Lecture 5

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Let's remember the idealized and simplified block-diagram structure a discrete-time control system (See Fig. 5.1)

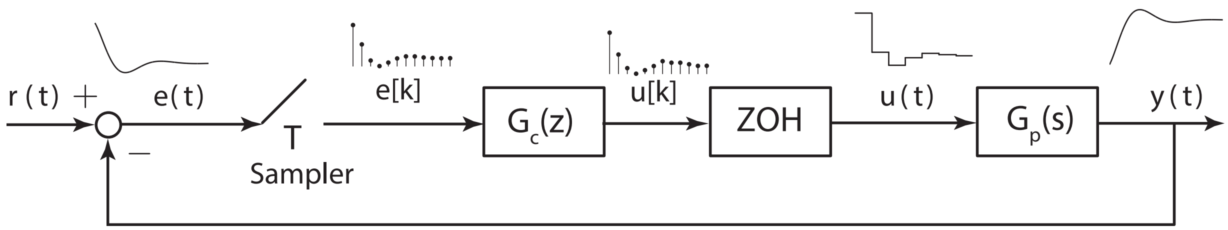


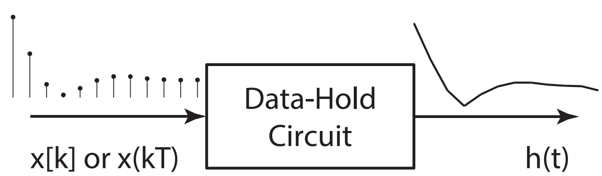
Figure 5.1: Block diagram of an LTI discrete-time control system

Loop contains both continuous-time and discrete-time signals and blocks.

- We can treat the system as a completely discrete-time system. We technically restrict ourselves into sampled time instants (in this course we will fundementally follow this path)
- Alternatively, we can use continuous time signals (as much as possible) and deal with starred versions of signals and starred Laplace transform (In original–with star notation–notes, I sometimes follow this path).

5.1 Data Hold Operation

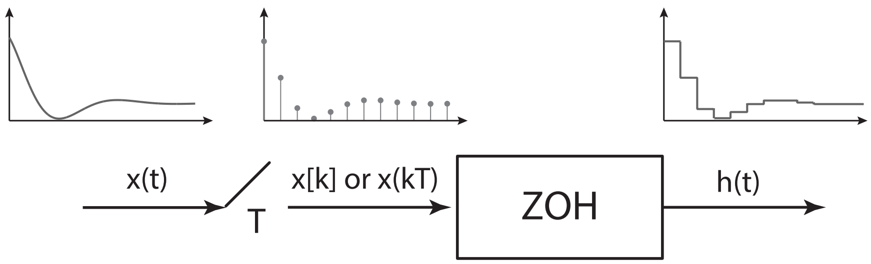
Data-Hold operation is an idealized model of a DAC device which converts a digital signal to an analog signal. In terms of the terminology used in this class, Data-Hold operation is the process of obtaining a CT signal $h(t)$ from a DT sequence. A general data-hold operation block circuit is shown below



Simplest and most dominantly used (I have never seen a practical usage of other hold operations) hold circuit/operation is the zero-order-hold (ZOH). Basically, at each time instant kT ZOH "samples" the input $x[k]$ or $x(kT)$ and "holds" this value at the output until the next sampling event. Mathematically,

$$h(kT + t) = x(kT) = x[k], \text{ for } 0 \leq t < T$$

The figure below illustrates a serios connection of an ideal CT-DT sampler and an ideal ZOH block.



Let's assume that $x(t)$ is a strictly causal signal, then from the definition of ZOH we can express $h(t)$ in terms of $x(t)$ (or $x[k]$, $x(kT)$) as

$$h(t) = x(0)\,[u(t) - u(t-T)] + x(T)\,[u(t-T) - u(t-2T)] + x(2T)\,[u(t-2T) - u(t-3T)] + \cdots$$
$$h(t) = \sum_{k=0}^{\infty} x(kT)\,[u(t-kT) - u(t-(k+1)T)]$$

where $u(t)$ is the unit-step function.

If we take the Laplace transform of $h(t)$, we obtain

$$\begin{aligned}\mathcal{L}\{h(t)\} &= \sum_{k=0}^{\infty} x(kT)\mathcal{L}\left\{[u(t-kT) - u(t-(k+1)T)]\right\} \\ &= \sum_{k=0}^{\infty} x(kT)\left[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}\right] \\ H(s) &= \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x[k]e^{-kTs}\end{aligned}$$

Z-transform & ZOH

When analyzing the discrete time control systems, we will (frequently) need to compute the Z-transform of sampled signals, for which the Laplace transform involves the term $\left(1 - e^{-Ts}\right)$.

Let $\mathcal{L}\{x(t)\} = X(s) = \left(1 - e^{-Ts}\right) G(s)$. Now let's analyze the z-transform of the sampled version of the signal, i.e. $X(z) = \mathcal{Z}\{x(kT)\}$. First let's find $x(t)$ from $X(z)$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\left(1 - e^{-Ts}\right) G(s)\right\} = \mathcal{L}^{-1}\left\{G(s)\right\} - \mathcal{L}^{-1}\left\{e^{-Ts}G(s)\right\}$$

Let $g(t) = \mathcal{L}^{-1}\left\{G(s)\right\}$ then

$$x(t) = g(t) - g(t-T)$$

$x(kT)$ and $x[k]$ takes the form

$$\begin{aligned}x(kT) &= g(kT) - g(kT-T) \\ x[k] &= \hat{g}[k] - \hat{g}[k-1]\end{aligned}$$

Then $X(z)$ takes the form

$$X(z) = \left(1 - z^{-1}\right) G(z)$$

where $G(z) = \mathcal{Z}\left\{\left[\mathcal{L}^{-1}\left\{G(s)\right\}\right]^*\right\}$ and $*$ is the sampling operation. In the textbook this notation is shortened to have $G(z) = \mathcal{Z}\left\{G(s)\right\}$. After that we have

$$X(z) = \left(1 - z^{-1}\right)\mathcal{Z}\left\{G(s)\right\}$$

Example 1. Obtain the z transform of $x(kT)$ where $T = 1$ and $X(s)$ is given as

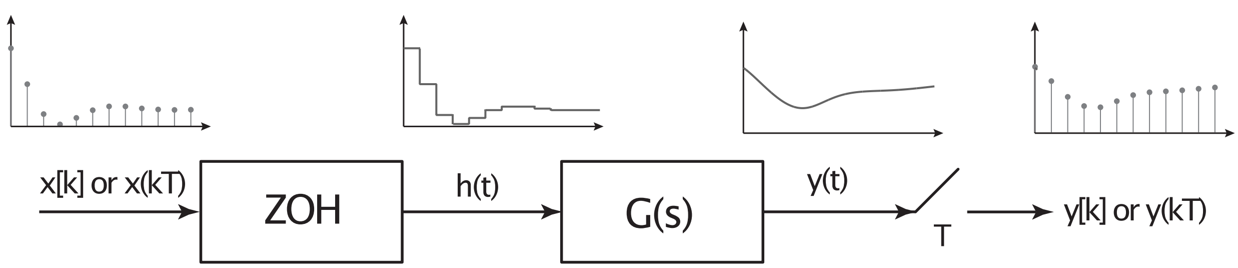
$$X(s) = \frac{1 - e^{-s}}{s}\frac{1}{s+1}$$

Solution:

$$\begin{aligned}
X(z) &= \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{1}{s(s+1)}\right\} = \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} \\
&= \frac{z-1}{z}\left(\frac{z}{z-1} - \frac{z}{z-e^{-1}}\right) = 1 - \frac{z-1}{z-e^{-1}} \\
X(z) &= \frac{1-e^{-1}}{z-e^{-1}}
\end{aligned}$$

5.2 Discretization of CT TF under ZOH and Ideal Sampling Operators

The figure below illustrates an open loop fundamental digital control control system that is composed of a CT plant, $G(s)$, a ZOH operator, and an ideal syncronous sampler. Our goal is to find a DT (z-domain) transfer function between the discrete-time input signal, $x[k]$ and the discrete-time output signal, $y[k]$, i.e. $fracY(z)X(z)$.



Let's first concentrate on input–output dynamics of $G(s)$

$$\begin{aligned}
Y(s) &= G(s)H(s) \\
&= G(s)\frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x[k]e^{-kTs} \\
&= \left(1 - e^{-Ts}\right)\frac{G(s)}{s}\sum_{k=0}^{\infty} x[k]e^{-kTs} \\
&= \left(1 - e^{-Ts}\right)\hat{G}(s)\sum_{k=0}^{\infty} x[k]e^{-kTs}
\end{aligned}$$

where $\hat{G}(s)$ is also the Laplace transform of the step-response of $G(s)$. Let $P(s) = \hat{G}(s) \sum_{k=0}^{\infty} x[k]e^{-kTs}$, and try to derive $P(z) = \mathcal{Z}\{P(s)\}$. First take the inverse Laplace transform of the expression

$$
\begin{aligned}
p(t) &= \mathcal{L}^{-1}\left\{\hat{G}(s)\sum_{k=0}^{\infty} x[k]e^{-kTs}\right\} = \sum_{k=0}^{\infty} x[k]\mathcal{L}^{-1}\left\{\hat{G}(s)e^{-kTs}\right\} \\
&= \sum_{k=0}^{\infty} x[k]\hat{g}(t-kT)
\end{aligned}
$$

If we limit ourselves to causal $g(t)$ case and sample $p(t)$, we will obtain

$$
p(nT) = p[k] = \sum_{k=0}^{n} x[k]\hat{g}((n-k)\,T) = \sum_{k=0}^{n} x[k]\hat{g}[n-k]
$$

Note that this is the expression of the discrete-time convolution, and thus we can infer the followings

$$
\begin{aligned}
p(nT) &= \hat{g}(nT) * x(nT) = x(nT) * \hat{g}(nT) \\
p[n] &= \hat{g}[n] * x[n] = x[n] * \hat{g}[n] \\
P(z) &= \hat{G}(z)X(z)
\end{aligned}
$$

If we use the derivation that we found previously regarding the Z-transform of sampled signals, for which the Laplace transform involves the term $\left(1 - e^{-Ts}\right)$, we can compute $Y(z)$ as

$$
\begin{aligned}
Y(z) &= \left(1 - z^{-1}\right) P(z) = \left(1 - z^{-1}\right)\hat{G}(z)X(z) \\
G_d(z) = \frac{Y(z)}{X(z)} &= \left(1 - z^{-1}\right)\hat{G}(z) \quad \text{where } \hat{G}(z) = \mathcal{Z}\left\{\frac{G(s)}{s}\right\}
\end{aligned}
$$

We call $G_d(z)$ as the discretized transfer function of $G(s)$ under ZOH and ideal sampling operators. The result is pretty interesting: the impulse response of the "discretized" system is obtained by sampling the step response function of original the continuous time system.