EE502 - Linear Systems Theory II **Spring 2032**

Lecture 7

Lecturer: Assoc. Prof. M. Mert Ankarali

7.1 Discrete-Time Linear Time Varying State Space Models

State-space representation of a (causal & finite dimensional) LTV DT system is given by

$$\begin{aligned}
\text{Let } x[k] \in \mathbb{R}^n \ , \ y[k] \in \mathbb{R}^m \ , \ u[k] \in \mathbb{R}^r, \\
x[k+1] = A[k]x[k] + B[k]u[k], \\
y[k] = C[l]x[k] + Du[k], \\
\text{where } G[k] \in \mathbb{R}^{n\times n} \ , \ B[k] \in \mathbb{R}^{n\times r} \ , \ C[k] \in \mathbb{R}^{m\times n} \ , \ D[d] \in \mathbb{R}^{m\times r}
\end{aligned}$$

Let's first assume that $u[k] = 0$, and find un-driven response.

$$\begin{aligned}
x[k+1] &= A[k]x[k] \\
y[k] &= C[k]x[k]
\end{aligned}$$

Unlike LTV-CT systems we easily can compute the response iteratively

$$\begin{aligned}
&x[0] = Ix[0] \quad , \quad y[0] = C[0]x[0] \\
&x[1] = A[0]x[0] \quad , \quad y[1] = C[1]x[1] \\
&x[2] = A[0]x[1] = A[1]A[0]x[0] \quad , \quad y[2] = C[2]x[2] \\
&x[3] = A[2]x[2] = A[3]A[1]A[0]x[0] \quad , \quad y[3] = C[3]x[3] \\
&\quad \vdots \\
&x[k] = A[k-1]x[k-1] = A[k-1]A[k-2]\cdots A[1]A[0]x[0] \quad , \quad y[k] = \quad , \quad y[k] = C[k]x[k] \\
&x[k] = \prod_{i=0}^{k-1} A[k-1-i]
\end{aligned}$$

Motivated by the LTI case, we define the **state transition matrix**, which relates the state of the undriven system at time k to the state at an earlier time m

$$x[k] = \Phi[k,m]x[m] \ , \ k \geq m \ , \ \text{where}$$

$$\Phi[k,m] = \begin{cases} \prod_{i=0}^{k-1} A[k-1-i] \ , & k > m \\ I & , \ k = m \end{cases}$$

Note that state-transition matrix satisfies following important properties undriven system at time k to the state at an earlier time m

$$\begin{aligned}
\Phi[k,k] &= I \\
x[k] &= \Phi[k,0]x[0] \\
\Phi[k+1,m] &= A[k]\Phi[k,m]
\end{aligned}$$

as you can see, the state-transition matrix satisfies the discrete dynamical state equations. Now let's consider input-only state response (i.e. $x[0] = 0$).

$$x[k+1] = A[k]x[k] + B[k]u[k]$$

$$x[1] = B[0]u[0] = \Phi[1,1]B[0]u[0]$$

$$x[2] = A[1]x[1] + B[1]u[1] = A[1]B[0]u[0] + B[1]u[1] = \Phi[2,1]B[0]u[0] + \Phi[2,2]B[1]u[1]$$

$$\begin{aligned}
x[3] &= A[2]x[2] + B[2]u[2] = A[2]A[1]B[0]u[0] + A[2]B[1]u[1] + B[2]u[2] \\
&= \Phi[3,1]B[0]u[0] + \Phi[3,2]B[1]u[1] + \Phi[3,3]B[2]u[2] \\
x[4] &= A[3]x[3] + B[3]u[3] = A[3]A[2]A[1]B[0]u[0] + A[3]A[2]B[1]u[1] + A[3]B[2]u[2] + B[3]u[3] \\
&= \Phi[4,1]B[0]u[0] + \Phi[4,2]B[1]u[1] + \Phi[4,3]B[2]u[2] + \Phi[4,4]B[3]u[3] \\
&\vdots \\
x[k] &= \Phi[k,1]B[0]u[0] + \Phi[k,2]B[1]u[1] + \cdots + \Phi[k,k-1]B[k-2]u[k-2] + \Phi[k,k]B[k-1]u[k-1] \\
&= \left[\begin{array}{c|c|c|c|c} \Phi[k,1]B[0] & \Phi[k,2]B[1] & \cdots & \Phi[k,k-1]B[k-2] & \Phi[k,k]B[k-1] \end{array}\right] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix} \\
&= \sum_{j=0}^{k-1} \Phi[k,j+1]B[j]u[j] \\
&= \Gamma[k,0]\mathcal{U}[k,0]
\end{aligned}$$

where

$$\Gamma[k,0] = \left[\begin{array}{c|c|c|c|c} \Phi[k,1]B[0] & \Phi[k,2]B[1] & \cdots & \Phi[k,k-1]B[k-2] & \Phi[k,k]B[k-1] \end{array}\right]$$

$$\mathcal{U}[k,0] = \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix}$$