ROB501 - Fundamentals & Emerging Topics in Robotics - Digital Control Systems

Lecture 1

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1.1 Big Picture

In this phase course, the main focus will be on continuous-time systems (plants) that are controlled (sampled and actuated) by a digital computer interface. Such a discrete-time control system consists of four major parts as illustrated in Fig. 1.2,

1. *The plant* is a continuous-time dynamical system
2. Analog-to-Digital Converter (ADC)
3. Controller (μP), a microprocessor/microcontroller with a "real-time" OS
4. Digital-to-Analog Converter (DAC)

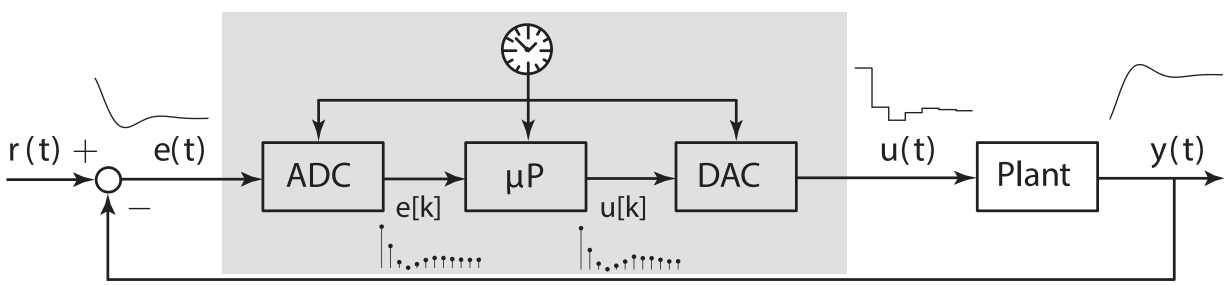

Figure 1.1: Block diagram of a digital control system

Most of the time, the plant is modeled as a "smooth" continuous dynamical system. In this course, we will cover only LTI systems. Thus, we will assume that (unless otherwise is given) the plant is a continuous-tıme LTI plant model with a transfer function of $G_p(s)$ for which both the input and output are continuous-time signals.

The "digital" blocks inside the closed-loop block diagram structure are the ADC, the Controller, and the DAC. It is generally assumed (design requirement) that all blocks share a common "hard real-time" clock.

A general ADC is a device that converts an analog signal to a digital signal. In this course, we will model the ADC block as an *ideal sampler* for which the input is a continuous-time signal, $e(t)$, and the output is a discrete-time signal, $e[k]$, where the relation between the continuous- and discrete-time signals are given as

$$e(kT) = e[k], \;\; k \in \mathbb{Z}^+,$$

where constant T is the *sampling time*.

The microcontroller/microprocessor processes some set of digital input signals to produce some set of digital output signals. The outputs are defined at only some specified instances determined by the real-time clock.

In this course, we will model the μP block as an ideal discrete-time LTI system for which both the input and output are discrete-time signals, with a transfer function of $G_c(z)$.

The DAC is a device that converts a digital signal to an analog signal. In this course, we assume that it is an ideal *Hold* element for which the input signal is a discrete-time signal, whereas the output is a continuous-time signal. The most commonly used *Hold* system is ZOH (Zero-Order-Hold) which is a mapping defined by the following relation

$$u(t) = u[k], \text{ for} t \in [kT, (k+1)T)$$

Higher-order hold operators exist, but they are extremely rarely used in practice.

The idealized and simplified block-diagram structure is given in Fig.

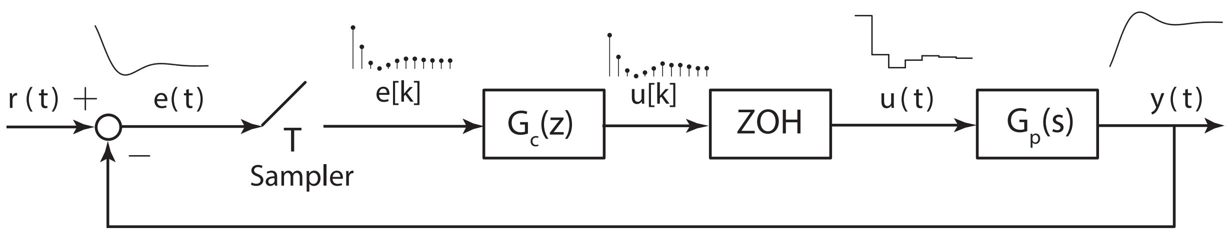

Figure 1.2: Block diagram of an LTI discrete-time control system

Major challenge: Loop contains both continuous-time and discrete-time parts.

Fundamental Discrete-Time Signals & Systems Concepts

1.2 Z-transform

Z-transform of a (causal) discrete time signal $x[k]$ is given by

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k] z^{-k}$$

If $x[k]$ is a sampled signal from a continuous time signal $x(t)$ with a sampling time of T, we (abuse of notation) also use the following notation

$$X(z) = \mathcal{Z}\{x(kT)\} = \mathcal{Z}\{x^*(t)\}$$

Z-transforms of elementary functions

We assume that all signals are causal thus $t \in \mathbb{R}^+$ and $k \in \mathbb{Z}^+$

Unit-step function $x(t) = 1$ and thus $x(kT) = x[k] = 1$, the Z-transform is given by

$$X(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

Unit-ramp function $x(t) = t$ and thus $x(kT) = x[k] = kT$, the Z-transform is given by

$$\begin{aligned}
X(z) &= \sum_{k=0}^{\infty} kTz^{-k} = T(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots) = Tz(z^{-2} + 2z^{-3} + 3z^{-4} + \cdots) \\
&= Tz\frac{d}{dz}\left(\int (z^{-2} + 2z^{-3} + 3z^{-4} + \cdots)dz\right) = Tz\frac{d}{dz}\left(-(z^{-1} + z^{-2} + z^{-3} + \cdots)\right) \\
&= Tz\frac{d}{dz}\left(\frac{-1}{z-1}\right) = \frac{Tz}{(z-1)^2} = \frac{Tz^{-1}}{(1-z^{-1})^2}
\end{aligned}$$

Exponential sequence $x[k] = a^k$

$$X(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} \left(\frac{z}{a}\right)^{-k} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

Exponential function $x(t) = e^{bt}$ and thus $x(kT) = x[k] = e^{bTk}$

$$X(z) = \sum_{k=0}^{\infty} e^{bTk} z^{-k} = \sum_{k=0}^{\infty} (e^{bT})^k z^{-k} = \frac{1}{1 - e^{bT}z^{-1}} = \frac{z}{z - e^{bT}}$$

Cosine function $x(t) = \cos(\omega t)$, and thus $x(kT) = x(k) = \cos(\omega Tk)$

$$\begin{aligned}
\cos(\omega Tk) = \frac{1}{2}\left(e^{j\omega Tk} + e^{-j\omega Tk}\right) X(z) &= \frac{1}{2}\left(\frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}}\right) = \frac{1}{2}\frac{z(z - e^{-j\omega T}) + z(z - e^{j\omega T})}{(z - e^{-j\omega T})(z - e^{j\omega T})} \\
&= \frac{1}{2}\frac{2z^2 - z(e^{-j\omega T} + e^{j\omega T})}{z^2 - z(e^{-j\omega T} + e^{j\omega T}) + 1} = \frac{z^2 - z\cos(\omega T)}{z^2 - z2\cos(\omega T) + 1} \\
&= \frac{1 - z^{-1}\cos(\omega T)}{1 - z^{-1}2\cos(\omega T) + z^{-2}}
\end{aligned}$$

Properties and Theorems of the Z-transform

Linearity

$$x[k] = \alpha f[k] + \beta g[k] \ \to X(z) = \alpha F(z) + \beta G(z) \ ,\forall \ \alpha,\ \beta,\ f[k],\ \&\ g[k]$$

Multiplication by a^k

$$\begin{aligned}\mathcal{Z}\{a^k x[k]\} &= \sum_{k=0}^{\infty} a^k x[k] z^{-k} = \sum_{k=0}^{\infty} x[k](z/a)^{-k} \\ \mathcal{Z}\{a^k x[k]\} &= X(z/a)\end{aligned}$$

Complex translation theorem

Let $y(t) = e^{-at}x(t)$ and $X(z) = \mathcal{Z}\{x(kT)\}$, then

$$\mathcal{Z}\{y(kT)\} = \mathcal{Z}\{e^{-aTk}x(kT)\} = X(e^{aT}z)$$

Shifting theorem

Let $x(t)$ be a causal CT signal, thus we have $x(t) = 0$ for $t < 0$. Similarly, associated sampled DT signal has the property of $x[k] = 0$ for $k < 0$. For the sake of simplicity lets work on the sampled (i.e. DT) signal. Let

$$\mathcal{Z}\{x^*(t)\} = \mathcal{Z}\{x[k]\} = X(z)$$

Shifting right by N (Causal shifting): Let $y[k] = x[k-N]$, then

$$\mathcal{Z}\{y[k]\} = \sum_{k=0}^{\infty} y[k]z^{-k} = \sum_{k=0}^{\infty} x[k-N]z^{-k} = \sum_{k=N}^{\infty} x[k-N]z^{-k}$$

Let $k = m + N$ then

$$\begin{aligned}\mathcal{Z}\{y[k]\} &= \sum_{m=0}^{\infty} x[m]z^{-(m+N)} = z^{-N}\sum_{m=0}^{\infty} x[m]z^{-m} \\ \mathcal{Z}\{x[k-N]\} &= z^{-N}X(z)\end{aligned}$$

Shifting left by N (Non-causal shifting) & Bilateral Z transform: Let $y[k] = x[k+N]$,

$$\begin{aligned}\mathcal{Z}\{x[k+N]\} &= \sum_{k=-\infty}^{\infty} x[k+N]z^{-k} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m-N)} = z^{N}\sum_{m=-\infty}^{\infty} x[m]z^{-m} \\ \mathcal{Z}\{x[k+N]\} &= z^{N}X(z)\end{aligned}$$

Shifting left by N (Non-causal shifting) & Unilateral Z transform: Let $y[k] = x[k+N]$,

$$\mathcal{Z}\{x[k+N]\} = \sum_{k=0}^{\infty} x[k+N]z^{-k}$$

Let $k = m - N$ then

$$\mathcal{Z}\{x[k+N]\} = \sum_{m=N}^{\infty} x[m]z^{-(m-N)} = z^N \sum_{m=N}^{\infty} x[m]z^{-m} = z^N \left(\sum_{k=0}^{\infty} x[k]z^{-k} - \sum_{k=0}^{N-1} x[k]z^{-k} \right)$$
$$\mathcal{Z}\{x[k+N]\} = z^N \left(X(z) - \sum_{k=0}^{N-1} x[k]z^{-k} \right)$$

From this equation we can obtain

$$\mathcal{Z}\{x[k+1]\} = zX(z) - zx[0]$$
$$\mathcal{Z}\{x[k+2]\} = z^2X(z) - z^2x[0] - zx[1]$$
$$\vdots$$

Example 1. Let $u[k]$ be the unit-step function. Compute $\mathcal{Z}\{u[k-1]\}$ both directly and using the shifting property.

$$\mathcal{Z}\{u[k-1]\} = \frac{z^{-1}}{1 - z^{-1}}$$

Example 2. Let $y[k] = \sum\limits_{n=0}^{k} x[k]$ where $k \in \mathbb{Z}^+$. Compute $Y(z)$ in terms of $X(z)$ using the shifting theorem.

$$Y(z) = \frac{1}{1 - z^{-1}} X(z)$$

Initial Value Theorem Let $X(z) = \mathcal{Z}\{x[n]\}$ and if the following limit exists, then the initial value of $x[0]$ or $x(0)$ is given by

$$x[0] = \lim_{z \to \infty} X(z)$$

Indeed the proof is very easy

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \left[\sum_{k=0}^{\infty} x(k)z^{-k} \right] = \lim_{z\to\infty} \left[x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots \right] = x(0)$$

Final Value Theorem

Let's assume that $x(kT)$ or $x[k]$ is a convergent sequence (DT signal). Then the final value theorem states that

$$\lim_{k\to\infty} x[k] = \lim_{z\to 1} (1 - z^{-1})X(z)$$

Proof: Let's take the Z transform of $x[k] - x[k-1]$

$$\mathcal{Z}\{x[k] - x[k-1]\} = \sum_{k=0}^{\infty} (x[k] - x[k-1])\, z^{-k}$$
$$X(z) - X(z)z^{-1} = \left(x[0]\left(1 - z^{-1}\right) + x[1]\left(z^{-1} - z^{-2}\right) + x[2]\left(z^{-2} - z^{-3}\right) + x[3]\left(z^{-3} - z^{-4}\right) + \cdots \right) + \lim_{k\to\infty} x[k]z^{-k}$$
$$\lim_{z\to 1} X(z)\left(1 - z^{-1}\right) = (0 + 0 + \cdots) + \lim_{z\to 1}\lim_{k\to\infty} x[k]z^{-k}$$
$$\lim_{z\to 1} X(z)\left(1 - z^{-1}\right) = \lim_{k\to\infty} x[k]$$

Complex Differentiation Theorem

Consider

$$\frac{d}{dz}X(z) = \frac{d}{dz}\left[\sum_{k=0}^{\infty} x[k]z^{-k}\right] = \sum_{k=0}^{\infty} x[k]\frac{d}{dz}z^{-k} = \sum_{k=0}^{\infty}(-k)x[k]z^{-k-1}$$

$$-z\frac{d}{dz}X(z) = \sum_{k=0}^{\infty} kx[k]z^{-k}$$

$$-z\frac{d}{dz}X(z) = \mathcal{Z}\{kx[k]\}$$

In general

$$(-z)^m\frac{d}{dz^m}X(z) = \mathcal{Z}\{k^m x[k]\}$$

Example 3. Find the Z-transform of the unit ramp function, $r[k] = k$, $k \in \mathbb{Z}^+$ by applying the Complex Differentiation Theorem to the Z-transform of the unit step function.

Solution:

$$\mathcal{Z}\{r[k]\} = \mathcal{Z}\{ku[k]\}$$

$$R(z) = (-z)\frac{d}{dz}U(z) = (-z)\frac{d}{dz}U(z) = (-z)\frac{d}{dz}\left(\frac{z}{z-1}\right) = (-z)\left(\frac{1}{z-1} - \frac{z}{(z-1)^2}\right)$$

$$= \frac{z^2}{(z-1)^2} - \frac{z}{z-1} = \frac{z^2 - z(z-1)}{(z-1)^2}$$

$$R(z) = \frac{z}{(z-1)^2}$$

Real Convolution Theorem Let $f[k]$ and $g[k]$ are causal signals and associated Z transforms are $F(z)$ and $G(z)$ respectively. The DT convolution operator is defined as

$$f[n] * g[n] = \sum_{k=0}^{n} f[n-k]g[k]$$

Real Convolution Theorem states that

$$\mathcal{Z}\{f[n] * g[n]\} = F(z)G(z)$$

Proof

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} f[n-k]g[k]\right]z^{-n}$$

Since we know that $f[m] = 0$ for $m < 0$, we can stretch the upper limit of the sum as

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty} f[n-k]g[k]\right]z^{-n} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} f[n-k]g[k]z^{-n}$$

Let $n = m + k$ then

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{k=0}^{\infty} \sum_{m=-k}^{\infty} f[m]g[k]z^{-m}z^{-k} = \sum_{k=0}^{\infty} g[k]z^{-k} \sum_{m=0}^{\infty} f[m]z^{-m}$$
$$\mathcal{Z}\{f[n] * g[n]\} = F(z)G(z)$$

1.3 The Inverse Z-transform

1. Direct division method
2. Z-tranform tables & partial-fraction expansion
3. "Simulation" method

Direct division

Direct division (or long division) method uses the fact that $X(z)$ can be expressed as

$$X(z) = \sum_{k=0}^{\infty} x[k]z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$$

The goal is finding the power series expansion of $X(z)$ using the long division approach. Here we assume that $X(z)$ cane be represnted as a ratio of two polynomials in z (or z^{-1})

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n} = \frac{b_0 z^{-n+m} + b_1 z^{-n+m-1} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

For the direct division method it is easer to work when the polynomials are written interms of powers of z^{-1}.

Example 4. Find the inverse Z-transform of $X(z) = \frac{z^{-1}}{1-2z^{-1}+z^{-2}}$.

$$\begin{array}{r|l}
z^{-1} & 1 - 2z^{-1} + z^{-2} \\
\hline
z^{-1} - 2z^{-2} + z^{-3} & z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \ldots \\
\hline
2z^{-2} - z^{-3} & \\
2z^{-2} - 4z^{-3} + 2z^{-4} & \\
\hline
3z^{-3} - 2z^{-4} & \\
3z^{-3} - 6z^{-4} + 3z^{-5} & \\
\hline
4z^{-4} - 3z^{-5} & \\
\vdots &
\end{array}$$

Thus,

$$X(z) = 0 + 1z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + ...$$
$$\downarrow$$
$$x[k] = 0\delta[k] + 1\delta[k-1] + 2\delta[k-2] + 3\delta[k-3] + 4\delta[k-4] + ... = k$$

Partial Fraction Expansion

In most applications $X(z)$ can be re-written in terms of poles and zeros as

$$X(z) = b_0 \frac{(z-z_1)\cdots(z-z_m)}{(z-p_1)\cdots(z-p_n)} \quad (m \leq n)$$

Specific (but extremely common) case

$$\frac{X(z)}{z} = \sum_{i=1}^{n} \frac{a_i}{(z-p_i)}$$

where all poles are distinct and simple order. We can compute each a_i using

$$a_i = \lim_{z \to p_i} \left[(z-p_i) \frac{X(z)}{z} \right]$$

Example 5. Find the inverse Z-transform of $X(z) = \frac{(1-b)z}{(z-1)(z-b)}$. Solution:

$$\begin{aligned}
\frac{X(z)}{z} &= \frac{(1-b)}{(z-1)(z-b)} = \frac{a_1}{z-1} + \frac{a_2}{z-b} \\
a_1 &= \lim_{z \to 1} \left[(z-1) \frac{X(z)}{z} \right] = 1 \\
a_2 &= \lim_{z \to b} \left[(z-b) \frac{X(z)}{z} \right] = -1 \\
X(z) &= \frac{z}{z-1} - \frac{z}{z-b} \\
x[k] &= 1 - b^k
\end{aligned}$$

Now let's assume that $\frac{X(z)}{z}$ has double pole at p_1 and all other poles are distinct

$$\frac{X(z)}{z} = \frac{c_1}{z-p_1} + \frac{c_2}{(z-p_1)^2} + \cdots$$

It is easy to show that

$$c_2 = \lim_{z \to p_1} \left[(z-p_1)^2 \frac{X(z)}{z} \right]$$

It is also possible to show that

$$c_1 = \lim_{z \to p_1} \left\{ \frac{d}{dz} \left[(z-p_1)^2 \frac{X(z)}{z} \right] \right\}$$

Example 6. Find the inverse Z-transform $X(z) = \frac{2z^2-3z}{(z-1)^2}$. Solution:

$$\begin{aligned}
\frac{X(z)}{z} &= \frac{c_1}{z-1} + \frac{c_2}{(z-1)^2} \\
c_1 &= \lim_{z\to 1} \frac{d}{z}\left[(z-1)^2\frac{X(z)}{z}\right] = 2 \\
c_2 &= \lim_{z\to 1} \left[(z-1)^2\frac{X(z)}{z}\right] = -1 \\
x[k] &= 2 - k
\end{aligned}$$

Example 7. Find the inverse Z-transform $X(z) = \frac{(1-b)}{(z-1)(z-b)}$. Solution:

$$\begin{aligned}
X(z) &= \frac{(1-b)}{(z-1)(z-b)} = \frac{a_1}{z-1} + \frac{a_2}{z-b} \\
a_1 &= \lim_{z\to 1} [(z-1)X(z)] = 1 \\
a_2 &= \lim_{z\to b} [(z-b)X(z)] = -1 \\
X(z) &= z^{-1}\left(\frac{z}{z-1} - \frac{z}{z-b}\right) \\
x[k] &= [1 - b^{k-1}]u[k-1]
\end{aligned}$$

Example 8. Find the inverse Z-transform $X(z) = \frac{z^2-2}{(z-1)(z-2)}$. Solution:

$$\begin{aligned}
X(z) &= \frac{z^2-2}{z^2-3z+2} = 1 + \frac{3z-4}{z^2-3z+2} \\
X(z) &= 1 + \frac{a_1}{z-1} + \frac{a_2}{z-2} \\
a_1 &= \lim_{z\to 1} [(z-1)X(z)] = 1 \\
a_2 &= \lim_{z\to 2} [(z-2)X(z)] = 2 \\
X(z) &= 1 + \frac{1}{z-1} + \frac{2}{z-2} = \frac{1}{z-1} + \frac{z}{z-2} \\
x[k] &= 1 + 2^k - \delta[k]
\end{aligned}$$

1.4 Difference Equations

In discrete-time domain, we have difference equations that replaces differential equations. We are mainly interested in LTI systems, that are represented by linear constant coefficient difference equations. Let $x[k]$ and $y[k]$ be the input and output respectively, then an LTI difference equation can be expressed as

$$\begin{aligned}
a_0y[k] + a_1y[k-1] + ... + a_Ny[k-N] &= b_0x[k] + ... + b_Mx[k-M] \\
\sum_{n=1}^{N} a_n y[k-n] &= \sum_{n=1}^{M} b_n x[k-n]
\end{aligned}$$

Unlike ODEs difference equations are very easy to solve computationally or simulate in computer environment. Let's consider the following first-order difference equation

$$y[k] = \frac{1}{2}y[k-1] + x[k] \quad, x[k] = 0 \ \&\ y[k] = 0, \ \text{for} k < 0$$

Let's "simulate" the difference equation for $x[k] = \delta[k]$.

$$
\begin{aligned}
y[0] &= \frac{1}{2}y[-1] + x[0] = 0 + 1 = 1 \\
y[1] &= \frac{1}{2}y[0] + x[1] = \frac{1}{2} + 0 = \frac{1}{2} \\
y[2] &= \frac{1}{2}\frac{1}{2} = \frac{1}{4} \\
y[3] &= \frac{1}{2}\frac{1}{4} = \frac{1}{8} \\
&\vdots \\
y[k] &= \left(\frac{1}{2}\right)^k
\end{aligned}
$$

Now let's simulate for $x[k] = u[k]$

$$
\begin{aligned}
y[0] &= 0 + 1 = 1 \\
y[1] &= \frac{1}{2} + 1 \\
y[2] &= \frac{1}{4} + \frac{1}{2} + 1 \\
y[3] &= \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1 \\
&\vdots \\
y[k] &= \frac{1}{2^k} + \cdots + \frac{1}{2} + 1 = 2 - \left(\frac{1}{2}\right)^k
\end{aligned}
$$

This is a great method for "simulating" using a computational approach, but in general it may be very hard to get a closed form expression. The most basic solution method is solving the difference equation directly in time domain by trying to find a "basis" for the solution space similar to the operation in ODEs. We try sequences/signals of the form λ^k , $k > 0$ to find a solution form for the homogeneous equation. Let's apply this method for the first-order difference equation above

$$
\begin{aligned}
y[k] = \lambda^k \ \rightarrow y[k] - \frac{1}{2}y[k-1] &= 0 \\
\lambda^k - \frac{\lambda^{k-1}}{2} &= 0 \\
\lambda^{k-1}\left(\lambda - \frac{1}{2}\right) &= 0 \\
\lambda - \frac{1}{2} &= 0
\end{aligned}
$$

Where the last equation is the characteristic equation of the difference equation. Since the characteristic equation has one root only, we obtain a solution of the form

$$
y[k] = y_h[k] + y_p[k] = C\left(\frac{1}{2}\right)^k + y_p[k]
$$

Let's assume that for $x[k] = u[k]$ particular solution has the form $y_p[k] = A$ for $k > 0$ then

$$
A = \frac{1}{2}A + 1 \ \rightarrow A = 2
$$

Now let's find C using the fact that $y[k] = 0$ for $k < 0$

$$y[0] = \frac{1}{2}y[-1] + x[0] \ \rightarrow \ y[0] = 1$$
$$1 = C\left(\frac{1}{2}\right)^0 + 2 \rightarrow C = -1$$

Then the solution can be written as

$$y[k] = -\left(\frac{1}{2}\right)^k + 2$$

Example 1.1 Find the general form of the homogeneous solution for the following difference equation

$$y[k] - 3y[k-1] + 2y[k-2] = x[k]$$

Solution:

$$\lambda^2 - 3\lambda + 2 = 0$$
$$\lambda_1 = 1 \ \& \ \lambda_2 = 2$$
$$y[k] = C_1 + C_2 2^k \ , k > 0$$

Example 1.2 Now let's assume that $y[k] = 0$ for $k < 0$ and $x[k] = 3^k$, then find $y[k]$ for $k \geq 0$.

Solution: First let's find a particular solution. Let's assume that $y_p[k] = A3^k$, then

$$A3^k - 3A3^{k-1} + 2A3^{k-2} = 3^k \ \rightarrow \ A = 9/2$$
$$y_p[k] = 4.5 \ 3^k$$

Now let's try to find C_1 and C_2

$$y[k] - 3y[k-1] + 2y[k-2] = x[k]$$
$$y[0] = x[0] \ \rightarrow \ C_1 + C_2 = -3.5$$
$$y[1] - 3y[0] = x[1] \ \rightarrow \ C_1 + C_2 2 = -7.5$$
$$C_1 = 0.5 \ \& \ C_2 = -4$$
$$y[k] = 0.5 - 4\,2^k + 4.5\,3^k \ , k > 0$$

What about repeated roots? Possible mini project question

Example 2 Find the general form of the homogeneous solution for the following difference equation

$$y[k] + 4y[k-2] = x[k]$$

Solution:

$$\lambda^2 + 4 = 0 \ \rightarrow \ \lambda_{1,2} = \pm 2j$$
$$y[k] = C_1(2j)^k + C_2(-2j)^k = C_1 2^k e^{jk\pi/2} + C_2 2^k e^{-jk\pi/2}$$
$$y[k] = \bar{C}_1 2^k \frac{e^{jk\pi/2} + e^{-jk\pi/2}}{2} + \bar{C}_2 2^k \frac{e^{jk\pi/2} - e^{jk\pi/2}}{2j}$$
$$y[k] = \bar{C}_1 2^k \cos(k\pi/2) + \bar{C}_2 2^k \sin(k\pi/2)$$

How we can generalize this to arbitrary complex conjugate roots? Possible mini project question

What is the home message? Similar to ODEs time domain solution of difference equations is generally "messy".

Z-transform & Difference Equations

Difference Equations to Z-transform

Let's consider the following difference equation with $y[n]$ and $x[n]$ be the strictly causal input–output pair.

$$a_0 y[k] + a_1 y[k-1] + ... + a_N y[k-N] = b_0 x[k] + ... + b_M x[k-M]$$

Now let's assume that $\mathcal{Z}\{x[k]\} = X(z)$ and $\mathcal{Z}\{y[k]\} = Y(z)$. If we take the Z-transform fo the both sides of the equation by applying the shifting theorem we obtain

$$\begin{aligned}
a_0 Y(z) + a_1 z^{-1} Y(z) + ... + a_N z^{-N} Y(z) &= b_0 X(z) + ... + b_M z^{-M} X(z) \\
(a_0 + a_1 z^{-1} + ... + a_N z^{-N}) Y(z) &= (b_0 + b_1 z^{-1} + ... + b_M z^{-M}) X(z) \\
\frac{Y(z)}{X(z)} = G(z) &= \frac{b_0 + b_1 z^{-1} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + ... + a_N z^{-N}} \\
&= z^{N-M} \frac{b_0 z^M + b_1 z^{M-1} + ... + b_M}{a_0 z^N + a_1 z^{N-1} + ... + a_N}
\end{aligned}$$

Under "zero initial conditions" if we can find $X(z)$, then we can compute $Y(z)$ using $Y(z) = G(z)X(z)$. After that we can take the inverse z-transform and compute $y[k]$.

Example 3.1 Compute $y[k]$ using the Z-transform method

$$\begin{aligned}
y[k] &= \frac{1}{2} y[k-1] + x[k] \\
y[k] &= 0, \text{ for } k < 0 \ \& \ x[k] = \delta[k]
\end{aligned}$$

Solution:

$$\begin{aligned}
Y(z) &= \frac{1}{2} Y(z) z^{-1} + X(z) \ \rightarrow \ \frac{Y(z)}{X(z)} = G(z) = \frac{z}{z - 1/2} \\
Y(z) &= \frac{z}{z - 1/2} \ \rightarrow \ y[k] = \left(\frac{1}{2}\right)^k
\end{aligned}$$

Example 3.2 Now let's compute $y[k]$ for $x[k] = u[k]$

$$\begin{aligned}
Y(z) &= G(z)X(z) \ \rightarrow \ Y(z) = \frac{z^2}{(z - 1/2)(z - 1)} \\
Y(z) &= -\frac{z}{z - 1/2} + 2\frac{z}{z - 1} \\
y[k] &= 2 - \left(\frac{1}{2}\right)^k
\end{aligned}$$

Example 4 For the following difference equation, compute $y[k]$ for $x[k] = 3^k u[k]$

$$y[k] - 3y[k-1] + 2y[k-2] = x[k]$$

Solution:

$$Y(z)(1-3z^{-1}+2z^{-2}) = X(z) \;\rightarrow\; G(z) = \frac{z^2}{z^2-3z+2} = \frac{z^2}{(z-1)(z-2)}$$
$$Y(z) = \frac{z^3}{(z-1)(z-2)(z-3)} = 0.5\frac{z}{z-1} - 4\frac{z}{z-2} + 4.5\frac{z}{z-3}$$
$$y[k] = \left(0.5 - 4\;2^k + 4.5\;3^k\right) u[k]$$

Z-transform to Difference Equations

Sometimes the Z-domain transfer function of a system is given, and we may be supposed to find the difference equation representation. Let's assume that we have a general transfer function that can be represented in terms of ratio of two polynomials in z or z^{-1} as given below

$$\frac{Y(z)}{X(z)} = G(z) = \frac{b_0 + b_1 z^{-1} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + ... + a_N z^{-N}} = z^{N-M}\frac{b_0 z^M + b_1 z^{M-1} + ... + b_M}{a_0 z^N + a_1 z^{N-1} + ... + a_N}$$

In his case, I prefer to work with the polynomials that are written in terms of z^{-1}. Let's manipulate the Z-domain equation to obtain

$$Y(z)(a_0 + a_1 z^{-1} + ... + a_N z^{-N}) = X(z)(b_0 + b_1 z^{-1} + ... + b_M z^{-M})$$
$$a_0 Y(z) + a_1 z^{-1} Y(z) + ... + a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + ... + b_M z^{-M} X(z)$$

Let's assume that $\mathcal{Z}^{-1}\{Y(z)\} = y[k]$ and $\mathcal{Z}^{-1}\{X(z)\} = x[k]$. If we take the inverse Z-transform of both sides by applying the shifting theorem we obtain

$$a_0 y[k] + a_1 y[k-1] + ... + a_N y[k-N] = b_0 x[k] + b_1 x[k-1] + ... + b_M x[k-M]$$

We can use this conversion to "simulate" a given discrete time transfer function or realizing the given system (it may be a filter or controller) to implement on an embedded platform.

It can also be used for computationally finding the inverse Z-transform of a given z-domain rational function. The next example will illustrate this feature.

Example 5 Find a computational solution for the inverse Z-transform of $H(z) = \frac{z^{-1}}{1-2z^{-1}+z^{-2}}$ by using the conversion from Z-domain transfer function to difference equation concept.

Solution: Let's assume that $H(z)$ is a "transfer function" not an arbitrary z-domain function. Then $\mathcal{Z}^{-1}\{H(z)\} = h[k]$ becomes the impulse response of the "system". Thus we can assume some imaginary input-output pair $y[n]$ and $x[n]$ where

$$\frac{Y(z)}{X(z)} = H(z)$$

If we can find a difference equation realization for $H(z)$ then we can simulate the difference equation by assuming $x[k] = \delta[k]$ (i.e. unit impulse input). So let's find a realization for the given $H(z)$ as

$$\frac{Y(z)}{X(z)} = \frac{z^{-1}}{1-2z^{-1}+z^{-2}}$$
$$Y(z) - 2z^{-1}Y(z) + z^{-2}Y(z) = z^{-1}X(z)$$
$$y[k] - 2y[k-1] + y[k-2] = x[k-1]$$

Now let's simulate the above equation for $x[k] = \delta[k]$

$$
\begin{aligned}
y[k] &= 2y[k-1] - y[k-2] + x[k-1] \\
y[0] &= 2y[-1] - y[-2] + x[-1] = 0 \\
y[1] &= 2y[0] - y[-1] + x[0] = 1 \\
y[2] &= 2y[1] - y[0] + x[1] = 2 \\
y[3] &= 2y[2] - y[1] + x[2] = 3 \\
y[4] &= 4 \\
&\dots \\
y[k] &= k
\end{aligned}
$$