EBU4375 Signals and Systems Theory

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More on Basic Time Signals

- Rectangular Pulse Function
- Signum Function
- Ramp Function
- Sinc Function
- CT Unit Impulse Function (recap)
- DT Unit Impulse Sequence (recap)
- Representation of Signals using Impulse Sequence (DT Signals)
- Representation of Signals using Impulse Function (CT Signals)

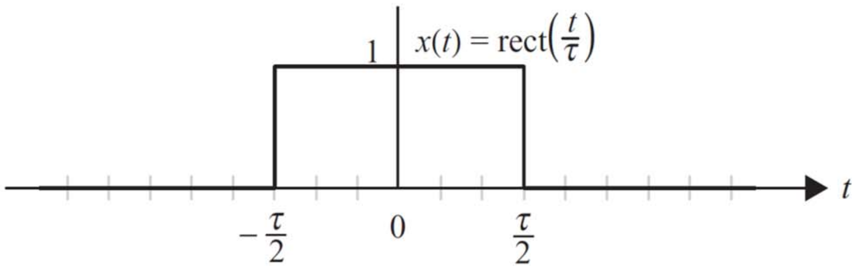
Basic Time Signals – Rectangular Pulse Function

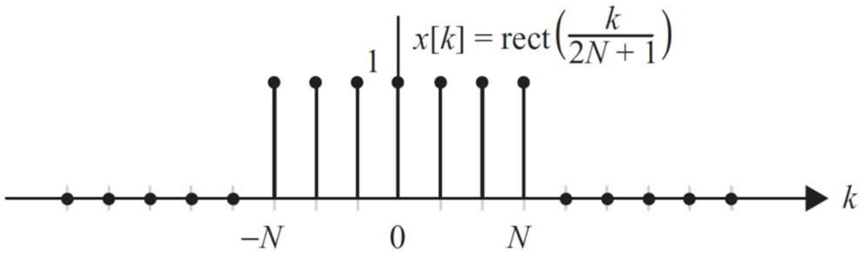
The CT rectangular pulse $\text{rect}(t/\tau)$ is defined as follows:

$$\text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & |t| \le \tau/2 \\ 0 & |t| > \tau/2 \end{cases}$$

The DT rectangular pulse $\text{rect}(k/(2N+1))$ is defined as follows:

$$\text{rect}\left(\frac{k}{2N+1}\right) = \begin{cases} 1 & |k| \le N \\ 0 & |k| > N \end{cases}$$





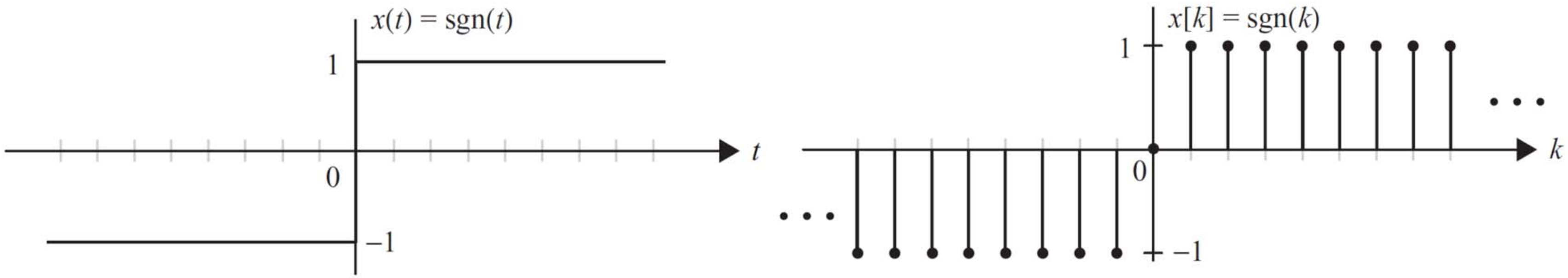
Basic Time Signals – Signum Function

The *signum* (or *sign*) function, denoted by sgn(t), is defined as follows:

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0. \end{cases}$$

The DT signum function, denoted by sgn(k), is defined as follows:

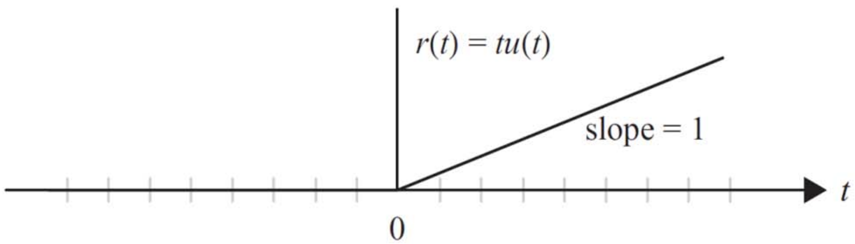
$$\operatorname{sgn}[k] = \begin{cases} 1 & k > 0 \\ 0 & k = 0 \\ -1 & k < 0 \end{cases}$$

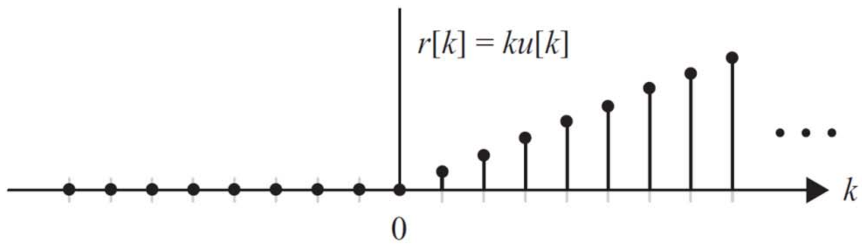


Basic Time Signals – Ramp Function

$$r(t) = tu(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0, \end{cases}$$

$$r[k] = ku[k] = \begin{cases} k & k \geq 0 \\ 0 & k < 0, \end{cases}$$

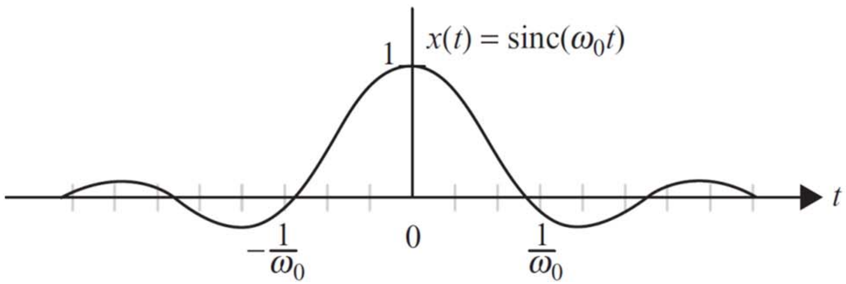


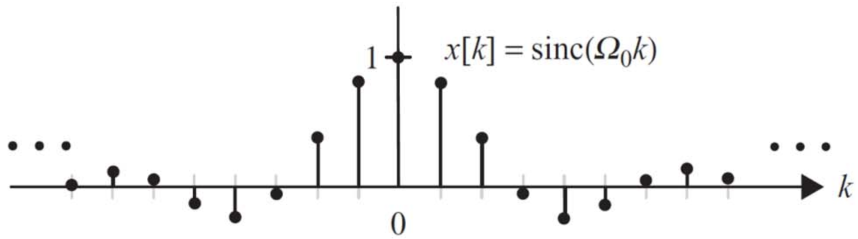


Basic Time Signals – Sinc Function

$$\operatorname{sinc}(\omega_0 t) = \frac{\sin(\pi \omega_0 t)}{\pi \omega_0 t}$$

$$\operatorname{sinc}(\Omega_0 k) = \frac{\sin(\pi \Omega_0 k)}{\pi \Omega_0 k}$$



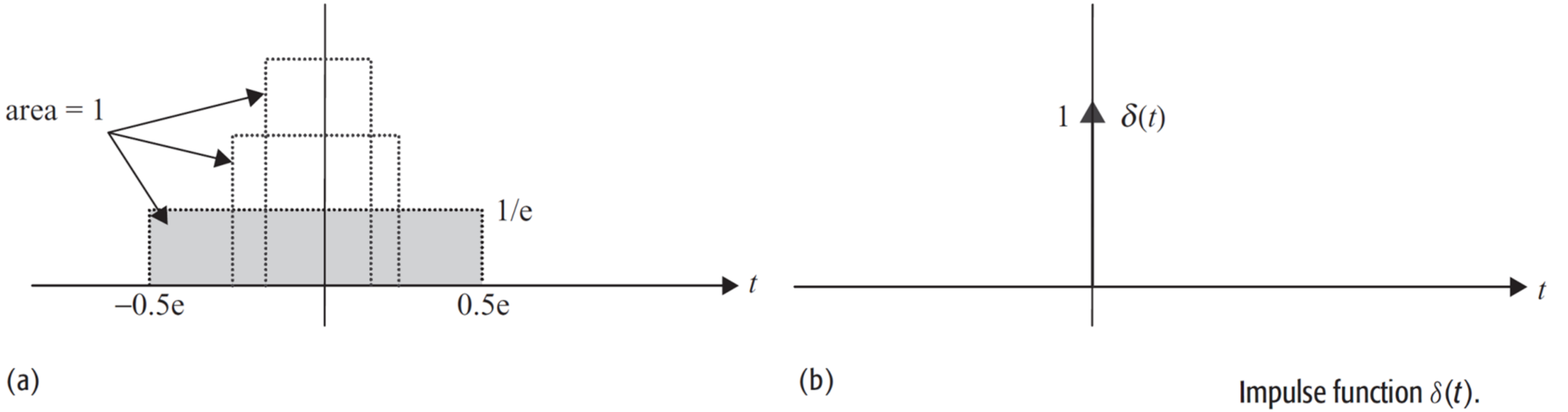


Basic Time Signals – CT Unit Impulse Function (Recap)

The *unit impulse* function $\delta(t)$, also known as the *Dirac delta* function or simply the *delta* function, is defined in terms of two properties as follows:

(1) amplitude $\quad \delta(t) = 0, \quad t \neq 0;$

(2) area enclosed $\quad \displaystyle\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1.$



Impulse function $\delta(t)$.
(a) Generating the impulse function $\delta(t)$ from a rectangular pulse. (b) Notation used to represent an impulse function.

Basic Time Signals – CT Unit Impulse Function (Recap)

Properties of impulse function

(i) The impulse function is an even function, i.e. $\delta(t) = \delta(-t)$.

(ii) Integrating a unit impulse function results in one, provided that the limits of integration enclose the origin of the impulse. Mathematically,

$$\int_{-T}^{T} A\delta(t - t_0)\mathrm{d}t = \begin{cases} A & \text{for } -T < t_0 < T \\ 0 & \text{elsewhere.} \end{cases}$$

(iii) The scaled and time-shifted version $\delta(at + b)$ of the unit impulse function is given by

$$\delta(at + b) = \frac{1}{a}\delta\left(t + \frac{b}{a}\right).$$

(iv) When an arbitrary function $\phi(t)$ is multiplied by a shifted impulse function, the product is given by

$$\phi(t)\delta(t - t_0) = \phi(t_0)\delta(t - t_0).$$

Basic Time Signals – CT Unit Impulse Function (Recap)

In other words, multiplication of a CT function and an impulse function produces an impulse function, which has an area equal to the value of the CT function at the location of the impulse. Combining properties (ii) and (iv), it is straightforward to show that

$$\int_{-\infty}^{\infty} \phi(t)\delta(t - t_0)\mathrm{d}t = \phi(t_0).$$

(v) The unit impulse function can be obtained by taking the derivative of the unit step function as follows:

$$\delta(t) = \frac{\mathrm{d}u}{\mathrm{d}t}.$$

(vi) Conversely, the unit step function is obtained by integrating the unit impulse function as follows:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\mathrm{d}\tau.$$

Basic Time Signals – DT Unit Impulse Sequence (Recap)

- recall the *unit impulse* signal: a sequence that is zero everywhere except at sample 0:

$$\delta(n) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}$$

- any discrete-time signal can be represented as a sum of scaled and shifted unit impulses, i.e.

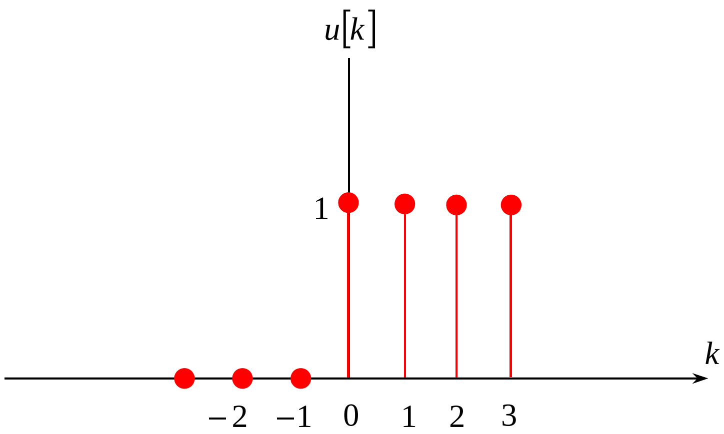
$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

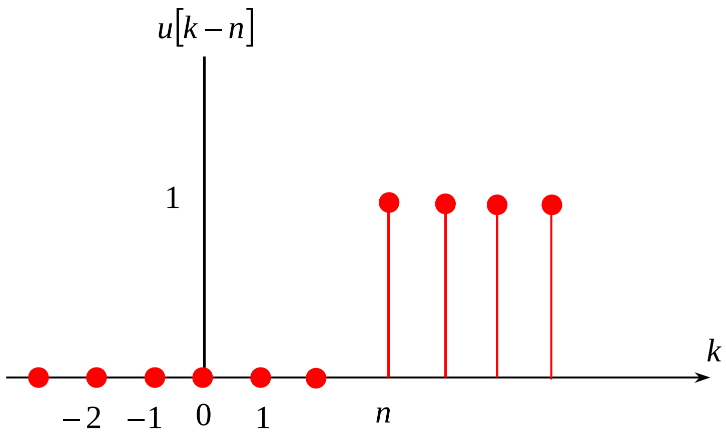
- this may seem complicated, but will become handy in a minute!

Basic Time Signals – DT Unit Impulse Sequence (Recap)

The DT impulse function, also referred to as the Kronecker delta function or the DT unit sample function, is defined as follows:

$$\delta[k] = u[k] - u[k-1] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0. \end{cases}$$





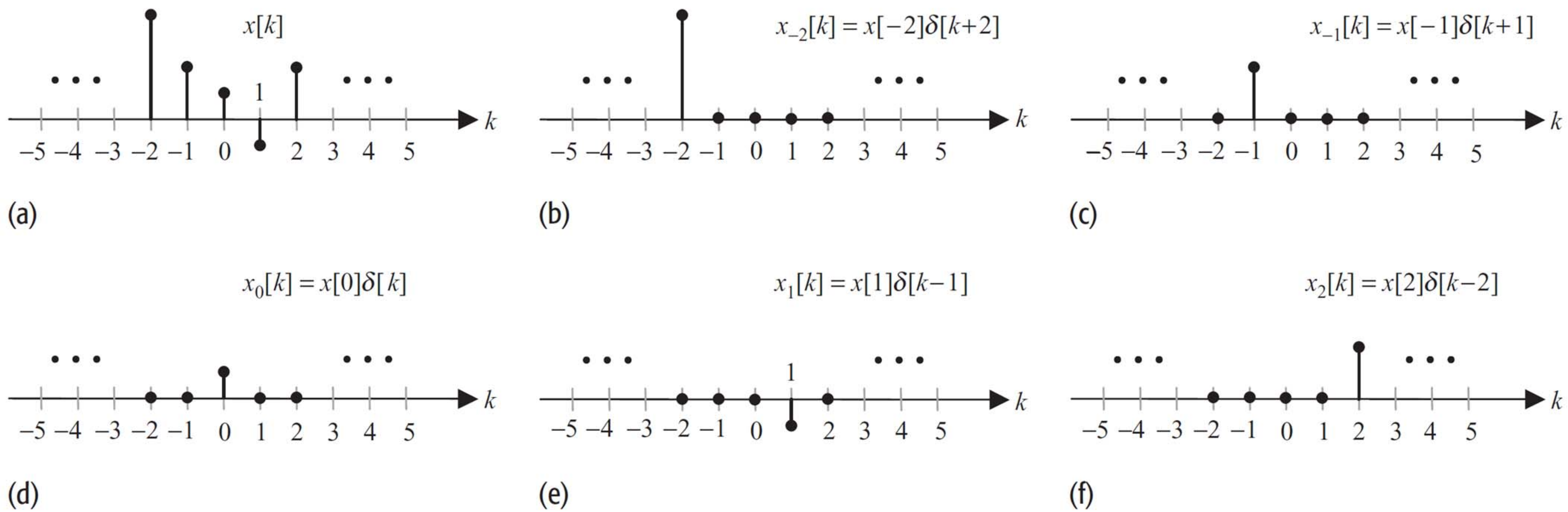
Basic Time Signals – Representation of Signals using Impulse Sequence (DT Signals)

In other words,

$$x_m[k] = x[m]\delta[k-m]$$

In terms of $x_m[k]$, the DT sequence $x[k]$ is, therefore, represented by

$$\begin{aligned} x[k] &= \cdots + x_{-2}[k] + x_{-1}[k] + x_0[k] + x_1[k] + x_2[k] + \cdots \\ &= \cdots + x[-2]\delta[k+2] + x[-1]\delta[k+1] + x[0]\delta[k] \\ &\quad + x[1]\delta[k-1] + x[2]\delta[k-2] + \cdots , \end{aligned}$$



Basic Time Signals – Representation of Signals using Impulse Sequence (DT Signals)

which reduces to

$$x[k] = \sum_{m=-\infty}^{\infty} x[m]\delta[k-m]$$

Basic Time Signals – Representation of Signals using Impulse Function (CT Signals)

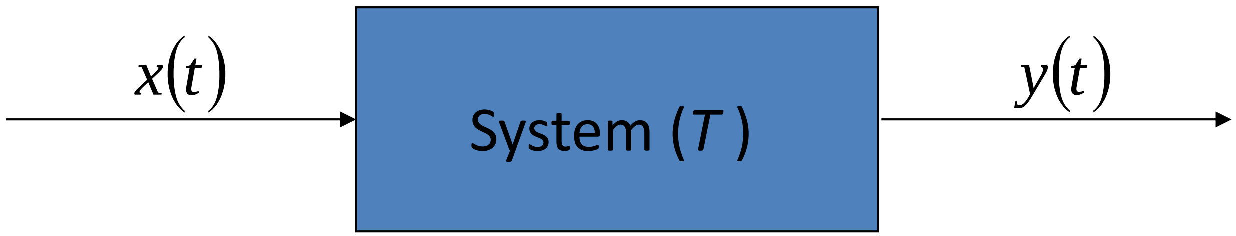
Similarly, a continuous-time signal $x(t)$ may be expressed as

$$x(t) = \int_{-\infty}^{\infty} d\tau \, x(\tau)\delta(t-\tau)$$

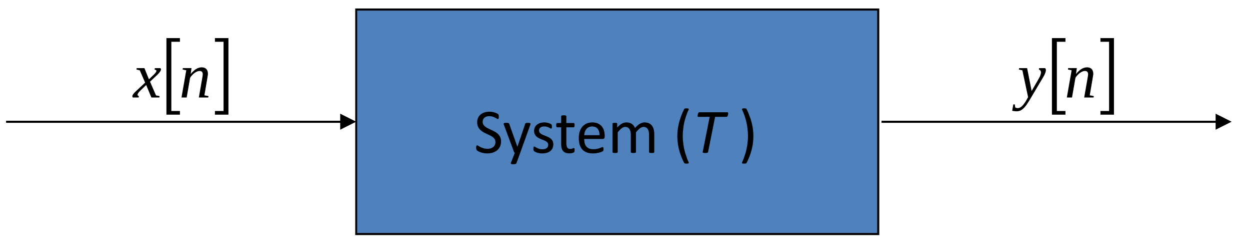
System Classification and Properties

- Continuous-time and Discrete-time Systems
- Systems with and without memory
- Causal and Non-causal Systems
- Linear and Nonlinear Systems
- Time-invariant and Time-variant Systems
- Linear Time-invariant Systems
- Feedback Systems

Continuous-time and Discrete-time Systems



Continuous-time System



Discrete-time System

Systems with and without Memory

(T) is without memory if the output only depends on the input at the corresponding time-mark, i.e.

$$y(t_i) \leftrightarrow x(t_i)$$

e.g. a resistor is <u>memoryless</u> in that the input $x(t) = i(t)$ produces an output (response) $y(t) = v(t)$ related through the system (transfer) function by

$$v(t) = Ri(t)$$

e.g. a capacitor has <u>memory</u> in that, again using the input as $x(t) = i(t)$ and the output as $y(t) = v(t)$; these are related through the system (transfer) function by

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} d\tau\, i(\tau)$$

An example of memory in a discrete-system, whose input and output sequences are related by

$$y(n) = \sum_{k=-\infty}^{n} x[k]$$

Causal and Non-causal Systems

- A system (T) is *causal* if its output $y(t)$ at an arbitrary time $t = \tau$ depends only on the input $x(t)$ for $t \leq \tau$; i.e. the present-time output depends only on the present and/or past inputs: not on future values.

- Simply, you cannot have an output before and input

- Examples of non-causal systems are:

$$y(t) = x(t+1)$$
$$y[n] = x[-n]$$

NB:

All memoryless systems are causal, but not vice-versa.

Linear and Nonlinear Systems

- If the operator (T) satisfies the following two conditions then it is *linear* and represents a *linear* system:

1. <u>The Addition Rule:</u>

$$(T)x_1 = y_1 \quad and \quad (T)x_2 = y_2 \quad then \quad (T)\{x_1 + x_2\} = y_1 + y_2$$

for any signals $x_1 \quad and \quad x_2$

2. <u>Scaling Rule:</u> $(T)\{\alpha x\} = \alpha y$ for any signal x and any scale-factor α

- Any system not satisfying these conditions is classified *nonlinear*

- Conditions 1. and 2. may be combined into the single condition

$$(T)\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 y_1 + \alpha_2 y_2$$

where α_1, α_2 are arbitrary scalars.

Time-invariant and Time-variant Systems

A system is *time-invariant* if a time-shift (advance or retardation (or delay)) at the input causes an *identical* shift at the output. So for a continuous-time system, time-invariance exists if:

$$(T)\{x(t \pm \tau)\} = y(t \pm \tau) \quad ; \quad \tau \in \Re \qquad (2)$$

For a discrete-time system, the system is time- or shift-invariant if

$$(T)\{x[n \pm k]\} = y[n \pm k] \quad ; \quad k \in Z \qquad (3)$$

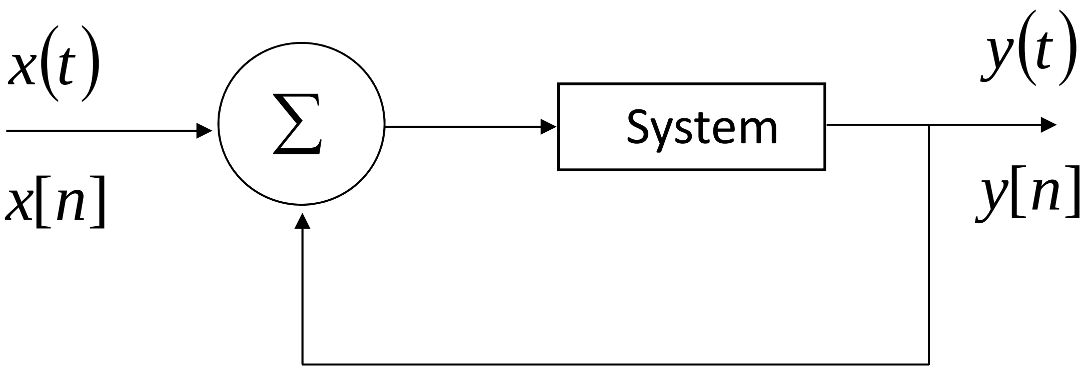
- A system not satisfying equations (2) and (3) is time-varying..

Linear Time-invariant Systems (LTI)

An LTI system posses together attributes of *linearity* and *shift-invariance*.

Feedback Systems

An important system-class in which output is fed back and added to the input



System Feedback

Response and the Convolution

- Response to an Impulse **(CT Systems)**
- Response to a General Input **(CT Systems)**
- Convolution Integral **(CT Systems)**
- Convolution Algebra **(CT Systems)**
- Response to an Impulse **(DT Systems)**
- Response to a General Input **(DT Systems)**
- Convolution Sum **(DT Systems)**
- Sequence Convolution Algebra **(DT Systems)**

Response to an Impulse (CT Systems)

The impulse-response (IR) $h(t)$ of a continuous-time LTI system is defined to be the response following excitation by the signal $\delta(t)$ i.e.

$$h(t) = T\{\delta(t)\}$$

system response to the impulse

"impulse" excitation

Transfer Function

Response to a General Input (CT Systems)

Recall the earlier result that a general signal could be expressed by

$$x(t) = \int_{-\infty}^{\infty} d\tau \, x(\tau)\delta(t-\tau)$$

Since the system is **linear**, the response $y(t)$ to an excitation $x(t)$ can be written as

$$y(t) = T\{x(t)\} = T\left\{ \int_{-\infty}^{\infty} d\tau \, x(\tau)\delta(t-\tau) \right\}$$

$$= \int_{-\infty}^{\infty} d\tau \, x(\tau) T\{\delta(t-\tau)\} \qquad (1)$$

Time-invariance implies

$$h(t-\tau) = T\{\delta(t-\tau)\} \qquad (2)$$

$$(2) \rightarrow (1) \Rightarrow y(t) = \int_{-\infty}^{\infty} d\tau \, x(\tau) h(t-\tau) \qquad (3)$$

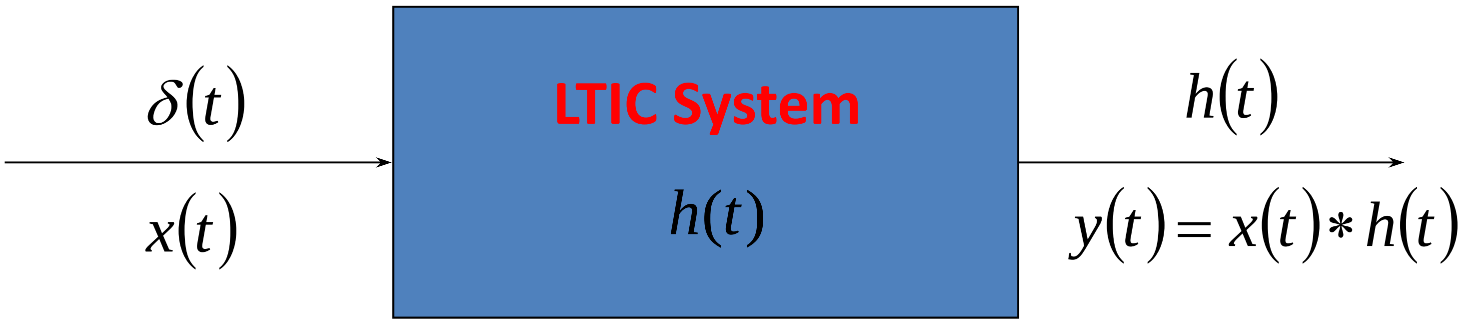
(3) says that the continuous-time response of an LTI system is entirely characterised by it's impulse response $h(t)$.

Convolution Integral (CT Systems)

Equation (3) ***<u>defines</u>*** the convolution operation, i.e.

$$y(t) = x(t) * h(t) \equiv \int_{-\infty}^{\infty} d\tau \, x(\tau) h(t-\tau) \qquad (4)$$

so that (4) is the *convolution integral*.



Convolution Algebra (CT Systems)

Commutation: $$x(t)*h(t)=h(t)*x(t)$$

Association: $$\{x(t)*h_1(t)\}*h_2(t)=x(t)*\{h_1(t)*h_2(t)\}$$

Distribution: $$x(t)*\{h_1(t)+h_2(t)\}=x(t)*h_1(t)+x(t)*h_2(t)$$

Goto http://mathworld.wolfram.com/Convolution.html to get a dynamic appreciation of convolution.

Response to an Impulse (DT Systems)

The impulse-response (IR) $h[n]$ of a discrete-time LTI system is defined to be the response following excitation by the signal $\delta[n]$ i.e.

$$h[n] = T\{\delta[n]\}$$

Response to a General Input (DT Systems)

From earlier we know

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

So for a **linear** system we can write

$$y[n]=T\{x[n]\}=T\left\{\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]\right\}=\sum_{k=-\infty}^{\infty}x[k]T\{\delta[n-k]\}$$

Time-invariance means

$$T\{\delta[n-k]\}=h[n-k]$$

$$\therefore\ y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k] \qquad (5)$$

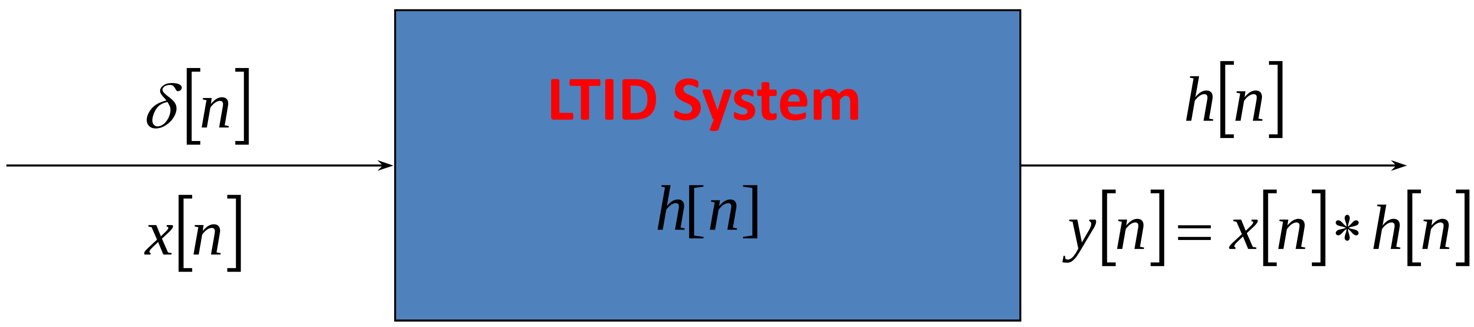
(5) Shows that a Discrete-time (DT) LTI system is completely characterised by its IR, $h[n]$.

Convolution Sum (DT Systems)

(5) Defines a convolution of two sequences, namely, $x[n]$ and $y[n]$; i.e.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad (6)$$

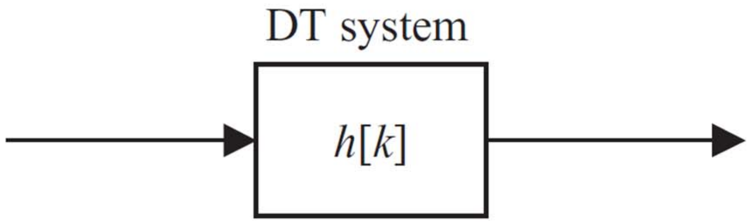
(6) is the convolution sum. In summary then



Convolution Sum (DT Systems)

$x[m]\delta[k-m]$

$x[m]h[k-m]$



$$x[k]=\sum_{m=-\infty}^{\infty}x[m]\delta[k-m]$$

$$y[k]=\sum_{m=-\infty}^{\infty}x[m]\,h[k-m]=x[m]*h[m]$$

Sequence Convolution Algebra (DT Systems)

Commutation: $$x[n]*h[n]=h[n]*x[n]$$

Association: $$\{x[n]*h_1[n]\}*h_2[n]=x[n]*\{h_1[n]*h_2[n]\}$$

Distribution: $$x[n]*\{h_1[n]*h_2[n]\}=x[n]*h_1[n]+x[n]*h_2[n]$$

Goto http://mathworld.wolfram.com/Convolution.html to get a dynamic appreciation of convolution.

Examples on LTID Systems

Definition *The impulse response $h[k]$ of an LTID system is the output of the system when a unit impulse $\delta[k]$ is applied at the input of the LTID system.*

$$\delta[k] \rightarrow h[k]$$

Note that an LTID system satisfies the linearity and the time-shifting properties. Therefore, if the input is a scaled and time-shifted impulse function $a\delta[k - k_0]$, the output of the DT system is also scaled by a factor of a and

$$a\delta[k - k_0] \rightarrow ah[k - k_0]$$

Example

Consider the LTID systems with the following input–output relationships:

$$y[k] = x[k - 1] + 2x[k - 3]$$

Calculate the impulse response. Also, determine the output response when the input is given by $x[k] = 2\delta[k] + 3\delta[k - 1]$.

Examples on LTID Systems

Solution

(i) The impulse response of a system is the output of the system when the input sequence $x[k] = \delta[k]$. Therefore, the impulse response $h[k]$ of system (i) can be obtained by substituting $y[k]$ by $h[k]$ and $x[k]$ by $\delta[k]$ in Eq. (10.8). In other words, the impulse response for system (i) is given by

$$h[k] = \delta[k-1] + 2\delta[k-3].$$

To evaluate the output response resulting from the input sequence $x[k] = 2\delta[k] + 3\delta[k-1]$, we use the linearity and time-invariance properties of the system. The outputs resulting from the two terms $2\delta[k]$ and $3\delta[k-1]$ in the input sequence are as follows:

$$2\delta[k] \rightarrow 2h[k] = 2\delta[k-1] + 4\delta[k-3]$$

and

$$3\delta[k-1] \rightarrow 3h[k-1] = 3\delta[k-2] + 6\delta[k-4].$$

Examples on LTID Systems

Applying the superposition principle, the output $y[k]$ to input $x[k] = 2\delta[k] + 3\delta[k-1]$ is given by

$$2\delta[k] + 3\delta[k-1] \rightarrow 2h[k] + 3h[k-1]$$

or

$$\begin{aligned} y[k] &= (2\delta[k-1] + 4\delta[k-3]) + (3\delta[k-2] + 6\delta[k-4]) \\ &= 2\delta[k-1] + 3\delta[k-2] + 4\delta[k-3] + 6\delta[k-4]). \end{aligned}$$

Examples on LTID Systems

Example

The impulse response of an LTID system is given by $h[k] = 0.5^k u[k]$. Determine the output of the system for the input sequence $x[k] = \delta[k-1] + 3\delta[k-2] + 2\delta[k-6]$.

Solution

Because the system is LTID, it satisfies the linearity and time-shifting properties. The individual responses to the three terms $\delta[k-1]$, $3\delta[k-2]$, and $2\delta[k-6]$ in the input sequence $x[k]$ are given by

$$\delta[k-1] \rightarrow h[k-1] = 0.5^{k-1}u[k-1],$$
$$3\delta[k-2] \rightarrow 3h[k-2] = 3 \times 0.5^{k-2}u[k-2],$$

and

$$2\delta[k-6] \rightarrow 2h[k-6] = 2 \times 0.5^{k-6}u[k-6].$$

Examples on LTID Systems

Applying the principle of superposition, the overall response to the input sequence $x[k]$ is given by

$$y[k] = h[k-1] + 3h[k-2] + 2h[k-6].$$

Substituting the value of $h[k] = 0.5^k u[k]$ results in the output response:

$$y[k] = 0.5^{k-1}u[k-1] + 3 \times 0.5^{k-2}u[k-2] + 2 \times 0.5^{k-6}u[k-6] .$$