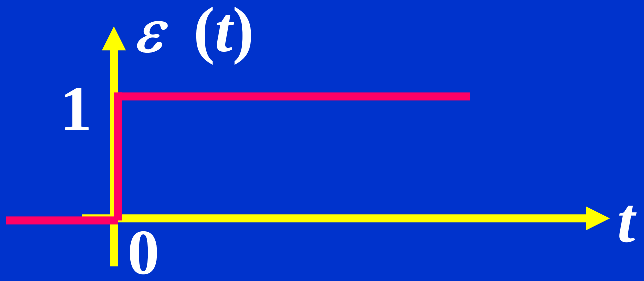
11.4　一阶电路的阶跃和冲激响应

11.4.1　一阶电路的阶跃响应

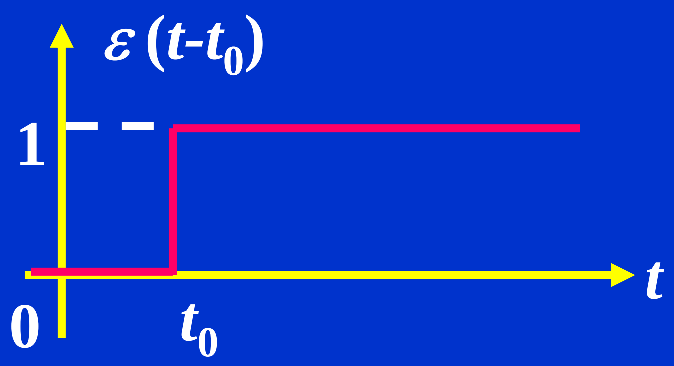
1. 单位阶跃函数

- 定义

$$\varepsilon(t)=\begin{cases}0 & (t<0)\\ 1 & (t>0)\end{cases}$$



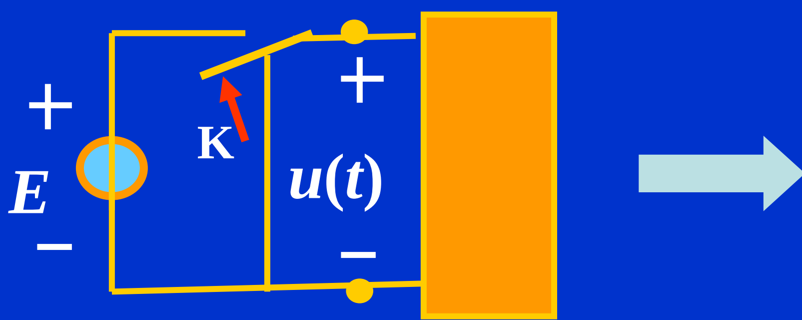
- 单位阶跃函数的延迟



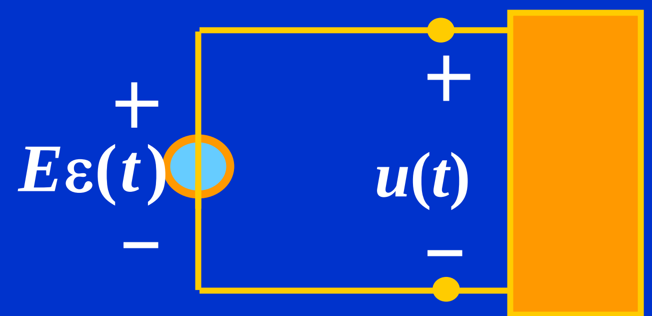
$$\varepsilon(t-t_0)=\begin{cases}0 & (t<t_0)\\ 1 & (t>t_0)\end{cases}$$

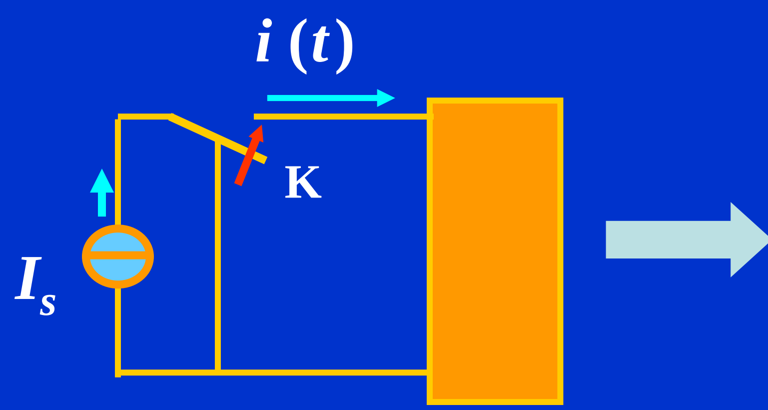
● 单位阶跃函数的作用

（1）在电路中模拟开关的动作

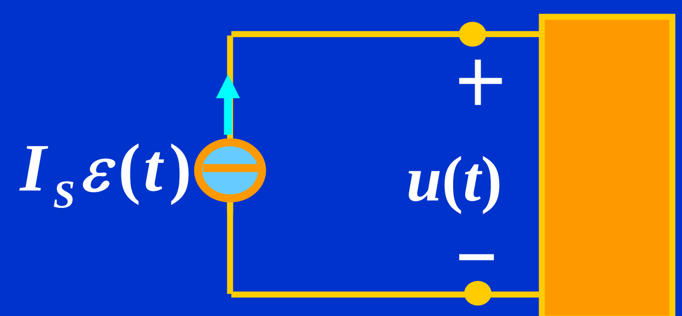


$t = 0$合闸　$u(t) = E\,\varepsilon(t)$

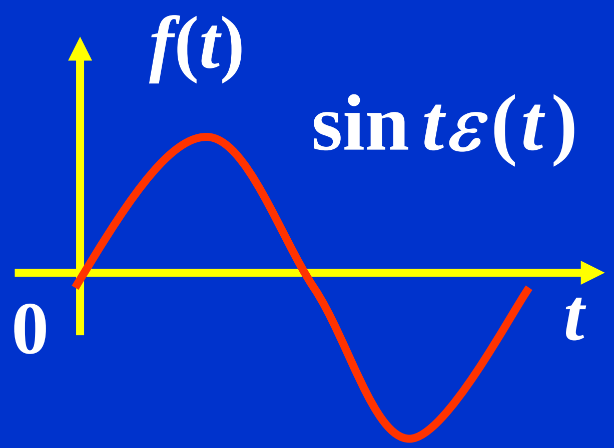


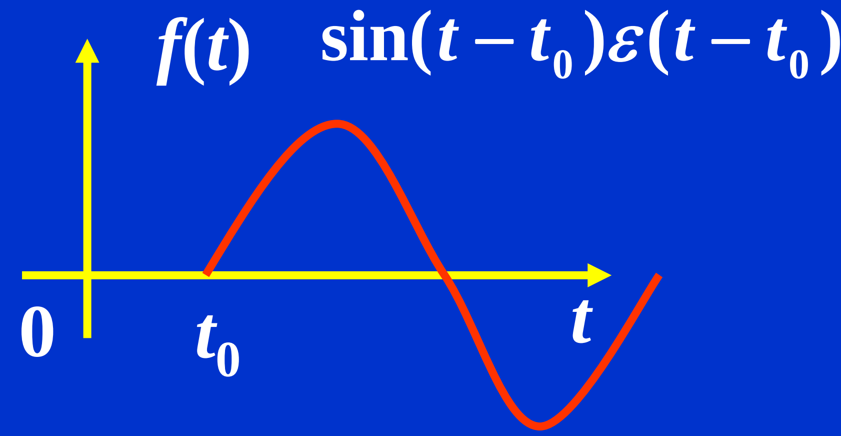


$t = 0$合闸　$i(t) = I_s\,\varepsilon(t)$

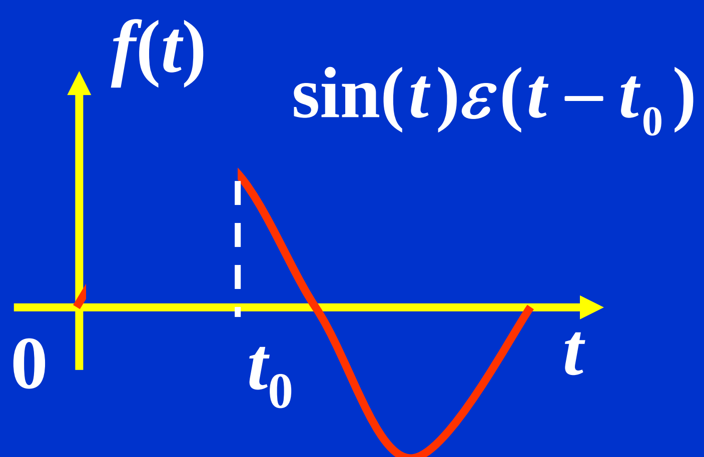


（2）延迟一个函数



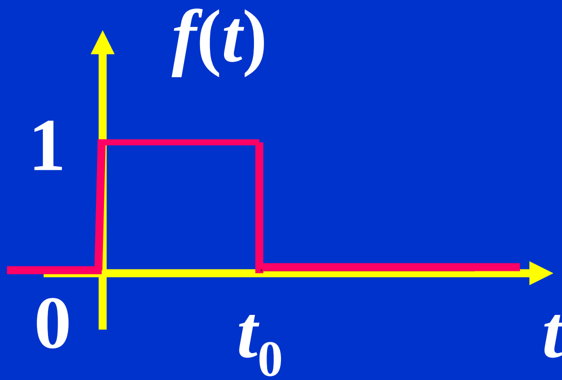


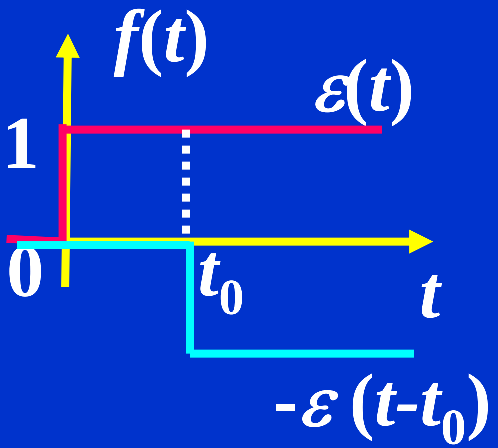
（3）起始一个函数



● 用单位阶跃函数表示复杂的信号

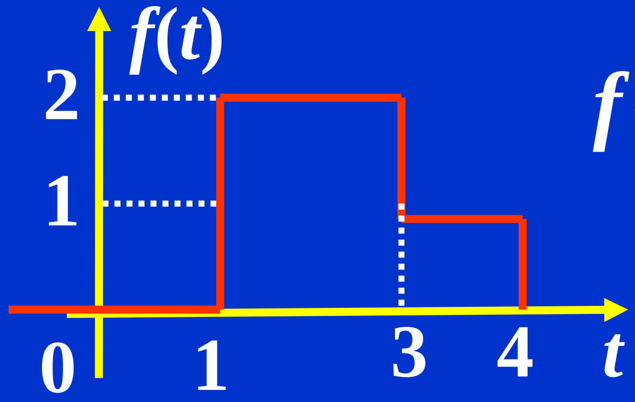
例 1





$$f(t)=\varepsilon(t)-\varepsilon(t-t_0)$$

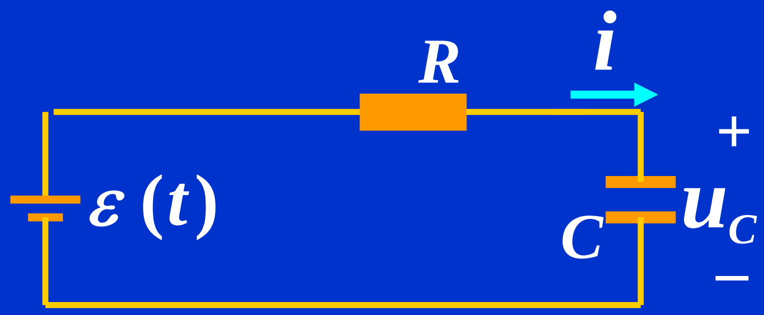
例 2

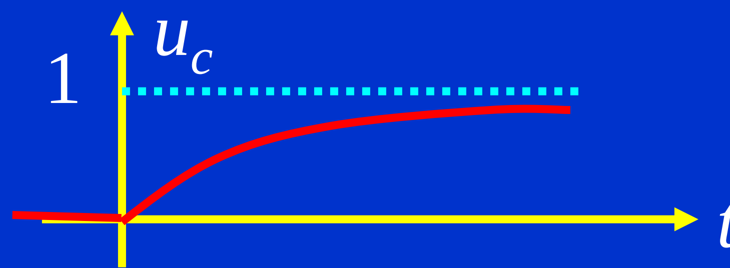


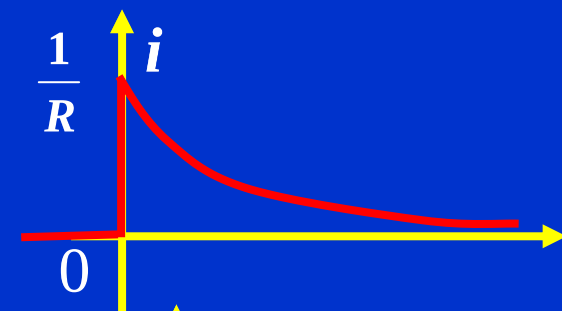
$$f(t)=2\varepsilon(t-1)-\varepsilon(t-3)-\varepsilon(t-4)$$

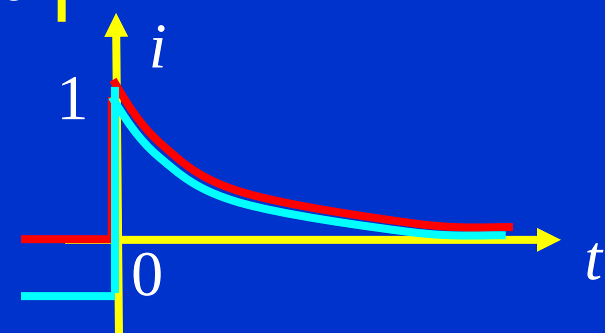
2. 一阶电路的阶跃响应

阶跃响应 → 激励为单位阶跃函数时，电路中产生的零状态响应。







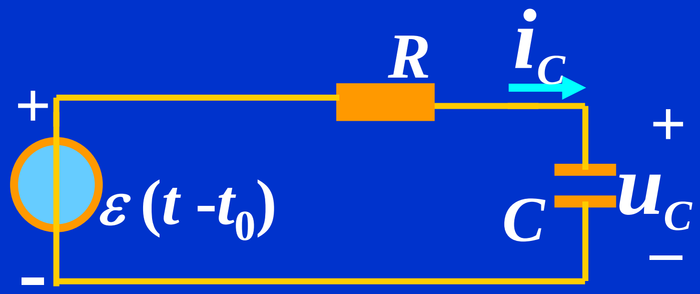


$$u_C(0^-)=0$$

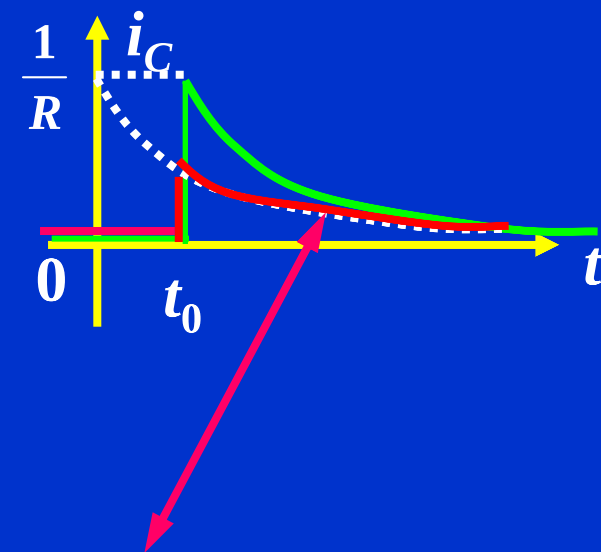
$$u_C(t)=(1-e^{-\frac{t}{RC}})\varepsilon(t)$$

$$i(t)=\frac{1}{R}e^{-\frac{t}{RC}}\varepsilon(t)$$

注意 $i=e^{-\frac{t}{RC}}\varepsilon(t)$ 和 $i=e^{-\frac{t}{RC}}$ $t\ge 0$ 的区别



激励在 $t=t_0$ 时加入，
则响应从$t=t_0$开始。

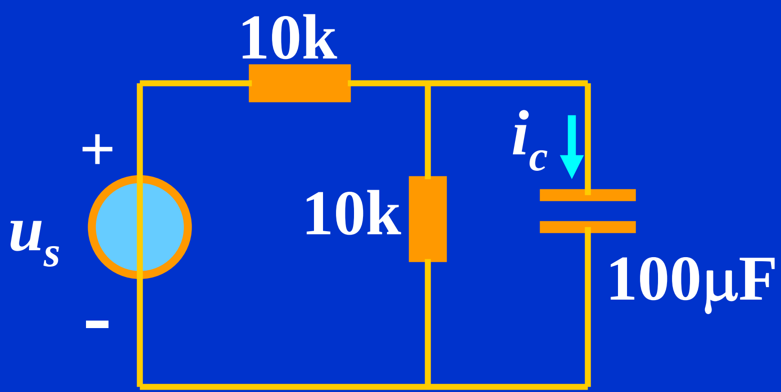


$$i_C=\frac{1}{R}e^{-\frac{t-t_0}{RC}}\varepsilon(t-t_0)$$

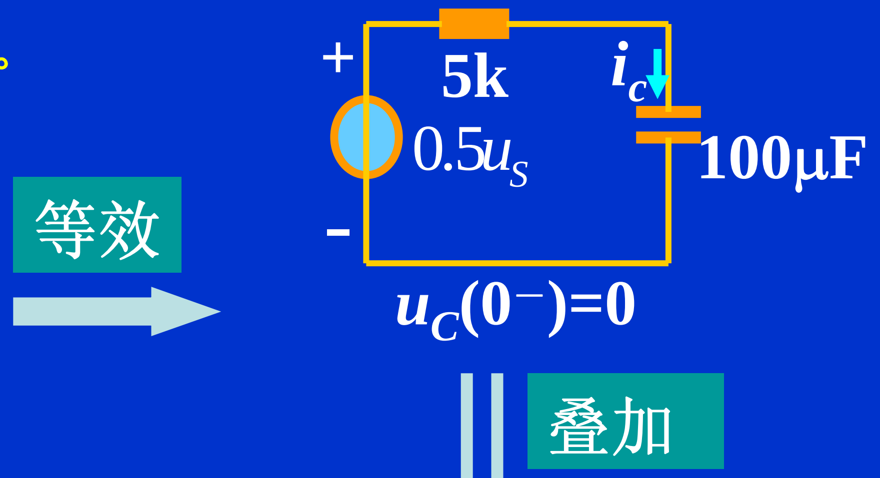
注意

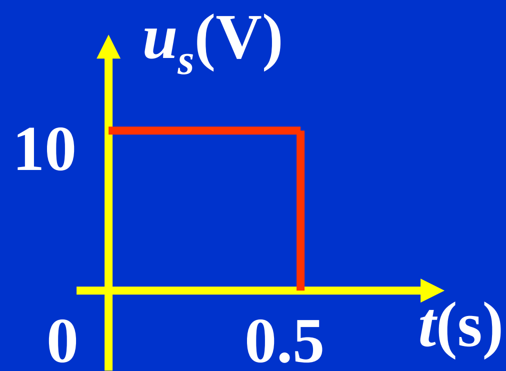
不要写为 $\frac{1}{R}e^{\frac{-t}{RC}}\varepsilon(t-t_0)$

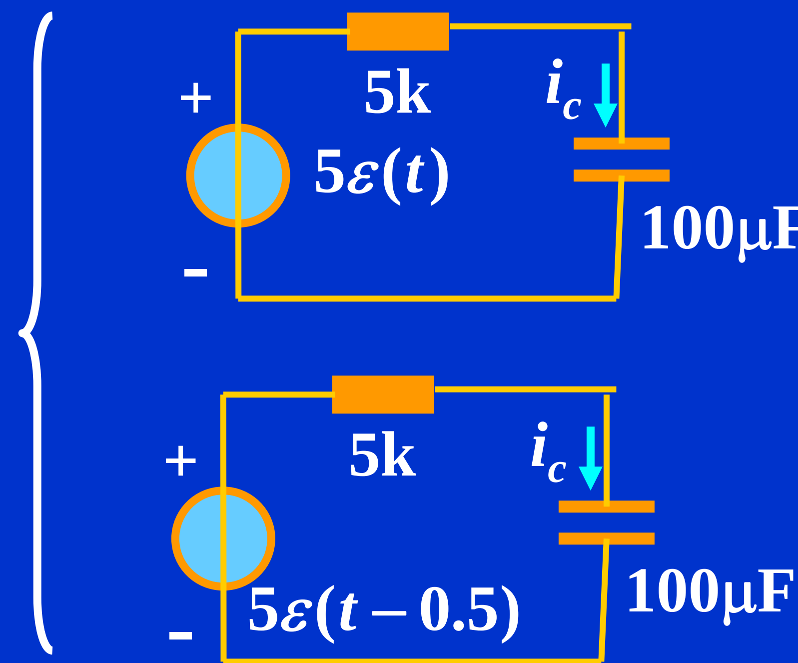
例 求图示电路中电流$i_C(t)$。



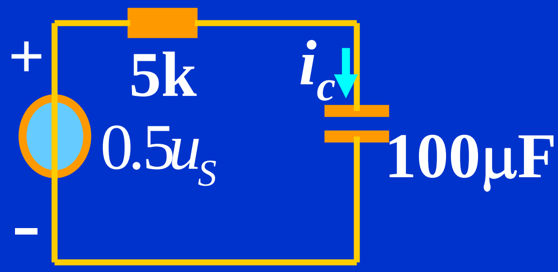
$u_C(0^-)=0$







$$u_S = 10\varepsilon(t) - 10\varepsilon(t-0.5)$$



$u_C(0^-)=0$

$$\tau = RC = 100\times10^{-6}\times5\times10^{-3} = 0.5\text{s}$$

阶跃响应为:

$$u_C(t) = (1-e^{-2t})\varepsilon(t)$$

$$i_C = C\frac{\mathrm{d}u_C}{\mathrm{d}t} = \frac{1}{5}e^{-2t}\varepsilon(t)\ \ \text{mA}$$

由齐次性和叠加性得实际响应为:

$$i_C = 5[\frac{1}{5}e^{-2t}\varepsilon(t) - \frac{1}{5}e^{-2(t-0.5)}\varepsilon(t-0.5)]$$

$$= e^{-2t}\varepsilon(t) - e^{-2(t-0.5)}\varepsilon(t-0.5)\quad \text{mA}$$

$$i_C = e^{-2t}\varepsilon(t) - e^{-2t}\varepsilon(t-0.5) + e^{-2t}\varepsilon(t-0.5) - e^{-2(t-0.5)}\varepsilon(t-0.5)$$

$$= e^{-2t}[\varepsilon(t) - \varepsilon(t-0.5)] + e^{-2t}\varepsilon(t-0.5) - e^{-2(t-0.5)}\varepsilon(t-0.5)$$

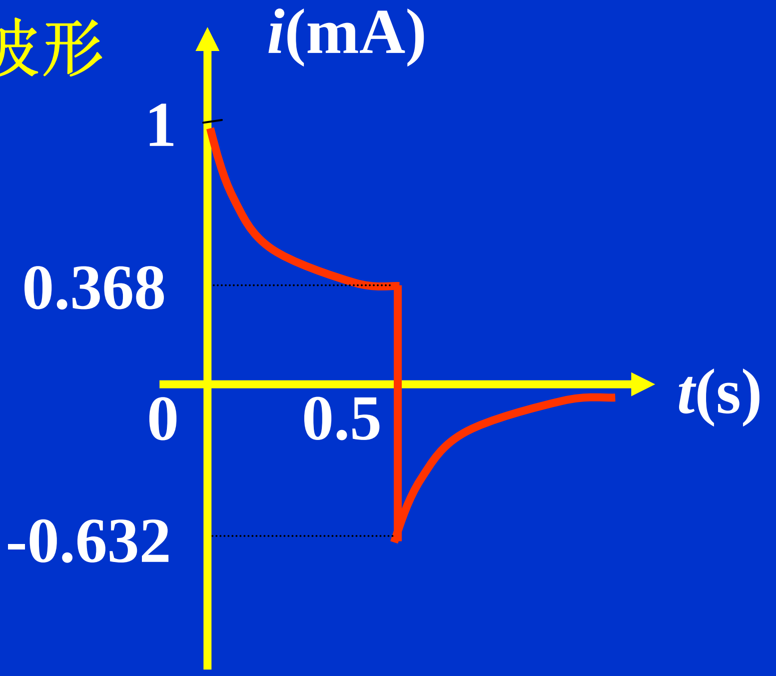
$$= e^{-2t}[\varepsilon(t) - \varepsilon(t-0.5)] + e^{-1}e^{-2(t-0.5)}\varepsilon(t-0.5) - e^{-2(t-0.5)}\varepsilon(t-0.5)$$

$$= e^{-2t}[\varepsilon(t) - \varepsilon(t-0.5)] - 0.632\,e^{-2(t-0.5)}\varepsilon(t-0.5)$$

分段表示为

$$i(t) = \begin{cases} e^{-2t} \ \text{mA} & (0 < t < 0.5\text{s}) \\ -0.632e^{-2(t-0.5)} \ \text{mA} & (t > 0.5\text{s}) \end{cases}$$

波形

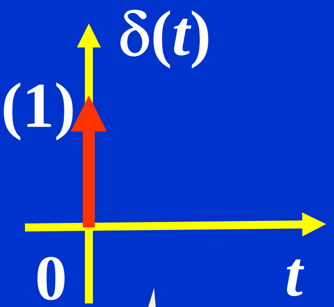


11.4.2 一阶电路的冲激响应

1. 单位冲激函数

● 定义

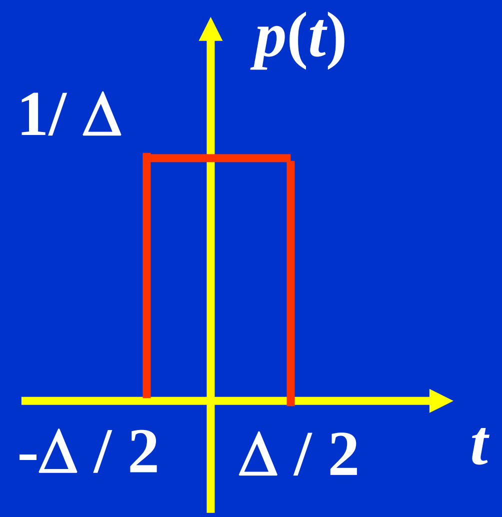
$$\begin{cases} \delta(t) = 0 \quad (t \neq 0) \\ \int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1 \end{cases}$$



$$p(t) = \frac{1}{\Delta}[\varepsilon(t+\frac{\Delta}{2}) - \varepsilon(t-\frac{\Delta}{2})]$$

$$\Delta \to 0 \qquad \frac{1}{\Delta} \to \infty$$

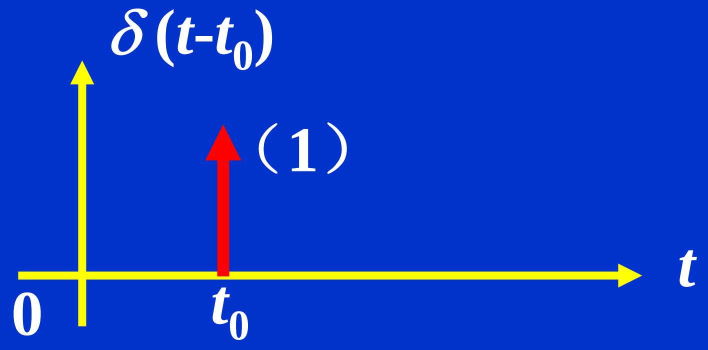
$$\lim_{\Delta \to 0} p(t) = \delta(t)$$



单位脉冲函数的极限

● 单位冲激函数的延迟

$$\begin{cases} \delta(t-t_0)=0 \quad (t\neq t_0) \\ \int_{-\infty}^{\infty}\delta(t-t_0)\mathrm{d}t=1 \end{cases}$$



● 单位冲激函数的性质

（1）冲激函数对时间的积分等于阶跃函数。

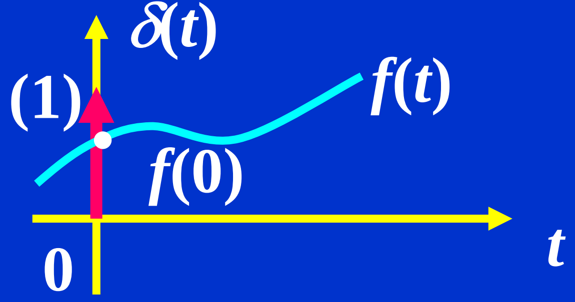
$$\int_{-\infty}^{t}\delta(t)\mathrm{d}t=\begin{cases}0 & t<0^- \\ 1 & t>0^+\end{cases}=\varepsilon(t) \longrightarrow \frac{d\varepsilon(t)}{dt}=\delta(t)$$

2. 冲激函数的筛分性

$$\int_{-\infty}^{\infty} f(t)\delta(t)\mathrm{d}t = f(0)\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = f(0)$$

$$f(0)\delta(t)$$

同理有: $\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\mathrm{d}t = f(t_0)$



例

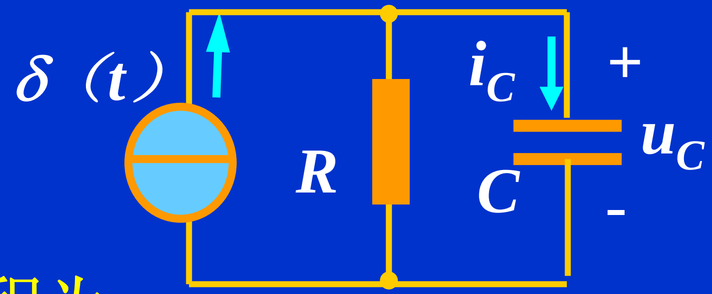
$$\int_{-\infty}^{\infty} (\sin t + t)\delta\left(t-\frac{\pi}{6}\right)\mathrm{d}t$$

$$= \sin\frac{\pi}{6} + \frac{\pi}{6} = \frac{1}{2} + \frac{\pi}{6} = 1.02$$

* $f(t)$在 t_0 处连续

2. 一阶电路的冲激响应

冲激响应 → 激励为单位冲激函数时，电路中产生的零状态响应。



例1. 分二个时间段来考虑冲激响应。

(1). t 在 $0^- \to 0^+$间　电容充电，方程为：

$u_C(0^-)=0$

$$C\frac{du_c}{dt}+\frac{u_c}{R}=\delta(t)$$

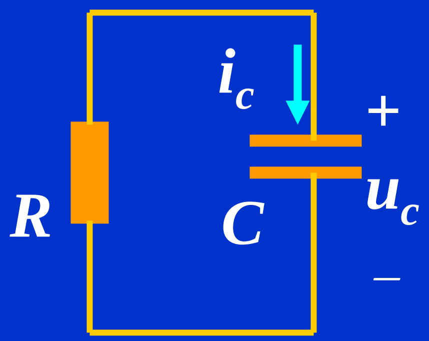
u_c不是冲激函数，否则**KCL**不成立

$$\int_{0^-}^{0^+} C\frac{du_c}{dt}\mathrm{d}t+\overbrace{\int_{0^-}^{0^+}\frac{u_c}{R}\mathrm{d}t}^{0}=\int_{0^-}^{0^+}\delta(t)\mathrm{d}t=1$$

$$\longrightarrow \quad C[u_c(0^+)-u_c(0^-)]=1 \qquad u_c(0^+)=\frac{1}{C}\neq u_C(0^-)$$

电容中的冲激电流使电容电压发生跃变

(2). $t>0^{+}$ 为零输入响应（RC放电）

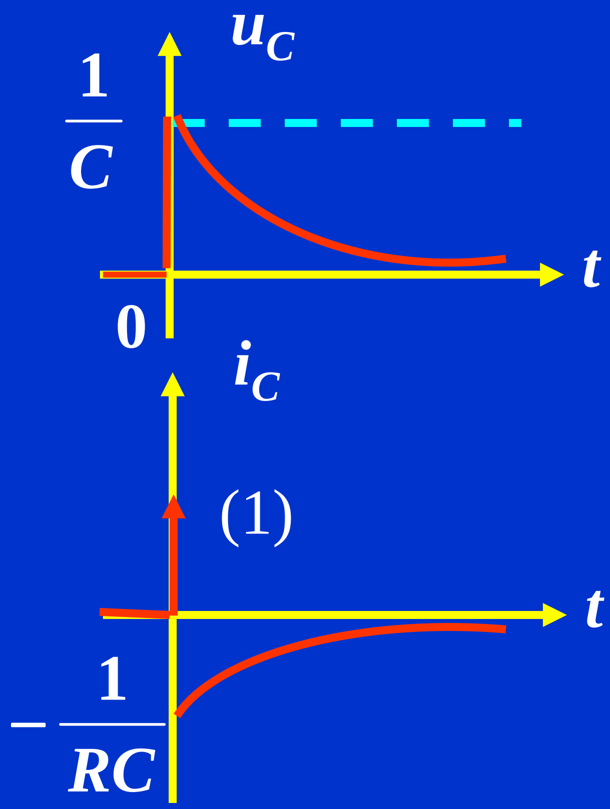


$$u_c=\frac{1}{C}e^{-\frac{t}{RC}}\qquad t\geq 0^{+}$$

$$\boxed{u_c(0^{+})=\frac{1}{C}}$$

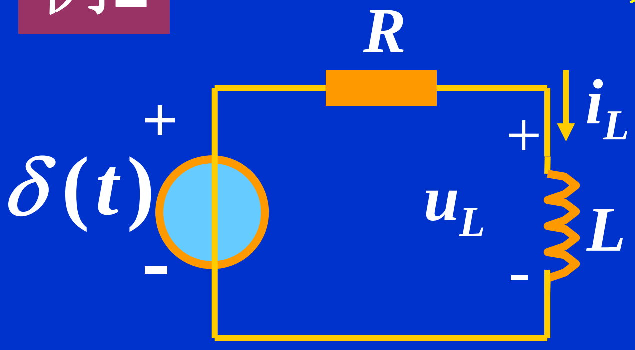
$$i_c=-\frac{u_c}{R}=-\frac{1}{RC}e^{-\frac{t}{RC}}\qquad t\geq 0^{+}$$

$$\begin{cases} u_c=\dfrac{1}{C}e^{-\frac{t}{RC}}\varepsilon(t) \\ i_c=\delta(t)-\dfrac{1}{RC}e^{-\frac{t}{RC}}\varepsilon(t) \end{cases}$$



例2

分二个时间段来考虑冲激响应。



$i_L(0^-) = 0$

(1). t 在 $0^- \rightarrow 0^+$间方程为：

$$Ri_L + L\frac{di_L}{dt} = \delta(t)$$

i_L不可能是冲激函数

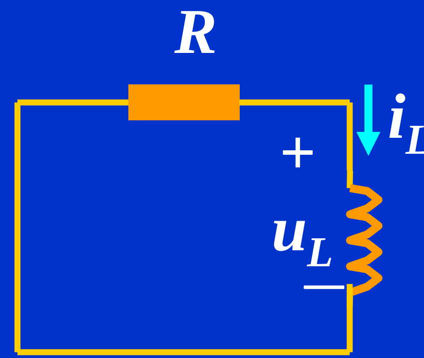
$$\cancelto{0}{\int_{0^-}^{0^+} Ri_L dt} + \int_{0^-}^{0^+} L\frac{di_L}{dt}dt = \int_{0^-}^{0^+} \delta(t)dt = 1$$

$$\longrightarrow \quad L\left[i_L(0^+) - i_L(0^-)\right] = 1 \qquad i_L(0^+) = \frac{1}{L} \neq i_L(0^-)$$

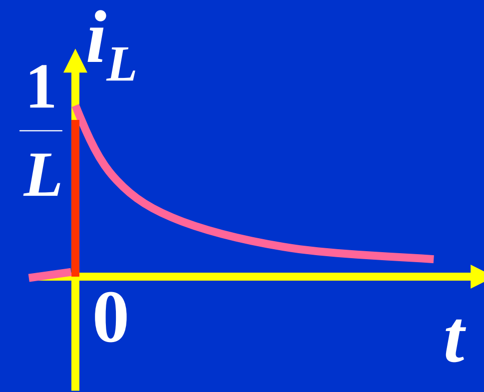
电感上的冲激电压使电感电流发生跃变

(2). $t > 0^+$ RL放电

$$\tau = \frac{L}{R} \qquad i_L(0^+) = \frac{1}{L}$$

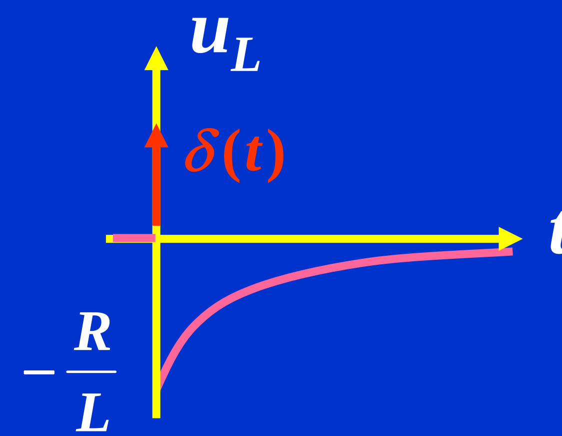


$$i_L = \frac{1}{L} e^{-\frac{t}{\tau}} \qquad t \geq 0^+$$



$$u_L = -i_L R = -\frac{R}{L} e^{-\frac{t}{\tau}} \qquad t \geq 0^+$$

$$\begin{cases} i_L = \dfrac{1}{L} e^{-\frac{t}{\tau}} \varepsilon(t) \\ u_L = \delta(t) - \dfrac{R}{L} e^{-\frac{t}{\tau}} \varepsilon(t) \end{cases}$$

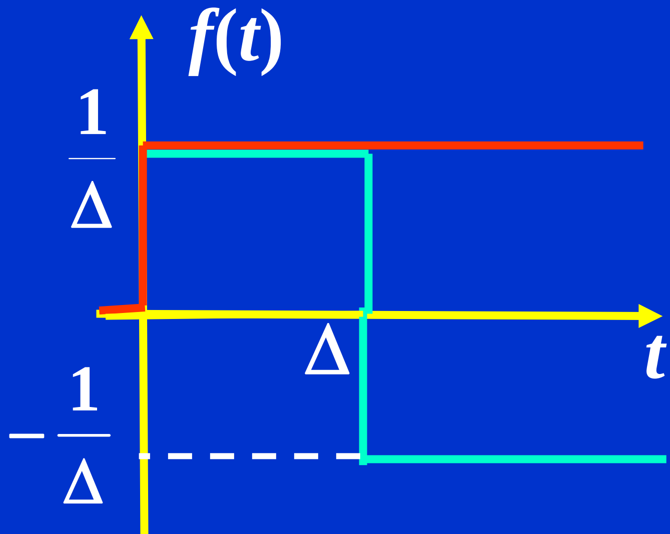


● 单位阶跃响应和单位冲激响应关系

单位阶跃	单位阶跃响应	
$\varepsilon(t)$	$s(t)$	$\delta(t)=\dfrac{d\varepsilon(t)}{dt}$
单位冲激	单位冲激响应	
$\delta(t)$	$h(t)$	$h(t)=\dfrac{d}{dt}s(t)$

证明：

$$\varepsilon(t) \longrightarrow \boxed{\text{零状态}} \longrightarrow s(t) \qquad \delta(t) \longrightarrow \boxed{\text{零状态}} \longrightarrow h(t)$$



$$f(t) = \frac{1}{\Delta}\varepsilon(t) - \frac{1}{\Delta}\varepsilon(t-\Delta)$$

$$\downarrow \qquad\qquad \downarrow$$

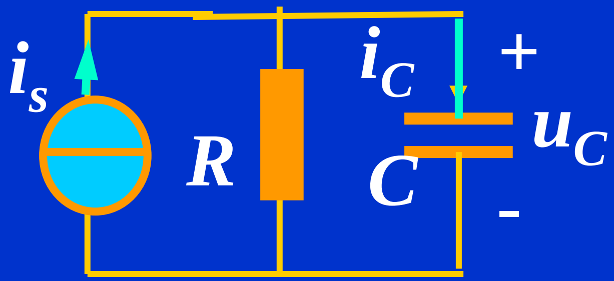
$$\frac{1}{\Delta}s(t) \qquad -\frac{1}{\Delta}s(t-\Delta)$$

$$h(t) = \lim_{\Delta\to 0}\frac{1}{\Delta}[s(t) - s(t-\Delta)] = \frac{d}{dt}s(t)$$

注

$s(t)$定义在（-∞，∞）整个时间轴

例1 求：$i_s(t)$为单位冲激时电路响应 $u_C(t)$和 $i_C(t)$



已知：$u_c(0^-) = 0$

先求单位阶跃响应，令：

$$i_S(t) = \varepsilon(t)$$

$u_C(0^+)=0$ $\quad$ $u_C(\infty)=R$ $\quad$ $\tau = RC$ $\quad$ $i_C(0^+)=1$ $\quad$ $i_C(\infty)=0$

$$u_C(t) = R(1-e^{-\frac{t}{RC}})\varepsilon(t) \qquad i_c = e^{-\frac{t}{RC}}\varepsilon(t)$$

再求单位冲激响应，令：$i_S(t) = \delta(t)$

$$u_C = \frac{d}{dt}R(1-e^{-\frac{t}{RC}})\varepsilon(t) = \underbrace{R(1-e^{-\frac{t}{RC}})\delta(t)}_{0} + \frac{1}{C}e^{-\frac{t}{RC}}\varepsilon(t)$$

$$= \frac{1}{C}e^{-\frac{t}{RC}}\varepsilon(t)$$

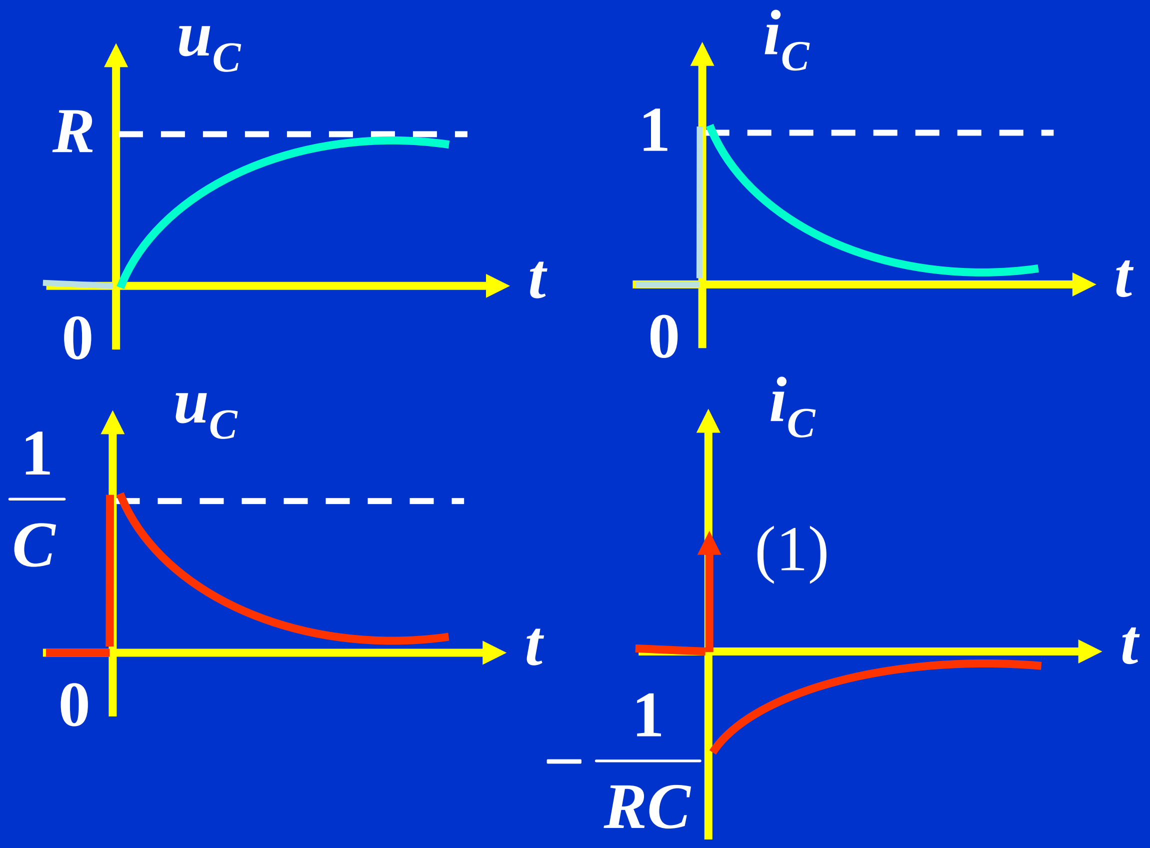
$$f(t)\delta(t) = f(0)\delta(t)$$

$$i_c = \frac{\mathrm{d}}{\mathrm{d}t}[e^{-\frac{t}{RC}}\varepsilon(t)] = e^{-\frac{t}{RC}}\delta(t) - \frac{1}{RC}e^{-\frac{t}{RC}}\varepsilon(t)$$

$$= \delta(t) - \frac{1}{RC}e^{-\frac{t}{RC}}\varepsilon(t)$$

阶跃响应

冲激响应



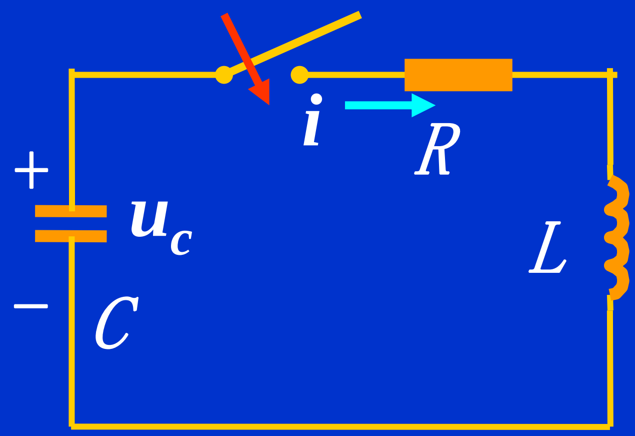
11.5 二阶电路的动态过程

● 重点:

1. 用经典法分析二阶电路的过渡过程;

2. 了解二阶电路的动态性质;

二阶电路的零输入响应



已知： $u_c(0^+)=U_0$ $i(0^+)=0$

电路方程：

$$LC\frac{d^2u_c}{dt}+RC\frac{du_c}{dt}+u_c=0$$

特征方程： $LCP^2+RCP+1=0$

特征根： $P=\frac{-R\pm\sqrt{R^2-4L/C}}{2L}=-\frac{R}{2L}\pm\sqrt{(\frac{R}{2L})^2-\frac{1}{LC}}$

2. 零输入响应的三种情况

$$P=\frac{-R\pm\sqrt{R^2-4L/C}}{2L}$$

$R>2\sqrt{\frac{L}{C}}$ 二个不等负实根 过阻尼

$R=2\sqrt{\frac{L}{C}}$ 二个相等负实根 临界阻尼

$R<2\sqrt{\frac{L}{C}}$ 二个共轭复根 欠阻尼

$(1)\ R > 2\sqrt{\frac{L}{C}}$

$$u_c = A_1 e^{p_1 t} + A_2 e^{p_2 t}$$

$$u_c(0^+) = U_0 \rightarrow A_1 + A_2 = U_0$$

$$i(0^+) = -C\frac{du_c}{dt}(0^+)$$

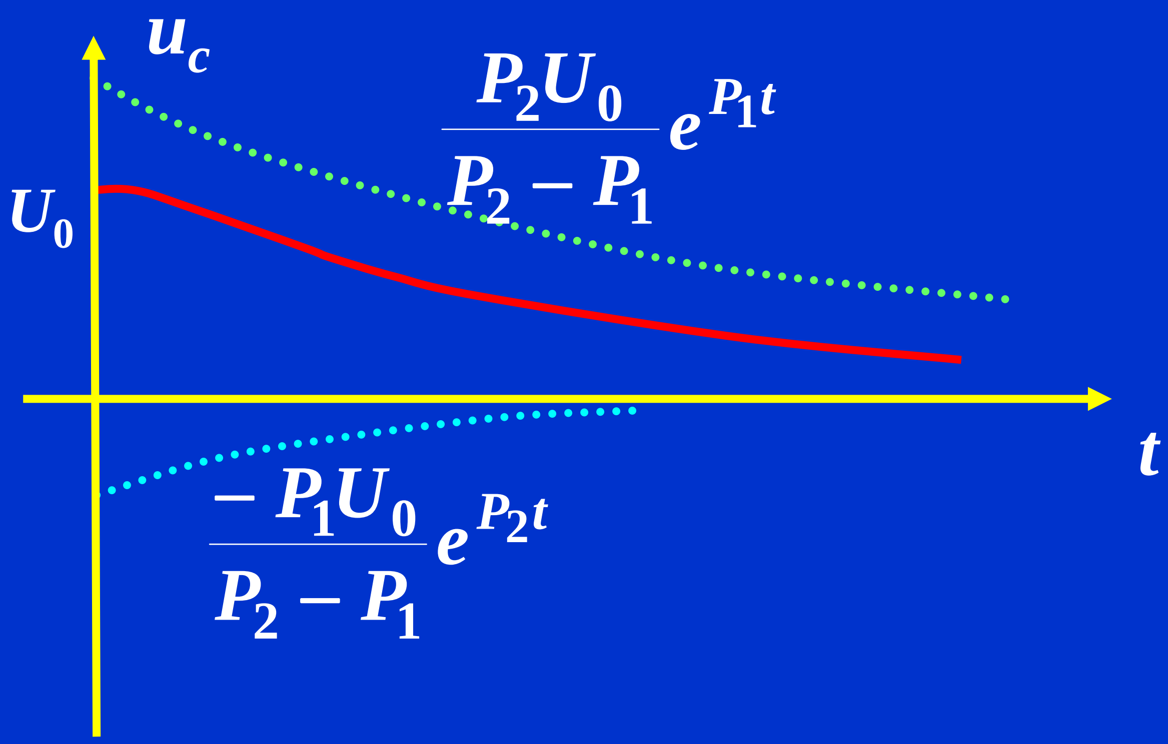
$$\rightarrow P_1 A_1 + P_2 A_2 = 0$$

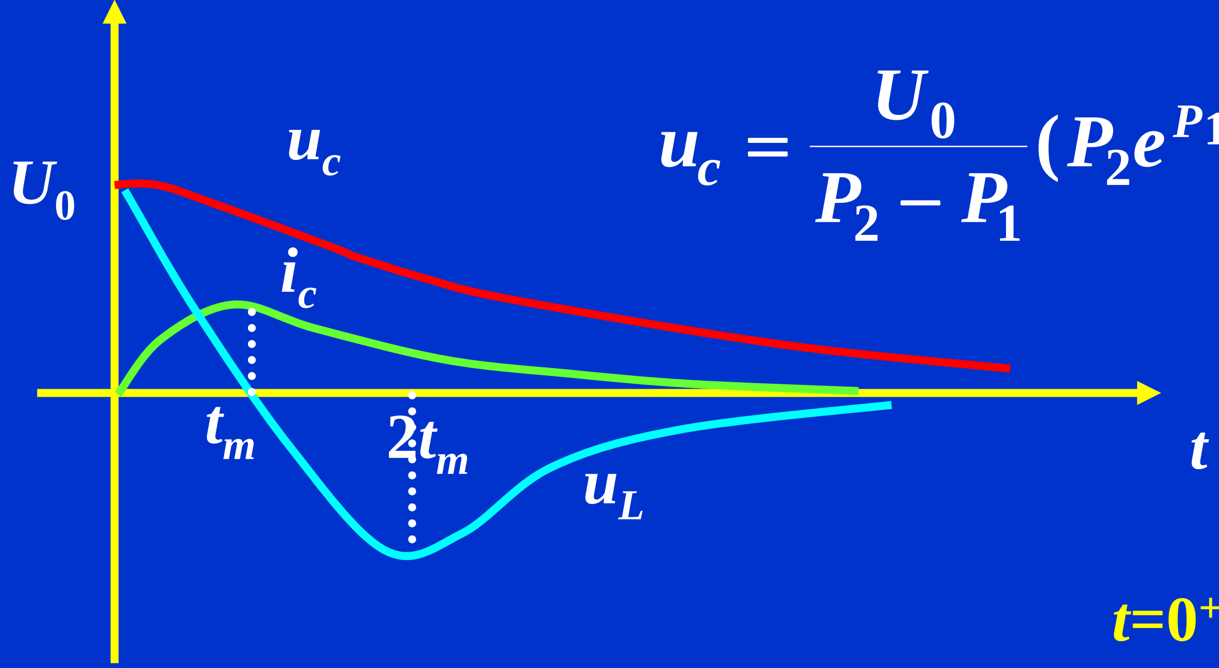
$$\longrightarrow \begin{cases} A_1 = \dfrac{P_2}{P_2 - P_1} U_0 \\ A_2 = \dfrac{-P_1}{P_2 - P_1} U_0 \end{cases}$$

$$u_c = \frac{U_0}{P_2 - P_1}(P_2 e^{P_1 t} - P_1 e^{P_2 t})$$

$$u_c = \frac{U_0}{P_2 - P_1}(P_2 e^{P_1 t} - P_1 e^{P_2 t})$$

设$|P_2|>|P_1|$





$$u_c = \frac{U_0}{P_2 - P_1}(P_2 e^{P_1 t} - P_1 e^{P_2 t})$$

$t=0^+$ $i_c=0$, $t=\infty$ $i_c=0$

$$i_c = -C\frac{du_c}{dt} = \frac{-U_0}{L(P_2 - P_1)}(e^{p_1 t} - e^{p_2 t})$$

$i_c>0$ $t = t_m$ 时i_c 最大

$$u_L = L\frac{di}{dt} = \frac{-U_0}{(P_2 - P_1)}(P_1 e^{p_1 t} - P_2 e^{p_2 t})$$

$0< t < t_m$ i增加，$u_L>0$

$t > t_m$ i减小，$u_L<0$

$t=2t_m$时 u_L 最大

$t = 0,\ u_L = U_0 \quad t = \infty, u_L = 0$

$$u_L = L\frac{di}{dt} = \frac{-U_0}{(P_2 - P_1)}(P_1 e^{p_1 t} - P_2 e^{p_2 t})$$

i_C为极值时的t_m即u_L=0时的 t,计算如下:

$$(P_1\ e^{p_1 t} - P_2\ e^{p_2 t}) = 0$$

$$\frac{P_2}{P_1} = \frac{e^{P_1 t_m}}{e^{P_2 t_m}}$$

$$t_m = \frac{\ell n \dfrac{p_2}{p_1}}{p_1 - p_2}$$

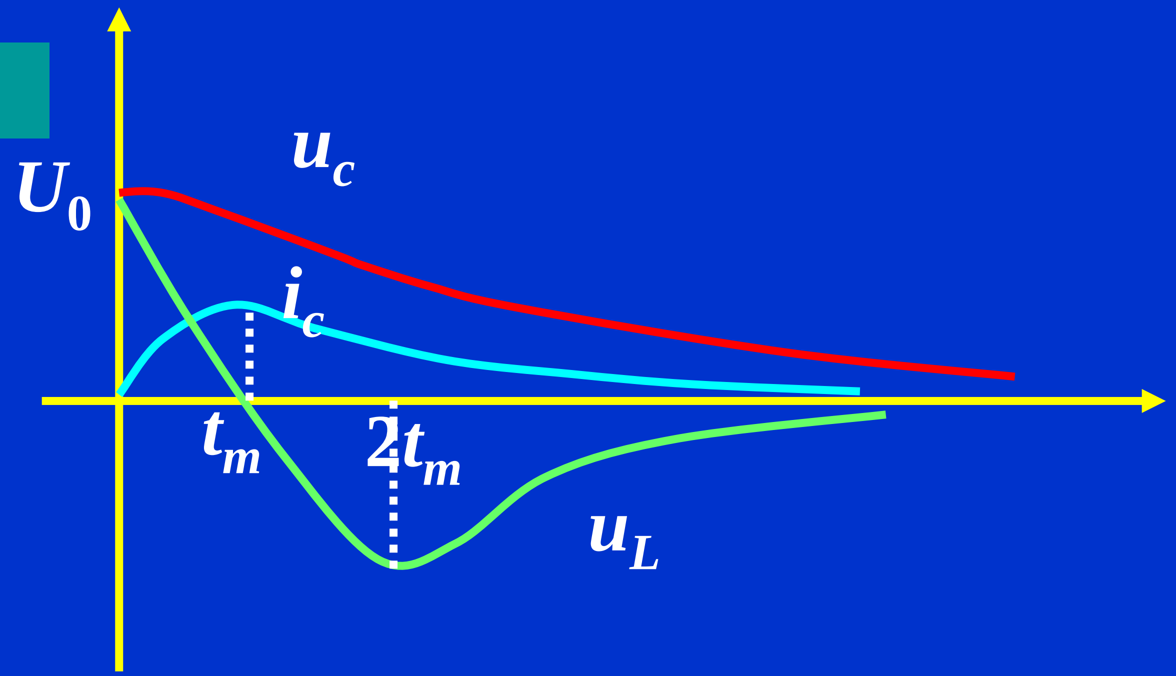
由du_L/dt可确定u_L为极小时的 t .

$$(P_1^{\ 2} e^{p_1 t} - P_2^{\ 2} e^{p_2 t}) = 0$$

$$t = \frac{2\ell n \dfrac{p_2}{p_1}}{p_1 - p_2}$$

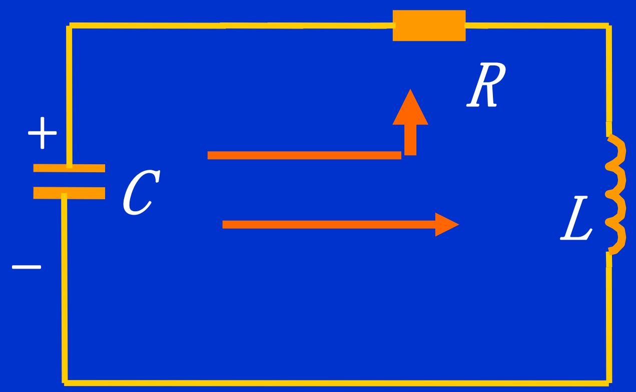
$$t = 2t_m$$

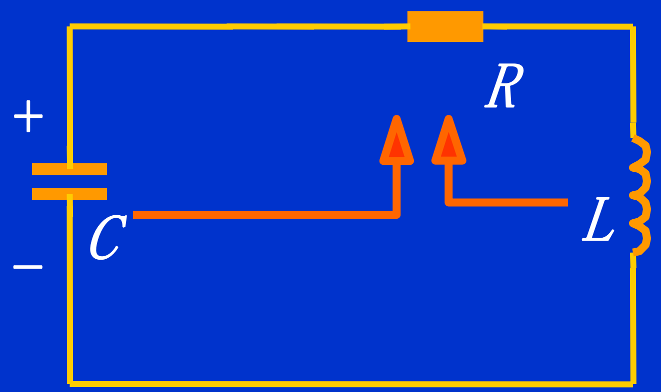
能量转换关系



$0 < t < t_m$　u_c减小，i 增加。

$t > t_m$　u_c减小，i 减小.





$$(2)\ R < 2\sqrt{\frac{L}{C}}$$

$$P = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

特征根为一对共轭复根

令：$\delta = \frac{R}{2L}$ (衰减系数)

$$\omega_0 = \sqrt{\frac{1}{LC}}$$ (谐振角频率)

则 $\omega = \sqrt{\omega_0^2 - \delta^2}$

(固有振荡角频率)

$$P = -\delta \pm j\omega$$

u_c的解答形式：

$$u_c = A_1 e^{p_1 t} + A_2 e^{p_2 t} = e^{-\delta(t)}(A_1 e^{j\omega t} + A_2 e^{-j\omega t})$$

经常写为：

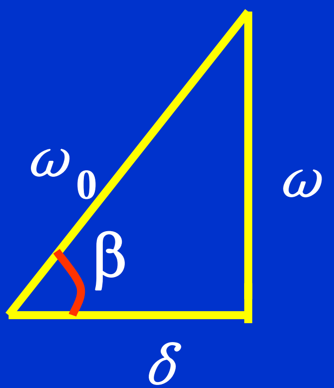
$$u_c = Ae^{-\delta t}\sin(\omega t + \beta)$$

A，β为待定常数

$$\text{由初始条件}\begin{cases} u_c(0^+) = U_0 \rightarrow A\sin\beta = U_0 \\ \dfrac{du_c}{dt}(0^+) = 0 \rightarrow A(-\delta)\sin\beta + A\omega\cos\beta = 0 \end{cases}$$

$$A = \frac{U_0}{\sin\beta} \quad , \quad \beta = arctg\frac{\omega}{\delta}$$

ω，ω_0，δ间的关系:



$$\sin\beta = \frac{\omega}{\omega_0} \qquad A = \frac{\omega_0}{\omega}U_0$$

$$u_c = \frac{\omega_0}{\omega}U_0 e^{-\delta t}\sin(\omega t + \beta)$$

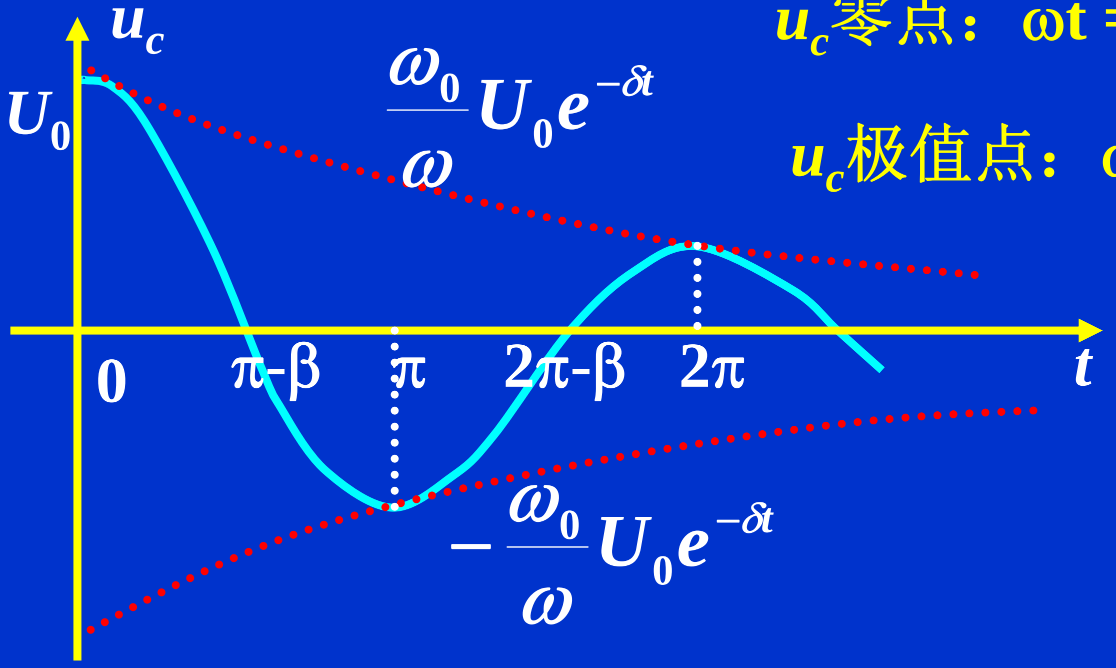
$$u_c = \frac{\omega_0}{\omega} U_0 e^{-\delta t} \sin(\omega t + \beta)$$

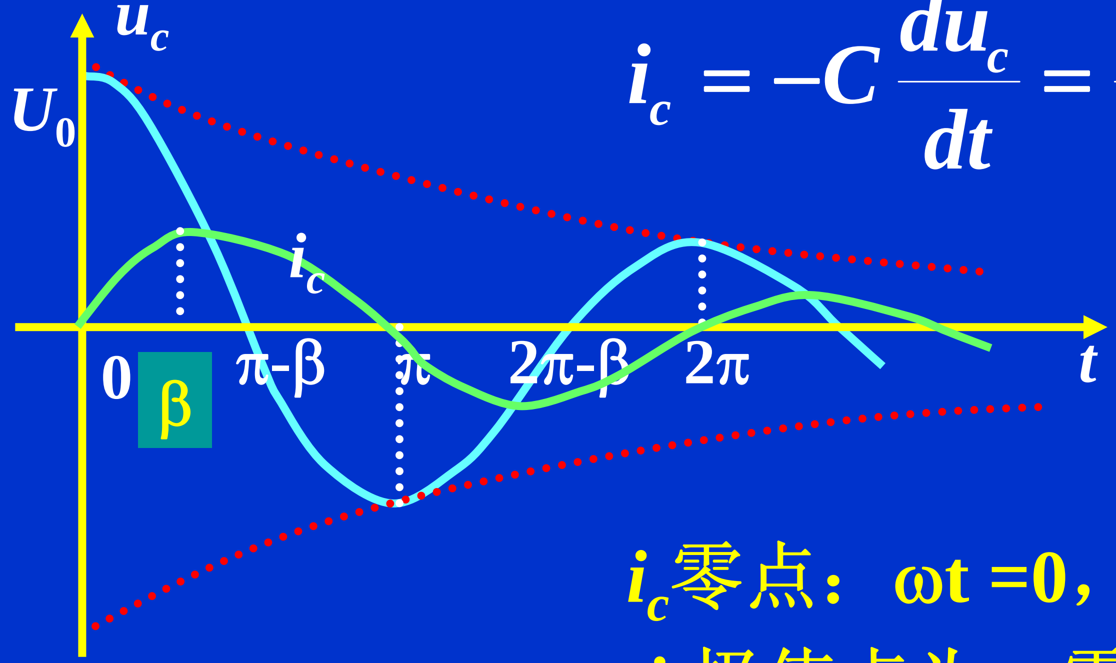
u_c是其振幅以$\pm\dfrac{\omega_0}{\omega}U_0$为包线依指数衰减的正弦函数。

t=0时 $u_c=U_0$

u_c零点：ωt = π-β，2π-β ... nπ-β

u_c极值点：ωt =0， π，2π ... nπ





$$i_c = -C\frac{du_c}{dt} = \frac{U_0}{\omega L} e^{-\delta t} \sin\omega t$$

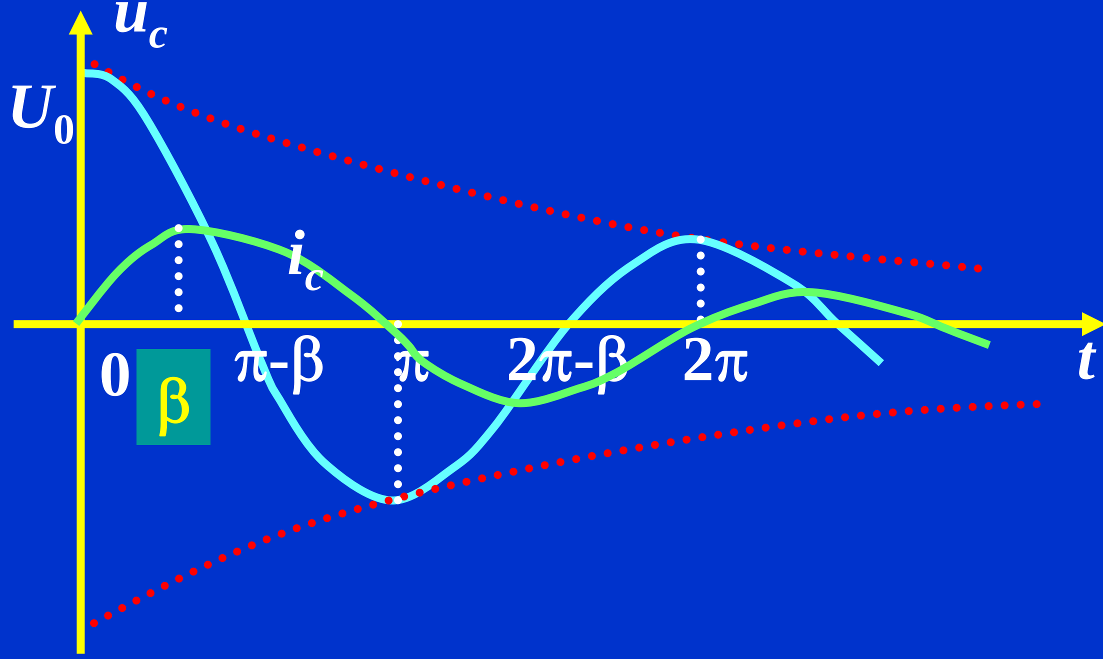
i_c零点：ωt =0，π，2π ... nπ ，

i_c极值点为u_L零点。

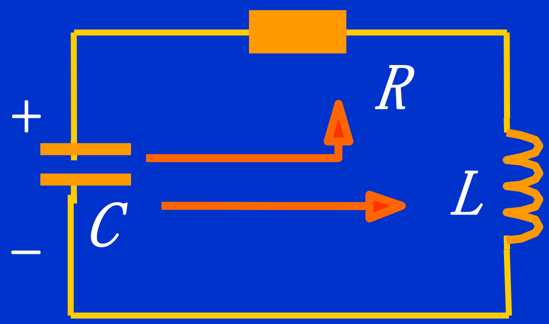
$$u_L = L\frac{di}{dt} = -\frac{\omega_0}{\omega} U_0 e^{-\delta t} \sin(\omega t - \beta)$$

u_L零点：ωt = β ，π+β，2π+β ... nπ+β

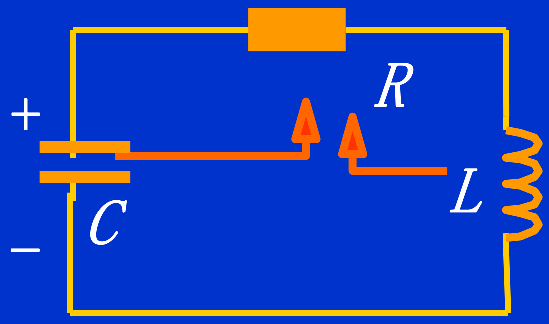
能量转换关系：



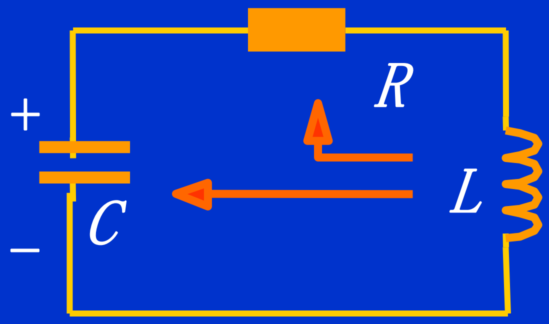
$0 < \omega t < \beta$



$\beta < \omega t < \pi-\beta$



$\pi-\beta < \omega t < \pi$



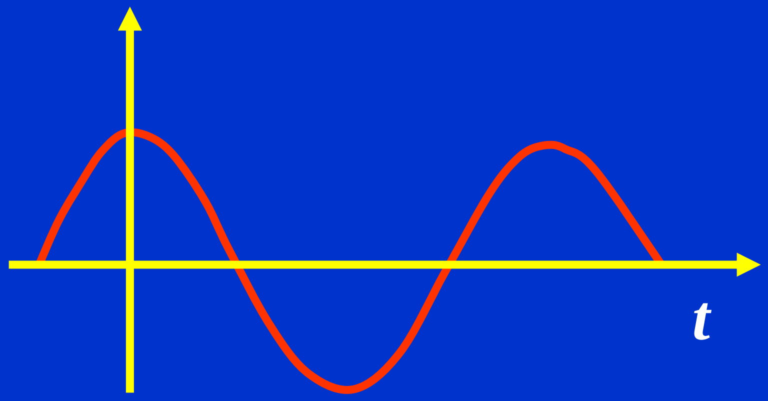
特例：R=0时

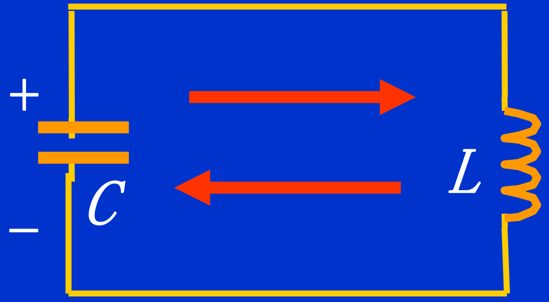
$$\text{则}\,\delta = 0\ ,\quad \omega = \omega_0 = \frac{1}{\sqrt{LC}}\ ,\ \beta = \frac{\pi}{2}$$

$$u_c = U_0 \sin(\omega t + 90^0) = u_L$$

$$i = \frac{U_0}{\omega L}\sin \omega t$$

$\longrightarrow$ 等幅振荡





$$\boxed{(3)\ R=2\sqrt{\frac{L}{C}}}$$

$$P_1=P_2=-\frac{R}{2L}=-\delta$$

$$u_c=A_1e^{-\delta t}+A_2te^{-\delta t}$$

由初始条件$\begin{cases} u_c(0^+)=U_0 \rightarrow A_1=U_0 \\ \dfrac{du_c}{dt}(0^+)=0 \rightarrow A_1(-\delta)+A_2=0 \end{cases}$

解出：

$$\begin{cases} A_1=U_0 \\ A_2=U_0\delta \end{cases}$$

$$\left.\begin{aligned} u_c&=U_0e^{-\delta t}(1+\delta t) \\ i_c&=-c\frac{du_c}{dt}=\frac{U_0}{L}te^{-\delta t} \\ u_L&=L\frac{di}{dt}=U_0e^{-\delta t}(1-\delta t) \end{aligned}\right\}\text{非振荡放电}$$

小结：

(1) 二阶电路含二个独立储能元件，是用二阶常微分方程所描述的电路。

(2) 二阶电路的性质取决于特征根，特征根取决于电路结构和参数，与激励和初值无关。

$$p=-\delta\pm\sqrt{\delta^2-\omega_0^2}$$

$R>2\sqrt{\dfrac{L}{C}}$ 或$\delta>\omega_0$ 过阻尼，非振荡放电 $\quad u_c=A_1e^{p_1t}+A_2e^{p_2t}$

$R=2\sqrt{\dfrac{L}{C}}$ 或 $\delta=\omega_0$ 临界阻尼，非振荡放电 $\quad u_c=A_1e^{-\delta t}+A_2te^{-\delta t}$

$R<2\sqrt{\dfrac{L}{C}}$ 或 $\delta<\omega_0$ 欠阻尼，振荡放电 $\quad u_c=Ae^{-\delta t}\sin(\omega t+\beta)$