Abelian extension and crossed modules for Lie Algebras

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DECLARATION

This work was carried out at AIMS Senegal in partial fulfilment of the requirements for a Master of Science Degree.

I hereby declare that except where due acknowledgement is made, this work has never been presented wholly or in part for the award of a degree at AIMS Senegal or any other University.

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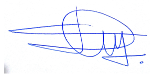
Supervisor: Andre Saint Eudes MIALEBAMA BOUESSO 

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DEDICATION

I dedicate this work to my family, the KAMDOUM family

Abstract

Let G be a Lie algebra and V a G-module.The Lie algebra cohomology spaces $H^0(G,V)$ and $H^1(G,V)$ can be easily computed by direct computation. The goal of this project is to compute $H^2(G,V)$ and $H^3(G,V)$. For the second cohomology space $H^2(G,V)$, we construct the group of equivalence classes of abelian extensions with kernel V and cokernel G, and then we prove that this group is in bijection with $H^2(G,V)$. For the third cohomology space $H^3(G,V)$, we construct the group of equivalence classes of crossed-modules with kernel V and cokernel G, and then we prove that this is in bijection with $H^3(G,V)$.

Contents

1. Introduction

In this work, our goal is to determine isomorphism near the second group and the third group of cohomology extracted on a complex constructed from a Lie algebra. In this projection, we establish these isomorphisms thanks to the equivalence classes of the abelian extensions for the second group of cohomology and the crossed modules for the third group of cohomology. Indeed, an abelian extension of a Lie algebra is a short exact sequence of Lie algebra:

$$0 \longrightarrow A \xrightarrow{i} G \xrightarrow{\pi} H \longrightarrow 0 \tag{1.0.1}$$

where A the Lie algebra is abelian and $ker\pi = A$ and $cokeri = H$.

Thus, the goal is to construct from a given abelian extension as 1.0.1, a cohomology class of $H^2(H, A)$ to identify or correspond to any element of $H^2(H, A)$ an abelian extension and vice verca. The same method is established by Grestenhaber in his article $Gerstenhaber$ (1966), where he calculates isomorphism near the third group of cohomology from the equivalence class of the crossed modules. Indeed, A crossed module of Lie algebras is the data of a homomorphism of Lie algebras $\mu : M \longrightarrow N$ together with an action η of N on M by derivations, denoted $\eta : N \longrightarrow der(M)$ or $m \longmapsto n.m$ for all $m \in M$ and all $n \in N$, such that:

i- $\mu(n.m) = [n, \mu(m)]$ for all $n \in N$ and all $m \in M$,

ii- $\mu(m).m' = [m, m']$ for all $m, m' \in m$.

In view of the diversity of these notions in algebra and even in other areas of mathematics, this work is the result of a motivation based essentially on the extent of applications of abelian extensions, crossed modules and calculation. (co)homological. Indeed, in homological algebra, the computation of the (co)homology of a sequence or a complex of given chain, makes it possible to evaluate the exactitude of the latter, which is very important notably in homological theory because abounds multiple properties. For most of the time the computation of the groups of (co)homology is not direct, it is generally done by isomorphic approach, because if one establishes a bijection between a well known space and whose properties are mastered with a group of homology one will be able to carry the properties of this space on the group of homology.

This is why this work proves to be very interesting because from the equivalence classes of an abelian extension, we identify the second group of cohomology and from the equivalence classes of the crossed modules. we identify the third group of cohomology, all this on Lie algebras, which are algebras with very interesting and applicable properties.

In order to achieve our goals, we have to split work into three main chapters presenting as follows:

- In the first chapter, we give or recall some basic notions that will allow us to better understand the substance of the work, namely the categories and functors, of the homological algebra with complex chains, exact sequences and groups of (co)homology.

Mainly based on the following authors: ($Rotman$, 2008), ($HiltonandStammbach$, 2012), ($Facchiniet\ al.$, 2004), ($Rotman$, 2008), ($BaezandCrans$, 2004), ($Weibel$, 1994b).

- In the second chapter, we establish the bijection between the equivalence class of an abelian extension with the second cohomology group on the Lie algebra, detailed as follows: a reminder on the groups, extension of groups and in this chapter also, we make a little bit a reminder on the theory of Lie algebras, in the end a section on the abelian extension for Lie algebras useful to evaluate the second group of cohomology. Here we rely mainly on the following authors: ($Hochschild$, 1954), ($Radul$, 1989), ($FialowskiandWagemann$, 2008), ($FeiginandFuchs$, 2000), ($FarnsteinerandStrade$, 1981), ($Bourbaki$, 2008), ($Neeb$, 2006).

- In the third chapter, we establish a bijection between the equivalence class of the crossed module and the third group of cohomology. This part is essentially drawn from Friedrien Wagemann in $Wagemann$ (2006) detail as follows: a reminder of the modules and some properties and the calculation of the third group of cohomology. Here the results come mainly from the following researchers: ($Wagemann$, 2006), ($Gerstenhaber$, 1964), ($Gerstenhaber$, 1966), ($AgrebaouiandFraj$, 2003).

2. Preliminaries

This chapter is inspired and drawn mainly from the following authors:

($Rotman$, 2008), ($HiltonandStammbach$, 2012), ($Facchiniet\ al.$, 2004), ($Rotman$, 2008), ($BaezandCrans$, 2004), ($Weibel$, 1994b)

2.1 Groups

2.1.1 Definitions and properties

Definition 2.1.1. (Law of internal composition on a set)

Let E be a set, we call the law of internal composition on E any application:

$$* : E \times E \longrightarrow E$$

One recalls that $E \times E = \{(x, y), x \in E, y \in E\}$.

Definition 2.1.2. (Group)

A group $(G, *)$ is the data of a non-empty set G and $*$ a law of internal composition of G such that:

- $*$ is associative
- $*$ has a neutral element

$$e : \exists e \in G,\ \forall x \in G,\ x * e = e * x = x$$

- Every element of G has a symmetrical element:

$$\forall x \in G,\ \exists y \in G,\ x * y = y * x = e$$

Remark.

We say that the group $(G, *)$ is commutative, or abelian, if $*$ is commutative. The set G is called the underlying set of the group.

Proposition 2.1.1. (Property of a group)

Let $(G, *)$ be a group, e its neutral element.

- e is unique and is its own symmetrical.
- Every element has a unique symmetrical.

- If x' is the symmetric of x, then x is the symmetric of x'.

Proof.

Suppose such an element e' exists. Uniqueness

- $e = e * e' = e'$

Symmetrical:

$$e * e = e$$

- Let $x \in G$, let y, z be two symmetric of x.

$$(y * x) * z = y * (x * z)$$
$$e * z = y * e$$
$$z = y$$

□

Definition 2.1.3. (Finite group)

Let $(G, *)$ be a group, we say that the group is finite if G is. We Call $CardG$ the order of the group, we write $|G|$.

We observe that $|G| \geq 1$ for any group $(G, *)$

Definition 2.1.4. (Image by homomorphism)

A group G' is said homomorphic image of the group G if there exists a surjective group morphism $f : G \longrightarrow G'$

Definition 2.1.5. (Subgroup)

A set $H \subseteq G$ is a subgroup of G if it satisfies the following conditions:

- H is not empty;
- H is a stable subset of $(G, *)$;
- $x^{-1} \in H$ for all $x \in H$.

Example 2.1.1.

We have $H = \{e\}$ and G are the subgroup de G.

Definition 2.1.6. (Morphism (or homomorphism) of groups)

A morphism (or homomorphism) of groups from G to G' is an application $\rho : G \longrightarrow G'$ such that

$$\rho(x.y) = \rho(x) \circ \rho(y), \ for\ all\ x, y \in G. \tag{2.1.1}$$

It is said of a morphism of G towards itself that it is an endomorphism of G. We note $Hom(G, G')$ the set of morphisms from G to G', and $End(G)$ the set of endomorphisms of G.

Proposition 2.1.2.

If $\rho : G \longrightarrow G'$ and $\psi : G' \longrightarrow G''$ are group morphisms, then:

- $\rho(e) = e'$;
- $\rho(x^{-1}) = \rho(x)^{-1}$ for all $x \in G$;

Proof.

- We observe that $\rho(e)e' = \rho(e) = \rho(ee) = \rho(e)\rho(e)$, and therefore $\rho(e)\rho(e) = \rho(e)e'$. Multiplying on the left by the inverse of $\rho(e)$, we find $\rho(e) = e'$
- Consider $x \in G$. We have: $\rho(x)\rho(x)^{-1} = e' = \rho(e) = \rho(xx^{-1}) = \rho(x)\rho(x)^{-1})$, which implies $\rho(x^{-1}) = \rho(x)^{-1}$.

□

2.1.2 Action on a group

Definition 2.1.7. (Action on a group)

Let G be a group and E a set. An action (on the left) of G on E is the data of a law of external composition (on the left)

$$\begin{array}{rcl} G \times E & \rightarrow & E \\ (g, x) & \mapsto & g.x \end{array}$$

verifying the following conditions:

- $e.x = x$, where e is the neutral element of G and $x \in E$;
- $(g_1.g_2).x = g_1.(g_2.x)$ for all $g_1, g_2 \in G$ and $x \in E$.

In this case we also say that G operates (on the left) on E, or that G acts on E. The set E is called a G-set.

Example 2.1.2.

- **Translation left**

  The multiplication on the left of G provides G with the structure of a G-set called translation on the left. Indeed, the law of external composition

  $$\begin{array}{rcl} G \times G & \rightarrow & G \\ (g, x) & \mapsto & gx \end{array}$$

  check for all $x \in E$ and $g_1, g_2 \in G$ that $ex = x$ and $(g_1g_2)x = g_1(g_2x)$ (the law of G is associative).

- **Action by conjugation**

  The action by conjugation of G is defined as follows:

$$\begin{array}{rcl} G \times G & \to & G \\ (g, x) & \mapsto & g.x = gxg^{-1} \end{array}$$

  The action by conjugation is an action because for all $x \in E$ and $g_1, g_2 \in G$ we have $e.x = exe = x$ and $(g_1g_2).x = (g_1g_2)x(g_1g_2)^{-1} = g_1g_2xg_2^{-1}g_1^{-1} = g_1(g_2xg_2^{-1})g_1^{-1} = g_1.(g_2.x)$.

2.1.3 Quotient groups

Definition 2.1.8. (Constructing the quotient of a group)

Here, $(G, .)$ denotes a group and $(H, .)$ denotes a subgroup of G. We also consider the relation, if x and y are elements of G:

$$x\mathcal{R}y \Longleftrightarrow x^{-1}.y \in H$$

Remark.

The relation defined by $x\mathcal{R}y \Longleftrightarrow x^{-1}.y \in H$ is a relation of equivalence.

Definition 2.1.9. (Equivalence classes)

We denote G/H the set $G/\mathcal{R}$ of equivalence classes of the relation $\mathcal{R}$ on G.

Proposition 2.1.3.

Let $x \in G$. The equivalence class of x for the relation

$$x\mathcal{R}y \Longleftrightarrow y.x^{-1} \in H$$

is the set $xH = \{xh; y \in H\}$.

Proof.

Let $y \in G$ be equivalent to x for the relation. Then there is $h \in H$ such that $x^{-1}y = h$. And so there is an element of Hx. Conversely, if there is an eminence xH, it is clear that $y\mathcal{R}x$. $\square$

Definition 2.1.10. (Class on the left)

The set xH is called class on the left of the element x of G.

Remark.

We could also have defined our relationship of equivalence by:

$$x\mathcal{R}y \Longleftrightarrow y.x^{-1} \in H$$

In this case, the equivalence class of an element x of G would have been given by the set Hx.

Definition 2.1.11. (Class on the right)

The set Hx is called class on the right of the element x of G.

Proposition 2.1.4.

If H is a finite subgroup of G and if x and y are two elements of G then the equivalence classes (left or right) of x and y for the relation have same number of elements and this number is equal to the cardinal of H.

Proof.

Let x be an element of G. Set

$$f : H \longrightarrow xH$$

$$h \longmapsto f(h) = xh$$

f is injective because if h and h' are elements of H such that $f(h) = f(h')$ then we have the equality $x.h = x.h'$ and x being elementary of group G, this implies, by multiplying on the left each of the members of the previous equality by x^{-1} that $h = h'$. f is also surjective because if there is an element of xH, then there exists $h \in H$ such that $y = x.h$ and so $y = f(h)$. f both injective and surjective, it is bijective. This proves that H and xH have even number of elements. But if there is an element of G, yH and H will also have the same number of elements. So xH and yH have even cardinal. □

Definition 2.1.12. (Structure of the quotient set of a group)

Note that, if x is an element of G, we will write $\bar{x}$ the equivalence class of x in G/H. $\bar{x}$ will then be a representative of the equivalence class $\bar{x}$.

We will define an internal law $\perp$ on G/H by: If x and y are elements of G then:

$$\bar{x} \perp \bar{y} = \bar{x.y}$$

When no confusion is to be feared, we will note the internal law of G/H of the same way as that of G. This law is the one induced by G on G/H.

2.2 Rings and Modules

2.2.1 Rings

Definition 2.2.1. (Ring)

A ring $(A, +, .)$ Is the data of a set A and of two internal laws $+, .$ verifying:

1- $(A, +)$ is an abelian group.

2- The multiplication . is associative and has a neutral element (noted 1).

3- . is distributive with respect to $+$: for all $x, y, z \in A$, we have

$$x(y+z) = xy + xz \text{ and } (y+z)x = yx + zx$$

Example 2.2.1.

1- The null ring $\{0\}$.

2- $(Z, +, .)$, $(Z/nZ, +, .)$ are commutative rings. such that,
we denote Z/nZ the quotient of Z by the following equivalence relation $\mathcal{R}$, defined on Z by:

$$x\mathcal{R}y \Longleftrightarrow x - y \in nZ$$

Noting $\bar{x}$ the equivalence class of x,

$$Z/nZ = \{\bar{x}, x \in Z\}$$

Z/nZ is provided with the commutative group law:

$$\begin{array}{cccccc} * : & Z/nZ & \times & Z/nZ & \to & Z/nZ \\ & (\overline{x}, \overline{y}) & & & \mapsto & x \overline{+} y \end{array}$$

Definition 2.2.2. (Invertible elements of a ring)

We call together invertible elements of a ring to the set of $x \in A$ such that there exists $y \in A$ with $xy = yx = 1$. It is a group for multiplication, generally noted as A^*.

Example 2.2.2.

1- $(Z/nZ)^*$ is the set of classes $\bar{m}$, with m first to n.

2- In a field K, we have by definition $K^* = K \setminus \{0\}$.

Definition 2.2.3. (Homomorphism (or morphism) of rings)

A homomorphism (or morphism) of rings $f : A \longrightarrow B$ is an application between two verifying rings:

1- $f(x+y) = f(x) + f(y)$.

2- $f(xy) = f(x)f(y)$

3- $f(1) = 1$.

Remark.

Note that the null application is not a ring morphism because it does not verify (3).

Definition 2.2.4. (Sub-ring)

A part of B is a sub-ring if $(B, +, .)$ Is a ring having the same unit as A. It is equivalent to saying that $1 \in B$, and that $(B, +)$ is a subgroup of $(A, +)$ which is stable by internal multiplication.

Let denote A a commutative ring.

2.2.2 Modules

Definition 2.2.5. (A-Module)

An A-module $(M, +, .)$ Is a set equipped with an internal law $+$ and an external law $A \times M \longrightarrow M$, $(\alpha, m) \longmapsto \alpha m$ verifying:

1- $(M, +)$ is an Abelian group.

2- In addition, we have the following four properties:

- $\alpha(m + m') = \alpha m + \alpha m'$
- $(\alpha + \beta)m = \alpha m + \beta m$
- $(\alpha\beta)m = \alpha(\beta m)$
- $1.m = m$

for all $\alpha, \beta \in A$ and all $m, m' \in M$.

Remark.

Since A is assumed to be commutative, there is no need to distinguish between modules on the left and on the right (for non commutative A, the third axiom would be different for a module on the right).

Definition 2.2.6. (Sub-module)

Let M be an A-module. An N sub-module of M is a subgroup of $(M, +)$ which is additionally stable for external multiplication by everything from A.

In other words, a part N of M is a sub-module if and only if contains 0, and if for all x, y of N and every α of A we have: $x + y \in N$ and $\alpha x \in N$.

Example 2.2.3.

- A is an A-module, the external operation being the multiplication in A.

- Any abelian group M can be considered as a Z-module for the external law: $\alpha m = m + m + ... + m$ (α terms) if $\alpha > 0$, $\alpha m = (-\alpha)(-m)$ if $\alpha < 0$ and $0.m = 0$.

Definition 2.2.7. (Homomorphism (or morphism) of A-modules)

A homomorphism (or morphism) of A-modules is a application $f : M \longrightarrow M'$ between two A-modules which verifies:

$$f(x+y) = f(x) + f(y) \text{ and } f(\alpha.x) = \alpha.f(x)$$

for all x, y of M and any α of A. We note $kerf := f^{-1}(\{0\})$ the kernel of f and $Imf := f(M)$ its image. Those are submodules of M, M' respectively.

Remark. There is a notion of modulus on the right:

- For every ring A, there exists a ring A^o or A^{opp} called ring opposite of A such that $A^o = A$, $+^o = +$ and $a \times^o b = ba$
- A morphism of rings AB^o is an additive application $f : A \longrightarrow B$, which sends 1 on 1 and satisfies $f(ab) = f(b)f(a)$.

  We call $f : A \longrightarrow B$ an antimorphism. So we have a correspondence between the modules on the left and right.

Example 2.2.4.

The additive group $(A, +)$ of a ring A is provided with a module structure on A.

Definition 2.2.8. (Morphism of commutative groups)

- An A-module morphism between two A modules M, N is a morphism of commutative groups $f : M \longrightarrow N$ such that for all $x \in M$ and $a \in A$, $f(ax) = af(x)$.
- The $Ker(f)$ kernel (resp., $Im(f)$ image) is the kernel (resp., Image) of the group morphism f.
- An n-multilinear (alternating) application is a $f : M_1 \times ... \times M_n \longrightarrow N$ such that for all i and $(x_j)_{j \neq i}$, $f(x_1, ..., x_{i-1}, ..., x_{i+1}, ..., x_n)$ is A-linear (and its cancels as soon as two variables are equal).

Example 2.2.5.

If A is a commutative ring, the set $hom_A(M, N)$ of the morphisms of A-modules of M in N is provided with a natural structure of A-module.

Definition 2.2.9. (Sub-A-module of module)

A sub-A-module of a module M is a subgroup $N \subset M$ such that for all $a \in A$ and $x \in N$, $ax \in N$.

Example 2.2.6.

The sub-A-modules of A are his ideals.

2.3 Categories and functors

2.3.1 Categories

Definition 2.3.1. (Categories)

A category C is the data:

(i) a class $ob(C)$ of objects of C;

(ii) for any pair (X, Y) of C objects of a set noted $Hom_C(X, Y)$ whose elements are called morphisms of X in Y of C (with the notation $f : X \longrightarrow Y$ for $f \in Hom_C(X, Y)$);

(iii) for any triplet (X, Y, Z) of C objects, an application:

$$\begin{array}{rcl} Hom_C(X, Y) \times Hom_C(Y, Z) & \to & Hom_C(X, Z) \\ (f, g) \mapsto g \circ f & & \end{array}$$

called composition of morphisms, which is associative, and such that for any object X of C, there exists an element $1_X \in Hom_C(X, X)$, called identity of X (sometimes also noted id_X), such that $\forall f \in Hom_C(X, Y)$, we have $f \circ 1_X = f$ and $\forall f \in Hom_C(Y, X)$, we have $1_X \circ f = f$.

Example 2.3.1.

- Ens: the category of sets. Morphisms are the applications, the composition is the composition of applications.
- Top: the category of topological spaces. Morphisms are the continuous applications
- Ab: the category of abelian groups. Morphisms are the morphisms of groups
- Mod: the category of A-modules, with A a ring. Morphisms are the morphisms of A-modules

Definition 2.3.2. (Small category)

A category C is called small if $ob(C)$ is a set.

Definition 2.3.3. (Opposite category)

Let C be a category. The opposite category of C is the category, denoted C^{op}, whose objects are the same as those of C and such that if X, Y are objects of C^{op}, we have:

$$Hom_{C^{op}}(X, Y) = Hom_C(Y, X).$$

Definition 2.3.4. (Subcategory)

Let C be a category. A subcategory C' of C is the data of a subclass $ob(C')$ of $ob(C)$ of objects of C', and for all objects X, Y of C' of a subset $Hom_{C'}(X, Y)$ of $Hom_C(X, Y)$, such as

(i) if X is an object of C', we have $1_X \in Hom_{C'}(X, X)$;

(ii) if X, Y, Z are objects of C' and if $f \in Hom_{C'}(X, Y)$, $g \in Hom_{C'}(Y, Z)$, we have $g \circ f \in Hom_{C'}(X, Z)$

A subcategory C' of C is called full if for all objects X, Y of C', we have $Hom_{C'}(X, Y) = Hom_C(X, Y)$

Example 2.3.2.

Ab, the category of abelian groups, is a full subcategory of Grp

Definition 2.3.5. (Initial - final object)

Let C be a category and X an object of C. We say that X is an initial object (resp. final object) of C if for any object A of C the set $Hom_C(X, A)$ (respectively $Hom_C(A, X)$) is reduced to an element.

Example 2.3.3.

- An initial (or final) object of C is a final object (resp. initial) of C^{op}
- Z is an initial object in Ann, the category of rings, while the null ring is a final object

Definition 2.3.6. (Monomorphism-Epimorphism-Isomorphism)

Let C be a category and $f \in Hom_C(X, Y)$. We say that f is a:

(i) monomorphism if $\forall g, h \in Hom_C(Z, X)$, we have:

$$f \circ g = f \circ h \Longrightarrow g = h$$

(ii) epimorphism if $\forall g, h \in Hom_C(Y, Z)$, we have

$$g \circ f = h \circ f \Longrightarrow g = h$$

(iii) isomorphism if there exists $g \in Hom_C(Y, X)$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$.

Example 2.3.4.

An initial (or final) object of a category is, if it exists, unique to isomorphism.

(Indeed: Let X and Y be initial objects of a category C, and let f, g be the unique morphisms respective of $Hom_C(X, Y)$ and $Hom_C(Y, X)$. Then we have $f \circ g = 1_Y$ and $g \circ f = 1_X$ because $Hom_C(Y, Y)$ and $Hom_C(X, X)$ are reduced to one element. The proof for the end objects is the same.)

2.3.2 Functors

Definition 2.3.7. (Functor)

A functor (covariant) F of a category C to a category C', $F : C \longrightarrow C'$, is the data

(i) For any object X of C of an object $F(X)$ of C'.

(ii) For any pair of objects (X, Y) of C and any $f \in Hom_C(X, Y)$, of an $F(f) \in Hom_{C'}(F(X), F(Y))$ such that:

- for every object X of C, we have $F(1_X) = 1_{F(X)}$;
- $\forall f \in Hom_C(X, Y)$, $\forall g \in Hom_C(Y, Z)$, we have $F(g \circ f) = F(g) \circ F(f)$

Example 2.3.5.

The identity functor $1_C : C \longrightarrow C$, $X \longmapsto X$, $f \longmapsto f$

If $F : C \longrightarrow C'$ and $G : C' \longrightarrow C''$ are functors, we define the compound functor $G \circ F : C \longrightarrow C''$ by $G \circ F(X) = G(F(X))$ for any object X of C, and $G \circ F(f) = G(F(f))$ for any morphism f of C.

Definition 2.3.8. (Contravariant functor)

A contravariant functor F from a category C to a category C' is a covariant functor $F : C \longrightarrow C'^{op}$.

Example 2.3.6.

Let X be an object of a category C. Then $Hom_C(-, X)$ is a contravariant functor of C in Ens, with $f : Y \longrightarrow Z$,

$$\begin{array}{rcl} \mathrm{Hom_C(-,X)(f) = - \circ f : Hom_C(X,Z)} & \rightarrow & \mathrm{Hom_C(X,Y)} \\ \mathrm{u} & \mapsto & u \circ f \end{array}$$

If $C = Mod(A)$, the functor $Hom_A(-, X)$ is a contravariant functor $Mod(A) \longrightarrow Ab$

Theorem 2.3.1.

Let $F : C \longrightarrow C'$ be a functor, and X, Y be objects of C. If X and Y are isomorphic, so $F(X)$ and $F(Y)$ are isomorphic.

Proof.

Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ be such that $f \circ g = 1_Y$ and $g \circ f = 1_X$. Then $1_F(Y) = F(1_Y) = F(f \circ g) = F(f) \circ F(g)$ and $1_{F(X)} = F(1_X) = F(g \circ f) = F(g) \circ F(f)$, hence $F(X)$ and $F(Y)$ are isomorphic. $\square$

Definition 2.3.9. (Morphims of functors)

Let $F, G : C \longrightarrow C'$ be two functors. A morphism of functors (or natural transformation) of F in $G, \phi : F \longrightarrow G$, is the data for each object X of C of a morphism $\phi_X \in Hom_{C'}(F(X), G(X))$ such that $\forall f \in Hom_C(X, Y)$, we have $G(f) \circ \phi_X = \phi_Y \circ F(f)$, that is, the following diagram is commutative

$$\begin{array}{ccc} F(X) & \xrightarrow{\phi_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\phi_Y} & G(Y) \end{array}$$

i.e $\phi_Y \circ F(f) = G(f) \circ \phi_X$

A morphism of functors ϕ is an isomorphism if ϕ_X is an isomorphism for any object X of C.

2.4 Complexes, homology, cohomology

2.4.1 Exact sequences of modules

Definition 2.4.1. (Exact sequences)

Let $f : M \longrightarrow N$ and $g : N \longrightarrow P$ be morphisms of A-modules. The following sequence

$$M \xrightarrow{f} N \xrightarrow{g} P$$

is said to be exact in N if $Ker(g) = Im(f)$. A sequence of morphisms of A-modules

$$M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow ... \longrightarrow M_n \longrightarrow M_{n+1}$$

is said to be exact if it is exact in $M_1, ..., M_n$.

Example 2.4.1.

- The sequence $0 \longrightarrow M \xrightarrow{f} N$ is exact (in M) if and only if f is injective.
- The sequence $M \xrightarrow{f} N \longrightarrow 0$ is exact (in N) if and only if f is surjective
- The sequence $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ is exact if and only if f is injective, g is surjective and $Ker(g) = Im(f)$

Theorem 2.4.1. *(Five Lemma)*

Consider the commutative diagram with exact lines of morphisms of A-modules following:

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "\phi_2"] & M_3 \arrow[r, "f_3"] \arrow[d, "\phi_3"] & M_4 \arrow[r, "f_4"] \arrow[d, "\phi_4"] & M_5 \arrow[d, "\phi_5"] \\ N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r, "g_3"] & N_4 \arrow[r, "g_4"] & N_5. \end{tikzcd} \tag{2.4.1}$$

(i) Suppose ϕ_1 surjective, ϕ_2 injective, ϕ_4 injective. Then ϕ_3 is injective.

(ii) Assume ϕ_2 surjective , ϕ_4 surjective and ϕ_5 injective . So ϕ_3 is surjective

(iii) If ϕ_2 and ϕ_4 are isomorphisms, ϕ_1 is surjective and ϕ_5 is injective, then ϕ_3 is a isomorphism.

Proof.

- Let $x_3 \in M_3$ be such that $\phi_3(x_3) = 0$. Then $g_3(\phi_3(x_3)) = 0 = \phi_4(f_3(x_3))$. ϕ_4 is injective so $f_3(x_3) = 0$, and by accuracy in M_3 there exists $x_2 \in M_2$ such that $x_3 = f_2(x_2)$. We have $0 = \phi_3(x_3) = \phi_3(f_2(x_2)) = g_2(\phi_2(x_2))$, so by exactness in N_2 there exists $y_1 \in N_1$ such that $\phi_2(x_2) = g_1(y_1)$. It exists, by surjectivity of $\phi_1, x_1 \in M_1$ such that $y_1 = \phi_1(x_1)$, and then $g_1(y_1) = g_1(\phi_1(x_1)) = \phi_2(f_1(x_1)) = \phi_2(x_2)$. By injectivity of ϕ_2 we have $f_1(x_1) = x_2$, so $x_3 = f_2(x_2) = f_2(f_1(x_1)) = 0$. So ϕ_3 is injective.
- Let $y_3 \in N_3$. By surjectivity of ϕ_4, there exists $x_4 \in M_4$ such that $\phi_4(x_4) = g_3(y_3)$ and we have $g_4(\phi_4(x_4)) = 0 = \phi_5(f_4(x_4))$. By injectivity of ϕ_5, we have $f_4(x_4) = 0$, and by exactitude in M_4 there exists $x_3 \in M_3$ such that $f_3(x_3) = x_4$. Then $g_3(y_3) = \phi_4(x_4) = \phi_4(f_3(x_3)) = g_3(\phi_3(x_3))$. So for accuracy in N_3, there is $y_2 \in N_2$ such that $y_3 - \phi_3(x_3) = g_2(y_2)$. Since ϕ_2 is surjective, there exists $x_2 \in M_2$ such that $y_2 = \phi_2(x_2)$, hence $y_3 - \phi_3(x_3) = g_2(\phi_2(x_2)) = \phi_3(f_2(x_2))$, and $y_3 = \phi_3(x_3 + f_2(x_2))$. ϕ_3 is therefore surjective. The third assertion is the combination of the first two.

□

2.4.2 Complex

Definition 2.4.2. (Z-graduated A-module)

A Z-graduated A-module is a sequence $M_* = (M_n)_{n\in Z}$ of A-modules. If $M_* = (M_n)_{n\in Z}$ and $N_* = (N_n)_{n\in Z}$ are Z-graduated A-modules, a morphism of Z-graduated A-modules $M_* \longrightarrow N_*$ is a sequence of A-linear applications $f_n : M_n \longrightarrow N_n$, $n \in Z$. The category obtained, denoted $Mod_Z(A)$, is called the category of Z-graduated A-modules.

Remark.

The data of a Z-graduated A-module is equivalent to the data of an A-module M equipped with a direct sum decomposition $M = \bigoplus_{n \in Z} M_n$, where the M_n are sub-A-modules of M.

Definition 2.4.3. (Linear mapping of degree k)

Let $k \in Z$. Let $M_* = (M_n)_{n \in Z}$ and $N_* = (N_n)_{n \in Z}$ be Z-graduated A-modules. A linear mapping of degree k of $M_* \in N_*$ is a sequence of A-linear applications $f_n : M_n \longrightarrow N_{n+k}$, $n \in Z$.

Definition 2.4.4. (Chain complex)

A chain complex (or simply complex) of A-modules is a couple $C = (C_*, d_*)$ where C_* is a Z-graduated A-module and $d_* : C_* \longrightarrow C_*$ is a linear application of degree -1, called differential such that for all $n \in Z$, we have $d_n \circ d_{n+1} = 0$ (that is to say $Im(d_{n+1}) \subset Ker(d_n)$).

$$C : ... \longrightarrow C_{n+1} \xrightarrow{d_n} C_n \xrightarrow{d_{n-1}} C_{n-1} \xrightarrow{d_{n-2}} C_{n-2} \longrightarrow ...$$

Definition 2.4.5. (Homology)

Let $C = (C_*, d_*)$ be a complex of A-modules. We ask, for all $n \in Z$,

- $Z_n(C) = Ker(d_n) \subset C_n$ is called a set of a n-cycles of C
- $B_n(C) = Im(d_{n+1}) \subset C_n$ is called a set of a n-bords of C
- $H_n(C) = Z_n(C)/B_n(C)$. $H_n(C)$ is called the nth A homology module of C (the nth homology group when $A = Z$).

Example 2.4.2.

Let $f : M \longrightarrow N$ and $g : N \longrightarrow P$ morphisms of A-modules such that $g \circ f = 0$. Then $... \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow ...$ is a complex, by putting $M_n = 0$ if $n < 0$ or $n > 2$, $M_0 = P$, $M_1 = N$, $M_2 = M$, and $d_n = 0$ if $n > 2$ or $n \leq 0$, $d_1 = g$, $d_2 = f$. If we denote C this complex, we have $H_0(C) = P/Im(g)$, $H_1(C) = Ker(g)/Im(f)$, $H_2(C) = Ker(f)$, and $H_n(C) = 0$ if $n \neq 0, 1, 2$.

Definition 2.4.6. (Morphism of complex)

Let $C = (C_*, d^C)$ and $D = (D_*, d^D)$ be two complexes. A morphism of complex $f : C \longrightarrow D$ is a linear application of degree 0 that switches to differentials:

$$\begin{tikzcd}
C_n \arrow[r, "f_n"] \arrow[d, "d_n^C"'] & D_n \arrow[d, "d_n^D"'] \\
C_{n-1} \arrow[r, "f_{n-1}"] & D_{n-1}
\end{tikzcd}$$

i.e $f_{n-1} \circ d_n^D = d_n^D \circ f_n$

Remark.

The set of chain complex form a category, where the morphisms are the morphisms of chain complex. We denote this category by $Comp(-)$.

Proposition 2.4.1. $Weibel$ $(1994b)$

Let $f : C \longrightarrow D$ be a morphism of complexes. So for all $n \in Z$, we have $f_n(Z_n(C)) \subset Z_n(D)$ and $f_n(B_n(C)) \subset B_n(D)$, and f induces a morphism of A-modules

$$\begin{array}{rccc} H_n(f): & H_n(C) & \to & H_n(D) \\ & [c] & \mapsto & [f_n(c)] \end{array}$$

In addition $H_n : Comp(A) \longrightarrow Mod(A)$ is a functor.

Definition 2.4.7. (Complex of cochains)

A complex of cochains (or simply complex) of A-modules is a couple $C = (C^*, d^*)$ where C^* is a Z-graduated A-module and $d^* : C^* \longrightarrow C^*$ is a linear application of degree $+1$ such that for all $n \in Z$, we have $d^{n+1} \circ d^n = 0$ (that is $Im(d^n) \subset Ker(d^{n+1})$)

$$C : ... \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} C^{n+2} \longrightarrow ...$$

Definition 2.4.8. (Cohomology)

Let $C = (C^*, d^*)$ be a complex of cochains of A-modules. We ask, for everything $n \in Z$,

- $Z^n(C) = Ker(d^n) \subset C^n$ is called the set of n-cocycles
- $B^n(C) = Im(d^{n-1}) \subset C^n$ is called the set of n-cobords
- $H^n(C) = Z^n(C)/B^n(C)$. $H^n(C)$ is called the nth A-cohomology module of C (the nth group of cohomology when $A = Z$).

2.4.3 Long exact sequence of homology

Definition 2.4.9. (Long exact sequence)

A sequence of complex morphism $C \xrightarrow{f} D \xrightarrow{g} E$ is said to be exact if for all $n \in Z$, the following $C_n \xrightarrow{f} D_n \xrightarrow{g} E_n$ is exact.

Proposition 2.4.2.

Let

$$C \xrightarrow{f} D \xrightarrow{g} E$$

an exact sequence of complexes. So for all $n \in Z$ the following sequence is exact

$$H_n(C) \xrightarrow{H_n(f)} H_n(D) \xrightarrow{H_n(g)} H_n(E)$$

Proof.

We have $H_n(g) \circ H_n(f) = H_n(g \circ f)$. Reciprocally, let $x \in Z_n(D)$ such that $H_n(g)([x]) = 0 = [g_n(x)]$. We thus have $g_n(x) \in B_n(E)$: there exists $y \in E_{n+1}$ such that $g_n(x) = d^E_{n+1}(y)$. By surjectivity of g_{n+1} there exists $z \in D_{n+1}$ such that $y = g_{n+1}(z)$, and so $g_n(x) = d^E_{n+1}(y) = d^E_{n+1}(g_{n+1}(z)) = g_n(d^D_{n+1}(z))$ and $x - d^D_{n+1}(z) \in Ker(g_n) = Im(f_n)$. There exists therefore $c \in C_n$ such that $x - d^D_{n+1}(z) = f_n(c)$.

We have $f_{n-1}(d^C_n(c)) = d^D_n(f_n(c)) = d^D_n(x - d^D_{n+1}(z)) = d^D_n(x) - d^D_n(d^D_{n+1}(z)) = 0$, therefore by injectivity of f_{n-1} we have $d^C_n(c) = 0$, and $c \in Z_n(C)$. Thus $[x] = [f_n(c)] = H_n(f)([c])$. $\square$

Theorem 2.4.2. *(Long exact sequence of homology)*

Let $0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0$ an exact sequence of complexes. There exists for all $n \in Z$, a morphism of A-modules $\nabla_n : H_n(E) \longrightarrow H_{n-1}(C)$ called connection morphism, such that the following long sequence is exact:

$$\dots \longrightarrow H_n(C) \xrightarrow{H_n(f)} H_n(D) \xrightarrow{H_n(g)} H_n(E) \xrightarrow{\nabla_n} H_{n-1}(C) \xrightarrow{H_{n-1}(f)} H_{n-1}(D) \longrightarrow \dots$$

Proof.

We have already established the exactitude in $H_n(D)$, and we must construct ∇_n. Let $z \in Z_n(E)$. We begin by associating with z an element $x \in Z_{n-1}(C)$.

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d^C_{n+1}"] & D_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d^D_{n+1}"] & E_{n+1} \arrow[r] \arrow[d, "d^E_{n+1}"] & 0 \\
0 \arrow[r] & C_n \arrow[r, "f_n"] \arrow[d, "d^C_n"] & D_n \arrow[r, "g_n"] \arrow[d, "d^D_n"] & E_n \arrow[r] \arrow[d, "d^E_n"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "d^C_{n-1}"] & D_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "d^D_{n-1}"] & E_{n-1} \arrow[r] \arrow[d, "d^E_{n-1}"] & 0 \\
0 \arrow[r] & C_{n-2} \arrow[r, "f_{n-2}"] & D_{n-2} \arrow[r, "g_{n-2}"] & E_{n-2} \arrow[r] & 0
\end{tikzcd} \tag{2.4.3}$$

Let $y \in D_n$ such that $g_n(y) = z$. We have $d^E_n(z) = 0 = d^E_n(g_n(y)) = g_{n-1}(d^D_n(y))$. There is therefore a unique $x \in C_{n-1}$ such that $f_{n-1}(x) = d^D_n(y)$. We have $f_{n-2}(d^C_{n-1}(x)) = d^D_{n-1}(f_{n-1}(x)) = d^D_{n-1}(d^D_n(y)) = 0$. By injectivity of f_{n-2} we have $d^C_{n-1}(x) = 0$ and so $x \in Z_{n-1}(C)$.

Let us show that $[x]$, the cohomology class of x, is independent of the choice of y. Let $y' \in D_n$ such that $g_n(y') = z = g_n(y)$. Then $y - y' \in Ker(g_n)$: there exists $a \in C_n$ such that

$y-y' = f_n(a)$. Is $x' \in C_{n-1}$ such that $f_{n-1}(x') = d_n^D(y')$. We have $f_{n-1}(x-x') = d_n^D(y-y') = d_n^D(f_n(a)) = f_{n-1}(d_n^C(a))$, where $x - x' = d_n^C(a)$ and $[x] = [x']$. So we built an application

$$\begin{array}{rccl} \alpha_n : & Z_n(E) & \to & H_{n-1}(C) \\ & z & \mapsto & [x] \end{array}$$

with $f_{n-1}(x) = d_n^D(y)$ and $g_n(y) = z$ which is A-linear.

Let $z \in B_n(E)$, that is $z = d_{n+1}^E(c)$ for $c \in E_{n+1}$. Let $b \in D_{n+1}$ such that $g_{n+1}(b) = c$. We have $g_n(d_{n+1}^D(b)) = d_{n+1}^E(g_{n+1}(b)) = z$. So we have $\alpha_n(x) = [x]$, for x such that $f_{n-1}(x) = d_n^D(d_{n+1}^D(b)) = 0$, hence $\alpha_n([x]) = 0$. Thus α_n induces a morphism of A-modules

$$\begin{array}{rccl} \nabla_n : & H_n(E) & \to & H_{n-1}(C) \\ & z & \mapsto & [x] \end{array}$$

with $f_{n-1}(x) = d_n^D(y)$ and $g_n(y) = z$

Let us show the accuracy in $H_n(E)$. Let $r \in Z_n(D)$. We have $\nabla_n \circ H_n(g)([r]) = \nabla_n([g_n(r)]) = [x]$ for $x \in C_{n-1}$ such that $f_{n-1}(x) = d_n^D(y)$ and $g_n(y) = g_n(r)$, where $f_{n-1}(x) = d_n^D(r) = 0$, and $\nabla_n \circ H_n(g)([r]) = 0$.

Conversely, let $z \in Z_n(E)$ such that $\nabla_n([z]) = 0 = [x]$, where $x \in Z_{n-1}(C)$ satisfies $f_{n-1}(x) = d_n^D(y)$ and $z = g_n(y)(y \in D_n)$. So there exists $c \in C_n$ such that $x = d_n^C(c)$, and we have $f_{n-1}(x) = f_{n-1}(d_n^C(c)) = d_n^D(f_n(c)) = d_n^D(y)$, hence $y - f_n(c) \in Z_n(D)$, with $z = g_n(y - f_n(c))$. Thus $[z] = H_n(g)([y - f_n(c)])$ for $y - f_n(c) \in Z_n(D)$, and $[z] \in Im(\nabla_n)$.

It remains to show the accuracy in $H_{n-1}(C)$. Let $z \in Z_n(E)$. We have $H_{n-1}(f)(\nabla_n([z])) = [f_{n-1}(x)]$ for $x \in C_{n-1}$ satisfying $f_{n-1}(x) = d_n^D(y)$, $z = g_n(y)$, $y \in D_n$. Thus $H_{n-1}(f)(\nabla_n([z])) = [d_n^D(y)] = 0$.

Reciprocally, let $c \in Z_{n-1}(C)$ such that $H_{n-1}(f)([c]) = 0 = [f_{n-1}(c)]$. So there is $r \in D_n$ such that $d_n^D(r) = f_{n-1}(c)$. Let $z = g_n(r) \in E_n$. We have $d_n^E(z) = d_n^E(g_n(r)) = g_{n-1}(d_n^D(r)) = g_{n-1}(f_{n-1}(c)) = 0$, hence $z \in Z_n(E)$, and by construction we have $[c] = \nabla_n([z])$. $\square$

3. Lie algebras cohomology

This chapter is inspired and drawn mainly from the following authors:

$(Hochschild, 1954)$, $(Radul, 1989)$, $(FialowskiandWagemann, 2008)$, $(FeiginandFuchs, 2000)$, $(FarnsteinerandStrade, 1981)$, $(Bourbaki, 2008)$, $(KasselandLoday, 1982)$, $(Neeb, 2006)$

3.1 Theory of Lie algebras

In this section we are going to give some basic definitions, examples and properties of Lie algebras.

Definition 3.1.1. (Lie Algebra)

A Lie algebra G over a field F is a F-vector space together with a F-bilinear map $[,]: G \times G \longrightarrow G$, the so-called Lie bracket, which satisfies the following two conditions for all $x, y, z \in G$:

(a) **Skew-symmetry:** $[x, y] = -[y, x]$,

(b) **Jacobi identity:** $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Remark.

In fact, if $char(F) \neq 2$ and $x, y \in G$, we have

$$[x, y] = -[y, x] \Longleftrightarrow [x, x] = 0.$$

Indeed, for "$\Longleftarrow$" one obtains

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x],$$

using the bilinearity of the bracket. Regarding $\Longrightarrow$ " we set $x = y$ and immediately arrive at $[x, x] = -[x, x]$ and thus, $2[x, x] = 0$. Since we required $char(F) \neq 2$, we conclude $[x, x] = 0$. Hence, if the field is of characteristic 2, only " $\Longrightarrow$ " holds.

Example 3.1.1.

Any vector space over a field can be turned into a Lie algebra if we endow it with the trivial Lie bracket, i.e. setting $[x, y] = 0$ for all its elements. Lie algebras with this property are called **abelian**. Another example is the direct sum $G \oplus H$ of two Lie algebras G, H. Here, the vector space is simply $G \times H$ and the bracket operation is performed componentwise, that is $[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2])$.

Definition 3.1.2. (Lie algebra homomorphism)

Let G, H be two Lie algebras over a field F. We define a Lie algebra homomorphism $\rho : G \longrightarrow H$ to be a linear map that preserves the bracket, that is, for $x, y \in G$ we require

$$\rho([x, y]) = [\rho(x), \rho(y)].$$

Let K, L be two subspaces of G. Define $[K, L]$ to be the subspace generated by the brackets $[x, y]$, $x \in K$,$y \in L$. Thus, each element of $[K, L]$ is a linear combination of brackets $[x_i, y_i]$ for $x_i \in K$, $y_i \in L$. This notation allows us to give the following definition.

Definition 3.1.3. (Lie subalgebra)

A subspace K of G is called a Lie subalgebra of G if:

$$[K, K] \subseteq K$$

i.e. if it is closed under the bracket. Then K itself becomes a Lie algebra in its own right with the inherited operations.

Moreover, $I \subseteq G$ is said to be an ideal or Lie ideal of G if

$$[G, I] = [I, G] \subseteq I,$$

written $I \trianglelefteq G$. In this definition we used that the bracket of subspaces is commutative. Indeed, $[x, y] \in [K, L] = -[y, x] \in [L, K]$, hence $[K, L] \in [L, K]$. Analogously one obtains $[L, K] \subseteq [K, L]$.

Let define now the quotient Lie algebra. If $I \trianglelefteq G$ we can construct the quotient algebra G/I by setting

$$[x + I, y + I] = [x, y] + I$$

for two equivalence classes $x + I, y + I$. Let $x' = x + I$, $y' = y + J$. Then,

$$[x', y'] = [x + I, y + J] = [x, y] + [x, J] + [I, y] + [I, J]$$

applying bilinearity of the bracket. This prompts the independence of the choice of the representative since $[x, J] + [I, y] + [I, J] \in I$ by the ideal property.

Theorem 3.1.1. *(Isomorphism theorems)*

Let $\rho : G \longrightarrow H$ be a Lie algebra homomorphism and let I, J be two ideals of G. Then:

(a) $G/ker\rho \cong im\rho$.

(b) $(G/I)/(J/I) \cong G/J$ *whenever* $I \subseteq J$.

(c) $(I + J)/J \cong I/(I \cap J)$.

Proof.

The proof is omitted since the isomorphisms are given in a canonical way and the procedure is analogous to the case of groups. □

Lemma 3.1.1.

Let A be an associative algebra over a field F, i.e. a F-vector space with an associative, bilinear map $(x, y) \longmapsto x.y$ for elements $x, y \in A$. Then $[x, y] := x.y - y.x$ defines a Lie algebra structure on A.

Proof.

By definition, $[x, y] = -[y, x]$ hence the bracket is skew-symmetric. The Jacobi identity is verified by expanding the product. □

Definition 3.1.4. (Representation of a Lie algebra)

A representation of a Lie algebra G is defined as a F-vector space V together with a Lie algebra homomorphism $\rho : G \longrightarrow GL(V)$. The representation is called faithful if ρ is injective.

Definition 3.1.5. (Adjoint endomorphism)

Let G be a Lie algebra over F. For $x \in G$ we define the adjoint endomorphism

$$ad(x) : G \longrightarrow G,$$
$$ad(x)(y) := [x, y].$$

Then the linear mapping $ad : G \longrightarrow GL(G)$ with $x \longmapsto ad(x)$ defines a representation, called the adjoint representation of G. Observe that:

$$\begin{aligned} \ker(ad(x)) &= \{x \in G : \ ad(x)(y) = 0 \qquad \forall y \in G\} \\ &= \{x \in G : \ [x, y] = 0 \qquad \forall y \in G\} \\ &=: Z(G) \end{aligned}$$

the center of the Lie algebra G.

It is important to add that the adjoint endomorphisms are derivations of g in the sense of the following definition.

Definition 3.1.6. (Derivation)

A linear map $D : G \longrightarrow G$ satisfying

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

for all $x, y \in G$ is called a derivation of the Lie algebra G. The set of all derivations of G is denoted by $der(G)$ and is easily seen to be a vector space. One actually has more structure:

Proposition 3.1.1.

Let G be a Lie algebra over F. Then the following assertions hold:

(a) The derivations $der(G)$ form a Lie subalgebra of $GL(G)$.

(b) For all $x \in G$ the adjoint endomorphism $ad(x) \in der(G)$.

(c) $ad(G)$ is a Lie ideal in $der(G)$.

Proof.

a) Regarding the first statement, we have to show that the commutator of two derivations is again a derivation. Let $D_1, D_2 \in der(G)$ and $x, y \in G$. Then one calculates

$$\begin{aligned} \mathrm{D_1D_2([x,y]) - D_2D_1([x,y])} &= \mathrm{D_1([D_2(x),y] + [x, D_2(y)]) - D_2([D_1(x),y] + [x, D_1(y)])} \\ &= \mathrm{[D_1D_2(x),y] + [D_2(x), D_1(y)] + [D_1(x), D_2(y)] + [x, D_1D_2(y)]} \\ &\quad \mathrm{-[D_2D_1(x),y] - [D_1(x), D_2(y)] - [D_2(x), D_1(y)] - [x, D_2D_1(y)]} \\ &= \mathrm{[D_1D_2(x) - D_2D_1(x), y] + [x, D_1D_2(y) - D_2D_1(y)].} \end{aligned}$$

b) In order to see the second assertion, let $x, y, z \in G$ and observe

$$\begin{aligned} \mathrm{ad(x)([y,z])} &= \mathrm{[x,[y,z]]} \\ &= \mathrm{-[y,[z,x]] - [z,[x,y]]} \\ &= \mathrm{[[x,y],z] + [y,[x,z]]} \\ &= \mathrm{[ad(x)(y), z] + [y, ad(x)(z)]} \end{aligned}$$

c) To prove the third item we show that for every $D \in der(G)$ we have $[D, ad(x)] = ad(D(x))$. To this end, compute

$$\begin{aligned} \mathrm{[D, ad(x)](y)} &= \mathrm{(D(x))(y) - (ad(x)D)(y)} \\ &= \mathrm{D([x,y]) - [x, D(y)]} \\ &= \mathrm{[D(x),y] + [x,D(y)] - [x, D(y)]} \\ &= \mathrm{[D(x),y]} \\ &= \mathrm{ad(D(x))(y)} \end{aligned}$$

□

Remark.

Notice that we can define derivations for any F-algebra A. In fact, one simply defines a linear map $D \in End(A)$ to be a derivation of A if it satisfies

$$D(x.y) = D(x).y + x.D(y)$$

for all $x, y \in A$.

In the following example we display the procedure to compute the adjoint representation of a given Lie algebra.

Definition 3.1.7. (Lie ring)

A Lie ring is an abelian group $(G, +)$ together with a Lie bracket $[,] : G \times G \longrightarrow G$ that satisfies

(a) Z-bilinearity,

(b) skew-symmetry,

(c) the Jacobi identity.

Remark.

Thus, any Lie algebra is a Lie ring if we consider it over an abelian group instead of a field.

3.2 Cohomology of Lie algebras

In this part, we define cohomology of Lie algebras. To this end, we define the Chevalley-Eilenberg chain complex. Let K be a field, G a Lie algebra over K, and V a G-module. Set

$$C^n(G, V) := Alt(G^n, V),\ n > 0, C^0(G, V) := V.$$

That is, $C^n(G, V)$ is the set of alternating n-linear maps $G^n \longrightarrow V$. These are the n-cochains of the Chevalley-Eilenberg complex. We define the differential $d : C^n(G, V) \longrightarrow C^{n+1}(G, V)$ as follows:

Given $c \in C^n(G, V)$, let $dc \in C^{n+1}(G, V)$ be given by

$$dc(x_1, ..., x_{n+1}) = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} c([x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+1}) - \sum_{i=1}^{n+1} (-1)^i x_i.c(x_1, ..., \hat{x}_i, ..., x_{n+1}).$$

for all $x_1, ..., x_{n+1} \in G$, where $x_i.c(x_1, ..., \hat{x}_i, ..., x_{n+1})$ denotes the action of $x_i \in G$ on $c(x_1, ..., \hat{x}_i, ..., x_{n+1}) \in V$ according to the G-module structure of V.

Proposition 3.2.1.

$(C^*(G, V), d)$ is a complex chain.

Proof.

We are going to show in fact that $d^2 = 0$.

We have by definition:

$$dc(x_1,...,x_{n+1}) =$$
$$\sum_{1\leq i<j\leq n+1}(-1)^{i+j}c([x_i,x_j],x_1,...,\hat{x}_i,...,\hat{x}_j,...,x_{n+1}) - \sum_{i=1}^{n+1}(-1)^i x_i.c(x_1,...,\hat{x}_i,...,x_{n+1}).$$

for all $x_1,...,x_{n+1} \in G$. So we have:

$$\begin{aligned}
d^2c(x_1,...,x_n) &= \sum_i (-1)^{i+1}x_i dc(x_1,\hat{x}_i,x_k) + \sum_{i<j}(-1)^{i+j}dc([x_i,x_j],x_1,\hat{x}_i,\hat{x}_j,...,x_k) && (1)\\
&= \sum_{r<i}(-1)^{i+r}(x_ix_r).c(x_1,\hat{x}_r,\hat{x}_i,x_k) + && (2)\\
&+ \sum_{i<r}(-1)^{i+r+1}(x_ix_r).c(x_1,\hat{x}_i,\hat{x}_r,x_k) && (3)\\
&+ \sum_{r<s<i}(-1)^{i+r+s+1}x_i.c([x_r,x_s]x_1,\hat{x}_r,\hat{x}_s,\hat{x}_i,x_k) && (4)\\
&+ \sum_{r<i<s}(-1)^{i+r+s}x_i.c([x_r,x_s]x_1,\hat{x}_r,\hat{x}_i,\hat{x}_s,x_k) && (5)\\
&+ \sum_{i<r<s}(-1)^{i+r+s+1}x_i.c([x_r,x_s]x_1,\hat{x}_i,\hat{x}_r,\hat{x}_s,x_k) && (6)\\
&+ \sum_{i<j}(-1)^{i+j+1}[x_i,x_j].c(x_1,\hat{x}_i,\hat{x}_j,x_k) && (7)\\
&+ \sum_{r<i<j}(-1)^{i+j+r}x_r.c([x_i,x_j],x_1,\hat{x}_r,\hat{x}_i,\hat{x}_j,x_k) && (8)\\
&+ \sum_{i<r<j}(-1)^{i+j+r+1}x_r.c([x_i,x_j],x_1,\hat{x}_i,\hat{x}_r,\hat{x}_j,x_k) && (9)\\
&+ \sum_{i<j<r}(-1)^{i+j+r}x_r.c([x_i,x_j],x_1,\hat{x}_s,\hat{x}_i,\hat{x}_j,x_k) && (10)\\
&+ \sum_{s<i<j}(-1)^{i+j+s}x_r.c([[x_i,x_j],x_s],x_1,\hat{x}_s,\hat{x}_i,\hat{x}_j,x_k) && (11)\\
&+ \sum_{i<s<j}(-1)^{i+j+s+1}c([[x_i,x_j],x_s],x_1,\hat{x}_i,\hat{x}_s,\hat{x}_j,x_k) && (12)\\
&+ \sum_{i<j<s}(-1)^{i+j+s}c([[x_i,x_j],x_s],x_1,\hat{x}_i,\hat{x}_j,\hat{x}_s,x_k) && (13)\\
&+ \sum_{r<s<i<j}(-1)^{i+j+r+s}c([x_r,x_s],[x_i,x_j],x_1,\hat{x}_r,\hat{x}_s,\hat{x}_i,\hat{x}_j,x_k) && (14)\\
&+ \sum_{r<i<s<j}(-1)^{i+j+r+s+1}c([x_r,x_s],[x_i,x_j],x_1,\hat{x}_r,\hat{x}_i,\hat{x}_s,\hat{x}_j,x_k) && (15)\\
&+ \sum_{r<i<j<s}(-1)^{i+j+r+s}c([x_r,x_s],[x_i,x_j],x_1,\hat{x}_r,\hat{x}_i,\hat{x}_j,\hat{x}_s,x_k) && (16)\\
&+ \sum_{i<r<s<j}(-1)^{i+j+r+s+1}c([x_r,x_s],[x_i,x_j],x_1,\hat{x}_i,\hat{x}_r,\hat{x}_j,\hat{x}_s,x_k) && (17)\\
&+ \sum_{i<r<j<s}(-1)^{i+j+r+s+1}c([x_r,x_s],[x_i,x_j],x_1,\hat{x}_i,\hat{x}_r,\hat{x}_j,\hat{x}_s,x_k) && (18)\\
&+ \sum_{i<j<r<s}(-1)^{i+j+r+s}c([x_r,x_s],[x_i,x_j],x_1,\hat{x}_i,\hat{x}_j,\hat{x}_r,\hat{x}_s,x_k) && (19)
\end{aligned}$$

So,
Since, $[x_i,x_r] = x_ix_r - x_rx_i$ *we have :* (2) *and* (3) *will cancel with* (7)
$-$ (4) *will cancel with* (10)
$-$ (5) *will cancel with* (9)
$-$ (6) *will cancel with* (8)
$-$ *Since* $[[x_i,x_j],x_s] = [[x_i,x_s],x_j] - [[x_j,x_s],x_i]$
$= [[x_i,x_s],x_j] + [[x_s,x_j],x_i]$
(11) *and* (8) *will cancel with* (12)

$$So\ we\ have,$$

$$d^2c(x_1, ..., x_n) = 0$$

Then, $(C^*(G, V), d)$ is a complex chain. □

Definition 3.2.1. (Chevalley-Eilenberg complex)

$(C^*(G, V), d)$ is called the Chevalley-Eilenberg complex of G with coefficients V, and the cohomology of G with coefficients in V, $H^*(G, V)$, is defined as the homology of the Chevalley-Eilenberg complex.

Indeed, let us evaluate the groups $H^0(G, V)$ and $H^1(G, V)$ to see how we can use the above complex of Chevalley-Eilenberg.

3.2.1 Cohomology groups: $H^0(G, V)$ and $H^1(G, V)$

Let consider the Chevalley-Eilenberg complex denoted by $(C^*(G, V), d)$, with the differential d given by the following:

$$dc(x_1, ..., x_{n+1}) =$$
$$\sum_{1\leq i<j\leq n+1} (-1)^{i+j} c([x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+1}) - \sum_{i=1}^{n+1} (-1)^i x_i.c(x_1, ..., \hat{x}_i, ..., x_{n+1}).$$

for all $x_1, ..., x_{n+1} \in G$,

So, we are using the Chevalley-Eilenberg complex $(C^*(G, V), d)$ we can split it as follow ::

$$... \xrightarrow{d^{-1}} C^0(G, V) \xrightarrow{d^0} C^1(G, V) \xrightarrow{d^1} C^2(G, V) \xrightarrow{d^2} C^3(G, V) \longrightarrow ...$$

1- Evaluation of $H^0(G, V)$

By definition we have:

$$H^0(G, V) = \frac{Z^0(G,V)}{B^0(G,V)} = \frac{ker(d^0 : C^0(G,V) \longrightarrow C^1(G,V))}{im(d^{-1})}$$

since $C^0(G, V) = V$ (by definition of Chevalley-Eilenberg complex $(C^*(G, V), d)$), we have:

$$H^0(G, V) = \frac{ker(d^0 : V \longrightarrow C^1(G,V))}{im(d^{-1})} = \underset{x \in G}{ker\{v \in V,\ such\ that\ v \longmapsto (x \longmapsto x.v = 0)\}}$$, with

according to the definition of the differential d of Chevalley-Eilenberg.

So, we conclude that,

$$H^0(G,V) = \frac{ker(d^0:V\longrightarrow C^1(G,V))}{im(d^{-1})} = ker\{v \in V,\ such\ that\ v \longmapsto (x \longmapsto x.v = 0)\}, \text{ with } x \in G$$

2- Evaluation of $H^1(G,V)$

By definition we have:

$$H^1(G,V) = \frac{Z^1(G,V)}{B^1(G,V)} = \frac{ker(d^1:C^1(G,V)\longrightarrow C^2(G,V))}{im(d^0:C^0(G,V)\longrightarrow C^1(G,V))}$$

since $C^0(G,V) = V$ (by definition of Chevalley-Eilenberg complex $(C^*(G,V),d)$), we have:

$$H^1(G,V) = \frac{Z^1(G,V)}{B^1(G,V)} = \frac{ker(d^1:C^1(G,V)\longrightarrow C^2(G,V))}{im(d^0:V\longrightarrow C^1(G,V))} = \frac{c:G\longrightarrow V\ such\ that\ \forall x,y\in G: c[x,y]-x.c(y)+y.c(x)=0}{\exists c:G\longrightarrow V\ such\ that\ \forall x\in G\ and\ v\in V, c_v(x):=x.v}$$

according to the definition of the differential d of Chevalley-Eilenberg.

So, we conclude that,

$$H^1(G,V) = \frac{Z^1(G,V)}{B^1(G,V)} = \frac{ker(d^1:C^1(G,V)\longrightarrow C^2(G,V))}{im(d^0:V\longrightarrow C^1(G,V))} = \frac{c:G\longrightarrow V\ such\ that\ \forall x,y\in G: c[x,y]-x.c(y)+y.c(x)=0}{\exists c:G\longrightarrow V\ such\ that\ \forall x\in G\ and\ v\in V, c_v(x):=x.v}$$

Hence, denoting by $der(G,V)$ the space of derivations (defined in the section of Theory of Lie algebra in this work) given as follow :

For a linear map $c : G \longrightarrow V$ such that for all $x, y \in G : c[x,y] = x.c(y) - y.c(x)$.

and denoting by $inder(G,V)$ the space of inner derivations given as follows:

For a linear map $c : G \longrightarrow V$ which is obtained as $c_v(x) := x.v$ for some $v \in V$

we have then the cohomology group $H^1(G,V)$ given by:

$$H^1(G,V) = \frac{der(G,V)}{inder(G,V)}$$

As we now know the first two groups of cohomology and know how they look like, what about the second group $H^2(G,V)$?

3.2.2 Second group of cohomology: $H^2(G,V)$

Comparing to the case of $H^0(G,V)$ and $H^1(G,V)$, by definition we have:

$$H^2(G,V) = \frac{Z^2(G,V)}{B^2(G,V)} = \frac{ker(d^2:C^2(G,V)\longrightarrow C^3(G,V))}{im(d^1:C^1(G,V)\longrightarrow C^2(G,V))}$$

Since $ker(d^2 : C^2(G,V) \longrightarrow C^3(G,V))$ and $im(d^1 : C^1(G,V) \longrightarrow C^2(G,V))$ are not known, we can not compute directly the group $H^2(G,V)$ explicitly as we did in the case of $H^0(G,V)$ and $H^1(G,V)$ (see 3.2.1). So, we are going to introduce the notion of abelian extension which will be very helpful to find or to evaluate $H^2(G,V)$.

3.3 Abelian extension for Lie algebra

Our goal in this section is to construct the bijection between the class of abelian extension of Lie algebra and the second group of cohomology. In fact, we are going to prove the following theorem.

Theorem 3.3.1.

The set of equivalence classes of abelian extensions with fixed cokernel H and fixed kernel A is in bijection with $H^2(H, A)$.

3.3.1 Definitions and properties

Definition 3.3.1. (Extension)

Let H and A be Lie algebras. An **extension** of H by A is a short exact sequence:

$$0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0,$$

of Lie algebras.

It is called **abelian**, if A is abelian, i.e. if $[.,.]_A \equiv 0$.

Definition 3.3.2. (Equivalent abelian extensions)

Two abelian extensions

$$0 \longrightarrow A \xrightarrow{i} G \xrightarrow{\pi} H \longrightarrow 0$$

and

$$0 \longrightarrow A \xrightarrow{i'} G' \xrightarrow{\pi'} H \longrightarrow 0,$$

such that G and G' have the same kernel A and the same cokernel H are called equivalent if there exists a morphism of Lie algebras $\Phi : G \longrightarrow G'$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "id_A"] & G \arrow[r] \arrow[d, "\Phi"] & H \arrow[r] \arrow[d, "id_H"] & 0 \\ 0 \arrow[r] & A \arrow[r] & G' \arrow[r] & H \arrow[r] & 0. \end{tikzcd} \tag{3.3.2}$$

is commutative.

Remark.

- An extension $0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0$, is often denoted only by $A \longrightarrow G \longrightarrow H$ or even $G \longrightarrow H$.
- $Ext(H, A)$ denotes the set of equivalence classes of extensions of H by A vector space A
- According to the five Lemma, ϕ is bijective.

Definition 3.3.3. (G-module)

Let V be a K-vector space and G a K-Lie algebra. V is called a **G-module**, if there exists a morphism of Lie algebras $\phi : G \longrightarrow End(V)$.

In fact, let G be a Lie algebra. A G-module is a K-vector space V together with a representation $\psi : G \longrightarrow End(V)$ of G, i.e. ψ is a Lie algebra homomorphism. In other words, a G-module is a K-vector space V and a linear action of G on V, $-,- : G \times V \longrightarrow V$, satisfying

$$[X, Y].v = X.(Y.v) - Y.(X.v) \text{ for all } X, Y \in G, v \in V.$$

Example 3.3.1. (G-module)

1- K with the trivial action, $G \longrightarrow End(K)$ the zero map.

2- G itself with the adjoint action; $ad : G \longrightarrow End(G)$, $ad(x)(y) = [x, y]$.

Lemma 3.3.1.

If $0 \longrightarrow A \longrightarrow G \xrightarrow{\pi} H \longrightarrow 0$ or $G \xrightarrow{\pi} H$ is an abelian extension of H by A, there exists a H-module structure on A.

Proof.

Let $s : H \longrightarrow G$ be a section of π i.e s is a linear map with $s \circ \pi = id_H$. It exists, as one can choose a basis on H and lift each basis vector to a vector in G.

Set

$$\alpha : H \times A \longrightarrow A$$

as $\alpha(x).a = [\alpha(x), a]_G$ for all $x \in H$, $a \in A$, where a is not distinguished from its image in G. So now, it remains to show:

- α is does not depend on the choice of the section s:

  Let $g_1, g_2 \in G$ with $\pi(g_1) = \pi(g_2) \Longrightarrow g_1 - g_2 \in ker\pi \cong A \Longrightarrow [g_1, a]_G - [g_2, a]_G = [g_1 - g_2, a]_G = 0$, as A is abelian.

- $[g_1, a] \in ker\pi \cong A$:

  This holds, as $ker\pi$ is an ideal in G.

□

Corollary.

Equivalent abelian extensions of H by A, $0 \longrightarrow A \longrightarrow G \xrightarrow{\pi} H \longrightarrow 0$ lead to the same module structure on A.

Proof.

Since the abelian extensions are equivalent, the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "id_A"] & G \arrow[r] \arrow[d, "\Phi"] & H \arrow[r] \arrow[d, "id_H"] & 0 \\ 0 \arrow[r] & A \arrow[r] & G' \arrow[r] & H \arrow[r] & 0. \end{tikzcd} \tag{3.3.1}$$

Choose linear sections $s$and s' of π and π'. Since ψ is the identity on A, it suffices to show:

$$\psi([s(x), a]) = [s'(x), a]$$

$\forall a \in A$, $x \in H$ as elements of A.

This holds, because $\psi([s(x), a]) = [\psi(s(x)), \psi(a)] = [\psi(s(x)), a]$ as the diagram commutes.

As shown above, the module structure does not depend on the choice of the section in a given extension, and $\psi \circ s$ defines another section of $G' \longrightarrow H$. □

3.3.2 Evaluation of $H^2(G, V)$

In fact, our vision is show that the group $H^2(G, V)$ is isomorphic with the class of equivalence of abelian extensions denoted by $Ext(G, V)$, i.e we are going to show the following theorem.

Theorem 3.3.2.

Let G be a Lie algebra and V an abelian Lie algebra. Then:

$$Ext(G, V) \cong H^2(G, V) = Z^2(G, V)/d(C^1(G, V))$$

Before trying to prove this theorem 3.3.2, let us start first with the following result.

Proposition 3.3.1.

Let $\alpha \in C^2(G, V)$. Then α leads to an abelian extension $G_\alpha = V \oplus G$ with the Lie bracket

$$[(v_1, g_1), (v_2, g_2)] = (g_1.v_2 - g_2.v_1 + \alpha(g_1, g_2), [g_1, g_2])$$

$\forall v_1, v_2 \in V$, $g_1, g_2 \in G$ of G by V if and only if $\alpha \in Z^2(G, V)$.

Proof.

By definition the Lie bracket is skew symmetric, but the Jacobi identity should hold, too. Writing down the first term of the Jacobi identity one gets, with $v_1, v_2, v_3 \in V$ and $g_1, g_2, g_3 \in G$

$$[[(v_1, g_1), (v_2, g_2)], (v_3, g_3)] = [g_1.v_2 - g_2.v_1 + \alpha(g_1, g_2), [g_1, g_2]), (v_3, g_3)]$$
$$= ([g_1, g_2].v_3 - g_3.(g_1.v_2 - g_2.v + \alpha(g_1, g_2) + \alpha([g_1, g_2], g_3), [[g_1, g_2], g_3])$$

If one takes the sum over the permutations, as in the Jacobi identity, the second component vanishes, as the Jacobi identity holds for the Lie bracket in G. If one focuses at the first component, one gets

$$[g_1, g_2].v_3 - g_3.(g_1.v_2 - g_2.v_1 + \alpha(g_1, g_2)) + \alpha([g_1, g_2], g_3)$$
$$+(g_3, g_1].v_2 - g_2.(g_3.v_1 - g_1.v_3 + \alpha(g_3, g_1)) + \alpha([g_3, g_1], g_2))$$
$$+([g_2, g_3].v_1 - g_1.(g_2.v_3 - g_3.v_2 + \alpha([g_2, g_3])) + \alpha([g_2, g_3], g_1))$$
$$= [g_1, g_2].v_3 - g_1.g_2.v_3 + g_2.g_1.v_3 + [g_3, g_1].v_2 - g_3.g_1.v_2 + g_1.g_3.v_2$$
$$+[g_2, g_3].v_1 - g_2.g_3.v_1 + g_3.g_2.v_1$$
$$+g_1.\alpha(g_2, g_3) - g_1.\alpha(g_1, g_3) + g_3.\alpha(g_1, g_2) - \alpha([g_1, g_2], g_3) + \alpha([g_1, g_3], g_2) - \alpha([g_2, g_3], g_1)$$

The first two rows of terms vanish by the definition of a Lie algebra action.

The third and fourth row of terms are exactly the equation one gets as the derivative $d\alpha$ of $\alpha \in C^2(G, V)$. The Jacobi identity holds if and only if it vanishes. This equivalent to α being a cocycle. Then $G_\alpha \longrightarrow G$ defines an extension of G by V. □

Proof. (Theorem 3.3.2)

- $\supseteq$)

  Let $\alpha \in Z^2(G, V)$. As shown above, this leads to an extension of G by V.

  It remains to show, that equivalent cocycles α, α' lead to equivalent extensions. Therefore one wants to define a Lie algebra homomorphism $\psi : G_\alpha \longrightarrow G_{\alpha'}$, which has to be of the form $\psi : (v, g) \longmapsto (v + \psi(g), g)$, as it has to be the identity on the second component, and you get the first component by linearity and Lie brackets. One computes by an similar calculation as above, that it commutes with the Lie brackets if and only if $\alpha = \alpha' + d\psi$, that means, if α and α' are in the same equivalence class.

  In this part of the proof, it is only necessary to know, that equivalent cocycles are mapped to the same equivalence class of extensions, but later also the other direction is needed, i.e. if G_α and $G_{\alpha'}$ are eqivalent, then α and α' are so as well via the equation between them above.

- $\subseteq$)

Let $A \xrightarrow{i} G \xrightarrow{\pi} H$ be an abelian extension. One wants to show, that it is equivalent to another abelian extension $H_\alpha \longrightarrow H$, where $\alpha \in Z^2(G, V)$.

Choose a section $s : H \longrightarrow G$, such that $\pi \circ s = id_H$, as above.

Let

$$\alpha(g, h) = [s(g), s(h)] - s([g, h])$$

for all $g, h \in H$. To make sure it is a cocycle, one has to check:

- $im(\alpha) \subset A \cong i(A)$, i.e. $\pi(\alpha(g, h)) = 0$: π is a Lie algebra homomorphism, and therefore $\pi(\alpha(g, h)) = [\pi(s(g)), \pi(s(h))] - \pi(s([g, h])) = 0$
- $\alpha \in Z^2(H, A)$, i.e. $d\alpha = 0$ (via computation, using skew symmetry and Jacobi identity for the Lie brackets, and the definition of the H-action on A). In fact, we have:

  Let $x \in H$ and $\alpha(y, z) \in A$.

  The product $x.\alpha(y, z)$ is an action of H on A. Let us define the action as follows:

  $$\begin{array}{rcl} H \times A & \to & A \\ (h, a) & \mapsto & h.a \end{array}$$

  which satisfy the following properties:

  * $e.a = a$, where e is the neutral element of H and $a \in A$;
  * $(h_1.h_2).a = h_1.(h_2.a)$ for all $h_1, h_2 \in H$ and $a \in A$.

  So, in particular for all $y, z \in H$ such that $\alpha(y, z) \in A$ (with $\alpha : H \times H \longrightarrow A$), we define the following map:

  $$\begin{array}{rcl} H \times A & \to & A \\ (x, \alpha(y, z)) & \mapsto & x.\alpha(y, z) \end{array}$$

  since $x \in H$ and the map $\pi : G \longrightarrow H$ is surjective (because the sequence $0 \longrightarrow A \xrightarrow{i} G \xrightarrow{\pi} H \longrightarrow 0$ is exact), then there exists $y \in G$ such that $x = \pi(y)$, in fact,

  $$x = \pi(s(x)) \text{ and } i(\alpha(y, z)) = \alpha(y, z)$$

  So, we have finally according to definition of our action, we can define the map by:

  $$\begin{array}{rcl} H \times A & \to & A \\ (x, \alpha(y, z)) & \mapsto & x.\alpha(y, z) = [s(x), \alpha(y, z)] \ \in \ A \end{array}$$

  We are going to show now that for $\alpha \in Z^2(H, A)$, $d\alpha = 0$ with d the differential of Chevalley-Eilenberg. So, Let prove that $d\alpha = 0$.

$$d\alpha = -\alpha([x,y],z) - \alpha([y,z],x) - \alpha([z,x],y) + x.\alpha(y,z) + y.\alpha(z,x) + z.\alpha(x,y) = -s[[x,y],z] + [s[x,y],s(z)] - s[[y,z],x] + [s[y,z],s(x)] - s[[z,x],y] + [s[z,x],s(y)] + [s(x),s[y,z]] - [s(x),[s(y),s(z)]] + [s(y),s[z,x]] - [s(y),[s(z),s(x)]] + [s(z),s[x,y]] - [s(z),[s(x),s(y)]]$$
$$= [s[x,y],s(z)] + [s[y,z],s(x)] + [s[z,x],s(y)] + [s(x),s[y,z]] + [s(y),s[z,x]] + [s(z),s[x,y]] = 0$$

So, $d\alpha = 0$.

Now we want to show, that $H_\alpha \longrightarrow H$ is equivalent to $A \longrightarrow G \longrightarrow H$:

Set $\psi : H_\alpha \cong V \times H \longrightarrow G$, $(v,g) \longmapsto i(v)+s(g)$. ψ is a bijective module homomorphism. ψ is as well a Lie algebra homomorphism from G_α to G as $\psi([(v,g),(v',g')]) = i(g.v') - i(g'.v) + i([s(g),s(g')]) - i(s[g,g']) + s[g,g'] = [i(v),s(g')] + [s(g),i(v')] + [s(g),s(g')]$ (use again the definition of the H-action on V) α only depends on the equivalence class of the given extension:

If two equivalent extensions lead to two different extensions H_α and H_α, these are equivalent, too. Due to the remark in the first part of the proof, then α and α' are equivalent.

□

Conclusion:

So, we have seen that the calculus of $H^0(G,V)$, $H^1(G,V)$ can be evaluated directly using the explicit definition of cohomology, but for the case of of $H^2(G,V)$ we realized that we can not fine it explicitly using the direct computation, so we proved that there exist a bijection between the class of equivalence abelian extensions and $H^2(G,V)$ according to the theorem 3.3.2. Now, what about the third group $H^3(G,A)$? How can we evaluate it?. That is the goal of the next chapter.

4. Crossed module for Lie algebras

This chapter is inspired and drawn mainly from the following authors:

$(Wagemann, 2006)$, $(Gerstenhaber, 1964)$, $(Gerstenhaber, 1966)$, $(AgrebaouiandFraj, 2003)$

Let G a Lie algebra and V a vector space. In this chapter, we are going to evaluate the third group of cohomology denoted by $H^3(G,V)$. In fact, as we have seen in the chapter 3, the computation of the two first groups $H^0(G,V)$ and $H^1(G,V)$ is direct, but for the second group $H^2(G,V)$ it is not possible to get it by the same process using just the explicit definition of cohomology, we saw in fact that, we can make a bijection between the class of equivalence of abelian extensions denoted by $Ext(G,V)$ and itself i.e $H^2(G,V)$ (see theorem 3.3.2). Now, our goal is to see how the third group $H^3(G,V)$ look like or how we can evaluate it. By definition, using the Chevalley-Eilenberg complex $(C^*(G,V),d)$ (see proposition 3.2.1) defined as follow:

$$... \xrightarrow{d^{-1}} C^0(G,V) \xrightarrow{d^0} C^1(G,V) \xrightarrow{d^1} C^2(G,V) \xrightarrow{d^2} C^3(G,V) \longrightarrow C^4(G,V) \longrightarrow ...$$

where for all n, the differential $d^n : C^n(G,V) \longrightarrow C^{n+1}(G,V)$ is define as follows:

Given $c \in C^n(G,V)$, let $d^n c \in C^{n+1}(G,V)$ be given by

$$d^n c(x_1,...,x_{n+1}) =$$
$$\sum_{1\leq i<j\leq n+1} (-1)^{i+j} c([x_i,x_j],x_1,...,\hat{x}_i,...,\hat{x}_j,...,x_{n+1}) - \sum_{i=1}^{n+1}(-1)^i x_i.c(x_1,...,\hat{x}_i,...,x_{n+1}).$$

for all $x_1,...,x_{n+1} \in G$, where $x_i.c(x_1,...,\hat{x}_i,...,x_{n+1})$ denotes the action of $x_i \in G$ on $c(x_1,...,\hat{x}_i,...,x_{n+1}) \in V$ according to the G-module structure of V.

we have:

$$H^3(G,V) = \frac{Z^3(G,V)}{B^3(G,V)} = \frac{ker(d^3 : C^3(G,V) \longrightarrow C^4(G,V))}{im(d^2 : C^2(G,V) \longrightarrow C^3(G,V))}$$

Since $ker(d^3 : C^3(G,V) \longrightarrow C^4(G,V))$ and $im(d^2 : C^2(G,V) \longrightarrow C^3(G,V))$ are not known, we can not compute directly the group $H^3(G,V)$ explicitly as we did in the case of $H^0(G,V)$ and $H^1(G,V)$ (see 3.2.1). So, we are going to introduce the notion of Crossed Modules which will be very helpful to find or to evaluate $H^3(G,V)$.

4.1 Definitions and properties

4.1.1 Definitions

Definition 4.1.1. (Crossed modules)

Let M, N two Lie algebras and

$$\begin{array}{rcl} \eta: N & \to & der(M) \\ n & \mapsto & \eta(n): M \to M \\ & & \quad m \mapsto \eta(n).m = n.m \end{array}$$

Where $der(M)$ is a set of all derivations of M, defined in 3.1.6.

A crossed module of Lie algebras is a homomorphism of Lie algebras $\mu : M \longrightarrow N$, such that:

i- $\mu(n.m) = [n, \mu(m)]$ for all $n \in N$ and all $m \in M$,

ii- $\mu(m).m' = [m, m']$ for all $m, m' \in M$.

Remark.

To each crossed module of Lie algebras $\mu : M \longrightarrow N$, one associates a four term exact sequence:

$$0 \longrightarrow V \xrightarrow{i} M \xrightarrow{\mu} N \xrightarrow{\pi} G \longrightarrow 0$$

where $ker\mu = V$ and $coker\mu = G$

Remark.

- According to the point $i)$ in definition 4.1.1, $im(\mu)$ is an ideal, and thus G is a Lie algebra.
- According to the point $ii)$ in definition 4.1.1, V is a central ideal of M, and in particular abelian

Definition 4.1.2. (Elementary equivalence)

Two crossed modules $\mu : M \longrightarrow N$ (with action η) and $\mu' : M' \longrightarrow N'$ (with action η') such that $ker(\mu) = ker(\mu') =: V$ and $coker(\mu) = coker(\mu') =: G$ are called elementary equivalent if there are morphisms of Lie algebras $\phi : M \longrightarrow M'$ and $\psi : N \longrightarrow N'$ which are compatible with the actions, meaning

$$\phi(\eta(n)(m)) = \eta'(\psi(n))(\phi(m)),$$

for all $n \in N$ and all $m \in M$, and such that the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d, "id_V"] & M \arrow[r] \arrow[d, "\Phi"] & N \arrow[r] \arrow[d, "\psi"] & G \arrow[r] \arrow[d, "id_G"] & 0 \\ 0 \arrow[r] & V \arrow[r] & M' \arrow[r] & N' \arrow[r] & G \arrow[r] & 0. \end{tikzcd} \tag{4.1.2}$$

Definition 4.1.3. (Sum of crossed modules)

Consider two crossed modules $\mu : M \longrightarrow N$ and $\mu' : M' \longrightarrow N'$ with isomorphic kernel and cokernel and their corresponding four term exact sequences

$$0 \longrightarrow V \xrightarrow{i} M \xrightarrow{\mu} N \xrightarrow{\pi} G \longrightarrow 0 \tag{4.1.1}$$

and

$$0 \longrightarrow V \xrightarrow{i'} M' \xrightarrow{\mu'} N' \xrightarrow{\pi'} G \longrightarrow 0 \tag{4.1.2}$$

The sum of this two crossed modules 4.1.1 and 4.1.2, denoted by $\mu \oplus \mu'$, is the following crossed module:

$$0 \longrightarrow V \xrightarrow{(i\oplus i')\circ\Delta} \frac{M \oplus M'}{K} \xrightarrow{\mu\oplus\mu'} N \oplus_G N' \xrightarrow{\pi} G \longrightarrow 0 \tag{4.1.3}$$

where, $K := \{(v, -v) \in V \oplus V\}$ is the kernel of the addition map $V \oplus V \longrightarrow V$. K can be considered as a subspace in $M \oplus M'$ via $i \oplus i'$. As V is central in M and M', K is an ideal of $M \oplus M'$.

$\Delta : V \longrightarrow V \oplus V$ is the diagonal followed by the quotient map $V \oplus V \longrightarrow (V \oplus V)/K$ identifies V and $(V \oplus V)/K$, and $N \oplus_G N' = \{(n, n') \in N \oplus N'/\pi(n) = \pi(n')\}$

Remark.

- Let $T : V \oplus V \longrightarrow V$. Since T is surjective, We have

  $$T(V \oplus V) = V$$

  and by the isomorphism theorem we have:

  $$\tfrac{V\oplus V}{K} \cong V$$

- From the set $N \oplus_G N' = \{(n, n') \in N \oplus N'/\pi(n) = \pi(n')\}$ defined in 4.1.3, we have that:

  $$\pi : N \oplus_G N' \longrightarrow G$$

  can be identified with $\frac{1}{2}(\pi+\pi') : N \oplus_G N' \longrightarrow G$ and $\frac{1}{2}(\pi+\pi')(n, n') = \frac{1}{2}[\pi(n)+\pi'(n)] = \pi(n)$

Notation: Let us denote by $crmod(G, V)$ the set of equivalence classes of Lie algebra crossed modules with respect to fixed kernel V and fixed cokernel G.

As we did in the case of extension abelian, we can consider the following exact short sequence :

$$0 \longrightarrow V \xrightarrow{i} M \xrightarrow{\mu} N \xrightarrow{\pi} G \longrightarrow 0 \tag{4.1.4}$$

where, the kernel of μ is V and the cokernel of μ is G

Let ρ, ρ' the sections of π. Recall that ρ is a section of π if the following diagram is commutative:

$$\begin{tikzcd} N \arrow[r, "\pi"] \arrow[d, "\rho"'] & G \\ G \arrow[ur, "\pi\circ\rho=id_G"'] & \end{tikzcd} \tag{4.1.4}$$

We are going to define the action of G on M as follow:

$$\begin{tikzcd} G \arrow[r, "\eta\circ\rho"] \arrow[d, "\rho"'] & der(M) \\ N \arrow[ur, "\eta"'] & \end{tikzcd} \tag{4.1.4}$$

such that :

$$\begin{array}{llllll} \eta\circ\rho: & G & \to & der(M) & & \\ & g & \mapsto & \eta\circ\rho(g): & M & \to M \\ & & & & m & \mapsto \eta\circ\rho(g).m = g.m \end{array}$$

So, for ρ, ρ' the sections of π, we have $\eta\circ\rho$ and $\eta\circ\rho'$ define as follow:

$$\begin{array}{l} \eta\circ\rho: \ G \ \to \ der(M) \\ \qquad g \mapsto \eta\circ\rho(g): \ M \ \to \ M \\ \qquad\qquad\qquad\qquad m \ \mapsto \eta\circ\rho(g)(m) \\ and \\ \\ \eta\circ\rho': \ G \ \to \ der(M) \\ \qquad g \mapsto \eta\circ\rho'(g): \ M \ \to \ M \\ \qquad\qquad\qquad\qquad m \ \mapsto \eta\circ\rho'(g)(m) \end{array}$$

We can observe by the following proposition that $\eta\circ\rho$ is an action.

Proposition 4.1.1.

Let $\lambda\in G$ and $\mu: M \longrightarrow N$ a crossed module with an action η. $\eta(\rho(\lambda))$ and $\eta(\rho'(\lambda))$ differ by the inner derivation $ad_{m'}$ for some $m'\in M$

Proof.

Let $\lambda\in G$.

As ρ and ρ' are the sections of π, we have:

$\pi\circ\rho = id_G$ and $\pi\circ\rho' = id_G$ i.e $\pi(\rho)-\pi(\rho')=0 \Longrightarrow \pi(\rho-\rho')=0$. Then, $\rho-\rho'\in ker\pi$

Since our sequence is exact, we have $ker\pi = Im\mu$, so $(\rho - \rho') \in ker\pi = Im\mu$ i.e $\exists m' \in M$ such that

$$\rho - \rho' = \mu(m')$$

So we have for all $m \in M$:

$$\eta(\rho(\lambda))(m) - \eta(\rho'(\lambda))(m) = \eta((\rho - \rho')(\lambda))(m)$$

$$= \eta(\mu(m'))(m) = [m', m].\text{(according to } (ii-) \text{ of the definition 4.1.1)}$$

so we conclude that the expression $\eta(\rho(\lambda))$ is well-defined up to inner derivation. □

Indeed, let us consider the failure α associated to ρ in such a way that π will be a morphism of Lie algebras. Let us consider α by the follow:

$$\forall x, y \in G \ \alpha(x, y) = [\rho(x), \rho(y)] - \rho([x, y])$$

Remark.

1- We have $\pi(\alpha(x, y)) = 0$.

   Indeed, we have:

   $\pi(\alpha(x, y)) = \pi([\rho(x), \rho(y)] - \rho([x, y])) = \pi([\rho(x), \rho(y)]) - \pi(\rho([x, y])) = [\pi(\rho(x)), \pi(\rho(y))] - \pi(\rho([x, y])) = [x, y] - [x, y] = 0$, since ρ is a section of π. This implies that $\alpha(x, y) \in ker\pi = Im\mu$ (since the sequence is exact). Then, there exists $\beta(x, y) \in M$ such that $\mu(\beta(x, y)) = \alpha(x, y)$.

2- We have $\eta \circ \rho$ an action also up to inner derivations. Indeed, considering $\alpha(x, y) = [\rho(x), \rho(y)] - \rho([x, y])$, and by the previous remark i.e there exists $\beta(x, y) \in M$ such that $\mu(\beta(x, y)) = \alpha(x, y)$, we have: for some $m \in M$,

   $$\eta([\rho(x), \rho(y)] - \rho([x, y]))(m) = \eta(\alpha(x, y))(m) = \eta(\mu(\beta(x, y)))(m) = [\beta(x, y), m],$$
   according to the second property of the definition 4.1.1.

   so, in this sense, an outer action is an action up to inner derivations.

Lemma 4.1.1.

$(crmod(G, V), \oplus)$ is a group

Proof.

Recall that $crmod(G, V)$ is a set of equivalence classes of Lie algebra crossed modules with respect to fixed kernel V and fixed cokernel G.

1) Let us show that $\oplus$ is associative i.e

Let $\mu_1 : M_1 \longrightarrow N_1$, $\mu_2 : M_2 \longrightarrow N_2$ and $\mu_3 : M_3 \longrightarrow N_3$ tree crossed modules in $crmod(G,V)$ it means that we can respectively associated to each of them the following sequences:

$$0 \longrightarrow V \longrightarrow M_1 \xrightarrow{\mu_1} N_1 \longrightarrow G \longrightarrow 0$$
$$0 \longrightarrow V \longrightarrow M_2 \xrightarrow{\mu_2} N_2 \longrightarrow G \longrightarrow 0$$
$$0 \longrightarrow V \longrightarrow M_3 \xrightarrow{\mu_3} N_3 \longrightarrow G \longrightarrow 0$$

Our goal here is to show that:

$$(\mu_1 \oplus \mu_2) \oplus \mu_3 = \mu_1 \oplus (\mu_2 \oplus \mu_3)$$

Now according to definition 4.1.3, we have $(\mu_1 \oplus \mu_2)$ which correspond to the following sequence:

$0 \longrightarrow \frac{V \oplus V}{K} \longrightarrow \frac{M_1 \oplus M_2}{K} \xrightarrow{\mu_1 \oplus \mu_2} \frac{N_1 \oplus_G N_2}{K} \longrightarrow \frac{G \oplus G}{K} \longrightarrow 0$

i.e since $\frac{V \oplus V}{K} \cong V$ and $\frac{G \oplus G}{K} \cong G$ acccording to the remark 4.1.1, we have:

$0 \longrightarrow V \longrightarrow \frac{M_1 \oplus M_2}{K} \xrightarrow{\mu_1 \oplus \mu_2} \frac{N_1 \oplus_G N_2}{K} \longrightarrow G \longrightarrow 0$, so we can easily get by the same way, the sequence associated to $(\mu_1 \oplus \mu_2) \oplus \mu_3$, which is:

$0 \longrightarrow \frac{V \oplus V}{K} \longrightarrow \frac{\frac{M_1 \oplus M_2}{K} \oplus M_3}{K} \xrightarrow{(\mu_1 \oplus \mu_2) \oplus \mu_3} \frac{\frac{N_1 \oplus_G N_2}{K} \oplus_G N_3}{K} \longrightarrow \frac{G \oplus G}{K} \longrightarrow 0$

i.e

$$0 \longrightarrow V \longrightarrow \frac{M_1 \oplus M_2 \oplus M_3}{K} \xrightarrow{(\mu_1 \oplus \mu_2) \oplus \mu_3} \frac{N_1 \oplus_G N_2 \oplus_G N_3}{K} \longrightarrow G \longrightarrow 0 \tag{4.1.5}$$

By the same process, we have also for $\mu_1 \oplus (\mu_2 \oplus \mu_3)$ the following sequence:

$$0 \longrightarrow V \longrightarrow \frac{M_1 \oplus M_2 \oplus M_3}{K} \xrightarrow{\mu_1 \oplus (\mu_2 \oplus \mu_3)} \frac{N_1 \oplus_G N_2 \oplus_G N_3}{K} \longrightarrow G \longrightarrow 0 \tag{4.1.6}$$

So, according to 4.1.5 and 4.1.6 we conclude that :

$$(\mu_1 \oplus \mu_2) \oplus \mu_3 = \mu_1 \oplus (\mu_2 \oplus \mu_3)$$

which implies that, $\oplus$ is associative.

2) Neutral element

Let $\mu : M \longrightarrow N$ the crossed module in $crmod(G,V)$ i.e we have the following sequence:

$$0 \longrightarrow V \xrightarrow{i} M \xrightarrow{\mu} N \xrightarrow{\pi} G \longrightarrow 0$$

Our goal here is to find a crossed module e in $crmod(G,V)$ such that: $\mu \oplus e = e \oplus \mu = \mu$ or such that $\mu \oplus e = e \oplus \mu$ is equivalent to μ in the sens of equivalence of crossed module, seen in the definition 4.1.2. Since we are finding e, the neutral element of $crmod(G,V)$ which is the set of equivalence classes of crossed modules with respect to the fixed kernel V and the cokernel G, we can define e by the following sequence:

$$0 \longrightarrow V \xrightarrow{id_V} V \xrightarrow{e} G \xrightarrow{id_G} G \longrightarrow 0$$

so since the sum of the both crossed module is given by

$$0 \longrightarrow V \xrightarrow{(id\oplus i)\circ\Delta} \frac{M\oplus V}{K} \xrightarrow{\mu\oplus e} N \oplus_G G \xrightarrow{\pi} G \longrightarrow 0$$

we have to show that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "(id\oplus i)\circ\Delta"] \arrow[d, "id_V"] & \frac{M\oplus V}{K} \arrow[r, "\mu\oplus e"] \arrow[d, "\phi"] & N \oplus_G G \arrow[r, "\pi"] \arrow[d, "\psi"] & G \arrow[r] \arrow[d, "id_G"] & 0 \\ 0 \arrow[r] & V \arrow[r, "i"] & M \arrow[r] & N \arrow[r, "\pi"] & G \arrow[r] & 0. \end{tikzcd} \tag{4.1.2}$$

With

$$\phi : \frac{M\oplus V}{K} \longrightarrow M \text{ such that for all } (\bar{x},\bar{y}) \in \frac{M\oplus V}{K}, \text{ we have}$$
$$\phi(\bar{x},\bar{y}) = x \in M$$

and

$$\psi : N \oplus_G G \longrightarrow N \text{ such that for all } (n,g) \in N \oplus_G G, \text{ we have}$$
$$\psi(n,g) = n \in N$$

These two maps are well-defined.

Let show that the previous diagram is commutative, we have:

- $\phi \circ [(i \oplus id_V) \circ \Delta](v) = i(v) \Longrightarrow \phi \circ [(i \oplus id_V) \circ \Delta] = i$
- $\mu \circ \phi(\bar{m,v}) = \mu(m)$
- $\psi(\mu \circ e)(\bar{m,v}) = \psi[\mu(m) \oplus e] = \psi(\mu(m), e) = \mu(m) \Longrightarrow \mu \circ \phi(\bar{m,v}) = \psi \circ (\mu \oplus e)(\bar{m,v})$.

  Also,

$$\pi \circ \psi(n,g) = \pi(n)$$
$$id \circ \pi(n,g) = \pi(n)$$

So, we conclude that the diagram is commutative, then $\mu \oplus e = e \oplus \mu$ is equivalent to μ. So the neutral element of $crmod(G,V)$ is $0 \longrightarrow V \longrightarrow V \longrightarrow G \longrightarrow G \longrightarrow 0$.

3) Inverse element

By the same way as we did for the neutral element, for a given crossed module $\mu : M \longrightarrow N$ define as follow:

$0 \longrightarrow V \xrightarrow{i} M \xrightarrow{\mu} N \xrightarrow{\pi} G \longrightarrow 0$, his inverse is $0 \longrightarrow V \xrightarrow{-i} M \xrightarrow{\mu} N \xrightarrow{\pi} G \longrightarrow 0$,

In fact, we have to show that the sum of the both crossed μ and his inverse is equivalent to the identity crossed module e, seen previously, in the sens of equivalence of crossed modules define in 4.1.2. So,we have to show that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, "id_V"] & \frac{(V\oplus M)}{(e\oplus(-i(v)))} \arrow[r] \arrow[d, "proj"] & N \oplus_G G \arrow[r] \arrow[d, "\pi"] & G \arrow[r] \arrow[d, "id_G"] & 0 \\
0 \arrow[r] & V \arrow[r, "id_V"] & V \arrow[r, "e"] & G \arrow[r] & G \arrow[r] & 0.
\end{tikzcd} \tag{4.1.2}$$

We observe by the same way than the case of neutral crossed module, this diagram is also commutative. So, we conclude that each element of $crmod(G, V)$ has an inverse.

So, according to $1), 2)$ and $3)$ we conclude that $(crmod(G, V), \oplus)$ is a group.

$\square$

4.1.2 Evaluation of $H^3(G, V)$

Actually, our goal in this chapter is to construct the isomorphim between the third group of cohomology and the set of equivalence of crossed module. So, the objection is to prove the following theorem.

Theorem 4.1.1.

There is an isomorphism of abelian groups

$$I : crmod(G, V) \cong H^3(G, V) \tag{4.1.7}$$

According to the remark 4.1.1, we have seen by the fact that our sequence 4.1.4 is exact, that there exists $\beta(x_1, x_2) \in M$ such that

$$\mu(\beta(x_1, x_2)) = \alpha(x_1, x_2) \tag{4.1.8}$$

Choosing a linear section σ on $im(\mu)$ i.e $\sigma \circ \mu = id$, we have from 4.1.8:

$$\sigma \circ \mu(\beta(x_1, x_2)) = \sigma \circ \alpha(x_1, x_2) = \beta(x_1, x_2)$$

$$\text{Since } \sigma \circ \mu = id \text{ i.e } \sigma \circ \mu(\beta(x_1, x_2)) = \beta(x_1, x_2)$$

So we have,

$$\beta(x_1, x_2) = \sigma(\alpha(x_1, x_2)) \tag{4.1.9}$$

showing that we can suppose β bilinear and skewsymmetric in x_1, x_2.

In order tho prove the theorem 4.1.1, let us give the following intermediate results.

Lemma 4.1.2.

Let d the formal expression of the Lie algebra cohomology boundary operator corresponding to cohomology of G with values in M, and $x_1, x_2, x_3 \in G$, we have:

$$\mu(d\beta(x_1, x_2, x_3)) = 0.$$

Proof.

By definition of d we have for x_1, x_2, x_3:

$$d\beta(x_1, x_2, x_3) = \sum_{cycl} \beta([x_1, x_2], x_3) - \sum_{cycl} \eta(\rho(x_1))\beta(x_2, x_3)$$

$$\begin{aligned}
\mu(d\beta(x_1, x_2, x_3)) &= \mu(\sum_{cycl} \beta([x_1, x_2], x_3) - \sum_{cycl} \eta(\rho(x_1))\beta(x_2, x_3)) \\
&= \sum_{cycl} \mu(\beta([x_1, x_2], x_3)) - \sum_{cycl} [\rho(x_1), \mu(\beta(x_2, x_3))] \\
&= \sum_{cycl} \alpha([x_1, x_2], x_3) - \sum_{cycl} [\rho(x_1), \mu(\beta(x_2, x_3))] \\
&= \sum_{cycl} \alpha([x_1, x_2], x_3) - \sum_{cycl} [\rho(x_1), \alpha(x_2, x_3)] \\
&= \sum_{cycl} [\rho([x_1, x_2]), \rho(x_3)] - \rho([[x_1, x_2], x_3])] \\
&- \sum_{cycl} [\rho(x_1), [\rho(x_2), \rho(x_3)]] + [\rho(x_1), \rho([x_2, x_3])] = 0.
\end{aligned}$$

$\square$

Remark.

The lemma 4.1.2 means that $d\beta(x_1, x_2, x_3) \in ker(\mu) = im(i) = i(V)$, i.e. there exists $\gamma(x_1, x_2, x_3) \in V$ such that

$$d\beta(x_1, x_2, x_3) = i(\gamma(x_1, x_2, x_3))$$

4.2 Computation of third degree cohomology space

In this section, we are going to show that γ defined in the remark 4.1.2 is a 3-cocycle of G with values in V and show the theorem 4.1.1.

4.2.1 γ is a 3-cocycle

The goal is to show that $d\gamma(x_1, x_2, x_3, x_4) = 0$, denoting by x_1, x_2, x_3, x_4 four elements of G and by d the Lie algebra coboundary operator of G with values in V. The expression for $d\gamma(x_1, x_2, x_3, x_4)$ is the sum of "action terms" and "bracket terms". So we have only to show that

$$i(d\gamma(x_1, x_2, x_3, x_4)) = 0.$$

We have:

$$i(d\gamma(x_1, x_2, x_3, x_4)) = di(\gamma(x_1, x_2, x_3, x_4)) = d \circ d\beta(x_1, x_2, x_3, x_4).$$

Since $d \circ d$ is not automatically zero, because of the fact that $\eta \circ \rho$ is not an action of G on M in general. Let us display here only some terms of it, while the other terms vanish as usual. The terms we choose are all "action terms of the action terms" and some "action terms of the bracket terms".

$$i(d\gamma(x_1, x_2, x_3, x_4)) = \sum_{1 \le i < j \le 4} (-1)^{i+j} \eta(\rho([x_i, x_j]))\beta(x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_4)$$
$$- \textstyle\sum_{i=1}^{4} (-1)^i \eta(\rho(x_i)) \sum_{l=1}^{3} (-1)^l \eta(\rho(z_l))\beta(z_1, ..., \hat{z}_l, ..., z_3).$$

Here, we denote by z_1, z_2, z_3 the three remaining x_r after having chosen x_i from the list. Now, the difference of acting by $\eta(\rho([x, y]))$ and acting by $\eta(\rho(x))\eta(\rho(y)) - \eta(\rho(y))\eta(\rho(x))$ is just the action by $\eta(\alpha(x, y))$. Calculating the differences of the actions by the bracket and the action of the single elements gives thus terms of the form

$$\eta(\alpha(x, y))\beta(u, v) = \eta(\mu(\beta(x, y)))\beta(u, v) = [\beta(x, y), \beta(u, v)],$$

Then,

$$[\beta(x_1, x_2), \beta(x_3, x_4)] - [\beta(x_2, x_3), \beta(x_4, x_1)] + [\beta(x_3, x_4), \beta(x_1, x_2)] - [\beta(x_4, x_1), \beta(x_2, x_3)] +$$
$$[\beta(x_4, x_2), \beta(x_1, x_3)] - [\beta(x_1, x_3), \beta(x_2, x_4)]$$

and vanishes.

4.2.2 The class of the cocycle γ in $C^3(G,V)$ does not depend on the choice of the section

a- Let ρ and ρ' be two sections of π. Denote by $\alpha(x,y)$ resp. $\alpha'(x,y)$, $\beta(x,y)$ resp. $\beta'(x,y)$, $\gamma(x,y,z)$ resp. $\gamma'(x,y,z)$ the elements of N, M and V constructed above with respect to ρ and ρ'. Here $x,y,z \in G$. By construction, we have:

$$\rho'(x) = \rho(x) + \delta(x),$$

for some linear map $\delta : G \longrightarrow ker(\pi) \subset N$.

But then α' may be written

$$\alpha'(x,y) = [\rho'(x),\rho'(y)] - \rho'([x,y]) = [(\rho+\delta)(x),(\rho+\delta)(y)] - (\rho+\delta)([x,y]) = \alpha(x,y) + [\rho(x),\delta(y)] + [\delta(x),\delta(y)] + [\delta(x),\delta(y)] - \delta([x,y]).$$

Observe that the expression $[\rho(x),\delta(y)]+[\delta(x),\rho(y)]-\delta([x,y])$ is just the formal coboundary $d\delta(x,y)$, where $\delta : G \longrightarrow N$ is considered as a cochain with values in N although lifting elements, N is in general not a G-module via the adjoint action.

As $d\delta(x,y)$ lies in the kernel of π, there exists $\epsilon(x,y) \in M$ such that $\mu \circ \epsilon = d\delta$, and as before, we may take ϵ bilinear. In the same way, as $[\delta(x),\delta(y)]$ is in $ker(\epsilon)$, there exists $\theta(x,y) \in M$ with $\mu(\theta(x,y)) = [\delta(x),\delta(y)]$. Therefore we get

$$\mu(\beta'(x,y)) = \mu(\beta(x,y)) + \mu(\epsilon(x,y)) + \mu(\theta(x,y)),$$

and hence there exists an element in $ker(\mu) = im(i)$, denoted $i(\zeta)$, such that

$$\beta'(x,y) = \beta(x,y) + \epsilon(x,y) + \theta(x,y) + i(\zeta)(x,y).$$

When applying in the next step d to all terms, the ζ will give a coboundary, because $d(i(\zeta)) = i(d^V\zeta)$. Let us treat the term $\epsilon(x,y)$. Observe that using the linear section σ on $im(\mu)$, we have $\epsilon = \sigma d\delta$. Actually, we have

$$\begin{aligned}
&\mu\sigma d\delta(x,y) - d\sigma\delta(x,y)\\
&= \mu\sigma(\rho(x).\delta(y) - \rho(y).\delta(x) + \delta([x,y])\\
&= -\mu(\eta(\rho(x))(\sigma\delta(y)) + \eta(\rho(y))(\sigma\delta(x)) - \sigma\delta([x,y])\\
&= 0,
\end{aligned}$$

using property $i-$ in the definition 4.1.1 of a crossed module. This means that the difference is in the kernel of μ, thus in the image of i, and replacing ϵ by $d\sigma\delta$ adds only another coboundary in the end. A similar reasoning applies to the term $\theta(x,y)$. In conclusion, we have shown that changing the section ρ results in changing the cocycle γ by a coboundary.

b- Now suppose that we chose two different linear sections σ and σ' of μ on $im(\mu)$, leading to different lifts $\beta = \sigma\alpha$ and $\beta' = \sigma'\alpha$. We have then $\beta - \beta' \in ker(\mu) = im(i)$, and we have already seen that this leads to the corresponding γ and γ' differing by a coboundary.

c- Two sections τ and τ' of i have to be the same; they are both inverses of the isomorphism i on its image.

So now, Let $\mu : M \longrightarrow N$ (with action η) and $\mu' : M' \longrightarrow N'$ (with action η') such that $ker(\mu) = ker(\mu') =: V$ and $coker(\mu) = coker(\mu') =: G$ be two elementary equivalent crossed modules. Then the corresponding cohomology classes $I([\mu]) = [\gamma]$ and $I([\mu']) = [\gamma']$ coincide in $H^3(G, V)$.

In fact, let us consider the couple (Φ, Ψ) the morphism rendering the two crossed modules elementary equivalent as we have seen in the definition 4.1.2.

Let γ (resp. γ') the cocycle associated to $\mu : M \longrightarrow N$ (resp. to $\mu' : M' \longrightarrow N'$) by choosing sections ρ, σ and τ (resp. ρ', σ' and τ'). Since ρ is a section of π, then, $\tilde{\rho'} := \Psi \circ \rho$ is a section of π'. Also, we have that α' and $\tilde{\alpha'} := \Psi \circ \alpha$ give the same cohomology class $[\gamma']$ represented by γ'.

So, let $\tilde{d}$ the formal Lie algebra coboundary operator with values in M' and with the formal action $\eta' \circ \Psi \circ \rho$. We have to compute the following expression:

$$(\tilde{d}(\sigma' \circ \Psi \circ \alpha) - \tilde{d}(\Phi \circ \beta))(x, y, z)$$

for $x, y, z \in G$ where . Let us consider $\tilde{\beta'} := \sigma' \circ \Psi \circ \alpha$. We have $(\tilde{\beta'} - \Phi \circ \beta)(x, y) \in ker(\mu')$, thus we introduce $f(x, y) \in V$ such that $(\tilde{\beta'} - \Phi \circ \beta)(x, y) = i' f(x, y)$ for all $x, y \in G$. Since (Φ, Ψ) satisfies the commutativity of the diagram given in definition 4.1.2 we have:

$$\tilde{d}(\Phi \circ \beta)(x, y, z) = \Phi(d\beta(x, y, z)).$$

By the same way we have also

$$(\tilde{d}i' f)(x, y, z) = i'(df(x, y, z)).$$

We finally obtain:

$$(\gamma' - \gamma)(x, y, z) = df(x, y, z),$$

showing that the two elementary equivalent crossed modules have the same cohomology class.

So, according to the theorem 3.3.1, we conclude that I is an isomorphism of abelian extension which is the map I associating to an equivalence class of crossed modules with kernel V and cokernel G a cohomology class in $H^3(G, V)$.

5. Conclusion

In the end, throughout this work we went to review two great results on the computation of (co)homology, especially those of the second and third group of (co)homology on Lie algebras. For us, this project aimed to establish the bijection that exists between the equivalence class of the Abelian extensions and the second (co)homology group on a Lie algebra and the isomorphism between the crossed module equivalence class. and the third group of (co)homology on a Lie algebra also. To this end, we began by recalling some preliminary notions about the homological algebra and the Lie algebra theory in order to provide insight into what would be developed later. In a continuity, recalling among other things what we can remember about the Abelian extensions we have, after a considerable development and computations around are able to establish the relation that exists between the equivalence class of the Abelian extensions and the second group from (co)homology through the theorem 3.3.1. Indeed, we mean by an abelian extension of Lie algebras, a short exact sequence of Lie algebras

$$0 \longrightarrow A \xrightarrow{i} G \xrightarrow{\pi} H \longrightarrow 0,$$

where the Lie algebra A is abelian and $ker\pi = A$, $cokeri = H$

In almost identical reasoning, we have established that there is an isomorphism between the equivalence class of the crossed modules and the third (co)homology group, as present in the theorem 4.1.1. Indeed as we have seen in the definition 4.1.1, a crossed module of Lie algebras is the data of a homomorphism of Lie algebras $\mu : M \longrightarrow N$ together with an action η of N on M by derivations, denoted $\eta : N \longrightarrow der(M)$ or $m \longmapsto n.m$ for all $m \in M$ and all $n \in N$, such that:

i- $\mu(n.m) = [n, \mu(m)]$ for all $n \in N$ and all $m \in M$,

ii- $\mu(m).m' = [m, m']$ for all $m, m' \in m$.

the idea has been to bring together an element of the crossed-module equivalence class and to show that from this element one can construct a class of (co)homology and vice verca this through well-defined actions and a particular attention on the different operations or operators used.

At the end of this work, we retain that in view of these results, their applicability in pure algebra to establish evidence of multiple theorems, is very large. In other words, they have enormous importance, especially in homological algebra, because knowing how to determine certain (co)homology groups, especially the first three groups of (co)homology, makes it possible to evaluate the accuracy of the short or long sequences, which has great utility in algebra generally. We find satisfaction after this project because knowing that for the most part the computation of groups of (co)homology is not always direct, the difficulties lie at the level of determining to the whole or to the space which will be in bijection with these groups so that we can just consider the latter and its properties in order to deport them on groups of (co)homology that we seek and handle them with ease. Thus, our next curiosity will be to look at how to get the other groups of (co)homology and if possible seek a generalization that would allow us to achieve very applicable results.

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