

## Lecture 13

*Lecturer: Asst. Prof. M. Mert Ankarali***Matrix Exponential,  $e^{At}$** 

Let's first review the matrix exponential,  $e^{At}$ . Let  $t \in \mathbb{R}$  and  $A \in \mathbb{R}^{n \times n}$ , then  $e^{At}$  defined as

$$e^{At} := I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots$$

$$e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

which converges for all  $t \in \mathbb{R}$  and  $A \in \mathbb{R}^{n \times n}$ .

Now let's review some properties

- **Claim:**

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

**Proof:**

$$\begin{aligned} \frac{d}{dt}e^{At} &= \frac{d}{dt} \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right) = \sum_{k=0}^{\infty} k \frac{t^{k-1}}{k!} A^k = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^k \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} A^{n+1} = A \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \left( \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \right) A \\ \frac{d}{dt}e^{At} &= Ae^{At} = e^{At}A \end{aligned}$$

- **Claim:** Let  $t_1, t_2 \in \mathbb{R}$  then

$$e^{At_1}e^{At_2} = e^{At_2}e^{At_1} = e^{A(t_1+t_2)}$$

**Proof:**

$$e^{At_1}e^{At_2} = \left( \sum_{k=0}^{\infty} \frac{t_1^k}{k!} A^k \right) \left( \sum_{j=0}^{\infty} \frac{t_2^j}{j!} A^j \right) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{t_1^k}{k!} \frac{t_2^j}{j!} A^{k+j}$$

Let  $n = k + j$  and  $j = n - k$ , then

$$e^{At_1}e^{At_2} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{t_1^k}{k!} \frac{t_2^{n-k}}{(n-k)!} A^n = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{t_1^k}{k!} \frac{t_2^{n-k}}{(n-k)!} \frac{n!}{n!} A^n = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{t_1^k}{k!} \frac{t_2^{n-k}}{(n-k)!} \binom{n}{k} A^n$$

Since for  $n < k$ ,  $\binom{n}{k} = 0$ ,

$$e^{At_1} e^{At_2} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{t_1^k t_2^{n-k}}{n!} \binom{n}{k} A^n = \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{\infty} t_1^k t_2^{n-k} \binom{n}{k}$$

Using *binomial* theorem we find

$$e^{At_1} e^{At_2} = \sum_{n=0}^{\infty} \frac{A^n}{n!} (t_1 + t_2)^n$$

$$e^{At_1} e^{At_2} = e^{A(t_1+t_2)}$$

Now let  $t_1 = t$  and  $t_2 = -t$ , then we have

$$e^{At} e^{-At} = e^{A(t-t)} = I \quad \rightarrow \quad (e^{At})^{-1} = e^{-At}$$

- **Claim:** Let  $A, B \in \mathbb{R}^{n \times n}$  and  $AB = BA$ , then

$$e^{At} e^{Bt} = e^{Bt} e^{At} = e^{(A+B)t}$$

**Proof:** Possible mini project question

Note that if  $AB \neq BA$  then

$$e^{At} e^{Bt} \neq e^{(A+B)t}$$

- **Claim:** Let  $P \in \mathbb{R}^{n \times n}$  and  $\det(P) \neq 0$ , then

$$e^{(P^{-1}AP)t} = P^{-1} e^{At} P$$

**Proof:** Possible mini project question

## Solution of CT State-Space Equations

CT state-space representation has the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ \dot{y}(t) &= Cx(t) + Du(t) \end{aligned}$$

where Let  $x(t) \in \mathbb{R}^n$ ,  $y(t) \in \mathbb{R}^p$ ,  $u(t) \in \mathbb{R}^q$

First consider the homogeneous solution, i.e.  $u(t) = 0$  and  $x(0) = x_0$ .

$$\begin{aligned} \dot{x}(t) &= Ax(t) \quad , \quad x(0) = x_0 \\ \dot{y}(t) &= Cx(t) \end{aligned}$$

Let's test if  $x(t) = e^{At}x_0$  is a solution of the homogeneous equation

$$\begin{aligned}x(0) &= e^{A0}x_0 = x_0 \\ \dot{x}(t) - Ax(t) &= (Ae^{At})x_0 - Ae^{At}x_0 = 0\end{aligned}$$

Now let's compute the forced response. First let's analyze the following derivative

$$\frac{d}{dt} [e^{-At}x(t)] = e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}[\dot{x}(t) - Ax(t)]$$

Now using this relation let's solve the state-space equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ \dot{x}(t) - Ax(t) &= Bu(t) \\ e^{-At}[\dot{x}(t) - Ax(t)] &= e^{-At}Bu(t) \\ \frac{d}{dt} [e^{-At}x(t)] &= e^{-At}Bu(t) \\ e^{-At}x(t) &= x(0) + \int_0^t e^{-A\tau}Bu(\tau)d\tau \\ x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\end{aligned}$$

Thus the solution of a system in state-space form can be written as

$$\begin{aligned}x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) &= Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)\end{aligned}$$

The function  $\Psi(t) = e^{At}$  is called the state-transition matrix of the system.

**Example:** Let's assume that system is a SISO system and  $u(t) = \delta(t)$  (unit-impulse function) and  $x_0 = 0$ , compute the impulse response of the system, i.e.  $y(t) = h(t)$ ,

$$\begin{aligned}h(t) &= \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau + D\delta(t) \\ &= Ce^{At}B + D\delta(t)\end{aligned}$$

## S-Domain Solution of CT State-Space Equations

First take the Laplace transform of state evaluation equation

$$\begin{aligned}\mathcal{L}[\dot{x}(t)] &= \mathcal{L}[Ax(t) + Bu(t)] \\ sX(s) - x(0) &= AX(s) + BU(s) \\ [sI - A]X(s) &= x(0) + BU(s) \\ X(s) &= [sI - A]^{-1}x_0 + [sI - A]^{-1}BU(s) \\ Y(s) &= C[sI - A]^{-1}x_0 + [C[sI - A]^{-1}B + D]U(s)\end{aligned}$$

If we relate time and s-domain solutions we obtain

$$\begin{aligned}e^{At} &= \mathcal{L}^{-1}[[sI - A]^{-1}] \\ h(t) &= \mathcal{L}^{-1}[C[sI - A]^{-1}B + D]U(s)\end{aligned}$$

## Discretization of CT State-Space Equations

Consider the CT system with the given state-space representation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

and suppose that the input is piece-wise constant over intervals of length  $T$ . That is

$$u(t) = u[k] \quad , \quad t \in (kT \text{ } (k+1)T)$$

i.e. input of the system is the output of a ZOH operator. Let's derive the the DT state-space equations with respect to the sampled-state  $x[k] = x(kT)$ .

Let's start with the state evolution equation

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Due to time-invariant nature of the dynamics, we can generalize the initial time of the equation as

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau \quad , \quad t > t_0$$

Now let  $t_0 = kT$  and  $t = (k+1)T$ ,

$$x((k+1)T) = e^{AT}x(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau$$

Since the  $u(t) = u[k]$  in this time interval

$$x[k+1] = e^{AT}x[k] + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bd\tau u[k]$$

Let  $\lambda = (k+1)T - \tau$ , then

$$\begin{aligned} x[k+1] &= e^{AT} x[k] - \int_T^0 e^{A\lambda} B d\lambda u[k] \\ &= [e^{AT}] x[k] + \left[ \left( \int_0^T e^{A\lambda} d\lambda \right) B \right] u[k] \end{aligned}$$

Given that

$$x[k+1] = Gx[k] + Hu[k]$$

$G$  and  $H$  matrices can be extracted as

$$\begin{aligned} G &= e^{AT} \\ H &= \left( \int_0^T e^{A\lambda} d\lambda \right) B \end{aligned}$$

**Claim:** If  $A$  is invertable then we also have

$$H = A^{-1} (e^{AT} - I) B = (e^{AT} - I) A^{-1} B$$

**Proof:** Possible mini project question

Now let's consider the output equation

$$\begin{aligned} y(t) &= Cx(t) + Dx(t) \\ y(kT) &= Cx(kT) + Dx(kT) \\ y[k] &= Cx[k] + Dx[k] \end{aligned}$$

It can be seen that output equation matrices are not affected from the discretization.

**Example:** Consider the following CT-plant transfer function.

$$\frac{Y(s)}{U(s)} = \frac{1}{s} + \frac{1}{s + \ln(2)}$$

Find a CT state-space representation for this system

**Solution:**

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 0 \\ 0 & -\ln(2) \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \end{aligned}$$

Compute the state-transition matrix

**Solution:**

$$e^{At} = \begin{bmatrix} e^{0t} & 0 \\ 0 & e^{-\ln(2)t} \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 \\ 0 & 0.5^t \end{bmatrix}$$

Discretize the CT State-Space equation under zero hold operation and ideal sampling of the defined state variables, with  $T = 1s$ .

**Solution:**

$$G = e^{AT} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \\ H = \left( \int_0^T e^{A\lambda} d\lambda \right) B = \left( \int_0^1 \begin{bmatrix} 1 & 0 \\ 0 & 0.5^\lambda \end{bmatrix} d\lambda \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \int_0^1 \begin{bmatrix} 1 \\ 0.5^\lambda \end{bmatrix} d\lambda \\ H = \begin{bmatrix} 1 \\ 0.721 \end{bmatrix}$$

Full DT state-space formulation takes the form

$$x[k+1] = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0.721 \end{bmatrix} u[k] \\ y[k] = \begin{bmatrix} 1 & 1 \end{bmatrix} x[k]$$

Compute the DT pulse transfer function  $Y(z)/U(z)$  **Solution:**

$$\frac{Y(z)}{U(z)} = C [zI - G]^{-1} H \\ = \begin{bmatrix} 1 & 1 \end{bmatrix} [zI - G]^{-1} \begin{bmatrix} 1 \\ 0.721 \end{bmatrix} \\ = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} z-1 & 0 \\ 0 & z-0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0.721 \end{bmatrix} \\ = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z-0.5} \end{bmatrix} \begin{bmatrix} 1 \\ 0.721 \end{bmatrix} \\ = \frac{1}{z-1} + \frac{0.721}{z-0.5}$$

Now discretize  $Y(s)/U(s)$  directly under ZOH operation

**Solution:**

$$\frac{Y(z)}{U(z)} = \mathcal{Z} \left[ \frac{1-e^{-s}}{s} \frac{Y(s)}{U(s)} \right] = \frac{1}{z-1} + \frac{0.721}{z-0.5}$$