EE502 - Linear Systems Theory II **Spring 2023**

Lecture 11

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11.1 Observability in CT-LTI Systems

In the context of observability of dynamical systems, it turns out that it is more convenient to think in terms of "un-observabile states" and then connect it to the concept of observability and fully observable systems. as reflected in the following definitions.

Definition: For an LTI contnous-time state-space representation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

A state x_u is said to be unobservable over $t \in [0, T)$, if with $x(0) = x_u$ and $\forall\, u(t)$ we get the same $y(t)$ as we would with $x(0) = 0$.

The set, $\bar{\mathcal{O}}_T$, of all unobservable states over $t \in [0, T)$ forms a vector space, $\bar{\mathcal{O}}_T \subset \mathbb{R}^n$, and the system is called fully observable over $t \in [0, T)$, if $\dim[\bar{\mathcal{O}}_T] = 0$.

Note for linear dynamical systems observability of a state and system is independent from $u(t)$, in that respect we will only analyze zero-input response of the system in our derivations.

Theorem: $x_u \in \bar{\mathcal{O}}_T \iff x_u \in \bar{\mathcal{O}}_\tau, \forall \tau > 0 \iff x_u \in \mathcal{N}[\mathbf{O}]$, where

$$\mathbf{O} = \begin{bmatrix} C \\ \hdashline CA \\ \hdashline CA^2 \\ \hdashline \vdots \\ \hdashline CA^{n-1} \end{bmatrix}$$

Let's first show that $x_u \in \mathcal{N}[\mathbf{O}] \iff x_u \in \bar{\mathcal{O}}_\tau, \forall \tau > 0$. Let $x_u \in \mathcal{N}[\mathbf{O}]$, then

$$\mathbf{O}x_u = 0 \rightarrow \begin{array}{c} Cx_u = 0 \\ \hdashline CAx_u = 0 \\ \hdashline CA^2x_u = 0 \\ \hdashline \vdots \\ \hdashline CA^{n-1}x_u = 0 \end{array}$$

Moreover, by Cayley-Hamilton theorem, we can also conclude that $CA^l x_u = 0,\ \forall l \in \mathbf{Z}^+$. Now let's analyze the zero-input response of the system with $x(0) = x_u$

$$x(\tau) = Ce^{A\tau}x_u = 0 \Rightarrow x_u \in \bar{\mathcal{O}}_\tau$$

and indeed it is true for all $\tau \in \mathbb{R}$. Now let's show that $x_u \in \bar{\mathcal{O}}_T \Rightarrow x_u \in \mathcal{N}[\mathbf{O}]$

$$x(t) = Ce^{At}x_u = 0, \forall t \in [0, \tau], \forall \tau \in \mathbb{R} \Rightarrow x_u \in \bar{\mathcal{O}}_\tau$$

Now let's show that $x_u \in \bar{\mathcal{O}_T} \Rightarrow x_u \in \mathcal{N}[\mathbf{O}]$. If $x_u \in \bar{\mathcal{O}_T}$, then

$$x(t) = Ce^{At}x_u = 0, \forall t \in [0,T] \Rightarrow \begin{array}{c} x(0) = 0 \Rightarrow Cx_u = 0 \\ \left[\frac{d}{dt}x(t)\right]_{t=0} = 0 \Rightarrow CAx_u = 0 \\ \left[\frac{d^2}{dt^2}x(t)\right]_{t=0} = 0 \Rightarrow CA^2x_u = 0 \\ \vdots \\ \left[\frac{d^{n-1}}{dt^{n-1}}x(t)\right]_{t=0} = 0 \Rightarrow CA^{n-1}x_u = 0 \end{array} \Rightarrow \mathbf{O}x_u = 0 \iff x_u \in \mathcal{N}[\mathbf{O}]$$

Similar to the reachability, we show that for CT-LTI systems observability and unobservable (and observable) subspace are independent of time.

11.1.1 CT Observability Grammian

For a CT-LTI system, observability Grammian is defined as

$$\mathbf{Q}(t) = \int_0^t e^{A^T(t-\tau)}C^TCe^{A(t-\tau)}d\tau$$

Theorem: $\mathcal{N}[\mathbf{Q}(t)] = \mathcal{N}[\mathbf{O}]\ \forall t > 0$.

Proof: Let's first show that $\mathcal{N}[\mathbf{O}] \subset \mathcal{N}[\mathbf{Q}(t)]\ \forall t > 0$. If $x_u \in \mathcal{N}[\mathbf{O}]$, then we know that $CA^lx_u = 0,\ \forall l \in \mathbb{Z}^{\geq 0}$. Let's anlayze if x_u is in the null-space of $\mathbf{Q}(t)$

$$\mathbf{Q}(t)x_u = \int_0^t e^{A^T(t-\tau)}C^TCe^{A(t-\tau)}x_u d\tau = 0 \Rightarrow x_u \in \mathcal{N}[\mathbf{Q}(t)]\ \forall t > 0$$

Now let's show that $\mathcal{N}[\mathbf{Q}(t)] \subset \mathcal{N}[\mathbf{O}],\ \forall t > 0$. Let $x_u \in \mathcal{N}[\mathbf{Q}(t)]$, then

$$\mathbf{Q}(t)x_u = 0 \Rightarrow x_u^T\mathbf{Q}(t)x_u \iff \int_0^t x_u^Te^{A^T(t-\tau)}C^TCe^{A(t-\tau)}x_u d\tau = 0 \iff Ce^{A(t-\tau)}x_u = 0\,\forall\tau \in [0,t]$$

Then we know that

$$\begin{aligned} [Ce^{A\eta}x_u]_{\eta=0} = 0 &\Rightarrow Cx_u = 0 \\ \frac{d}{d\eta}[Ce^{A\eta}x_u]_{\eta=0} = 0 &\rightarrow CAx_u = 0 \\ \frac{d^2}{d\eta^2}[Ce^{A\eta}x_u]_{\eta=0} = 0 &\Rightarrow CA^2x_u = 0 \\ &\vdots \\ \frac{d^{n-1}}{d\eta^{n-1}}[Ce^{A\eta}x_u]_{\eta=0} = 0 &\Rightarrow CA^{n-1}x_u = 0 \\ &\Rightarrow \mathbf{O}x_u = 0 \Rightarrow x_u \in \mathcal{N}[\mathbf{O}] \Rightarrow \mathcal{N}[\mathbf{Q}(t)] \subset \mathcal{N}[\mathbf{O}]\,\forall t > 0 \end{aligned}$$

Ex 11.1 *Show that*

if $\dot{x} = Ax$ is asymptotically stable, then observability Grammian at $t \to \infty$, $Q := \mathbf{Q}_\infty$, satisfies the following Lyapunov equation

$$AQ + QA^T = -C^T C$$

, and this Lyapunov equation has a (unique) positive-definite solution for Q, if and only if, (A, C) is fully observable.

11.1.2 Futher Results in CT-LTI Observability

In view of duality, we can use our reachability results to immediately derive various conclusions, tests, standard and canonical forms, etc., for observable and unobservable systems. The reader is highly encouraged to derive the following results, theorems, and claims by referring to the dual results in the reachability lecture.

Result 1: The unobservable sub-space, $\mathcal{N}[\mathbf{O}]$ is A invariant, i.e. $x \in \mathcal{N}[\mathbf{O}] \;\Rightarrow\; Ax \in \mathcal{N}[\mathbf{O}]$.

Result 2: (A, C) pair is unobservable $\iff$ $Cv = 0$ for some right eigenvector of A, i.e. $Av = \lambda v \iff$

$$\text{rank}\left[\begin{array}{c} \lambda I - A \\ \hline C \end{array}\right] = n,\ \forall \lambda \in \mathcal{C}$$

Result 3: Observability is invariant under state/similarity transformation.

11.1.3 Standard Form of Unobservable Systems

Let $\dot{x} = Ax + Bu$, $y = Cx + Du$ be an unobservable system; then it can be convenient and practical to choose coordinates (via similarity transformation) to highlight unobservable and observable "spaces". Let

$$\begin{aligned} &\dim(\mathbf{O}) = \bar{o} \ \&\ T = \left[\begin{array}{c|c} T_2 & T_{\bar{o}} \end{array}\right] \text{ where } T_{\bar{o}} \in \mathbb{R}^{n \times \bar{o}} \ , T_2 \in \mathbb{R}^{n \times (n-\bar{o})} \ , \\ &T^{-1}AT = \bar{A} \ , \ CT = \bar{C} \end{aligned}$$

Let's choose a $T_{\bar{o}}$ such that $\text{Ra}(T_{\bar{o}}) = \mathcal{N}(\mathbf{O}) = \bar{\mathcal{O}}$, in other words columns of $T_{\bar{o}}$ span the null-space of the observability matrix (i.e. non-observable subspace). Similarly, let's choose T_2 such that columns of T are linearly independent. Let's analyze the similarity transformation.

$$\begin{aligned} AT = A\left[\begin{array}{c|c} T_2 & T_{\bar{o}} \end{array}\right] = T\bar{A} = \left[\begin{array}{c|c} T_2 & T_{\bar{o}} \end{array}\right]\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \\ AT_{\bar{o}} = T_2 A_{12} + T_{\bar{o}} A_{22} \end{aligned}$$

Note that the unobservable sub-space is A invariant $AT_{\bar{o}}$ must remain in $\text{Ra}(T_{\bar{o}}) = \mathcal{N}(\mathbf{O}) = \bar{\mathcal{O}}$, since T_2 is composed of linearly independent columns, in order $T_2 A_{12} + T_{\bar{o}} A_{22}$ remain in $\text{Ra}(T_{\bar{o}})$, $A_{12} = \mathbf{0}$. Now let's analyze the effect of transformation on the output matrix.

$$\begin{aligned} \bar{C} = CT = C\left[\begin{array}{c|c} T_2 & T_{\bar{o}} \end{array}\right] = \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right] \\ C_2 = CT_{\bar{o}} = 0 \end{aligned}$$

thus a standard unobservable form takes the form

$$\begin{aligned} \dot{\bar{x}} &= \left[\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & A_{22} \end{array}\right]\bar{x} + \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right]u \\ y &= \left[\begin{array}{c|c} C_1 & 0 \end{array}\right]x + Du \end{aligned}$$

11.2 Observability in DT-LTI Systems

Definition: For LTI a discrete-time state-space representation

$$x[k+1] = A[k]x + B[k]u$$
$$y[k] = C[k]x + D[k]u$$

A state x_u is said to be m-step unobservable if with $x[0] = x_u$ and $\forall\, u[k]\, k \in [0, m)$ we get the same $y[k]\, k \in [0, m)$ as we would with $x[0] = 0$.

The set, $\bar{\mathcal{O}_m}$, of all m-step unobservale states forms a vecotor space, $\bar{\mathcal{O}_m} \subset \mathbb{R}^n$.

Similar to the CT-LTI case, we will only analyze zero-input response of the system in our observability-related derivations.

If $x_u \in \bar{\mathcal{O}_m}$, then

$$y[k] = CA^k x_u = 0\ ,\ \forall k \in [0, m)\,, \Rightarrow \begin{array}{c} Cx_u = 0 \\ CAx_u = 0 \\ \vdots \\ CA^{m-1}x_u = 0 \end{array} \Rightarrow \begin{bmatrix} C \\ CAx_u \\ \vdots \\ CA^{m-1}x_u \end{bmatrix} x_u = \mathbf{O}_m x_u = 0$$

This implies that x_u is m-step unobservable if and only if $x_u \in \mathcal{N}[\mathbf{O}_m]$

Theorem: $\mathcal{N}[\mathbf{O}_m] \subset \mathcal{N}[\mathbf{O}_n] = \mathcal{N}[\mathbf{O}_i]$ for $m < n < l$.

Corollary: If x_u is n-step observable, then it is i-step observable $\forall k \in \mathbb{Z}^+$.

In this context for DT systems, unobservable space is defined as $\bar{O} = \mathcal{N}[\mathbf{O}_n]$ and the system is fully observable if and only if

$$\bar{O} = 0 \iff \dim(\mathcal{N}[\mathbf{O}_n]) = 0 \iff \text{rank}[\mathbf{O}_n] = n$$

11.2.1 DT Observability Grammian

m-step observability Grammian is defined as

$$Q_m = \mathbf{O}_m^T \mathbf{O} = \sum_{i=0}^{m-1} (A^i)^T C^T C A^i$$

Theorem: $\mathcal{N}[\mathbf{O}_m] = \mathcal{N}[\mathbf{Q}_m]$.

Let $x[k+1] = Ax[k]$ be asymptotically stable, then $Q = \mathbf{Q}_\infty$ is well defined and satisfies the Lyapunov equation below

$$A^T Q + QA = -C^T C$$

and this Lyapunov equation has a (unique) positive-definite solution for Q, if and only if, (A, C) is fully observable.

Corollary: Total observability theorem

- System is fully observable $\iff$ (A, C) pair is observable $\iff$

- rank$[\mathbf{O}] = n$, where $\mathbf{O} = \mathbf{O}_n \iff$
- $Cv \neq 0\,,\, \forall v \in \{v | v \neq 0\,,\, Av = \lambda v,\ \lambda \in \mathcal{C}\} \iff$
- rank $\left[\begin{array}{c} \lambda I - A \\ \hline C \end{array}\right] = n,\ \forall \lambda \in \mathcal{C}$