EE502 - Linear Systems Theory II **Spring 2023**

Lecture 13

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13.1 State Feedback & Stabilizability

The state-feedback based control-policies for LTI systems starts with the assumption that we have "access" to the all of the states of the systems either via direct measurement or through some observer/estimator/tracker. In that context a family of state-feedback controllers for CT- and DT-LTI systems can be constructed as

$$u(t) = \gamma r(t) - Kx(t) \ \& \ u[k] = \gamma r[k] - Kx[k]$$

where $r(t)$ can be considered as the reference signal (most of the time it is), γ is a feed-forward scaling factor, and K is the state-feedback gain. Now let's find a state-space representation for dynamics of the closed-loop system for both CT- and DT-LTI systems under state-feedback rule proposed above

$$\begin{aligned} \dot{x} &= Ax + B\left(\gamma r(t) - Kx(t)\right) \Rightarrow \dot{x} = (A - BK)\,x + \gamma B r \\ x[k+1] &= Ax + B\left(\gamma r[k] - Kx[k]\right) \Rightarrow x[k+1] = (A - BK)\,x[k] + \gamma B r[k] \end{aligned}$$

In both cases the closed loop system and input matrices takes the following form

$$A_c = A - BK \ , \ B_c = \gamma B r$$

A key question in this domain is that can I find a K such that eigenvalues of A_c is located at arbitrary desired locations.

Theorem: (Eigenvalue/Pole Placement) Given (A, B), $\exists K$ s.t.

$$\begin{aligned} \det\left[\lambda I - (A - BK)\right] &= \lambda^n + a_{n-1}^* \lambda^{n-1} + \cdots + a_1^* \lambda + a_0^* \\ \forall \mathcal{A} &= \{a_0^*, \, a_1^* \cdots a_{n-1}^*\}, \, a_i^* \in \mathbb{R} \end{aligned}$$

if and only if (A, B) is reachable.

Proof: For a general complete proof we need to show that reachability of (A, B) is necessary and sufficient.

Proof of necessity: Let's assume that (A, B) not reachable and $\exists(\lambda_u, w_u^T)$ pair such that $w_u^T A = w_u^T \lambda_u$ and $w_u^T = 0$. Now check weather w_u^T is a left eigenvector of A_c

$$\begin{aligned} w_u^T A_c &= w_u^T (A - BK) = w_u^T A - w_u^T BK = w_u^T \lambda_u - 0 = w_u^T \lambda_u \\ w_u^T B_c &= w_u^T B \gamma = 0 \end{aligned}$$

Here not only we showed that λ_u can not be moved hence it is not possible to locate the poles arbitrary locations, we also showed that state-feedback rules does not affect the reachability.

Proof of sufficiency: We will only show the sufficiency for a multi-input case, i.e. $B \in \mathbb{R}^{n \times 1}$, however the reader should not the fact that for a complete proof multi-input case also needs to be analyzed. Let's assume that (A, B) is reachable and we know that reachability is invariant under similarity transformations, i.e.

$$\begin{aligned} &z = T^{-1}x \, , \, \det(T) \neq 0 \Rightarrow \dot{z} = \bar{A}z + \bar{B}u \\ &\bar{A} = T^{-1}AT \ , \ T^{-1}B \end{aligned}$$

and $(\bar{A}, \bar{B})$ is reachable. Noe let's choose T such that

$$T = \mathbf{R} = \left[\begin{array}{c|c|c|c} B & AB & \cdots & A^{n-1}B \end{array} \right]$$