

Lecture 12

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12.1 State-Space Models of Dynamical Systems

For a causal dynamical system, in order to compute output at a given time, t_0 (or k_o for DT systems), we need to know “only” the input signal over $(-\infty, t_o]$ (or $(-\infty, k_o]$ for DT systems). This requires a lot of data/information (indeed infinite amount of data), can we summarize it with something more manageable? For example using some latent variables stored in a vector, a.k.a. *state-vector*.

State property of DT state-space models: Given the state vector $x[k_0]$ and input $u[k_0]$ at an arbitrary time k_0 , we can compute the present output, $y[k_0]$, and next state $x[k_0 + 1]$.

State property of CT state-space models: Given the initial time, t_0 and state $x(t_0)$ and input $u(t)$ for $t_0 \leq t \leq t_f$ (with t_0 & t_f arbitrary), we can compute the output $y(t)$ for $t_0 \leq t \leq t_f$ and the state $x(t)$ for $t_0 \leq t \leq t_f$.

In other words $x(t_0)$ (and $x[k]$ for DT systems) summarizes the whole input history, $t \in (-\infty, t_0)$ (or $k \in (-\infty, k_0)$) in a compact (preferably *finite-dimensional*) memory package, for the purpose of predicting the future output (and states)

Note that both definitions are not limited to LTI state-space models. Nonlinear and time-varying state-space models also are based on this definition. Indeed CT state-definition is more general and indeed also applies for DT state-space models.

Note that the choice of state-variables is not unique (and there exist infinite possible of *realizations*), however, there are some options that are preferable to others (minimal representations, canonical forms, practical benefits etc.)

When a state-space representation includes minimum possible of number of state variables, the representation is called minimal.

12.1.1 State-Space Representations of LTI, LTV, & Non-Linear Dynamical Systems

LTI Systems

State-space representation of a (causal & finite dimensional) LTI CT system is given by

$$\begin{aligned} \text{Let } x(t) \in \mathbb{R}^n, \quad y(t) \in \mathbb{R}^m, \quad u(t) \in \mathbb{R}^r, \\ \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \\ \text{where } A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times r}, \quad C \in \mathbb{R}^{m \times n}, \quad D \in \mathbb{R}^{m \times r} \end{aligned}$$

State-space representation of a (causal & finite dimensional) LTI DT system is given by

$$\begin{aligned} \text{Let } x[k] &\in \mathbb{R}^n, \quad y[k] \in \mathbb{R}^m, \quad u[k] \in \mathbb{R}^r, \\ x[k+1] &= Gx[k] + Hu[k], \\ y[k] &= Cx[k] + Du[k], \\ \text{where } G &\in \mathbb{R}^{n \times n}, \quad H \in \mathbb{R}^{n \times r}, \quad C \in \mathbb{R}^{m \times n}, \quad D \in \mathbb{R}^{m \times r} \end{aligned}$$

LTV Systems

State-space representation of a (causal & finite dimensional) LTV CT system is given by

$$\begin{aligned} \text{Let } x(t) &\in \mathbb{R}^n, \quad y(t) \in \mathbb{R}^q, \quad u(t) \in \mathbb{R}^p, \\ \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t) + D(t)u(t), \\ \text{where } A(t) &\in \mathbb{R}^{n \times n}, \quad B(t) \in \mathbb{R}^{n \times p}, \quad C(t) \in \mathbb{R}^{q \times n}, \quad D(t) \in \mathbb{R}^q \end{aligned}$$

State-space representation of a (causal & finite dimensional) LTV DT system is given by

$$\begin{aligned} \text{Let } x[n] &\in \mathbb{R}^n, \quad y[n] \in \mathbb{R}^q, \quad u[n] \in \mathbb{R}^p, \\ x[n+1] &= A[n]x[n] + B[n]u[n], \\ y[n] &= C[n]x[n] + D[n]u[n], \\ \text{where } A[n] &\in \mathbb{R}^{n \times n}, \quad B[n] \in \mathbb{R}^{n \times p}, \quad C[n] \in \mathbb{R}^{q \times n}, \quad D[n] \in \mathbb{R}^q \end{aligned}$$

Non-Linear Systems

State-space representation of a (causal & finite dimensional) non-linear CT system is given by

$$\begin{aligned} \text{Let } x(t) &\in \mathbb{R}^n, \quad y(t) \in \mathbb{R}^q, \quad u(t) \in \mathbb{R}^p, \\ \dot{x}(t) &= F(x(t), u(t)), \\ y(t) &= H(x(t), u(t)), \end{aligned}$$

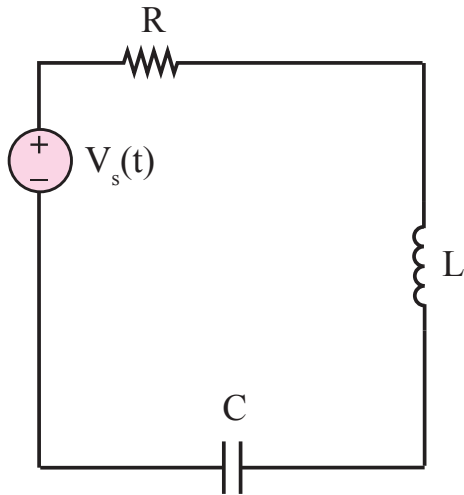
State-space representation of a (causal & finite dimensional) non-linear DT system is given by

$$\begin{aligned} \text{Let } x[n] &\in \mathbb{R}^n, \quad y[n] \in \mathbb{R}^q, \quad u[n] \in \mathbb{R}^p, \\ x[n+1] &= F(x[n], u[n]), \\ y[n] &= H(x[n], u[n]), \end{aligned}$$

12.1.2 Modeling of Example CT and DT Systems

Ex 1: Series RLC Circuit

Given than input is $u(t) = V_s(t)$ and output os $y(t) = V_C(t)$, first find an ODE description of the given dynamical circuit.



$$\begin{aligned}
 V_L + V_R + V_C &= V_s(t) \\
 L \frac{dI}{dt} + RI + V_C &= V_s(t) \\
 L \frac{d}{dt} \left(C \frac{dV_C}{dt} \right) + R \left(C \frac{dV_C}{dt} \right) + V_C &= V_s(t) \\
 LC \ddot{V}_C + RC \dot{V}_C + V_C &= V_s(t) \\
 \ddot{y} + \frac{R}{L} \dot{y} + \frac{1}{LC} y &= \frac{1}{LC} u
 \end{aligned}$$

Now, find the transfer function representation of the system for the given input-output pair.

$$\begin{aligned}
 \mathcal{L} \left\{ \ddot{y} + \frac{R}{L} \dot{y} + \frac{1}{LC} y \right\} &= \mathcal{L} \left\{ \frac{1}{LC} u \right\} \\
 s^2 Y(s) + s \frac{R}{L} Y(s) + \frac{1}{LC} Y(s) &= \frac{1}{LC} U(s) \\
 G(s) = \frac{Y(s)}{U(s)} &= \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}
 \end{aligned}$$

Find a state-space representation of the system.

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$, then

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= -\frac{1}{LC} x_1 - \frac{R}{L} x_2 + \frac{1}{LC} u
 \end{aligned}$$

If we put the equations in state-space form, we obtain

$$\begin{aligned}
 \dot{x} &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u \\
 y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x
 \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Now let, $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} V_C \\ I \end{bmatrix}$, then

$$\begin{aligned}
 \dot{z}_1 &= \frac{1}{C} z_2 \\
 \dot{z}_2 &= -\frac{1}{L} z_1 - \frac{R}{L} z_2 + \frac{1}{L} u
 \end{aligned}$$

If we put the equations in state-space form, we obtain

$$\dot{z} = \begin{bmatrix} 0 & \frac{1}{\bar{C}} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} z + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} z$$

Show that both representations will produce the same transfer function ?

Ex 2: State-Space Realization of a Third Order FIR Systems

A third order FIR (Finite impulse response) filter has the following difference equation and transfer function

$$y[k] = b_0 u[k] + b_1 u[k-1] + b_2 u[k-2] + b_3 u[k-3]$$

$$Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}) U(z)$$

From inspection it is easy to see that we need at least three memory (unit delay) blocks to construct a the realization. Fig. 12.1 also provides the block-diagram realization of a third order FIR filter. Let

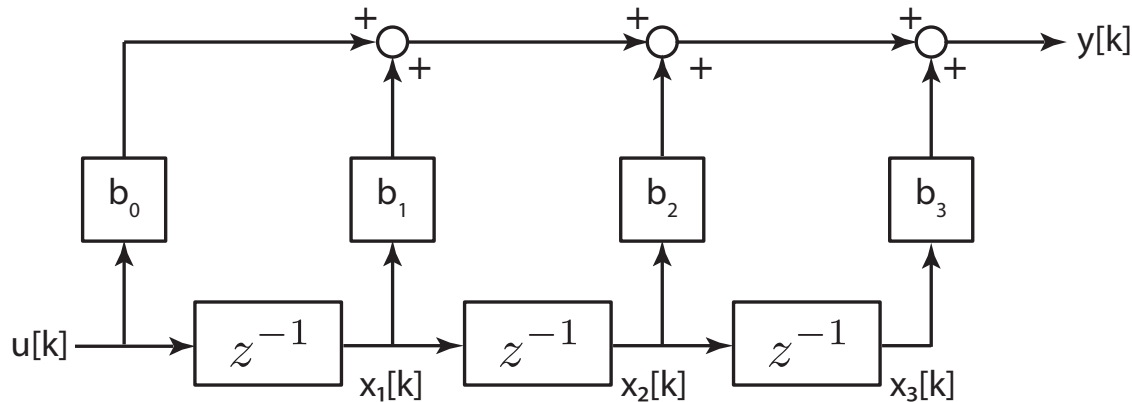


Figure 12.1: Block diagram realization of a third order FIR system

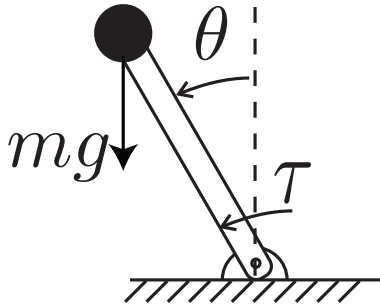
$$x[k] = \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \end{bmatrix} = \begin{bmatrix} u[k-1] \\ u[k-2] \\ u[k-3] \end{bmatrix}, \text{ then}$$

$$x[k+1] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u[k]$$

$$y = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} x[k] + [b_0] u[k]$$

Ex 3: Pendulum

Given than input is $u(t) = \tau(t)$ and output os $y(t) = \theta(t)$, find a state-space model of the pendulum dynamics.



$$ml^2\ddot{\theta} = \tau(t) + mgl \sin(\theta)$$

$$\text{Let } x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{g}{l} \sin(x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$f(x, u) = \begin{bmatrix} x_2 \\ \frac{g}{l} \sin(x_1) + \frac{1}{ml^2} u \end{bmatrix}$$

Ex 4: Predator-Prey Model Consider an island populated primarily by goats and foxes. Goats survive and breed by consuming the island's sources, while foxes survive and breed by consuming goats. To build a DT state-space model (based on behavioral observations) let's define the following state variables

$$x_1[k] : \text{\#goats}$$

$$x_2[k] : \text{\#foxes}$$

State-equation for the population of goats can be modeled as

$$x_1[k+1] = gx_1[k] - c_{fg}x_1[k]x_2[k]$$

where $g > 1$ (which models geometric growth rate of goats), and $c_{fg} > 0$. Note that $-c_{fg}x_1[k]x_2[k]$ models the negative effect of fox population on goat population. On the other hand state-equation for the population of foxes can be modeled as

$$x_2[k+1] = fx_2[k] - c_{gf}x_1[k]x_2[k]$$

where $0 < f < 1$ (which models geometric decay rate of foxes), and $c_{gf} > 0$. Note that $c_{gf}x_1[k]x_2[k]$ models the positive effect of goat population on fox population. If we combine both of the state equations we obtain

$$x[k+1] = \begin{bmatrix} gx_1[k+1] - c_{fg}x_1[k]x_2[k] \\ fx_2[k] + c_{gf}x_1[k]x_2[k] \end{bmatrix}$$

Note that as constructed this is an autonomous dynamical system (no external input)

12.1.3 Linearization of Non-linear Dynamical Systems

Consider a non-linear CT dynamical system represented in state-space form

$$\text{Let } x(t) \in \mathbb{R}^n, \quad y(t) \in \mathbb{R}^q, \quad u(t) \in \mathbb{R}^p,$$

$$\dot{x}(t) = F(x(t), u(t)),$$

$$y(t) = H(x(t), u(t)),$$

Suppose that we are interested in finding an approximate linear dynamical system model around a nominal point (equilibrium), (x_o, u_o, y_o) that solves the equation constraints, i.e.

$$0 = F(x_o, u_o),$$

$$y(t) = H(x_o, u_o),$$

Since we are interested in the dynamics around the nominal solution, we define set of “small” *perturbation* variables; $\delta x(t) = x(t) - x_o$, $\delta u(t) = u(t) - u_o$ & $\delta y(t) = y(t) - y_o$, . If we perform a (multivariate) first order Taylor series expansion, we can find the linearized state-space model as

$$\begin{aligned}\dot{\delta x} &\approx \left(\left[\frac{\partial F(x, u)}{\partial x} \right]_{(x_o, u_o)} \right) \delta x + \left(\left[\frac{\partial F(x, u)}{\partial u} \right]_{(x_o, u_o)} \right) \delta u, \\ \delta &\approx \left(\left[\frac{\partial H(x, u)}{\partial x} \right]_{(x_o, u_o)} \right) \delta x + \left(\left[\frac{\partial H(x, u)}{\partial u} \right]_{(x_o, u_o)} \right) \delta u,\end{aligned}$$

where

$$\begin{aligned}A &= \left(\left[\frac{\partial F(x, u)}{\partial x} \right]_{(x_o, u_o)} \right), \quad B = \left(\left[\frac{\partial F(x, u)}{\partial u} \right]_{(x_o, u_o)} \right) \\ C &= \left(\left[\frac{\partial H(x, u)}{\partial x} \right]_{(x_o, u_o)} \right), \quad D = \left(\left[\frac{\partial H(x, u)}{\partial u} \right]_{(x_o, u_o)} \right)\end{aligned}$$

Ex 3-2: Linearization(s) of the Pendulum Model

Compute the approximate linear model of the pendulum around $(x, u) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0 \right)$

$$A = \left(\left[\frac{\partial F(x, u)}{\partial x} \right]_{(x_o, u_o)} \right) = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix}, \quad B = \left(\left[\frac{\partial F(x, u)}{\partial u} \right]_{(x_o, u_o)} \right) = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Compute the approximate linear model of the pendulum around $(x, u) = \left(\begin{bmatrix} \pi \\ 0 \end{bmatrix}, 0 \right)$

$$A = \left(\left[\frac{\partial F(x, u)}{\partial x} \right]_{(x_o, u_o)} \right) = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix}, \quad B = \left(\left[\frac{\partial F(x, u)}{\partial u} \right]_{(x_o, u_o)} \right) = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Compute the approximate linear model of the pendulum around $(x, u) = \left(\begin{bmatrix} -\pi/2 \\ 0 \end{bmatrix}, mgl \right)$

$$A = \left(\left[\frac{\partial F(x, u)}{\partial x} \right]_{(x_o, u_o)} \right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \left(\left[\frac{\partial F(x, u)}{\partial u} \right]_{(x_o, u_o)} \right) = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Now suppose that we are interested in finding an approximate linear dynamical system model for a non-linear discrete-time dynamical system. Actually the process is exactly same (since we are still utilizing Taylor series expansion around a nominal point), except that the definition of nominal solution is different. Let's assume that a nominal point (equilibrium), (x_o, u_o, y_o) solves the equation constraints of the non-linear discrete-time dynamical system, then we know that

$$\begin{aligned}x_o &= F(x_o, u_o), \\ y_o &= H(x_o, u_o),\end{aligned}$$

Computation of state-space matrices and definition of perturbation variables are completely same for the CT case.

Ex 4-2: Linearization of the Predator-Prey Model

Let $[g, f, c_{fg}, c_{gf}] = [2, 0.5, 0.1, 0.05]$, first compute the equilibrium point of the dynamical system

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 2\bar{x}_1 - 0.1\bar{x}_1\bar{x}_2 \\ 0.5\bar{x}_2 + 0.05\bar{x}_1\bar{x}_2 \end{bmatrix} \rightarrow x_0 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Now linearize the dynamics around the equilibrium and derive the approximate DT linear state-space representation

$$A = \left(\left[\frac{\partial F(x)}{\partial x} \right]_{(x_o)} \right) = \begin{bmatrix} 2-1 & -1 \\ 0.5 & 0.5+0.5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0.5 & 1 \end{bmatrix}$$

Let's go back to non-linear CT time dynamical models. Sometimes, we want to obtain a *Linear* dynamical system model around a nominal trajectory (not a point) which still satisfies the constraints of the dynamical system. In such a case nominal solution takwa the form $(x_o(t), u_o(t), y_o(t))$ and still satisfies the equation constraints, i.e.

$$\dot{x}_o(t) = F(x_o(t), u_o(t)), \quad y_o(t) = H(x_o(t), u_o(t)), \quad \forall t \in \mathbb{R}$$

We define set of “small” *perturbation* variables in a similar way; $\delta x(t) = x(t) - x_o(t)$, $\delta u(t) = u(t) - u_o(t)$ & $\delta y(t) = y(t) - y_o(t)$. Indeed formulation of the (multivariate) first order Taylor series expansion is (almost) exactly same, and we can find the linerized state-space model as

$$\begin{aligned} \dot{\delta x}(t) &\approx \left(\left[\frac{\partial F(x, u)}{\partial x} \right]_{(x_o(t), u_o(t))} \right) \delta x + \left(\left[\frac{\partial F(x, u)}{\partial u} \right]_{(x_o(t), u_o(t))} \right) \delta u(t), \\ \delta y(t) &\approx \left(\left[\frac{\partial H(x, u)}{\partial x} \right]_{(x_o(t), u_o(t))} \right) \delta x(t) + \left(\left[\frac{\partial H(x, u)}{\partial u} \right]_{(x_o(t), u_o(t))} \right) \delta u(t), \end{aligned}$$

where

$$\begin{aligned} A(t) &= \left(\left[\frac{\partial F(x, u)}{\partial x} \right]_{(x_o(t), u_o(t))} \right), \quad B(t) = \left(\left[\frac{\partial F(x, u)}{\partial u} \right]_{(x_o(t), u_o(t))} \right) \\ C(t) &= \left(\left[\frac{\partial H(x, u)}{\partial x} \right]_{(x_o(t), u_o(t))} \right), \quad D(t) = \left(\left[\frac{\partial H(x, u)}{\partial u} \right]_{(x_o(t), u_o(t))} \right) \end{aligned}$$

Note that in such a case state-space matrices (almost surely) potentially becomes time-dependent and hence linearization leads to a LTV (Linear-Time-Varying) state-space model.

Ex 3-3: Linearization(s) of the Pendulum Model Around a Trajectory

Let's go back to the simple pendulum model. We want to “control” the system such that it rotates with a constant angular velocity and analyze the dynamics around the nominal trajectory. In that respect nominal trajectory for the state variables can take the following form

$$x_o(t) = \begin{bmatrix} \theta_o(t) \\ \dot{\theta}_o(t) \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Compute the nominal solution for the input that satisfies the nominal state-trajectories

$$u_0(t) = mgl \sin(t)$$

Compute the approximate linear model of the pendulum around $(x_o(t), u_o(t))$

$$A(t) = \left(\left[\frac{\partial F(x, u)}{\partial x} \right]_{(x_o, u_o)} \right) = \begin{bmatrix} 0 & 1 \\ g/l \cos(t) & 0 \end{bmatrix}, \quad B = \left(\left[\frac{\partial F(x, u)}{\partial u} \right]_{(x_o, u_o)} \right) = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

We can see that system matrix $A(t)$ is now time-dependent thus the approximate system is an LTV system. Indeed we can see that $A(t)$ is a periodic function, this is a special class of LTV system and belongs to the group of LTP (Linear Time Periodic) systems.