

## Lecture 6

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## 6.1 Modal Decomposition of State-Space Models

### 6.1.1 Zero Input Response

Let's consider autonomous LTI CT and DT state-space models

In linear systems theory course, we are interested in matrix polynomials, specifically

$$\begin{aligned}\dot{x} &= Ax \\ x[k+1] &= Ax[k]\end{aligned}$$

Let  $x_0 = \alpha v_i$ , where  $v_i$  is an eigenvector of  $A$  associated with eigenvalue  $\lambda_i$ , we can then find the solution for both systems

$$\begin{aligned}x(t) &= e^{At}x_0 = \alpha e^{\lambda_i t}v_i \\ x[k] &= A^k x_0 = \alpha \lambda_i^k v_i\end{aligned}$$

Now let's assume that  $A$  is diagonalizable, then we now that there exist a set of  $n$  linearly independent eigenvectors  $\mathcal{V} = \{v_1, \dots, v_n\}$ . Thus, we can write any initial condition,  $x_0 \in \mathbb{R}^n$ , as a linear combination of eigenvectors, i.e.

$$x_0 = \sum_{i=1}^n \alpha_i v_i$$

if we use this (modal) decomposition we can find the solutions of DT and CT equations as

$$\begin{aligned}x(t) &= e^{At}x_0 = \sum_{i=1}^n \alpha_i e^{\lambda_i t} v_i \\ x[k] &= A^k x_0 = \sum_{i=1}^n \alpha_i \lambda_i^k v_i\end{aligned}$$

where  $e^{\lambda_i t} v_i$  ( $\lambda_i^k v_i$  in DT case) is called a “mode” of the system. Now let's try to find  $\{\alpha_1, \dots, \alpha_n\}$  via

diagonalization of  $A$

$$A = V\Lambda V^{-1} = V \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ & 0 & & \ddots \\ & & & & \lambda_n \end{bmatrix} V^{-1}, \text{ where}$$

$$V = [v_1 \ \cdots \ v_n], \text{ where } Av_i = \lambda_i v_i$$

$$V^{-1} = \bar{V} = \begin{bmatrix} \bar{v}_1^T \\ \vdots \\ \bar{v}_n^T \end{bmatrix}, \text{ where } \bar{v}_i^T A = \lambda_i \bar{v}_i^T$$

$$\bar{V}V = V\bar{V} = I, \ \bar{v}_i^T v_i = 1, \ \bar{v}_i^T v_j = 0 \text{ for } i \neq j$$

Now let's compute the zero-input responses for an arbitrary  $x_0$

$$x(t) = e^{At}x_0 = Ve^{\Lambda t}V^{-1}x_0 = [v_1 \ \cdots v_n] \begin{bmatrix} e^{\lambda_1 t} \bar{v}_1^T x_0 \\ \vdots \\ e^{\lambda_n t} \bar{v}_n^T x_0 \end{bmatrix} = \sum_{i=1}^n v_i e^{\lambda_i t} (\bar{v}_i^T x_0) \rightarrow \alpha_i = \bar{v}_i^T x_0$$

$$x[k] = V\Lambda^k V^{-1}x_0 = [v_1 \ \cdots v_n] \begin{bmatrix} \lambda_1^k \bar{v}_1^T x_0 \\ \vdots \\ \lambda_n^k \bar{v}_n^T x_0 \end{bmatrix} = \sum_{i=1}^n v_i \lambda_i^k (\bar{v}_i^T x_0) \rightarrow \alpha_i = \bar{v}_i^T x_0$$

Based on these results, we can see that in order to excite the  $i^{th}$  mode the system, we need  $\bar{v}_i^T x_0 \neq 0$ . If we treat initial conditions as inputs this generates a reachability/controllability argument. Let's also analyze the output response

$$y(t) = Cx(t) = Ce^{At}x_0 = \sum_{i=1}^n (Cv_i) e^{\lambda_i t} (\bar{v}_i^T x_0)$$

$$y[k] = Cx[k] = \sum_{i=1}^n (Cv_i) \lambda_i^k (\bar{v}_i^T x_0)$$

We can see that if  $Cv_i = 0$ , then we can not observe the  $i^{th}$  mode at the output  $\forall x_0 \in \mathbb{R}^n$ . Thus we can conclude that in order to have a fully observable system all modes needs to be observable, i.e. i.e.  $Cv_i \neq 0 \ \forall i \in \{1, \dots, n\}$ .

Now let's try to find a similar solution for systems where matrix  $A$  can not be diagonalizable. For the sake of clarity let's focus on matrices that is composed of a single Jordan block

$$A = GJG^{-1} = G \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots \\ \vdots & & \ddots & & \\ & & & \ddots & 1 \\ 0 & & & 0 & \lambda \end{bmatrix} G^{-1}$$

where

$$\begin{aligned}
 G &= \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \\
 Ag_1 &= \lambda g_1 \rightarrow (A - \lambda I)g_1 = 0 \\
 Ag_2 &= \lambda g_2 + g_1 \rightarrow (A - \lambda I)g_2 = g_1, \text{ note } (A - \lambda I)^2 g_2 = 0 \text{ \& } (A - \lambda I)g_2 \neq 0 \\
 Ag_3 &= \lambda g_3 + g_2 \rightarrow (A - \lambda I)g_3 = g_2, \text{ note } (A - \lambda I)^3 g_3 = 0 \text{ \& } (A - \lambda I)^2 g_3 \neq 0 \\
 &\vdots \\
 Ag_n &= \lambda g_n + g_{n-1} \rightarrow (A - \lambda I)g_n = g_{n-1}, \text{ note } (A - \lambda I)^n g_n = 0 \text{ \& } (A - \lambda I)^{n-1} g_n \neq 0
 \end{aligned}$$

and we also know that

$$\begin{aligned}
 G^{-1} &= \bar{G} = \begin{bmatrix} \bar{g}_1^T \\ \vdots \\ \bar{g}_n^T \end{bmatrix} \\
 \bar{G}G &= G\bar{G} = I, \quad \bar{g}_i^T g_i = 1, \quad \bar{g}_i^T g_j = 0 \text{ for } i \neq j
 \end{aligned}$$

Let  $x_0 = \alpha_1 g_1$ , i.e. the eigenvector of  $A$ , then we can find the responses as

$$\begin{aligned}
 x(t) &= e^{At} g_1 = G e^{Jt} G^{-1} g_1 \alpha_1 \\
 &= \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!} e^{\lambda t} & \frac{t^{n-1}}{(n-1)!} e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!} e^{\lambda t} \\ \vdots & & \ddots & & & \vdots \\ 0 & & & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} \\ 0 & & \cdots & & e^{\lambda t} & te^{\lambda t} \\ 0 & & \cdots & & 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
 &= \alpha_1 e^{\lambda t} g_1
 \end{aligned}$$

$$x[k] = G J^k G^{-1} x_0 = \alpha_1 \lambda^k g_1$$

the format of the solution associated with  $g_1$  seems to be exactly same with diagonal case (since  $g_1$  is an eigenvector). This implies that, if we choose an initial condition inside the eigenvector space, the response stays in this space and acts like a “first-order” system. Now, let  $x_0 = \alpha_2 g_2$ , i.e. a first order generalized eigenvector of  $A$ , then we can find the responses as

$$\begin{aligned}
 x(t) &= G e^{Jt} G^{-1} g_2 \alpha_2 \\
 &= \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix} \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!} e^{\lambda t} & \frac{t^{n-1}}{(n-1)!} e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!} e^{\lambda t} \\ \vdots & & \ddots & & & \vdots \\ 0 & & & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} \\ 0 & & \cdots & & e^{\lambda t} & te^{\lambda t} \\ 0 & & \cdots & & 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_2 \\ \vdots \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix} \begin{bmatrix} \alpha_2 t e^{\lambda t} \\ \alpha_2 e^{\lambda t} \\ \vdots \\ 0 \end{bmatrix} = \alpha_2 (t e^{\lambda t} g_1 + e^{\lambda t} g_2)
 \end{aligned}$$

$$\begin{aligned}
x[k] &= GJ^k G^{-1} g_2 \alpha_2 \\
&= \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix} \begin{bmatrix} \lambda^k & \binom{k}{1} \lambda^{k-1} & \binom{k}{2} \lambda^{k-2} & \cdots & \binom{k}{n-2} \lambda^{k-n+2} & \binom{k}{n-1} \lambda^{k-n+1} \\ 0 & \lambda^k & \binom{k}{1} \lambda^{k-1} & \binom{k}{2} \lambda^{k-2} & \cdots & \binom{k}{n-2} \lambda^{k-n+2} \\ \vdots & & \ddots & & & \vdots \\ & & & \lambda^k & \binom{k}{1} \lambda^{k-1} & \binom{k}{2} \lambda^{k-2} \\ 0 & & \cdots & & \lambda^k & \binom{k}{1} \lambda^{k-1} \\ 0 & & \cdots & & 0 & \lambda^k \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_2 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix} \begin{bmatrix} \alpha_2 k \lambda^{k-1} \\ \alpha_2 \lambda^k \\ \vdots \\ 0 \end{bmatrix} = \alpha_2 (k \lambda^{k-1} g_1 + \lambda^k g_2)
\end{aligned}$$

We can observe that the response acts like a “second-order” (critically-damped) response. Moreover, the response does not stay inside the span of the generalized eigenvector, i.e.  $\text{Span}\{g_2\}$ , instead it navigates inside the span of the eigenvector and  $g_2$ , i.e.  $\text{Span}\{g_1, g_2\} = \mathcal{N}(A - \lambda I)^2$ . Now, let  $x_0 = \alpha_i g_i$ ,  $0 \leq i \leq n$ , i.e. order generalized eigenvector of order  $i$ , then we can find the responses as

$$\begin{aligned}
x(t) &= G e^{Jt} G^{-1} g_2 \alpha_2 = \alpha_i e^{\lambda t} \sum_{j=1}^i g_j \frac{t^{i-j}}{(i-j)!} \\
x[k] &= G J^k G^{-1} g_i \alpha_i = \alpha_i \sum_{j=1}^i g_j \binom{k}{i-j} \lambda^{k-i+j}
\end{aligned}$$

Similar to the second-order case, we can see that response acts like an  $i^{th}$  order dynamical system, and trajectories stay inside,  $\text{Span}\{g_1, \dots, g_i\} = \mathcal{N}(A - \lambda I)^i$ . In this context, in order to excite all higher order modes of the system projection of the initial condition to the sub-space spanned highest order generalized eigenvector has to be non-zero. Now, let's try to find general solution for an arbitrary  $x_0$ . We can write any  $x_0 \in \mathbb{R}^n$  as a linear combination of  $\mathcal{G} = \{g_1, g_2, \dots, g_n\}$ , thus we have

$$\begin{aligned}
x_0 &= \sum_{i=1}^n \alpha_i g_i \\
x(t) &= \sum_{i=1}^n \alpha_i e^{\lambda t} \sum_{j=1}^i g_j \frac{t^{i-j}}{(i-j)!} \\
x[k] &= \sum_{i=1}^n \alpha_i \sum_{j=1}^i g_j \binom{k}{i-j} \lambda^{k-i+j}
\end{aligned}$$

where

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = G^{-1} x_0 = \begin{bmatrix} \bar{g}_1^T \\ \bar{g}_2^T \\ \vdots \\ \bar{g}_n^T \end{bmatrix} x_0 \rightarrow \alpha_i = \bar{g}_i^T x_0$$

Based on these results, we can see that in order to excite all of the modes associated with a Jordan block of size  $n$ , we need  $\alpha_n \bar{g}_n^T x_0 \neq 0$ . If we treat initial conditions as inputs this generates a reachability/controllability argument. Thus in order for this Jordan block to be reachable/controllable, we need to excite highest order mode (generalized eigenvector).

**Ex 6.1** Let

$$\dot{x} = Ax \text{ where } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

derive the solution of  $x(t)$  using modal decomposition for an arbitrary  $x_0 \in \mathbb{R}^3$

**Solution:** We know that Jordan canonical form of matrix  $A$  has the form

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the transformation matrices that leads to this Jordan form are

$$G = \begin{bmatrix} g_1 & g_2 & v \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

where  $g_1$  and  $v$  are eigenvectors and  $g_2$  is the single generalized eigenvector associated with  $g_1$ . If we follow the derivation in this lecture note, we can first find all individual components of the responses as

$$\begin{aligned} x_{g_1}(t) &= \alpha_{g_1} e^t g_1 \\ x_{g_2}(t) &= \alpha_{g_2} (te^t g_1 + e^t g_2) \\ x_v(t) &= \alpha_v e^t v \end{aligned}$$

where the combined solution and  $\alpha_*$ 's can be derived using

$$\begin{aligned} x(t) &= x_{g_1}(t) + x_{g_2}(t) + x_v(t) = e^t ((\alpha_{g_1} + t\alpha_{g_2})g_1 + \alpha_{g_2}g_2 + \alpha_v v) \\ \begin{bmatrix} \alpha_{g_1} \\ \alpha_{g_2} \\ \alpha_v \end{bmatrix} &= G^{-1} x_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} x_0 \end{aligned}$$