EBU4375 Signals and Systems Theory

Dr Maged Elkashlan

Queen Mary
University of London

What is a signal?

- In our context the signal will be a function of time
- That is, a physical variable that changes with time
- The physical variable could be pressure, temperature, flow-rate, etc.
- These variables can be converted to electrical signals (voltage or current) using a transducer (sensor)

What is a signal?

- ~ 90% engineering/physics involves study or application of vibrations and waves of many forms

acoustics, fluid mechanics, optics, electromagnetics, astronomy, quantum mechanics, information theory

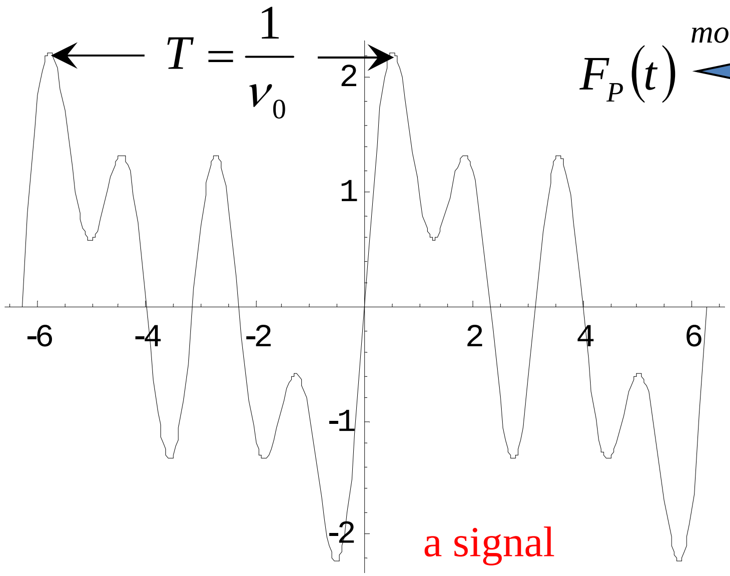
a **signal** and its **spectrum**

What is a signal?

e.g. 1

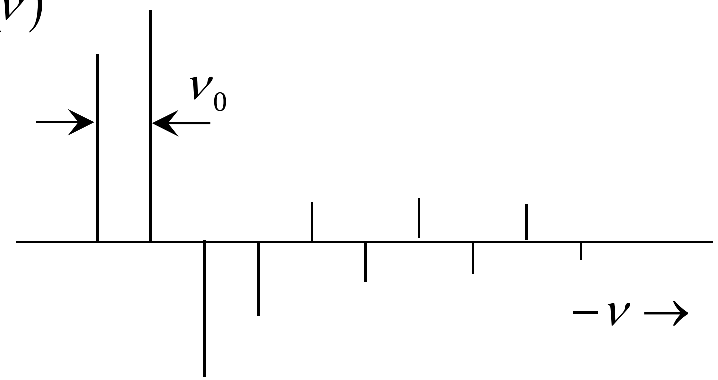
- a musician plays a <u>steady</u> note (for example ($\nu_0 = 256\,Hz$) on a violin
- a microphone (or generally a transducer) produces a voltage $V(t)$, proportional to the instantaneous air pressure $F_P(t)$
- $F_P(t)$ can be displayed as a <u>periodic</u> signal on an oscilloscope

$$F_P(t) \xleftrightarrow{\textit{modular}\text{ transform}} A(\nu)$$



a signal

and



its spectrum

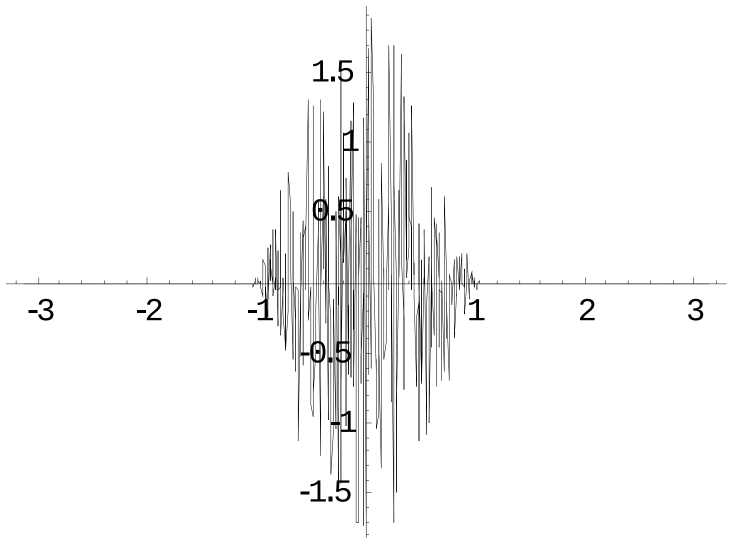
- $F_P(t)$ not a boring sinusoid, i.e. it has *harmonics* or *overtones* ($\nu_m = m\nu_0\,;\quad m \in N$)

- $F_P(t)$ can be analysed to reveal the amplitudes $A_m(\nu_m)$ and phases $\phi_m(\nu_m)$ of the *overtones*
- *phase* describes *retardation* of one wave (or vibration), with respect to another

What is a signal?

e.g. 2

- suppose the sound now is aperiodic (e.g. a drumbeat, a clap or a crash)
- to describe $F_P(t)$ now requires not just $\{A_m, \phi_m\}$, but a *continuous* range of frequencies



crash spectrum: all frequencies present (each in an infinitesimal amount)

Classification of Signals and Properties

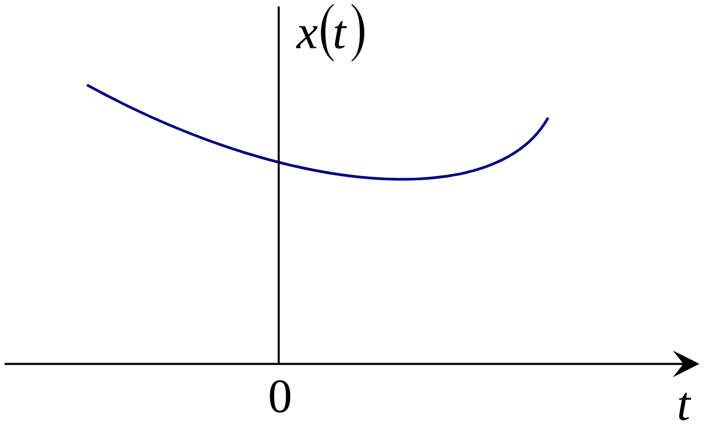
• *methods* used for *processing* a signal or *analysing* the response of a system to a signal significantly depend on the characteristic attributes of the signal.

• certain techniques apply to only specific types of signals – <u>hence the need for classification</u>

- continuous-time (CT) & discrete-time (DT)
- analogue and digital
- periodic and aperiodic
- deterministic and stochastic (random)
- even and odd
- energy and power

Continuous-Time (CT) and Discrete-Time (DT) Signals

Continuous-time signals

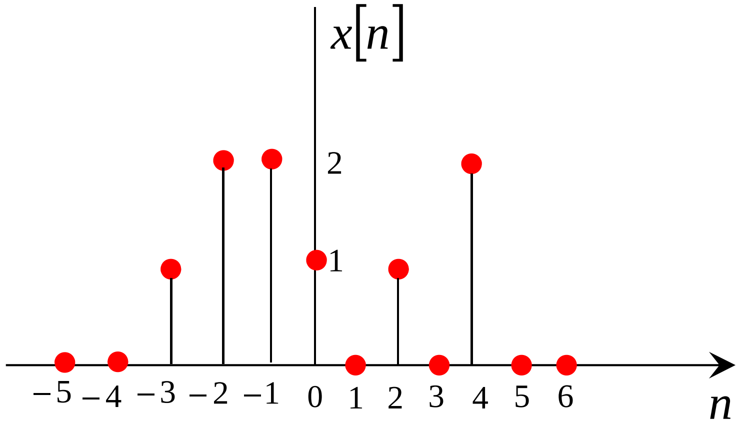


t continuous variable $\Longrightarrow$

$x(t)$ continuous-time signal

and

Discrete-time signals



t discrete variable $\Longrightarrow$

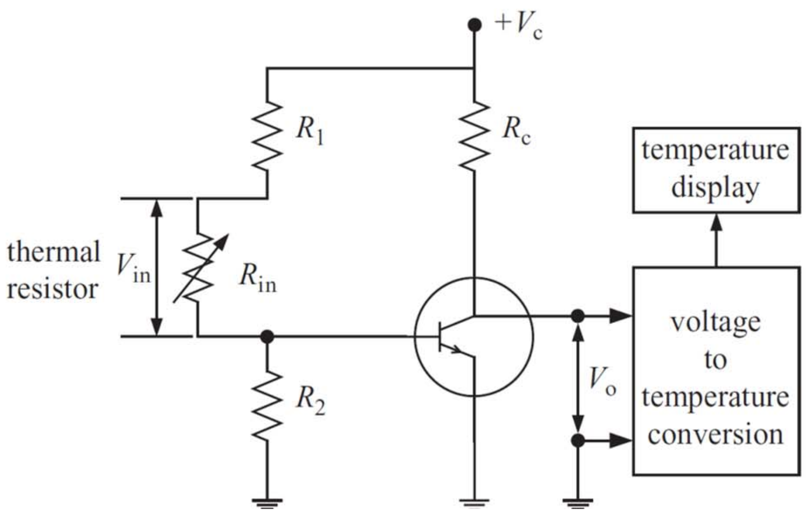
$x(t)$ discrete-time signal

$x(t)$ defined at discrete-times leads to it being identified as a sequence of numbers $\{x_n\}$ or $x[n]$; $n \in Z$

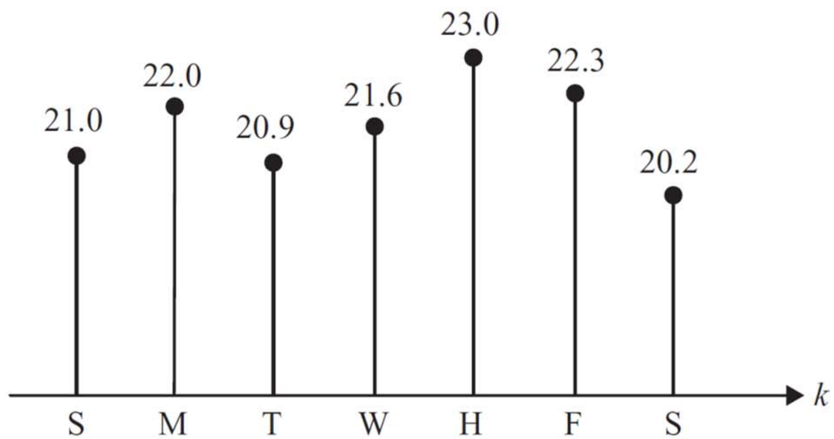
Continuous-Time (CT) and Discrete-Time (DT) Signals

a DT signal is denoted with square parenthesis as

$$x[kT], \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots,$$



a digital thermometer



output signals

Continuous-Time (CT) and Discrete-Time (DT) Signals

- a DT signal $x[n]$ may represent a phenomenon for which the *independent* variable is *inherently* discrete, e.g. sun-set, sun-rise, tidal-patterns etc..
- $x[n]$ may otherwise result from *sampling* a *continuous-time* signal resulting in:

$$x(t_0), x(t_1), \cdots x(t_n), \cdots \equiv x[0], x[1], \cdots x[n], \cdots \quad or \quad x_0, x_1, \cdots, x_n, \cdots$$

- when sampling at regular intervals with a uniform sampling period T_S, then

$$x_n = x[n] = x(nT_S)$$

- A DT signal is typically defined in one of two ways:
    1. a rule is used for generating the n^{th} term e.g.

$$x[n] = x_n = \begin{cases} \dfrac{1}{2^n} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

or

$$\{x_n\} = \left\{1, \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}, \cdots\right\}$$

Continuous-Time (CT) and Discrete-Time (DT) Signals

2. explicit listing of the terms in the sequence e.g.

$$\{x_n\} = \{\cdots, 0, 0, 1, 2, 2, \underset{\uparrow}{1}, 0, 1, 0, 2, 0, 0, \cdots\}$$

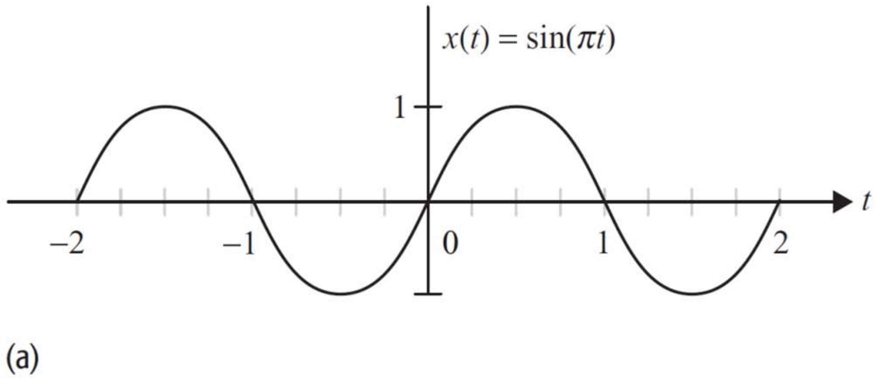
or $$\{x_n\} = \{1, 2, 2, \underset{\uparrow}{1}, 0, 1, 0, 2\}$$

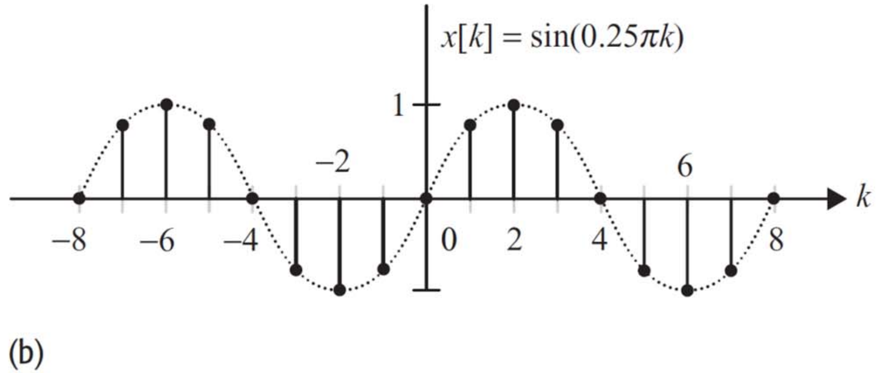
The $\uparrow$ marks the position of the 0^{th} (i.e. $n = 0$), term.
By convention no marker fixes $n = 0$ as the 0^{th} term,
and $x[n] = 0$, $n < 0$

Sequence Algebra:

$$\{c_n\} = \{a_n\} \pm \{b_n\} \rightarrow c_n = a_n \pm b_n$$
$$\{c_n\} = \{a_n\}\{b_n\} \quad \rightarrow c_n = a_n b_n$$
$$\{c_n\} = \alpha\{a_n\} \quad \rightarrow c_n = \alpha a_n \qquad \alpha = \text{constant}$$

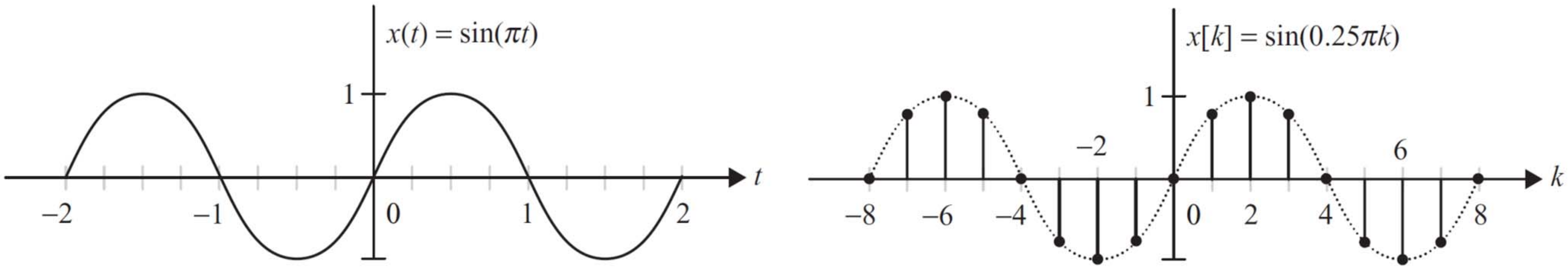
Continuous-Time (CT) and Discrete-Time (DT) Signals





(a) CT sinusoidal signal $x(t)$ specified in Example 1.1; (b) DT sinusoidal signal $x[k]$ obtained by discretizing $x(t)$ with a sampling interval $T = 0.25$ s.

Continuous-Time (CT) and Discrete-Time (DT) Signals



(a) (b)

Example

Consider the CT signal $x(t) = \sin(\pi t)$.

Discretize the signal using a sampling interval of $T = 0.25$ s, and sketch the waveform of the resulting DT sequence for the range $-8 \le k \le 8$.

Continuous-Time (CT) and Discrete-Time (DT) Signals

Solution

By substituting $t = kT$, the DT representation of the CT signal $x(t)$ is given by

$$x[kT] = \sin(\pi k \times T) = \sin(0.25\pi k).$$

For $k = 0, \pm1, \pm2, \ldots$, the DT signal $x[k]$ has the following values:

$x[-8] = x(-8T) = \sin(-2\pi) = 0,$

$x[-7] = x(-7T) = \sin(-1.75\pi) = \frac{1}{\sqrt{2}},$

$x[-6] = x(-6T) = \sin(-1.5\pi) = 1,$

$x[-5] = x(-5T) = \sin(-1.25\pi) = \frac{1}{\sqrt{2}},$

$x[-4] = x(-4T) = \sin(-\pi) = 0,$

$x[-3] = x(-3T) = \sin(-0.75\pi) = -\frac{1}{\sqrt{2}},$

$x[-2] = x(-2T) = \sin(-0.5\pi) = -1,$

$x[-1] = x(-T) = \sin(-0.25\pi) = -\frac{1}{\sqrt{2}},$

$x[0] = x(0) = \sin(0) = 0.$

$x[1] = x(T) = \sin(0.25\pi) = \frac{1}{\sqrt{2}},$

$x[2] = x(2T) = \sin(0.5\pi) = 1,$

$x[3] = x(3T) = \sin(0.75\pi) = \frac{1}{\sqrt{2}},$

$x[4] = x(4T) = \sin(\pi) = 0,$

$x[5] = x(5T) = \sin(1.25\pi) = -\frac{1}{\sqrt{2}},$

$x[6] = x(6T) = \sin(1.5\pi) = -1,$

$x[7] = x(7T) = \sin(1.75\pi) = -\frac{1}{\sqrt{2}},$

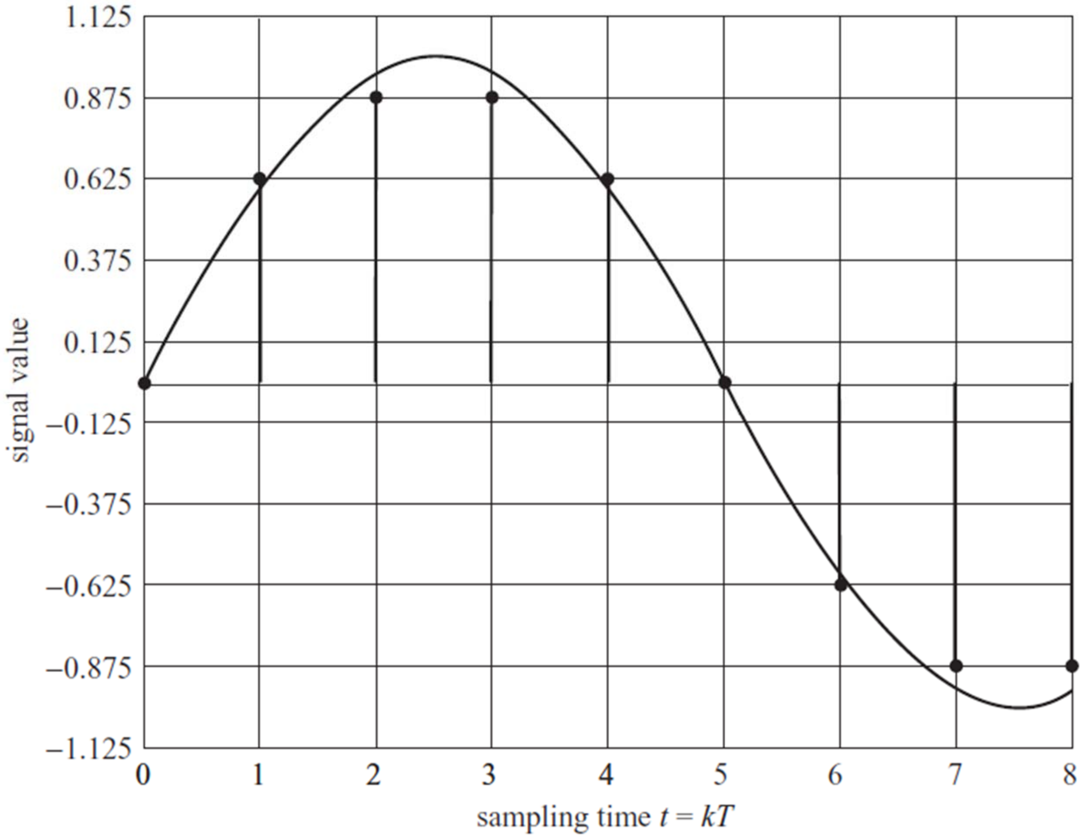
$x[8] = x(8T) = \sin(2\pi) = 0,$

Analogue (continuous-valued) and Digital (discrete-valued) Signals

- if a continuous-time signal $x(t)$ *can take on <u>any</u> value* in the continuous interval (a,b): $a \to -\infty, b \to \infty$, then the continuous-time signal is called *analogue*.

- if a discrete-time signal $x[n]$ *can take on only a <u>bounded</u>* number of quantum values then the discrete-time signal is called *digital*.

Analogue (continuous-valued) and Digital (discrete-valued) Signals



Analog signal with its digital approximation. The waveform for the analog signal is shown with a line plot; the quantized digital approximation is shown with a stem plot.

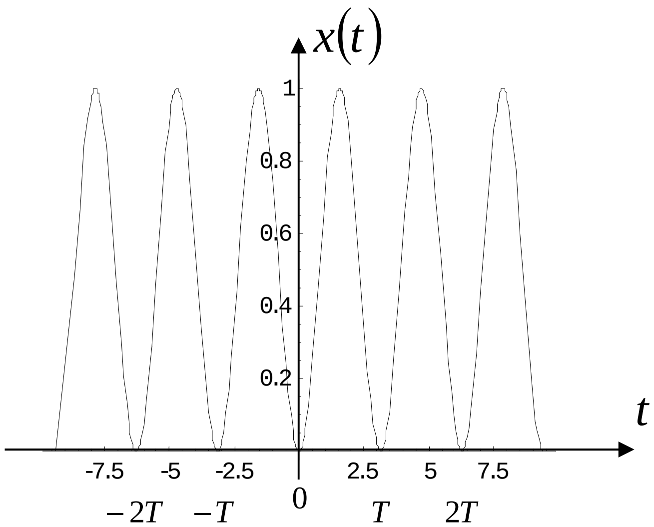
Periodic and Aperiodic Signals (CT Signals)

A continuous-time signal $x(t)$ is periodic with period T , if there is $T > 0$ such that

$$x(t + mT) = x(t): \quad m \in Z \quad \text{for all } t$$

The smallest +ve value of T for which the periodicity holds is the fundamental period T_0

- Where T_0 does not exist $x(t)$ is termed *aperiodic*.



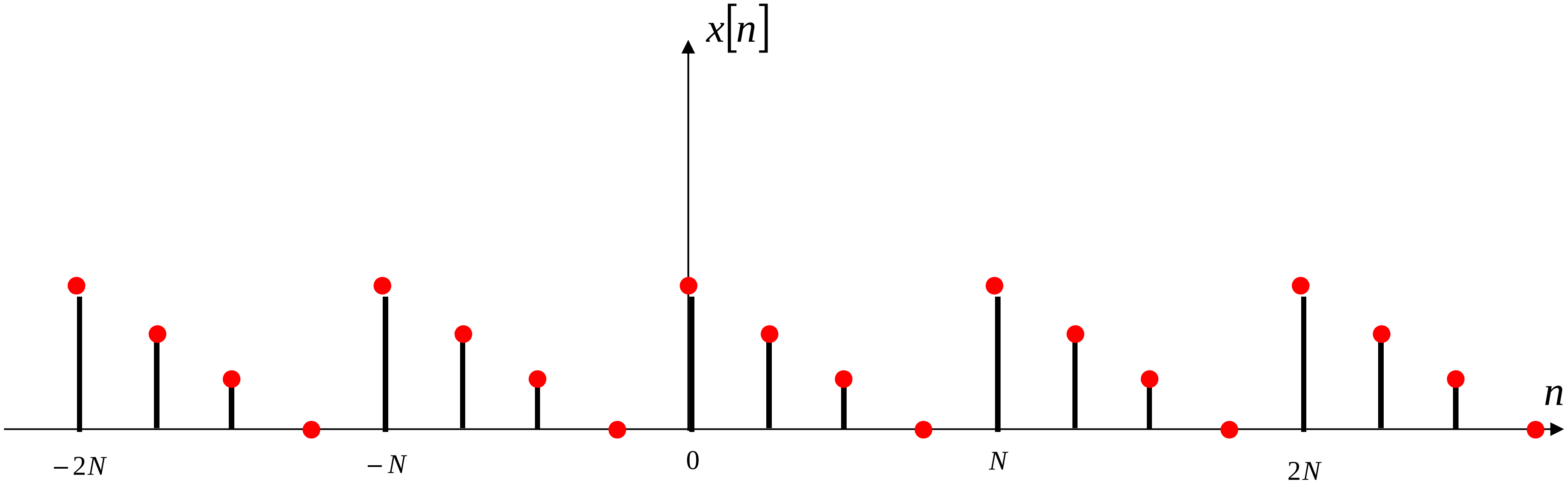
Periodic and Aperiodic Signals (DT Signals)

A <u>discrete-time signal</u> $x[n]$ is periodic with integer-period N , if there is $N > 0$ such that

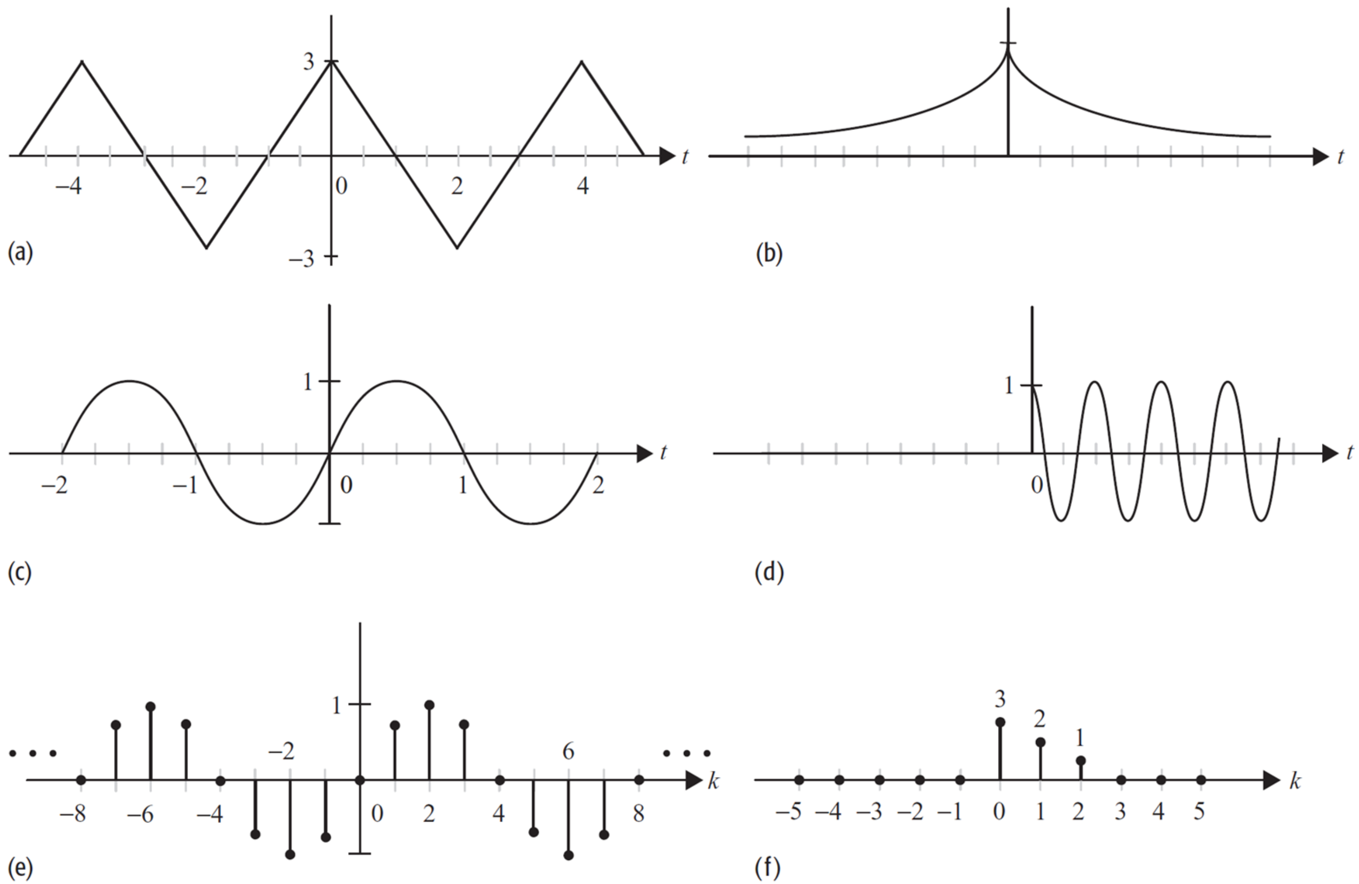
$$x[n+mN] = x[n]: \quad m \in Z \quad \text{for all } n$$

The smallest +ve value of N for which the periodicity holds is the fundamental period N_0

- Where N_0 does not exist $x[n]$ is termed *aperiodic*



Periodic and Aperiodic Signals (Examples of CT and DT Signals)



Examples of periodic ((a), (c), and (e)) and aperiodic ((b), (d), and (f)) signals. The line plots (a) and (c) represent CT periodic signals with fundamental periods T_0 of 4 and 2, while the stem plot (e) represents a DT periodic signal with fundamental period $K_0 = 8$.

Periodic and Aperiodic Signals (CT and DT Signals)

Mathematically, the fundamental frequency is

$$f_0 = \frac{1}{T_0}, \text{ for CT signals,} \quad \text{or} \quad f_0 = \frac{1}{K_0}, \text{ for DT signals}$$

the *angular frequency* is

$$\omega_0 = \frac{2\pi}{T_0}, \text{ for CT signals,} \quad \text{or} \quad \Omega_0 = \frac{2\pi}{K_0}, \text{ for DT signals.}$$

Periodic and Aperiodic Signals (Example of CT Signals)

Example

Let $x_1(t) = \cos(\pi t / 2)$ and $x_2(t) = \cos(\pi t / 3)$

What are the fundamental periods of $x_1(t)$ and $x_2(t)$

What is the fundamental period of the sum $x_1(t) + x_2(t)$

Deterministic and Stochastic (Random) Signals

• *deterministic* signals have values that are *completely specified* at any given time (e.g. a sinusoid produced by a function-generator).

• *stochastic* signals take on random values at any given time and necessitate a statistical description (e.g. thermal noise in any device above absolute zero).

Even and Odd Signals (CT and DT Signals)

A CT signal $x_e(t)$ is said to be an even signal if

$$x_e(t) = x_e(-t).$$

Conversely, a CT signal $x_o(t)$ is said to be an odd signal if

$$x_o(t) = -x_o(-t).$$

A DT signal $x_e[k]$ is said to be an even signal if

$$x_e[k] = x_e[-k].$$

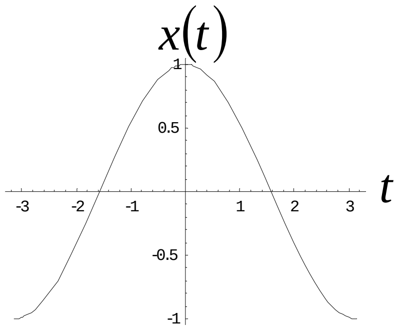
Conversely, a DT signal $x_o[k]$ is said to be an odd signal if

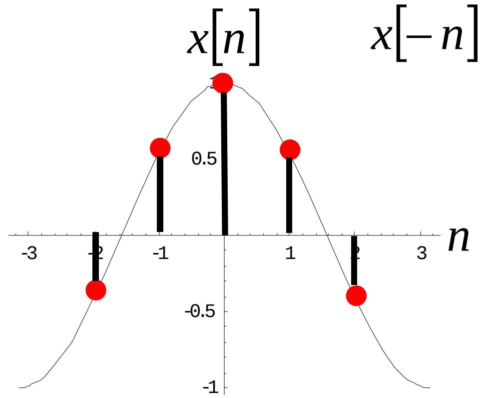
$$x_o[k] = -x_o[-k].$$

Even and Odd Signals (CT and DT Signals)

A signal $x(t)$ or $x[n]$ is termed an *even* signal if:

$$x(-t) = x(t)$$
$$x[-n] = x[n]$$

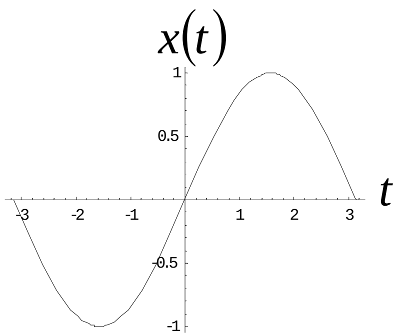


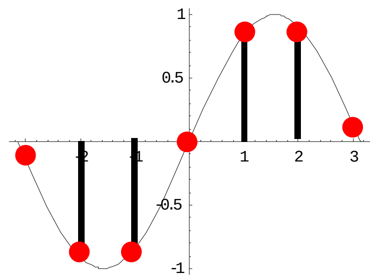


i.e. reflection symmetry in the ordinate axis

A signal $x(t)$ or $x[n]$ is termed an *odd* signal if:

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$





i.e. inversion symmetry

Even and Odd Signals (CT and DT Signals)

Any signal $x(t)$ or $x[n]$ can be decomposed as the sum of an even $(x_e(t), x_e[n])$ and odd $(x_o(t), x_o[n])$ signal, i.e.:

$$x(t) = x_e(t) + x_o(t)$$
$$x[n] = x_e[n] + x_o[n]$$

where

$$x_e(t) = \frac{1}{2}\{x(t) + x(-t)\} \qquad x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$$
$$x_e[n] = \frac{1}{2}\{x[n] + x[-n]\} \qquad x_o[n] = \frac{1}{2}\{x[n] - x[-n]\}$$

Energy and Power Signals (CT and DT Signals)

The *total energy* of a CT signal is its energy calculated over the interval $t = [-\infty, \infty]$. Likewise, the total energy of a DT signal is its energy calculated over the range $k = [-\infty, \infty]$. The expressions for the total energy are therefore given by the following:

CT signals $$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t;$$

DT sequences $$E_x = \sum_{k=-\infty}^{\infty} |x[k]|^2$$

Energy and Power Signals (CT and DT Signals)

Since power is defined as energy per unit time, the *average power* of a CT signal $x(t)$ over the interval $t = (-\infty, \infty)$ and of a DT signal $x[k]$ over the range $k = [-\infty, \infty]$ are expressed as follows:

CT signals
$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \mathrm{d}t$$

DT sequences
$$P_x = \frac{1}{2K+1} \sum_{k=-K}^{K} |x[k]|^2$$

Energy and Power Signals (CT and DT Signals)

A signal $x(t)$, or $x[k]$, is called an *energy signal* if the total energy E_x has a non-zero finite value, i.e. $0 < E_x < \infty$. On the other hand, a signal is called a *power signal* if it has non-zero finite power, i.e. $0 < P_x < \infty$. Note that a signal cannot be both an energy and a power signal simultaneously. The energy signals have zero average power whereas the power signals have infinite total energy.

1. $x(t)$, $x[n]$ are said to be an energy signal or sequence if $0 < E < \infty$, <u>and</u> $P = 0$

2. $x(t)$, $x[n]$ are said to be a power signal or sequence if $0 < P < \infty \Rightarrow E = \infty$

Energy and Power Signals (CT and DT Signals)

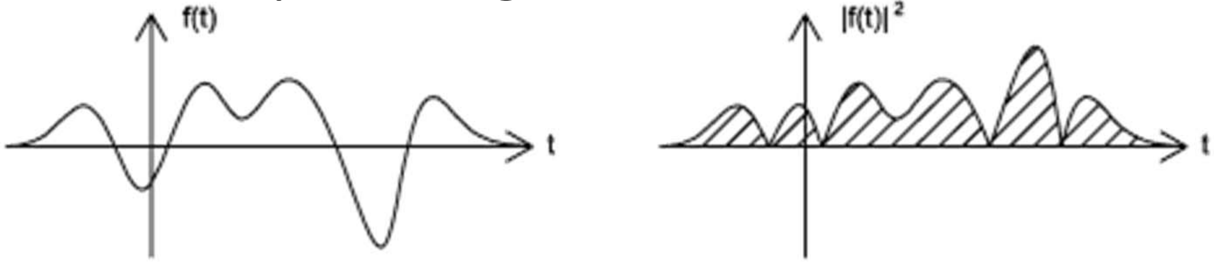
NB:

A periodic signal is a power signal if its energy content per period is finite, and then the average power of this signal need only be calculated over a period.

Energy and Power Signals (CT and DT Signals)

• Since we often think of signal as a function of varying amplitude through time, it seems to reason that a good measurement of the strength of a signal would be the area under the curve.

• This suggests either squaring the signal or taking its absolute value, then finding the area under that curve. It turns out that what we call the ***energy*** of a signal is the area under the squared signal.

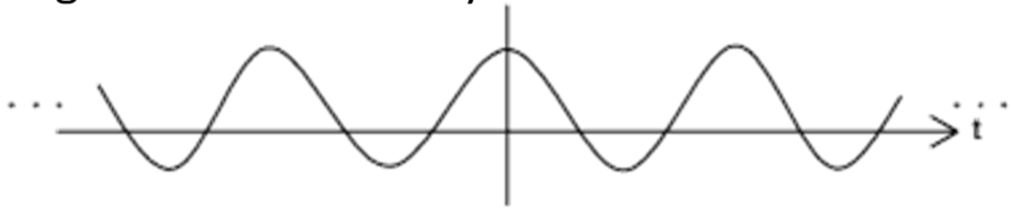


The energy of this signal is the shaded region.

$$E = \int_{-\infty}^{\infty} dt \left| f(t) \right|^2 \quad joules$$

Energy and Power Signals (CT and DT Signals)

• Our definition of energy seems reasonable, and it is. However, what if the *signal does not decay*?



A simple, common signal with infinite energy.

• In this case we have infinite energy for any such signal. Does this mean that a fifty hertz sine wave feeding into your headphones is as strong as the fifty hertz sine wave coming out of your power outlet? Obviously not. This is what leads us to the idea of ***signal power***.

• Power is a time average of energy (energy per unit time). This is useful when the energy of the signal goes to infinity.

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \left| f(t) \right|^2 \quad watts$$

Energy and Power Signals (CT and DT Signals)

- **"Energy signals" have finite energy.**
- **"Power signals" have finite power.**

Are all energy signals also power signals?

No, any signal with finite energy will have zero power.

Are all power signals also energy signals?

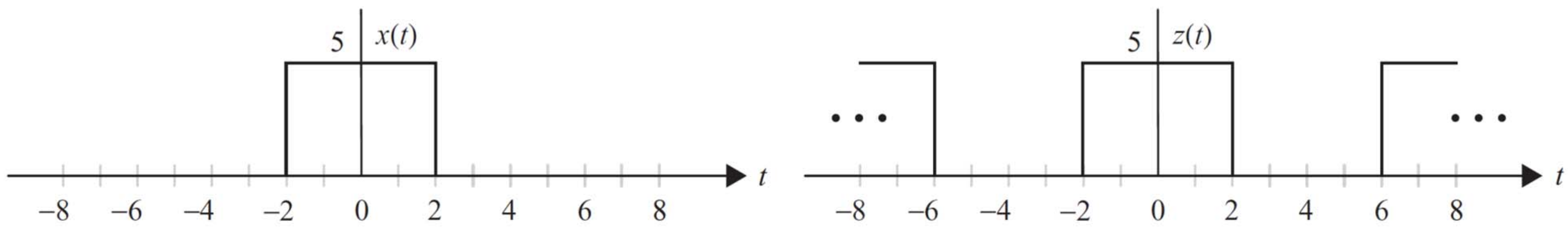
No, any signal with finite power will have infinite energy.

Are *all* signals either energy or power signals?

No. Any infinite-duration, increasing-magnitude function will not be either. (eg $f(t) = t$ is neither)

Energy and Power Signals (Example of CT Signals)

Example



(a) (b)

Consider the CT signals shown in Figs. (a) and (b). Calculate the instantaneous power, average power, and energy present in the two signals. Classify these signals as power or energy signals.

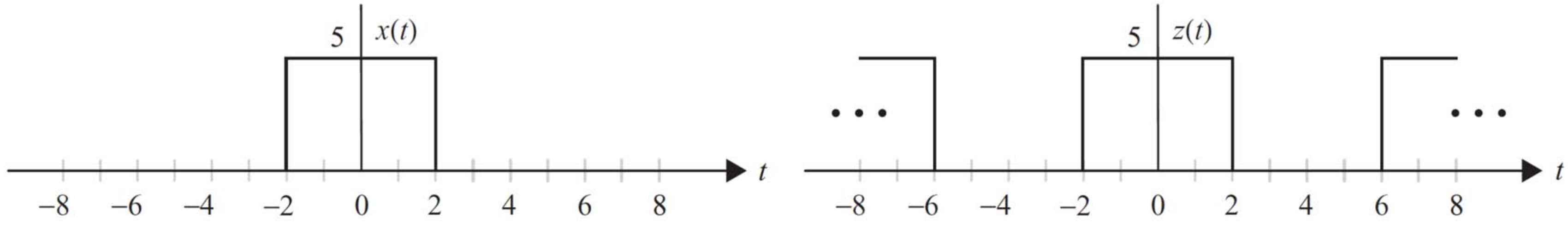
Solution

(a) The signal $x(t)$ can be expressed as follows:

$$x(t) = \begin{cases} 5 & -2 \le t \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Energy and Power Signals (Example of CT Signals)

Example



(a) (b)

instantaneous power $$P_x(t) = \begin{cases} 25 & -2 \le t \le 2 \\ 0 & \text{otherwise;} \end{cases}$$

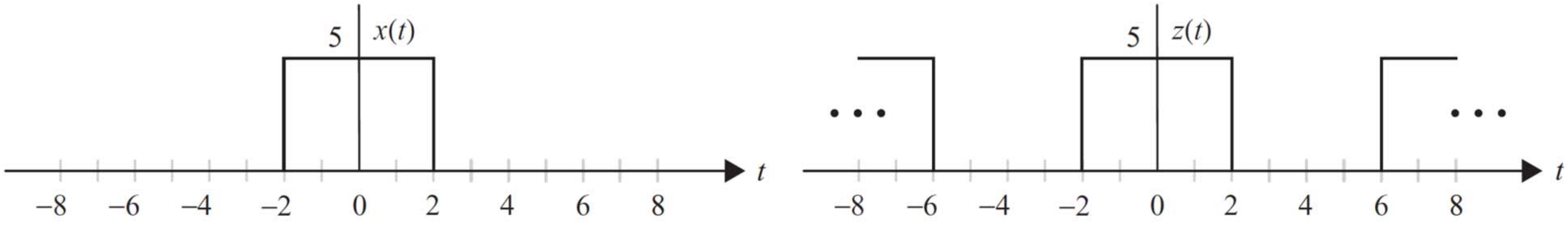
energy $$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \int_{-2}^{2} 25 \, \mathrm{d}t = 100;$$

average power $$P_x = \lim_{T \to \infty} \frac{1}{T} E_x = 0.$$

Because $x(t)$ has finite energy ($0 < E_x = 100 < \infty$) it is an energy signal.

Energy and Power Signals (Example of CT Signals)

Example



(a) (b)

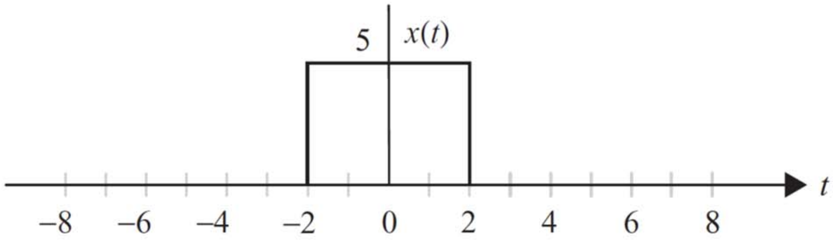
(b) The signal $z(t)$ is a periodic signal with fundamental period 8 and over one period is expressed as follows:

$$z(t) = \begin{cases} 5 & -2 \le t \le 2 \\ 0 & 2 < |t| \le 4, \end{cases}$$

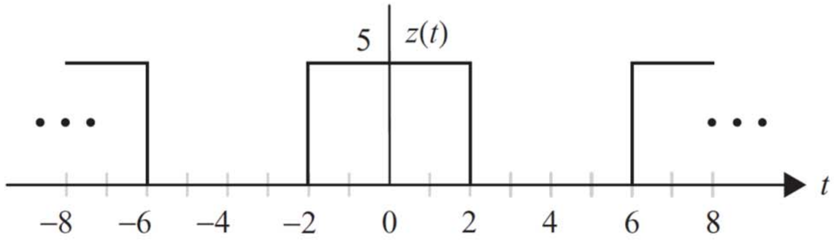
with $z(t+8) = z(t)$.

Energy and Power Signals (Example of CT Signals)

Example



(a)



(b)

instantaneous power $$P_z(t) = \begin{cases} 25 & -2 \le t \le 2 \\ 0 & 2 < |t| \le 4 \end{cases}$$

and $P_z(t+8) = P_z(t)$;

average power $$P_z = \frac{1}{8}\int_{-4}^{4} |z(t)|^2 \, dt = \frac{1}{8}\int_{-2}^{2} 25 \, dt = \frac{100}{8} = 12.5;$$

energy $$E_z = \int_{-\infty}^{\infty} |z(t)|^2 \, dt = \infty.$$

Because the signal has finite power ($0 < P_z = 12.5 < \infty$), $z(t)$ is a power signal.