

Lecture 16

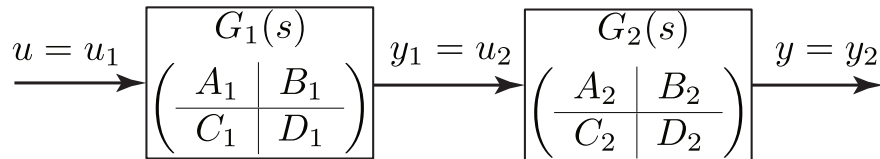
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16.1 Minimality of Interconnected Systems

In this section we shall examine the conditions under which minimality is lost when minimal subsystems are interconnected in various configurations,

16.1.1 Series - Cascade Connection

Consider the following system structure where two sub systems, with transfer functions $G_1(s)$ and $G_2(s)$ and associated minimal representations $\left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right)$ and $\left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right)$, connected in series/cascade configuration.



The transfer function of the connection is simply equal to $G(s) = G_2(s)G_1(s)$. Let x_1 and x_2 state-variables of the sub-systems, then natural choice of the state variable for the series connection is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Under this definition the state-space representation for the whole system can be found as

$$A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline B_2 C_1 & A_2 \end{array} \right], B = \left[\begin{array}{c} B_1 \\ B_2 D_1 \end{array} \right], C = [D_2 C_1 \mid C_2], D = [D_2 D_2]$$

Clearly eigenvalues of A are the combination of the eigenvalues of A_1 and A_2 and the poles of the system. Let's analyze the observability of the connection via PBH test.

$$\left[\begin{array}{c|c} \lambda I - A & 0 \\ \hline C & \lambda I - A_2 \end{array} \right] = \left[\begin{array}{c|c} \lambda I - A_1 & 0 \\ \hline -B_2 C_1 & \lambda I - A_2 \end{array} \right]$$

If we remember from the observability lecture that (A, C) (whole state-space model) pair is unobservable if and only if $\left[\begin{array}{c|c} \lambda I - A & 0 \\ \hline C & \lambda I - A_2 \end{array} \right]$ loses rank for some λ , which can only happen if λ is an eigenvalue of A . Let's assume that λ_2 is an eigenvalue of A_2 but not eigenvalue of A_1 . Then $\left[\begin{array}{c|c} \lambda I - A & 0 \\ \hline C & \lambda I - A_2 \end{array} \right]_{\lambda=\lambda_2}$ loses rank, if $\exists v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

such that

$$\left[\frac{\lambda_2 I - A}{C} \right] v = 0 \Rightarrow \left[\begin{array}{c|c} \lambda_2 I - A_1 & 0 \\ -B_2 C_1 & \lambda_2 I - A_2 \end{array} \right] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow v_1 = 0 \& \left[\frac{\lambda_2 I - A}{C_2} \right] v_2 = 0$$

which contradicts with the fact that (A_2, C_2) is observable since both individual sub-system representations are minimal. In that respect $\left[\frac{\lambda I - A}{C} \right]$ can loose rank only at an eigenvalue of A_1 (i.e. a pole of A_2). Let's λ_1 is an eigenvalue of A_1 . Then $\left[\frac{\lambda I - A}{C} \right]_{\lambda=\lambda_1}$ loses rank, if $\exists v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that

$$\begin{aligned} \left[\frac{\lambda_1 I - A}{C} \right] v = 0 &\Rightarrow \left[\begin{array}{c|c} \lambda_1 I - A_1 & 0 \\ -B_2 C_1 & \lambda_1 I - A_2 \end{array} \right] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow v_1 \neq 0 \& A v_1 = \lambda_1 v_1 \text{ and} \\ \left[\frac{\lambda_1 I - A_2}{C_2} \mid \frac{-B_2 C_1}{D_2 C_1} \right] \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} = 0 &\Rightarrow \left[\frac{\lambda_1 I - A_2}{C_2} \mid \frac{-B_2}{D_2} \right] \begin{bmatrix} v_2 \\ C_1 v_1 \end{bmatrix} = 0 \end{aligned}$$

Note that $\left[\frac{\lambda_1 I - A_2}{C_2} \mid \frac{-B_2}{D_2} \right] \begin{bmatrix} v_2 \\ C_1 v_1 \end{bmatrix} = 0$ implies that λ_1 is a **right** zero of $G_2(s)$ where v_2 and $C_1 v_1$ is the associated state-zero-direction and *input-zero-direction* respectively.

If we summarize the results, cascaded system is unobservable if $\exists(\lambda_1, v_1)$ and $v_2 \neq 0$ such that (λ_1, v_1) is an eigenvalue-eigenvector pair of A_1 and λ_1 is a **right** zero of $G_2(s)$ with v_2 and $C_1 v_1$ as the state-zero-direction and *input-zero-direction* respectively.

Now let's analyze the reachability of the connection via PBH test.

$$\left[\lambda I - A \mid B \right] = \left[\begin{array}{c|c|c} \lambda I - A_1 & 0 & B_1 \\ -B_2 C_1 & \lambda I - A_2 & B_2 D_1 \end{array} \right]$$

If we remember from the reachability lecture that (A, B) (whole state-space model) pair is unreachable if and only if $\left[\lambda I - A \mid B \right]$ losses rank for some λ , which can only happen if λ is an eigenvalue of A . Let's assume that λ_1 is an eigenvalue of A_1 but not eigenvalue of A_2 . Then $\left[\lambda I - A \mid B \right]_{\lambda=\lambda_1}$ loses rank, if $\exists w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ such that

$$\begin{aligned} \begin{bmatrix} w_1^T & w_2^T \end{bmatrix} \left[\lambda_1 I - A \mid B \right] = 0 &\Rightarrow \begin{bmatrix} w_1^T & w_2^T \end{bmatrix} \left[\begin{array}{c|c|c} \lambda_1 I - A_1 & 0 & B_1 \\ -B_2 C_1 & \lambda_1 I - A_2 & B_2 D_1 \end{array} \right] = 0 \\ \Rightarrow w_2^T = 0 \&\& w_1^T \left[\lambda_1 I - A_1 \mid B_1 \right] = 0 \end{aligned}$$

which contradicts with the fact that (A_1, B_1) is reachable since both individual sub-system representations are minimal. Now let λ_2 is an eigenvalue of A_2 . Then $\left[\lambda I - A \mid B \right]_{\lambda=\lambda_2}$ loses rank, if $\exists w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ such that

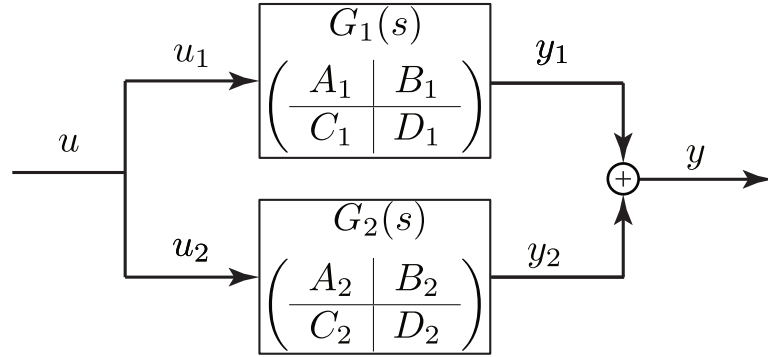
$$\begin{aligned} \begin{bmatrix} w_1^T & w_2^T \end{bmatrix} \left[\lambda_2 I - A \mid B \right] = 0 &\Rightarrow \begin{bmatrix} w_1^T & w_2^T \end{bmatrix} \left[\begin{array}{c|c|c} \lambda_2 I - A_1 & 0 & B_1 \\ -B_2 C_1 & \lambda_2 I - A_2 & B_2 D_1 \end{array} \right] = 0 \\ \Rightarrow w_2 \neq 0 \&\& w_2^T A_2 = w_2^T \lambda_2 \text{ and } \begin{bmatrix} w_1^T & w_2^T \end{bmatrix} \left[\begin{array}{c|c} \lambda_2 I - A_1 & B_1 \\ -B_2 C_1 & B_2 D_1 \end{array} \right] = 0 \Rightarrow \\ \begin{bmatrix} w_1^T & -(B_2^T w_2)^T \end{bmatrix} \left[\begin{array}{c|c} \lambda_2 I - A_1 & B_1 \\ C_1 & -D_1 \end{array} \right] = 0 &\Rightarrow \begin{bmatrix} w_1^T & -(B_2^T w_2)^T \end{bmatrix} \left[\begin{array}{c|c} \lambda_2 I - A_1 & -B_1 \\ C_1 & D_1 \end{array} \right] = 0 \end{aligned}$$

Note that $\begin{bmatrix} w_1^T & (-B_2^T w_2)^T \end{bmatrix} \left[\begin{array}{c|c} \lambda_2 I - A_1 & -B_1 \\ \hline C_1 & D_1 \end{array} \right] = 0$ implies that λ_2 is a **left** zero of $G_1(s)$ where w_1 and $(-B_2^T w_2)$ are the associated state-zero-direction and *input-zero-direction* respectively.

If we summarize the results, cascaded system is unreachable if $\exists(\lambda_2, w_2)$ and $w_1 \neq 0$ such that (λ_2, v_2) is a left eigenvalue-eigenvector pair of A_2 and λ_2 is a **left** zero of $G_1(s)$ with w_1 and $(-B_2^T w_2)$ as the state-zero-direction and *input-zero-direction* respectively.

16.1.2 Parallel Connection

Consider the following system structure where two sub systems, with transfer functions $G_1(s)$ and $G_2(s)$ and associated minimal representations $\left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right)$ and $\left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right)$, connected in parallel configuration.



The transfer function of the connection is simply equal to $G(s) = G_1(s) + G_2(s)$. Let x_1 and x_2 state-variables of the sub-systems, then natural choice of the state variable for the series connection is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Under this definition the state-space representation for the whole system can be found as

$$A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array} \right], B = \left[\begin{array}{c} B_1 \\ B_2 \end{array} \right], C = [C_1 \mid C_2], D = [D_1 + D_2]$$

Clearly eigenvalues of A are the combination of the eigenvalues of A_1 and A_2 and the poles of the system. Let's analyze the observability of the connection via one of the modal reachability tests.

$$\left[\begin{array}{c|c} \lambda I - A & \\ \hline C \end{array} \right] = \left[\begin{array}{c|c} \lambda I - A_1 & 0 \\ \hline 0 & \lambda I - A_2 \\ \hline C_1 & C_2 \end{array} \right]$$

It is easy to observe that combined system is always observable if A_1 and A_2 does not share any common eigenvalue. Thus a necessary (but not sufficient) condition such that parallel connection loses observability is that A_1 and A_2 will have at least 1 common eigenvalue. Let's assume that λ_c is a common eigenvalue/pole of both sub-systems. Then $\left[\begin{array}{c|c} \lambda I - A & \\ \hline C \end{array} \right]_{\lambda=\lambda_c}$ loses rank, if $\exists v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that

$$\left[\begin{array}{c|c} \lambda I - A & \\ \hline C \end{array} \right] v = 0 \iff \left[\begin{array}{c|c} \lambda I - A_1 & 0 \\ \hline 0 & \lambda I - A_2 \\ \hline C_1 & C_2 \end{array} \right] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \iff$$

$$A_1 v_1 = \lambda_c v_1, A_2 v_2 = \lambda_c v_2, \& C_1 v_1 + C_2 v_2 = 0$$

In other words combined system loses observability if and only if $\exists \lambda_c, v_1, \& v_2$ such that λ_c is a common eigenvalue of both systems, and there exists an (v_1, v_2) eigenvector pair (v_i is an eigenvector of A_i associated with λ_c) such that $C_1 v_1 + C_2 v_2 = 0$.

Now analyze the reachability of the connection via one of the modal reachability tests.

$$\left[\lambda I - A \mid B \right] = \left[\begin{array}{c|c|c} \lambda I - A_1 & 0 & B_1 \\ \hline 0 & \lambda I - A_2 & B_2 \end{array} \right]$$

It is easy to observe that combined system is always reachable if A_1 and A_2 does not share any common eigenvalue. Thus a necessary (but not sufficient) condition such that parallel connection loses reachability is that A_1 and A_2 will have at least 1 common eigenvalue. Let's assume that λ_c is a common eigenvalue/pole of both sub-systems Then $\left[\lambda I - A \mid B \right]_{\lambda=\lambda_c}$ loses rank, if $\exists w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ such that

$$w^T \left[\lambda_c I - A \mid B \right] = 0 \iff \begin{bmatrix} w_1^T & w_2^T \end{bmatrix} \left[\begin{array}{c|c|c} \lambda_c I - A_1 & 0 & B_1 \\ \hline 0 & \lambda_c I - A_2 & B_2 \end{array} \right] = 0 \iff$$

$$w_1^T A_1 = \lambda_c w_1^T, w_2^T A_2 = \lambda_c w_2^T, \& w_1^T B_1 + w_2^T B_2 = 0$$

In other words combined system loses reachability if and only if $\exists \lambda_c, w_1, \& w_2$ such that λ_c is a common eigenvalue of both systems, and there exists an (w_1, w_2) left-eigenvector pair (w_i is a left eigenvector of A_i associated with λ_c) such that $w_1^T B_1 + w_2^T B_2 = 0$.