EBU4375 Signals and Systems Theory

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Basic Time Signals

Basic Continuous-Time Signals

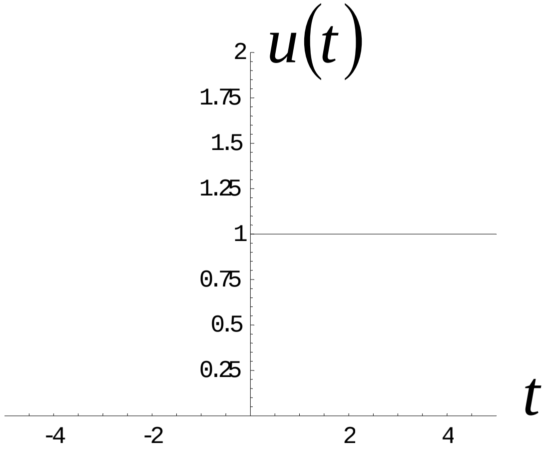
- The Unit-Step Function
- The Unit-Impulse Function
- Complex Exponential and Sinusoidal Signals

Basic Discrete-Time Signals

- The Unit-Step Sequence
- The Unit-Impulse Sequence
- Complex Exponential and sinusoidal Sequence

The Unit-Step Function (CT Signals)

- the unit (or Heaviside) step function is <u>defined</u> as

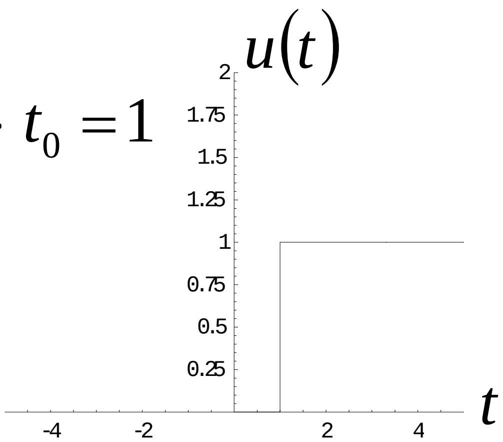


$$u(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$$

i.e. it's discontinuous at $t=0$

- the shifted (retarded) step function is similarly defined as

e.g. $t_0=1$

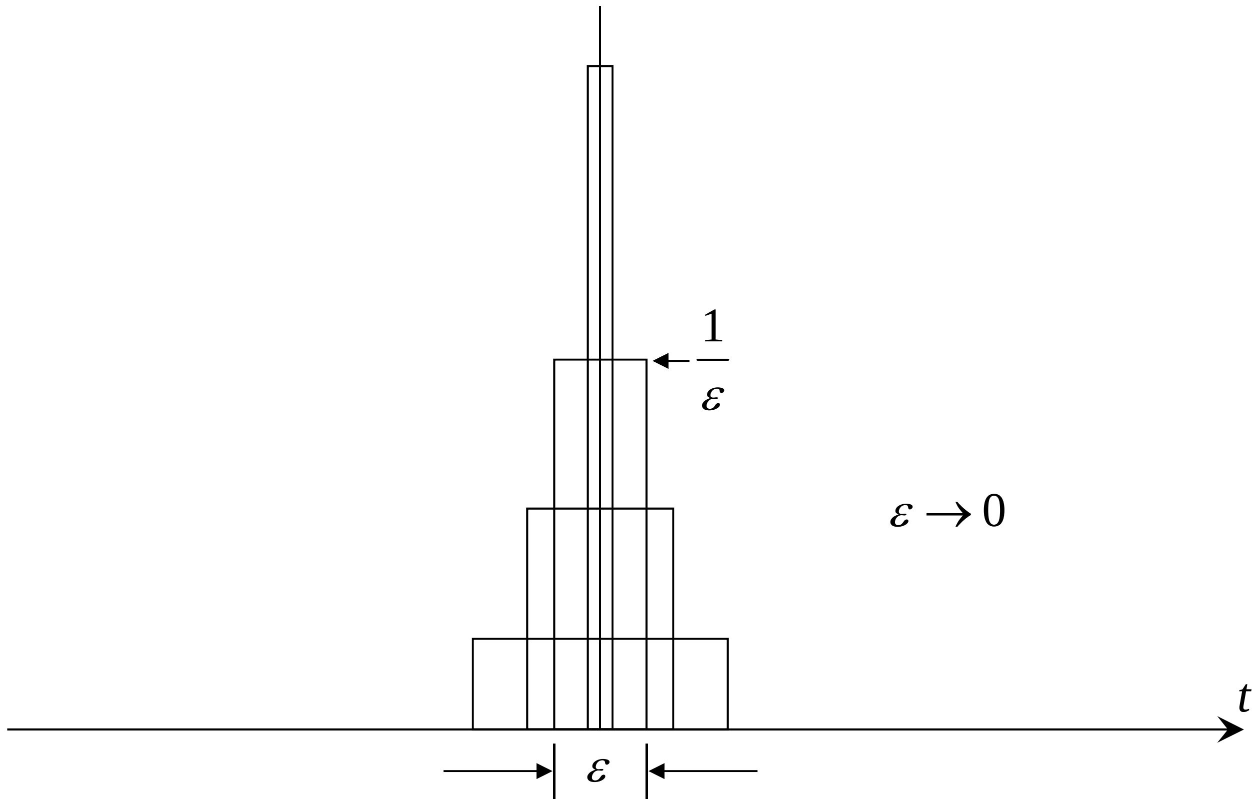


$$u(t-t_0)=\begin{cases}1 & t>t_0\\ 0 & t<t_0\end{cases}$$

The Unit-Impulse Function (CT Signals)

The unit-impulse (Dirac-delta) function is defined as

$$\delta(t) \equiv \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \quad \rightarrow \quad \int_{-\varepsilon}^{\varepsilon} dt\, \delta(t) \equiv 1$$

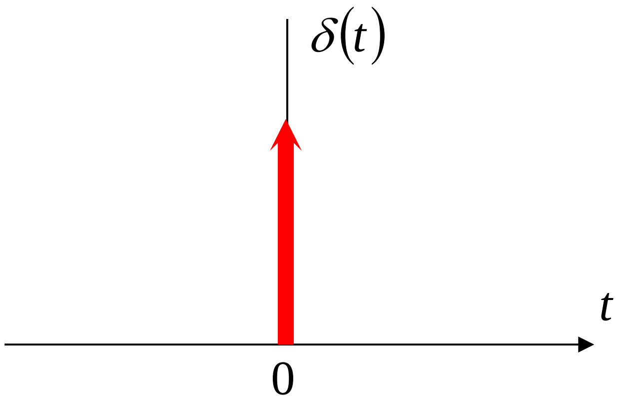


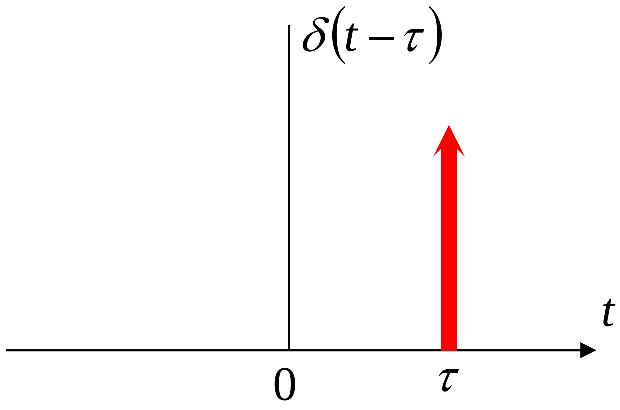
The Unit-Impulse Function (CT Signals)

- It is also defined by

$$\int_a^b dt\,\phi(t)\delta(t) \equiv \begin{cases} \phi(0) & a < 0 < b \\ 0 & a < b < 0 \text{ or } 0 < a < b \\ \text{undefined} & a = 0 \text{ or } b = 0 \end{cases}$$

- A delayed (retarded) delta function $\delta(t-\tau)$ is defined by $\int_{-\infty}^{\infty} dt\,\phi(t)\delta(t-\tau) = \phi(\tau)$ (1)





The Unit-Impulse Function (CT Signals)

Properties of $\delta(t)$:

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

$$\delta(-t) = \delta(t) \qquad (2)$$

$$x(t)\delta(t) = x(0)\delta(t) \qquad \text{(if } x(t) \text{ is continuous at } t = 0\text{)}$$

$$x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau) \qquad \text{(if } x(t) \text{ is continuous at } t = \tau\text{)}$$

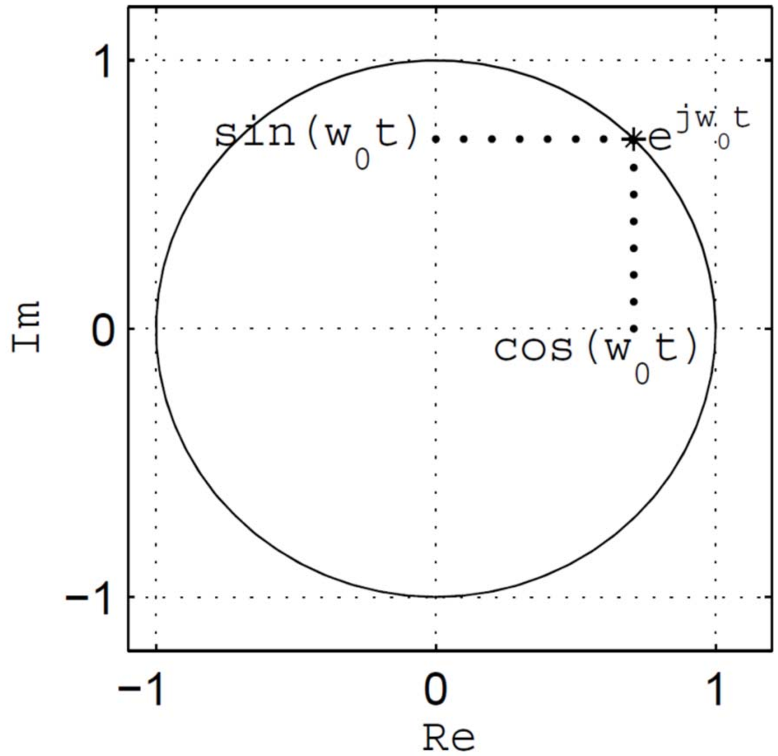
A continuous-time signal $x(t)$ may be expressed as (we prove this in the following lecture)

$$x(t) = \int_{-\infty}^{\infty} d\tau\, x(\tau)\delta(t-\tau)$$

Complex Exponential and Sinusoidal (CT Signals)

Euler's formula: $e^{jw_0 t} = \underbrace{\cos(w_0 t)}_{\mathrm{Re}\{e^{jw_0 t}\}} + j \underbrace{\sin(w_0 t)}_{\mathrm{Im}\{e^{jw_0 t}\}}$

where $j = \sqrt{-1}$, $w_0 \neq 0$ is real, and t is the time.



Complex Exponential and Sinusoidal (CT Signals)

Since

$$e^{jw_0\left(t+\frac{2\pi}{|w_0|}\right)} = e^{jw_0t}e^{j2\pi\frac{w_0}{|w_0|}} = e^{jw_0t}\underbrace{e^{j2\pi\mathrm{sign}(w_0)}}_{=1} = e^{jw_0t}$$

we have

$$e^{jw_0t} \text{ is periodic with fundamental period } \frac{2\pi}{|w_0|}$$

Note that

- $e^{j2\pi k} = 1$, for $k = 0, \pm 1, \pm 2, \ldots$.

Complex Exponential and Sinusoidal (CT Signals)

- $e^{jw_0 t}$ and $e^{-jw_0 t}$ have the same fundamental period
- Energy in $e^{jw_0 t}$: $\int_{-\infty}^{\infty} |e^{jw_0 t}|^2 dt = \int_{-\infty}^{\infty} 1.dt = \infty$
- Average Power in $e^{jw_0 t}$: $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |e^{jw_0 t}|^2 dt$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 1.dt = 1$$

Complex Exponential and Sinusoidal (CT Signals)

$$Ce^{at}$$

where C and a are complex numbers.

If

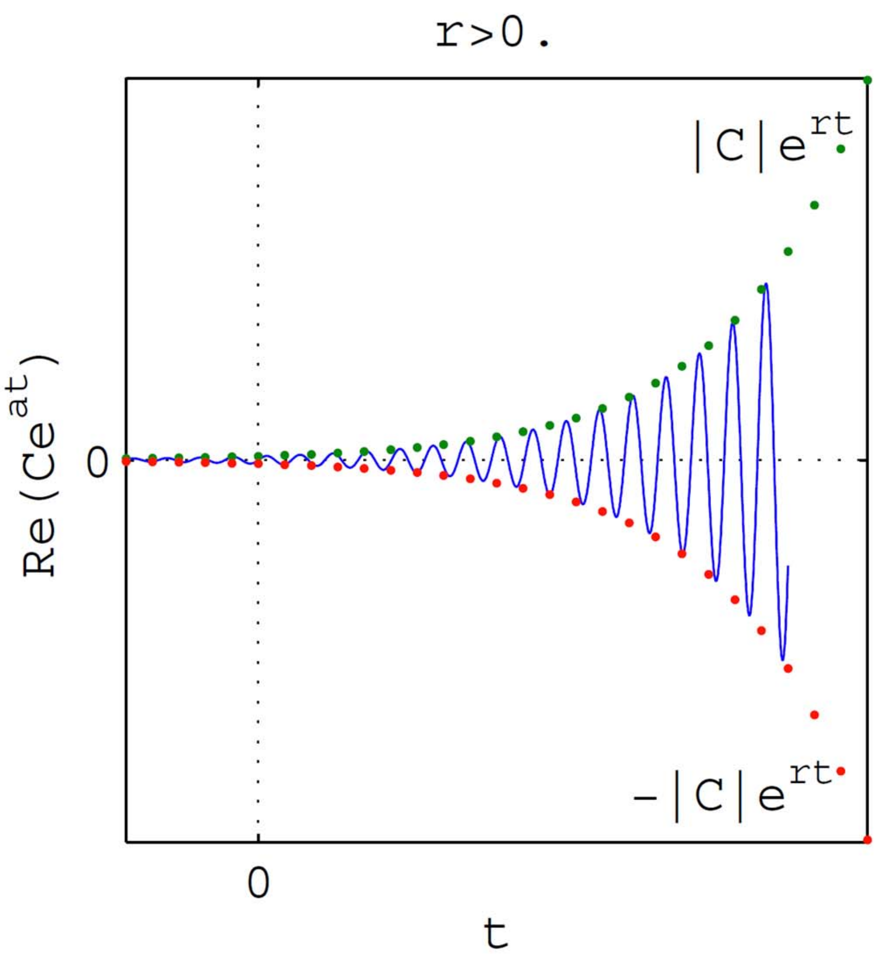
$$C = |C|e^{j\theta} \qquad \text{and} \qquad a = r + jw_0$$

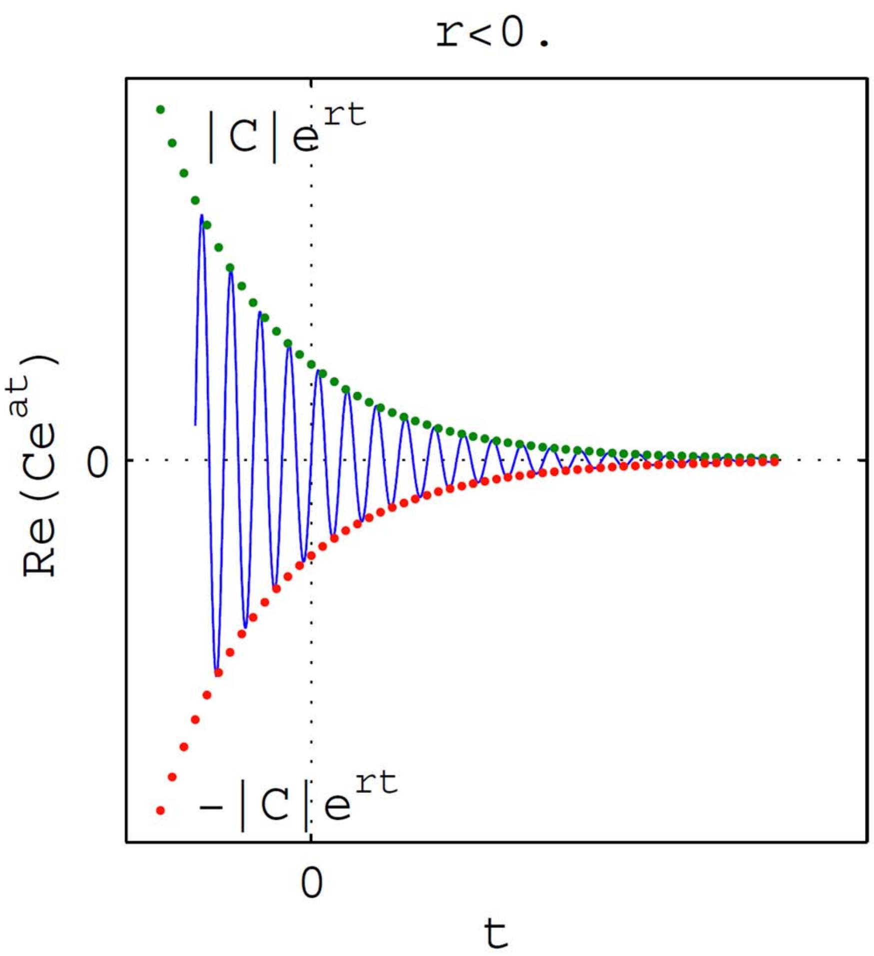
then

$$Ce^{at} = |C|e^{j\theta}e^{(r+jw_0)t} = |C|e^{rt}e^{j(w_0t+\theta)}$$

$$= \underbrace{|C|e^{rt}\cos(w_0t+\theta)}_{\mathrm{Re}(Ce^{at})} + j\underbrace{|C|e^{rt}\sin(w_0t+\theta)}_{\mathrm{Im}(Ce^{at})}$$

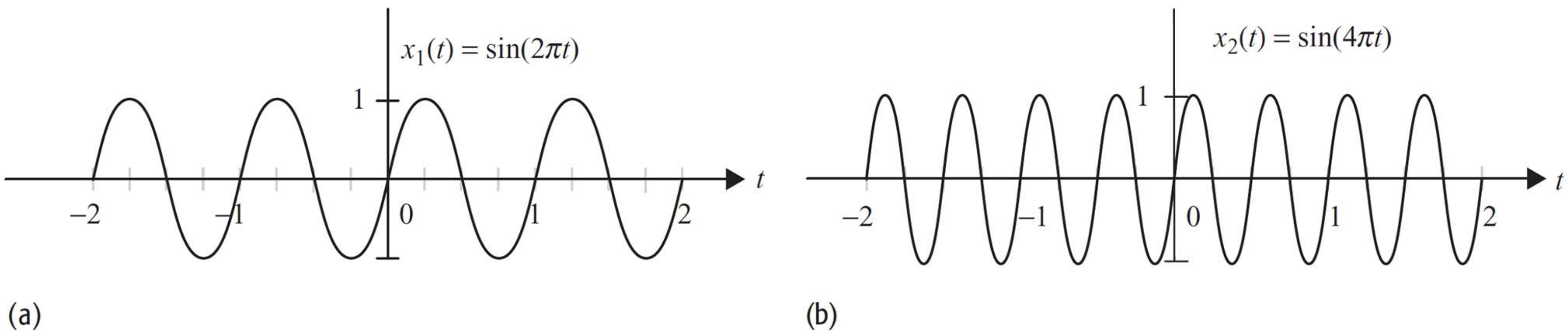
Complex Exponential and Sinusoidal (CT Signals)





Periodicity and Fundamental Period (CT Signals)

Consider two sinusoidal functions $x(t) = \sin(\omega_0 t + \theta)$ and $x_m(t) = \sin(m\omega_0 t + \theta)$. The fundamental angular frequencies of these two CT signals are given by ω_0 and $m\omega_0$ radians/s, respectively. In other words, the angular frequency of the signal $x_m(t)$ is m times the angular frequency of the signal $x(t)$. In such cases, the CT signal $x_m(t)$ is referred to as the mth harmonic of $x(t)$.



Examples of harmonics.
(a) Waveform for the sinusoidal signal $x(t) = \sin(2\pi t)$; (b) waveform for its second harmonic given by $x_2(t) = \sin(4\pi t)$.

Periodicity and Fundamental Period (CT Signals)

Proposition *A signal $g(t)$ that is a linear combination of two periodic signals, $x_1(t)$ with fundamental period T_1 and $x_2(t)$ with fundamental period T_2 as follows:*

$$g(t) = ax_1(t) + bx_2(t)$$

is periodic iff

$$\frac{T_1}{T_2} = \frac{m}{n} = \textit{rational number}.$$

The fundamental period of $g(t)$ is given by $nT_1 = mT_2$ provided that the values of m and n are chosen such that the greatest common divisor (gcd) between m and n is 1.

Periodicity and Fundamental Period (CT Signals)

Example

Determine if the following signals are periodic. If yes, determine the fundamental period.

$$g_1(t) = 3\sin(4\pi t) + 7\cos(3\pi t);$$

Solution

$$\frac{T_1}{T_2} = \frac{1/2}{2/3} = \frac{3}{4}$$

fundamental period of $g_1(t)$ is given by $nT_1 = 4T_1 = 2\,\mathrm{s}$.

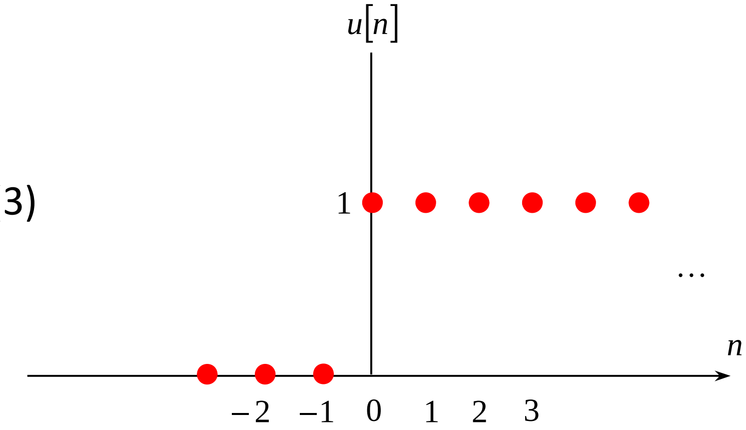
fundamental period of $g_1(t)$ can also be evaluated from $mT_2 = 3T_2 = 2\,\mathrm{s}$.

The Unit-Step Sequence (DT Signals)

- The unit-step sequence $u[n]$ is defined by

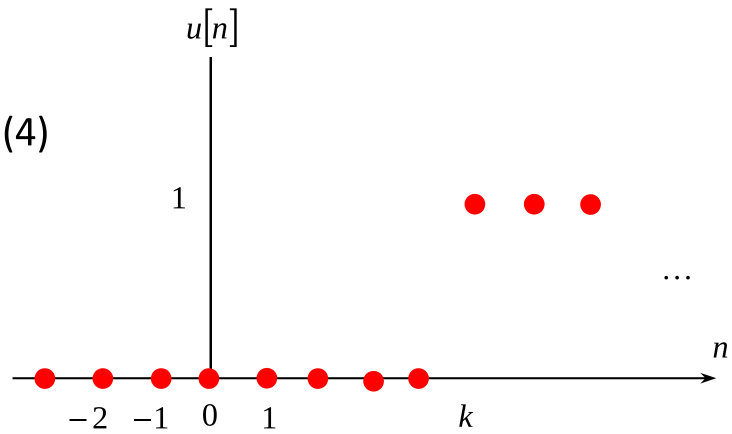
$$u[n] \equiv \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad (3)$$

Unlike $u(t)$, $u[n]$ <u>is</u> defined at $n = 0$



- The shifted unit-step sequence $u[n-k]$ is similarly defined by

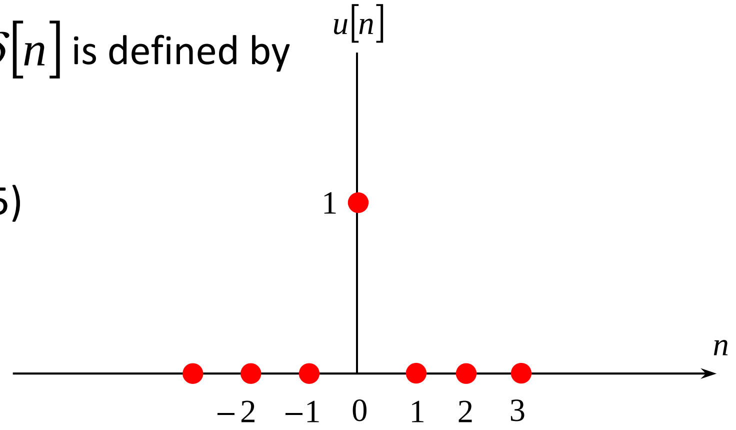
$$u[n-k] \equiv \begin{cases} 1 & n \geq k \\ 0 & n < k \end{cases} \qquad (4)$$



The Unit-Impulse Sequence (DT Signals)

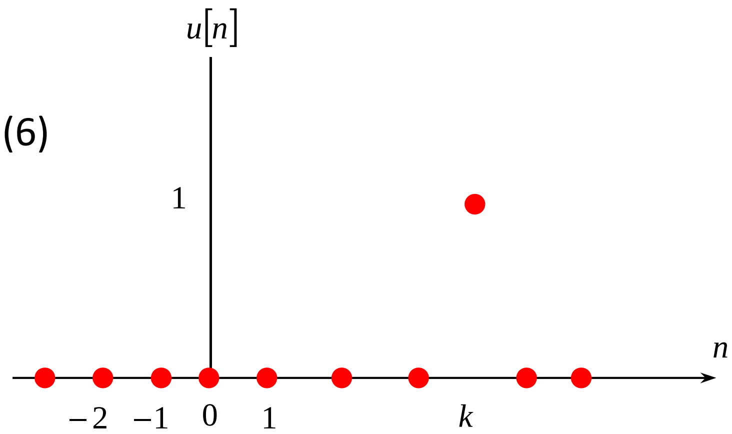
- The unit-impulse (or unit-sample) sequence $\delta[n]$ is defined by

$$\delta[n] \equiv \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \qquad (5)$$



- The shifted unit-impulse (sample) sequence $\delta[n-k]$ is similarly defined by

$$\delta[n-k] \equiv \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases} \qquad (6)$$



The Unit-Impulse Sequence (DT Signals)

- Unlike $\delta(t)$, $\delta[n]$ *is* readily defined. From (5) and (6) it is evident that

$$x[n]\delta[n] = x[0]\delta[n]$$
$$x[n]\delta[n-k] = x[k]\delta[n-k]$$

are the discrete-time counterparts of

$$x(t)\delta(t) = x(0)\delta(t)$$

$$x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau)$$

- from (3) and (4), $\delta[n]$ and $u[n]$ are related by

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

A discrete-time signal $x[n]$ may be expressed as (we prove this in the following lecture)

$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]}$$

Complex Exponential and Sinusoidal (DT Signals)

$$x[n] = e^{jw_0 n}$$

$$\begin{aligned}
e^{jw_0 n} \text{ is periodic} \quad &\Leftrightarrow \quad e^{jw_0 n} = e^{jw_0 (n+M)} \text{ for some integer } M > 0 \\
&\Leftrightarrow \quad e^{jw_0 M} = 1 \text{ for some integer } M > 0 \\
&\Leftrightarrow \quad w_0 M = 2\pi m \text{ for some integers } m, M > 0 \\
&\Leftrightarrow \quad \frac{w_0}{2\pi} \text{ is rational.}
\end{aligned}$$

Complex Exponential and Sinusoidal (DT Signals)

- If $\frac{w_0}{2\pi} = \frac{m}{M}$ for some integers m and M which have no common factors, then the fundamental period is

$$M = \frac{2m\pi}{w_0}$$

Periodicity and Fundamental Period (DT Signals)

Examples

1) Is $x[n] = e^{jn2\pi/3} + e^{jn3\pi/4}$ periodic? If it is periodic, what's its fundamental period?

For $e^{jn2\pi/3}$, $w_0/(2\pi) = 1/3$, so $e^{jn2\pi/3}$ is periodic with fundamental period 3.

For $e^{jn3\pi/4}$, $w_0/(2\pi) = 3/8$, so $e^{jn3\pi/4}$ is periodic with fundamental period 8.

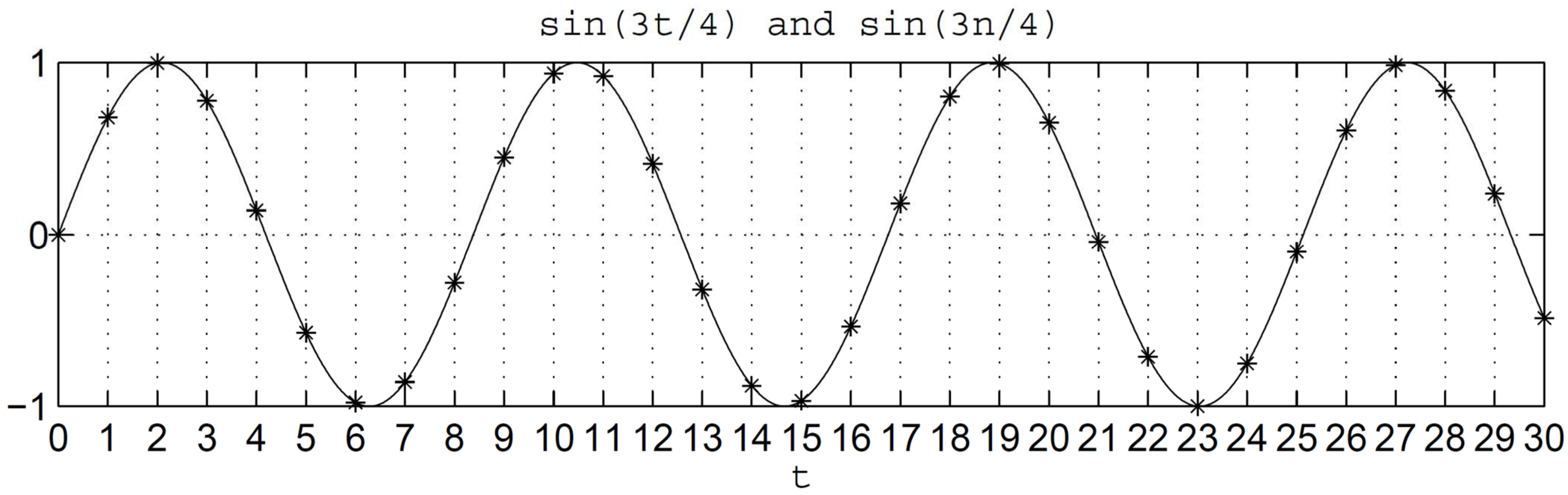
$x[n]$ is periodic with fundamental period $24 = lcm(3, 8)$.

Periodicity and Fundamental Period (DT Signals)

Examples

2) Is $x[n] = \sin(3n/4)$ periodic? If it is periodic, what's its fundamental period?

Since $\frac{w_0}{2\pi} = \frac{3}{8\pi}$ is irrational, $x[n]$ is not periodic



Periodicity and Fundamental Period (DT Signals)

Examples

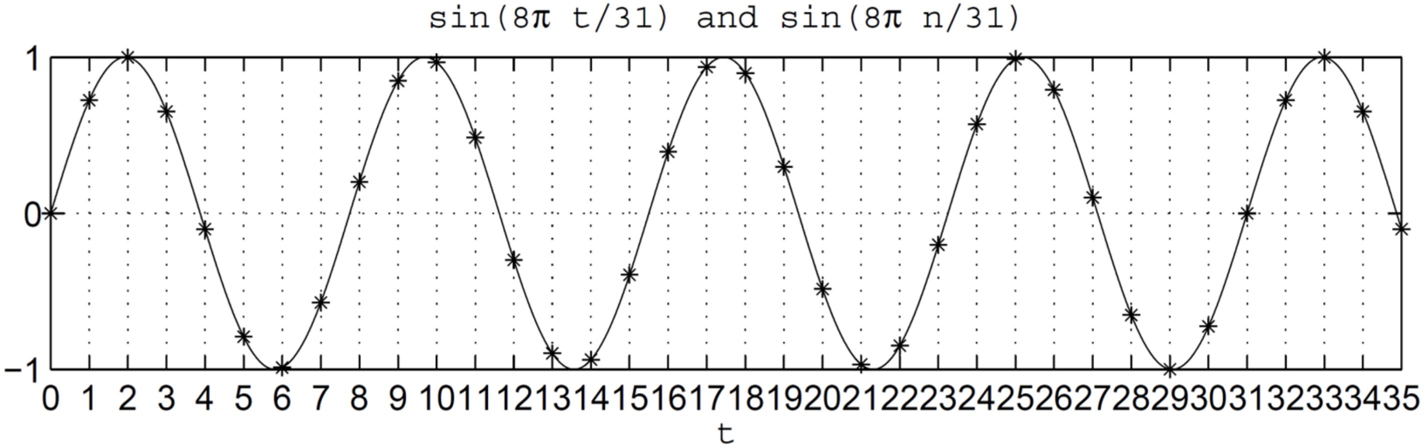
3) Is $x[n] = \sin(8\pi n/31)$ periodic? If it is periodic, what's its fundamental period?

Since $w_0/(2\pi) = 4/31$, $x[n]$ is periodic with fundamental period 31

$$x[0] = x[31] = 0$$

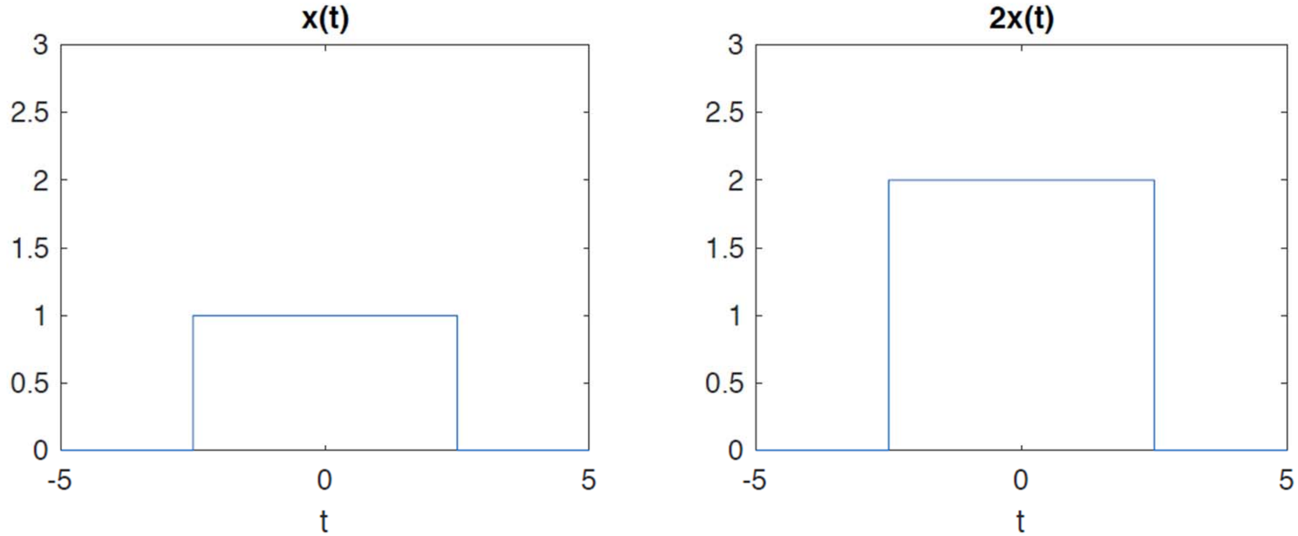
Note that the continuous-time signal $\sin(8\pi t/31)$ has fundamental period $31/4$

But $x[n]$ has no $31/4-$th sample and it misses 0 between $x[7]$ and $x[8]$



Operations – Amplitude Scaling (CT Signals)

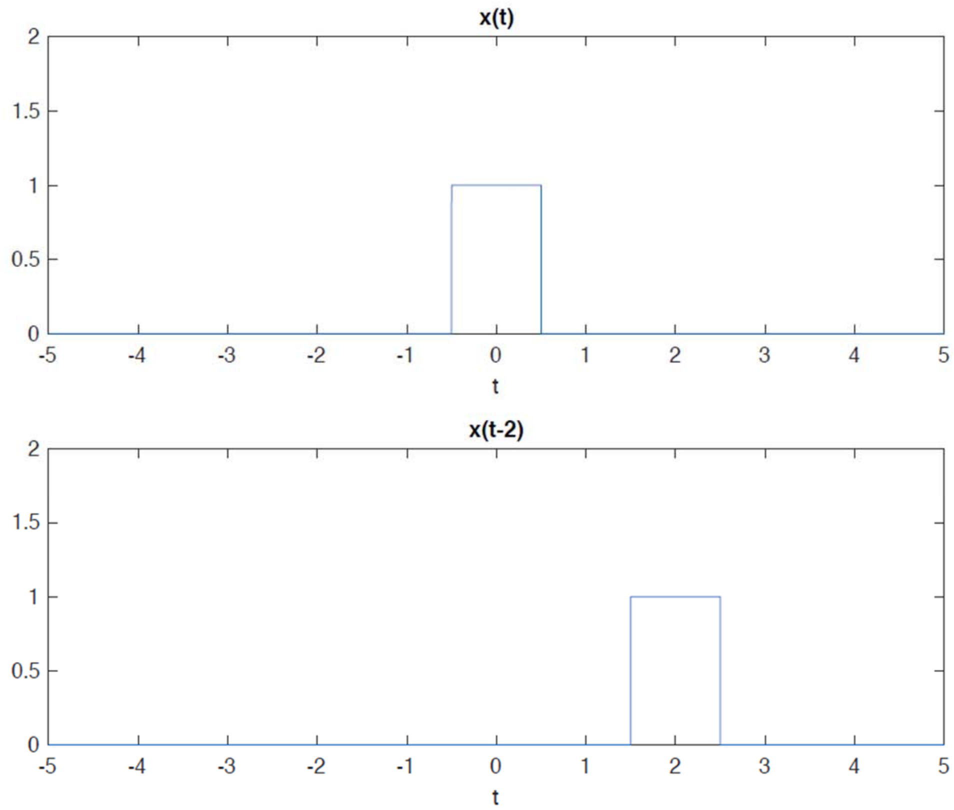
Given a CT signal $x(t)$, scaling consist of multiplying it by a scalar value a, producing the new signal $y(t) = ax(t)$.



Scaling is defined in an analogous way for DT signals.

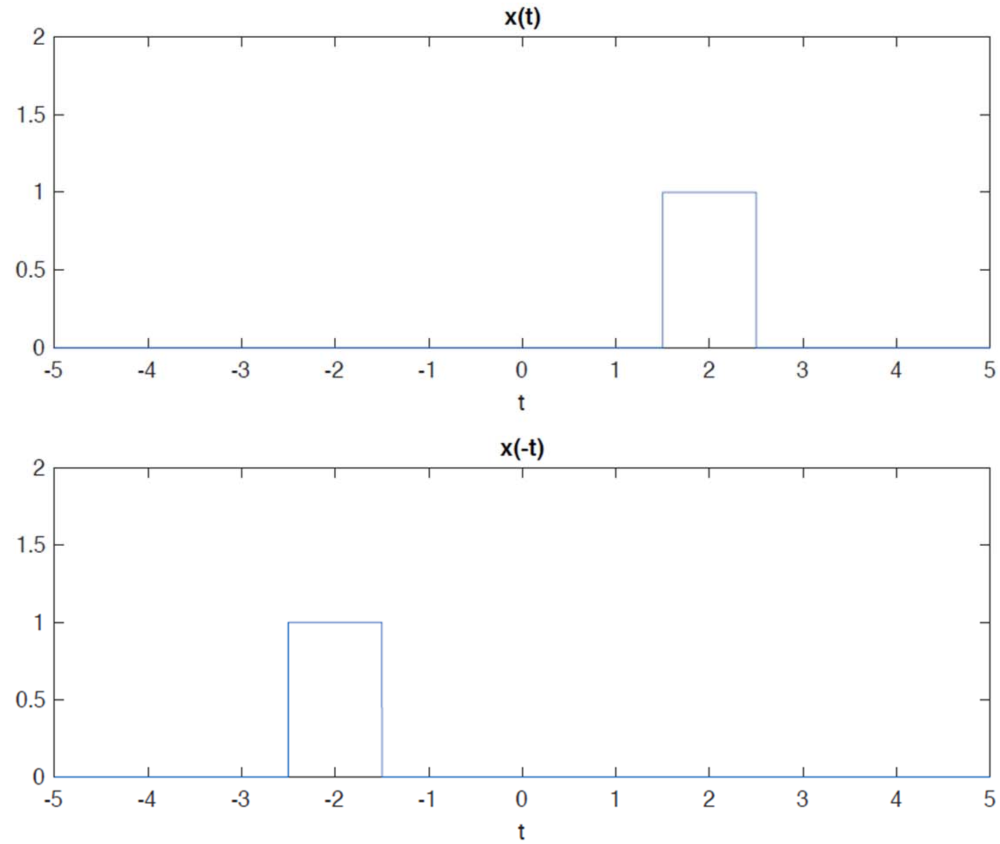
Operations – Time Shift (CT Signals)

Given a CT signal $x(t)$, time shifting by t_0 units of time produces the new signal $y(t) = x(t - t_0)$ (DT shifting is defined in a similar way).



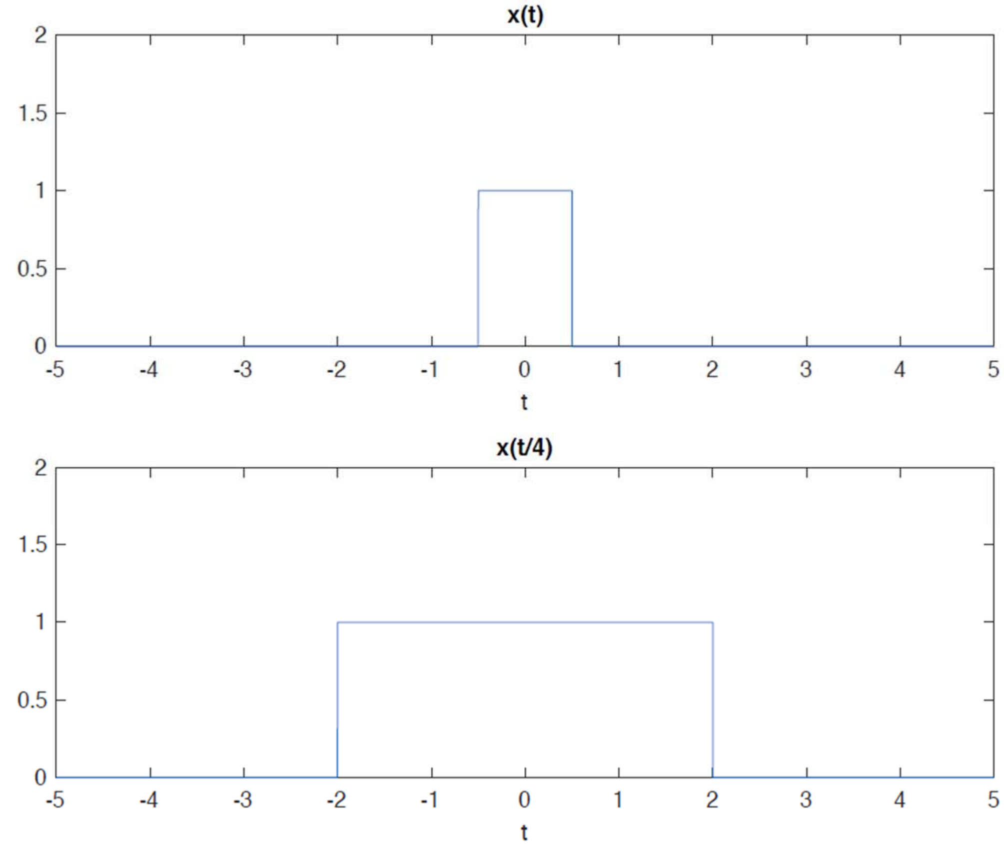
Operations – Time Inversion/Reversal (CT Signals)

Time reversal *flips* the time axis producing the signal $y(t) = x(-t)$.



Operations – Time Scaling (CT Signals)

Time scaling **expands** or **compresses** the time axis. Signal $y(t) = x(at)$ is a compressed version of $x(t)$ if $|a| > 1$, and an expanded version if $|a| < 1$.



Operations – Combined Time Operations (CT Signals)

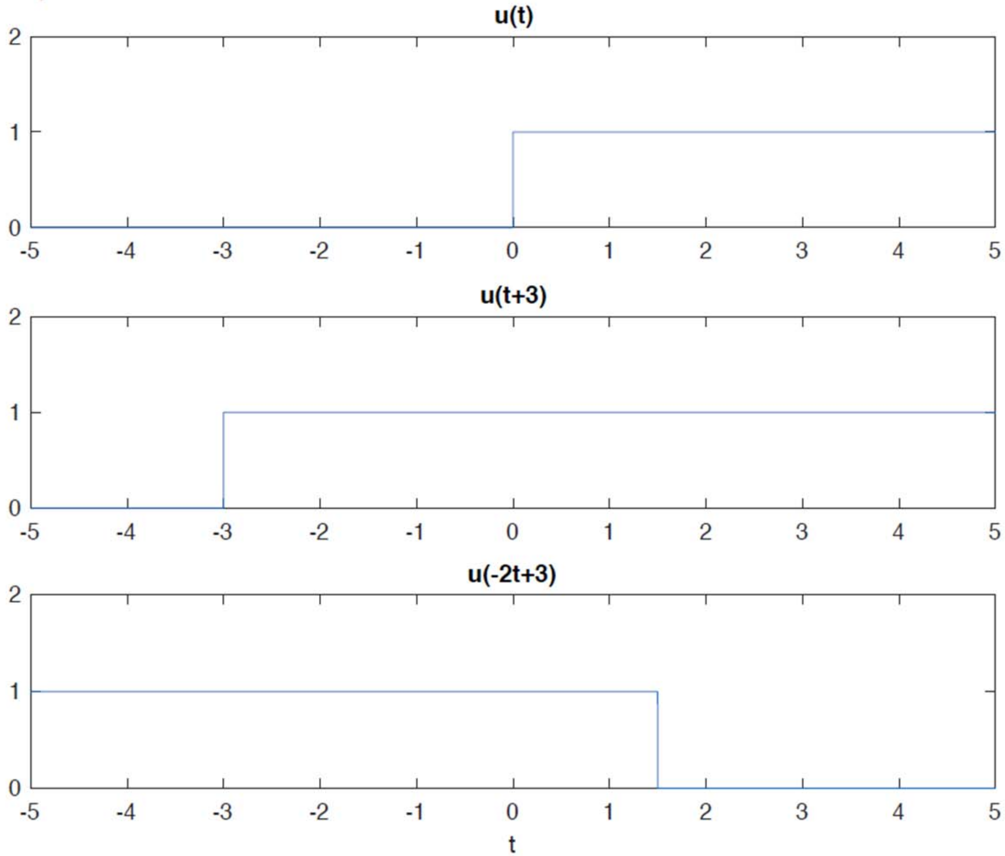
Consider the signal $x(t) = u(-2t + 3)$. In order to obtain $x(t)$ we will take the following steps:

- Define the time-shift $y(t) = u(t + 3)$.
- Define the time scaling and reverse $z(t) = y(-2t)$.

As we can see, $z(t) = y(-2t) = u(-2t + 3)$ and therefore $x(t) = z(t)$.

Operations – Combined Time Operations (CT Signals)

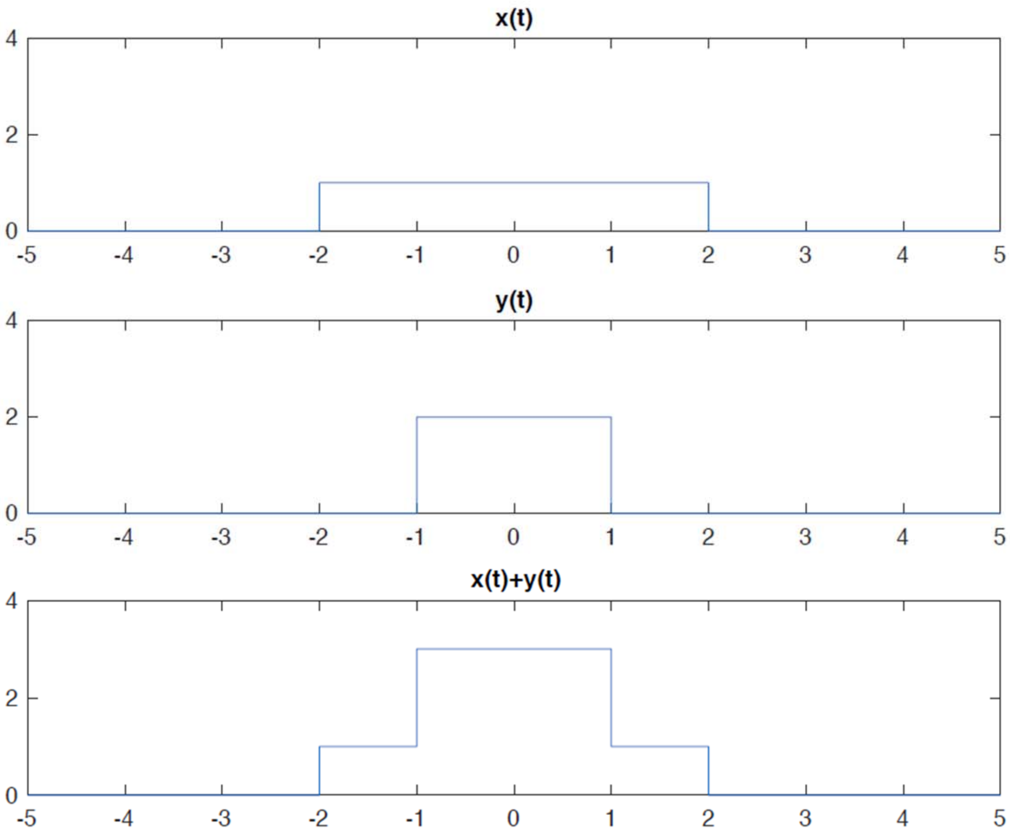
In general, we can obtain the signal $y(t) = x(-at + t_0)$ by shifting $x(t)$ first and then by scaling and time reversing the result. Graphically, the signal $u(-2t + 3)$ can be obtained as follows:



You can see that $u(-2(1.5) + 3) = u(0)$.

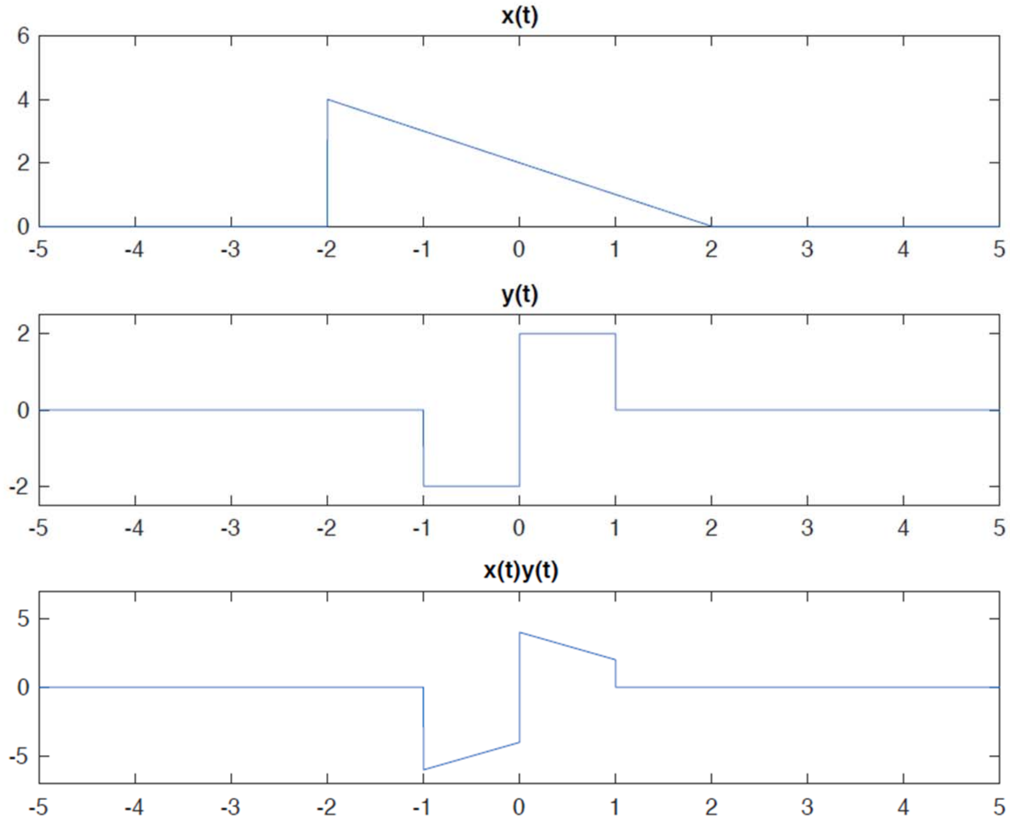
Operations – Sum (CT Signals)

Adding two signals $x(t)$ and $y(t)$ means adding their values each time instant.



Operations – Product (CT Signals)

Similarly, we multiply signals by multiplying their values each time instant.

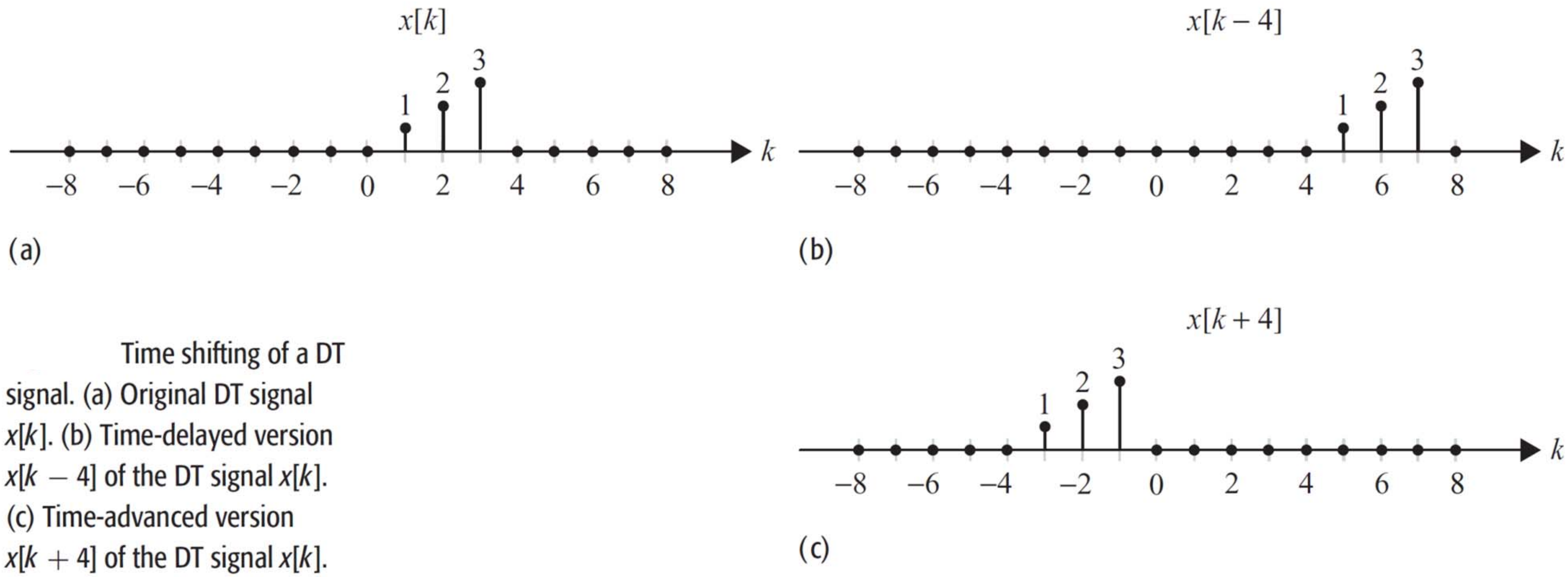


Operations – Time Shift (DT Signals)

When a DT signal $x[k]$ is shifted by m time units, the delayed signal $\phi[k]$ is expressed as

$$\phi[k] = x[k + m]$$

If $m < 0$, the signal is said to be delayed in time.

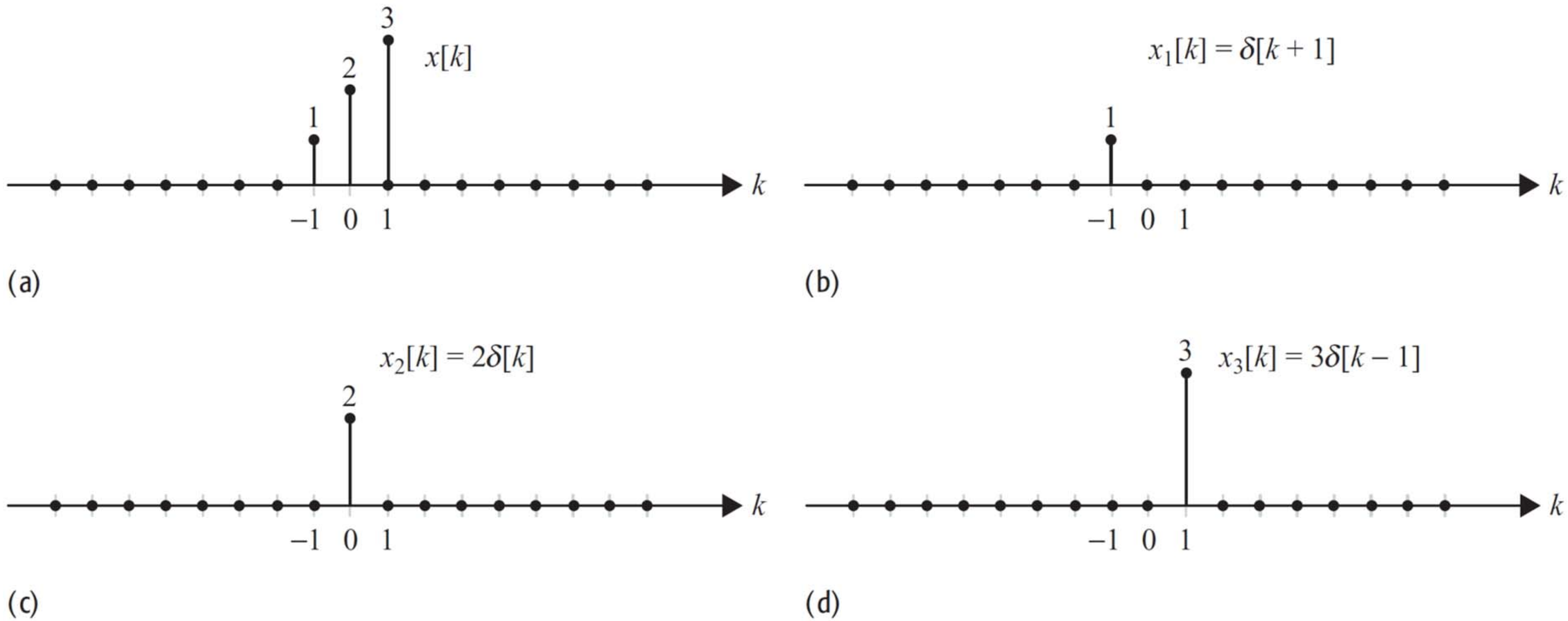


Time shifting of a DT signal. (a) Original DT signal $x[k]$. (b) Time-delayed version $x[k-4]$ of the DT signal $x[k]$. (c) Time-advanced version $x[k+4]$ of the DT signal $x[k]$.

Operations – Time Shift (DT Signals)

Example

Represent the DT sequence shown in (a) as a function of time-shifted DT unit impulse functions.



$$x[k] = \delta[k+1] + 2\delta[k] + 3\delta[k-1]$$

Operations – Time Inversion/Reversal (DT Signals)

$$y(n) = x(-n)$$

positive time switches to negative time and vice versa

Example

Sketch the time-inverted version of the following DT sequence:

$$x[k] = \begin{cases} 1 & -4 \leq k \leq -1 \\ 0.25k & 0 \leq k \leq 4 \\ 0 & \text{elsewhere,} \end{cases}$$

Operations – Time Inversion/Reversal (DT Signals)

Solution

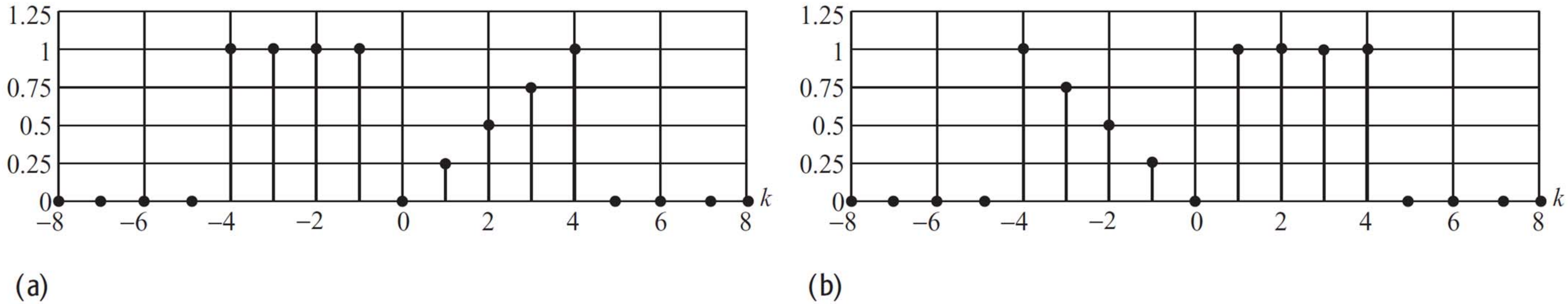
To derive the expression for the time-inverted signal $x[-k]$, substitute $k = -m$

$$x[-m] = \begin{cases} 1 & -4 \le -m \le -1 \\ -0.25m & 0 \le -m \le 4 \\ 0 & \text{elsewhere.} \end{cases}$$

Simplifying the above expression and expressing it in terms of the independent variable k yields

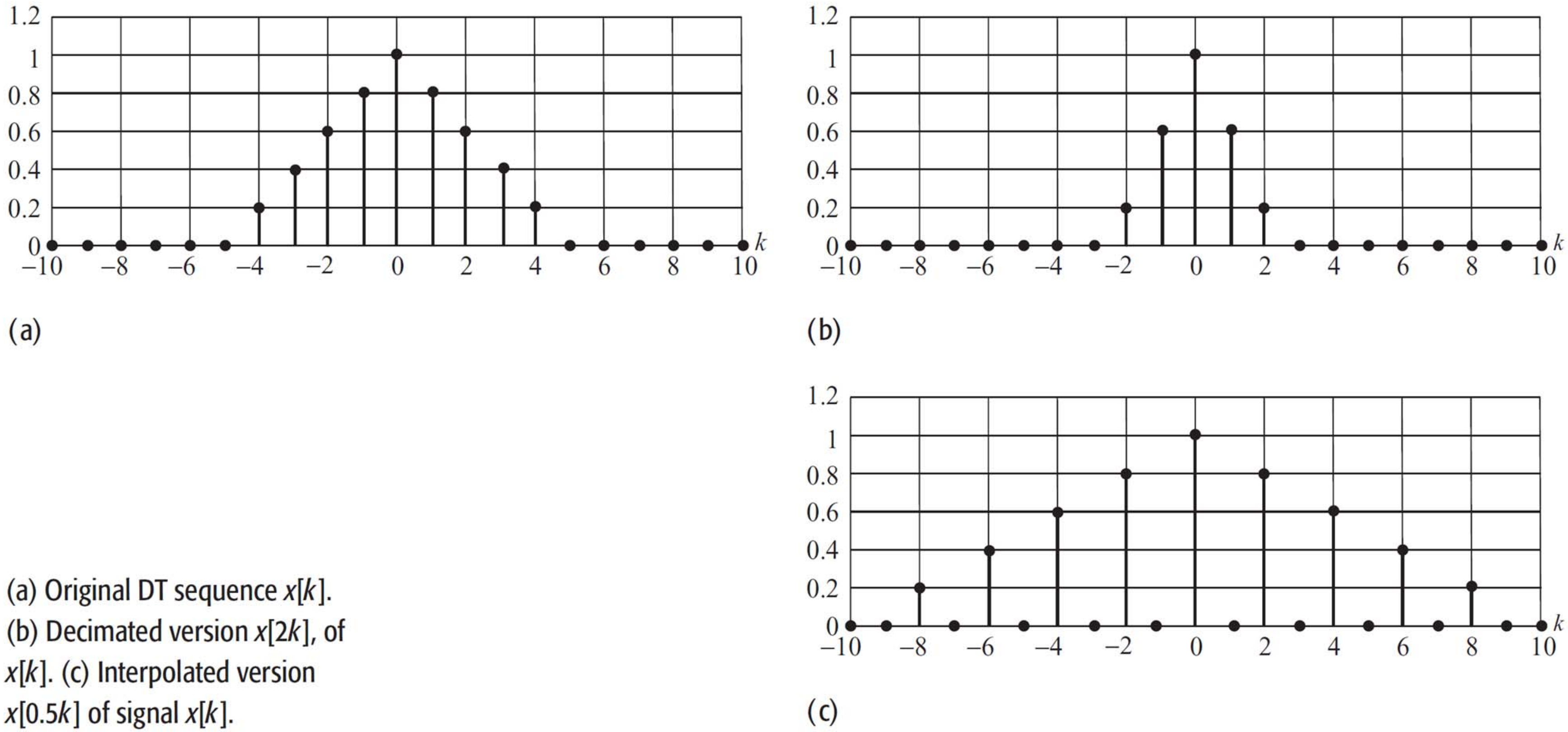
$$x[-m] = \begin{cases} 1 & 1 \le m \le 4 \\ -0.25m & -4 \le \quad m \le 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Operations – Time Inversion/Reversal (DT Signals)



(a) Original CT sequence $x[k]$
(b) Time-inverted version $x[-k]$

Operations – Time Scaling (DT Signals) also known as Decimation and Interpolation

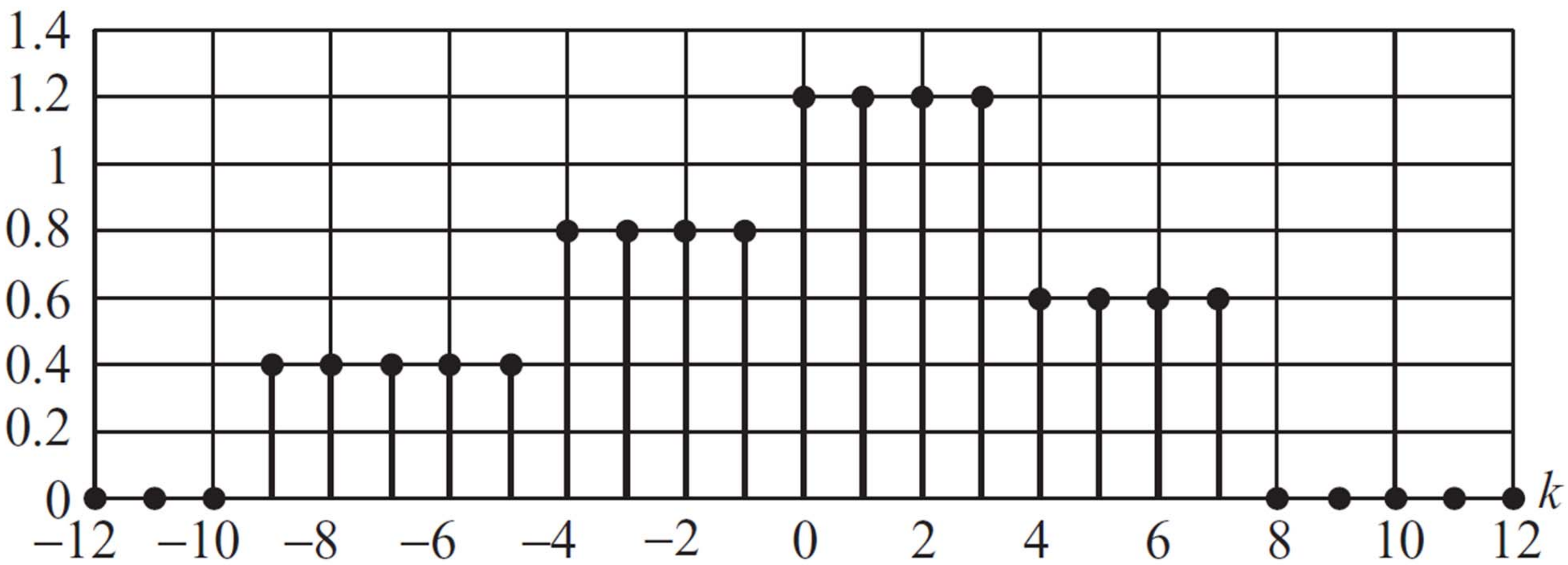


(a) Original DT sequence $x[k]$. (b) Decimated version $x[2k]$, of $x[k]$. (c) Interpolated version $x[0.5k]$ of signal $x[k]$.

Operations – Combined Time Operations (DT Signals)

Example

Sketch the waveform for $x[-15 - 3k]$ for the DT sequence $x[k]$



Operations – Combined Time Operations (DT Signals)

Solution

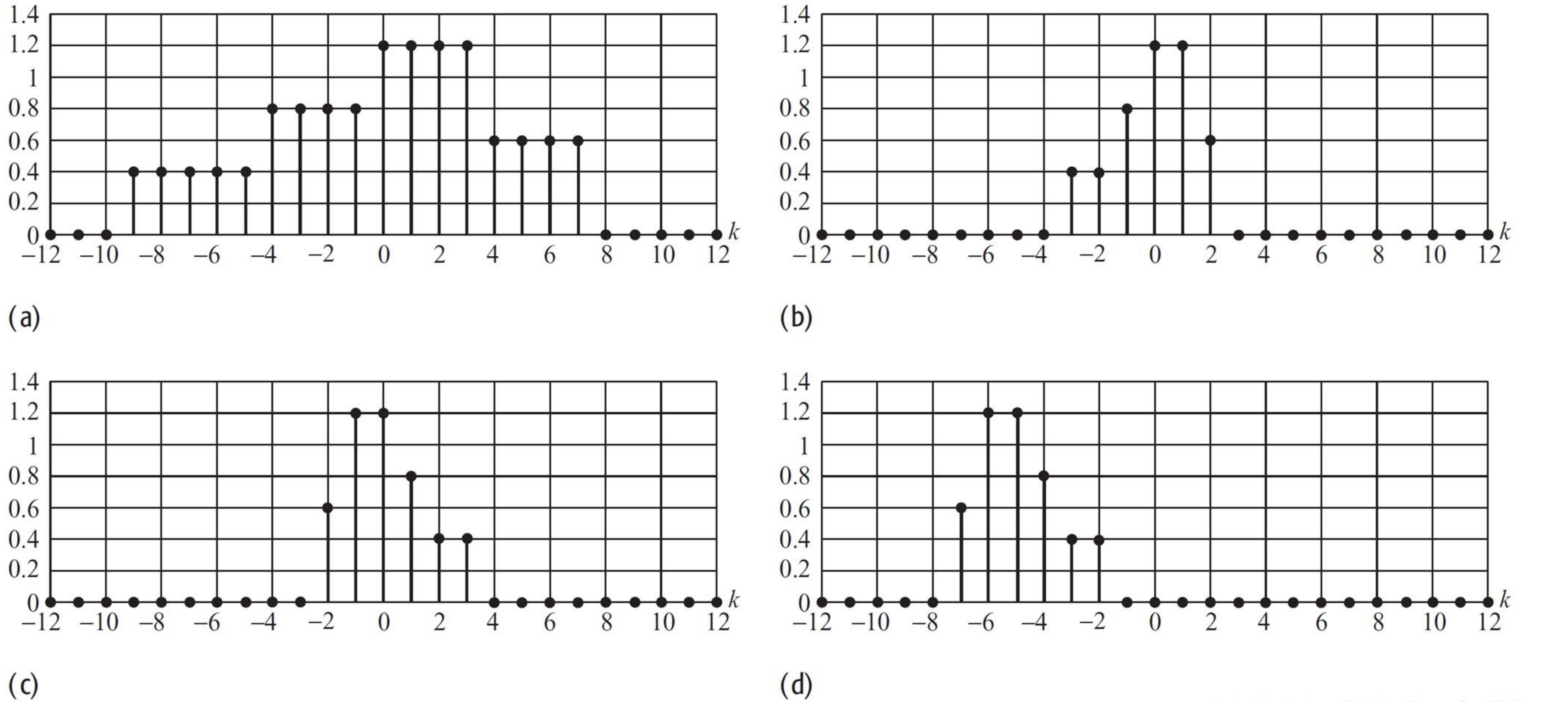
Express $x[-15 - 3k] = x[-3(k + 5)]$ and follow steps (i)–(iii) as outlined below.

(i) Compress $x[k]$ by a factor of 3 to obtain $x[3k]$.

(ii) Time-reverse $x[3k]$ to obtain $x[-3k]$.

(iii) Shift $x[-3k]$ towards the left-hand side by five time units to obtain $x[-3(k + 5)] = x[-15 - 3k]$.

Operations – Combined Time Operations (DT Signals)



(a) Original DT signal $x[k]$.
(b) Time-scaled version $x[3k]$.
(c) Time-inverted version $x[-3k]$ of (b). (d) Time-shifted version $x[-15-3k]$ of (c).