Steady State Error & Error Constants

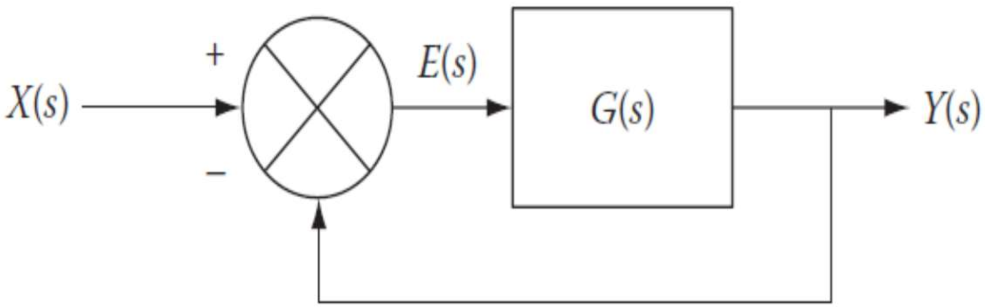
Source of info for week 2:

- Error Analysis: Theory from Dorf’s book, practice problem from Chapter 3 of the other book*
- Transient Response: Majority taken from the Richard Dorf’s book. + Selected parts of Chapter 4*.
- Stability: Selected part of chapter 5*
- Root Locus Analysis: Selected part of Chapter 6* [also some part from Richard Dorf’s book]
- * **Book:** Control System Problems Formulas, solutions, and simulation tools

Introduction

- The steady-state error $e_{ss}(t)$ is a factor that determines the operation of control systems.
- It is observed at the output of the system after the end of the transient response period.
- More specifically, the value of $e_{ss}(t)$ characterizes the final value of the error as a difference between the final value of the input $x(t)$ and the final value of the system response $y_{ss}(t)$.
- Generally speaking, the steady-state error of a stable closed-loop control system is much smaller than the associated error of an open-loop control system.

- Consider the following system:



- For the signal *E*(*s*) it holds that, $E(s) = X(s) - Y(s)$
- The quotient $\frac{E(s)}{X(s)}$ is called **error transfer function** and is defined by the following equation:

$$\frac{E(s)}{X(s)} = 1 - \frac{Y(s)}{X(s)} = \frac{1}{1+G(s)}$$

- The error $e(t)$ is $$e(t) = x(t) - y(t)$$

- The steady-state error $e_{ss}(t)$ is computed by

$$e_{ss}(t) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \overset{(3.3)}{=} \frac{sX(s)}{1 + G(s)}$$

- The above relationship is valid, provided that the roots of the equation sE(s) = 0 have negative real parts.
- The steady-state error of a system depends on the input signal

Position Error

1. The first case is that of a step input signal $x(t) = Au(t)$.
   In this case, the steady-state error is called *position error* and is given by

$$e_{ss}(t) = \frac{A}{1 + \lim_{s \to 0} G(s)} = \frac{A}{1 + k_p}$$

   where the term k_p is called *position error constant* and is given by

$$k_p = \lim_{s \to 0} G(s)$$

2. The second case is that of a ramp input signal $x(t) = A \cdot t$
   In this case, the steady-state error is called *velocity error* and is given by

$$e_{ss}(t) = \frac{A}{\lim_{s \to 0} sG(s)} = \frac{A}{k_v}$$

   where the term k_v is called *velocity error constant* and is given by

$$k_v = \lim_{s \to 0} sG(s)$$

3. The third case is that of a parabolic input signal $x(t)=\frac{1}{2}At^2$

In this case, the steady-state error is called *acceleration error* and is given by

$$e_{ss}(t)=\frac{A}{\lim_{s\to 0} s^2G(s)}=\frac{A}{k_a}$$

where the term k_a is called *acceleration error constant* and is given by

$$k_a=\lim_{s\to 0} s^2G(s)$$

- Finally, the steady-state error for any input signal $x(t)$ is computed as follows:

$$E(s) = \frac{1}{1+G(s)} \cdot X(s)$$

- By denoting $F(s) = \frac{1}{1+G(s)}$, above relation becomes $E(s) = F(s)G(s)$

- From the convolution property of Laplace transform, the error $e(t)$ is given by the convolution integral

$$e(t) = \int_0^t f(\tau)x(t-\tau)d\tau$$

- Applying Taylor series to the integral, and taking into account that

$$e_{ss}(t) = \lim_{t\to\infty} e(t)$$

- we obtain a general expression for the steady-state error for any input signal

$$e_{ss}(t) = \sum_{k=0}^{\infty} \frac{c_k}{k!} x_{ss}^{(k)}(t)$$

Types of Control System

- From, the following relation for generalised equation

$$G(s) = \frac{K \prod_{i=1}^{M} (s + z_i)}{s^N \prod_{k=1}^{Q} (s + p_k)},$$

- The value of S^N defines the type.

Steady-State Errors

		Steady-State Errors		
System Type	**Error Constants**	**Step Input** $x(t) = Au(t)$	**Ramp Input** $x(t) = A \cdot t$	**Parabolic Input** $x(t) = \frac{1}{2} A \cdot t^2$
0	Position: k_p const. Velocity: $k_v = 0$ Acceleration: $k_a = 0$	$\frac{A}{1+k_p}$	∞	∞
1	Position: $k_p \to \infty$ Velocity: k_v const. Acceleration: $k_a = 0$	0	$\frac{A}{k_v}$	∞
2	Position: $k_p \to \infty$ Velocity: $k_v \to \infty$ Acceleration: k_a const.	0	0	$\frac{A}{k_a}$

Error Constant and Transient Response
Practice Problem

Error Constant

k_p is called *position error constant* and is given by

$$k_p = \lim_{s \to 0} G(s)$$

k_v is called *velocity error constant* and is given by

$$k_v = \lim_{s \to 0} sG(s)$$

k_a is called *acceleration error constant* and is given by

$$k_a = \lim_{s \to 0} s^2 G(s)$$

- Compute the error constants (position, velocity, and acceleration) for the unity feedback systems with open-loop transfer functions:

$$G(s) = \frac{10}{(0.1s+1)(0.5s+1)}$$

$$G(s) = \frac{20}{s(s+2)(s+5)}$$

$$G(s) = \frac{100(s+1)}{s^2(s^2+4s+5)}$$

$$G(s) = \frac{k(s+2)}{s^3(s^2+2s+5)}$$

a. For $G(s)=\dfrac{10}{(0.1s+1)(0.5s+1)}$, the error constants are

$$k_p=\lim_{s\to 0}G(s)=\lim_{s\to 0}\frac{10}{(0.1s+1)(0.5s+1)}=10$$

$$k_v=\lim_{s\to 0}sG(s)=\lim_{s\to 0}\frac{10s}{(0.1s+1)(0.5s+1)}=0$$

$$k_a=\lim_{s\to 0}s^2G(s)=\lim_{s\to 0}\frac{10s^2}{(0.1s+1)(0.5s+1)}=0$$

b. For $G(s)=\dfrac{20}{s(s+2)(s+5)}$, the error constants are

$$k_p=\lim_{s\to 0} G(s)\to\infty$$

$$k_v=\lim_{s\to 0} sG(s)=2$$

$$k_a=\lim_{s\to 0} s^2 G(s)\to 0$$

c. For $G(s) = \dfrac{100(s+1)}{s^2(s^2+4s+5)}$, the error constants are

$$k_p = \lim_{s \to 0} G(s) \to \infty$$

$$k_v = \lim_{s \to 0} sG(s) \to \infty$$

$$k_a = \lim_{s \to 0} s^2 G(s) = 20$$

For $G(s)=\dfrac{k(s+2)}{s^3(s^2+2s+5)}$, the error constants are

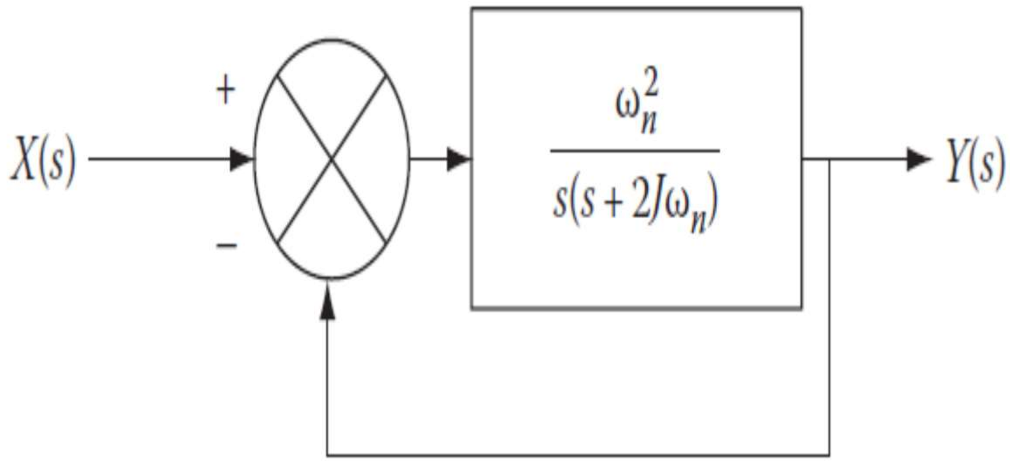
$$k_p=\lim_{s\to 0} G(s)\to\infty$$

$$k_v=\lim_{s\to 0} sG(s)\to\infty$$

$$k_a=\lim_{s\to 0} s^2G(s)\to\infty$$

Transient Response Practice Problem

The input of the system that is depicted in the following figure is a unit-step function. Given that $J = 0.6$ and $\omega_n = 5\,\text{rad/s}$, compute ω_d, t_r, t_p, M_p, and t_s.



1. $$t_r = \frac{1}{\omega_n \sqrt{1-J^2}} \tan^{-1}\left(-\frac{\sqrt{1-J^2}}{J} \right)$$ Rise time

2. $$t_p = \frac{\pi}{\omega_n \sqrt{1-J^2}}$$ Peak time

3. $$t_s = \frac{\pi}{J\omega_n} \quad \text{for } (\pm 2\%)$$ Settling time

or

$$t_s = \frac{3}{J\omega_n} \quad \text{for } (\pm 5\%)$$

4. $$M_p = \frac{y_m - y_f}{y_f} \cdot 100\% \quad \text{or} \quad M_p\% = 100 e^{-\frac{J\pi}{\sqrt{1-J^2}}}$$ Percent overshoot

Damped natural frequency $\omega_d = \omega_n \sqrt{1-J^2}$

Solution

- Note: there is slight notational difference between two books

The transfer function of the system is

$$G(s) = \frac{Y(s)}{X(s)} = \frac{\omega_n^2}{s^2 + 2J\omega_n s + \omega_n^2}$$

It is a second-order system with a unit-step input; hence,

$$\omega_d = \omega_n\sqrt{1-J^2} = 4 \text{ rad/s}$$

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\sqrt{1-J^2}/J\right) = \frac{\pi - \tan^{-1}\left(\sqrt{1-J^2}/J\right)}{\omega_d} = 0.553 \text{ s}$$

$$t_p = \frac{\pi}{\omega_d} = 0.785 \text{ s}$$

$$M_p = e^{-\frac{J\pi}{\sqrt{1-J^2}}} = 0.095$$

The percent overshoot and the settling time are computed as follows:

$$M_p\% = 9.5\%$$

$$t_s = \frac{4}{J\omega_n} = 1.33 \text{ s} \quad (\text{for } 2\%)$$

or

$$t_s = \frac{3}{J\omega_n} = 1 \text{ s} \quad (\text{for } 5\%)$$

The transfer function of a control system is

$$G(s) = \frac{k}{s^2 + 10s + k} = \frac{Y(s)}{X(s)}$$

where k is the gain of the system. Suppose that the input signal is the unit-step function and compute for k = 10, 100, and 1000

a. The undamped natural frequency ω_n.

b. The damping ratio J.

c. The damped natural frequency ω_d.

d. The roots of the characteristic equation $s_{1,2}$.

e. The maximum value of the gain k in order to have real negative roots of the characteristic equation.

f. The maximum percent overshoot M_p%.

g. The time response of the system $y(t)$.

h. Discuss the influence of the amplifier gain k upon the specifications of the system.

It is a second-order system with a unit-step input; therefore,

a. $$\omega_n = \sqrt{k} \Rightarrow \begin{array}{l} \text{For } k = 10, \quad \omega_n = 3.16 \text{ rad/s} \\ \text{For } k = 100, \quad \omega_n = 10 \text{ rad/s} \\ \text{For } k = 1000, \quad \omega_n = 31.6 \text{ rad/s} \end{array}$$

b. $$J = \frac{10}{2\omega_n} \Rightarrow \left. \begin{array}{l} \text{For } k = 10, \quad J = 1.58 \\ \text{For } k = 100, \quad J = 0.5 \\ \text{For } k = 1000, \quad J = 0.158 \end{array} \right\}$$

c. $$\omega_d = \omega_n \sqrt{1 - J^2} \Rightarrow \left. \begin{array}{l} \text{For } k = 10, \quad \omega_d \text{ is not defined as } J > 1 \\ \text{For } k = 100, \quad \omega_d \simeq 8.66 \text{ rad/s} \\ \text{For } k = 1000, \quad \omega_d \simeq 31.2 \text{ rad/s} \end{array} \right\}$$

d. $$s_{1,2} = -J\omega_n \pm \omega_n\sqrt{J^2-1} \Rightarrow \left.\begin{array}{l} \text{For } k=10, \quad \begin{cases} s_1 = -1.125 \\ s_2 = -8.875 \end{cases} \\ \text{For } k=100, \quad s_{1,2} = -5 \pm j8.66 \\ \text{For } k=1000, \quad s_{1,2} = -5 \pm j31.2 \end{array}\right\}$$

e. From the characteristic equation, we get

$$s^2 + 10s + k = 0 \Rightarrow s_{1,2} = \frac{-10 \pm \sqrt{100-4k}}{2}$$

For real and negative roots, it must hold that

$$100 - 4k \geq 0 \Rightarrow k \leq 25 \Rightarrow k_{max} = 25$$

$$\text{f. } M_p\% = 100 \cdot e^{-\frac{J\pi}{\sqrt{1-J^2}}} \Rightarrow \left.\begin{array}{l} \text{For } k = 10, \quad M_p\% \text{ is not defined} \\ \text{For } k = 100, \quad M_p\% \simeq 16.3\% \\ \text{For } k = 1000, \quad M_p\% \simeq 60\% \end{array}\right\}$$

h. i. If k increases, then the damping ratio decreases and the percent overshoot increases.

ii. The settling time for the 2% requirement is

$$t_s = \frac{4}{J\omega_n} = \begin{cases} 0.80115 \text{ s} & \text{for } k = 10 \\ 0.8 \text{ s} & \text{for } k = 100 \\ 0.80115 \text{ s} & \text{for } k = 1000 \end{cases}$$

The fastest settling time is for $k = 100$. In general, the settling time is minimized for $J = 0.707$.