Frequency Response Techniques in Feedback Control Systems

Let's assume $u(t)$, $y(t)$, and $G(t)$ represents the input, output, and transfer function representation of an input-output continuous time system.

In order to characterize frequency response of a dynamical system, the test signal is

$$u(t) = e^{j\omega t}$$

which is an artificial complex periodic signal with a frequency of ω. If we assume that the system is "stable" or system is a part of closed loop system and closed loop behavior is stable then at steady state we have

$$\begin{aligned} y_{ss}(t) &= G(j\omega)e^{j\omega t} \\ &= |G(j\omega)|e^{i\omega t + \angle[G(j\omega)]} \\ &= Me^{i\omega t+\theta} \end{aligned}$$

In other words complex periodic signal is scaled and phase shifted based on the following operators

$$\begin{aligned} M &= |G(j\omega)| \\ \theta &= \angle G(j\omega) \end{aligned}$$

It is very easy to show that for a general real time domain signal $u(t) = \sin(\omega t + \phi)$, the output $y(t)$ at steady state is computed via

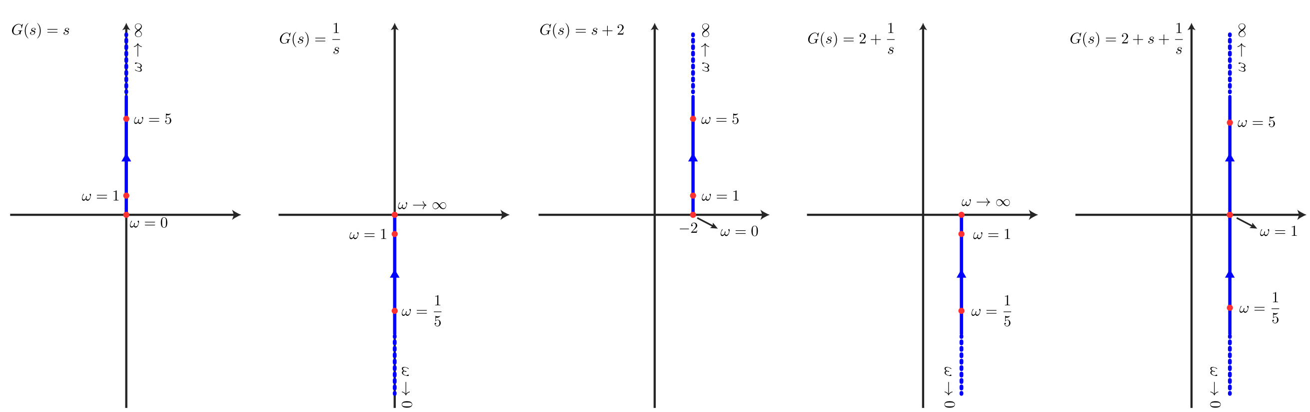
$$y_{ss}(t) = M\sin(\omega t + \phi + \theta)$$

2. Plotting Frequency Response: Polar Plot

We can consider the frequency response function $G(j\omega)$ as a mapping from positive $j\omega$ axis to a curve in the complex plane. In polar plot, we simply draw the frequency response function starting from $\omega = 0$ (or $\omega \to 0^+$) to $\omega \to \infty$ on the complex plane.

Ex: Let's draw the polar plots of

$$G_1(s) = s \quad , \quad G_2(s) = \frac{1}{s} \quad , \quad G_3(s) = s+2$$
$$G_4(s) = 2 + \frac{1}{s} \quad , \quad G_5(s) = 2 + s + \frac{1}{s}$$



Ex: Draw the polar plots of

$$G_1(s) = \frac{1}{s+1} \quad , \quad G_2(s) = \frac{s}{s+1}$$

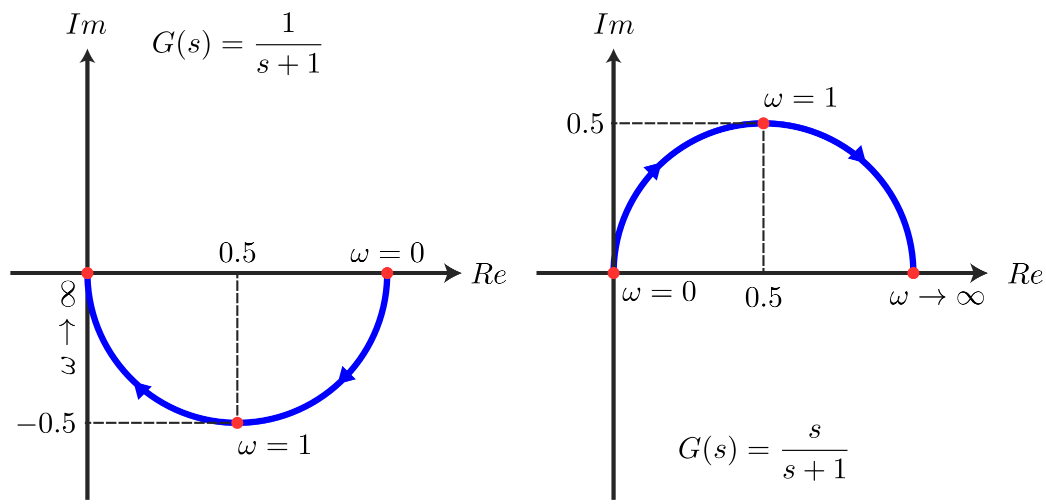
Let's analyze $G_1(j\omega)$ for $\omega \in [0, \infty)$

$$\begin{aligned} G_1(j\omega) &= \frac{1}{j\omega+1} = \frac{1-j\omega}{\omega^2+1} = \frac{1}{\omega^2+1} - \frac{\omega}{\omega^2+1}j \\ |G_1(j\omega)| &= \frac{1}{\sqrt{1+\omega^2}} \\ \angle[G_1(j\omega)] &= \arctan(-\omega) \end{aligned}$$

Now let's analyze $G_2(j\omega)$ for $\omega \in [0, \infty)$

$$\begin{aligned} G_2(j\omega) &= \frac{j\omega}{j\omega+1} = \frac{j\omega+\omega^2}{\omega^2+1} = \frac{\omega^2}{\omega^2+1} + \frac{\omega}{\omega^2+1}j \\ |G_2(j\omega)| &= \sqrt{\frac{\omega^2}{1+\omega^2}} \\ \angle[G_2(j\omega)] &= \arctan(1/\omega) \end{aligned}$$

Polar plots of $G_1(s)$ and $G_2(s)$ are illustrated below.



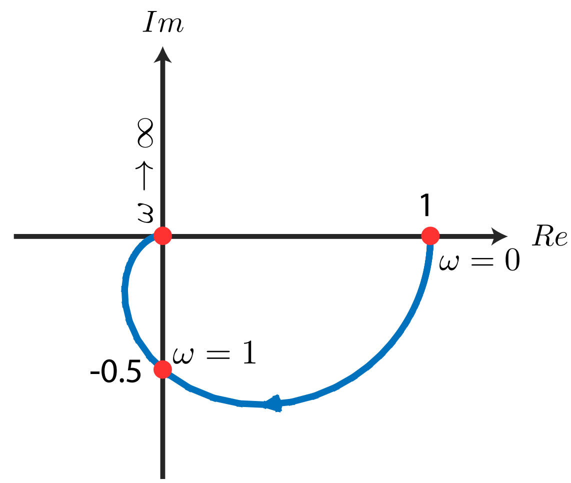
Ex: Draw the polar plot of $G(s) = \frac{1}{(s+1)^2}$

$$\begin{aligned} G(j\omega) &= \frac{1}{(j\omega+1)^2} = \frac{(-j\omega+1)^2}{(\omega^2+1)^2} \\ &= \left[\left(1-\omega^2\right) + j(-2\omega)\right]\frac{1}{(\omega^2+1)^2} \end{aligned}$$

Some important points and associated features on the polar plot can be computed as

$$\begin{aligned} &\omega \to 0 \ \Rightarrow G(j\omega) = 1 \\ &\omega \to 1 \ \Rightarrow G(j\omega) = -0.5j \\ &\omega \to \infty \ \Rightarrow |G(j\omega)| \to 0 \quad \& \quad \angle[G(j\omega)] \to -\pi \end{aligned}$$

Resultant polar plot is illustrated below



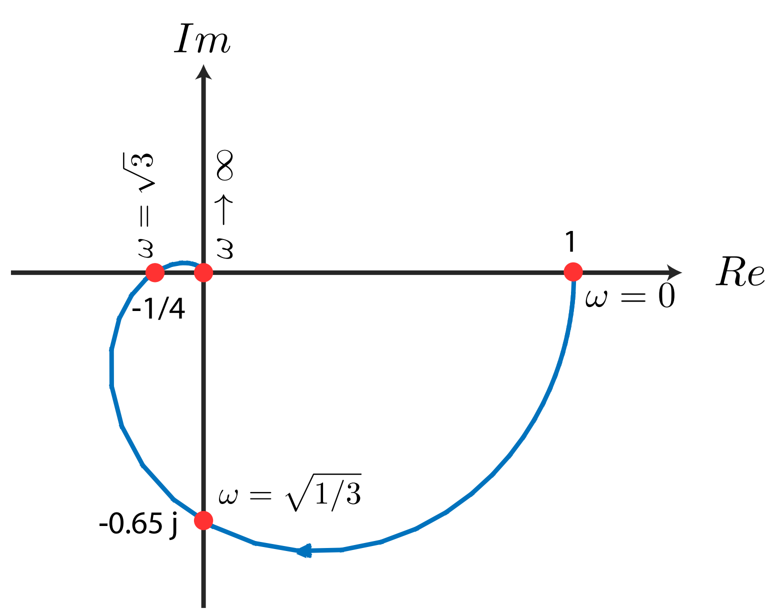
Ex: Draw the polar plot of $G(s) = \frac{1}{(s+1)^3}$

$$
\begin{aligned}
G(j\omega) &= \frac{1}{(j\omega+1)^3} = \frac{(-j\omega+1)^3}{(\omega^2+1)^3} \\
&= \left[\left(1-3\omega^2\right) + j(\omega^3 - 3\omega)\right] \frac{1}{(\omega^2+1)^3}
\end{aligned}
$$

Some important points and associated features on the polar plot can be computed as

$$
\begin{aligned}
&\omega \to 0 \ \Rightarrow G(j\omega) = 1 \\
&\omega \to \sqrt{1/3} \ \Rightarrow G(j\omega) = -0.65j \\
&\omega \to \sqrt{3} \ \Rightarrow G(j\omega) = -1/8 \\
&\omega \to \infty \ \Rightarrow |G(j\omega)| \to 0 \quad \& \quad \angle[G(j\omega)] \to \pi/2
\end{aligned}
$$

Resultant polar plot is illustrated below



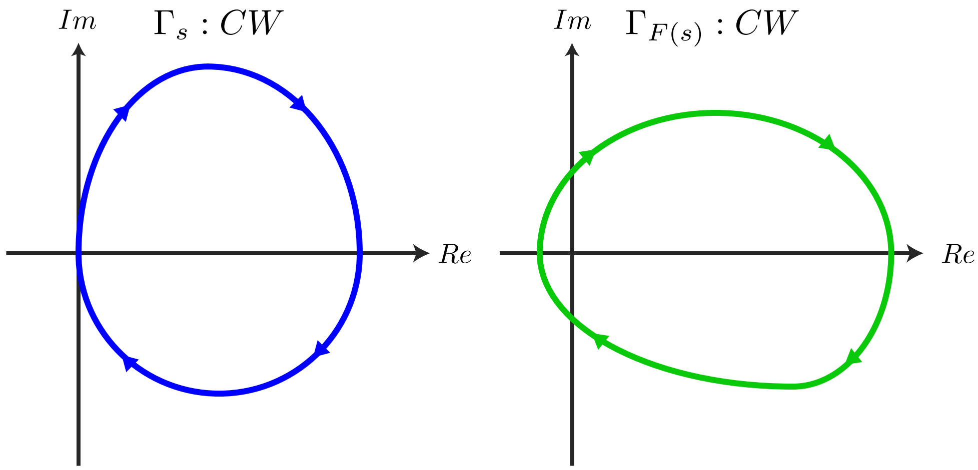
2. Nyquist Contour & Nyquist Plot

Nyquist plot is another tool that we use to to investigate the stability and robustness of a feedback system. The technique utilizes the frequency response characteristics of a system.

Definition: A contour Γ_s is a closed path with a direction in a complex plane.

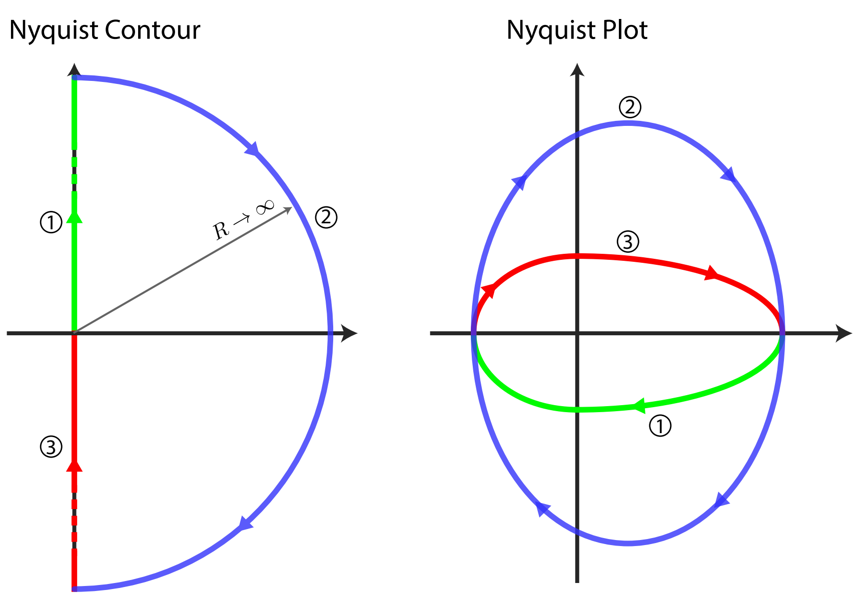
Remark: A continuous function $F(s)$ maps a contour Γ_s in $s-$plane to another contour $\Gamma_{F(s)}$ in $F(s)$ plane.

The figure below illustrates a clock-wise contour Γ_s and its map $\Gamma_{F(s)}$ which is also clock-wise in this example.



Let's consider an LTI transfer function $G(s)$ that has *no zeros or poles on the imaginary axis.* Nyquist contour/path, Γ_s, is defined in a way that it covers the whole open-right half plane (i.e. whole unstable region).

As illustrated in the Figure below, Nyquist contour is technically a half-circle for which the radius, $R \to \infty$. After that, one can draw the Nyquist plot, which is the mapped contour $\Gamma_{G(s)}$. Figure below illustrates a Nyquist contour and associated Nyquist plot.



Ex: Let's draw the Nyquist plot of $G(s) = \frac{1}{s+1}$

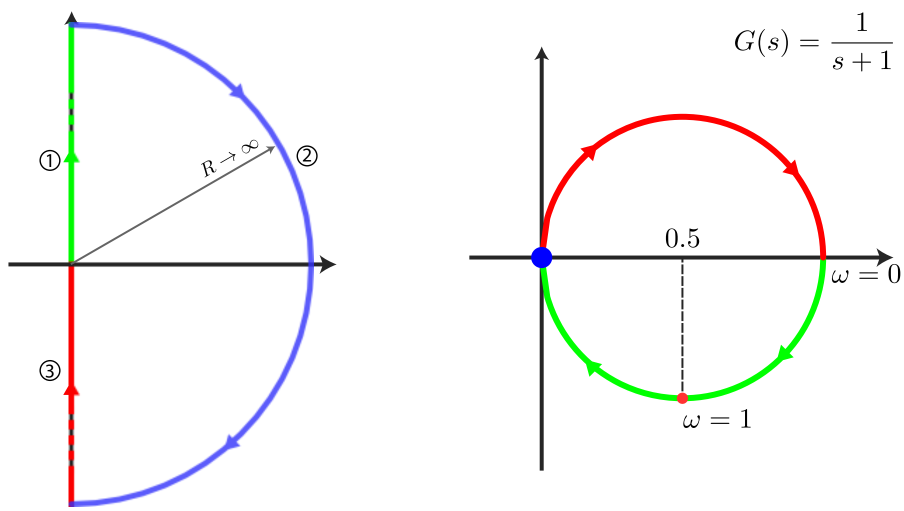
Solution: Based on the Nyquist contour we have three major paths.

1. This part corresponds to the polar plot that we already solved previously. We previously plotted $G(j\omega)$, where $\omega : 0 \to \infty$ on the complex plane.
2. This is the mapping of the infinite radius circular path on Nyquist contour. In this case if we write s in polar form, we get $s = Re^{j\theta}$ where $\theta : \pi/2 \to -\pi/2$. Then we can derive that

$$G\left(Re^{j\theta}\right) \approx \frac{1}{Re^{j\theta}} = \frac{e^{j(-\theta)}}{R} \quad \Rightarrow \quad |G\left(Re^{j\theta}\right)| \approx 0$$

3. Last path (mapping of negative imaginary axis) is simply the conjugate of polar plot with reverse direction.

If we follow the procedure, we obtain the following Nyquist plot.



Ex: Let's draw the Nyquist plot of $G(s) = \frac{1}{(s+1)^2}$

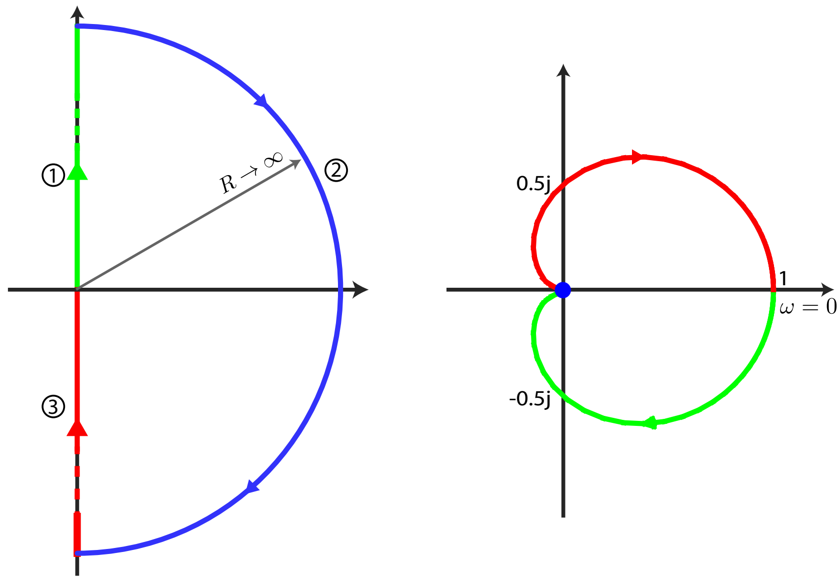
Solution: Let's analyze the Nyquist paths

1. Mapping of first path corresponds to the polar plot that we covered in the previously.
2. Mapping of the infinite radius circular path. Let $s = Re^{j\theta}$ where $\theta : \pi/2 \to -\pi/2$. Then we can derive that

$$G\left(Re^{j\theta}\right) \approx \frac{1}{R^2 e^{j2\theta}} = \frac{e^{j(-2\theta)}}{R^2} \qquad \Rightarrow |G\left(Re^{j\theta}\right)| \approx 0$$

3. Last path (mapping of negative imaginaty axis) is again the conjugate of polar plot with reverse direction.

If we follow the procedure, we obtain the following Nyquist plot.



Ex: Draw the Nyquist plot of $G(s) = \frac{1}{(s+1)^3}$

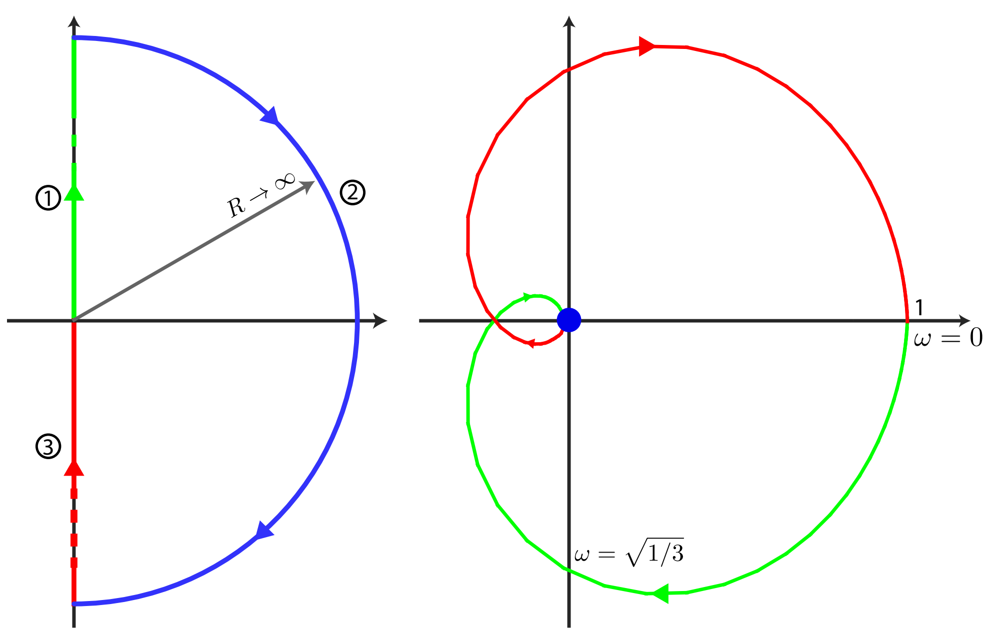
Solution: First let's analyze the Nyquist paths

1. This is the polar plot that we have solved previously.
2. Mapping of the infinite radius circular path on Nyquist contour. Let $s = Re^{j\theta}$ and $\theta : \pi/2 \to -\pi/2$. Then we can derive that

$$G\left(Re^{j\theta}\right) \approx \frac{1}{R^3 e^{j3\theta}} = \frac{e^{j(-3\theta)}}{R^3} \quad \Rightarrow \quad |G\left(Re^{j\theta}\right)| \approx 0$$

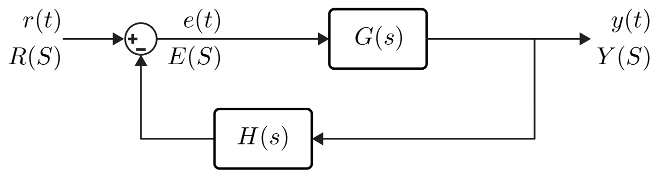
3. Last path (mapping of negative imaginary axis) is again the conjugate of polar plot with reverse direction.

If we follow the procedure, we obtain the following Nyquist plot.



3. Nyquist Stability Criterion for Feedback Systems

In control theory and its applications Nyqist plot and Nyquist stability test are dominantly used for analyzing feedback topologies. The figure below illustrates the fundamental feedback system topology for a SISO system



We know that the closed-loop transfer function, $T(s)$, for this system has the following form

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{G(s)}{1 + G_{OL}(s)}$$

where $G_{OL}(s)$ is the open-loop transfer function for the given topology. We know that poles of $T(s)$ are the roots that satisfy $1 + G_{OL}(s) = 0$.

In order to characterize and "quantify" the stability of the whole closed system, we will utilize the Nyquist plot of the open-loop transfer function, $G_{OL}(s)$.

In this part (frequency response analysis) of the course, we will limit to a specific (but very important and general) class of systems and assume that

- $G_{OL}(s)$ is a *minimum-phase* system, i.e.
  - No poles/zeros in the Open Right Half Plane
  - $\lim_{\omega \to \infty} [\frac{G_{OL}(s)}{s}]_{s=\jmath\omega} = 0$
- The feed-back system is Type $0 - 2$ (i.e. no integrator of order larger than 3 in the open-loop transfer function).
- Polar plot of $G_{OL}(j\omega)$ crosses the negative real-axis at most once.

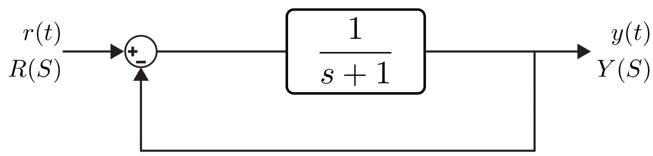
Under these assumptions, Nyquist stability criterion is reduced to the following definition

Def: A closed-loop system is BIBO stable, if the Nyquist plot of the open-loop transfer function neither encircle nor intersect the point $(-1 + 0j)$ point on the complex plane.

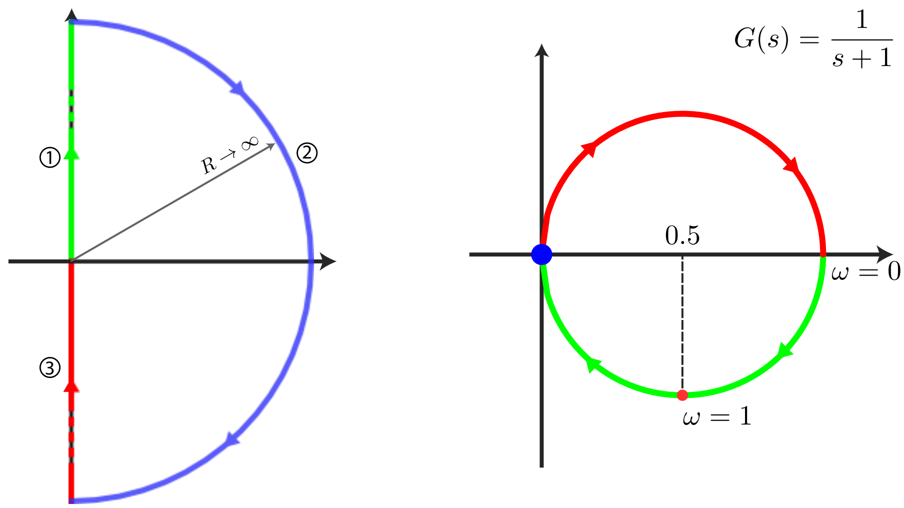
Def: T(s) is BIBO stable, if the Nyquist plot of $G_{OL}(s)$ does not encircle $(\mathbf{-1 + 0j})$

Indeed, it very easy to understand why the system becomes BIBO unstable, when the Nyquist plot intersect the $(-1+0j)$ point. If Nyquist plot intersects $(-1+0j)$, it means that $\exists\, \omega \in \mathbb{R}$ such that $G(j\omega) = -1$ and $1+G(j\omega) = 0$, which further implies that the closed-loop system has at least one pole on the imaginary axis.

Ex: Analyze the stability of the following feedback system using Nyquist plot.

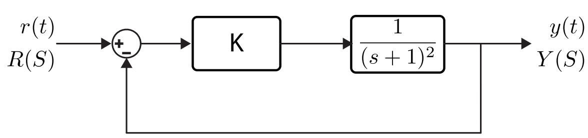


Solution: For this given system $G_{OL}(s) = \frac{1}{s+1}$. In the previous lecture we already derived the Nyquist plot for $\frac{1}{s+1}$ which is illustrated below



It is clear that Nyquist plot of $G_{OL}(s)$ neither encircles nor intersect the $(-1 + 0j)$ point, thus the closed-loop system is BIBO stable.

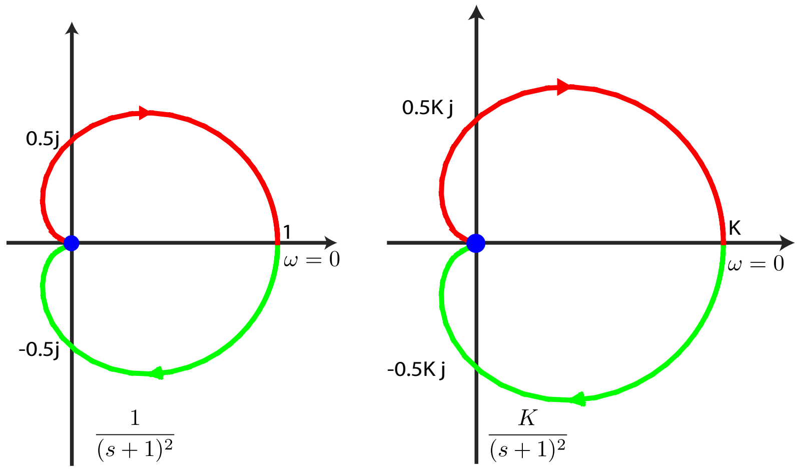
Ex: Find the range of positive K values that makes the following closed-loop system stable.



Soln: We would like to test the stability of the closed-loop system for different values of K, moreover we would like to derive the range of $K > 0$ values that makes the closed-loop system stable. Indeed, we don't need to re-draw the Nyquist plot for each K that we want to test.

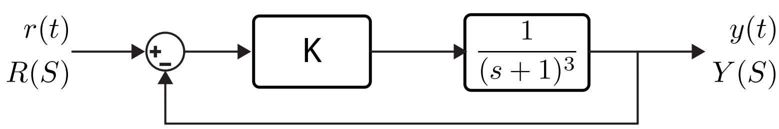
Instead, we will simply analyze the effect of a positive gain K, on the Nyquist plot. Let Γ_1 be the Nyquist plot of $G_{OL}(s)$ when $K = 1$. Since K is a simple positive gain it only radially scales the Nyquist plot without affecting the phase, i.e. Nyquist plot for an arbitrary gain K will be $\Gamma_K = K\Gamma_1$. Since we are interested in only the encirclement of $(-1 + j)$, it would be sufficient to concentrate of the intersection of Nyquist plot with the negative real axis.

We already derived the Nyquist plot for $\frac{1}{(s+1)^2}$. We can easily obtain the Nyquist plot of $\frac{K}{(s+1)^2}$ by scaling. Both Nyquist plots are illustrated below.

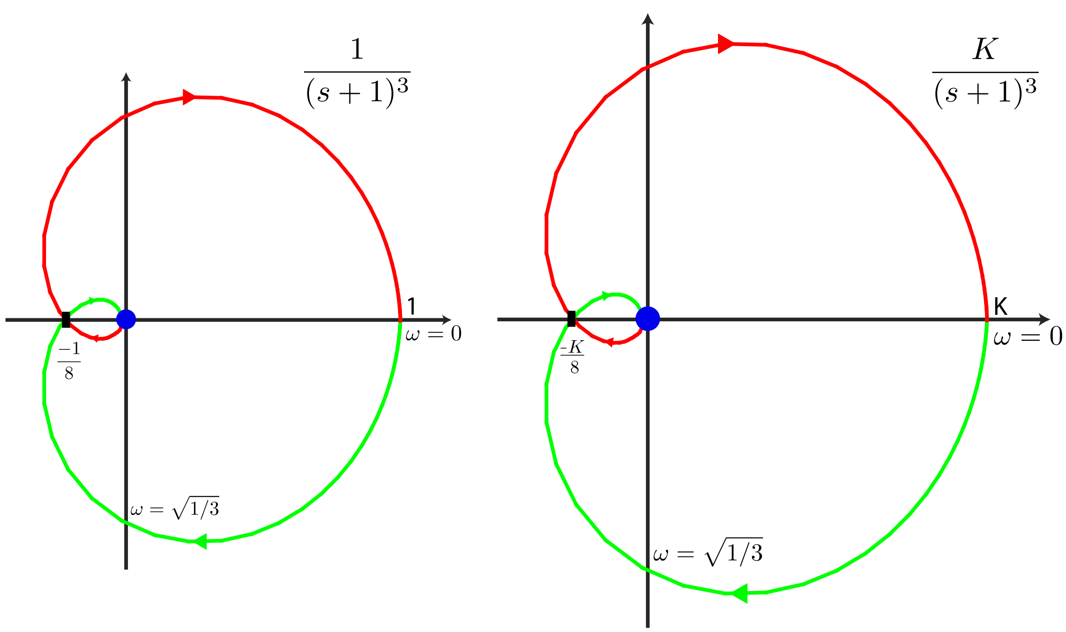


We can see from the derived Nyquist plots that the closed-system would be stable for all values of K.

Ex: Find the range of K values that makes the following closed-loop system stable.



Solution: In the previous lecture we already derived the Nyquist plot for $\frac{1}{(s+1)^2}$. Now let's illustrate the Nyquist plots of both $\frac{1}{(s+1)^2}$ and $\frac{K}{(s+1)^2}$



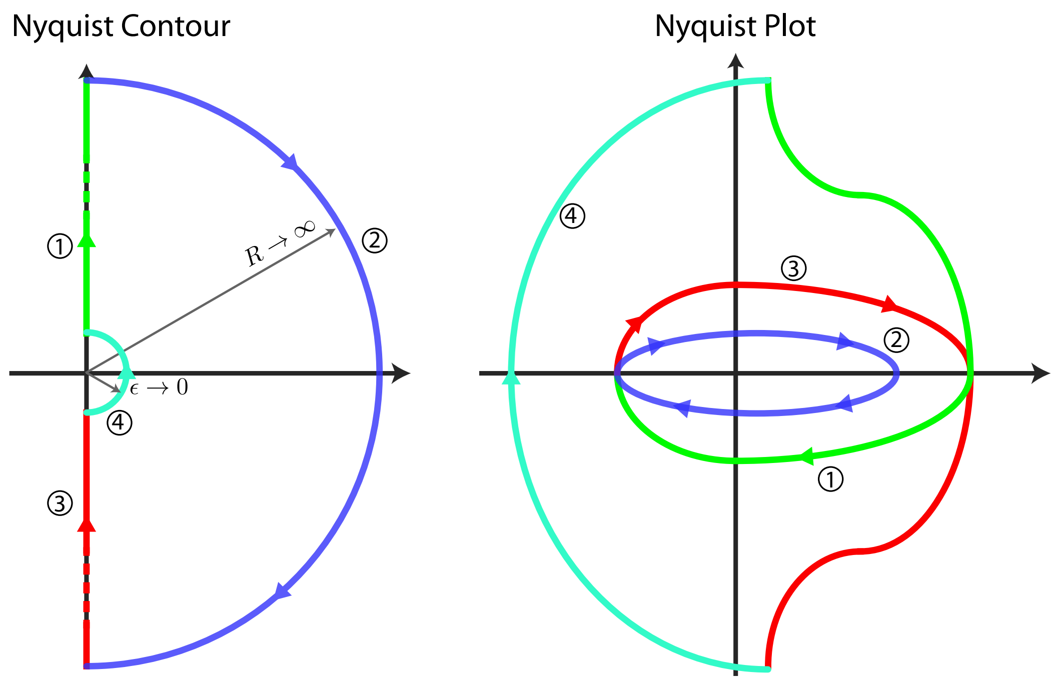
As we can see from the Nyquist plot pf $\frac{K}{(s+1)^3}$, intersects the negative real axis at the point of $(-K/8+0j)$. We can easily see that when $K = 8$, Nyquist plot intersects the $(-1+0j)$ point, and when $K > 8$, Nyquist plot encircles the $(-1+0j)$. As a results, the closed loop system is BIBO stable for $K \in (0, 8)$.

Nyquist Plot and Stability Test with Open-Loop Poles/Zeros on the Imaginary Axis

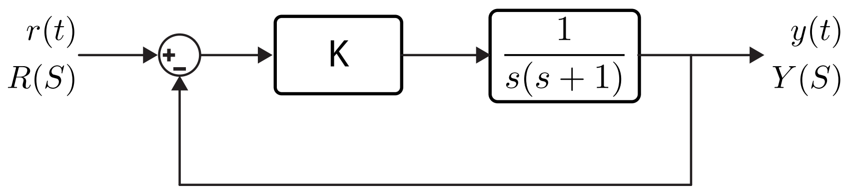
If you remember, when we introduced Nyquist contour and Nyquist stability test, we assumed that there is no pole/zero on the imaginary axis (in addition to other assumptions). However, it is especially common to have poles at the origin since it corresponds to the simple integrator. In this course, we will explicitly cover the case when there exists a pole or zero at the origin.

Let's assume that $G_{OL}(s)$ has a pole (or zero, or double pole) at the origin. We simply modify the Nyquist contour by adding an infinitesimal notch at the origin to the original Nyquist contour.

The figure below illustrates this modified Nyquist contour and an illustrative Nyquist plot.



Ex: Find the range of positive K values that makes the following closed-loop system stable.



Solution: For this given system, when $K = 1$ open-loop transfer function takes the form $G_{OL}(s) = \frac{1}{s(s+1)}$. Note, there exist a pole at the origin, thus we need to utilize the modified Nyquist Contour. In the modified Nyquist contour there exist 4 major paths (as opposed to three paths in classical Nyquist contour), and we need to draw the Nyquist plot based on these 4 paths.

1. This is the polar plot, where we need to plot $G(j\omega)$, where $\omega : 0 \to \infty$.

$$G(j\omega) = \frac{1}{j\omega(j\omega + 1)} = \frac{-j - \omega}{\omega(\omega^2 + 1)} = \frac{-1}{\omega^2 + 1} + \frac{-1/\omega}{\omega^2 + 1} j$$

Some observations about polar plot

$$\begin{aligned} Re\{G(j\omega)\} < 0 \quad &\& \quad Im\{G(j\omega)\} < 0 \quad , \forall \omega > 0 \\ \lim_{\omega \to 0} Re\{G(j\omega)\} = -1 \quad &\& \quad \lim_{\omega \to 0} Re\{G(j\omega)\} = -\infty \\ \lim_{\omega \to \infty} |G(j\omega)| = 0 \quad &\& \quad \lim_{\omega \to \infty} \angle[G(j\omega)] = -\pi \end{aligned}$$

2. This part is the mapping of in the infinite radius half-circle part of the Nyquist contour. Let $s = Re^{j\theta}$ and $\theta : \pi/2 \to -\pi/2$. Then we can derive that

$$\begin{aligned} &G\left(Re^{j\theta}\right) \approx \frac{1}{R^2 e^{j2\theta}} = \frac{e^{j(-2\theta)}}{R^2} \\ &\Rightarrow |G\left(Re^{j\theta}\right)| \approx 0 \end{aligned}$$

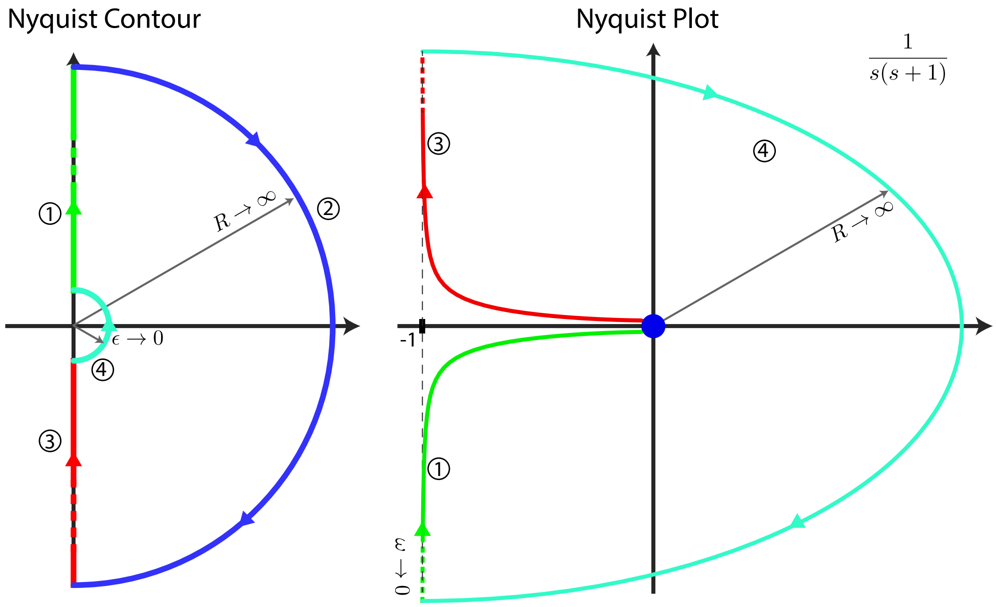
Note that when $\theta : \pi/2 \to -\pi/$, the infinite-small contour around origin rotates in CCW direction.

3. This part is simply the conjugate of polar plot with reverse direction.

4. This part is new. We need to deal with this path since there is a pole at the origin. Let's write s in polar coordinates $s = \epsilon e^{j\phi}$, where $\epsilon \to 0$ and $\phi : -\pi/2 \to \pi/2$ (CCW direction). Then we can derive that

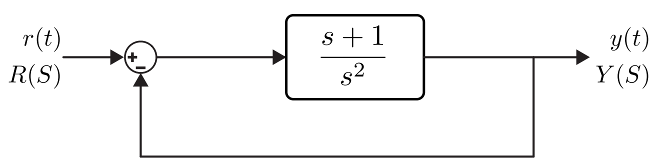
$$\begin{aligned} &G\left(\epsilon e^{j\phi}\right) \approx \frac{1}{\epsilon e^{j\phi}} = Re^{j(-\phi)} \\ &\Rightarrow |G\left(\epsilon e^{j\phi}\right)| \approx R \ll 1 \quad , \quad \angle[G\left(\epsilon e^{j\phi}\right)] \approx -\phi \end{aligned}$$

As a result, we obtain the following Nyquist plot for $\frac{1}{s(s+1)}$



Since gain K only scales the Nyquist plot, we can clearly see that Nyquist plot never circulates $-\mathbf{1}$ point for all possible values of $K > 0$. Thus, the system is stable $\forall\ K > 0$.

Ex: Draw the Nyquist plot of the open-loop transfer function for the following feedback topology and apply Nyquist stability test



Solution: For this given system, open-loop transfer function has the form $G_{OL}(s) = \frac{s+1}{s^2}$. Note, there exist a repeated pole at the origin, thus we need to utilize the modified Nyquist Contour. In the modified Nyquist contour there exist 4 major paths, and we need to draw the Nyquist plot based on these 4 paths.

1. This is the polar plot, where we need to plot $G(j\omega)$, where $\omega : 0 \to \infty$.

$$G(j\omega) = \frac{j\omega + 1}{(j\omega)^2} = -\frac{1}{\omega^2} - \frac{1}{\omega}j$$

One can easily see that

$$Re\{G(j\omega)\} < 0 \quad \& \quad Im\{G(j\omega)\} < 0 \quad , \forall \omega > 0$$

This implies that polar plot is only located in the third quadrant of the complex plane. Now let's analyze the behavior of the polar plot for infinitely small ω, i.e. $\omega = \frac{1}{R}$, where $R \to \infty$

$$Re\{G(j\omega)\} = -R^2 \quad \Rightarrow \quad \lim_{R\to\infty} Re\{G(j\omega)\} = -\infty$$
$$Im\{G(j\omega)\} = -R \quad \Rightarrow \quad \lim_{R\to\infty} Im\{G(j\omega)\} = -\infty$$

As once can see both Real and Imaginary parts goes to $-\infty$ when $\omega \to 0$. However real part goes faster due to square term. Thus the phase of the polar when $\omega \to 0$ can be find as

$$\lim_{\omega\to 0} \frac{Im\{G(j\omega)\}}{Re\{G(j\omega)\}} = 0 \quad \Rightarrow \quad \lim_{\omega\to 0} \angle[G(j\omega)] = \arctan(0) - \pi = -\pi$$

Now let's analyze the behavior of the polar plot for infinitely large ω, i.e. $\omega = R$, where $R \to \infty$

$$Re\{G(j\omega)\} = -\frac{1}{R^2} \quad \Rightarrow \quad \lim_{R\to\infty} Re\{G(j\omega)\} = -0$$
$$Im\{G(j\omega)\} = -\frac{1}{R} \quad \Rightarrow \quad \lim_{R\to\infty} Im\{G(j\omega)\} = -0$$

As once can see both the Real and Imaginary parts goes to 0 when $\omega \to \infty$. However real part goes faster due to square term. Thus the phase of the polar when $\omega \to \infty$ can be find as

$$\lim_{\omega\to 0} \frac{Im\{G(j\omega)\}}{Re\{G(j\omega)\}} = \infty \quad \Rightarrow \quad \lim_{\omega\to 0} \angle[G(j\omega)] = \arctan(\infty) - \pi = -\frac{\pi}{2}$$

2. This part is the mapping of in the infinite radius half-circle part of the Nyquist contour. Let $s = Re^{j\theta}$ and $\theta : \pi/2 \to -\pi/2$. Then we can derive that

$$G\left(Re^{j\theta}\right) \approx \frac{Re^{j\theta}}{R^2 e^{j2\theta}} = \frac{e^{j(-2\theta)}}{R^2} \quad \Rightarrow \quad |G\left(Re^{j\theta}\right)| \approx 0$$

Note that whole path is mapped to a single "point" at the origin.

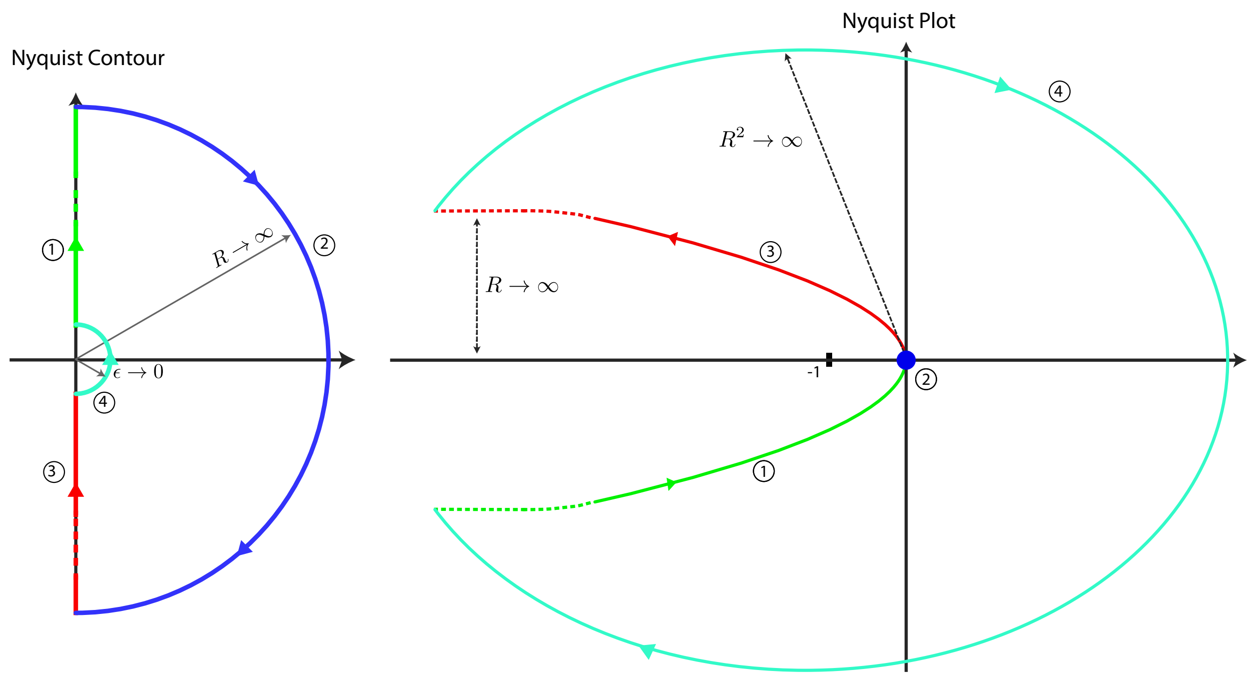
3. This part is simply the conjugate of polar plot with reverse direction.

4. Let's write s in polar coordinates $s = \epsilon e^{j\phi}$, where $\epsilon \to 0$ and $\phi : -\pi/2 \to \pi/2$ (CCW direction). Then we can derive that

$$G\left(\epsilon e^{j\phi}\right) \approx \frac{1}{\epsilon^2 e^{j2\phi}} = R^2 e^{j(-2\phi)}$$
$$\Rightarrow |G\left(\epsilon e^{j\phi}\right)| \approx R \to \infty \quad , \quad \angle[G\left(\epsilon e^{j\phi}\right)] \approx -2\phi$$

As a result, the mapping of this contour is an infinitely large almost full circle in CW direction.

If we combine all four paths the Nyquist plot of $\frac{s+1}{s^2}$ takes the following form



We can see that Nyquist plot never circulates $-\mathbf{1}$ point. Thus, the closed-loop system is BIBO stable.