EBU4375 Signals and Systems Theory

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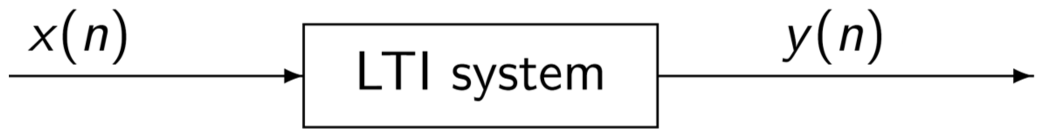
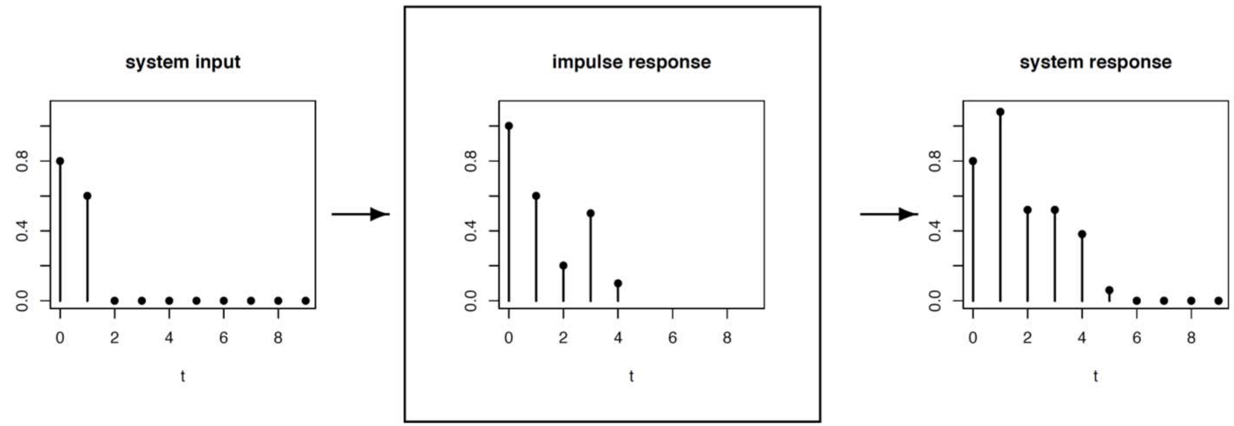
Convolution (DT Systems)

Convolution (DT Systems)

- we previously found that convolution with the impulse response gets us the system output:





- but *how* to calculate $y(n)$ from $x(n)$ and $h(n)$?
- that is: how do we calculate the convolution $y(n) = x(n) * h(n)$?

Convolution (DT Systems)

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$
$$= \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots$$

The sequences $h(k)$ and $x(k)$ are interchangeable.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$
$$= \ldots + x(-1)h(n+1) + x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + \ldots$$

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = \ldots + x(-1)h(1) + x(0)h(0) + x(1)h(-1) + x(2)h(-2) +$$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = \ldots + x(-1)h(2) + x(0)h(1) + x(1)h(0) + x(2)h(-1) +$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = \ldots + x(-1)h(3) + x(0)h(2) + x(1)h(1) + x(2)h(0) + \ldots$$

$\ldots$

Convolution (DT Systems)

Example

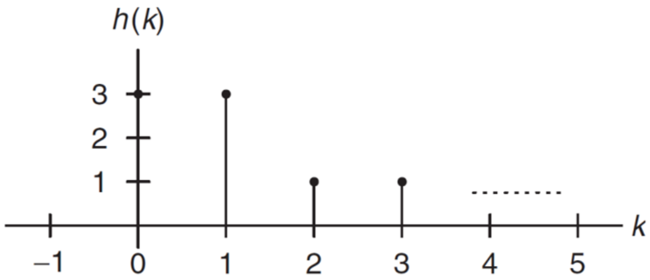
Given a sequence,

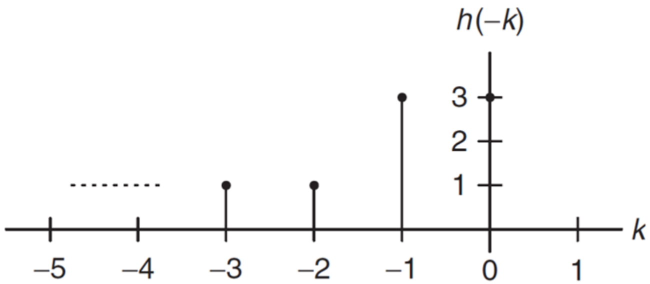
$$h(k) = \begin{cases} 3, & k = 0,1 \\ 1, & k = 2,3 \\ 0 & elsewhere \end{cases}$$

where k is the time index or sample number,

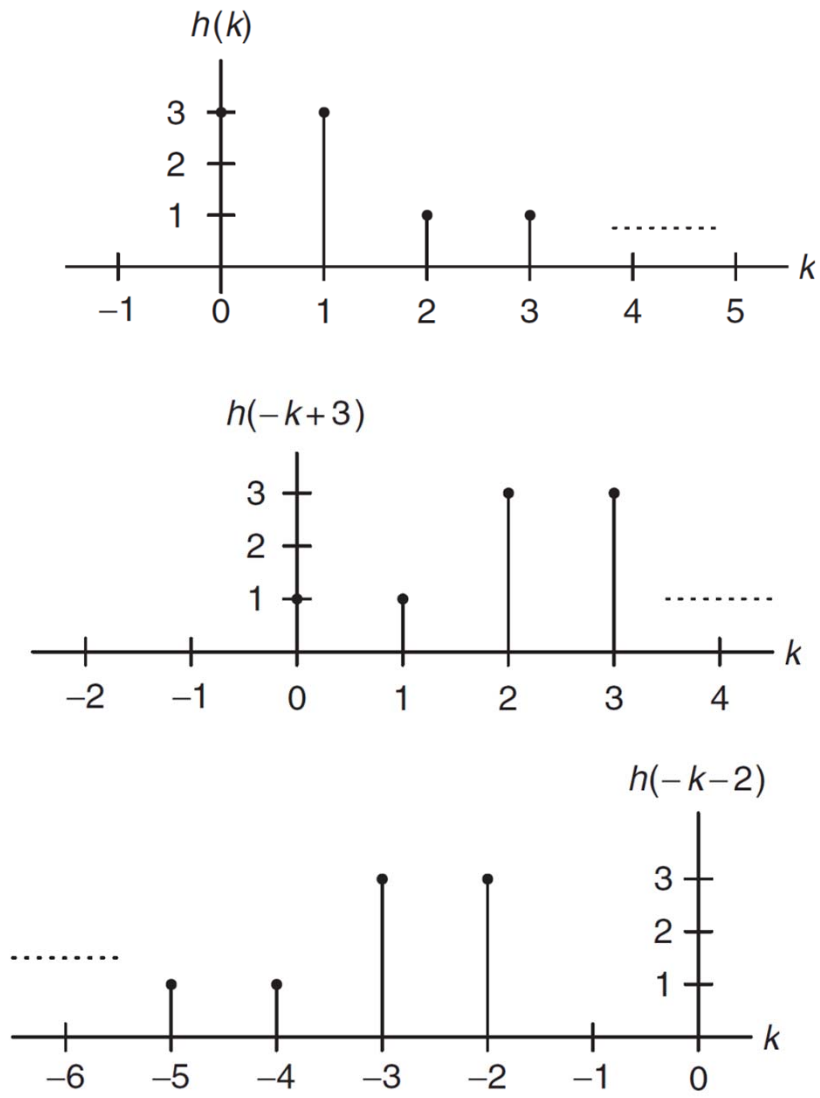
a. Sketch the sequence $h(k)$ and reversed sequence $h(-k)$.

b. Sketch the shifted sequences $h(-k+3)$ and $h(-k-2)$.





Convolution (DT Systems)



Convolution (DT Systems) - Digital convolution using the graphical method

Digital convolution using the graphical method

Step 1. Obtain the reversed sequence $h(-k)$.

Step 2. Shift $h(-k)$ by $|n|$ samples to get $h(n-k)$. If $n \geq 0$, $h(-k)$ will be shifted to the right by n samples; but if $n < 0$, $h(-k)$ will be shifted to the left by $|n|$ samples.

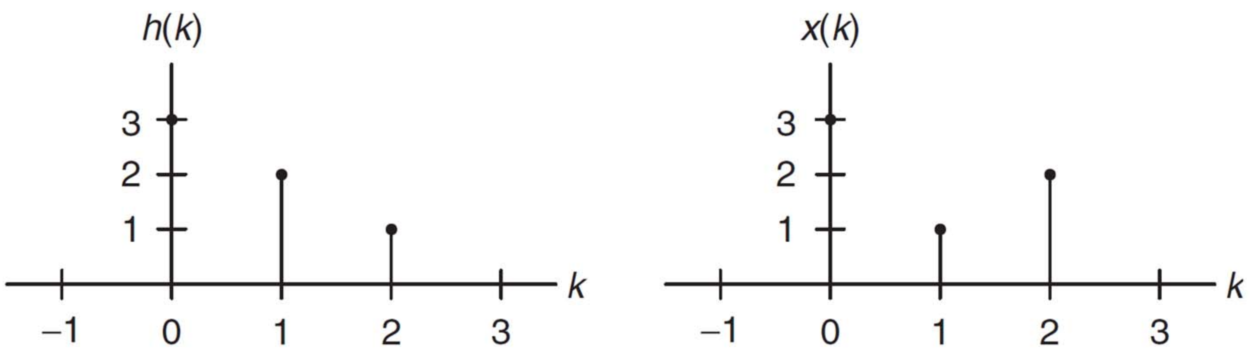
Step 3. Perform the convolution sum that is the sum of the products of two sequences $x(k)$ and $h(n-k)$ to get $y(n)$.

Step 4. Repeat steps 1 to 3 for the next convolution value $y(n)$.

Convolution (DT Systems) - Digital convolution using the graphical method

Example

Using the following sequences



evaluate the digital convolution

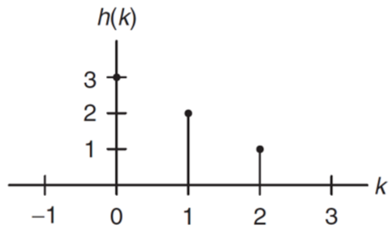
a. By the graphical method.

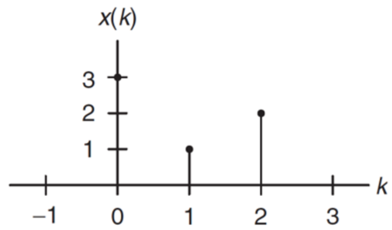
b. By applying the formula directly.

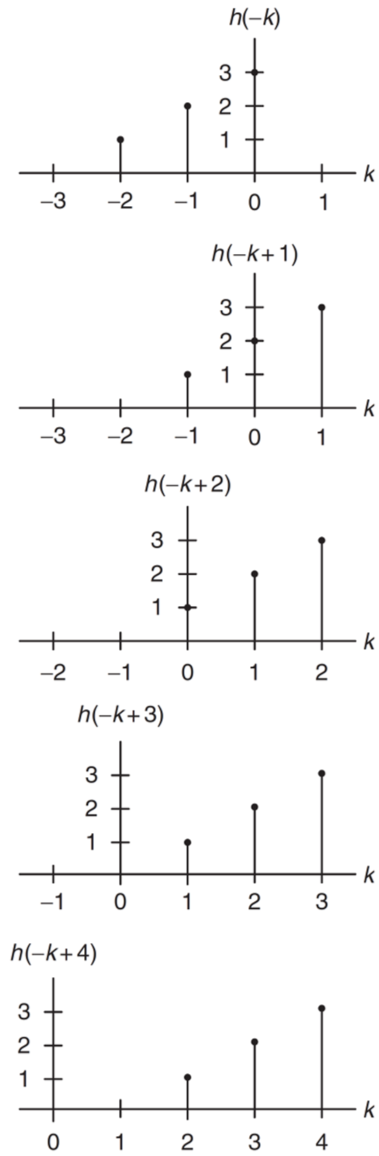
Convolution (DT Systems) - Digital convolution using the graphical method

Solution:

a. To obtain $y(0)$, we need the reversed sequence $h(-k)$; and to obtain $y(1)$, we need the reversed sequence $h(1-k)$, and so on.







sum of product of $x(k)$ and $h(-k)$: $y(0) = 3 \times 3 = 9$

sum of product of $x(k)$ and $h(1-k)$: $y(1) = 1 \times 3 + 3 \times 2 = 9$

sum of product of $x(k)$ and $h(2-k)$: $y(2) = 2 \times 3 + 1 \times 2 + 3 \times 1 = 11$

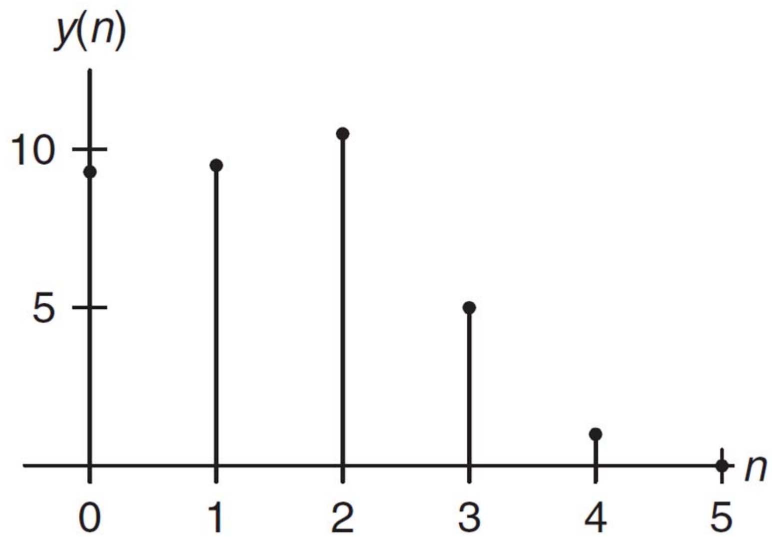
sum of product of $x(k)$ and $h(3-k)$: $y(3) = 2 \times 2 + 1 \times 1 = 5$

sum of product of $x(k)$ and $h(4-k)$: $y(4) = 2 \times 1 = 2$

sum of product of $x(k)$ and $h(5-k)$: $y(n) = 0$ for $n > 4$, since sequences $x(k)$ and $h(n-k)$ do not overlap.

Convolution (DT Systems) - Digital convolution using the graphical method

sum of product of $x(k)$ and $h(-k)$: $y(0) = 3 \times 3 = 9$
sum of product of $x(k)$ and $h(1-k)$: $y(1) = 1 \times 3 + 3 \times 2 = 9$
sum of product of $x(k)$ and $h(2-k)$: $y(2) = 2 \times 3 + 1 \times 2 + 3 \times 1 = 11$
sum of product of $x(k)$ and $h(3-k)$: $y(3) = 2 \times 2 + 1 \times 1 = 5$
sum of product of $x(k)$ and $h(4-k)$: $y(4) = 2 \times 1 = 2$
sum of product of $x(k)$ and $h(5-k)$: $y(n) = 0$ for $n > 4$, since sequences $x(k)$ and $h(n-k)$ do not overlap.



Convolution (DT Systems) - Digital convolution using the graphical method

Applying

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

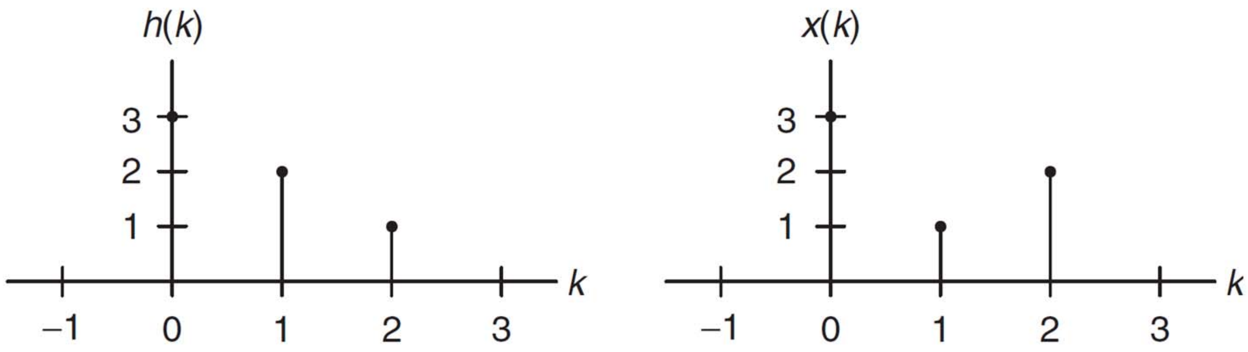
$$= \ldots + x(-1)h(n+1) + x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + \ldots$$

we get

$$y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2)$$

$$n = 0,\ y(0) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) = 3 \times 3 + 1 \times 0 + 2 \times 0 = 9$$
$$n = 1,\ y(1) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 3 \times 2 + 1 \times 3 + 2 \times 0 = 9$$
$$n = 2,\ y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$$
$$n = 3,\ y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = 3 \times 0 + 1 \times 1 + 2 \times 2 = 5$$
$$n = 4,\ y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = 3 \times 0 + 1 \times 0 + 2 \times 1 = 2$$
$$n \geq 5,\ y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) = 3 \times 0 + 1 \times 0 + 2 \times 0 = 0$$

Convolution (DT Systems) - Digital convolution using the table method



Convolution sum using the table method

k:	-2	-1	0	1	2	3	4	5	
$x(k)$:			**3**	**1**	**2**				
$h(-k)$:	1	2	3						$y(0) = 3 \times 3 = 9$
$h(1-k)$		1	2	3					$y(1) = 3 \times 2 + 1 \times 3 = 9$
$h(2-k)$			1	2	3				$y(2) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$
$h(3-k)$				1	2	3			$y(3) = 1 \times 1 + 2 \times 2 = 5$
$h(4-k)$					1	2	3		$y(4) = 2 \times 1 = 2$
$h(5-k)$						1	2	3	$y(5) = 0$ (no overlap)

Convolution (DT Systems) - Digital convolution using the table method

Example

Given the following two rectangular sequences,

$$x(n) = \begin{cases} 1 & n = 0,1,2 \\ 0 & otherwise \end{cases} \text{ and } h(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1,2 \\ 0 & otherwise \end{cases}$$

a. Convolve them using the table method.

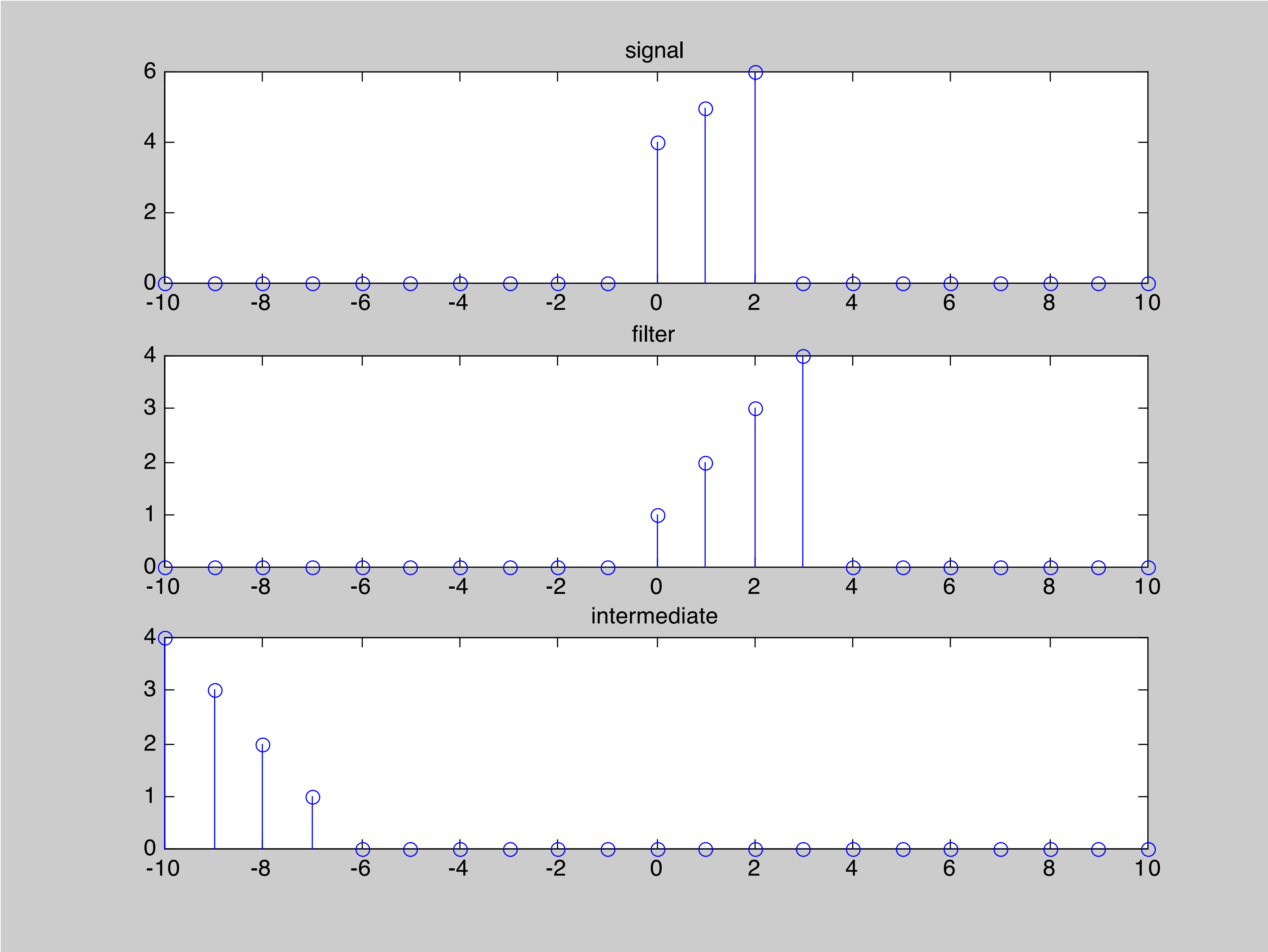
k:	-2	-1	0	1	2	3	4	5	...	
$x(k)$:			**1**	**1**	**1**				...	
$h(-k)$:	1	1	0							$y(0) = 0$ (no overlap)
$h(1-k)$		1	1	0						$y(1) = 1 \times 1 = 1$
$h(2-k)$			1	1	0					$y(2) = 1 \times 1 + 1 \times 1 = 2$
$h(3-k)$				1	1	0				$y(3) = 1 \times 1 + 1 \times 1 = 2$
$h(4-k)$					1	1	0			$y(4) = 1 \times 1 = 1$
$h(n-k)$						1	1	0		$y(n) = 0$, $n \geq 5$ (no overlap) Stop

Convolution (DT Systems) – Example using the graphical method and the table method

- $\{x(n)\} = \{\dots, 0, \underline{4}, 5, 6, 0, \dots\}$,
  $\{h(n)\} = \{\dots, 0, \underline{1}, 2, 3, 4, 0, \dots\}$ (position zero underlined)
- calculate $\{y(n)\} = \{x(n) * h(n)\} = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$
- need only calculate for $0 \leq n \leq (2+3)$
- $\{y(n)\} = \{\dots, 0, \underline{4}, 13, 28, 43, 38, 24, 0, \dots\}$

Note:

- if both x and h have finite duration, then convolution sum is non-zero only from $(n_{\text{begin } h} + n_{\text{begin } x})$ to $(n_{\text{end } h} + n_{\text{end } x})$


signal
filter
intermediate
6
4
3
2
1
0
-10
-8
-6
-4
-2
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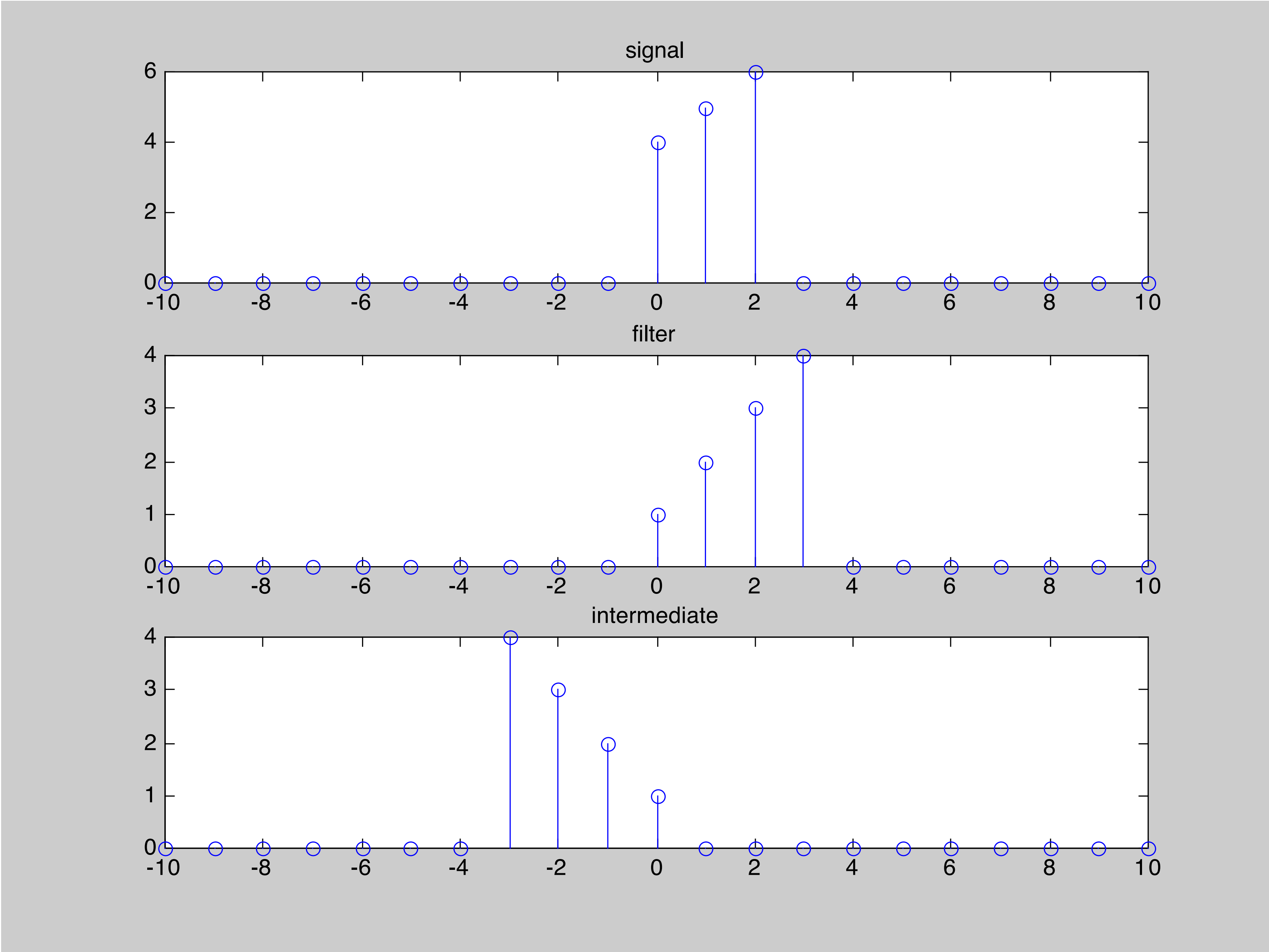
2
4
6
8
10



signal
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0
2
4
6
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filter
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1
0
-10
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-4
-2
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intermediate
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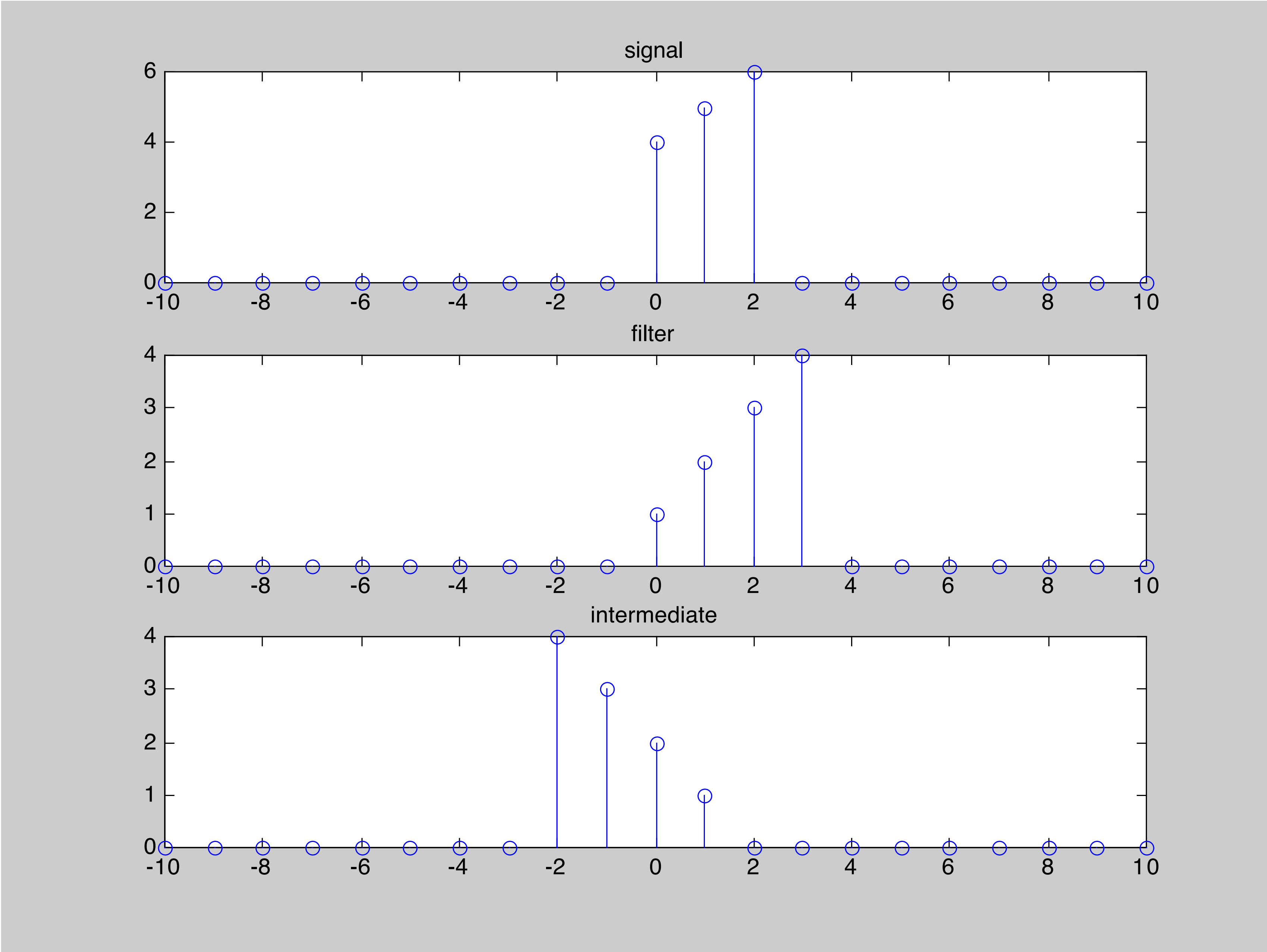
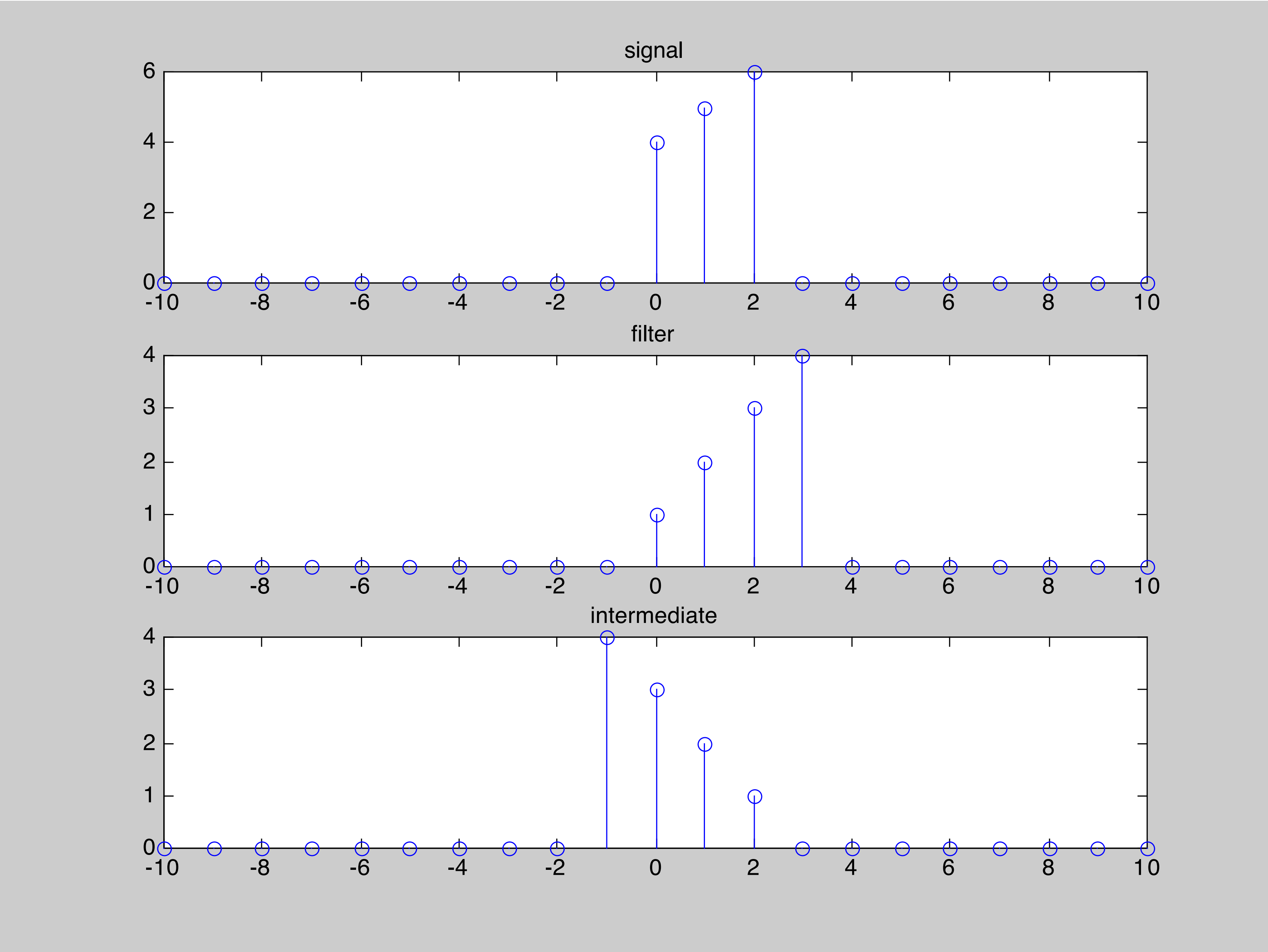

signal
filter
intermediate


signal
filter
intermediate
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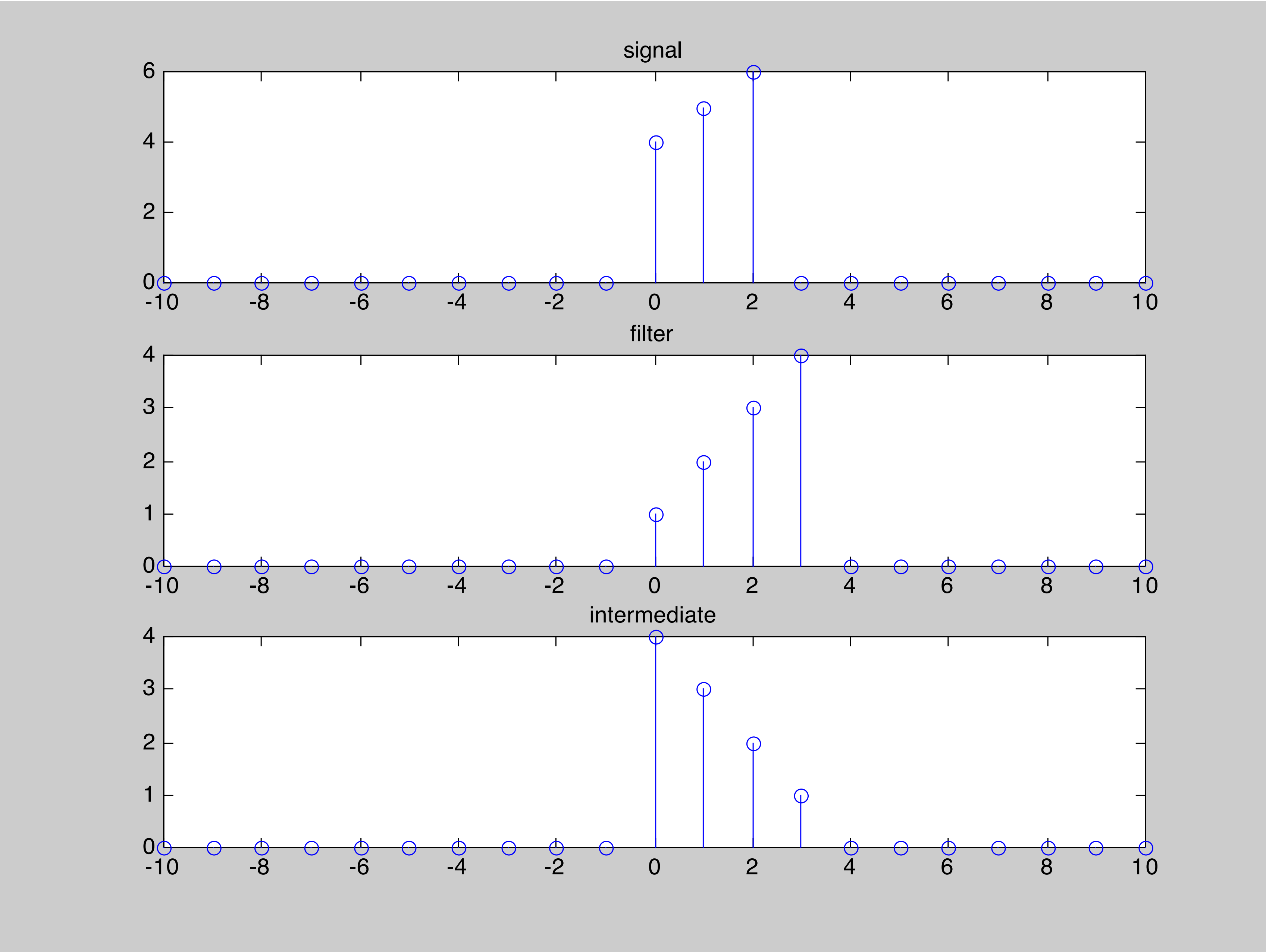
signal
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intermediate
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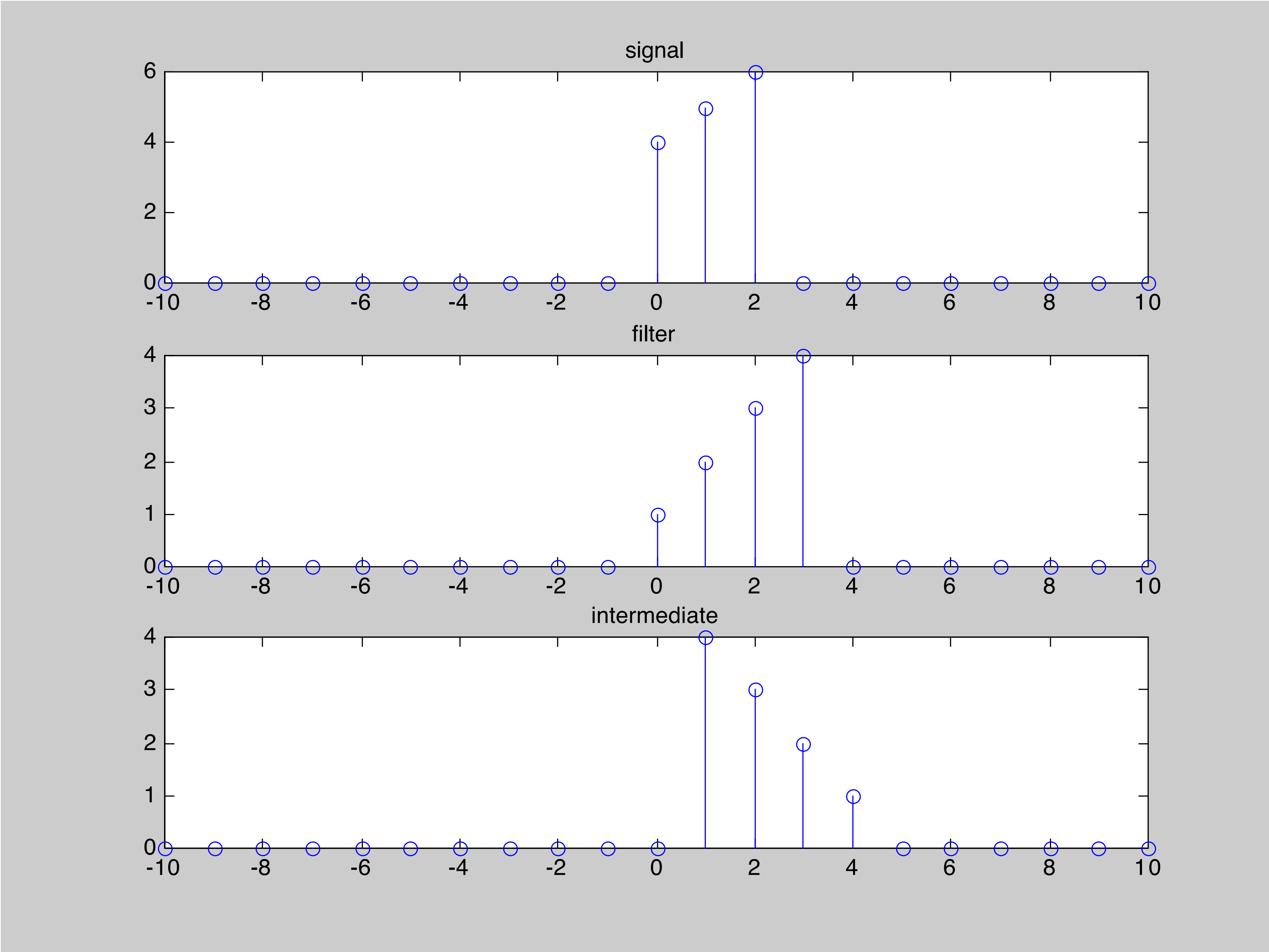
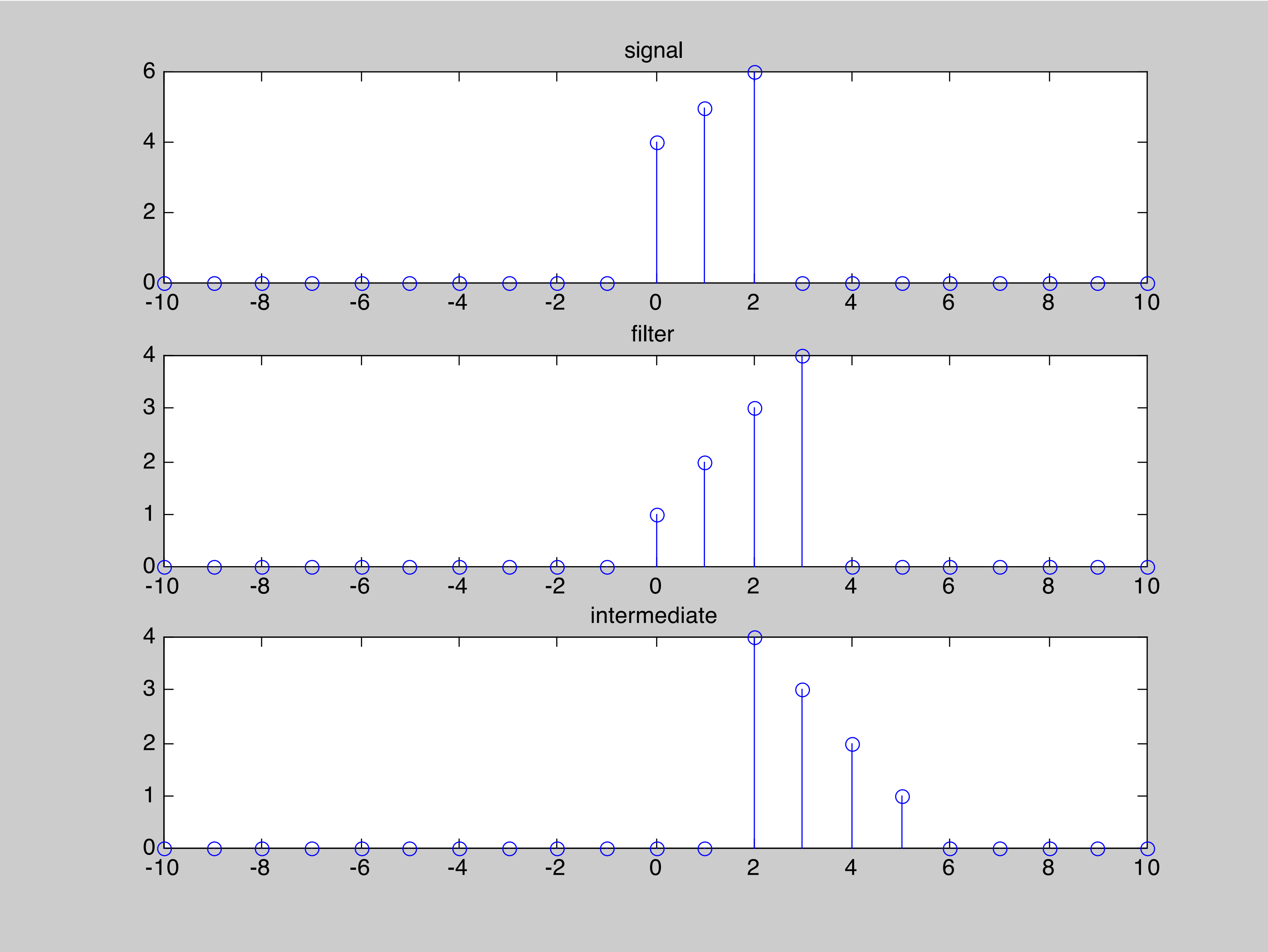

signal
filter
intermediate



signal
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filter
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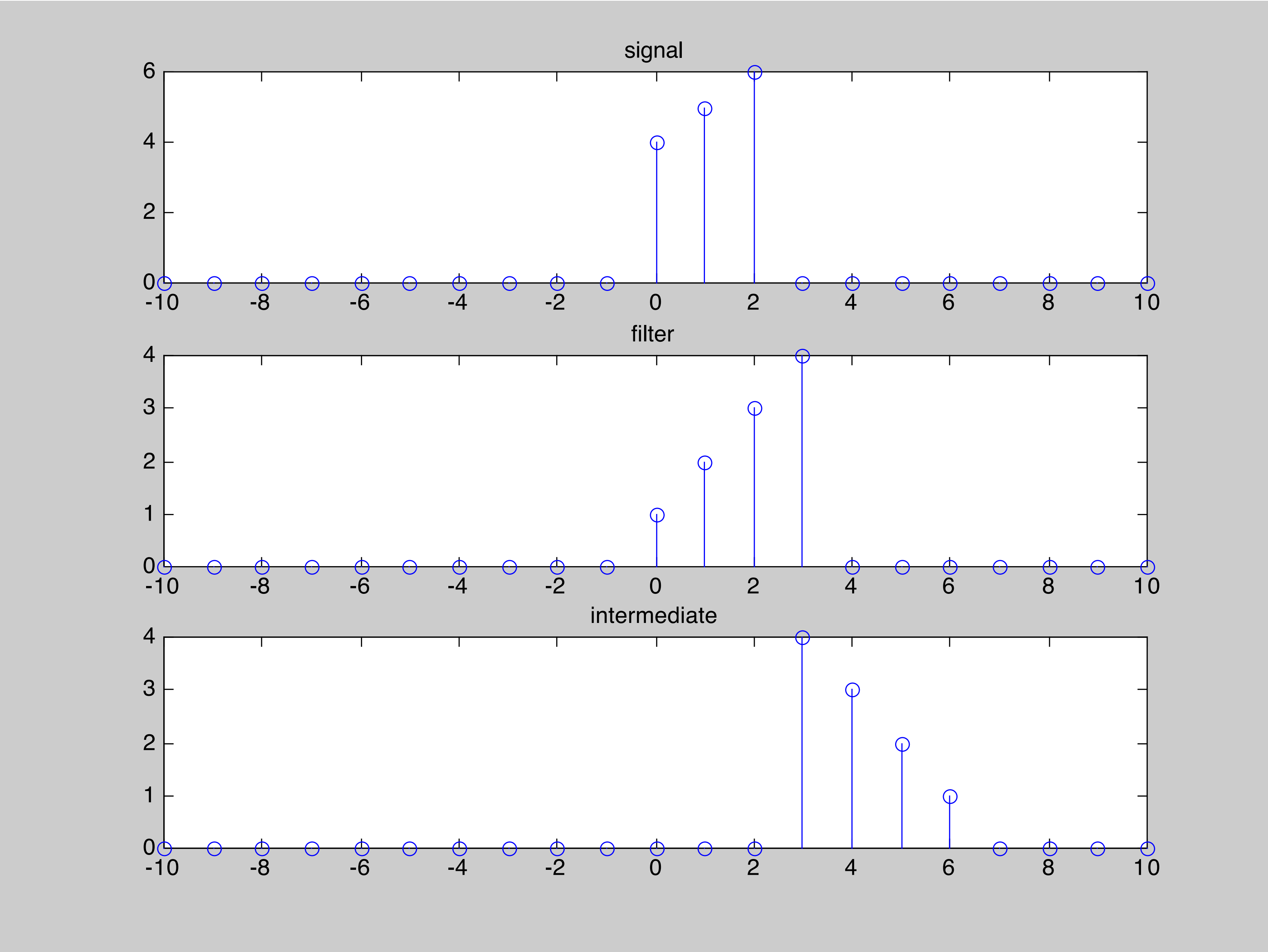
signal
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filter
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intermediate
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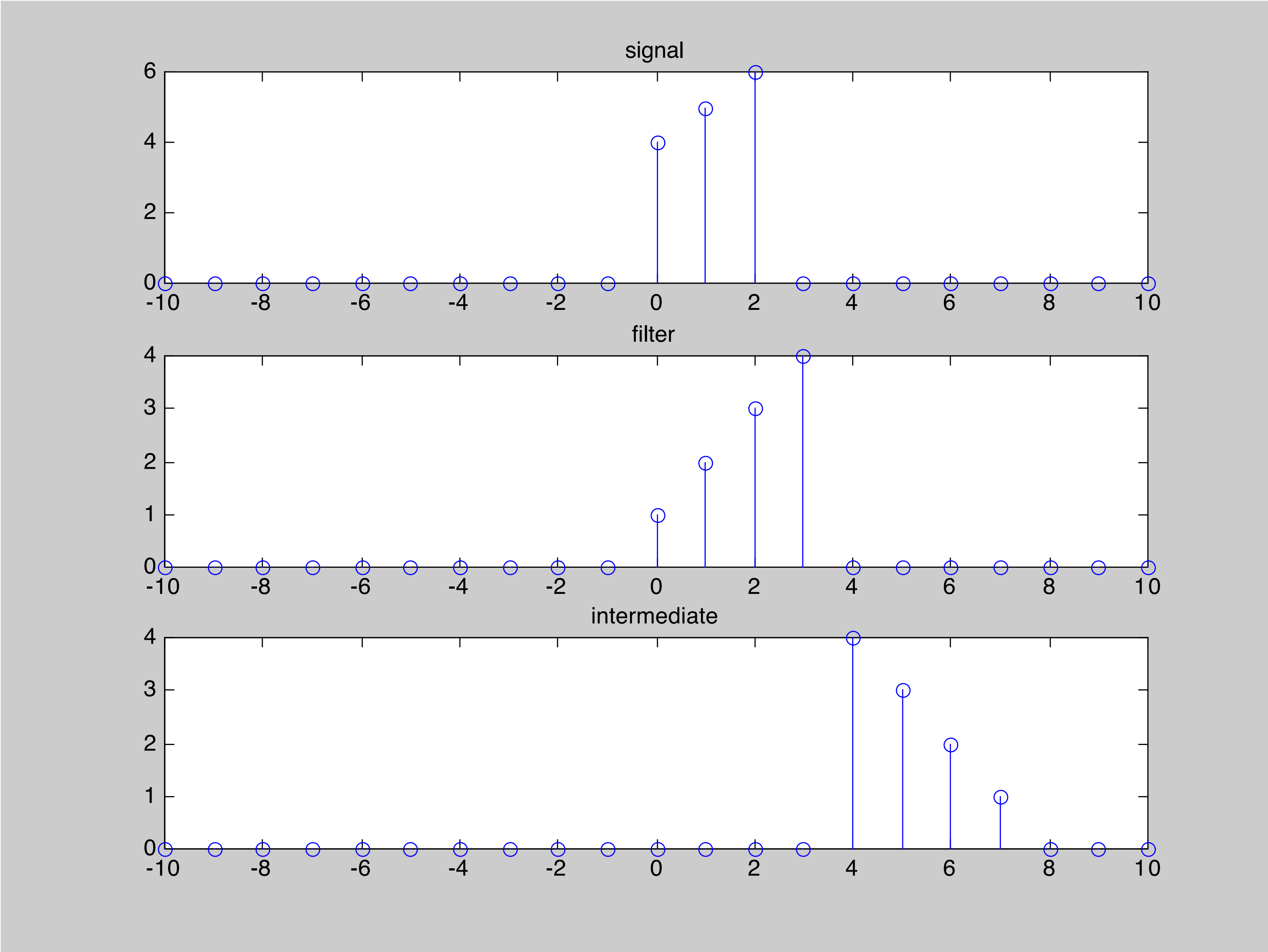
signal
filter
intermediate
6
4
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1
0
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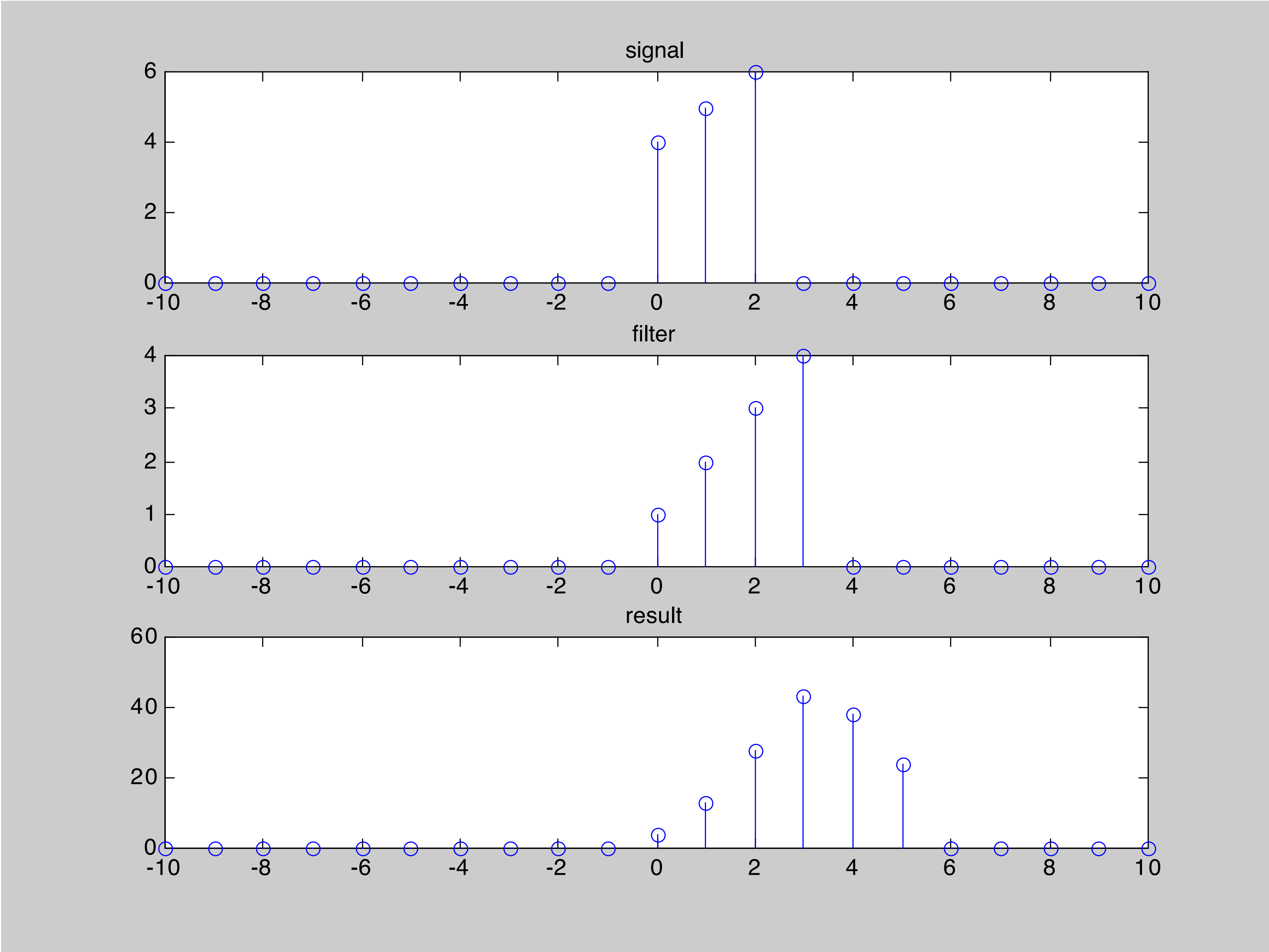
-8
-6
-4
-2
0
2
4
6
8
10


signal
filter
result
-10 -8 -6 -4 -2 0 2 4 6 8 10
0 2 4 6
0 1 2 3 4
0 20 40 60

Quick convolution by hand – example

- $\{x(n)\} = \{\ldots, 0, \underline{4}, 5, 6, 0, \ldots\}$
- $\{h(n)\} = \{\ldots, 0, \underline{1}, 2, 3, 4, 0, \ldots\}$

x(n)				4	5	6				
n										y(n)
0										
1										
2										
3										
4										
5										

Quick convolution by hand – example

- $\{x(n)\} = \{\ldots, 0, \underline{4}, 5, 6, 0, \ldots\}$
- $\{h(n)\} = \{\ldots, 0, \underline{1}, 2, 3, 4, 0, \ldots\}$

x(n)				4	5	6				
n										y(n)
0	4	3	2	1						
1		4	3	2	1					
2			4	3	2	1				
3				4	3	2	1			
4					4	3	2	1		
5						4	3	2	1	

Quick convolution by hand – example

- $\{x(n)\} = \{\ldots, 0, \underline{\mathit{4}}, 5, 6, 0, \ldots\}$
- $\{h(n)\} = \{\ldots, 0, \underline{\mathit{1}}, 2, 3, 4, 0, \ldots\}$

x(n)				4	5	6				
n										y(n)
0	4	3	2	1						4
1		4	3	2	1					13
2			4	3	2	1				28
3				4	3	2	1			43
4					4	3	2	1		38
5						4	3	2	1	24

Convolution (DT Systems) - Theorem

Theorem

Convolution is the time domain equivalent to multiplication in the frequency domain: if

$$Y(\omega) = X(\omega) \times H(\omega)$$

then

$$y(n) = x(n) * h(n)$$