EE502 - Linear Systems Theory II **Spring 2023**

Lecture 15

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15.1 Poles & Zeros of MIMO Systems

15.1.1 Poles & Zeros of SISO Systems

Let $G(s)$ (or $G(z)$ in DT case) and $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ are the transfer function and a *minimal* state-space representation of a SISO LTI system.

p_0 is a pole of the system if

- $\lim_{s\to p_0} G(s) = \infty$
- p_0 is an eigenvalue of A

whereas z_0 is a pole of the system if

- $\lim_{s\to z_0} G(s) = 0$
- steady-state part of the zero state response to $u(t) = e^{z_0 t}$

$$y_{ss}(t) = C(z_0 I - A)^{-1} B e^{z_0 t} = 0$$

15.1.2 Poles of MISO Systems

Unlike MIMO zeros, definition and derivation of MIMO poles is much more straightforward

Let $G(s)$ (or $G(z)$ in DT case) and $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ are the transfer function matrix and a *minimal* state-space representation of a MIMO LTI system.

p_0 is a pole of the system if

- $\exists (i,j)$ s.t. $\lim_{s\to p_0} G_{ij}(s) = \infty$
- $\lim_{s\to p_0} ||G(s)|| = \infty$
- p_0 is an eigenvalue of A

in other words in the context of transfer function matrix p_0 is a pole of the system if it is a pole of any entry of $G(s)$. Understanding and derivation of the multiplicities of a pole based on transfer function matrix is a little bit tricky and not very intuitive for the context and scope of the class. Thus, we can simply state that we can find the algebraic and geometric multiplicity of a pole based on Jordan decomposition of A provided that state-space representation is minimal.

Ex 15.1

$$G(s) = \begin{bmatrix} \frac{s+1}{(s+2)^2} & 0 & 0 \\ 0 & \frac{s}{(s+1)(s+2)} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can clearly see that $p_1 = -2$ and $p_2 = -2$ are the poles of the system. Note that $z = -1$ also a zero of the firs entry of the transfer function matrix (and indeed it is a zero of the system). This states that a MIMO system can have a pole and a zero at the same location.

15.1.3 Zeros of MISO Systems

If we generalised the definition of zero in SISO systems to the MIMO systems using the same logic with MIMO poles, we can see that resultant definition would not be useful and somewhat unsatisfactory. Let's consider the transfer function matrix in the previous example. In this example all non-diagonal elements are zero and hence if we define a zero such that $\exists(i,j)$ s.t. $\lim_{s\to z_0} G_{ij}(s) = 0$, then any $z_0 \in \mathbb{C}$ would be a zero of the system. In that context, we need more useful and satisfactory definition(s) for MIMO zeros.

Definition: Let $z_0 \neq p_i, \forall p_i \in \mathcal{P}$, where $\mathcal{P}$ is teh set of all poles of the MIMO system $G(s)$ with m inputs and q outputs. z_0 is a zero of the system if $H(s)$ drops rank at $s = z_0$.

Naturally let's assume that $G(s)$ is full rank, then we can also observe that

- if $m \leq p$, $G(s)$ drops rank at $s = z_0 \iff \exists u_0 \in \mathbb{R}^m \,\&\, u_0 \neq 0$ such that $G(z_0)u_0 = 0$
- if $q \leq m$, $G(s)$ drops rank at $s = z_0 \iff \exists u_0^T \in \mathbb{R}^q \,\&\, u_0 \neq 0$ such that $u_0^T G(z_0) = 0$

These definitions provide us the zeros that are different from poles of the system.

Ex 15.2

$$G(s) = \begin{bmatrix} \frac{s+1}{(s+2)^2} & \frac{s+3}{s+4} \\ 0 & \frac{s}{s+2} \end{bmatrix}$$

In this example we can see that $G(s)$ is full rank for *most* $s \in \mathbb{C}$. The matrix drops rank at locations $z_0 \in \{0, -1\}$, however rank is preserved for -3 even if $G_{12}(-3) = 0$. Thus zeros of the system are $\{0, -1\}$.

The major limitation of this definition is that it is limited to zeros that are different from the poles of the system. So we need a more general definition of zeros

Definition Let's assume that $G(s)$ is a full rank (for *most* $s \in mathbbC$ transfer function matrix with m inputs and q outputs. $z_0 \in \mathbb{C}$ is a zero of the system if

- if $m \leq p$, $\exists u(s) \neq 0$ such that $\lim_{s\to z_0}[G(s)u(s)] = 0$
- if $p \leq m$, $\exists u(s) \neq 0$ such that $\lim_{s\to z_0}[u(s)^T G(s)] = 0$

Ex 15.3

$$G(s) = \begin{bmatrix} 1 & \frac{s+1}{s+2} \\ 0 & 1 \end{bmatrix}$$

Let $u(s) = \begin{bmatrix} 1 \\ s+2 \end{bmatrix}$, then

$$\lim_{s \to -2} [G(s)u(s)] = \lim_{s \to -2} \begin{bmatrix} s+2 \\ s+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

this $s = -2$ is both a pole and zero location for $G(s)$

Corollary: Let $G(s)$ be square transfer function matrix and it is invertible for *most* $s \in \mathbb{C}$, then zeros of the systems are the poles of $G(s)^{-1}$.

Let's analyse the previous example

$$G(s)^{-1} = \begin{bmatrix} 1 & -\frac{s+1}{s+2} \\ 0 & 1 \end{bmatrix}$$

$$\lim_{s \to -2} ||G(s)|| = \infty \Rightarrow z_0 = -2 \text{ is zero of } G(s)$$