EBU5375 Signals and systems: Fourier series of discrete-time periodic signals

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Agenda

Quick review

The notion of frequency in discrete-time signals

Fourier series representation of discrete-time periodic signals

Important properties of Fourier series

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The notion of frequency in discrete-time signals

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The notion of frequency in CT

If $x(t)$ has high-frequency components

(a) The **amplitude** of $x(t)$ is **high**.
(b) The **amplitude** of $x(t)$ **changes quickly**.
(c) Signal $x(t)$ contain **many different frequency components**.

LTI filters and the convolution theorem

$$x(t),\ X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow y(t),\ Y(\omega)$$

$$y(t) = x(t) \star h(t) \overset{FT}{\Longleftrightarrow} Y(\omega) = X(\omega)H(\omega)$$

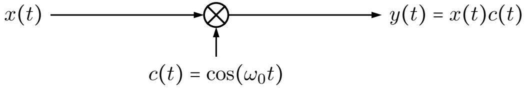
If $H(\omega)$ is a high-pass filter

(a) Signal $y(t)$ **has the high-frequency components** of $x(t)$.

(b) Signal $y(t)$ **has new high-frequencies** components that $x(t)$ doesn't have.

(c) Both (a) and (b).

Nonlinear filters and the modulation theorem



The frequency components of $y(t)$

(a) Have a **higher frequency** than the components of $c(t)$.

(b) Have a **lower frequency** than the components of $c(t)$.

(c) Can be **both (a) and (b)**.

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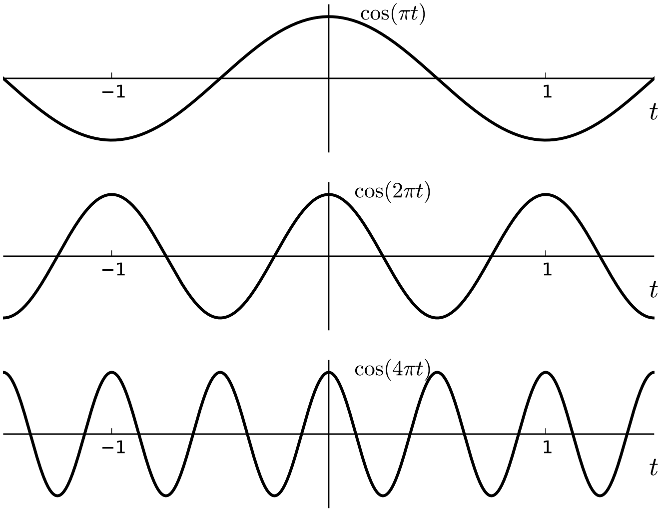
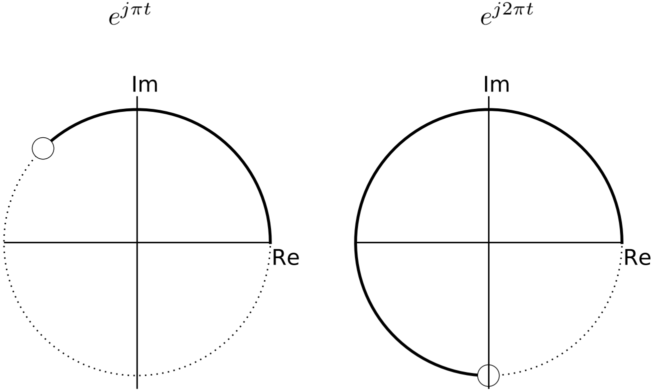
Continuous-time sinusoidal signals: Periodicity



Continuous-time complex exponentials: Periodicity

$$t = \frac{3}{4}$$



Discrete-time sinusoidal and complex exponentials: Periodicity

(**In DT the angular frequency is Ω, whereas in CT it is ω, sorry!**).

Consider the DT sinusoidal signal $x_2[n] = \cos(j\Omega n)$. This signal is:

(a) Always periodic.
(b) Never periodic.
(c) Periodic for some values of Ω.

Consider the DT complex exponential $x_1[n] = e^{j\Omega n}$. This signal is:

(a) Always periodic.
(b) Never periodic.
(c) Periodic for some values of Ω.

Discrete-time complex exponentials: Periodicity

Consider the discrete time complex exponentials $x[n] = e^{j\Omega n}$. If $x[n]$ is periodic with period N then

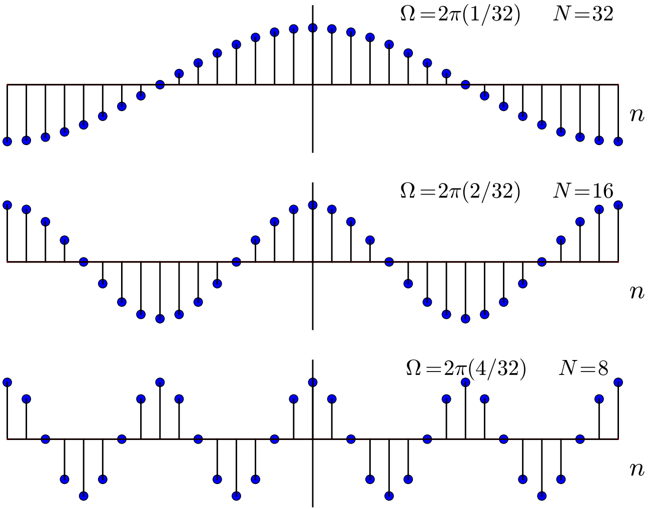
$$x[n] = x[n+N]$$

Not every value of Ω produces a periodic signal. If we assume that $x[n]$ is periodic,

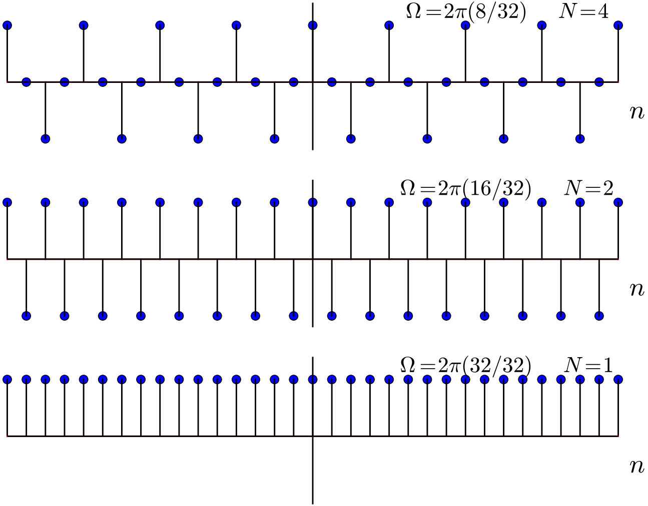
$$\begin{aligned} x[n+N] &= e^{j\Omega(n+N)} \\ &= e^{j\Omega n} e^{j\Omega N} \\ &= x[n] \end{aligned}$$

hence we need that $\Omega N = 2\pi k \longrightarrow \Omega = 2\pi \frac{k}{N}$. Similarly, sinusoidal signals are periodic if and only if $\Omega = 2\pi \frac{k}{N}$.

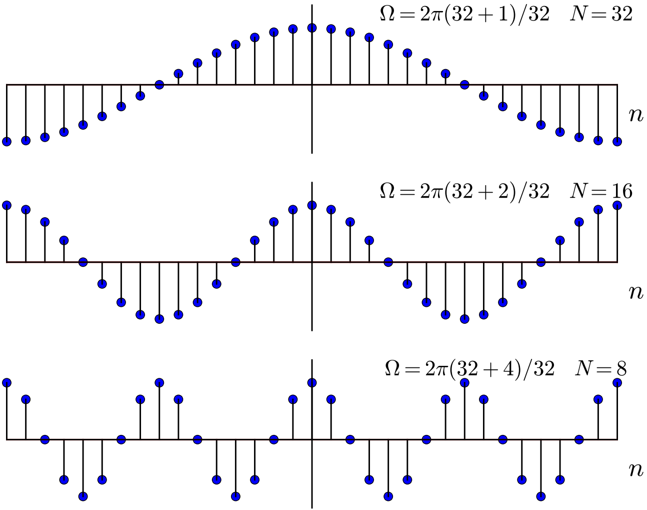
Discrete-time sinusoidal signals



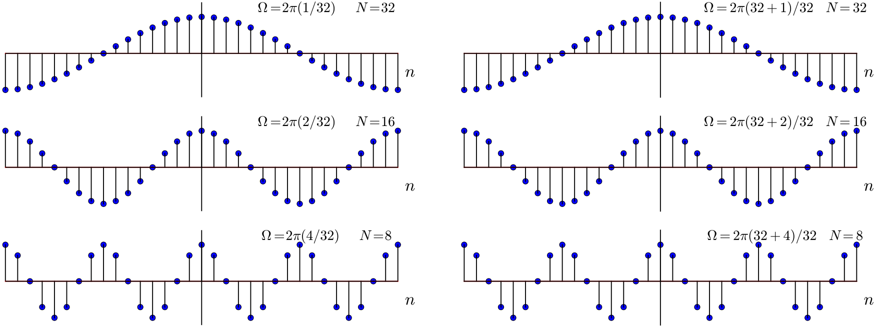
Discrete-time sinusoidal signals



Discrete-time sinusoidal signals

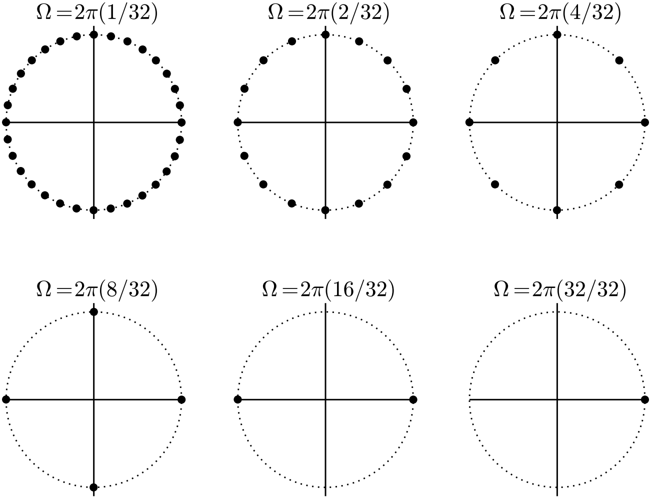


Discrete-time sinusoidal signals

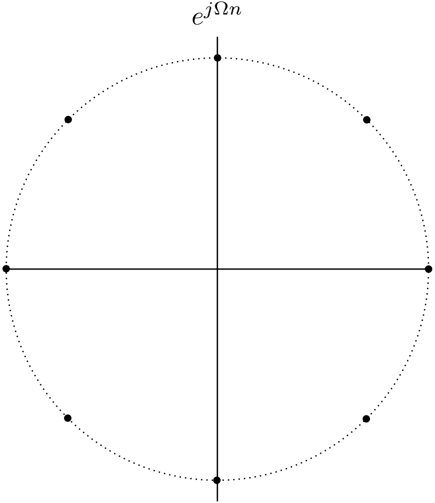


We have found **some sinusoidal signals with different frequencies that are identical**. Why is that?

Discrete-time complex exponentials



Discrete-time complex exponentials



Ω
$2\pi\frac{1}{8}$
$2\pi\left[\frac{1}{8}+1\right]$
$2\pi\left[\frac{1}{8}+2\right]$
$2\pi\left[\frac{1}{8}+3\right]$
$2\pi\left[\frac{1}{8}+4\right]$
$\ldots$

As in the case with the sinusoidal signals, we have found **some complex exponentials with different frequencies that are identical**.

Discrete-time complex exponentials

Consider the discrete time complex exponentials $x_1[n] = e^{j\Omega_1 n}$ and $x_2[n] = e^{j\Omega_2 n}$, where $\Omega_2 = \Omega_1 + 2\pi$. The angular frequency of $x_2[n]$ is higher than the angular frequency of $x_1[n]$, $\Omega_2 > \Omega_1$.

However, $x_1[n]$ and $x_2[n]$ are the same signal:

$$\begin{aligned} x_2[n] &= e^{j\Omega_2 n} \\ &= e^{j(\Omega_1 + 2\pi)n} \\ &= e^{j\Omega_1 n} e^{j2\pi n} \\ &= e^{j\Omega_1 n} \\ &= x_1[n] \end{aligned}$$

In discrete-time, **all the possible complex exponentials that can be generated are within any interval of frequencies of size** 2π, for instance $[-\pi, \pi]$.

Discrete-time complex exponentials

We have seen that

- In order for a complex exponential to be periodic, its angular frequency Ω must be such that $\Omega = 2\pi\frac{k}{N}$, where N is the period (whenever k and N have no factors in common).
- The frequencies Ω_1 and $\Omega_2 = \Omega_1 + 2\pi$ produce the same signal, since they visit the same points in the complex plane.

We can conclude that **there only exist N different complex exponentials of period N**, namely

$$0, \quad 2\pi\frac{1}{N}, \quad 2\pi\frac{2}{N}, \quad \dots \quad , \quad 2\pi\frac{N-1}{N}$$

For instance, $\Omega = 2\pi\frac{2N+2}{N}$ produces the same signal as $\Omega = 2\pi\frac{2}{N}$ and $\Omega = -2\pi\frac{2}{N}$ produces the same signal as $\Omega = 2\pi\frac{N-2}{N}$

Summary

CT complex exponentials	DT complex exponentials
Always periodic	Only periodic for $\Omega = 2\pi k/N$, k, N integers
Different frequencies produce different signals	Frequencies within an interval of size 2π produce different signals
There exist infinite complex exponentials with period T, namely those of frequencies $\frac{2\pi}{T}$, $2\frac{2\pi}{T}$,$3\frac{2\pi}{T}$,...	There only exist N complex exponentials with period N, namely those of frequencies $\frac{2\pi}{N}$, $2\frac{2\pi}{N}$,...,$N\frac{2\pi}{N}$

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Important properties of Fourier series

Fourier series in CT

A Fourier series is a representation of a **periodic signal** as a **linear combination** of **harmonically related complex exponentials**. By *harmonically related* we mean that their frequencies can be expressed as an integer multiple of the fundamental frequency.

For instance, in continuous-time, a periodic signal $x_T(t)$ with period T can be expressed as

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

where $\omega_0 = 2\pi/T$ is the fundamental frequency, $k\omega_0$ are its harmonics and a_k are its coefficients.

Fourier series in DT

What's different in discrete-time? Just the fact that **there are only N different complex exponentials with period N**!

Hence, the Fourier series representation of a periodic discrete-time signal $x_N[n]$ with period N is

$$x_N[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n}$$

where $\Omega_0 = 2\pi/N$ is the fundamental frequency, $k\Omega_0$ are its harmonic frequencies and a_k are its coefficients.

This equation is, of course, a **synthesis equation**.

Fourier series: Determining the coefficients I

How do we obtain the coefficients of the Fourier series of a discrete-time periodic signal $x_N[n]$?

One approach would be to solve the following system of N equations and N unknowns (a_k):

$$
\begin{aligned}
x_N[0] &= \sum_{k=\langle N \rangle} a_k \\
x_N[1] &= \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0} \\
x_N[2] &= \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 2} \\
&\dots \\
x_N[N-1] &= \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 (N-1)}
\end{aligned}
$$

Fourier series: Determining the coefficients I

In matrix form, the resulting system of linear equations is:

$$\begin{bmatrix} x_N[0] \\ x_N[1] \\ x_N[2] \\ \vdots \\ x_N[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j\Omega_0} & e^{j2\Omega_0} & & e^{j(N-1)\Omega_0} \\ 1 & e^{j\Omega_0 2} & e^{j2\Omega_0 2} & & e^{j(N-1)\Omega_0 2} \\ \vdots & & & \ddots & \vdots \\ 1 & e^{j\Omega_0(N-1)} & e^{j2\Omega_0(N-1)} & \cdots & e^{j(N-1)\Omega_0(N-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-1} \end{bmatrix}$$

Fourier series: Determining the coefficients II

Another option is using an **analysis equation**. Analysis equations use the fact that harmonically related exponentials are **orthogonal**, so that

$$
\begin{aligned}
\sum_{n=\langle N\rangle} e^{jk_1\Omega_0 n}\left(e^{jk_2\Omega_0 n}\right)^* &= \sum_{n=\langle N\rangle} e^{jk_1\Omega_0 n} e^{-jk_2\Omega_0 n} \\
&= \sum_{n=\langle N\rangle} e^{j(k_1-k_2)\Omega_0 n} \\
&= \begin{cases} N & \text{if } k_1-k_2 = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Fourier series: Determining the coefficients II

So we know that

$$
\begin{aligned}
x_N[n] &= \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}, \\
\sum_{n=\langle N\rangle} e^{jk_1\Omega_0 n}(e^{jk_2\Omega_0 n})^* &= \begin{cases} N & \text{if } k_1 - k_2 = 0, \pm N, \pm 2N, \dots \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Let us calculate

$$
\begin{aligned}
\sum_{n=\langle N\rangle} x_N[n](e^{jm\Omega_0 n})^* &= \sum_{n=\langle N\rangle} \left[\sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n} \right] e^{-jm\Omega_0 n} \\
&= a_m N
\end{aligned}
$$

Hence

$$
a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x_N[n] e^{-jk\Omega_0 n}
$$

Fourier series of periodic signals: summary

Continuous-time, $\omega_0 = \frac{2\pi}{T}$		Discrete-time, $\Omega_0 = \frac{2\pi}{N}$
$a_k = \dfrac{1}{T}\displaystyle\int_{\langle T\rangle} x_T(t)e^{-jk\omega_0 t}dt$	**Analysis**	$a_k = \dfrac{1}{N}\displaystyle\sum_{n=\langle N\rangle} x_N[n]e^{-jk\Omega_0 n}$
$x_T(t) = \displaystyle\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	**Synthesis**	$x_N[n] = \displaystyle\sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}$

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Parseval's relation

The average power of a periodic signal $x_N[n]$ can be calculated both in the time domain and by using the coefficients of its Fourier series:

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

The coefficient a_k of the Fourier series of $x_N[n]$ tells us *how much* of the harmonic frequency $k\omega_0$ there is in the signal.

Linearity and time shifting

Consider two periodic signals $x_N[n]$ and $y_N[n]$ with period N and Fourier coefficients a_k and b_k, respectively. Then:

- The signal $z_N[n] = Ax_N[n] + By_N[n]$ is periodic with period N and its Fourier coefficients are $c_k = Aa_k + Bb_k$.
- The signal $v_N[n] = x_N[n - n_0]$ is periodic with period N and its Fourier coefficients are $d_k = e^{jk\Omega_0 n_0} a_k$.

Fourier series and LTI systems

$$x_N(n) \longrightarrow \boxed{H(\Omega)} \longrightarrow y_N(n)$$

$$x_N[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n} \longrightarrow y_N[n] = \sum_{k=\langle N\rangle} H(k\Omega_0) a_k e^{jk\Omega_0 n}$$

Hence, the Fourier coefficients of $y_N[n]$ are $b_k = a_k H(k\Omega_o)$. In words, they are a filtered version of the coefficients a_k.