

2011 年 解:

状态变量: x_k 表示留给项目 $k..n$ 的投资额, 其中 n 为项目总个数, $k=1..n$.

决策变量: u_k 表示投给项目 k 的投资额.

允许决策集合:

$$D_k(x_k) = \{u_k \mid 0 \leq u_k \leq x_k\}$$

状态转移方程:

$$x_{k+1} = x_k - u_k$$

递推关系式:

$$\begin{cases} f_k(x_k) = \max_{u_k \in D_k(x_k)} \{g_k(u_k) + f_{k+1}(x_k - u_k)\} & k = n - 1, \dots, 1 \\ f_n(x_n) = g_n(x_n) \end{cases}$$

其中, $g_k(u_k)$ 表示项目 k 的投资额为 u_k 时的盈利.

针对本题, $n = 3$, x_k 最大取 8

手工详解过程:

1. 初始化 $k = 3$

$$\begin{aligned} f_3(0) &= 0; f_3(1) = 4; f_3(2) = 26; f_3(3) = 40; f_3(4) = \\ &45; f_3(5) = 50; f_3(6) = 51; f_3(7) = 52; f_3(8) = 53. \end{aligned}$$

x_3	0	1	2	3	4	5	6	7	8
$f_3(x_3)$	0	4	26	40	45	50	51	52	53

2. $k = 2$

$$f_2(0) = \max\{g_2(0) + f_3(0)\} = 0 + 0 = 0;$$

$$f_2(1) = \max\left\{\begin{matrix} g_2(0) + f_3(1), \\ g_2(1) + f_3(0) \end{matrix}\right\} = \max\left\{\begin{matrix} 0 + 4, \\ 5 + 0 \end{matrix}\right\} = 5;$$

$$f_2(2) = \max\left\{\begin{matrix} g_2(0) + f_3(2), \\ g_2(1) + f_3(1), \\ g_2(2) + f_3(0) \end{matrix}\right\} = \max\left\{\begin{matrix} 0 + 26, \\ 5 + 4, \\ 15 + 0 \end{matrix}\right\} = 26;$$

$$\begin{aligned} f_2(3) &= \max\left\{\begin{matrix} g_2(0) + f_3(3), g_2(1) + f_3(2), \\ g_2(2) + f_3(1), g_2(3) + f_3(0) \end{matrix}\right\} \\ &= \max\left\{\begin{matrix} 0 + 40, 5 + 26, \\ 15 + 4, 40 + 0 \end{matrix}\right\} = 40; \end{aligned}$$

$$f_2(4)$$

$$= \max \left\{ \begin{array}{l} g_2(0) + f_3(4), g_2(1) + f_3(3), \\ g_2(2) + f_3(2), g_2(3) + f_3(1), \\ \mathbf{g_2(4) + f_3(0)} \end{array} \right\} = \max \left\{ \begin{array}{l} 0 + 45,5 + 40, \\ 15 + 26,40 + 4, \\ \mathbf{60 + 0} \end{array} \right\}$$

$$= \mathbf{60};$$

$$f_2(5) = \max \left\{ \begin{array}{l} g_2(0) + f_3(5), g_2(1) + f_3(4), \\ g_2(2) + f_3(3), g_2(3) + f_3(2), \\ g_2(4) + f_3(1), g_2(5) + f_3(0) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} 0 + 50,5 + 45, \\ 15 + 40,40 + 26, \\ 60 + 4,70 + 0 \end{array} \right\} = 70;$$

$$f_2(6)$$

$$= \max \left\{ \begin{array}{l} g_2(0) + f_3(6), g_2(1) + f_3(5), \\ g_2(2) + f_3(4), g_2(3) + f_3(3), \\ g_2(4) + f_3(2), g_2(5) + f_3(1), \\ g_2(6) + f_3(0) \end{array} \right\} = \max \left\{ \begin{array}{l} 0 + 51,5 + 50, \\ 15 + 45,40 + 40, \\ 60 + 26,70 + 4, \\ 73 + 0 \end{array} \right\}$$

$$= 86;$$

$$f_2(7) = \max \left\{ \begin{array}{l} g_2(0) + f_3(7), g_2(1) + f_3(6), \\ g_2(2) + f_3(5), g_2(3) + f_3(4), \\ g_2(4) + f_3(3), g_2(5) + f_3(2), \\ g_2(6) + f_3(1), g_2(7) + f_3(0) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} 0 + 52,5 + 51, \\ 15 + 50,40 + 45, \\ 60 + 40,70 + 26, \\ 73 + 4,74 + 0 \end{array} \right\} = 100;$$

$$f_2(8)$$

$$= \max \left\{ \begin{array}{l} g_2(0) + f_3(8), g_2(1) + f_3(7), \\ g_2(2) + f_3(6), g_2(3) + f_3(5), \\ g_2(4) + f_3(4), g_2(5) + f_3(3), \\ g_2(6) + f_3(2), g_2(7) + f_3(1), \\ g_2(8) + f_3(0) \end{array} \right\} = \max \left\{ \begin{array}{l} 0 + 53, 5 + 52, \\ 15 + 51, 40 + 50, \\ 60 + 45, 70 + 40, \\ 73 + 26, 74 + 4, \\ 75 + 0 \end{array} \right\}$$

$$= 110.$$

x_2	0	1	2	3	4	5	6	7	8
$f_2(x_2)$	0	5	26	40	60	70	86	100	110

3. $k = 1$

$$f_1(8) = \max \left\{ \begin{array}{l} g_1(0) + f_2(8), g_1(1) + f_2(7), \\ g_1(2) + f_2(6), g_1(3) + f_2(5), \\ \mathbf{g_1(4) + f_2(4)}, g_1(5) + f_2(3), \\ g_1(6) + f_2(2), g_1(7) + f_2(1), \\ g_1(8) + f_2(0) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} 0 + 110, 5 + 100, \\ 15 + 86, 40 + 70, \\ \mathbf{80 + 60}, 90 + 40, \\ 95 + 26, 98 + 5, \\ 100 + 0 \end{array} \right\} = 140$$

x_1	0	1	2	3	4	5	6	7	8
$f_1(x_1)$	0	5	26	40	80	90	106	120	140

最终结果：给项目 1 投资 4 万元，项目 2 投资 4 万元，项目 3 不投资，将获得最大利润 140 万元。

2012 年 解：设阶段序数 k 表示月份，状态变量 x_k 为第 k 个月初拥有的单位产品数量，亦为第 $k-1$ 月末时的单位产品数量，决策变量 u_k 为第 k 个月生产的单位产品数量， c_k 为第 k 月份需要的产品数量，这里 x_k 和 u_k 均取离散变量。

状态转移方程为：

$$x_{k+1} = x_k + u_k - c_k, \quad k = 1, 2, 3, 4; \text{ 且 } x_1 = 0.$$

k 段允许决策集合为：

$$D_k(x_k) = \{\max(0, c_k - x_k) \leq u_k \leq 6\}, k = 1, 2, 3;$$

当 $k=4$ 时, $u_k = c_k - x_k$ 。

设 $v_k(x_k, u_k)$ 为第 k 月的成本费, 单位为(千元), 则

$$v_k = 0.5 * x_k + u_k + I(u_k),$$

$$I(u_k) = \begin{cases} 3, & u_k > 0 \\ 0, & u_k = 0 \end{cases}$$

故指标函数为

$$V_{1,4} = \sum_{k=1}^4 v_k$$

令 $f_k(x_k)$ 表示为由 x_k 出发采用最优生产方案到第 4 个月结束这段期间的产品成本。

根据最优化原理, 则有递推公式:

$$\begin{cases} f_k(x_k) = 0, & k = 5 \\ f_k(x_k) = \min_{u_k \in D_k(x_k)} \{0.5 * x_k + u_k + I(u_k) + f_{k+1}(x_k + u_k - c_k)\}, & k = 1, 2, 3, 4 \end{cases}$$

其中:

$$c_k = \begin{cases} 2, & k = 1 \\ 3, & k = 2 \\ 2, & k = 3 \\ 4, & k = 4 \end{cases}$$

逆序计算的详细步骤如下:

(1) 当 $k=4$ 时,

$$f_4(x_4) = \min_{u_4 \in D_4(x_4)} \{0.5 * x_4 + u_4 + I(u_4)\} = \begin{cases} 2 & x_4 = 4, u_4 = 0 \\ 5.5 & x_4 = 3, u_4 = 1 \\ 6 & x_4 = 2, u_4 = 2 \\ 6.5 & x_4 = 1, u_4 = 3 \\ 7 & x_4 = 0, u_4 = 4 \end{cases}$$

(2) 当 $k=3$ 时, 因为 $x_4 = x_3 + u_3 - c_3 = x_3 + u_3 - 2 \leq 4$, 且 $u_3 = x_4 - x_3 + 2 \in [0, 6]$, 所以有:

当 $x_3 = 0, u_3 = (6, 5, 4, 3, 2)$, 此时 $f_3(x_3) = \min(11, 13.5, 13, 12.5, 12) =$

11, 在 $u_3 = 6, u_4 = 0$ 处取得最小值。

当 $x_3 = 1, u_3 = (5, 4, 3, 2, 1)$, 此时 $f_3(x_3) = \min(10.5, 13, 12.5, 12, 11.5) =$

10.5, 在 $u_3 = 5, u_4 = 0$ 处取得最小值。

当 $x_3 = 2, u_3 = (4, 3, 2, 1, 0)$, 此时 $f_3(x_3) = \min(10, 12.5, 12, 11.5, 8) =$

8, 在 $u_3 = 0, u_4 = 4$ 处取得最小值。

当 $x_3 = 3, u_3 = (3, 2, 1, 0)$, 此时 $f_3(x_3) = \min(9.5, 12, 11.5, 8) = 8$, 在 $u_3 = 0, u_4 = 4$ 处取得最小值。

当 $x_3 = 4, u_3 = (2, 1, 0)$, 此时 $f_3(x_3) = \min(9, 11.5, 8) = 8$, 在 $u_3 = 0, u_4 = 4$ 处取得最小值。

当 $x_3 = 5, u_3 = (1, 0)$, 此时 $f_3(x_3) = \min(8.5, 8) = 8$, 在 $u_3 = 0, u_4 = 4$ 处取

得最小值。

当 $x_3 = 6, u_3 = (0)$, 此时 $f_3(x_3) = \min(5) = 5$, 在 $u_3 = 0, u_4 = 0$ 处取得最小值。

(3) 当 $k=2$ 时, 因为 $x_3 = x_2 + u_2 - c_2 = x_2 + u_2 - 3 \leq 6, u_2 = x_3 - x_2 + 3 \in [0, 6]$, 且 $x_2 \leq 6$ 所以有:

当 $x_2 = 0, u_2 = (6, 5, 4, 3)$ 时, $f_2(x_2) = \min(17, 16, 17.5, 17) = 16$, 在 $u_2 = 5, u_3 = 0, u_4 = 4$ 处取得最小值。

当 $x_2 = 1, u_2 = (6, 5, 4, 3, 2)$ 时, $f_2(x_2) = \min(17.5, 16.5, 15.5, 17, 16.5) = 15.5$, 且在 $u_2 = 4, u_3 = 0, u_4 = 4$ 处取得最小值。

当 $x_2 = 2, u_2 = (6, 5, 4, 3, 2, 1)$ 时, $f_2(x_2) = \min(18, 17, 16, 15, 16.5, 16) = 15$, 在 $u_2 = 3, u_3 = 0, u_4 = 4$ 处取得最小值。

当 $x_2 = 3, u_2 = (6, 5, 4, 3, 2, 1, 0)$ 时, $f_2(x_2) = \min(15.5, 17.5, 16.5, 15.5, 14.5, 16, 12.5) = 12.5$, 且在 $u_2 = 0, u_3 = 6, u_4 = 0$ 处取得最小值。

当 $x_2 = 4, u_2 = (5, 4, 3, 2, 1, 0)$ 时, $f_2(x_2) = \min(15, 17, 16, 15, 14, 12.5) = 12.5$, 且在 $u_2 = 0, u_3 = 5, u_4 = 0$ 处取得最小值。

当 $x_2 = 5, u_2 = (4, 3, 2, 1, 0)$ 时, $f_2(x_2) = \min(14.5, 16.5, 15.5, 14.5, 10.5) = 10.5$, 且在 $u_2 = 0, u_3 = 0, u_4 = 4$ 处取得最小值。

当 $x_2 = 6, u_2 = (3, 2, 1, 0)$ 时, $f_2(x_2) = \min(14, 16, 15, 14.5, 11) = 11$, 且在 $u_2 = 0, u_3 = 0, u_4 = 4$ 处取得最小值。

(4) 当 $k=1$ 时, 因为 $x_1 = 0, x_2 = x_1 + u_1 - c_1 = u_1 - 2 \leq 6, u_1 = x_2 + 2 \in [0, 6]$, 所以有:

当 $u_1 = (6, 5, 4, 3, 2)$, $f_1(x_1) = \min(21.5, 20.5, 22, 21.5, 21) = 20.5$, 且在 $u_1 = 5, u_2 = 0, u_3 = 6, u_4 = 0$ 处取得最小值。

综上所述, 最优的库存方案为: 第一月生产 5 单位产品, 第二月和第四月不生产, 第三月生产 6 单位产品。