

Lecture 19

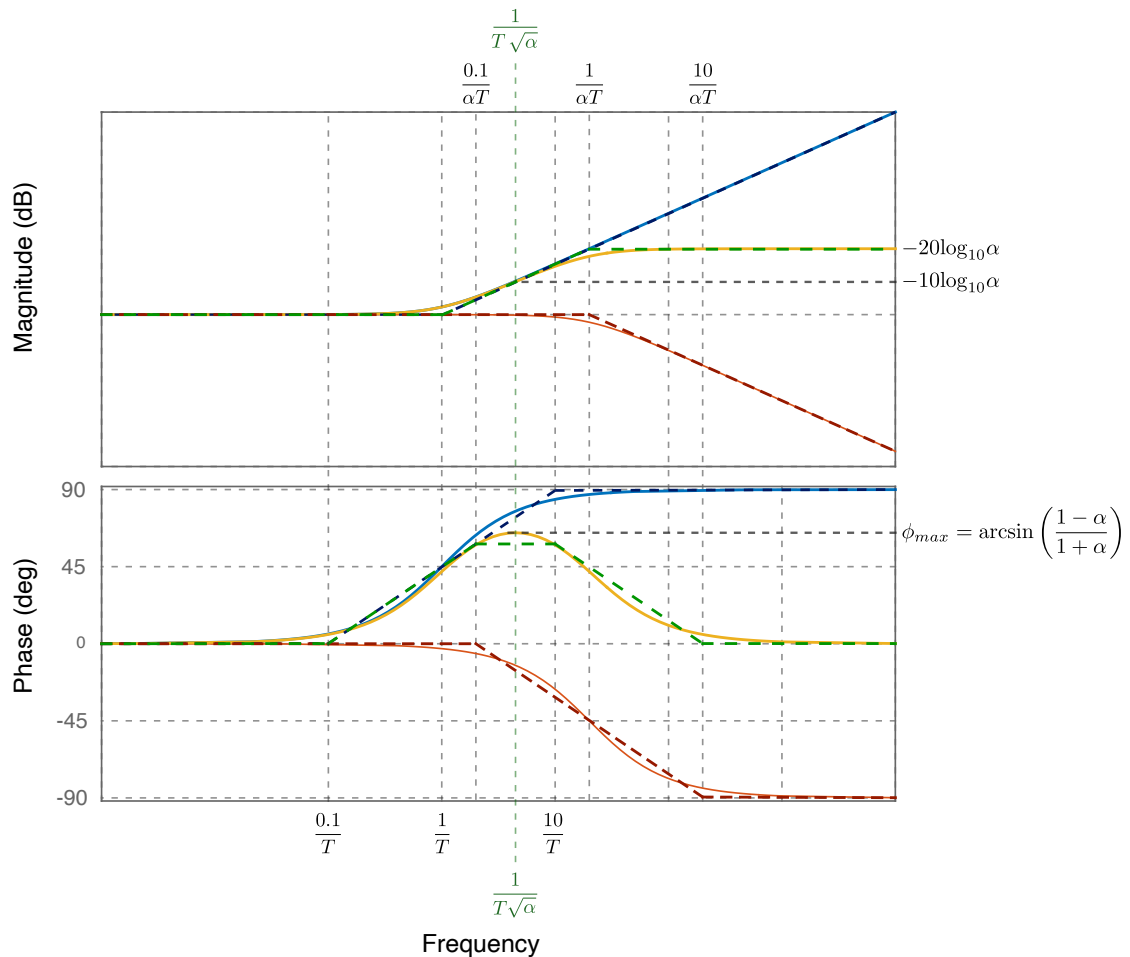
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19.1 Lead Compensator Design

The lead-compensator is a controller which has the form of a first-order high-pass filter

$$G_c(s) = K_c \frac{Ts + 1}{T\alpha s + 1} \quad \alpha \in (0, 1)$$

In general, we first design K_c based on the steady-state requirements of the system, and then design T and α based on the phase-margin requirement. First let's illustrate the bode-plots of a unity gain lead-compensator to understand how we can utilize its properties for the design process.



We can see that (both from the actual and approximate bode-plots), phase-lead compensator is a type of high-pass filter for which the cut-off (mid) frequency is $\omega_c = \frac{1}{\sqrt{\alpha}T}$. The low and high frequency gains are respectively 0 dB and $-20\log_{10}\alpha$ dB. The lead-compensator phase, ϕ_{max} , peaks at its cut-off frequency, and basically we will try to use this positive maximum phase shift to improve the phase margin of the feedback system.

Let's derive a formula for ϕ_{max} , which we will need during the design phase

$$\begin{aligned}
 \phi_{max} &= \angle[G_c(j\omega_c)] = \angle[1 + T\omega_c j] - \angle[1 + \alpha T\omega_c j] = \arctan(T\omega_c) - \arctan(\alpha T\omega_c) \\
 &= \arctan\left(\frac{1}{\sqrt{\alpha}}\right) - \arctan(\sqrt{\alpha}) \\
 &= \pi/2 - 2\arctan(\sqrt{\alpha}) \\
 \sin \phi_{max} &= \cos(2\arctan(\sqrt{\alpha})) = 1 - 2[\sin(\arctan(\sqrt{\alpha}))]^2 = 1 - 2\frac{\alpha}{1+\alpha} \\
 &= \frac{1-\alpha}{1+\alpha} \\
 \phi_{max} &= \arcsin\left(\frac{1-\alpha}{1+\alpha}\right) \Rightarrow \alpha = \frac{1 - \sin \phi_{max}}{1 + \sin \phi_{max}}
 \end{aligned}$$

As expected when $\alpha = 1$, $\phi_{max} = 0$ since numerator and denominator time constants becomes equal in this case. Accordingly, we can see that

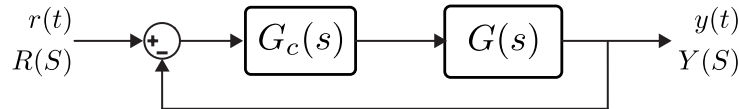
$$\alpha \searrow \Rightarrow \phi_{max} \nearrow$$

Theoretical maximum value for ϕ_{max} is 90° , however practically $\phi_{max} < 75^\circ$ for analog lead compensator circuits. Another important factor that we will need to pay attention is the gain-shift of the lead-compensator at the cut-off frequency

$$M_{dB}(j\omega_c) = -10\log_{10}\alpha \quad \& \quad |G(j\omega_c)| = \frac{1}{\sqrt{\alpha}}$$

We will illustrate the design process on an example

Ex: Consider the following feedback system illustrated below. It is given that $G(s) = \frac{1}{s(s+1)}$ and we want to design a lead-compensator, $G_c(s) = K_c \frac{Ts+1}{T\alpha s+1}$, such that unit-ramp steady-state error satisfies, $e_{ss} = 0.1$ and phase-margin of the compensated system satisfies $\phi_m^* = [45^\circ, 55^\circ]$.



Solution:

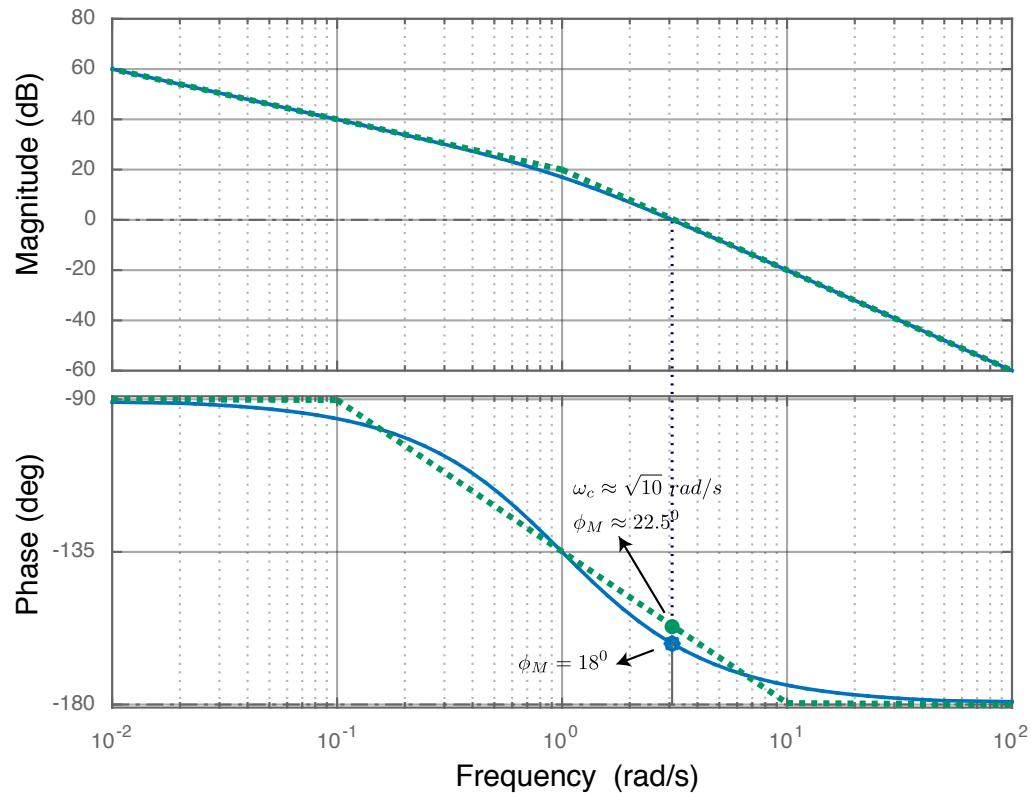
Step 1: We design/compute K_c based on the steady-state requirement on the unit-ramp error.

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K_c} = 1 \rightarrow K_c = 10 \quad (19.1)$$

Step 2: Embed K_c into $G(s)$, i.e.

$$\bar{G}(s) = K_c G(s) = \frac{10}{s(s+1)},$$

draw the bode-plot for $\bar{G}(s)$, and compute the gain-crossover frequency, ω_{gc} , and the phase margin, ϕ_M , for the uncompensated $\bar{G}(s)$. The bode plot of $\bar{G}(s)$ and its approximation is illustrated in the Figure below.



If we concentrate on the approximate bode-plots, we can estimate the gain-crossover frequency and the phase-margin as

$$\begin{aligned}\omega_{gc} &\approx \sqrt{10} \text{ rad/s} = 3.16 \text{ rad/s} \\ \phi_m &\approx 22.5^\circ\end{aligned}$$

If we compute the actual values from the actual Bode plot, we obtain

$$\begin{aligned}\omega_{gc} &= 3.08 \text{ rad/s} \\ \phi_m &\approx 18^\circ\end{aligned}$$

We can see that approximation is pretty good for the ω_c , however a little crude for ϕ_m . So let's compute $\angle[G(j\omega_{gc})]$ for $\omega_{gc} \approx \sqrt{10}$ and estimate the phase-margin based on this frequency

$$\begin{aligned}\angle[G(j\omega_{gc})] &= \angle[G(j\sqrt{10})] = -90^\circ - \arctan(\sqrt{10}) = -162.5^\circ \\ \phi_M &\approx 17.5^\circ\end{aligned}$$

Step 3: Compute the required phase increment, $\Delta\phi$ to be added by the compensator and compute α

$$\begin{aligned}\phi_{max} &\approx \Delta\phi = \phi_M^* - \phi_M + \epsilon \\ \epsilon &= 5^0 - 10^0\end{aligned}$$

So for the given problem, we can compute ϕ_{max} and α as

$$\begin{aligned}\phi_{max} &\approx 47.5^0 - 17.5^0 + 7^0 = 37^0 \\ \alpha &= \frac{1 - \sin \phi_{max}}{1 + \sin \phi_{max}} \approx \frac{1}{4}\end{aligned}$$

Step 4: Estimate the “new” gain-crossover frequency, $\hat{\omega}_{gc}$, and place the peak point of the lead-compensator, at this estimated $\hat{\omega}_{gc}$.

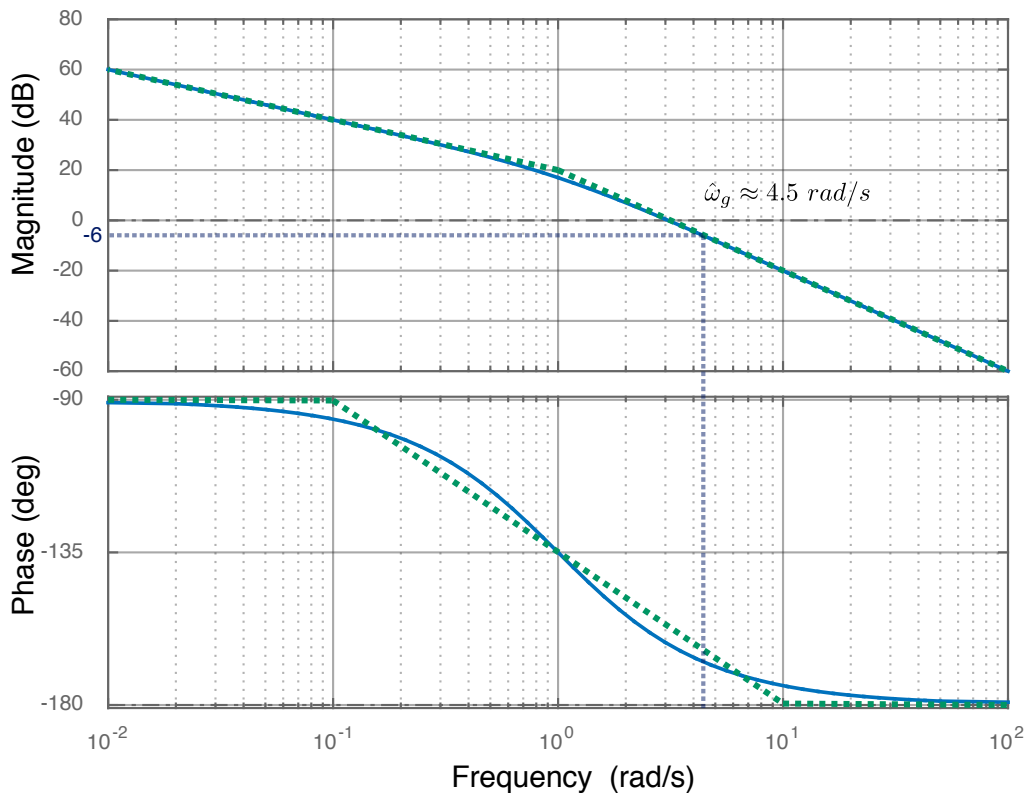
We already know that, lead-compensator at its center/cutoff frequency shifts the bode magnitude by $-10\log_{10}\alpha$ (or increases the gain by $\frac{1}{\sqrt{6}}$) and this causes a shift in gain-crossover frequency. For this reason, we can estimate the new gain-crossover frequency as the point where the bode magnitude of $\bar{G}(s)$ crosses $10 \log_{10}\alpha$, i.e.

$$M_{dB}(j\hat{\omega}_{gc}) = 10 \log_{10}\alpha \quad \text{or} \quad |G(j\hat{\omega}_{gc})| = \sqrt{\alpha}$$

In our problem,

$$M_{dB}(j\hat{\omega}_{gc}) \approx -6 \text{ dB} \quad \text{or} \quad |G(j\hat{\omega}_{gc})| = \frac{1}{2}$$

We can indeed estimate the new gain-crossover frequency graphically from the bode-plot. The figure below, illustrates how we can find the new-gain crossover frequency graphically



where $\hat{\omega}_{gc} \approx 4.5 \text{ rad/s}$. We can also estimate the new-gain crossover frequency numerically

$$\begin{aligned} |G(j\hat{\omega}_{gc})| &= \frac{1}{2} \\ \frac{100}{\hat{\omega}_{gc}^2 (\hat{\omega}_{gc}^2 + 1)} &= \frac{1}{4} \\ \hat{\omega}_{gc}^4 + \hat{\omega}_{gc}^2 - 400 &= 0 \\ \hat{\omega}_{gc} &\approx 400^{1/4} = 4.47 \text{ rad/s} \end{aligned}$$

which is very close to the graphical estimation.

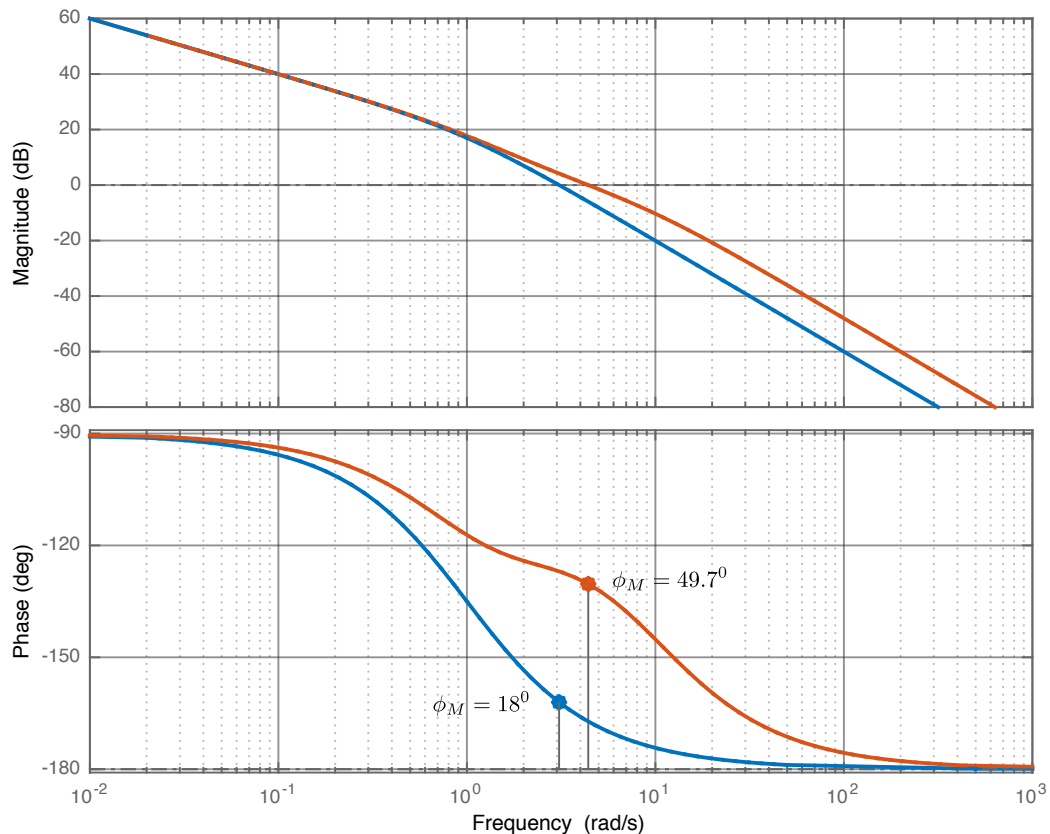
Step 5: Compute T using $\hat{\omega}_{gc}$ and check if the lead-compensator meets the phase-margin requirement. Otherwise, repeat the process with a higher $\Delta\phi$ angle.

$$\hat{\omega}_{gc} = \omega_c = \frac{1}{\sqrt{\alpha}T} \quad \Rightarrow \quad T = \frac{1}{\sqrt{\alpha}\hat{\omega}_{gc}}$$

In our example

$$T = \frac{1}{\frac{4.5}{2}} \approx 0.45 \quad \Rightarrow \quad G_c(s) = 10 \frac{0.45s + 1}{0.1125s + 1} = 40 \frac{s + 2.25}{s + 9}$$

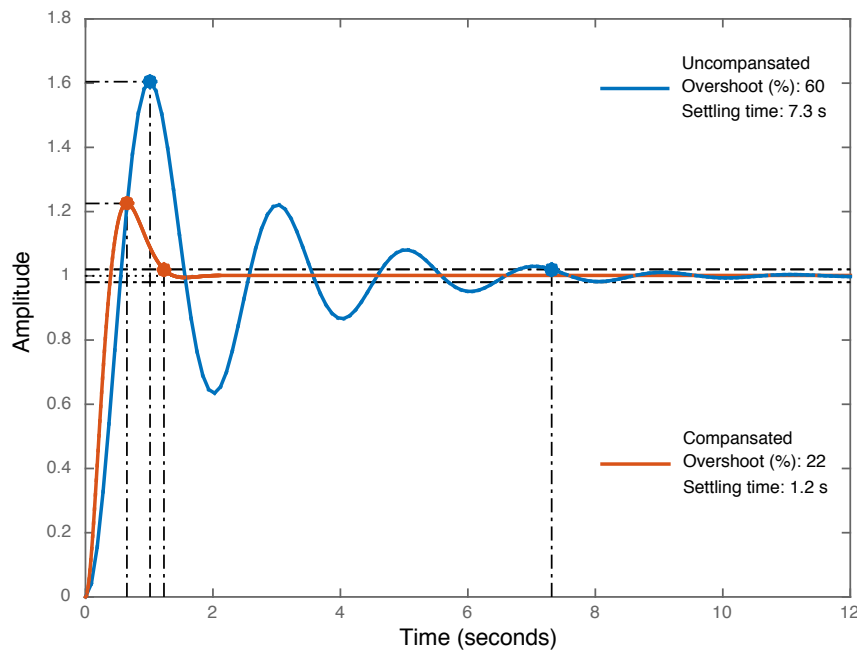
The Figure below illustrates the bode plots of both compensated and uncompensated ($\bar{G}(s) = K_c G(s)$) systems. Compensated systems has a phase margin of $\phi_m = 49.7^\circ$ which meets the requirements.



Now let's compute new phase margin directly using the new gain-crossover frequency

$$\begin{aligned}\phi_M &= -180^\circ + \angle[G_c(j4.45 \text{ rad/s})\bar{G}(j4.45 \text{ rad/s})] = -180^\circ + 37^\circ + \angle[\bar{G}(j4.45 \text{ rad/s})] \\ &= -180^\circ + 37^\circ - 90^\circ - \angle[j4.45 + 1] = -180^\circ + 37^\circ - 90^\circ - 77^\circ \\ &= 50^\circ\end{aligned}$$

which is consistent with the phase-margin on the bode-plot. In Figure below, we compare the closed-loop step responses of both uncompensated and compensated closed-loop transfer functions. We can clearly see that, the lead-compensator substantially improves both the settling time and over-shoot performance.



Phase-Lead Design Guideline:

1. Design/compute K_c based on the steady-state requirements
2. Embed K_c into $G(s)$, i.e. $\bar{G}(s) = K_c G(s)$, then draw the bode-plots for $\bar{G}(s)$, and compute the gain-crossover frequency, ω_{gc} , and the phase margin, ϕ_M , for the uncompensated $\bar{G}(s)$.
3. Compute the required phase increment, $\Delta\phi$ to be added by the compensator and design/compute α
4. Estimate the “new” gain-crossover frequency, $\hat{\omega}_{gc}$, and place the peak point of the lead-compensator, at this estimated $\hat{\omega}_{gc}$.
5. Check the phase-margin if it does not meet the requirements increase $\Delta\phi$ and repeat the process.

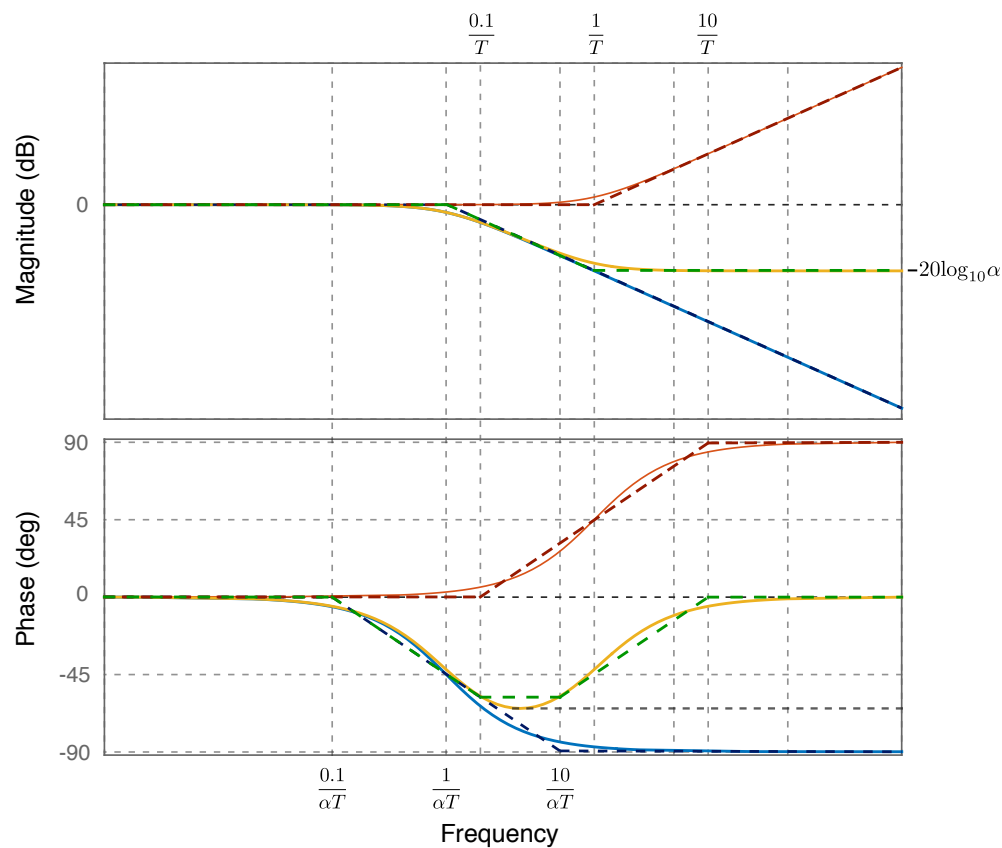
19.2 Lag Compensator Design

The lag-compensator is a controller which has the form of a first-order low-pass filter

$$G_c(s) = K_c \frac{Ts + 1}{T\alpha s + 1} \quad \alpha \in (1, \infty)$$

In general, we first design K_c based on the steady-state requirements of the system, then design α based on the phase-margin requirement, and finally choose a T such that phase-lag of the compensator does not interfere with the gain-crossover frequency.

First let's illustrate the bode-plots of a unity gain lag-compensator to understand how we can utilize its properties for the design process.



In lag-compensator design, we basically use the negative-gain shift of the compensator in the high-frequency region and we try to push the low and mid frequency region to the left (in frequency axis) such that they don't interfere with the gain-crossover frequency. For this reason, design process easier compared to the lead-compensator.

We will illustrate the lag-compensator design process on an the same example

Ex: Consider the feedback system that we analyzed previously in lead compensatory case. Plant is same, $G(s) = \frac{1}{s(s+1)}$. However now we want to design a lag-compensator, $G_c(s) = K_c \frac{Ts+1}{T\alpha s+1}$, $\alpha \in (1, \infty)$, such that unit-ramp steady-state error satisfies, $e_{ss} < 0.1$ and phase-margin of the compensated system satisfies $\phi_m^* > 40^\circ$.

Solution:

Step 1: Same as the lead-design, we design/compute K_c based on the steady-state requirement on the unit-ramp error.

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K_c} = 1 \rightarrow K_c = 10 \quad (19.2)$$

Step 2: Embed K_c into $G(s)$, i.e.

$$\bar{G}(s) = K_c G(s) = \frac{10}{s(s+1)},$$

draw the bode-plot for $\bar{G}(s)$.

Step 3: Compute/find the required gain-crossover frequency, ω_{gc}^* , based on the required phase-margin, ϕ_M^* , and compute/find the magnitude of $G(s)$ at ω_{gc}^* , i.e. $|\bar{G}(j\omega_{gc}^*)|$ or $20\log_{10}|\bar{G}(j\omega_{gc}^*)|$.

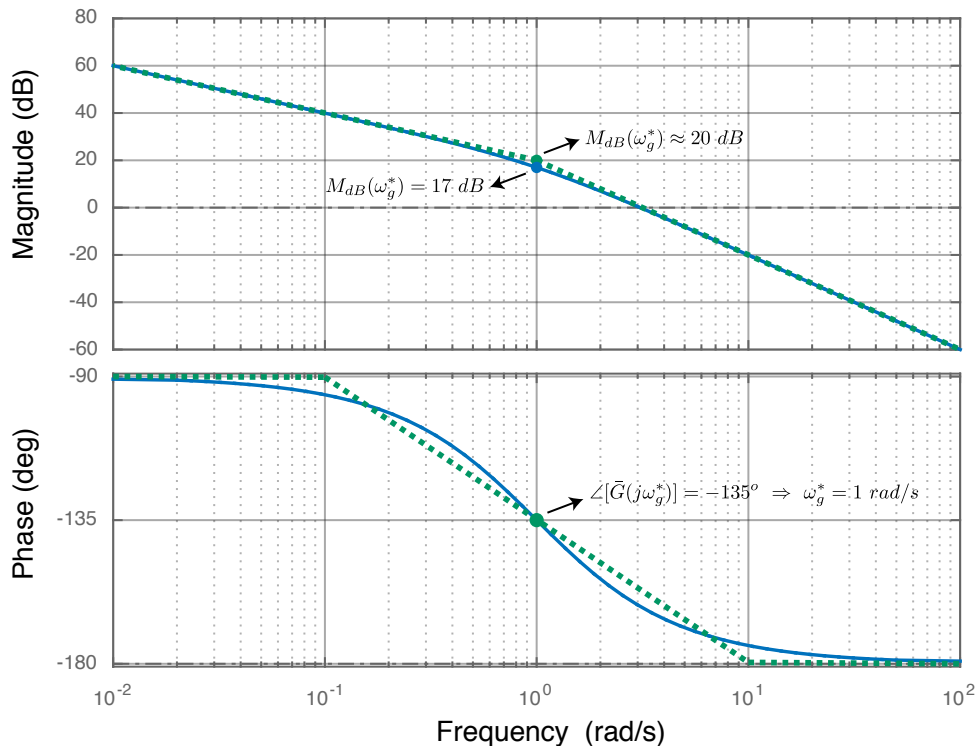
$$\begin{aligned} \angle[\bar{G}(j\omega_{gc}^*)] &= -180^\circ + \phi_M^* + \epsilon \\ \epsilon &\approx 5^\circ \end{aligned}$$

Note that at high-frequency region, lag-compensator acts like negative gain shift on magnitude plot while not affecting the phase response. Such a gain shift technically changes the gain-crossover frequency, which can be potentially used for changing the phase margin.

For our example, compensated gain-crossover frequency can be computed as

$$\angle[\bar{G}(j\omega_{gc}^*)] = -135^\circ \Rightarrow \omega_{gc}^* \approx 1 \text{ rad/s}$$

The bode plot of $\bar{G}(s)$, required gain-crossover frequency, and the magnitude of $\bar{G}(s)$ at the desired gain-crossover frequency are illustrated in the Figure below.



From the bode-plots we can observe that at the desired $\omega_{gc}^* \approx 1 \text{ rad/s}$, bode-plot approximation has a magnitude of 20 dB , whereas the magnitude in the actual bode plot is approximately 17 dB . In this example, let's use the magnitude of the approximation in the next Step.

Step 4: Compute α to compensate the the magnitude at the new-gain crossover frequency

$$20\log_{10}\alpha = 20\log_{10}|\bar{G}(j\omega_{gc}^*)| \quad \text{or} \quad \alpha = |\bar{G}(j\omega_{gc}^*)|$$

In our example, α , can be computed as

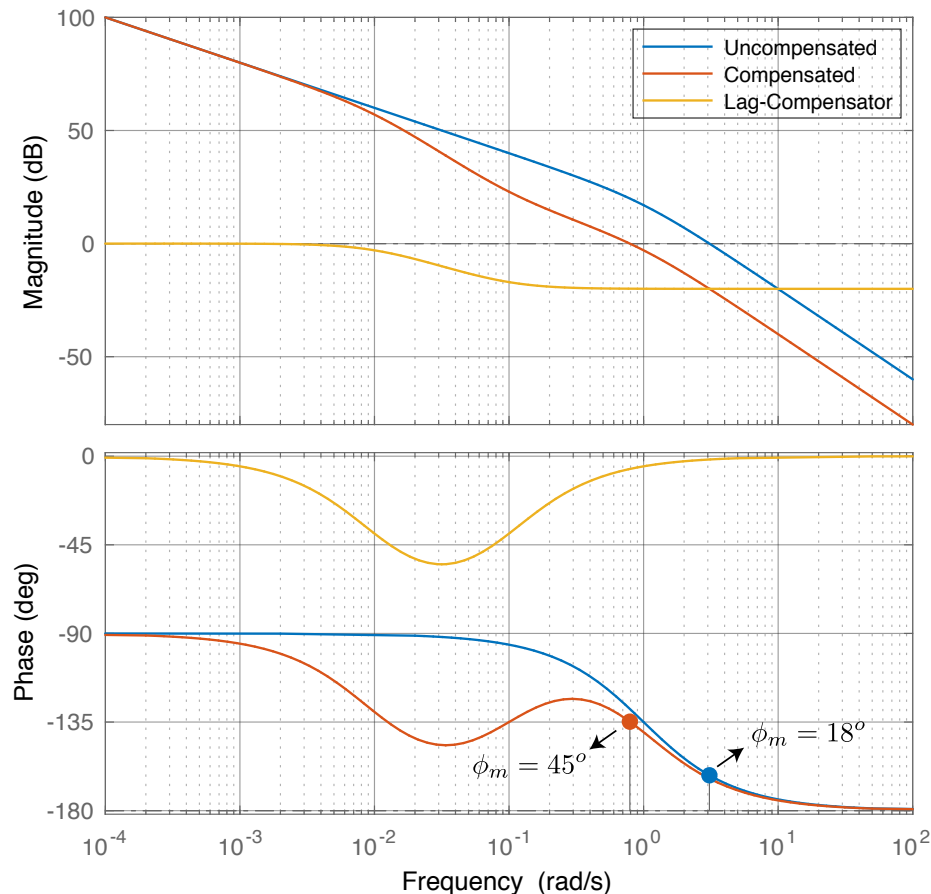
$$20\log_{10}\alpha \approx 20 \text{ dB} \Rightarrow \alpha \approx 10$$

Step 4: Choose T such that " $10/T \leq \omega_{gc}^*$ ". Note that $10/T$ is the frequency where the phase of the compensator approximately re-approaches to zero. This is required in order the negative phase bump of the compensator does not affect the phase-margin.

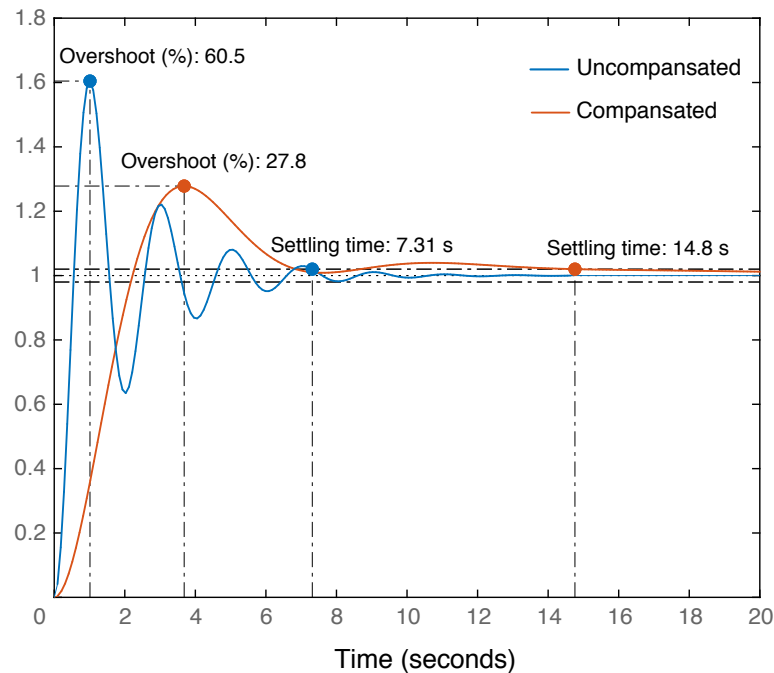
$$\frac{1}{T} \approx 0.1\omega_{gc}^* \Rightarrow T \approx 10$$

$$G_c(s) = 10 \frac{10s + 1}{100s + 1} = \frac{s + 0.1}{s + 0.01}$$

The Figure below illustrates the bode plots of the uncompensated ($\bar{G}(s) = K_c G(s)$) system, designed lag-compensator, and the compensated system. Compensated systems has a phase margin of $\phi_m = 45^\circ$ which meets the requirements.



In Figure below, we compare the closed-loop step responses of both uncompensated and compensated closed-loop transfer functions. We can see that, similar to the the lead-compensator, lag-compensator improves the over-shoot performance. However, the settling-time of the compensated system is approximately two times of the original system. This is major the drawback of the lag-compensator.



The reason behind the reduced convergence speed performance is that new gain-crossover frequency is less than the original gain-crossover frequency. PM is closely related with over-shoot performance. On the other hand gain-crossover frequency determines the band-width of the system which is closely related with the rise and settling times.

Phase-Lag Design Guideline:

1. Design/compute K_c based on the steady-state requirements
2. Compute/find the required gain-crossover frequency, ω_{gc}^* , based on the required phase-margin, ϕ_M^* , and compute/find the magnitude of $G(s)$ at ω_{gc}^*
3. Compute α to compensate the the magnitude at the new-gain crossover frequency
4. Chose T at such that " $10/T \leq \omega_{gc}^*$ ". This is required in order the negative phase bump of the compensator does not affect the phase-margin. Note that $10/T$ is the frequency where the phase of the compensator approximately re-approaches to zero.
5. Check the phase-margin if it does not meet the requirements change $\Delta\phi$ and repeat the process.