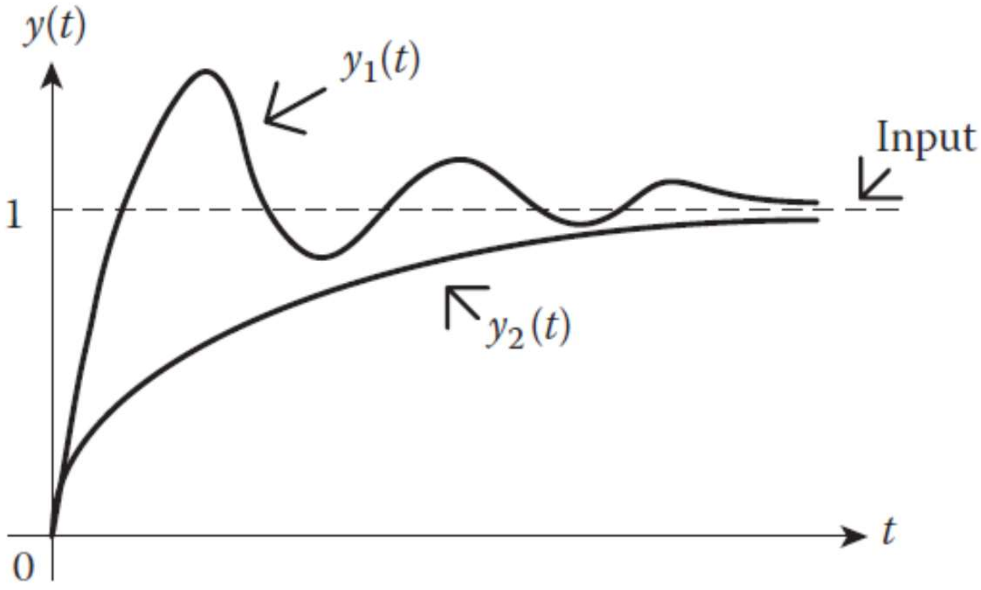
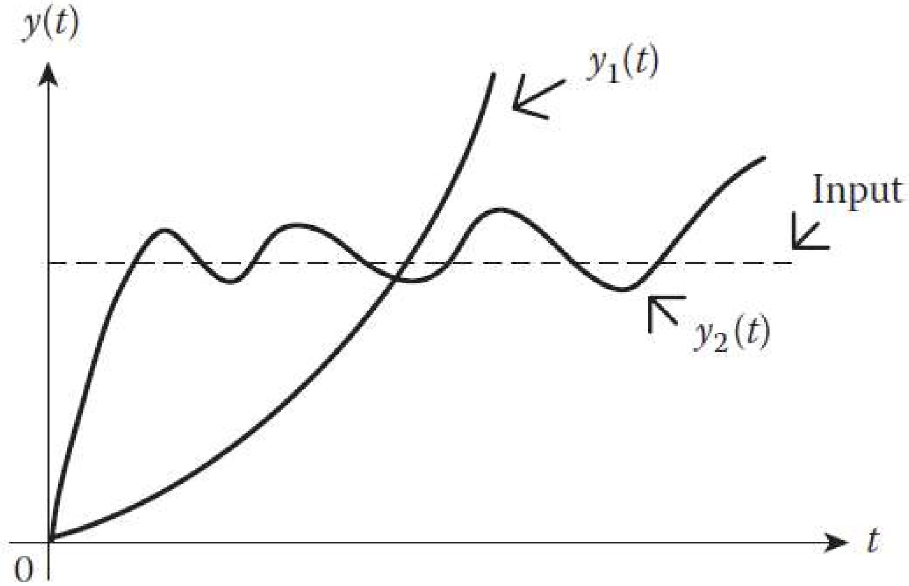
Stability of Control Systems

Introduction

- A system is **bounded-input bounded-output (BIBO)** stable, if every bounded input results in a bounded output.
- The output of a stable system is kept within the admissible boundaries
- whereas the output of an unstable system can, theoretically, increase to infinity
- linear time-invariant system is **stable**, if the poles of the closed-loop system are in the **left-half *s*-plane**, that is, the poles have real negative parts
- On the other hand, if at least one pole is in the right-half *s*-plane, the system is unstable
- The response of an automatic control system is related to the **roots** of the characteristic equation (poles) of the transfer function:
  - If the poles are in the left-half *s*-plane, then the response of the system to various disturbance signals is decreasing.
  - If there are poles on the imaginary $j\omega$ axis or in the right-half *s*-plane, then the response of the system to a disturbance input is constant or increasing.



y(t)
y1(t)
Input
1
y2(t)
t
0



y(t)
y1(t)
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y2(t)
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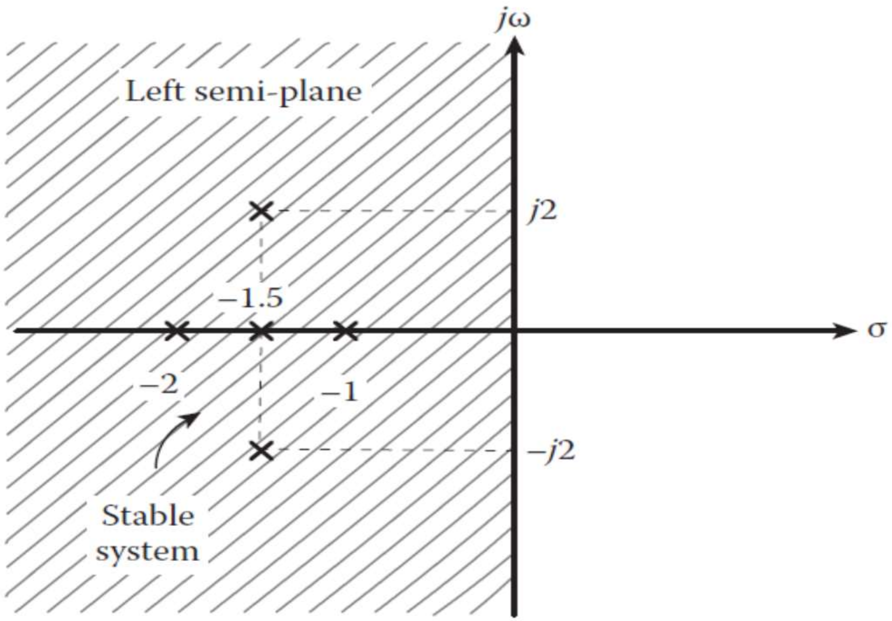
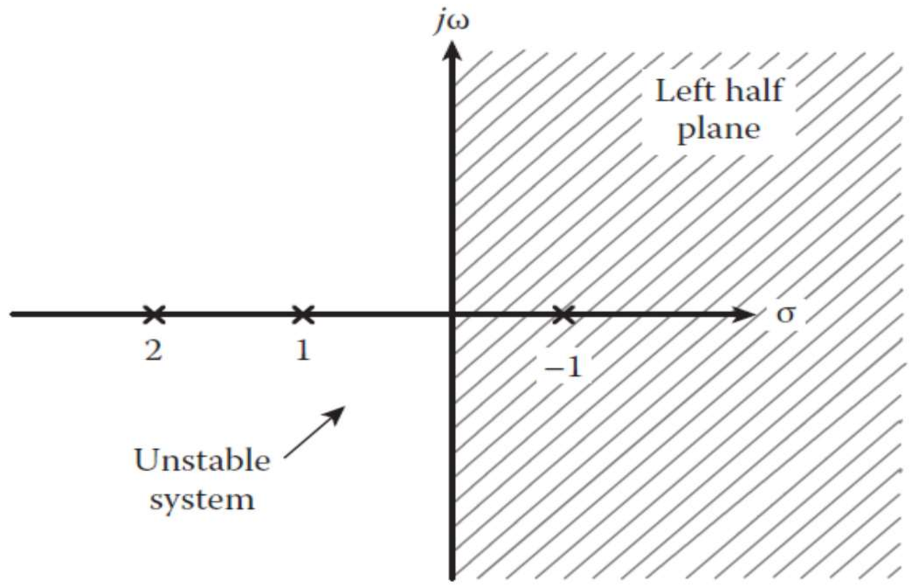
0



jω
Left semi-plane
j2
−1.5
σ
−2
−1
−j2
Stable
system



jω
Left half
plane
σ
2
1
−1
Unstable
system

Algebraic Stability Criteria

- **Routh's Stability Criterion**
- Routh's stability criterion determines the number of the poles of the transfer function, which are in the right-half *s*-plane. Recall that a pole in the right-half *s*-plane results in system instability.

$$a_0 s^5 + a_1 s^4 + a_2 s^3 + a_3 s^2 + a_4 s + a_5 = 0 \qquad \text{where } a_i \in R..$$

- To ensure that the roots of above Equation have no positive real parts, all coefficients a_i must be of the same sign

$$
\begin{array}{c|ccc}
s^5 & a_0 & a_2 & a_4 \\
s^4 & a_1 & a_3 & a_5 \\
s^3 & b_1 & b_2 & \\
s^2 & c_1 & c_2 & \\
s^1 & d_1 & & \\
s^0 & e_1 & &
\end{array}
$$

$$
b_1 = -\frac{\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1}, \quad b_2 = -\frac{\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1}
$$

$$
c_1 = -\frac{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1}, \quad c_2 = -\frac{\begin{vmatrix} a_1 & a_5 \\ b_1 & 0 \end{vmatrix}}{b_1} = a_5
$$

$$
d_1 = -\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}{c_1}, \quad e_1 = -\frac{\begin{vmatrix} c_1 & c_2 \\ d_1 & 0 \end{vmatrix}}{d_1} = c_2
$$

According to Routh's stability criterion, a system is stable if the terms of the first column of Routh's tabulation (i.e., a_0, a_1, b_1, c_1, d_1, e_1) are of the **same sign**.

$$\text{C.E.: } a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0 = 0$$

Routh's Criterion

$$\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

$$b_1 = -\frac{\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}, \quad b_2 = -\frac{\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}}, \cdots$$

$$c_1 = -\frac{\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}}{b_1}, \quad c_2 = -\frac{\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}}{b_1}, \cdots$$

The system is stable if $a_n, a_{n-1}, b_1, c_1, \ldots\ldots > 0$

- The number of roots of the characteristic equation, which are in the right-half *s*-plane, is equal to the number of sign changes of the coefficients in the first column of Routh's tabulation.
- If a system satisfies Routh's criterion, which means it is absolutely stable, then it is desirable to define its relative stability. It follows that the relative damping of every root of the characteristic equation must be examined. The greater the distance of a pole from the $j\omega$ axis, the greater is its relative stability.

a. If **a term of the first column is zero**, while all the other terms of the row are nonzero or do not exist, then the zero term is replaced by a very small number, which is of the same sign as the previous terms of the first column. The procedure continues as normal. Alternatively, the characteristic polynomial can be multiplied by $s + m$, where $m > 0$ and $-m$ is not a root of the characteristic equation.

b. If **all terms of a row in Routh's array are zero**, then the array is completed by replacing the zero terms by the terms of the differentiated auxiliary equation of the previous row.

c. If **two (or more) rows have zero terms**, then the system is unstable and the characteristic polynomial has two real poles with multiplicity 2.

d. In order to find the **marginal value of K that yields stability**, it is sufficient to suppose that the term of the s^1 row is zero and to solve the equation for $K = K_c$.

e. In order to find the **critical frequency of oscillation of the system**, it is sufficient to solve the auxiliary equation of the s^2 row for $\omega = \omega_c$.

Problems to Solve

- Determine the stability of the systems with the following characteristic equations:

a. $s^5 + 3s^4 + 7s^3 + 20s^2 + 6s + 15 = 0$

b. $2s^4 + s^3 + 3s^2 + 5s + 10 = 0$

c. $s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 = 0$

d. $s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$

$$s^5 + 3s^4 + 7s^3 + 20s^2 + 6s + 15 = 0$$

$$\begin{array}{c|ccc}
s^5 & 1 & 7 & 6 \\
s^4 & 3 & 20 & 15 \\
s^3 & 1/3 & 1 & \\
s^2 & 11 & 15 & \\
s^1 & 6/11 & & \\
s^0 & 15 & &
\end{array}$$

There is no change of sign in the first column of Routh's tabulation. Hence, the system is stable.

$$2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

$$\begin{array}{c|ccc}
s^4 & 2 & 3 & 10 \\
s^3 & 1 & 5 & \\
s^2 & -7 & 10 & \\
s^1 & 6.43 & & \\
s^0 & 10 & &
\end{array}$$

There are two sign changes in the first column of Routh's tabulation ($1 \to -7 \to 6.43$);
hence, the characteristic equation has two roots in the right-half *s*-plane,
and the **system is unstable**.

$$s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 = 0$$

$$
\begin{array}{c|ccc}
s^5 & 1 & 2 & 11 \\
s^4 & 2 & 4 & 10 \\
s^3 & \varepsilon & 6 & \\
s^2 & \dfrac{4\varepsilon - 12}{\varepsilon} & 10 & \\
s^1 & 6 + \dfrac{10}{12}\varepsilon^2 & & \\
s^0 & 10 & &
\end{array}
$$

The first term in row s_3 was zero and it is replaced by a very small number ε (where, $\varepsilon > 0$ and $\lim \varepsilon \to 0$).

We have $4\varepsilon - 12/\varepsilon < 0$ and

$6 + (10/12)\varepsilon^2 > 0$.

Hence, the system is unstable, and the characteristic equation has two roots in the right-half *s*-plane.

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

$$\begin{array}{c|ccc} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 8 & 96 & \\ s^2 & 24 & -50 & \\ s^1 & 112.7 & & \\ s^0 & -50 & & \end{array}$$

All coefficients of row s^3 were zero, thus, they have been replaced by the terms of the differentiated auxiliary equation of row s^4.

At the first column of Routh's tabulation, there is a change of sign; therefore, the characteristic equation has one root in the right-half s-plane.

$$2s^4 + 48s^2 - 50 = 0 \Rightarrow \frac{d}{dt}(2s^4 + 48s^2 - 50) = 0 \Rightarrow 8s^3 + 96s = 0$$

When is the system stable? $s^4 + 7s^3 + 15s^2 + (25+k)s + 2k = 0$

$$
\begin{array}{c|ccc}
s^4 & 1 & 15 & 2k \\
s^3 & 7 & 25+k & \\
s^2 & \dfrac{80-k}{7} & 2k & \\
s^1 & \dfrac{(80-k)(25+k)-98k}{80-k} & & \\
s^0 & 2k & &
\end{array}
$$

For the system to be stable, it must hold that

1. $\dfrac{80-k}{7} > 0 \Rightarrow k < 80$
2. $\dfrac{(80-k)(25+k)-98k}{80-k} > 0 \Rightarrow -71.1 < k < 28.1$
3. $2k > 0 \Rightarrow k > 0$