EE502 - Linear Systems Theory II **Spring 2023**

Lecture 12

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12.1 The Kalman Decomposition

In reachability and observability lectures, we derived two types of standards forms, specifically for unreachable systems and unobservable systems (separately). Now our goal is to propose a general standard form for a unreachable and unobservable system, based on the Kalman decomposition. The process is exactly same for bot DT and CT systems, thus we will present the decomposition for only CT systems. Let

$$\dot{x} = Ax + Bu \ , \ y = Cx + Du \ \& \ x \in \mathbb{R}^n$$

Let's assume that system is neither reachable, nor observable and

$$\begin{aligned} \text{rank}[\mathbf{R}] = r < n \ , \ \text{range}[\mathbf{R}] = \mathcal{R} \\ \dim[\mathcal{N}(\mathbf{O})] = \bar{o} > 0 \ , \ \bar{\mathcal{O}} = \mathcal{N}(\mathbf{O}) \end{aligned}$$

Let's consider the following similarity transformation

$$\hat{A} = T^{-1}AT \ , \ \hat{B} = T^{-1}B \ , \ \hat{C} = CT \ \& \ D = D$$

Let

$$T = \left[\begin{array}{c|c|c|c} T_{r\bar{o}} & T_{ro} & T_{\bar{r}\bar{o}} & T_{\bar{r}o} \end{array} \right]$$

Let's define sub-matrices as follows:

1. Let $\mathcal{R}\bar{\mathcal{O}} = \mathcal{R} \cap \bar{\mathcal{O}}$, i.e. $x \in \mathcal{R}\bar{\mathcal{O}} \Rightarrow x \in \mathcal{R} \ \& \ x \in \bar{\mathcal{O}}$. Choose $T_{r\bar{o}}$ such that columns of $T_{r\bar{o}}$ forms a basis for $\mathcal{R}\bar{\mathcal{O}}$.
2. Choose a T_{ro} such that Ra$\left[\begin{array}{c|c} T_{r\bar{o}} & T_{ro} \end{array} \right] = \mathcal{R} = \text{Ra}[\mathbf{R}]$, i.e. columns of T_{ro} complements $T_{r\bar{o}}$ in the reachable sub-space and
3. Choose a $T_{\bar{r}\bar{o}}$ such that Ra$\left[\begin{array}{c|c} T_{r\bar{o}} & T_{\bar{r}\bar{o}} \end{array} \right] = \bar{\mathcal{O}} = \mathcal{N}[\mathbf{O}]$, i.e. columns of $T_{\bar{r}\bar{o}}$ complements $T_{r\bar{o}}$ in the unobservable sub-space
4. Choose $T_{\bar{r}o}$ such that Ra$[T] = \mathbb{R}^n$

Let's remember the important sub-spaces invariant under A and some critical features that will be helpful for constructing the Kalman decomposition

$$\begin{array}{c} A\mathcal{R} \subset \mathcal{R} \\ A\bar{\mathcal{O}} \subset \bar{\mathcal{O}} \end{array} \Rightarrow A\mathcal{R}\bar{\mathcal{O}} \subset \mathcal{R}\bar{\mathcal{O}}$$

$$\text{Ra}[B] \subset \mathcal{R}$$

$$\bar{\mathcal{O}} \subset \mathcal{N}[C]$$

Let's analyze the similarity transformation of the system matrix.

$$AT = TA$$

$$A\left[\begin{array}{c|c|c|c} T_{r\bar{o}} & T_{ro} & T_{\bar{r}\bar{o}} & T_{\bar{r}o} \end{array}\right] = \left[\begin{array}{c|c|c|c} T_{r\bar{o}} & T_{ro} & T_{\bar{r}\bar{o}} & T_{\bar{r}o} \end{array}\right] \left[\begin{array}{c|c|c|c} A_{11} & A_{12} & A_{13} & A_{14} \\ \hline A_{21} & A_{22} & A_{23} & A_{24} \\ \hline A_{31} & A_{32} & A_{33} & A_{34} \\ \hline A_{41} & A_{42} & A_{43} & A_{44} \end{array}\right]$$

Let's expand $AT_{r\bar{o}}$

$$AT_{r\bar{o}} = \left[\begin{array}{c|c|c|c} T_{r\bar{o}} & T_{ro} & T_{\bar{r}\bar{o}} & T_{\bar{r}o} \end{array}\right] \left[\begin{array}{c} A_{11} \\ \hline A_{21} \\ \hline A_{31} \\ \hline A_{41} \end{array}\right]$$

Since Ra$[T_{r\bar{o}}]$ is invariant under A (i.e. $A\mathcal{R}\bar{\mathcal{O}} \subset \mathcal{R}\bar{\mathcal{O}}$), $A_{i1} = 0, \forall\, i > 1$.

Now let's focus on $A\left[\begin{array}{c|c} T_{r\bar{o}} & T_{ro} \end{array}\right]$

$$A\left[\begin{array}{c|c} T_{r\bar{o}} & T_{ro} \end{array}\right] = \left[\begin{array}{c|c|c|c} T_{r\bar{o}} & T_{ro} & T_{\bar{r}\bar{o}} & T_{\bar{r}o} \end{array}\right] \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \\ \hline 0 & A_{32} \\ \hline 0 & A_{42} \end{array}\right]$$

Since Ra$\left[\begin{array}{c|c} T_{r\bar{o}} & T_{ro} \end{array}\right]$ is invariant under A (i.e. $A\mathcal{R} \subset \mathcal{R}$), $A_{32} = 0$ and $A_{42} = 0$.

Now let's focus on $A\left[\begin{array}{c|c} T_{r\bar{o}} & T_{\bar{r}\bar{o}} \end{array}\right]$

$$A\left[\begin{array}{c|c} T_{r\bar{o}} & T_{\bar{r}\bar{o}} \end{array}\right] = \left[\begin{array}{c|c|c|c} T_{r\bar{o}} & T_{ro} & T_{\bar{r}\bar{o}} & T_{\bar{r}o} \end{array}\right] \left[\begin{array}{c|c} A_{11} & A_{13} \\ \hline 0 & A_{23} \\ \hline 0 & A_{33} \\ \hline 0 & A_{43} \end{array}\right]$$

Since Ra$\left[\begin{array}{c|c} T_{r\bar{o}} & T_{\bar{r}\bar{o}} \end{array}\right]$ is invariant under A (i.e. $A\bar{\mathcal{O}} \subset \bar{\mathcal{O}}$), $A_{23} = 0$ and $A_{43} = 0$.

As a results $\hat{A}$ takes the form

$$\hat{A} = \left[\begin{array}{c|c|c|c} A_{11} & A_{12} & A_{13} & A_{14} \\ \hline 0 & A_{22} & 0 & A_{24} \\ \hline 0 & 0 & A_{33} & A_{34} \\ \hline 0 & 0 & 0 & A_{44} \end{array}\right] = \left[\begin{array}{c|c|c|c} A_{r\bar{o}} & A_{12} & A_{13} & A_{14} \\ \hline 0 & A_{ro} & 0 & A_{24} \\ \hline 0 & 0 & A_{\bar{r}\bar{o}} & A_{34} \\ \hline 0 & 0 & 0 & A_{\bar{r}o} \end{array}\right]$$

Now let's focus on input matrix transformation.

$$B = T\hat{B} = \left[\begin{array}{c|c|c|c} T_{r\bar{o}} & T_{ro} & T_{\bar{r}\bar{o}} & T_{\bar{r}o} \end{array}\right] \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}$$

Since Ra$(B) \subset \mathcal{R} =$ Ra$\left[\begin{array}{c|c} T_{r\bar{o}} & T_{ro} \end{array}\right]$, $B_3 = 0$ and $B_4 = 0$ and thus $\hat{B}$ takes the form

$$\hat{B} = \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \\ 0 \\ 0 \end{bmatrix}$$

Now let's focus on input matrix transformation.

$$
\begin{aligned}
CT &= \hat{C} \\
C \left[\begin{array}{c|c|c|c} T_{r\bar{o}} & T_{ro} & T_{\bar{r}\bar{o}} & T_{\bar{r}o} \end{array}\right] &= \left[\begin{array}{c|c|c|c} C_1 & C_2 & C_3 & C_4 \end{array}\right] \\
\left[\begin{array}{c|c|c|c} CT_{r\bar{o}} & CT_{ro} & CT_{\bar{r}\bar{o}} & CT_{\bar{r}o} \end{array}\right] &= \left[\begin{array}{c|c|c|c} C_1 & C_2 & C_3 & C_4 \end{array}\right] \\
\left[\begin{array}{c|c|c|c} 0 & CT_{ro} & 0 & CT_{\bar{r}o} \end{array}\right] &= \left[\begin{array}{c|c|c|c} C_1 & C_2 & C_3 & C_4 \end{array}\right]
\end{aligned}
$$

and thus $\hat{C}$ takes the form

$$
\hat{C} = \left[\begin{array}{c|c|c|c} 0 & C_{ro} & 0 & C_{\bar{r}o} \end{array}\right]
$$

Based on Kalman decomposition, we can obtain reachable and observable (minimal) sub-system, reachable only sub-system, and observable only system.

- Reachable and observable sub-system

$$
\begin{aligned}
\dot{x}_{ro} &= A_{ro}x_{ro} + B_{ro}u \\
y &= C_{ro}x_{ro} + Du
\end{aligned}
$$

- Reachable only (but not fully observable) sub-system

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} x_{r\bar{o}} \\ x_{ro} \end{bmatrix} &= \left[\begin{array}{c|c} A_{r\bar{o}} & A_{12} \\ \hline 0 & A_{ro} \end{array}\right] \begin{bmatrix} x_{r\bar{o}} \\ x_{ro} \end{bmatrix} + \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \end{bmatrix} u \\
y &= \left[\begin{array}{c|c} 0 & C_{ro} \end{array}\right] \begin{bmatrix} x_{r\bar{o}} \\ x_{ro} \end{bmatrix} + Du
\end{aligned}
$$

- Observable only (but not fully reachable) sub-system

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} x_{ro} \\ x_{\bar{r}o} \end{bmatrix} &= \left[\begin{array}{c|c} A_{ro} & A_{24} \\ \hline 0 & A_{\bar{r}o} \end{array}\right] \begin{bmatrix} x_{ro} \\ x_{\bar{r}o} \end{bmatrix} + \begin{bmatrix} B_{ro} \\ 0 \end{bmatrix} u \\
y &= \left[\begin{array}{c|c} C_{ro} & C_{\bar{r}o} \end{array}\right] \begin{bmatrix} x_{ro} \\ x_{\bar{r}o} \end{bmatrix} + Du
\end{aligned}
$$

Theorem: A state-space representation is minimal if and only if it is both fully reachable and observable.

Corollary: All minimal representations are algebraically equivalent.

Ex 12.1 *Let*

$$
\begin{aligned}
x[k+1] &= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \end{bmatrix} x[k] + \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u \\
y[k] &= \begin{bmatrix} 0 & 1 & 1 & 2 \end{bmatrix} x
\end{aligned}
$$

Obtain Kalman decomposition and extract reachable and observable (minimal) sub-system, reachable only sub-system, and observable only system, and non-reachable and non-observable sub-system. Let's first find the reachability matrix and reachable subspace

$$\mathbf{R} = \left[\begin{array}{c|c|c|c} A^3B & A^2B & AB & B \end{array}\right] = \begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$$\mathcal{R} = \text{Ra}[\mathbf{R}] = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Now let's find the observability matrix and non-observable subspace

$$\mathbf{O} = \left[\begin{array}{cccc} C & CA & CA^2 & CA^3 \end{array}\right] = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\mathcal{O}} = \mathcal{N}[\mathbf{O}] = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

One can see the intersection of reachable and unobservable sub-spaces is one-dimensional and

$$\mathcal{R}\bar{\mathcal{O}} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Based on these findings, we can construct the transformation matrix as

$$T = \left[\begin{array}{c|c|c|c} T_{r\bar{o}} & T_{ro} & T_{\bar{r}\bar{o}} & T_{\bar{r}o} \end{array}\right] = \left[\begin{array}{c|c|c|c} 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right]$$

$$T^{-1} = \left[\begin{array}{c|c|c|c} 1/2 & 1/4 & 1/4 & 0 \\ -1/2 & 1/4 & 1/4 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

Now let's find the transformed state-space matrices and find the required decompisition

$$\hat{A} = T^{-1}AT = \left[\begin{array}{c|c|c|c} A_{r\bar{o}} & A_{12} & A_{13} & A_{14} \\ \hline 0 & A_{ro} & 0 & A_{24} \\ \hline 0 & 0 & A_{\bar{r}\bar{o}} & A_{34} \\ \hline 0 & 0 & 0 & A_{\bar{r}o} \end{array}\right] = \left[\begin{array}{c|c|c|c} 0 & 2 & 1 & 3/2 \\ \hline 0 & 0 & 0 & 1/2 \\ \hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right],$$

$$\hat{B} = T^{-1}B = \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{c} 0 \\ \hline 1 \\ \hline 0 \\ \hline 0 \end{array}\right]$$

$$\hat{C} = CT = \left[\begin{array}{c|c|c|c} 0 & C_{ro} & 0 & C_{\bar{r}o} \end{array}\right] = \left[\begin{array}{c|c|c|c} 0 & 4 & 0 & 2 \end{array}\right]$$

We can then extract the reachable and observable (minimal), reachable only, observable only, and non-reachable and non-observable sub-systems as

- Reachable and observable sub-system

$$x_{ro}[k+1] = [0]x_{ro}[k] + [1]u[k]$$
$$y[k] = [4]x_{ro}[k]$$

- Reachable only sub-system

$$\begin{bmatrix} x_{r\bar{o}}[k+1] \\ x_{ro} \end{bmatrix} [k+1] = \left[\begin{array}{c|c} 0 & 2 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} x_{r\bar{o}}[k] \\ x_{ro} \end{bmatrix} [k] + \left[\begin{array}{c} 0 \\ 1 \end{array}\right] u[k]$$
$$y[k] = \left[\begin{array}{c|c} 0 & 4 \end{array}\right] \begin{bmatrix} x_{r\bar{o}}[k] \\ x_{ro}[k] \end{bmatrix}$$

- Observable only (but not fully reachable) sub-system

$$\begin{bmatrix} x_{ro}[k+1] \\ x_{\bar{r}o}[k+1] \end{bmatrix} = \left[\begin{array}{c|c} 0 & 1/2 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} x_{ro}[k] \\ x_{\bar{r}o}[k] \end{bmatrix} + \left[\begin{array}{c} 1 \\ 0 \end{array}\right] u[k]$$
$$y[k] = \left[\begin{array}{c|c} 4 & 2 \end{array}\right] \begin{bmatrix} x_{ro}[k] \\ x_{\bar{r}o}[k] \end{bmatrix}$$

- Non-observable and non-reachable sub-system

$$x_{\bar{r}\bar{o}}[k+1] = [0]x_{\bar{r}\bar{o}}[k] + [0]u[k]$$
$$y[k] = [0]x_{\bar{r}\bar{o}}[k]$$