rank \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 3 & 0 & 2 \\ 1 & 0 & -1 \\ 9 & 0 & 1 \\ -2 & 0 & -3 \end{bmatrix} = 2 < 3 = n$$

Therefore, the system is not observable.

Summary

Signal sampling	$f^*(t)=\sum_{k=0}^{+\infty} f(kT)\delta(t-kT)$ $\quad F^*(s)=\sum_{k=0}^{+\infty} f(kT)e^{-kTs}$
Zero order hold	$G_{oh}(s)=\dfrac{F_h(s)}{F^*(s)}=\dfrac{1-e^{-Ts}}{s}$
Differential equation	$y(k)+2y(k-1)=u(k)-u(k-1)$
Digital PID control	$\Delta u(k)=d_0e(k)+d_1e(k-1)+d_2e(k-2)$
Discrete controllability	$rankS_c=rank\begin{bmatrix}B & AB & A^2B & \cdots & A^{n-1}B\end{bmatrix}=n$
Discrete observability	$rankS_o=rank\begin{bmatrix}C\\ CA\\ \vdots\\ CA^{n-1}\end{bmatrix}=n$