B. Fourier Transform Pair:

The function $X(\Omega)$ defined by Eq. (*6.23*) is called the *Fourier transform* of $x[n]$, and Eq. (*6.26*) defines the *inverse Fourier transform* of $X(\Omega)$. Symbolically they are denoted by

$$X(\Omega) = \mathscr{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \tag{6.27}$$

$$x[n] = \mathscr{F}^{-1}\{X(\Omega)\} = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega \tag{6.28}$$

and we say that $x[n]$ and $X(\Omega)$ form a Fourier transform pair denoted by

$$x[n] \leftrightarrow X(\Omega) \tag{6.29}$$

Equations (*6.27*) and (*6.28*) are the discrete-time counterparts of Eqs. (*5.31*) and (*5.32*).

C. Fourier Spectra:

The Fourier transform $X(\Omega)$ of $x[n]$ is, in general, complex and can be expressed as

$$X(\Omega) = |X(\Omega)| e^{j\phi(\Omega)} \tag{6.30}$$

As in continuous time, the Fourier transform $X(\Omega)$ of a nonperiodic sequence $x[n]$ is the frequency-domain specification of $x[n]$ and is referred to as the *spectrum* (or *Fourier spectrum*) of $x[n]$. The quantity $|X(\Omega)|$ is called the *magnitude spectrum* of $x[n]$, and $\phi(\Omega)$ is called the *phase spectrum* of $x[n]$. Furthermore, if $x[n]$ is real, the amplitude spectrum $|X(\Omega)|$ is an even function and the phase spectrum $\phi(\Omega)$ is an odd function of Ω.

D. Convergence of $X(\Omega)$:

Just as in the case of continuous time, the sufficient condition for the convergence of $X(\Omega)$ is that $x[n]$ is absolutely summable, that is,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \tag{6.31}$$

E. Connection between the Fourier Transform and the z-Transform:

Equation (*6.27*) defines the Fourier transform of $x[n]$ as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \tag{6.32}$$

The z-transform of $x[n]$, as defined in Eq. (*4.3*), is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{6.33}$$

Comparing Eqs. (*6.32*) and (*6.33*), we see that if the ROC of $X(z)$ contains the unit circle, then the Fourier transform $X(\Omega)$ of $x[n]$ equals $X(z)$ evaluated on the unit circle, that is,

$$X(\Omega) = X(z)|_{z=e^{j\Omega}} \tag{6.34}$$

Note that since the summation in Eq. (*6.33*) is denoted by $X(z)$, then the summation in Eq. (*6.32*) may be denoted as $X(e^{j\Omega})$. Thus, in the remainder of this book, both $X(\Omega)$

Table 6-1. Properties of the Fourier Transform

Property	Sequence	Fourier transform
	$x[n]$	$X(\Omega)$
	$x_1[n]$	$X_1(\Omega)$
	$x_2[n]$	$X_2(\Omega)$
Periodicity	$x[n]$	$X(\Omega + 2\pi) = X(\Omega)$
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(\Omega) + a_2 X_2(\Omega)$
Time shifting	$x[n - n_0]$	$e^{-j\Omega n_0} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 n} x[n]$	$X(\Omega - \Omega_0)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Time reversal	$x[-n]$	$X(-\Omega)$
Time scaling	$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n = km \\ 0 & \text{if } n \neq km \end{cases}$	$X(m\Omega)$
Frequency differentiation	$nx[n]$	$j\dfrac{dX(\Omega)}{d\Omega}$
First difference	$x[n] - x[n-1]$	$(1 - e^{-j\Omega})X(\Omega)$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\pi X(0)\delta(\Omega) + \dfrac{1}{1 - e^{-j\Omega}} X(\Omega)$
		$\lvert\Omega\rvert \le \pi$
Convolution	$x_1[n] * x_2[n]$	$X_1(\Omega) X_2(\Omega)$
Multiplication	$x_1[n] x_2[n]$	$\dfrac{1}{2\pi} X_1(\Omega) \otimes X_2(\Omega)$
Real sequence	$x[n] = x_e[n] + x_o[n]$	$X(\Omega) = A(\Omega) + jB(\Omega)$
		$X(-\Omega) = X^*(\Omega)$
Even component	$x_e[n]$	$\text{Re}\{X(\Omega)\} = A(\Omega)$
Odd component	$x_o[n]$	$j\,\text{Im}\{X(\Omega)\} = jB(\Omega)$
Parseval's relations		

$$\sum_{n=-\infty}^{\infty} x_1[n] x_2[n] = \frac{1}{2\pi} \int_{2\pi} X_1(\Omega) X_2(-\Omega)\, d\Omega$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2\, d\Omega$$

Table 6-2. Common Fourier Transform Pairs

$x[n]$	$X(\Omega)$
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\Omega n_0}$
$x[n]=1$	$2\pi\delta(\Omega), \lvert\Omega\rvert \le \pi$
$e^{j\Omega_0 n}$	$2\pi\delta(\Omega-\Omega_0), \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\cos \Omega_0 n$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\sin \Omega_0 n$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$u[n]$	$\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$-u[-n-1]$	$-\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$-a^n u[-n-1], \lvert a\rvert > 1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$(n+1)a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{\left(1-ae^{-j\Omega}\right)^2}$
$a^{\lvert n\rvert}, \lvert a\rvert < 1$	$\dfrac{1-a^2}{1-2a\cos\Omega+a^2}$
$x[n]=\begin{cases}1 & \lvert n\rvert \le N_1 \\ 0 & \lvert n\rvert > N_1\end{cases}$	$\dfrac{\sin\left[\Omega\left(N_1+\frac{1}{2}\right)\right]}{\sin(\Omega/2)}$
$\dfrac{\sin Wn}{\pi n}, 0 < W < \pi$	$X(\Omega)=\begin{cases}1 & 0 \le \lvert\Omega\rvert \le W \\ 0 & W < \lvert\Omega\rvert \le \pi\end{cases}$
$\sum_{k=-\infty}^{\infty} \delta[n-kN_0]$	$\Omega_0 \sum_{k=-\infty}^{\infty} \delta(\Omega-k\Omega_0), \Omega_0=\dfrac{2\pi}{N_0}$

6.5 THE FREQUENCY RESPONSE OF DISCRETE-TIME LTI SYSTEMS

A. Frequency Response:

In Sec. 2.6 we showed that the output $y[n]$ of a discrete-time LTI system equals the convolution of the input $x[n]$ with the impulse response $h[n]$; that is,

$$y[n]=x[n]*h[n] \tag{6.67}$$

Applying the convolution property (6.58), we obtain

$$Y(\Omega)=X(\Omega)H(\Omega) \tag{6.68}$$