ROB501 - Fundamentals & Emerging Topics in Robotics - Digital Control Systems

Lecture 7

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7.1 State-Space Representation of DT Systems

State-space representation of a (causal & finite dimensional) LTI CT system is given by

$$\begin{aligned}
&\text{Let } x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R}^m \ , \ u(t) \in \mathbb{R}^r, \\
&\dot{x}(t) = Ax(t) + Bu(t), \\
&y(t) = Cx(t) + Du(t), \\
&\text{where } A \in \mathbb{R}^{n \times n} \ , \ B \in \mathbb{R}^{n \times r} \ , \ C \in \mathbb{R}^{m \times n} \ , \ D \in \mathbb{R}^{m \times r}
\end{aligned}$$

State-space representation of a (causal & finite dimensional) LTI DT system is given by

$$\begin{aligned}
&\text{Let } x[k] \in \mathbb{R}^n \ , \ y[k] \in \mathbb{R}^m \ , \ u[k] \in \mathbb{R}^r, \\
&x[k+1] = Gx[k] + Hu[k], \\
&y[k] = Cx[k] + Du[k], \\
&\text{where } G \in \mathbb{R}^{n \times n} \ , \ H \in \mathbb{R}^{n \times r} \ , \ C \in \mathbb{R}^{m \times n} \ , \ D \in \mathbb{R}^{m \times r}
\end{aligned}$$

Depending on the values of m and r we have

- $m = r = 1$, the system represents a SISO system
- $m > 1 \ , \ r < 1$, the system represents a MIMO system
- $m = 1 \ , \ r > 1$, the system represents a MISO system
- $m > 1 \ , \ r = 1$, the system represents a SIMO system

for both CT and DT cases.

State property of CT state-space models: Given the initial time, t_0 and state $x(t_0)$ and input $u(t)$ for $t_0 \leq t < t_f$ (with t_0 & t_f arbitrary), we can compute the output $y(t)$ for $t_0 \leq t \leq t_f$ and the state $x(t)$ for $t_0 \leq t \leq t_f$.

State property of DT state-space models: Given the state vector $x[k]$ and input $u[k]$ at an arbitrary time k, we can compute the the present output, $y[k]$, and next state $x[k+1]$.

Note that both definitions are not limited to LTI state-space models. Nonlinear and time-varying state-space models also are based on this definition.

When a state-space representation includes minimum number of state variables, the representation is called minimal.

7.2 State-Space Realizations

We will first cover the state-space realization of a specific case and then cover general case and canonical forms.

Let's assume that system that we would like to represent is a third order DT system with the following transfer function and difference equation

$$
\begin{aligned}
Y(z) &= \frac{1}{z^3 + a_1 z^2 + a_2 z + a_3} U(z) \\
y[k+3] &= -a_1 y[k+2] - a_2 y[k+1] - a_3 y[k] + u[k]
\end{aligned}
$$

We can directly label the states on the difference equation and then derive the individual state-update equations

$$
\begin{aligned}
x_3[k+1] &= -a_1 x_3[k] - a_2 x_2[k] - a_3 x_1[k] + u[k] \\
x_1[k+1] &= x_2[k] \\
x_2[k+1] &= x_3[k]
\end{aligned}
$$

whereas the output equation simply takes has the form $y[k] = x_3[k]$. If we gather these equations, we can obtain the state space form

$$
\begin{aligned}
\mathbf{x}[k+1] &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[k] \\
y[k] &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

This state-space realization is valid only for the cases with static numerator dynamics.

7.2.1 Canonical State-Space Realizations of (SISO) DT Systems

Now we will examine (more) general cases and canonical forms. In this course, we assume that $D = 0$ and hence limit ourselves to the cases in which order of the numerator is less then the order of the denominator.

Let's assume that the system that we would like to represent is a third order DT system with the following difference equation and transfer function

$$
\begin{aligned}
Y(z) &= \frac{b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3 z} U(z) \\
y[k+3] &= -a_1 y[k+2] - a_2 y[k+1] - a_3 y[k] + b_1 u[k+2] + b_2 u[k+1] + b_3 u[k]
\end{aligned}
$$

Reachable/Controllable Canonical Form

In this realization technique, we will use the fact that the system is a LTI system and decompose the Z domain expression as

$$
Y(z) = \frac{b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3 z} U(z) = (b_1 z^2 + b_2 z + b_3) \frac{1}{z^3 + a_1 z^2 + a_2 z + a_3 z} U(z)
$$

Note that

$$X_1(z) = \frac{1}{z^3 + a_1 z^2 + a_2 z + a_3 z} U(z)$$

where $X_1(z) = \mathcal{Z}\{x_1[k]\}$ of the special case that we covered previously. If we analyze the numerator dynamics, we can see that

$$Y(z) = (b_1 z^2 + b_2 z + b_3) X_1(z) = b_1 X_3(z) + b_2 X_2(z) + b_3 X_1(z)$$

We can see that we have exact same state-update equation with special case, but a different output equation

$$\begin{aligned} \mathbf{x}[k+1] &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[k] \\ y[k] &= \begin{bmatrix} b_3 & b_2 & b_1 \end{bmatrix} \end{aligned}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_1[k] \end{bmatrix} \quad , \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} b_3 & b_2 & b_1 \end{bmatrix}$$

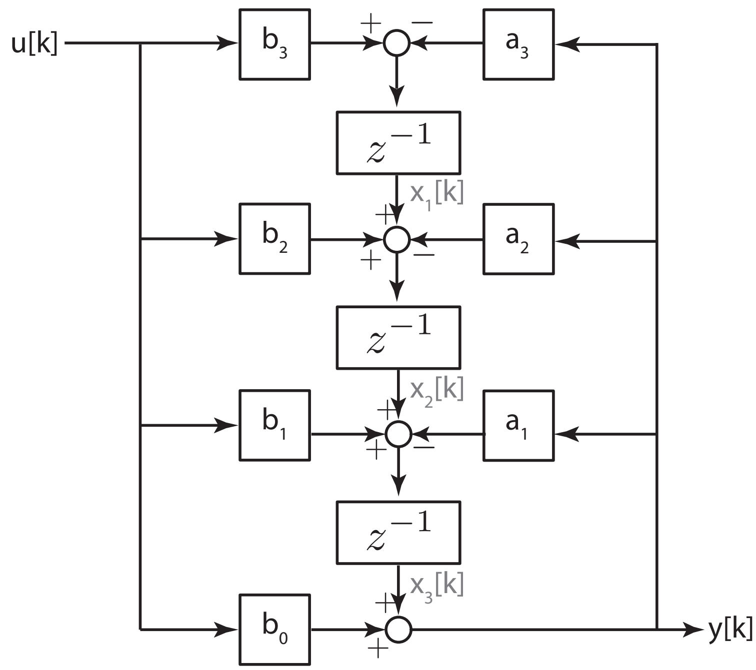
The form obtained with this approach is called reachable/controllable canonical form.

For a general n^{th} order system reachable/controllable canonical form has the following A ,B ,C ,& D matrices

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} b_n & b_{n-1} & \cdots & b_2 & b_1 \end{bmatrix} \end{aligned}$$

Observable Canonical Form

We also learnt a different type of canonical minimal realization which is illustrated in the Figure below



If we label the signals as given in the Figure, state evolution equations can be derived as

$$\begin{aligned}
X_1(z) &= (b_3U(z) - a_3\,(U(z)b_0 + X_3(z)))\,z^{-1} &\rightarrow\quad & x_1[k+1] = b_3u[k] - a_3\,(u[k]b_0 + x_3[k]) \\
X_2(z) &= (b_2U(z) + X_1(z) - a_2\,(U(z)b_0 + X_3(z)))\,z^{-1} &\rightarrow\quad & x_2[k+1] = b_2u[k] + x_1[k] - a_2\,(u[k]b_0 + x_3[k]) \\
X_3(z) &= (b_1U(z) + X_2(z) - a_1\,(U(z)b_0 + X_3(z)))\,z^{-1} &\rightarrow\quad & x_3[k+1] = b_1u[k] + x_2[k] - a_1\,(u[k]b_0 + x_3[k])
\end{aligned}$$

where as output equation can be derived as

$$y[k] = b_0u[k] + x_3[k]$$

If we combine the state and output equations, we can obtain the state space form as

$$\begin{aligned}
\mathbf{x}[k+1] &= \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}\mathbf{x}[k] + \begin{bmatrix} b_3 - b_0a_3 \\ b_2 - b_0a_2 \\ b_1 - b_0a_1 \end{bmatrix}u[k] \\
y[k] &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} + b_0u[k]
\end{aligned}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_1[k] \end{bmatrix} \quad , \quad A = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} b_3 - b_0a_3 \\ b_2 - b_0a_2 \\ b_1 - b_0a_1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad , \quad D = b_0$$

The form obtained with this approach is called observable canonical form.

For a general n^{th} order system observable canonical form has the following A , B , C , & D matrices

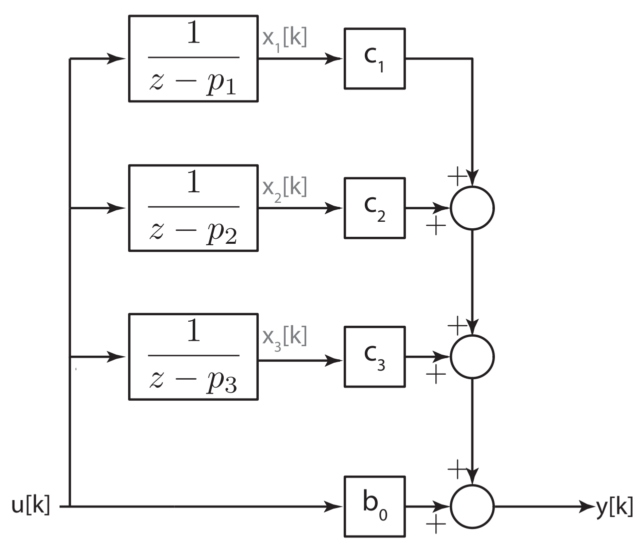
$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} (b_n - b_0 a_n) \\ (b_{n-1} - b_0 a_{n-1}) \\ \vdots \\ (b_2 - b_0 a_2) \\ (b_1 - b_0 a_1) \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad , \quad D = b_0$$

Diagonal Canonical Form

If the pulse transfer function of the system has distinct poles, we can expand it using partial fraction expansion

$$\begin{aligned} Y(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} X(z) \\ &= \left(b_0 + \frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} + \frac{c_3}{z - p_3} \right) X(z) \end{aligned}$$

It is possible to realize this expanded form in block diagram form as given in the Figure below.



Now let's concentrate on the candidate "state variables" and try to write state evaluation equations

$$\begin{aligned} X_1(z) &= \frac{1}{z - p_1} U(z) \quad &\rightarrow \quad x_1[k+1] = p_1 x_1[k] + u[k] \\ X_2(z) &= \frac{1}{z - p_2} U(z) \quad &\rightarrow \quad x_2[k+1] = p_2 x_2[k] + u[k] \\ X_3(z) &= \frac{1}{z - p_3} U(z) \quad &\rightarrow \quad x_3[k+1] = p_3 x_3[k] + u[k] \end{aligned}$$

where as output equation can be derived as

$$y[k] = b_0 u[k] + c_1 x_1[k] + c_2 x_2[k] + c_3 x_3[k]$$

If we combine the state and output equations, we can obtain the state space form as

$$\mathbf{x}[k+1] = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} + b_0 u[k]$$

where

$$\mathbf{x} = \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_1[k] \end{bmatrix} \quad , \quad A = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \quad , \quad D = b_0$$

The form obtained with this approach is called diagonal canonical form. Obviously, this form is not applicable for systems that has repeated roots.

For a general n^{th} order system with distinct roots diagonal canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} p_1 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & p_n \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix} \quad , \quad D = b_0$$

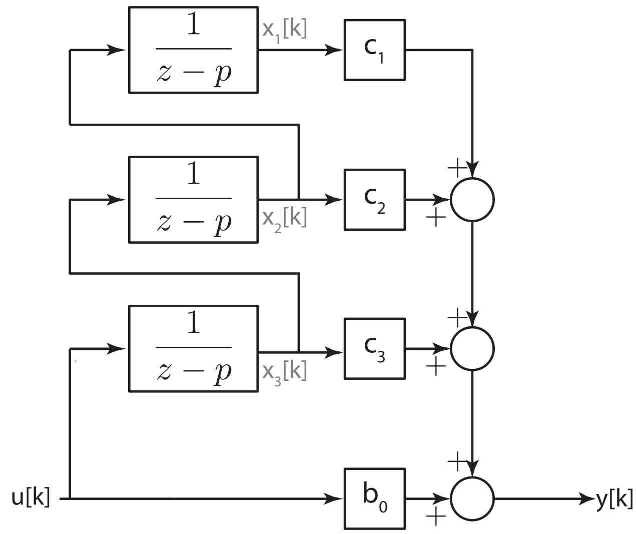
Jordan Canonical Form

Generalization of diagonal canonical farm is called Jordan canonical form which handles repeated roots.

In Jordan form the distinct roots has the same structure with Diagonal canonical form. Let's assume that the 3^{rd} order pulse transfer function has three repeated roots. In this case, we can expand it using partial fraction expansion

$$\begin{aligned} Y(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} X(z) \\ &= \left(b_0 + \frac{c_1}{(z-p)^3} + \frac{c_2}{(z-p)^2} + \frac{c_3}{z-p} \right) X(z) \end{aligned}$$

It is possible to realize this expanded form in block diagram form as given in the Figure below.



Now let's concentrate on the candidate "state variables" and try to write state evaluation equations

$$X_1(z) = \frac{1}{z-p}X_2(z) \quad \rightarrow \quad x_1[k+1] = p\ x_1[k] + x_2[k]$$
$$X_2(z) = \frac{1}{z-p}X_3(z) \quad \rightarrow \quad x_2[k+1] = p\ x_2[k] + x_3[k]$$
$$X_3(z) = \frac{1}{z-p}U(z) \quad \rightarrow \quad x_3[k+1] = p\ x_3[k] + u[k]$$

where as output equation can be derived as

$$y[k] = b_0u[k] + c_1x_1[k] + c_2x_2[k] + c_3x_3[k]$$

If we combine the state and output equations, we can obtain the state space form as

$$\mathbf{x}[k+1] = \begin{bmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{bmatrix}\mathbf{x}[k] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u[k]$$
$$y[k] = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} + b_0u[k]$$

where

$$\mathbf{x} = \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_1[k] \end{bmatrix} \quad , \quad A = \begin{bmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \quad , \quad D = b_0$$

A, B, & C forms a Jordan block.

For a general n^{th} order system a Jordan block with m repeated roots inside a stat-space representation in

Jordan canonical form looks like

$$A = \left[\begin{array}{c|ccccc|c} \ddots & & & & & & \\ \hline & \bar{p} & 1 & \cdots & 0 & 0 & \\ & 0 & \bar{p} & \cdots & 0 & 0 & \\ & & & \ddots & & & \\ & 0 & 0 & \cdots & \bar{p} & 1 & \\ & 0 & 0 & \cdots & 0 & \bar{p} & \\ \hline & & & & & & \ddots \end{array}\right] \quad , \quad B = \left[\begin{array}{c} \vdots \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \hline \vdots \end{array}\right]$$

$$C = \left[\begin{array}{c|ccccc|c} \cdots & c_1 & c_2 & \cdots & c_{n-1} & c_n & \cdots \end{array}\right]$$

Similarity Transformations

Consider the state-space representation of a given DT system

$$\begin{aligned} x[k+1] &= Gx[k] + Hu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

Let's define a new "state-vector" $\hat{x}$ such that

$$\begin{aligned} Px[k] = \hat{x}[k] \quad & \text{where} \\ P \in \mathbb{R}^{n \times n} \quad & \det(P) \neq 0 \end{aligned}$$

Then we can transform the state-space equations using P as

$$\begin{aligned} P^{-1}\hat{x}[k+1] &= GP^{-1}\hat{x}[k] + Hu[k] \quad , \quad y[k] = CP^{-1}\hat{x}[k] + Du[k] \\ \hat{x}[k+1] &= PGP^{-1}\hat{x}[k] + PHu[k] \quad , \quad y[k] = CP^{-1}\hat{x}[k] + Du[k] \end{aligned}$$

The "new" state-space representation is obtained as

$$\begin{aligned} \hat{x}[k+1] &= \hat{G}\hat{x}[k] + \hat{H}u[k] \\ y[k] &= \hat{C}x[k] + \hat{D}u[k] \\ \hat{G} &= PGP^{-1} \ , \ \hat{H} = PH \ , \ \hat{C} = CP^{-1} \ , \ \hat{D} = D \end{aligned}$$

Since there exist infinitely many non-singular $n \times n$ matrices, for a given LTI DT system, there exist infinitely many different but equivalent state-space representations.

Example: Show that $A \in \mathbb{R}^{n \times n}$ and $P^{-1}AP$, where $P\mathbb{R}^{n \times n}$ and $\det(P) \neq 0$, have the same characteristic equation

Solution:

$$\begin{aligned} \det\left(\lambda I - P^{-1}AP\right) &= \det\left(\lambda P^{-1}IP - P^{-1}AP\right) \\ &= \det\left(P^{-1}\left(\lambda I - A\right)P\right) \\ &= \det\left(P^{-1}\right)\det\left(\lambda I - A\right)\det\left(P\right) \\ &= \det\left(P^{-1}\right)\det\left(P\right)\det\left(\lambda I - A\right) \\ \det\left(\lambda I - P^{-1}AP\right) &= \det\left(\lambda I - A\right) \end{aligned}$$

Obtaining Transfer Functions from a State-Space Representation

Let's consider the following general state-space form

$$\begin{aligned} x[k+1] &= Gx[k] + Hu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

In order to obtain transfer function form, we assume that initial conditions are zero. Under this assumption, let's take z-transform of both equations

$$\begin{aligned} zX(z) &= GX(z) + HU(z) \quad , \quad Y(z) = CX(z) + DU(z) \\ (zI - G)\, X(z) &= HU(z) \\ X(z) &= (zI - G)^{-1}\, HU(z) \\ Y(z) &= C\,(zI - G)^{-1}\, HU(z) + DU(z) \\ Y(z) &= \left[C\,(zI - G)^{-1}\, H + D\right] U(z) \\ T(z) &= \left[C\,(zI - G)^{-1}\, H + D\right] \end{aligned}$$

If the system is a SISO system, then $T(z)$ is a transfer function, where as for MIMO case $T(z)$ becomes a *transfer function matrix*. Note that $(zI - G)^{-1}$ is invertible for all $z \in \mathbb{C}$ except the eigenvalues of G.

Example: Let p be a pole of $T(z)$, show that p is also an eigenvalue of G.

Solution: Let

$$T(z) = \frac{n(z)}{d(z)}$$

If p is a pole of $T(z)$, then $d(z)|_p = 0$. Now let's analyze the dependence of $T(z)$ to the state-space form.

$$\begin{aligned} T(z) &= \left[C\,(zI - G)^{-1}\, H + D\right] \\ (zI - G)^{-1} &= \frac{\mathrm{Adj}\,(zI - G)}{\det\,(zI - G)} \\ T(z) &= \frac{C\mathrm{Adj}\,(zI - G)\, H + D\det\,(zI - G)}{\det\,(zI - G)} \end{aligned}$$

If p is a pole of $T(z)$, then

$$\det\,(zI - G)\,|_{z=p} = 0$$

Obviously p is an eigenvalue of G.

Invariance of Transfer Functions Under Similarity Transformation

Consider the two different state-space representations

$$\begin{aligned} x[k+1] &= Gx[k] + Hu[k] \qquad & \hat{x}[k+1] &= \hat{G}\hat{x}[k] + \hat{H}u[k] \\ y[k] &= Cx[k] + Du[k] \qquad & y[k] &= \hat{C}x[k] + \hat{D}u[k] \end{aligned}$$

where they are related with the following similarity transformation

$$Px[k] = \hat{x}[k] \ , \ \hat{G} = PGP^{-1} \ , \ \hat{H} = PH \ , \ \hat{C} = CP^{-1} \ , \ \hat{D} = D$$

Let's compute the transfer function for the second representation

$$\begin{aligned}
\hat{T}(z) &= \left[\hat{C}\left(zI - \hat{G}\right)^{-1}\hat{H} + \hat{D}\right] \\
&= \left[CP^{-1}\left(zI - PGP^{-1}\right)^{-1}PH + D\right] \\
&= \left[CP^{-1}\left(P\left(zI - G\right)P^{-1}\right)^{-1}PH + D\right] \\
&= \left[CP^{-1}P\left(zI - G\right)^{-1}P^{-1}PH + D\right] \\
&= \left[C\left(zI - G\right)^{-1}H + D\right] \\
\hat{T}(z) &= T(z)
\end{aligned}$$

Solution of Discrete-Time State-Space Equations

Let's first assume that $u[k] = 0$, and find un-driven (homogeneous) response.

$$\begin{aligned} x[k+1] &= Gx[k] \\ y[k] &= Cx[k] \end{aligned}$$

Unlike CT systems we can compute the response iteratively

$$\begin{aligned} x[1] &= Gx[0] \\ x[2] &= Gx[1] = G^2 x[0] \\ x[3] &= Gx[2] = G^3 x[0] \\ &\vdots \\ x[k] &= Gx[k-1] = G^k x[0] \quad , \quad y[k] = CG^k x[0] \end{aligned}$$

It is easy to see that for $k, p \in \mathbb{Z}$ where $k > p$

$$x[k] = G^{k-p} x[p]$$

Let $\Psi(k) = G^k$, then this matrix of functions solves the homogeneous difference equation

$$x[k+1] = Gx[k]$$

$$\begin{aligned} x[k] &= \Psi[k]x[0] \\ x[k] &= \Psi[k-p]x[p] \\ x[k+m] &= \Psi[k+m-m]x[m] = \Psi[k] \end{aligned}$$

$\Psi[k]$ is called the state-transition matrix. Now let's consider input-only state response (i.e. $x[0] = 0$).

$$x[k+1] = Gx[k] + Hu[k]$$

$$\begin{aligned}
x[1] &= Hu[0] \\
x[2] &= Gx[1] + Hu[1] = GHu[0] + Hu[1] \\
x[3] &= Gx[2] + Hu[2] = G^2Hu[0] + GHu[1] + Hu[2] \\
x[4] &= Gx[3] + Hu[3] = G^3Hu[0] + G^2Hu[1] + GHu[2] + Hu[3] \\
&\vdots \\
x[k] &= Gx[k-1] + Hu[k-1] \\
&= G^{k-1}Hu[0] + G^{k-2}Hu[1] + \cdots + GHu[k-2] + Hu[k-1] \\
&= \left[\begin{array}{c|c|c|c|c} G^{k-1}H & G^{k-2}H & \cdots & GH & H \end{array}\right] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix} \\
&= \sum_{j=0}^{k-1} G^{k-j-1}Hu[j] \\
&= \sum_{j=0}^{k-1} G^{j}Hu[k-j-1]
\end{aligned}$$

Given that $\Psi[k] = G^k$

$$\begin{aligned}
x[k] &= \sum_{j=0}^{k-1} \Psi[k-j-1]Hu[j] \\
&= \sum_{j=0}^{k-1} \Psi[j]Hu[k-j-1]
\end{aligned}$$

If we combine homegeneous and driven responses we can simply obtain

$$\begin{aligned}
x[k] &= \Psi[k]x[0] + \sum_{j=0}^{k-1} \Psi[k-j-1]Hu[j] \\
&= \Psi[k]x[0] + \sum_{j=0}^{k-1} \Psi[j]Hu[k-j-1]
\end{aligned}$$

whereas output at time k has the form

$$\begin{aligned}
y[k] &= C\Psi[k]x[0] + C\left(\sum_{j=0}^{k-1} \Psi[k-j-1]Hu[j]\right) + Du[k] \\
&= C\Psi[k]x[0] + C\left(\sum_{j=0}^{k-1} \Psi[j]Hu[k-j-1]\right) + Du[k]
\end{aligned}$$

Z-domain Solution of State-Space Equations

We already computed the transfer function under zero initial conditions.

$$Y(z) = \left[C\,(zI - G)^{-1}\,H + D\right] U(z)$$

Now let's compute the response to initial condition in Z-domain.

$$\begin{aligned}
\mathcal{Z}\,[x[k+1]] &= \mathcal{Z}\,[Gx[k]] \\
zX(z) - zx[0] &= GX(z) \\
(zI - G)\,X(z) &= zX(z) \\
X(z) &= z\,(zI - G)^{-1}\,x[0]
\end{aligned}$$

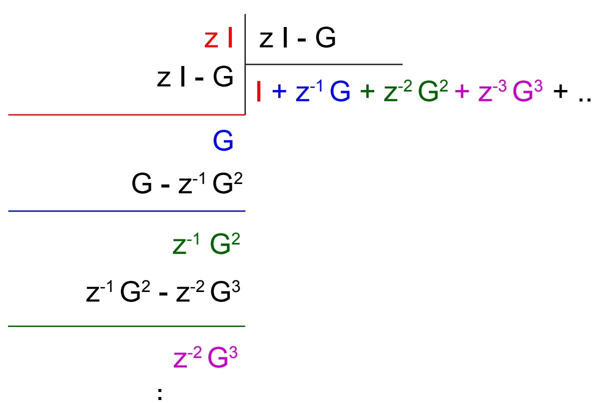
Similarly $Y(z)$ takes the form

$$Y(z) = zC\,(zI - G)^{-1}\,x[0]$$

We can also observe that

$$\begin{aligned}
\mathcal{Z}\,[\Psi[k]] = \mathcal{Z}[G^k] &= z\,(zI - G)^{-1} \\
\mathcal{Z}^{-1}[z\,(zI - G)^{-1}] &= \Psi[k] = G^k
\end{aligned}$$

If we expand $z\,(zI - G)^{-1}$ by long "division" we can also observe the relation in z-domain and time doman expressions from a different perspective



$$\begin{aligned}
z\,(zI - G)^{-1} &= I + z^{-1}G + z^{-2}G^2 + z^{-3}G^3 + \cdots \\
\mathcal{Z}^{-1}\left[z\,(zI - G)^{-1}\right] &= I\,\delta[k] + G\,\delta[k-1] + G^2\,\delta[k-2] + G^3\,\delta[k-3] + \cdots
\end{aligned}$$

Example: Consider the following state-space representation

$$x[k+1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} x[k]$$

- Compute the closed form expression $\Psi[k]$ using the time expression

  Solution: The state-space representation is in Diagonal canonical form

$$\Psi[k] = G^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix}^k = \begin{bmatrix} 1^k & 0 & 0 \\ 0 & (1/2)^k & 0 \\ 0 & 0 & (-1)^k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^k & 0 \\ 0 & 0 & (-1)^k \end{bmatrix}$$

- Compute the closed form expression $\Psi[k]$ using the z-domain solution method

  Solution:

$$\begin{aligned}
\Psi[k] &= \mathcal{Z}^{-1}\left[z\,(zI-G)^{-1}\right] \\
&= \mathcal{Z}^{-1}\left[z\left(\begin{bmatrix} z-1 & 0 & 0 \\ 0 & (z-1/2) & 0 \\ 0 & 0 & z+1 \end{bmatrix}\right)^{-1}\right] \\
&= \mathcal{Z}^{-1}\left[\begin{bmatrix} \frac{z}{z-1} & 0 & 0 \\ 0 & \frac{z}{z-1/2} & 0 \\ 0 & 0 & \frac{z}{z+1} \end{bmatrix}\right] \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^k & 0 \\ 0 & 0 & (-1)^k \end{bmatrix} \quad \text{for} k \geq 0
\end{aligned}$$

- Compute the impulse response of the system from the time domain solution

  Solution:

$$\begin{aligned}
x[k] &= G^{k-1}Hu[0] \quad \text{for } k>0 \\
y[k] &= CG^{k-1}H \quad \text{for } k>0 \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^{k-1} & 0 \\ 0 & 0 & (-1)^{k-1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ (1/2)^{k-1} \\ (-1)^{k-1} \end{bmatrix} \\
y[k] &= 1 + 2(1/2)^{k-1} + 3(-1)^{k-1} \quad \text{for } k>0
\end{aligned}$$

- Compute the transfer function $\frac{Y(z)}{U(z)}$

Solution:

$$
\begin{aligned}
T(z) &= C\,(zI-G)^{-1}\,H \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} z-1 & 0 & 0 \\ 0 & z-1/2 & 0 \\ 0 & 0 & z+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{z-1} & 0 & 0 \\ 0 & \frac{1}{z-1/2} & 0 \\ 0 & 0 & \frac{1}{z+1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{z-1} \\ \frac{1}{z-1/2} \\ \frac{1}{z+1} \end{bmatrix} \\
T(z) &= \frac{1}{z-1} + \frac{2}{z-1/2} + \frac{3}{z+1}
\end{aligned}
$$

- Compute the inverse Z-transform of the transfer function

  Solution:

$$
\begin{aligned}
t[k] &= \mathcal{Z}^{-1}\left[\frac{1}{z-1} + \frac{2}{z-1/2} + \frac{3}{z+1}\right] \\
&= \left(1 + 2(1/2)^{k-1} + 3(-1)^{k-1}\right) h[k-1]
\end{aligned}
$$

where $h[k]$ is the unit step function