EE502 - Linear Systems Theory II	Spring 2032

Lecture 8

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8.1 Internal (Lyapunov) Stability

In internal stability, we are interested in un-driven (zero-input response) part of the dynamical system and solely focus on state evolution dynamics, i.e. autonomous part of the dynamical system. CT and DT non-linear autonomous systems can simply be expressed by

$$x \in \mathbb{D} \subset \mathbb{R}^n$$
$$\dot{x} = F(x, t)$$
$$x[k+1] = F(x[k]], k)$$

For non-linear systems, in order to define and analyze the stability of a dynamical system, we need to define equilibrium points (or nominal solutions), since we will technically analyze the stability around such points. An equilibrium point for CT and DT non-linear systems are defined as

$$x_e \in \mathbb{D}$$
$$CT : 0 = F(x_e, t) \ \forall t > t_0$$
$$DT : x_e = F(x_e, k) \ \forall k > k_0$$

Obviously if a dynamical system at time t_0 (or k_0) starts from an an equilibrium point, $x(t_0) = x_e$ (or $x[k_o] = x_e$), it will remain on the equilibrium point $\forall \ t \geq t_0$ (or $\forall \ k \geq k_0$). A non-linear system can have a single equilibrium point, $x_e \in \mathcal{E}$, $\text{card}(\mathcal{E}) = 1$, have multiple finite number of equilibria, $x_e \in \mathcal{E}$, $\text{card}(\mathcal{E}) = n_e < \infty$, or infinite number of equilibrium points, $x_e \in \mathcal{E}$, $\text{card}(\mathcal{E}) = \infty$.

Ex 8.1 *Show that for an LTI dynamical system, set of equilibrium points define a vector space. Then characterize this vector space.*

Definition: Without loss of generality, let's assume that the equilibrium point that is point of interest is located at the origin $x_e = 0$.

1. The system is called *stable in the sense of Lyapunov (s.i.s.L)* around $x_e = 0$ if it satisfies

$$\forall \epsilon > 0, \ \exists \delta_L(\epsilon) \ s.t. \ if \ ||x(t_0)|| < \delta_L \ \rightarrow ||x(t)|| < \epsilon \ \forall t \geq t_0$$

2. The system is called *asymptotically stable* around around $x_e = 0$ if it is *stable in the sense of Lyapunov (s.i.s.L)* around $x_e = 0$ and *locally attractive*, i.e.

$$\exists \delta_a \ s.t. \ if \ ||x(t_0)|| < \delta_a \ \rightarrow \lim_{t \to \infty} ||x(t)|| = 0$$

3. The system is called *exponentially stable* around around $x_e = 0$ if it is *asymptotically stable* around $x_e = 0$ and satisfies

$$\exists \delta_e > 0, \, \alpha > 0, \, \sigma > 0 \ s.t. \ if \ ||x(t_0)|| < \delta_e \ \rightarrow ||x(t)|| \leq \alpha ||x(t_0)|| e^{-\sigma t} \ \forall t \geq t_0$$

Remark: If above stability conditions are satisfied $\forall\, t_0 \in \mathbb{R}$, then we call the system around the equilibrium *uniformly s.i.s.L*, *uniformly asymptotically stable*, and *uniformly exponentially stable* respectively. The difference between uniform and non-uniform stability is (slightly) important for only time-varying non-linear systems. Thus we will not use uniform stability definition in this course.

Remark: Note that as you can see the internal stability definitions, *s.i.s.L*, *asymptotic stability*, and *exponentially stability*, are all local stability definitions defined in the neighborhood of x_e. If a stability definition holds for all initial conditions, i.e. $x(t_0) \in \mathbb{D}$, then we use the terms *globally s.i.s.L*, *globally asymptotically stable*, and *globally exponentially stable*.

Ex 8.2 *Consider the pendulum dynamics*

$$\dot{x} = \begin{bmatrix} x_2 \\ -sin(x_1) \end{bmatrix} \;, \text{ where } \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

Analyze the stability of the dynamics around $x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ using Lyapunov's stability definitions.

Solution: We know that a mechanical pendulum is an energy-conserving system since there is no dissipative or active element. In that respect, at any time instant, we can write total energy as

$$E(x) = \frac{1}{2}x_2^2 + 1 - \cos x_2 \;, \text{ note } E(0) = 0$$

Let $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$ and $||x(0)||_2 < \epsilon$, $\epsilon \in \mathbb{R}^+$, then we know that $(x_1(0)^2 + x_2(0)^2) < \epsilon^2$. Due to the conservation of energy, we also know that

$$\begin{aligned} & E(x(t)) = E(x(0)) \\ & \frac{1}{2}x_2(t)^2 + 1 - \cos x_2(t) = \frac{1}{2}x_2(0)^2 + 1 - \cos x_2(0) < \frac{1}{2}x_2(0)^2 + \frac{1}{2}x_1(0)^2 < \frac{\epsilon^2}{2} \\ & \frac{1}{2}x_2(0)^2 + 1 - \cos x_2(0) < \frac{\epsilon^2}{2} \;\rightarrow\; x_2(0)^2 < \epsilon^2 + 4 \end{aligned}$$

We already know that $x_1 = \theta$ and it is bounded since $\theta \in \mathbb{S}$, so

$$x_1(t)^2 \leq 1 \;\rightarrow\; (x_1(t)^2 + x_2(t)^2) < \epsilon^2 + 5 \;\rightarrow\; ||x(t)||_2 < \sqrt{\epsilon^2 + 5} = \delta(\epsilon)$$

This shows that the dynamics around the equilibrium is *stable in the sense of Lyapunov* (globally). Note that since $E(t) = E(0)\,\forall t > 0$, $||x(t)||_2 \neq 0 \forall t > 0$, thus the system around the equilibrium is not asymptotically stable (local or global), and hence not exponentially stable.

8.1.1 Internal Stability of LTI Systems

Lyapunov's stability definitions may not be very useful for analyzing the stability of non-linear systems, however we can easily derive necessary and sufficient conditions for stability. One should also note the in LTI systems (and linear systems in general) we are interested in the stability of the origin $x_e = 0$. Autonomous CT and DT LTI systems are expressed by the following matrix differential and difference equations

$$\begin{aligned} \dot{x} &= Ax \\ x[k+1] &= Ax[k] \end{aligned}$$

and we know the analytical solutions to the zero-input responses have the following forms

$$x(t) = e^{At}x_0$$
$$x[k] = A^k x_0$$

It is easier to analyze the internal stability using Jordan decomposition of the system matrix A,

$$x(t) = Ge^{Jt}G^{-1}x_0 \;\rightarrow\; \left[G^{-1}x(t)\right] = e^{Jt}\left[G^{-1}x_0\right] \;\rightarrow\; \alpha(t) = e^{Jt}\alpha_0$$
$$x[k] = GJ^kG^{-1}x_0 \;\rightarrow\; \alpha[k] = J^k\alpha_0$$

Note that since G and G^{-1} finite and invertible matrices, we know that

$$||x|| = 0 \iff ||G^{-1}x|| = 0 \iff x = 0$$
$$||x|| < M_1 < \infty \iff ||G^{-1}x|| < M_2 < \infty \text{ where} M_1, M_2 \in \mathbb{R}$$
$$||x|| \rightarrow \infty \iff ||G^{-1}x|| \rightarrow \infty$$

We know that e^{Jt} and J^k has the following block diagonal form

$$e^{Jt} = \begin{bmatrix} e^{J_1t} & & & & \\ & e^{J_2t} & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & e^{J_nt} \end{bmatrix}$$

$$J^k = \begin{bmatrix} J_1^k & & & & \\ & J_2^k & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & J_n^k \end{bmatrix}$$

where an individual block associated with a Jordan block of A has the following form

$$e^{J_it} = \begin{bmatrix} e^{\lambda_it} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda_it} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda_it} & \frac{t^{n-1}}{(n-1)!}e^{\lambda_it} \\ 0 & e^{\lambda_it} & te^{\lambda_it} & \frac{t^2}{2!}e^{\lambda_it} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda_it} \\ \vdots & & \ddots & & & \vdots \\ & & & e^{\lambda_it} & te^{\lambda_it} & \frac{t^2}{2!}e^{\lambda_it} \\ 0 & & \cdots & & e^{\lambda_it} & te^{\lambda_it} \\ 0 & & \cdots & & 0 & e^{\lambda_it} \end{bmatrix}$$

$$J_i^k = \begin{bmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} & \cdots & \binom{k}{n-2}\lambda_i^{k-n+2} & \binom{k}{n-1}\lambda_i^{k-n+1} \\ 0 & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} & \cdots & \binom{k}{n-2}\lambda_i^{k-n+2} \\ \vdots & & \ddots & & & \vdots \\ & & & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} \\ 0 & & \cdots & & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} \\ 0 & & \cdots & & 0 & \lambda_i^k \end{bmatrix}$$

Based on this decomposition we can derive the stability conditions

1. • A CT LTI system is *asympotitically & exponentially stable*
   $\iff \lim_{t\to\infty} e^{J_i t} = 0 \ \forall i \iff \text{Re}\{\lambda_i\} < 0 \ \forall i$
   • A DT LTI system is *asympotitically & exponentially stable*
   $\iff \lim_{k\to\infty} J_i^k = 0 \ \forall i \iff |\lambda_i| < 1 \ \forall i$
2. • A CT LTI system is *s.i.s.L*
   $\iff \lim_{t\to\infty} e^{J_i t} = M_i < \infty \ \forall i \iff \{\text{Re}\{\lambda_i\} < 0 \text{ or } \{\text{Re}\{\lambda_i\} = 0 \text{ and } J_i \in \mathbb{R}\}\} \ \forall i$
   • A DT LTI system is *s.i.s.L*
   $\iff \lim_{k\to\infty} J_i^k = M_i < \infty \ \forall i \iff \{|\lambda_i| < 1 \text{ or } \{|\lambda_i| = 1 \text{ and } J_i \in \mathbb{R}\}\} \ \forall i$
3. • A CT LTI system is *unstable*
   $\iff \exists i \ \lim_{t\to\infty} e^{J_i t} = \infty \iff \{\text{Re}\{\lambda_i\} > 0 \text{ or } \{\text{Re}\{\lambda_i\} = 0 \text{ and } J_i \in \mathbb{R}^{n\times n}, n > 1\}\} \ \forall i$
   • A DT LTI system is *unstable*
   $\iff \exists i \ \lim_{k\to\infty} J_i^k = \infty \iff \exists i\{|\lambda_i| > 1 \text{ or } \{|\lambda_i| = 1 \text{ and } J_i \in \mathbb{R}^{n\times n}, n > 1\}\}$