

## Lecture 3

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### 3.1 State-Space Representation to Frequency Domain

In this lecture we will cover the conversion from state-space representations to frequency domain representations ( $s$ -domain for CT systems and  $z$ -domain for DT systems) and analyze the connections between two representations.

#### 3.1.1 CT State-Space to $s$ -domain

Note that a SS representation of an  $n^{th}$  order CTI-LTI system has the form below.

$$\begin{aligned} \text{Let } x(t) &\in \mathbb{R}^n, \quad y(t) \in \mathbb{R}^q, \quad u(t) \in \mathbb{R}^p, \\ \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ \text{where } A &\in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times p}, \quad C \in \mathbb{R}^{q \times n}, \quad D \in \mathbb{R}^q \end{aligned}$$

In order to convert state-space to frequency domain, we start with taking the Laplace transform of the both sides of the state-equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ sX(s) - x_0 &= AX(s) + BU(s) \\ sX(s) - AX(s) &= x_0 + BU(s) \\ (sI - A)X(s) &= x_0 + BU(s) \\ X(s) &= (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s) \end{aligned}$$

Now let's concentrate on the output equation

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ Y(s) &= C(sI - A)^{-1}x_0 + [C(sI - A)^{-1}B + D]U(s) \end{aligned}$$

where  $C(sI - A)^{-1}x_0$  corresponds to the initial-condition response and when  $u(t) = 0$  we have

$$\begin{aligned} Y(s) &= [C(sI - A)^{-1}B + D]U(s) \\ G(s) &= C(sI - A)^{-1}B + D \end{aligned}$$

where  $G(s)$  is called the **transfer function matrix** which has the following form for a general  $p$ -input- $q$ -output MIMO system

$$G(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1p}(s) \\ \vdots & & \vdots \\ G_{q1}(s) & \cdots & G_{qp}(s) \end{bmatrix}$$

**Definiton:**  $G_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)}$  is classified as follows

- $G_{ij}(s)$  is *proper*  $\Leftrightarrow \deg(n_{ij}(s)) \leq \deg(d_{ij}(s)) \Leftrightarrow G_{ij}(\infty) = \lim_{s \rightarrow \infty} G_{ij}(s) = C$  where  $|C| < \infty$
- $G_{ij}(s)$  is *strictly proper*  $\Leftrightarrow \deg(n_{ij}(s)) < \deg(d_{ij}(s)) \Leftrightarrow G_{ij}(\infty) = \lim_{s \rightarrow \infty} G_{ij}(s) = 0$
- $G_{ij}(s)$  is *bi-proper*  $\Leftrightarrow \deg(n_{ij}(s)) = \deg(d_{ij}(s)) \Leftrightarrow G_{ij}(\infty) = C$  where  $|C| < \infty$  &  $C \neq 0$
- $G_{ij}(s)$  is *improper*  $\Leftrightarrow \deg(n_{ij}(s)) > \deg(d_{ij}(s)) \Leftrightarrow |G_{ij}(\infty)| \rightarrow \infty$

**Remark:**  $G_{ij}(s)$  is strictly proper  $\forall(i, j)$  iff  $D = \mathbf{0}$

$$G(s) = C(sI - A)^{-1}B = \frac{C \text{Adj}(sI - A)B}{\text{Det}(sI - A)}, \text{ where}$$

$$\det(\text{Det}(sI - A)) = n$$

$$\text{Adj}(sI - A) = [\text{Cofactor}(sI - A)]^T$$

Let  $\text{Cofactor}(sI - A) = Co$  then  $Co_{ij} = (-1)^{i+j} \text{Det}(M_{ij})$ , where  $\text{Det}(M_{ij})$  is called the minor of  $(sI - A)_{ij}$  and is the determinant of the submatrix formed by deleting the  $i$ th row and  $j$ th column. Note that  $\deg(Co_{ij}) \leq (n - 1) \forall(i, j)$  which implies that  $G_{ij}(s)$  is strictly proper.

**Definition:**

- A scalar  $\lambda \in \mathbb{C}$  is called a pole of  $G_{ij}(s)$  if  $|G_{ij}(\lambda)| \rightarrow \infty$
- A scalar  $\gamma \in \mathbb{C}$  is called a zero of  $G_{ij}(s)$  if  $|G_{ij}(\gamma)| = 0$

**Definition:** Two polynomials are said to be coprime if they have no common root.

**Remark:**

- $\lambda \in \mathbb{C}$  is a pole of  $G_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)}$  if  $d_{ij}(s)$  and  $n_{ij}(s)$  are coprime and  $d_{ij}(\lambda) = 0$
- $\lambda \in \mathbb{C}$  is a zero of  $G_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)}$  if  $d_{ij}(s)$  and  $n_{ij}(s)$  are coprime and  $n_{ij}(\lambda) = 0$

It seems that if we can find  $\Phi(t) = \mathcal{L}\{(sI - A)^{-1}\}$ , it would be helpful to find both the initial condition respons and forced response in time-domain. Let's first expand  $(sI - A)^{-1}$  by long "division"

$$\begin{array}{r|l}
 & sI - A \\
 \hline
 I - s^{-1}A & s^{-1}I + s^{-2}A + s^{-3}A^2 + s^{-4}A^3 \\
 \hline
 & s^{-1}A \\
 & s^{-1}A - s^{-2}A^2 \\
 \hline
 & s^{-2}A^2 \\
 & s^{-2}A^2 - s^{-3}A^3 \\
 \hline
 & s^{-3}A^3 \\
 & \vdots
 \end{array}$$

If we follow the path we can find that

$$(sI - A)^{-1} = s^{-1}I + s^{-2}A + s^{-3}A^2 + s^{-4}A^3 + s^{-5}A^4 + \dots$$

$$\Psi(t) = \mathcal{L}^{-1} \left[ (sI - A)^{-1} \right] = I + At + A^2 \frac{t^2}{2} + A^3 \frac{t^3}{3!} + A^4 \frac{t^4}{4!} + \dots = e^{At}$$

We can then find initial condition only response and impulse response matrix of the system using  $\Psi(t) = e^{At}$

- Initial Condition Only Response :  $x(t) = e^{At}x_0$
- Impulse Response Matrix :  $G(t) = Ce^{At}B + D\delta(t)$

In that respect general solution can be written as

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

### 3.1.2 DT State-Space to $z$ -domain

Note that a SS representation of an  $n^{th}$  order DTI-LTI system has the form below.

$$\begin{aligned} \text{Let } x[k] &\in \mathbb{R}^n, \quad y[k] \in \mathbb{R}^q, \quad u[k] \in \mathbb{R}^p, \\ x[k+1] &= Ax[k] + Bu[k], \\ y[k] &= Cx[k] + Du[k], \\ \text{where } A &\in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times p}, \quad C \in \mathbb{R}^{q \times n}, \quad D \in \mathbb{R}^q \end{aligned}$$

In order to convert state-space to frequency domain, we start with taking the  $Z$ -transform of the both sides of the state-equation, where  $Z$ -transform of a unilateral (causal) discrete time signal  $w[k]$  is given by

$$W(z) = \mathcal{Z}\{w[k]\} = \sum_{k=0}^{\infty} w[k]z^{-k}$$

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ zX(z) - zx[0] &= AX(z) + BU(z) \\ zX(z) - AX(z) &= zx[0] + BU(z) \\ (zI - A)X(z) &= zx[0] + BU(z) \\ X(z) &= z(zI - A)^{-1}x[0] + (zI - A)^{-1}BU(z) \end{aligned}$$

I recommend to those of you not familiar with  $Z$ -transform operation on difference equations to read *shifting theorem* in EE402 Lecture Notes (Lecture # 2), indeed going over the whole Lecture would be very helpful.

Now let's concentrate on the output equation

$$\begin{aligned} y[k] &= Cx[k] + Du[k] \\ Y(z) &= zC(zI - A)^{-1}x[0] + \left[ C(zI - A)^{-1}B + D \right] U(z) \end{aligned}$$

where  $zC(sI - A)^{-1}x[0]$  corresponds to the initial-condition response and when  $u[k] = 0$  we have

$$Y(z) = \left[ C(zI - A)^{-1}B + D \right] U(z)$$
$$G(z) = C(zI - A)^{-1}B + D$$

similar to the CT case  $G(z)$  is called the **transfer function matrix**. Note that resultant frequency domain solution in DT systems is very similar to the solution in CT systems (except the initial condition response). Without a big surprise state-space to transfer function related definitions CT systems generally holds also for DT systems, Such as *properness*, *poles*, *zeros* etc.