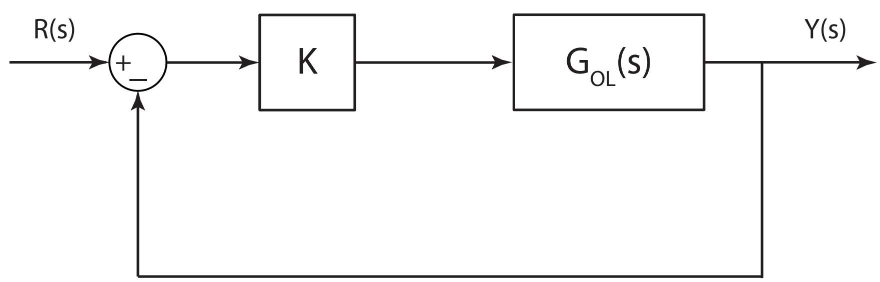
ROB501 - Fundamentals & Emerging Topics in Robotics - Digital Control Systems

Lecture 6

Lecturer: Asst. Prof. M. Mert Ankarali

6.1 Root Locus in Discrete-Time & Digital Control Systems

For continuous time systems the root locus diagram illustrates the location of roots/poles of a closed loop LTI systems, with respect to gain parameter K (can be considered as a P controller). The basic closed-loop topology is used for deriving the root-locus rules, however we know that many different topologies can be reduced to this from.



The closed loop transfer function of this basic control system is

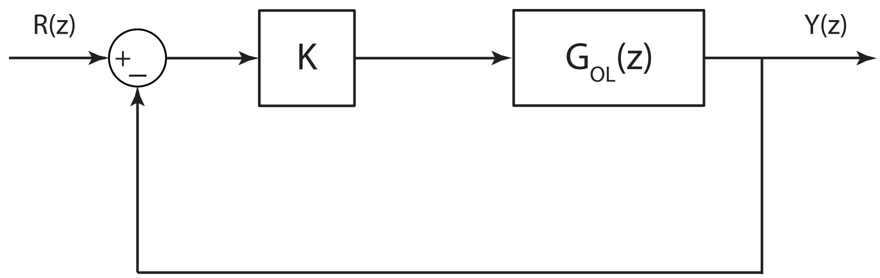
$$\frac{Y(s)}{R(s)} = \frac{KG_{OL}(s)}{1 + KG_{OL}(s)}$$

where the poles of the closed loop system are the roots of the characteristic equation

$$1 + KG_{OL}(s) = 0$$
$$1 + K\frac{n(s)}{d(s)} = 0$$

In 302 we learned the rules such that we can derive the qualitative and quantitive structure of root locus paths for **positive** gain K that solves the equation above.

In discrete time systems, similar to the CT case we use the root locus diagram to illustrate the location of roots/poles of a closed loop DT-LTI systems, with respect to a gain parameter K (can be considered as a P controller). The basic discrete-time closed-loop topology is used for deriving the root-locus rules, however we know that many different DT topologies can be reduced to this from.



The closed loop transfer function of this basic control system is

$$\frac{Y(z)}{R(z)} = \frac{KG_{OL}(z)}{1 + KG_{OL}(z)}$$

where the poles of the closed loop system are the roots of the characteristic equation

$$1 + KG_{OL}(z) = 0$$
$$1 + K\frac{n(z)}{d(z)} = 0$$

I think, it is obvious that fundamental equation that relates the gain K and roots/poles is exactly same for both CT and DT systems. This means that same rules are directly applied for CT systems.

However, even if we have same diagram for CT and DT systems the meaning and interpretation of the diagram is fundamentally different. Because, the effects of pole locations are different in CT and DT systems.

Angle and Magnitude Conditions

Let's analyze the characteristic equation

$$KG_{OL}(z) = -1 \quad \text{,or} \quad K\frac{n(z)}{d(z)} = -1$$
$$|KG_{OL}(z)| = 1 \quad \text{,or} \quad \left|K\frac{n(z)}{d(z)}\right| = 1$$
$$\angle[KG_{OL}(z)] = \pi(2k+1),\ k \in \mathbb{Z} \quad \text{,or} \quad \angle\left[K\frac{n(z)}{d(z)}\right] = \pi(2k+1),\ k \in \mathbb{Z}$$

For a given K, z values that satisfy both magnitude and angle conditions are located on the root loci.

Rules and procedure for constructing root loci

1. Characteristic equation, zeros and poles of the Open-Loop pulse transfer function.

$$1 + KG_{OL}(z) = 0$$
$$1 + K\frac{n(z)}{d(z)} = 0$$
$$1 + K\frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)} = 0$$

2. Root loci has N separate branches.

3. Root loci starts from poles of $G_{OL}(z)$ and

   (a) M branches terminates at the zeros of $G_{OL}(z)$

   (b) $N-M$ branches terminates at ∞ (implicit zeros of $G_{OL}(z)$)

   It is relatively easy to undestand this

$$\begin{aligned} d(z) + Kn(z) &= 0 \\ K \to 0 \ \to \ d(z) &= 0 \\ K \to \infty \ \to \ n(z) &= 0 \end{aligned}$$

4. Root loci on the real axis determined by open-loop zeros and poles. $z = \sigma \in \mathbb{R}$ then,

$$\begin{aligned} |KG_{OL}(\sigma)| &= 1 \\ \text{Sign}[G_{OL}(\sigma)] &= -1 \end{aligned}$$

   We can always find a K that satisfy the magnitude condition, so angle condition will determine which parts of real axis belong to the root locus.

   We can first see that complex conjugate zero/pole pairs has not effect, then for the remaining ones we can derive the following condition

$$\text{Sign}[G_{OL}(\sigma)] = \prod_{i=1}^{M} \text{Sign}[\sigma - z_i] \prod_{j=1}^{N} \text{Sign}[\sigma - p_j] = -1$$

   which means that for ODD number of poles + zeros Sign$[\sigma - p_i]$ and Sign$[\sigma - z_i]$ must be negative for satisfying this condition for that particular σ to be on the root-locus. We can summarize the rule as

   If the test point σ on real axis has ODD numbers of poles and zeros in its right, then this point is located on the root-locus.

5. Asymptotes

   - $N-M$ branches goes to infinity. Thus, there exist $N-M$ many asymptotes
   - For large z we can have the following approximation

$$\begin{aligned} &K\frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)} \approx \frac{K}{z^{N-M}} \\ &\angle\left[\frac{K}{z^{N-M}}\right] = -(N-M)\angle[z] = \pi(2k+1),\ k \in \mathbb{Z} \\ &\phi_a = \frac{\pm\pi(2k+1)}{N-M},\ k \in \{1, \cdots, N-M\} \end{aligned}$$

   - Real axis intercept σ_a can be computed as

$$\sigma_a = \frac{\sum p_i - \sum z_i}{N-M}$$

     This can be derived via a different approximation (see textbook)

6. Breakaway and break-in points on real axis. When z is real $z = \sigma,\ \sigma \in \mathbb{R}$, we can have

$$1 + KG_{OL}(\sigma) = 0 \quad \rightarrow \quad K(\sigma) = \frac{-1}{G_{OL}(\sigma)}$$

Note that break-in and breakaway points corresponds to double roots. Thus, σ_b is a break-away or break-in point if

$$\left[\frac{dK(\sigma)}{d\sigma}\right]_{\sigma=\sigma_b} = 0 \quad \text{or} \quad \left[\frac{dG_{OL}(\sigma)}{d\sigma}\right]_{\sigma=\sigma_b} = 0$$
$$K(\sigma) > 0$$

7. Angle of departure (or arrival) from open-loop complex conjugate poles (or to open-loop complex conjugate zeros)

Let's assume that p^* is a complex conjugate pole of $G_{OL}(z)$, then let's define a $P(z)$ such that

$$P^*(z) = (z - p^*)G_{OL}(z)$$

We know that for $K = 0$, the root locus is located at p^*. If we add a very small $K = \delta K$, then pole/root locus moves to $p^* + \delta z$. If we evaluate the phase condition of the root locus on this new "unknown" point

$$\begin{aligned}
\angle[KG_{OL}(z)]_{z=p*+\delta z} &= \pm\pi \\
\angle\left[\frac{P^*(z)}{z-p^*}\right]_{z=p^*+\delta z} &= \pm\pi \\
\angle\left[P^*(p^*+\delta z)\right] - \angle\left[\delta z\right] &= \pm\pi \\
\theta_d = \angle\left[\delta z\right] &= \pm\pi + \angle\left[P^*(p^*)\right]
\end{aligned}$$

Geometrically speaking, $\angle\left[P^*(p^*)\right]$ stands for **# of angles from the zeros to this specific pole** – **# of angles from all other remaining poles to this specific pole**.

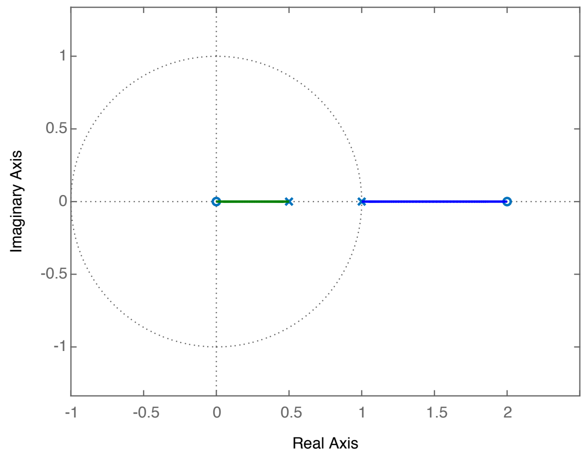
A similar condition can be derived for angle of arrival to complex conjugate zeros.

$$\begin{aligned}
\theta_a &= \pm\pi - \angle\left[P^*(z^*)\right] \\
P^*(z) &= (z - z^*)G_{OL}(z)
\end{aligned}$$

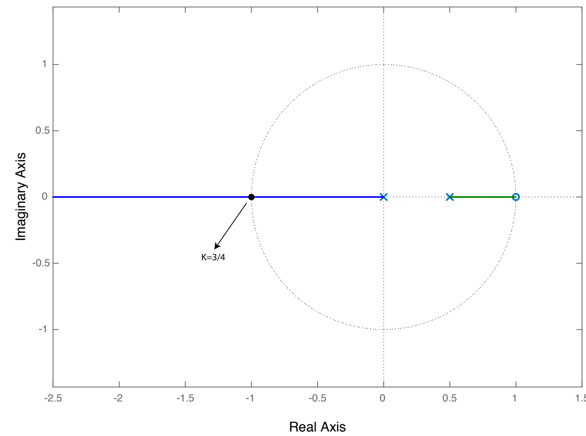
where z^* is a complex conjugate zero of $G_{OL}(z)$.

Examples

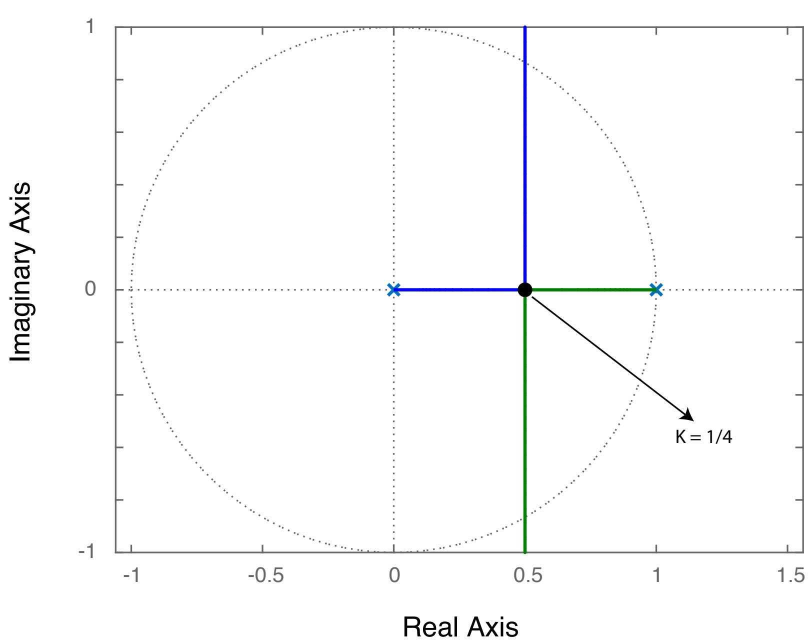
$$G_{OL}(z) = \frac{z(z-2)}{(z-1)(z-0.5)}$$



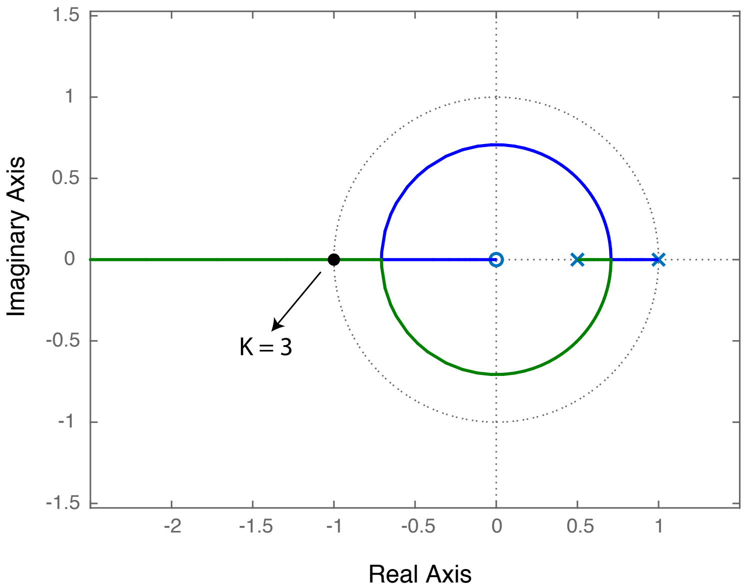
$$G_{OL}(z) = \frac{z-1}{z(z-0.5)}$$



$$G_{OL}(z) = \frac{1}{z(z-1)}$$



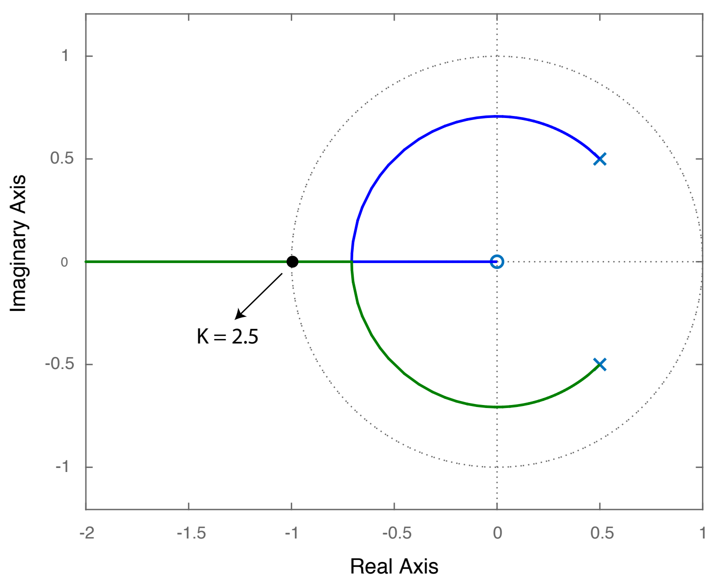
$$G_{OL}(z) = \frac{z}{(z-0.5)(z-1)}$$



$$\sigma_{ba} = \frac{\sqrt{2}}{2} \quad , \quad \sigma_{bi} = -\frac{\sqrt{2}}{2}$$

K = 0.086 K = 2.9

$$G_{OL}(z) = \frac{z}{z^2 - z + 1/2}$$



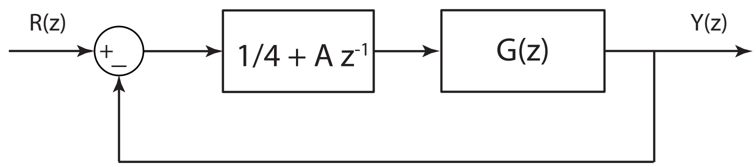
$$\theta_d = \pm\frac{3\pi}{4}$$

$$\sigma_{bi} = -\frac{\sqrt{2}}{2}$$

K = 2.4

6.1.1 Root-locus with respect to different parameters

Let's consider the following discrete-time system where the plant has a transfer function of $G(z)$ and controller is a first-order FIR filter (low-pass) with a transfer function of $G_c(z) = 1/4 + Az^{-1}$. We want to derive the location of closed-loop poles with respect to the parameter A, which does not directly fit to the classical form.



Let's first compute the closed-loop PTF and analyze the characteristic equation.

$$\begin{aligned}\frac{Y(z)}{R(z)} &= \frac{(0.25 + Az^{-}1)\,G(z)}{1 + (0.25 + Az^{-}1)\,G(z)} \\ 1 + \left(0.25 + Az^{-1}\right) G(z) &= 0 \\ 1 + 0.25G(z) + Az^{-1}G(z) &= 0\end{aligned}$$

If we divide the characteristic equation by $1 + 0.25G(z)$ we obtain

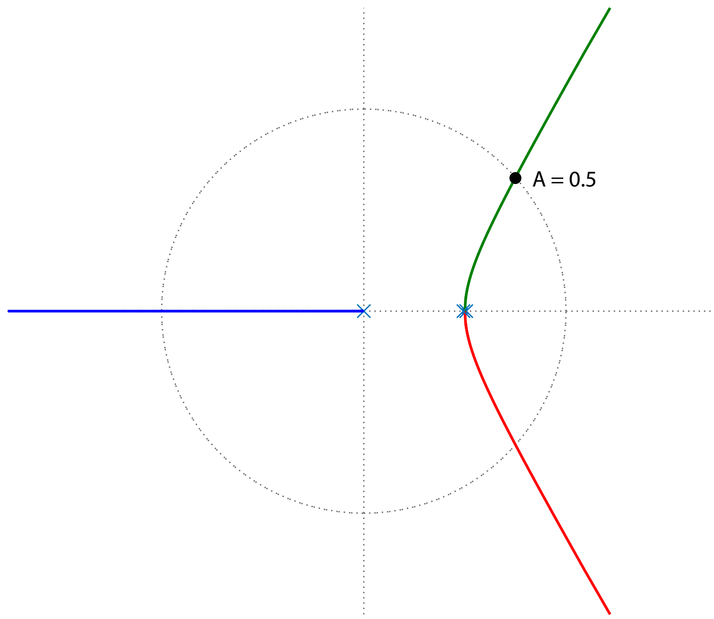
$$\begin{aligned}1 + A\frac{z^{-1}G(z)}{1 + 0.25G(z)} &= 0 \\ 1 + A\bar{G}_{OL}(z) &= 0\end{aligned}$$

Now if we consider as $\bar{G}_{OL}(z)$ as the open-loop transfer function and draw the root-locus then we would derive the dependence of the roots to the parameter A.

Let's assume that $G(z) = \frac{1}{z(z-1)}$. Then for this system, we can compute

$$\begin{aligned}\bar{G}_{OL}(z) &= \frac{z^{-1}G(z)}{1 + 0.25G(z)} = \frac{\frac{1}{z^2(z-1)}}{1 + \frac{0.25}{z(z-1)}} = \frac{\frac{1}{z^2(z-1)}}{\frac{z^2 - z + 0.25}{z(z-1)}} \\ &= \frac{1}{z(z^2 - z + 0.25)}\end{aligned}$$

Root-locus of the system w.r.t parameter A is given below. It can be seen that as A increases, dominant system poles deviates from the origin and eventually becomes unstable at $A = 0.5$. Technically this is a simple low-pass filter which may be inevitable in many closed-loop control systems. However, as we decrease the cut-off frequency (by increasing A) we push the poles towards the unit circle thus making the system less stable.


A = 0.5