EE502 - Linear Systems Theory	**Spring 2013**

Lecture 9

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9.1 Externall Input-Output Stability

9.1.1 Signal Norms

A continuous time bilateral signal is a mapping defined by $f : \mathbb{R} \mapsto \mathbb{R}^n$ (or for unilateral case $f : \mathbb{R}^{\geq 0} \mapsto \mathbb{R}^n$), whereas discrete time bilateral signal is a mapping defined by $g : \mathbb{Z} \mapsto \mathbb{R}$ (or for unilateral case $g : \mathbb{Z}^{\geq 0} \mapsto \mathbb{R}$). Graphical Examples

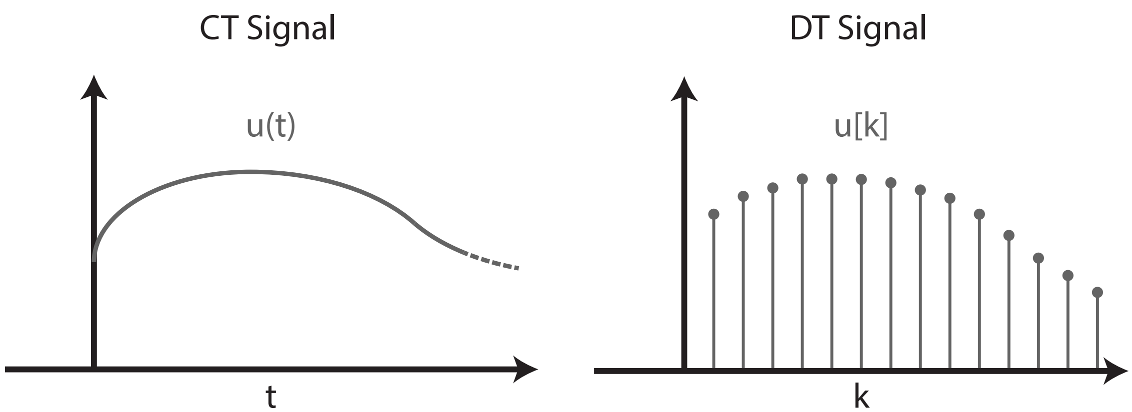

Figure 9.1: CT vs DT Signal

∞-norm

In the chacarterization and analysis of input-output stability of linear dynamical systems, most commonly used norm concept is the ∞-norm which is technically e measure of peak magnitude over time. For scalar signals ∞-norm is defined as

$$\begin{aligned} ||f||_\infty &\triangleq \sup_k |f(k)| \quad \text{(DT)} \\ &\triangleq \sup_t |f(t)| \quad \text{(CT)} \end{aligned}$$

The "sup" denotes the *supremum* or *least upper bound*, the value that is approached arbitrarily closely but never (i.e., at any finite time) exceeded. Note that this is the natural standard ∞-norm definition for finite-dimensional vectors to the infinite dimensional case, i.e. DT and CtT signals. Let's remember the ∞-norm of an n-dimensional vector,

$$||v||_\infty \triangleq \max_{i \in [1,n]} |v_i| \ , \ \text{where} v \in \mathbb{R}^n,$$

A scalar signal, $f(.)$ is called *bounded* if $||f||_\infty = M < \infty$ and that is the fundamental signal measure adopted in BIBO stability.

For multi-variate signals, we add a ned "dimension" in addition to the time dimension, thus in such a case we define ∞-norm as

$$\begin{aligned} ||f||_\infty &\triangleq \sup_k ||f(k)||_\infty \quad \text{(DT)} \\ &\triangleq \sup_t ||f(t)||_\infty \quad \text{(CT)} \end{aligned}$$

The space of all signals with finite ∞-norm are generally denoted by ℓ_∞ and $\mathcal{L}_\infty$ for DT and CT signals respectively. For multi-variate case, the dimension of the vector may be explicitly added as ℓ_∞^n and $\mathcal{L}_\infty^n$.

∞-norms of some example CT and DT uni-lateral signals (i.e. $t \geq 0$ and $k \geq 0$)

$$\begin{aligned} f(t) = 1\,,\ ||f||_\infty = 1 \quad &- \quad g[k] = 1\,,\ ||g||_\infty = 1 \\ f(t) = t\,,\ ||f||_\infty = \infty \quad &- \quad g[k] = k\,,\ ||g||_\infty = \infty \\ f(t) = e^t\,,\ ||f||_\infty = \infty \quad &- \quad g[k] = 2^k\,,\ ||g||_\infty = \infty \\ f(t) = 1 - e^t\,,\ ||f||_\infty = 1 \quad &- \quad g[k] = 1 - 0.5^k\,,\ ||g||_\infty = 1 \\ f(t) = \delta(t)\,,\ ||f||_\infty = \infty \quad &- \quad g[k] = \delta[k]\,,\ ||g||_\infty = 1 \end{aligned}$$

2-norm

2-norm of a signal is the most fundamental measure of signal in optimal control theory and it can be considered as the square root of the "energy" of the signal. For scalar signals 2-norm is defined as

$$\begin{aligned} ||f||_2 &\triangleq \left[\sum_k (f[k])^2\right]^{\frac{1}{2}} \quad \text{(DT)} \\ &\triangleq \left[\int_t (f[t])^2 dt\right]^{\frac{1}{2}} \quad \text{(CT)} \end{aligned}$$

For multivariate signals, we adopt the inner product and obtain

$$\begin{aligned} ||f||_2 &\triangleq \left[\sum_k (f[k])^T f[k]\right]^{\frac{1}{2}} = \left[\sum_k ||f[k]||_2^2\right]^{\frac{1}{2}} \quad \text{(DT)} \\ &\triangleq \left[\int_t (f(t))^T f(t) dt\right] = \left[\sum_k ||f(t)||_2^2\right]^{\frac{1}{2}} \quad \text{(CT)} \end{aligned}$$