EE402 - Discrete Time Systems	**Spring 2018**

Lecture 10

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10.1 Reachability & Controllability of DT-LTI Systems

For LTI a discrete time state-space representation

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

- A state x_r is said to be m-step **reachable**, if there exist an input sequence, $u[k], k \in \{0, 1, \cdots m-1\}$, that transfers the state vector $x[k]$ from the origin (i.e. $x[0] = 0$) to the state x_r in m number of steps, i.e. $x[m] = x_r$.
- A state x_d is said to be m-step **controllable**, if there exist an input sequence, $u[k], k \in \{0, 1, \cdots m-1\}$, that transfers the state vector $x[k]$ from the initial state x_c (i.e. $x[0] = x_c$) to the origin in m number of steps, i.e. $x[m] = 0$.

Note that

- the set $\mathcal{R}_m$ of all m-step reachable states is a linear (sub)space: $\mathcal{R}_m \subset \mathbb{R}^n$
- the set $\mathcal{C}_m$ of all m-step controllable states is a linear (sub)space: $\mathcal{C}_m \subset \mathbb{R}^n$

Let's characterize $\mathcal{R}_m$ and then try to generalize the reachability concept. When $x[0] = 0$, the solution of $x[m]$ is given by

$$x[m] = \left[\begin{array}{c|c|c|c|c} A^{m-1}B & A^{m-2}B & \cdots & AB & B \end{array} \right] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[m-2] \\ u[m-1] \end{bmatrix}$$

Let

$$\begin{aligned} \mathbf{R}_m &= \left[\begin{array}{c|c|c|c|c} A^{m-1}B & A^{m-2}B & \cdots & AB & B \end{array} \right] \\ \mathbf{U}_m &= \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[m-2] \\ u[m-1] \end{bmatrix} \end{aligned}$$

then if a state x_r is reachable at k steps, it should satisfy the following equation for some $\mathbf{U}_m$.

$$\mathbf{M}_m \mathbf{U}_m = x_m$$

In order this matrix equation to have a solution x_r should be in the range space of $\mathbf{M}_m$.

$$x_r \in \mathrm{Ra}(\mathbf{M}_m)$$

Thus m-step reachable sub-space is simply equal to range space of $\mathcal{R}_k$

$$\mathrm{Ra}(\mathbf{R}_m) = \mathcal{R}_m$$

Theorem: For $k < n < l$

$$\mathcal{R}_k \subset \mathcal{R}_n = \mathcal{R}_l$$
$$\mathrm{Ra}(\mathbf{R}_k) \subset \mathrm{Ra}(\mathbf{R}_n) = \mathrm{Ra}(\mathbf{R}_l)$$

Proof: It is fairly easy to observe that

$$\mathcal{R}_i \subset \mathcal{R}_{i+1}$$
$$\mathrm{Ra}(\mathbf{R}_i) \subset \mathrm{Ra}(\mathbf{R}_{i+1})$$

since we add a new column (or columns for multi-input systems) to $\mathbf{R}_i$, thus it can only increase the dimension of the range-space. Thus we can conclude that

$$\mathcal{R}_k \subset \mathcal{R}_n \subset \mathcal{R}_l$$
$$\mathrm{Ra}(\mathbf{R}_k) \subset \mathrm{Ra}(\mathbf{R}_n) \subset \mathrm{Ra}(\mathbf{R}_l)$$

In order prove $\mathcal{R}_n = \mathcal{R}_l$, we simply use the Cayley-Hamilton theorem. Based on Cayley-Hamilton theorem

$$A^n = -a_1 A^{n-1} - \cdots - a_{n-1} A - a_n I$$
$$A^n B = -a_1 A^{n-1} B - \cdots - a_{n-1} AB - a_n B$$

which shows that $A^n B$ is linearly dependent to previous columns and thus

$$\mathcal{R}_n = \mathcal{R}_l$$
$$\mathrm{Ra}(\mathbf{R}_n) = \mathrm{Ra}(\mathbf{R}_l)$$

This theorem shows that if x_r is reachable in n steps then it is reachable for $l > n$ steps, similarly, if it is not reachable in n steps then it is reachable for $l > n$ steps. In this context, the sub-space of states reachable in n-steps, $\mathcal{R}_n$ is referred as the reachable subspace of (A, N), and will be denoted simply by $\mathcal{R}$ and $\mathbf{R} = \mathbf{R}_k$ will be system wide the reachability matrix. The system is termed a (fully) reachable system if

$$\mathrm{rank}(\mathbf{R}) = n$$
$$\mathrm{Ra}(\mathbf{R}) = \mathcal{R} = \mathbb{R}^n$$

Ex 10.1 *Solve the following problems regarding controllable sub-space*

- Show that $\mathcal{R} \subset \mathcal{C}$, $\forall (A, B)$, however $\mathcal{C} \subset \mathcal{R}$ not necessarily true $\forall (A, B)$.
- Similar to the reachable subspace, characterize the controllable subspace
- Derive conditions such that $\mathcal{R} = \mathcal{C}$

10.1.1 Reachability Gramian

An alternative characterization of $\mathbf{R}$ is using reachability Gramian (which is more critical for CT systems). m-step reachability Gramian, $\mathbf{P}_m$, is defined as

$$\mathbf{P}_m = \mathbf{R}_m {\mathbf{R}_m}^T = \sum_{i=0}^{k-1} A^i B B^T \left(A^T\right)^i \tag{10.1}$$

Note that $\mathcal{P}_m$ is a symmetric positive semi-definite matrix.

Lemma: $\mathcal{R}_m = \mathrm{Ra}(\mathbf{R}_m) = \mathrm{Ra}(\mathbf{P}_m)$

Proof: Let's fits show that $\mathrm{Ra}(\mathbf{P}_m) \subset \mathrm{Ra}(\mathbf{R}_m)$. If $x \in \mathrm{Ra}(\mathbf{P}_m)$, then $\exists v \in \mathbb{R}^n$ s.t. $x = \mathbf{P}_m v$ then

$$x = \mathbf{P}_m v = \mathbf{R}_m {\mathbf{R}_m}^T v = \mathbf{R}_m y \;\Rightarrow\; x \in \mathrm{Ra}(\mathbf{R}_m) \;\Rightarrow\; \mathrm{Ra}(\mathbf{P}_m) \subset \mathrm{Ra}(\mathbf{R}_m)$$

Now let's show that $\mathrm{Ra}(\mathbf{R}_m) \subset \mathrm{Ra}(\mathbf{P}_m)$. We know that

$$\mathrm{Ra}(\mathbf{R}_m) \subset \mathrm{Ra}(\mathbf{P}_m) \iff \mathrm{Ra}^{\perp}(\mathbf{P}_m) \subset \mathrm{Ra}^{\perp}(\mathbf{R}_m)$$

So we can equivalently show that $\mathrm{Ra}^{\perp}(\mathbf{P}_m) \subset \mathrm{Ra}^{\perp}(\mathbf{R}_m)$. Let $q \in \mathrm{Ra}^{\perp}(\mathbf{P}_m)$, then

$$\begin{aligned} q^T \mathbf{P}_m = \mathbf{0} \;\Rightarrow\; q^T \mathbf{P}_m q = 0 \;&\iff\; q^T \mathbf{R}_m {\mathbf{R}_m}^T q = 0 \;\iff\; ({\mathbf{R}_m}^T q)^T ({\mathbf{R}_m}^T q) = 0 \\ &\iff\; {\mathbf{R}_m}^T q = \mathbf{0} \;\iff\; q^T \mathbf{R}_m = \mathbf{0}^T = \mathbf{0} \\ &\Rightarrow\; q \in \mathrm{Ra}^{\perp}(\mathbf{R}_m) \;\Rightarrow\; \mathrm{Ra}^{\perp}(\mathbf{P}_m) \subset \mathrm{Ra}^{\perp}(\mathbf{R}_m) \end{aligned}$$

This completes the proof. As a result of this lemma, full reachable subspace $\mathcal{R} = \mathrm{Ra}(\mathbf{P}_l)$ for any $l \geq 0$. As a result we can make the following conclusions

- (A, B) pair is fully reachable $\iff$ $\dim\left[\mathrm{Ra}(\mathbf{P}_l)\right] = n$ for any $l \geq n$
- (A, B) pair is fully reachable $\iff$ $\det\left[\mathbf{P}_l\right] \neq 0$ for any $l \geq n$

If $x[k+1] = Ax[k]$ is asymptotically stable, then $\mathbf{P}_\infty = \lim_{k\to\infty} \sum_{i=0}^{k-1} A^i B B^T \left(A^T\right)^i \triangleq P$ is well defined and P satisfies the following Lyapunov equation

$$APA^T - P = -BB^T$$

To understand the derivation of this, refer to the Quadratic Lyapunov Functions for LTI systems section in Lecture Notes 8.