EE302 - Feedback Systems **Spring 2019**

Lecture 14

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14.1 Frequency Response Techniques in Control Systems

Let's assume $u(t)$, $y(t)$, and $G(t)$ represents the input, output, and transfer function representation of an input-output continuous time system.

In order to characterize frequency response of a dynamical system, the test signal is

$$u(t) = e^{j\omega t}$$

which is an artificial complex periodic signal with a frequency of ω. The Laplace transform of $u(t)$ takes the form

$$U(s) = \mathcal{L}\{e^{j\omega t}\} = \frac{1}{s - j\omega}$$

Response of the system in s-domain is given by

$$Y(s) = G(s)U(s) = G(s)\frac{1}{s - j\omega}$$

Assuming that $G(s)$ is a rational transfer function we can perform a partial fraction expansion

$$\begin{aligned} Y(s) &= \frac{a}{s - j\omega} + [\text{terms due to the poles of } G(s)] \\ a &= \lim_{s \to j\omega} [(s - j\omega)Y(s)] = G(j\omega) \\ Y(s) &= \frac{G(j\omega)}{s - j\omega} + [\text{terms due to the poles of } G(s)] \end{aligned}$$

Taking the inverse Laplace transform yields

$$y(t) = G(j\omega)e^{j\omega t} + \mathcal{L}^{-1}[\text{terms due to the poles of } G(s)]$$

If we assume that the system is "stable" or system is a part of closed loop system and closed loop behavior is stable then at steady state we have

$$\begin{aligned} y_{ss}(t) &= G(j\omega)e^{j\omega t} \\ &= |G(j\omega)|e^{i\omega t + \angle[G(j\omega)]} \\ &= Me^{i\omega t + \theta} \end{aligned}$$

In other words complex periodic signal is scaled and phase shifted based on the following operators

$$\begin{aligned} M &= |G(j\omega)| \\ \theta &= \angle G(j\omega) \end{aligned}$$

It is very easy to show that for a general real time domain signal $u(t) = \sin(\omega t + \phi)$, the output $y(t)$ at steady state is computed via

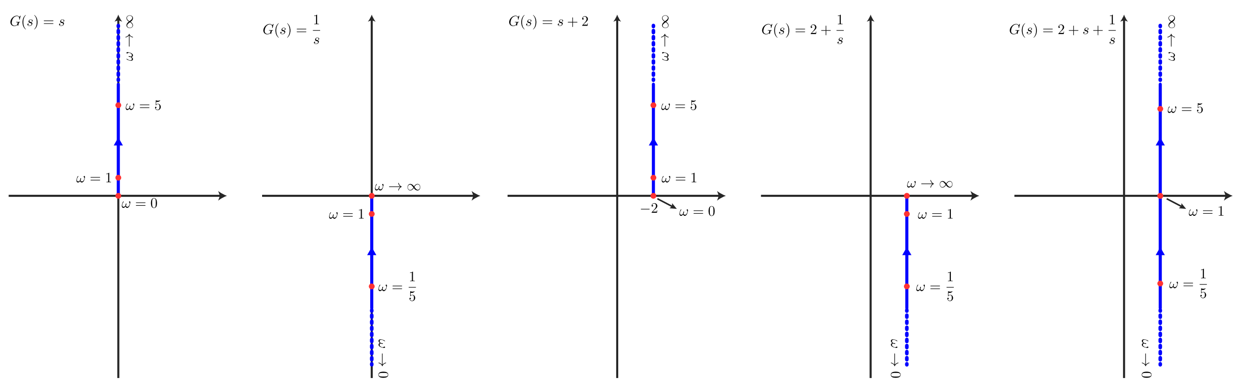
$$y_{ss}(t) = M \sin(\omega t + \phi + \theta)$$

14.1.1 Plotting Frequency Response: Polar Plot

We can consider the frequency response function $G(j\omega)$ as a mapping from positive $j\omega$ axis to a curve in the complex plane. In polar plot, we draw the frequency response function starting from $\omega = 0$ (or $\omega \to 0^+$) to $\omega \to \infty$.

Let's draw the polar plots of

$$G_1(s) = s \quad , \quad G_2(s) = \frac{1}{s} \quad , \quad G_3(s) = s + 2$$
$$G_4(s) = 2 + \frac{1}{s} \quad , \quad G_5(s) = 2 + s + \frac{1}{s}$$



Now let's draw the polar plots of

$$G_1(s) = \frac{1}{s+1} \quad , \quad G_2(s) = \frac{s}{s+1}$$

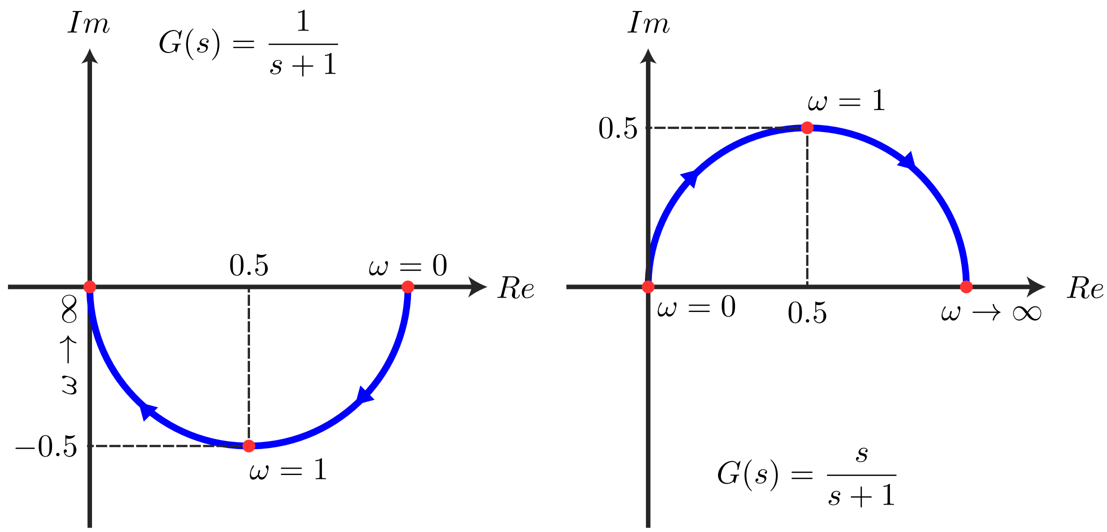
Let's analyze $G_1(j\omega)$ for $\omega \in [0, \infty)$

$$G_1(j\omega) = \frac{1}{j\omega + 1} = \frac{1 - j\omega}{\omega^2 + 1} = \frac{1}{\omega^2 + 1} - \frac{\omega}{\omega^2 + 1} j$$
$$|G_1(j\omega)| = \frac{1}{\sqrt{1 + \omega^2}}$$
$$\angle[G_1(j\omega)] = \arctan(-\omega)$$

Now let's analyze $G_2(j\omega)$ for $\omega \in [0, \infty)$

$$G_2(j\omega) = \frac{j\omega}{j\omega + 1} = \frac{j\omega + \omega^2}{\omega^2 + 1} = \frac{\omega^2}{\omega^2 + 1} + \frac{\omega}{\omega^2 + 1} j$$
$$|G_2(j\omega)| = \sqrt{\frac{\omega^2}{1 + \omega^2}}$$
$$\angle[G_2(j\omega)] = \arctan(1/\omega)$$

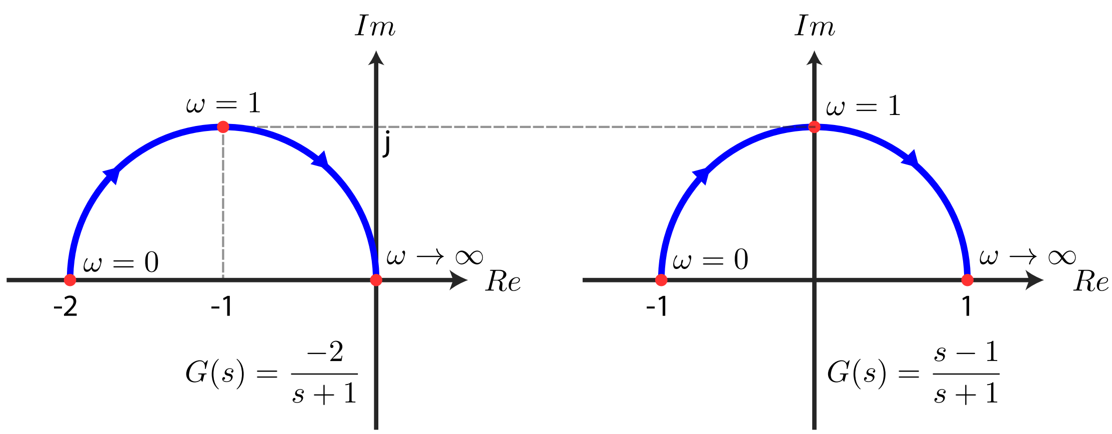
Polar plots of $G_1(s)$ and $G_2(s)$ are illustrated below.



Now let's draw the polar plot of $G(s) = \frac{s-1}{s+1}$ (note that there is a zero in open-right half plane). Note that

$$
\begin{aligned}
G(s) &= 1 - \frac{2}{s+1} \\
G(j\omega) &= 1 - 2\left(\frac{1}{\omega^2+1} - \frac{\omega}{\omega^2+1}j\right)
\end{aligned}
$$

Process of polar plot drawing of $G(s) = \frac{s-1}{s+1}$ is illustrated below



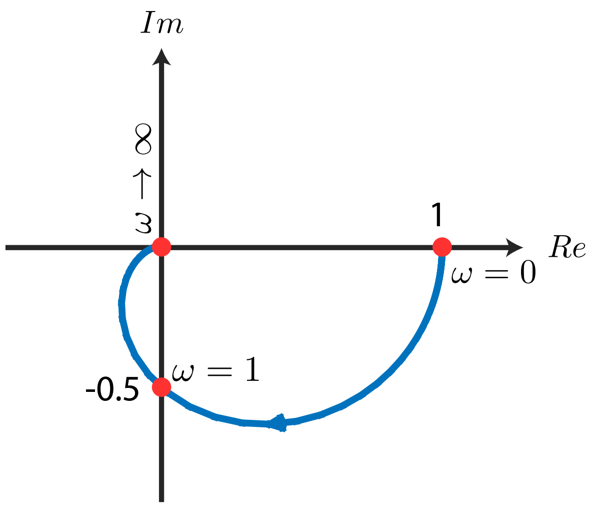
Now let's draw the polar plot of $G(s) = \frac{1}{(s+1)^2}$

$$
\begin{aligned}
G(j\omega) &= \frac{1}{(j\omega+1)^2} = \frac{(-j\omega+1)^2}{(\omega^2+1)^2} \\
&= \left[\left(1-\omega^2\right) + j(-2\omega)\right]\frac{1}{(\omega^2+1)^2}
\end{aligned}
$$

Some important points and associated features on the polar plot can be computed as

$$\begin{aligned}
&\omega \to 0 \ \Rightarrow G(j\omega) = 1\\
&\omega \to 1 \ \Rightarrow G(j\omega) = -0.5j\\
&\omega \to \infty \ \Rightarrow |G(j\omega)| \to 0 \quad \& \quad \angle[G(j\omega)] \to -\pi
\end{aligned}$$

Resultant polar plot is illustrated below



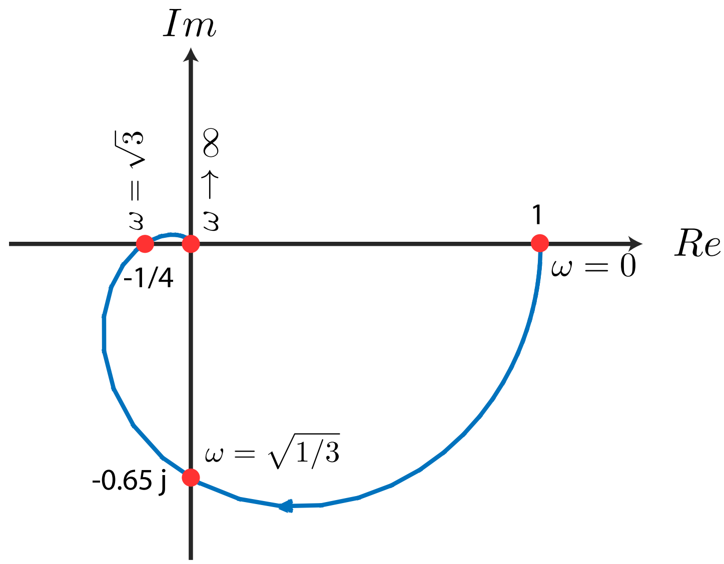
Now let's draw the polar plot of $G(s) = \frac{1}{(s+1)^3}$

$$\begin{aligned}
G(j\omega) &= \frac{1}{(j\omega+1)^3} = \frac{(-j\omega+1)^3}{(\omega^2+1)^3}\\
&= \left[\left(1-3\omega^2\right) + j(\omega^3 - 3\omega)\right]\frac{1}{(\omega^2+1)^3}
\end{aligned}$$

Some important points and associated features on the polar plot can be computed as

$$\begin{aligned}
&\omega \to 0 \ \Rightarrow G(j\omega) = 1\\
&\omega \to \sqrt{1/3} \ \Rightarrow G(j\omega) = -0.65j\\
&\omega \to \sqrt{3} \ \Rightarrow G(j\omega) = -1/4\\
&\omega \to \infty \ \Rightarrow |G(j\omega)| \to 0 \quad \& \quad \angle[G(j\omega)] \to \pi/2
\end{aligned}$$

Resultant polar plot is illustrated below


Im
Re
$\omega = \sqrt{3}$
$\omega \to \infty$
1
$\omega = 0$
-1/4
$\omega = \sqrt{1/3}$
-0.65 j