14.1   State Observer

Generally the full state measurment,$x(t)$ or $x[k]$, of a system is not accessible and *observers, estimators, filters*) have to be used to extract this information. The output, $y(t)$ or $y[k]$, represents the actual measurements which is a function of the input and the output. We will mainly focus on DT systems in this lecture however, all of the derivations and concepts are either same with the CT case or can be easily adopted with minor modifications. Let

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

A Luenberger observer is built using a "simulated" model of the system and the errors caused by the mismatched initial conditions $x_0 \neq \hat{x}_0$ (or other types of perturbations) are reduced by introducing output error feedback.

Let's assume that the state vector of the simulated system is $\hat{x}[k]$, then the state space equation of this synthetic system takes the form

$$\hat{x}[k+1] = A\hat{x}[k] + Bu[k]$$
$$\hat{y}[k] = C\hat{x}[k] + Du[k]$$

Note that since $u[k]$ is the input that is supplied by the controller, we assume that it is known apriori. If $x[0] = \hat{x}[0]$ and when there is no model mismatch or uncertainty in the system then we expect that $x[k] = \hat{x}[k]$ and $y[k] = \hat{y}[k]$ for all k. When $x[0] \neq \hat{x}[0]$, then we should observe a difference between the measured and predicted output $y[k] \neq \hat{y}[k]$. The core idea in Luenberger observer is feeding the error in the output prediction $y[k] - \hat{y}[k]$ to the simulated system via a linear feedback gain.

$$\hat{x}[k+1] = A\hat{x}[k] + Bu[k] + L\left(y[k] - \hat{y}[k]\right)$$
$$\hat{y}[k] = C\hat{x}[k] + Du[k]$$

In order to understand how a Luenberger observer works and to choose a proper observer gain L, we define an error signal $e[k] = x[k] - \hat{x}[k]$. The dynamics w.r.t $e[k]$ can be derived as

$$e[k+1] = x[k+1] - \hat{x}[k+1]$$
$$= (Ax[k] + Bu[k]) - (A\hat{x}[k] + Bu[k] + L\left(y[k] - \hat{y}[k]\right))$$
$$e[k+1] = (A - LC)\,e[k]$$

where $e[0] = x[0] - \hat{x}[0]$ denotes the error in the initial condition.

If the matrix $(A - LC)$ is stable then the errors in initial condition will diminish eventually. Moreover, in order to have a good observer/estimator performance the observer convergence should be sufficiently fast.

Theorem: (Observer Eigenvalue Placement) Given (A, C), $\exists L$ s.t.

$$\det\left[\lambda I - (A - LC)\right] = \lambda^n + a_{n-1}^*\lambda^{n-1} + \cdots + a_1^*\lambda + a_0^*$$
$$\forall \mathcal{A} = \{a_0^*,\, a_1^* \cdots a_{n-1}^*\},\, a_i^* \in \mathbb{R}$$

if and only if (A, C) is observable.

Proof of necessity: Let's assume that (A, C) not observable and $\exists(\lambda_u, v_u)$ pair such that $Av_u = v_u\lambda_u$ and $Cv_u = 0$. Now check weather v_u is a right eigenvector of $(A - LC)$

$$(A - LC)v_u = Av_u - LCv_U = Av_u = \lambda_u v_U$$
$$Cv_u = 0$$

Here not only we showed that λ_u can not be moved hence it is not possible to locate the observer poles to arbitrary locations, we also showed that state-observer rule does not affect the observability.

Proof of sufficiency: Note that if (A, C) pair is observable then (A^T, C^T) pair is reachable. Then we know that $\exists L^T$ s.t.

$$\det\left[\lambda I - (A^T - C^T L^T)\right] = \lambda^n + a_{n-1}^*\lambda^{n-1} + \cdots + a_1^*\lambda + a_0^*$$
$$\forall \mathcal{A} = \{a_0^*,\, a_1^* \cdots a_{n-1}^*\},\, a_i^* \in \mathbb{R}$$

and obviously

$$\det\left[\lambda I - (A^T - C^T L^T)\right] = \det\left[\lambda I - (A - LC)\right]$$

which completes the proof.

14.2   Detectability

The dual of stabilizability in the observability domain is called detectability.

Theorem The following statements are equal

- (A, C) is detectable $\iff$

- if unobservable modes of (A, C) are asymptotically stable $\iff$

- $\forall(\lambda_i, v_i)$ s.t. $Av_i = \lambda_i v_i$ $\iff$ and $\text{Re}\{\lambda_i\} \geq 0$ (or $|\lambda_i| \geq 0$), $v_i \notin \mathcal{N}(C)$ $\iff$

- $\text{rank}\left[\dfrac{A - \lambda I}{C}\right] = n,\, \forall \lambda \in \mathbb{C}$ s.t. $\text{Re}\{\lambda_i\} \geq 0$ for CT systems (or $\forall \lambda \in \mathbb{C}$ s.t. $|\lambda| \geq 1$ for DT systems)

- $\exists L$ such that $(A - LC)$ is asymptotically stable

14.3 Model Based Controllers: Closing the loop between the observer and state-feedback control low

In the state-feedback control policy, we defined the control law as

$$u[k] = \alpha r[k] - Kx[k]$$

However, as mentioned in the Observer section, in general, we don't have direct access to all states of the system. In this case, we learned how to design an Observer/Estimator that predicts the states of the system. In this respect, it is natural to assume that in a closed-loop system, the control policy that defines the input should depend on the estimated states.

$$u[k] = \alpha r[k] - K\hat{x}[k]$$

However, the critical question is how this coupling affects the closed-loop behavior, and an even deeper question is can we even use such a policy. The advantage of LTI systems is that state-feedback gain and observer gain can be separately designed, and we guarantee a stable closed-loop performance. In this section, we will analyze the coupled system. Equations of motion for the closed-loop observer & state-feedback based control system is given below

$$
\begin{aligned}
x[k+1] &= Ax[k] + Bu[k] \\
y[k] &= Cx[k] + Du[k] \\
\hat{x}[k+1] &= A\hat{x}[k] + Bu[k] + L\left(y[k] - \hat{y}[k]\right) \\
\hat{y}[k] &= C\hat{x}[k] + Du[k] \\
u[k] &= -K\hat{x}[k]
\end{aligned}
$$

If we eliminate $u[k]$ and $\hat{y}[k]$ we obtain following dynamical representation

$$
\begin{aligned}
x[k+1] &= Ax[k] - BK\hat{x}[k] + B\alpha r[k] \\
\hat{x}[k+1] &= A\hat{x}[k] - BK\hat{x}[k] + LC\left(x[k] - \hat{x}[k]\right) + B\alpha r[k] \\
y[k] &= Cx[k] - DK\hat{x}[k] + D\alpha r[k]
\end{aligned}
$$

Now let's replace $\hat{x}[k]$ with $e[k] = x[k] - \hat{x}[k]$

$$
\begin{aligned}
x[k+1] &= (A - BK)x[k] + BKe[k] + B\alpha r[k] \\
e[k+1] &= (A - LC)e[k] \\
y[k] &= (C - DK)x[k] + DKe[k] + D\alpha r[k]
\end{aligned}
$$

Now let's define a state for the whole system, $z[k] = \begin{bmatrix} x[k] \\ e[k] \end{bmatrix}$ then the state-space representation is given by

$$
\begin{aligned}
z[k+1] &= \begin{bmatrix} (A - BK) & BK \\ 0_{n \times n} & (A - LC) \end{bmatrix} z[k] + \begin{bmatrix} \alpha B \\ 0 \end{bmatrix} r[k] \\
y[k] &= \begin{bmatrix} (C - DK) & DK \end{bmatrix} z[k] + [\alpha D]r[k]
\end{aligned}
$$

The system matrix is in block diagonal form, and the eigenvalues of this new system, the matrix is found by taking the union of eigenvalues of $(A - BK)$ and eigenvalues of $(A - LC)$. Thus a separate pole-placement can be performed for the state-feedback controller and the observer.

Ex 14.1 *Consider the following open-loop digital control systems*
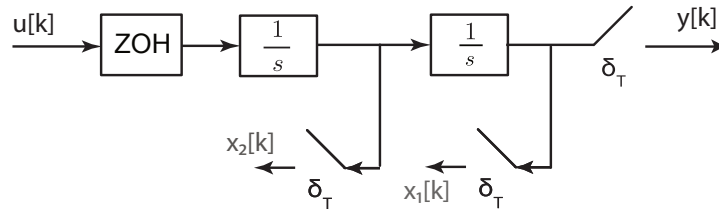
1. Find the discrete-time state-space representation between the discrete time input signal, $u[k]$ and the discrete time output signal, $u[k]$, given that state definition is $x[k] = \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix}$. Note that sampling time is generic for this problem T.

   Solution: The state update equation of the continuous-time plant has the following form

   $$\dot{x} = A_{ct}x + B_{ct}u \; , \; x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

   $$A_{ct} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \; , \; B_{ct} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   Since the ADC block is a simple ZOH operator we can find the system matrix of the discritized system using the state-transitions matrix of the CT-Plant

 $$A_{dt} = e^{A_{ct}T} = I + A_{ct}T + 0 + \cdots$$

   $$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

   Since A_{ct} is not invsertable, we need to derive B_{dt} using the following relation

   $$B_{dt} = \int_0^T e^{A_{ct}\tau} B_{ct} d\tau = \int_0^T \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau = \int_0^T \begin{bmatrix} \tau \\ 1 \end{bmatrix} d\tau = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

   Output equation is simply equal to $y[k] = x_1[k]$, and thus $C_{dt} = C_{ct} = C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $D_{dt} = D_{ct} = CD = \begin{bmatrix} 0 \end{bmatrix}$.

2. Show that following closed-loop PD type controller is indeed a form of state-feedback rule
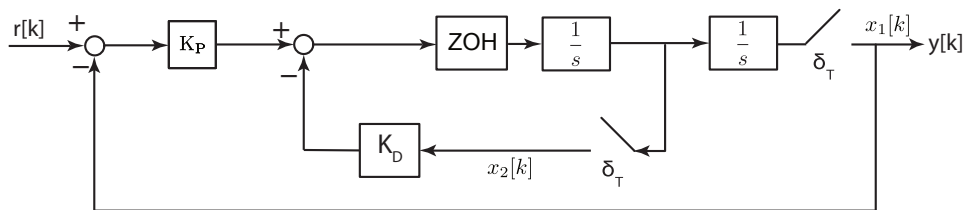
   Solution: Let's explicitly write $u[k]$

   $$u[k] = K_P(r[k] - x_1[k]) + K_D x_2[k] = K_P r[k] - \begin{bmatrix} K_P & K_D \end{bmatrix} x[k]$$
   $$K = \begin{bmatrix} K_P & K_D \end{bmatrix} \; , \; \alpha = K_P$$

3. Show that we can always find a (K_P, K_D) pair such that closed-loop poles are located at arbitrary desired locations.

Solution: We need to simply show that (A_{dt}, B_{dt}) pair is reachable

$$\mathbf{R} = \begin{bmatrix} A_{dt}B_{dt} & | & B_{dt} \end{bmatrix} = \begin{bmatrix} 3T^2/2 & | & T^2/2 \\ T & | & T \end{bmatrix} \Rightarrow \text{rank}[\mathbf{R}] = 2$$

thus (A_{dt}, B_{dt}) reachable and it implies that we can always find a (K_P, K_D) pair such that closed-loop poles are located at arbitrary desired locations.

4. Find (K_P, K_D) such that closed-loop systems shows dead-beat behavior

Solution: In such a case, we need to place the closed-loop eigenvalues at the origin

$$A_{cl} = A_{dt} - B_{dt}K = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \begin{bmatrix} K_P & K_D \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} K_P T^2/2 & K_D T^2/2 \\ K_P T & K_D T \end{bmatrix}$$

$$= \begin{bmatrix} 1 - K_P T^2/2 & T - K_D T^2/2 \\ -K_P T & 1 - K_D T \end{bmatrix}$$

$$\lambda I - A_{cl} = \begin{bmatrix} \lambda - 1 + K_P T^2/2 & -T + K_D T^2/2 \\ K_P T & \lambda - 1 + K_D T \end{bmatrix}$$

$$\det(\lambda I - A_{cl}) = \lambda^2 + \left(K_P T^2/2 + K_D T - 2\right)\lambda + \left(K_P T^2/2 - K_D T + 1\right)$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} T^2/2 & T \\ T^2/2 & -T \end{bmatrix} \begin{bmatrix} K_P \\ K_D \end{bmatrix} \Rightarrow \begin{bmatrix} K_P \\ K_D \end{bmatrix} = \frac{1}{T} \begin{bmatrix} T/2 & 1 \\ T/2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{T} \begin{bmatrix} 1/T & 1/T \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} K_P \\ K_D \end{bmatrix} = \begin{bmatrix} 1/T^2 \\ 3/(2T) \end{bmatrix} \Rightarrow K = \begin{bmatrix} 1/T^2 & 3/(2T) \end{bmatrix}$$

$$A_{cl} = \begin{bmatrix} 1/2 & T/4 \\ -1/T & -1/2 \end{bmatrix}$$