1) *Linear functions.*

a) Show that an inner product function, $f(x) = a^T x$, is linear.

b) Show that any scalar-valued linear function $f(x)$ satisfying superposition can be expressed as an inner product function, say $f(x) = a^T x$. Explicitly state the elements of a in terms of f.

a) $f(x) = a^T x$.

i) $f(x+y) = a^T(x+y) = a^T x + a^T y = f(x) + f(y)$

ii) $f(kx) = a^T(kx) = k \cdot a^T x = k f(x)$.

$\therefore$ $f(x)$ is linear func.

b) $x \triangleq \sum x_i e_i \Rightarrow f(x) = f(\sum x_i e_i)$

$\Rightarrow f(x) = \sum f(x_i e_i) = \sum x_i f(e_i)$ ($\because$ linear func).

$a \triangleq (f(e_1) \dots f(e_n)) \Rightarrow f(x) = a^T x$.

2) *Affine functions.*

a) Show that an inner product function plus a shift, $f(x) = a^T x + b$, is affine.

b) Show that any scalar-valued affine function $f(x)$ satisfying the restricted superposition (superposition defined for linear combination with coefficients that sum to 1) can be expressed as an inner product function plus a shift, say $f(x) = a^T x + b$. Explicitly state the elements of a and b in terms of f.

a) $\forall \alpha + \beta = 1 \in \mathbb{R} \Rightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$.

$\Rightarrow \alpha a^T x + \alpha b + \beta a^T y + \beta b \quad (\beta b + \alpha b = b)$.

$\Rightarrow \alpha a^T x + \beta a^T y + b = a^T(\alpha x + \beta y) + b$. $\square$

b) $f(x) = f\left(\sum x_i e_i\right) = f\left(\sum x_i e_i + \sum\left(\frac{1}{n} - x_i\right) x 0\right)$.

$= \sum x_i \{f(e_i) - f(0)\} + \frac{1}{n} \times n \times f(0)$.

$\therefore f(x) = a^T x + b$.

$b = f(0)$.

$a = (f(e_i) - f(0))$. n-vector.

3) *Cauchy-Schwarz inequality.* Show that any two vectors $a, b \in \mathbb{R}^n$ satisfy the following. Also state the condition under which the inequality is tight.

$$|a^T b| \leq \|a\| \|b\|.$$

$\lambda \in \mathbb{R}$. $\|a - \lambda b\|^2 \geq 0$.

$\Rightarrow \|a\|^2 - 2\lambda(a^T b) + \lambda^2 \|b\|^2 \geq 0$.

$f(\lambda) \triangleq \|a - \lambda b\|^2 \Rightarrow \frac{df(\lambda)}{d\lambda} = -2(a^T b) + 2\lambda\|b\|^2 = 0$.

$\therefore \lambda = \frac{a^T b}{\|b\|^2}$ $f(\lambda) \leq f(\alpha)$ $\forall \alpha \in \mathbb{R}$.

$\Rightarrow |a|^2 - 2\frac{(a^T b)^2}{|b|^2} + \left(\frac{a^T b}{|b|^2}\right)^2 |b|^2 \geq 0$.

$\Rightarrow \|a\|^2 \|b\|^2 \geq |a^T b|^2$ $\therefore |a||b| \geq |a^T b|$.

Where $a = kb$ or $b = ka$. for $\exists k \in \mathbb{R}$.

4) *Angle between two vectors.* Show that any two vectors $a, b \in \mathbb{R}^n$ satisfy the following, where θ is the angle between a and b. You may provide a proof for the two-dimension case, which easily generalizes to general n-dimension cases.

$$a^T b = \|a\| \|b\| \cos \theta.$$

Pf) for 2-D. $a \triangleq (a_1, a_2)$. $b \triangleq (b_1, b_2)$.

$\|a\| = \sqrt{a_1^2 + a_2^2}$, $|b| = \sqrt{b_1^2 + b_2^2}$.

$|a| = \|a\| (\cos\alpha, \sin\alpha)$. $|b| = \|b\| (\cos\beta, \sin\beta)$.

where $\alpha = \angle(a, e_1)$. $\beta = \angle(b, e_1)$.

$a^T b = |a||b| (\cos\alpha \cos\beta + \sin\alpha \sin\beta)$

, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha \sin\beta$. Let $\theta = \alpha - \beta$.

$\Rightarrow a^T b = |a||b| \cos\theta$.

5) *Parallelogram.* Draw two different vectors u and v out from the origin. Complete two more sides to make a parallelogram with diagonals $w = u + v$ and $z = u - v$. Show that $\|w\|^2 + \|z\|^2 = 2\|u\|^2 + 2\|v\|^2$.

$$|u+v|^2 + |u-v|^2 =$$

$$\Rightarrow |u|^2 + 2|u^T v| + |v|^2 + |u|^2 - 2|u^T v| + |v|^2$$

$$= |u|^2 + |v|^2$$

$$\|w\|^2 + \|z\|^2 = \|u\|^2 + \|v\|^2$$

a) **2.3** *Motion of a mass in response to applied force.*

$$s'(10) = s'(0) + \int_0^{10} F(u)\,du = s'(0) + \sum_i^{10} f_i$$

$$a \triangleq \mathbb{1}, \quad b \triangleq s'(0), \quad f = [f_1, f_2 \ldots f_{10}],$$

$$\Rightarrow a^T f + b.$$

$$s(10) = s(0) + \int_0^{10} s'(u)\,du.$$

$$= s + 10\, s'(0) + \frac{19}{2} f_1 + \cdots + \frac{1}{2} f_{10}.$$

$$c \triangleq \left(\frac{19}{2}, \frac{17}{2}, \ldots \frac{1}{2}\right), \quad d \triangleq s(0) + 10 s'(0)$$

$$c^T f + d.$$

b) **2.12** *Price change to maximize profit.*

$\hat{p} = \beta^T \lambda + p$.

(a) $\beta_3 < 0$ ∴ negative rel.

(b) $\exists \beta_i > 0 \Rightarrow \lambda_i = 0.01 \Rightarrow 0.01 = \frac{p_i^{new} - p_i}{p_i} \Rightarrow p_i^{new} = 1.01 p_i$

$\exists \beta_i < 0 \Rightarrow \lambda_i = -0.01 \Rightarrow -0.01 = \frac{p_i^{new} - p_i}{p_i} \Rightarrow p_i^{new} = 0.99 p_i$

∴ $\hat{p} = |\beta_i||\lambda_i| + p = 0.01|\beta_i| + p$

(c) $|\beta_a|, |\beta_b| \geq \beta_i \ (i \neq a, b)$

$\hat{p} = 0.01|\beta_a| + 0.01|\beta_b| + p = 0.01(|\beta_a| + |\beta_b|) + p$

c) **3.12** *Nearest point to a line.*

$$\|p-x\|^2 = \|(1-\theta)a+\theta b-x\|^2 = \|a-x+\theta(b-a)\|^2.$$

$$\frac{d}{d\theta}\left(\|a-x\|^2+2\theta(a-x)^T(b-a)+\theta^2\|b-a\|^2\right)$$

$$= 2\theta\|b-a\|^2+2(a-x)^T(b-a)=0. \Rightarrow \theta = \frac{(x-a)^T(b-a)}{\|b-a\|^2} \quad (b\neq a).$$

$$p = a+\frac{(x-a)^T(b-a)}{\|b-a\|}(b-a)$$

$$(p-x)^T(a-b) = \left(a+\frac{(x-a)^T(b-a)}{\|b-a\|^2}(b-a)-x\right)^T(a-b)$$

$$=(a-x)^T(a-b) - \frac{(x-a)^T(b-a)}{\|b-a\|^2}\|b-a\|^2$$

$$=0.$$