Numerical solution for the propagation of light along a spatial direction

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Abstract

In this short note, I introduce the equations pertaining to the numerical solution of the equations governing an electromagnetic wave propagating along one spatial direction.

1 The equations and their discretisation

We consider the system of partial differential equations:

$$
\begin{aligned}
\frac{\partial E}{\partial t}(t,z) &= -\frac{1}{\varepsilon_0}\frac{\partial H}{\partial z}(t,z),\\
\frac{\partial H}{\partial t}(t,z) &= -\frac{1}{\mu_0}\frac{\partial E}{\partial z}(t,z).
\end{aligned}
$$

Our task is to solve for the fields E and H in these equations over a closed square in (t,z)-space defined by $[0,T]\times[0,z_{\max}]$. Since the solution we seek is numerical, and for stability reasons, we will subdivide $[0,T]$ in $2N$ equal parts, respectively $[0,z_{\max}]$ in $2K$ equal parts, whose length we denote by Δt, respectively Δz. More specifically

$$
\Delta t = \frac{T}{2N}, \quad \Delta z = \frac{z_{\max}}{2K}
$$

We then want to define the fields over an "intertwined grid", i.e.,

$$
\begin{aligned}
E &: \{\Delta t, 3\Delta t, 5\Delta t, \dots, (2N-1)\Delta t\}\times\{0, 2\Delta z, 4\Delta z, \dots, 2K\Delta z\} \longrightarrow \mathbb{R},\\
H &: \{0, 2\Delta t, 4\Delta t, \dots, 2N\Delta t\}\times\{\Delta z, 3\Delta z, 5\Delta z, \dots, (2K-1)\Delta z\} \longrightarrow \mathbb{R}.
\end{aligned}
$$

In more concrete notation, we will have values for these fields at the following points:

$$
\begin{aligned}
&E\big((2n+1)\Delta t, 2k\Delta z\big), \quad n=0,1,\dots N-1, \quad k=0,1,\dots K,\\
&H\big(2n\Delta t, (2k+1)\Delta z\big), \quad n=0,1,\dots N, \quad k=0,1,\dots K-1.
\end{aligned}
$$

We're going to be interested in the finite differences:

$$
\begin{aligned}
\frac{\partial E}{\partial t}\big((2n+1)\Delta t, 2k\Delta z\big) &\approx \frac{E\big((2n+1)\Delta t, 2k\Delta z\big) - E\big((2n-1)\Delta t, 2k\Delta z\big)}{\Delta t}, \quad n=1,\dots N-1, \quad k=0,1,\dots K,\\
\frac{\partial E}{\partial z}\big((2n+1)\Delta t, 2k\Delta z\big) &\approx \frac{E\big((2n+1)\Delta t, 2k\Delta z\big) - E\big((2n+1)\Delta t, 2(k-1)\Delta z\big)}{\Delta z}, \quad n=0,1,\dots N-1, \quad k=1,\dots K,\\
\frac{\partial H}{\partial t}\big(2n\Delta t, (2k+1)\Delta z\big) &\approx \frac{H\big(2n\Delta t, (2k+1)\Delta z\big) - H\big(2(n-1)\Delta t, (2k+1)\Delta z\big)}{\Delta t}, \quad n=1,\dots N, \quad k=0,1,\dots K-1,\\
\frac{\partial H}{\partial z}\big(2n\Delta t, (2k+1)\Delta z\big) &\approx \frac{H\big(2n\Delta t, (2k+1)\Delta z\big) - H\big(2n\Delta t, (2k-1)\Delta z\big)}{\Delta z}, \quad n=0,1,\dots N, \quad k=1,\dots K-1.
\end{aligned}
$$

Inserting these approximations into the differential equation and simplifying, taking into account the domains of n and k, we get:

$$
\begin{aligned}
E\big((2n+1)\Delta t, 2k\Delta z\big) &= E\big((2n-1)\Delta t, 2k\Delta z\big) - \frac{\Delta t}{\varepsilon_0\Delta z}\Big(H\big(2n\Delta t, (2k+1)\Delta z\big) - H\big(2n\Delta t, (2k-1)\Delta z\big)\Big),\\
H\big(2n\Delta t, (2k+1)\Delta z\big) &= H\big(2(n-1)\Delta t, (2k+1)\Delta z\big) - \frac{\Delta t}{\mu_0\Delta z}\Big(E\big((2n+1)\Delta t, 2k\Delta z\big) - E\big((2n+1)\Delta t, 2(k-1)\Delta z\big)\Big),
\end{aligned}
$$

both of which hold for $n=1,\dots,N-1$ and $k=1,\dots,K-1$. Renormalising $\widetilde{E} := \sqrt{\frac{\varepsilon_0}{\mu_0}}E$, we get

$$
\begin{aligned}
\widetilde{E}\big((2n+1)\Delta t, 2k\Delta z\big) &= \widetilde{E}\big((2n-1)\Delta t, 2k\Delta z\big) - c\frac{\Delta t}{\Delta z}\Big(H\big(2n\Delta t, (2k+1)\Delta z\big) - H\big(2n\Delta t, (2k-1)\Delta z\big)\Big),\\
H\big(2n\Delta t, (2k+1)\Delta z\big) &= H\big(2(n-1)\Delta t, (2k+1)\Delta z\big) - c\frac{\Delta t}{\Delta z}\Big(\widetilde{E}\big((2n+1)\Delta t, 2k\Delta z\big) - \widetilde{E}\big((2n+1)\Delta t, 2(k-1)\Delta z\big)\Big),
\end{aligned}
$$

We take into account the stability conditions

$$
\Delta z \le \frac{\lambda}{10}, \quad \text{, e.g.,} \quad \Delta z = \frac{\lambda}{10} - \varepsilon,
$$

and then the *Courant condition*

$$
c\frac{\Delta t}{\Delta z} \le \frac{1}{2} \implies \Delta t \le \frac{1}{2c}\Delta z, \quad \text{e.g.,} \quad \Delta t = \frac{1}{2c}\Delta z - \varepsilon'
$$