Root Locus Analysis and Example

Additional Slides

Root Locus

- Root-locus analysis is a graphical method for examining how the roots of a system change under variation of a certain system parameter, commonly the gain of a feedback system.
- Consider the loop transfer function: $$G(s)H(s) = K\frac{P(s)}{Q(s)} \quad \text{(E.1)}$$

  where P(s) and Q(s) are polynomials of the complex variable s.
- The closed-loop transfer function that describes the dynamic behaviour of the system is
  $$\frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)H(s)} \overset{(6.1)}{=} \frac{G(s)Q(s)}{Q(s)+KP(s)} \quad \text{(E.2)}$$
- The roots of the characteristic equation are the **poles** of the closed-loop system. They can be computed by the relationship $Q(s)+KP(s)=0$ (E.3)

  where *K* is the gain of the system.

Root Locus

- The locations of the poles of the transfer function in the complex s-plane influence the transient response of the system and determine its stability.
- From relationship (E.3), we observe that every change in the value of the constant K results in the displacement of the poles in the complex plane.
- The root-locus diagram is a method for representing the poles of the closed-loop system on the s-plane, in relation to a system parameter (usually the gain K).
- From the root-locus diagram we obtain information about the stability and the overall behaviour of the system.
- The characteristic equation of the closed-loop system is $1+G(s)H(s)=0$ (E.3.1)

  Which can be rewritten as: $G(s)H(s)=-1$

Root Locus

- Continuing from the previous slide: $\Rightarrow |G(s)H(s)| = 1$ (E.4)
- And $\sphericalangle(G(s)H(s)) = (2\rho + 1)\pi, \quad \rho = 0, \pm 1, \pm 2, \ldots$ (E.5)
- Suppose that the open-loop transfer function is:

$$G(s)H(s) = K\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{E.6}$$

- So, the relationship (E.4) and (E.5) becomes:

$$|K|\frac{\prod_{j=1}^{m}|s+z_j|}{\prod_{i=1}^{n}|s+p_i|} = 1, \quad -\infty < K < \infty \tag{E.7}$$

- And

$$\sum_{j=1}^{m}\sphericalangle(s+z_j) - \sum_{i=1}^{n}\sphericalangle(s+p_i) = \begin{cases}(2\rho+1)\pi, & K \geq 0 \\ 2\rho\pi, & K < 0\end{cases}, \quad \rho = 0, \pm 1, \pm 2, \ldots \tag{E.8}$$

Root Locus

- The relationships (E.7) and (E.8) provide the magnitude-phase condition for the root locus.
- Once the root locus is drawn, the value of K for a specific point that corresponds to the root s1 can be determined from Equation (E.7).
- The root locus that fulfils the relationships (E.7) and (E.8) for $K \in (-\infty, 0)$ is called complementary root locus.

Designing a Root-Locus Diagram

- we will introduce a 10-step procedure for drawing the root-locus diagram of a control system:
- **STEP 1**: Branches start at the open-loop poles. The poles of $G(s)H(s)$ are called points of departure of the roots locus (RL).
- **STEP 2**: Branches end at the open-loop zeros or at infinity. These points are called points of arrival of the RL.
- **STEP 3**: The number of branches of the locus is equal to max(n, m), where m is the number of zeros and n is the number of the poles of $G(s)H(s)$.
- **STEP 4**: The root locus is symmetric to the real axis (horizontal axis).
- **STEP 5**: The intersection of the lines with the real axis can be found as:

$$\sigma_\alpha = \frac{\sum_{i=1}^{n}(p_i) - \sum_{j=1}^{m}(z_j)}{n-m} \tag{E9}$$

where $\sum_{i=1}^{n}(p_i)$ is the algebraic sum of the values of the poles of $G(s)H(s)$

$\sum_{j=1}^{m}(z_j)$ is the algebraic sum of the values of the zeros of $G(s)H(s)$

Designing a Root-Locus Diagram

- For large values of s, RL tends asymptotically to the lines that form the following angles with the real axis:

$$\sphericalangle\varphi_\alpha = \frac{(2\rho+1)\pi}{n-m}, \qquad \begin{array}{c}\rho = 0, 1, \ldots, |n-m|-1 \\ K \geq 0\end{array}$$

or

$$\sphericalangle\varphi_\alpha = \frac{2\rho\pi}{n-m}, \qquad \begin{array}{c}\rho = 0, 1, \ldots, |n-m|-1 \\ K \leq 0\end{array}$$

- **STEP 7**: Part of the real axis can be a segment of the RL if
  - For $K \geq 0$, the number of the poles and zeros that are at the right side of the segment is odd
  - For $K \leq 0$, the number of the poles and zeros that are at the right side of the segment is even
- **STEP 8**: The departure and arrival points are called **breakaway points** of the RL and can be found in two ways:

Designing a Root-Locus Diagram

- **STEP 8:** First Way: Taking differentiation of (E.3): $K = -\frac{Q(s)}{P(s)}$

$$\frac{dK}{ds} = 0 \overset{(6.14)}{\Rightarrow} -\frac{Q'(s)P(s) - P'(s)Q(s)}{P^2(s)} = 0 \Rightarrow$$

$$Q'(s)P(s) - P'(s)Q(s) = 0 \qquad \text{(E.10)}$$

- Every root of the Equation (E.10) is accepted as a breakaway point if it satisfies the condition (E.3.1) for any real value of K.
- **STEP 8:** Second Way: If the poles and zeros of $G(s)H(s)$ are real numbers, instead of (E.10) we can solve the following equation:

$$\sum_{i=1}^{n} \frac{1}{s - p_i} = \sum_{j=1}^{m} \frac{1}{s - z_j} \qquad \text{(E.11)}$$

$$\sphericalangle \varphi_d = (2\rho + 1)\pi - \left(\sum_{i=1}^{n} \varphi_{p_i} - \sum_{j=1}^{m} \varphi_{z_j} \right)$$

Designing a Root-Locus Diagram

- **STEP 9:** The angles of departure of the RL from a complex pole or the angles of arrival at a complex zero can be found as

$$\sphericalangle \varphi_d = (2\rho + 1)\pi - \left(\sum_{i=1}^{n} \varphi_{p_i} - \sum_{j=1}^{m} \varphi_{z_j} \right) \qquad \text{(E.12)}$$

- Where

$\sum_{i=1}^{n} \varphi_{p_i}$ is the algebraic sum of the angles formed by the poles and the relevant complex pole (or zero)

$\sum_{j=1}^{m} \varphi_{z_j}$ is the algebraic sum of the angles formed by the zeros and the relevant complex pole (or zero)

- **STEP 10:** The intersections of the root locus and the imaginary axis (vertical axis) are the points **$\pm j\omega_c$**, where the system from stable becomes unstable. They can be computed with the use of Routh's stability criterion.

Steps for Designing the Root-Locus Diagram

S/N	Formulas for Designing a Root-Locus Diagram	Remarks
1	$G(s)H(s) = K\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$	Open-loop transfer function
2	$\lvert K\rvert \frac{\prod_{i=1}^{m}\lvert s+z_i\rvert}{\prod_{j=1}^{n}\lvert s+p_j\rvert} = 1 \quad -\infty < K < \infty$	Magnitude condition for the points of the root locus
3	$\sum_{i=1}^{m} \sphericalangle(s+z_i) - \sum_{j=1}^{n} \sphericalangle(s+p_j) = \begin{cases} (2\rho+1)\pi, & K>0 \\ 2\rho\pi, & K<0 \end{cases}$ $\rho = \pm 1, \pm 2, \ldots$	Phase condition for the points of the root locus
4	$l = \max(m, n)$	Number of branches of the root locus
5(a)	$\sphericalangle\varphi_\alpha = \frac{(2\rho+1)\pi}{n-m}, \quad \begin{cases} \rho = 0, 1, \ldots, \lvert n-m\rvert - 1 \\ K \geq 0 \end{cases}$	Angles of asymptotes with the real axis for $K \geq 0$
5(b)	$\sphericalangle\varphi_\alpha = \frac{2\rho\pi}{n-m}, \quad \begin{cases} \rho = 0, 1, \ldots, \lvert n-m\rvert - 1 \\ K \leq 0 \end{cases}$	Angles of asymptotes with the imaginary axis for $K \leq 0$
6	$\sigma_\alpha = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n-m}$	Intersection of asymptotes with the real axis
7(a)	$\begin{cases} \frac{dK}{ds} = 0 \Rightarrow s_{b_i} \\ 1 + G(s_b)H(s_b) = 0 \quad \text{for } K \in R \end{cases}$	Computation of the breakaway points s_b (first way)
7(b)	$\sum_{i=1}^{n} \frac{1}{s-p_i} = \sum_{j=1}^{m} \frac{1}{s-z_j} p_i, \quad z_j \in R$	Computation of the breakaway points s_b (second way)
8	$\sphericalangle\varphi_d = (2\rho+1)\pi - \left(\sum_{i=1}^{n} \varphi_{p_i} - \sum_{j=1}^{m} \varphi_{z_j} \right)$	Angles of departure of the RL from complex poles or angles of arrival to complex zeros
9	$s_c = \pm j\omega_c$	Intersection points of the RL with the imaginary axis

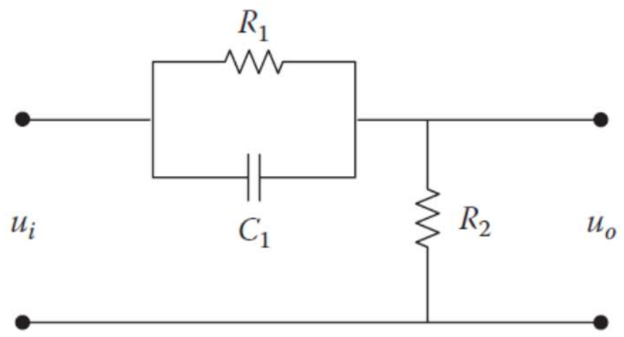
Design of a Control System with the Use of the Root Locus

- When designing a control system, we seek to adjust the time response and the frequency response to the technical requirements of the system.

- In doing so, we need to redistribute and add new poles or zeros in the open-loop transfer function G(s)H(s) of the system. For this purpose, we can introduce controllers to the system, as follows:
  - By connecting a controller in series with the control units of the system
  - By connecting a controller as a feedback loop in the system
  - By connecting a controller in parallel to one or more control units of the system

Phase-Lead Controller

- A phase-lead circuit is depicted in the below figure



- The transfer function of the circuit is

$$G(s) = \frac{V_o(s)}{V_i(s)} = \frac{R_2}{R_1 + R_2} \cdot \frac{sR_1C_1 + 1}{(R_1R_2C_1/(R_1 + R_2)) + 1} \qquad \text{(E.13)}$$

- It can be written as

$$G(s) = K\frac{(sT_1 + 1)}{(sT_2 + 1)} = \frac{1 + aTs}{a(1 + Ts)} \qquad \text{(E.14)}$$

- where

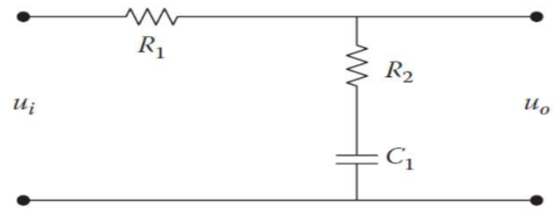
$$\left.\begin{aligned} &K = \frac{R_2}{R_1 + R_2}, \quad a = \frac{R_1 + R_2}{R_2} \\ &T_1 = R_1C_1 \\ &T_2 = \frac{R_1R_2C_1}{R_1 + R_2} = KT_1 = T \\ &T_1 > T_2 \end{aligned}\right\} \qquad \text{(E.15)}$$

The phase-lead controller is usually used to provide a sufficient phase margin for a system.
With the use of phase-lead controller, we achieve

- Reduction of the rise time T_s
- Stability
- Increase of the critical gain K_c and of the critical frequency of oscillation ω_c

Phase-Lag Controller

- A phase-lag circuit is shown in the following figure:



- The transfer function of the circuit is

$$G(s) = \frac{V_o(s)}{V_i(s)} = \frac{sR_2C_1 + 1}{s(R_1 + R_2)C_1 + 1} \qquad \text{(E.16)}$$

- It can be written as: $G(s) = \dfrac{(sT_1 + 1)}{(sT_2 + 1)} = \dfrac{1}{a}\dfrac{s+z}{s+p}$ (E.17)

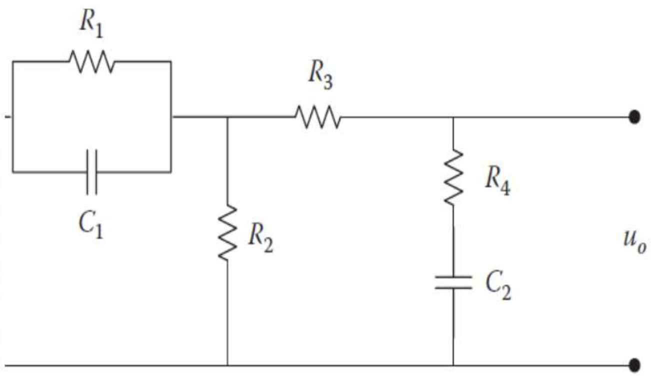
where
$$\left.\begin{aligned} &T_1 = R_2C_1 \\ &T_2 = (R_1 + R_2)C_1 \\ &T_1 < T_2 \\ &a = \frac{R_1 + R_2}{R_2}, \quad z = 1/T_1, \quad p = 1/T_2 \end{aligned}\right\} \qquad \text{(E.18)}$$

With the use of the phase-lag controller, we can reform the RL and determine the desired root locus in order to increase, for instance, its relative stability.
The effects of phase-lag compensation result in

- An increase of the rise time T_s
- An increase of the total static gain of the system
- The reduction of the steady state error e_{ss}

Lead-Lag Controller

- The circuit of a lead-lag controller is shown in the following figure:



- The transfer function of the system is:

$$G(s) = \frac{V_o(s)}{V_i(s)}$$

$$K = \frac{R_2(R_3+R_4)}{R_1R_2+R_1R_3+R_1R_4+R_2R_3+R_2R_4}$$

$$T_1 = R_1C_1$$

$$T_2 = R_4C_2$$

$$T_3 = (R_3+R_4)C_2$$

$$T_4 = \frac{R_1R_2(R_3+R_4)C_1}{R_1R_2+R_1R_3+R_1R_4+R_2R_3+R_2R_4}$$

- Which can be rewritten as:

$$= \frac{R_2(sR_4C_2+1)\cdots}{R_1R_2(R_3+R_4)C_1C_2\left[s+\frac{1}{(R_3+R_4)C_2}\right]\left[s+\frac{R_1R_2+R_1R_3+R_1R_4+R_2R_3+R_2R_4}{R_1R_2(R_3+R_4)C_1}\right]} \quad \text{(E.19)}$$

- Or,

$$G(s) = K\frac{(sT_1+1)(sT_2+1)}{(sT_3+1)(sT_4+1)} \quad \text{(E.20)}$$

Lead-Lag Controller

Where:

$$\left.\begin{aligned}
K &= \frac{R_2(R_3+R_4)}{R_1R_2+R_1R_3+R_1R_4+R_2R_3+R_2R_4} \\
T_1 &= R_1C_1 \\
T_2 &= R_4C_2 \\
T_3 &= (R_3+R_4)C_2 \\
T_4 &= \frac{R_1R_2(R_3+R_4)C_1}{R_1R_2+R_1R_3+R_1R_4+R_2R_3+R_2R_4}
\end{aligned}\right\} \qquad \text{(E.21)}$$

- Lead-lag controllers combine the advantages of the two controllers, but one has to be careful in the design in order to exploit the properties of the controller for different parts of the time response.

Lead-Lag Controller

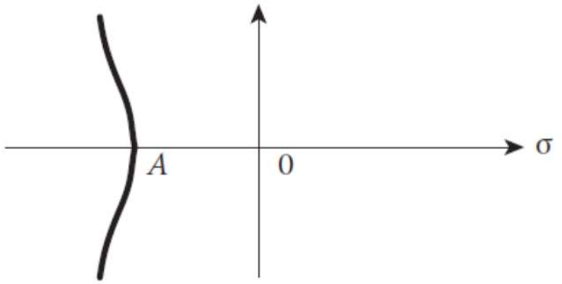
- The system compensation process with the use of root loci and controller configurations can be described as follows:

1. The system requirements are associated to the desired dominant roots.
2. The root locus of the system is drawn and we examine if the needed roots belong to the root locus.
3. We choose the most suitable controller and determine its transfer function.
4. The new pole is determined so that the angle condition is satisfied. This means that the angle of the location of the desired root must be 180° and thus the root belongs to the new root locus of the system that includes the controller.
5. The total gain of the system is computed for the desired root. We can also calculate the error constant of the compensated system.
6. If the error is not acceptable, then we repeat the design process.

Problems to Solve (1)

1. The following figure depicts with a bold line a segment of a root locus of the characteristic equation of a system with open-loop transfer function

$$GH(s) = \frac{K(s+(10/3))}{s(s+3)(s+6)}, \quad K > 0$$

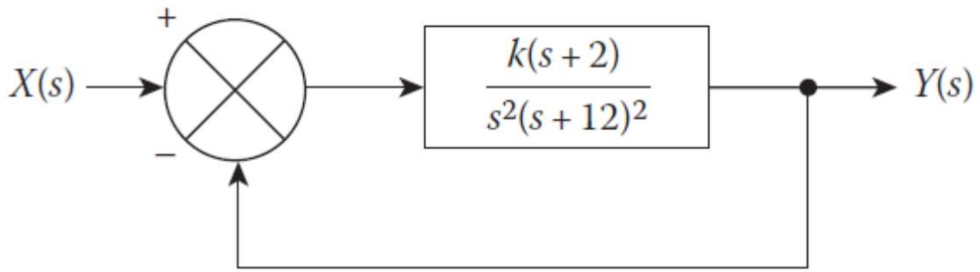


$$GH(s) = \frac{K(s+(10/3))}{s(s+3)(s+6)}, \quad K > 0$$

a) Plot the rest of the straight-line segments of the locus, which are on the real axis.
b) Mark the direction of the locus for every segment.
c) Find the abscissa of point A.
d) What is the value of K at the point A?
e) Find the asymptotes of the locus.
f) Discuss the stability of the system.

Problems to Solve (2)

2. For the system shown in the following figure, design the RL diagram of the characteristic equation for K > 0, and find out the values of K for which the system is stable.



3. The loop transfer function of a system is

$$G(s)H(s) = \frac{K}{s(s^2+4s+8)}, \quad K > 0$$

   a) Find the asymptotes and the angles of departure of the RL.
   b) Compute the critical value of K so that the closed-loop system is stable, and find the intersections of RL with the imaginary axis.
   c) Plot the RL of the characteristic equation of the system.

Problems to Solve (3)

4. Given the following loop transfer function sketch the RL of the characteristic equation for K > 0 and for K < 0.

$$GH(s) = \frac{K}{s(s+1)(s+3)(s+4)}$$

5. Plot the RL of the characteristic equation of the system when the loop transfer function is

$$G(s)H(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+10)}$$ for, K>0.

6. The loop transfer function of a system is given by

$$G(s)H(s) = \frac{K}{s(s+1)(s^2+4s+8)}, \quad K>0$$

   a) Find the asymptotes and the angles of departure of the RL.
   b) Find the breakaway points (if there are any) of the RL. Take into account that one root of the equation $4s^3 + 15s^2 + 24s + 8 = 0$ is $s = -1.6549 + j1.3432$.
   c) Find the critical value of K so that the system is stable.
   d) Plot the RL.