Laplace Transform Fundamentals

M. Mert ANKARALI

Laplace Transform

$$f(t) \to \mathcal{L}\{f(t)\} = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Basic Laplace Transform Pairs

Signal	$f(t)$	$F(s)$
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$\frac{1}{s}$
Unit Ramp	$tu(t)$	$\frac{1}{s^2}$
Exponential	$e^{-at}u(t)$	$\frac{1}{s+a}$
Damped Ramp	$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$
Sine	$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$
Damped Sine	$e^{-at}\sin(\omega t)u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
Damped Cosine	$e^{-at}\cos(\omega t)u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Laplace Transform Properties

Properties	Time Domain	Laplace Domain
Independent Variable	t	s
Signal Representation	$f(t)$	$F(s)$
Uniqueness	$\mathcal{L}^{-1}\{F(s)\} = f(t)$	$\mathcal{L}\{f(t)\} = F(t)$
Linearity	$Af_1(t) + Bf_2(t)$	$AF_1(s) + BF_2(s)$
Differentiation	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0-) - \dot{f}(0-)$
Time Shift	$f(t-T)$	$e^{-Ts}F(s)$
Final Value Theorem	$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$

Partial Fraction Expansion

$$Y(s) = \sum_{i=1}^{N} \frac{k_i}{s - p_i}$$

$$\forall i \ , \ p_i = \lim_{s \to pi} (s - p_i) Y(s)$$

For non-repeated roots

Partial Fraction Expansion

$$Y(s) = \left(\sum_{i=1}^{N} \frac{k_i}{s - p_i} \right) + \frac{k_{N+1}}{s - p_{N+1}} + \frac{k_{N+2}}{(s - p_{N+1})^2}$$

$$k_{N+1} = \lim_{s \to p_i} (s - p_{N+1})^2 Y(s)$$

$$k_{N+2} = \lim_{s \to p_i} \frac{d}{ds} \left((s - p_{N+1})^2 Y(s) \right)$$

For repeated (double) roots