Laplace Tutorial 9th and 10th October 2019 Solutions

Q1 Find the Laplace transform of $f(t) = \cosh(at)$.

Answer:

We know that $\cosh at = \frac{e^{at}+e^{-at}}{2}$ [1 mark]

and $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{dt} = F(s)$ [1 mark]

Now, $L\{\cosh at\} = \frac{1}{2}[L\{e^{at} + e^{-at}\}]$ [1 mark]

$= \frac{1}{2}[L(e^{at}) + L(e^{-at})]$ [1 mark]

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$ [1 mark]

$\therefore L[cosh\,at] = \dfrac{s}{s^2 - a^2}$ [1 mark]

Q2. A linear system is described by the following differential equation:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 2\frac{du}{dt} + u(t)$$

Find the system poles and zeros and plot them in the s-plane. Is the system stable?

Solution: From the differential equation the transfer function is

$$H(s) = \frac{2s+1}{s^2+5s+6}.$$ (1 mark)

which may be written in factored form

$$\begin{aligned} H(s) &= \frac{1}{2}\frac{s+1/2}{(s+3)(s+2)} \\ &= \frac{1}{2}\frac{s-(-1/2)}{(s-(-3))(s-(-2))}. \end{aligned}$$ (1 mark) (1 mark)

The system therefore has a single real zero at $s = -1/2$, and a pair of real poles at $s = -3$ and $s = -2$. (1 mark)

System is stable.

Q3. Calculate the Laplace transform of $f(t) = \sinh(at)$.

Answer:

We know that $\sinh at = \frac{e^{at} - e^{-at}}{2}$ [1 mark]

and $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, \mathrm{dt} = F(s)$ [1 mark]

Now, $L\{\sinh at\} = \frac{1}{2}[L\{e^{at} - e^{-at}\}]$ [1 mark]

$= \frac{1}{2}[L(e^{at}) - L(e^{-at})]$ [1 mark]

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$ [1 mark]

$\therefore L[sinh\, at] = \frac{a}{s^2 - a^2}$ [1 mark]

Q4. Calculate the inverse Laplace transform for $Y(s) = \frac{10}{s(s^2+5s+4)}$

Answer:

$$Y(s) = \frac{10}{s(s^2+5s+4)} = \frac{10}{s(s+1)(s+4)}.$$

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+4} + \frac{C}{s}.$$

$$A = Y(s)(s+1)\Big|_{s=-1} = \frac{10}{s(s+4)}\Bigg|_{s=-1} = -\frac{10}{3}$$

$$B = Y(s)(s+4)\Big|_{s=-4} = \frac{10}{s(s+1)}\Bigg|_{s=-4} = \frac{10}{12}$$

$$C = sY(s)\Big|_{s=0} = \frac{10}{(s+1)(s+4)}\Bigg|_{s=0} = \frac{10}{4}$$

$$Y(s) = 10\left[-\frac{1}{3(s+1)} + \frac{1}{12(s+4)} + \frac{1}{4s}\right].$$

$$x(t) = 10\left[-\frac{1}{3}e^{-t} + \frac{1}{12}e^{-4t} + \frac{1}{4}\right].$$

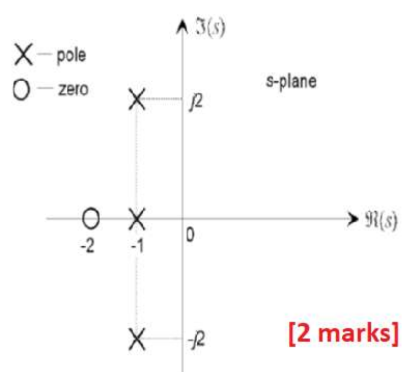
Q5. A system has a pair of complex conjugate poles, $p_1, p_2 = -1 \pm j2$, a single real zero, $z_1 = -4$, and a gain factor $K = 3$. Find the differential equation representing the system. Plot the poles and zeros and comment on stability.

Solution: The transfer function is

$$\begin{aligned} H(s) &= K\frac{s-z}{(s-p_1)(s-p_2)} \quad \textbf{[1 mark]} \\ &= 3\frac{s-(-4)}{(s-(-1+j2))(s-(-1-j2))} \quad \textbf{[1 mark]} \\ &= 3\frac{(s+4)}{s^2+2s+5} \quad \textbf{[1 mark]} \end{aligned}$$

and the differential equation is

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 3\frac{du}{dt} + 12u \quad \textbf{[1 mark]}$$



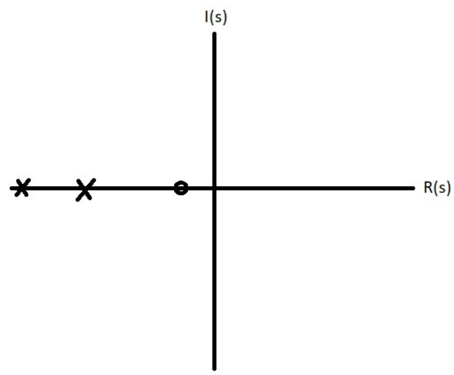
[2 marks]

Stable.

Q6. A linear system is described by the following differential equation:

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 12y = 3\frac{du}{dt} + 2u(t)$$

Determine the system poles and zeros and plot them in the s-plane. Is the system stable?


I(s)
R(s)

Q7. Calculate the inverse Laplace transform for $Y(s) = \frac{s+7}{(s^2+6s+1\ \ )}$

Answer:

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$$F(s) = \frac{s+7}{s^2+6s+13}$$

$$= \frac{s+7}{(s+3)^2+4} \quad [1\ \text{MARK}]$$

$$= \frac{s+3}{(s+3)^2+4} + \frac{4}{(s+3)^2+4} \quad [1\ \text{MARK}]$$

$$= \frac{s+3}{(s+3)^2+4} + 2 \cdot \frac{2}{(s+3)^2+4} \quad [1\ \text{MARK}]$$

REFERING TO LAPLACE TRANSFORM TABLE:

$$f(t) = e^{-3t} \cdot \cos 2t + 2e^{-3t} \sin 2t \quad [1\ \text{MARK}]$$

$$= e^{-3t}[\cos 2t + 2 \sin 2t] \quad [1\ \text{MARK}]$$

$$= e^{-3t} \cdot \sqrt{1^2+2^2} \cdot \cos\left(2t + \tan^{-1}\frac{2}{1}\right)$$

$$= \underline{2.24 \cdot e^{-3t} \cdot \cos(2t + 63.4^\circ)} \quad [2\ \text{MARKS}]$$