EE302 - Feedback Systems	**Spring 2019**

Lecture 16

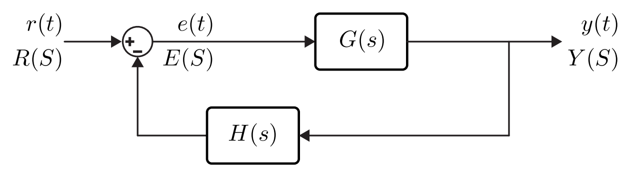
Lecturer: Asst. Prof. M. Mert Ankarali

16.1 Nyquist Stability Criterion for Feedback Systems

Even though we illustrated how to apply Nyquist stability criterion for feedforward systems in the previous lecture (for teaching the details of Nyquost plot and some basics), in control theory and applications Nyqist plot and Nyquist stability test are majorly used for analyzing feedback topologies.

16.1.1 Nyquist Stability for Feedback Systems

The figure below illustrates the fundamental feedback system topology for a SISO system



We know that the closed-loop transfer function, $T(s)$, for this system has the following form

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{G(s)}{1 + G_{OL}(s)}$$

where $G_{OL}(s)$ is the open-loop transfer function for the given topology. We know that poles of $T(s)$ are the roots that satisfy $1 + G_{OL}(s) = 0$. Now let's define an analytic function $F(s) = 1 + G_{OL}(s)$, analyze its relation with the open-loop and closed-loop transfer functions.

$$F(s) = 1 + G_{OL}(s) = 1 + \frac{N(s)}{D(s)} = \frac{D(s) + N(s)}{D(s)}$$

where

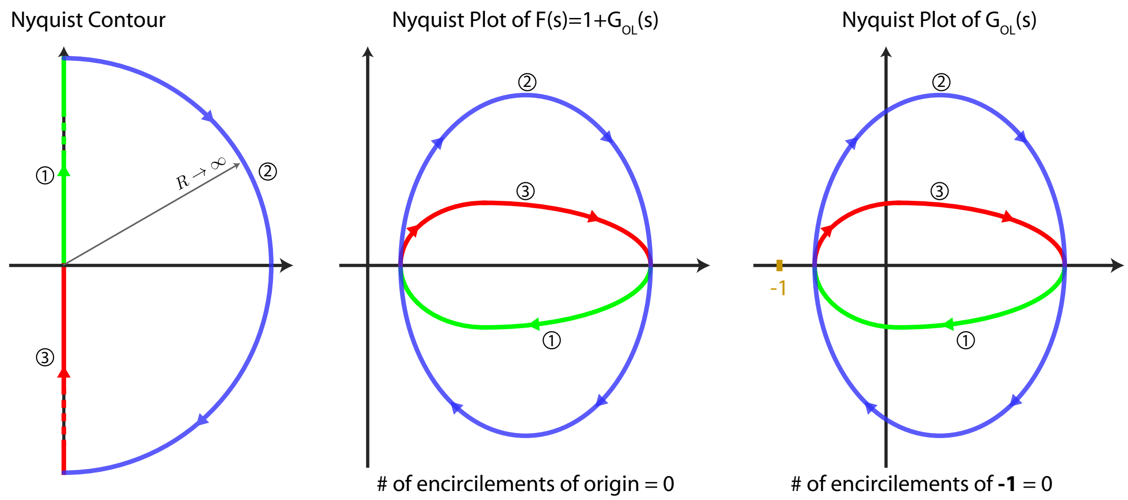
- Roots of $N(s)$ are open-loop zeros
- Roots of $D(s)$ are open-loop poles

Moreover

- **Poles** of $F(s)$ are the roots of $D(s)$, hence they constitutes the **open-loop poles**.
- **Zeros** of $F(s)$ are the poles of $T(s)$, hence they constitutes the **closed-loop poles**.

Note that open-loop zeros and poles are "known", and goal is to investigate/analyze the number of unstable poles of $T(s)$ which is equal to the number of zeros of $F(s)$ with positive real parts

Now let's assume that we derive Nyquist plots of $F(s) = 1 + G_{OL}(s)$ and $G_{OL}(s)$ and obtain $\Gamma_{F(s)}$ and $\Gamma_{G_{OL}(s)}$. The figure below provides an illustrative example of a Nyquist contour, Nyquist plot of $F(s)$, and Nyquist plot of $G_{OL}(s)$.



First simple observation that we have to pay attention is that in order to obtain $\Gamma_{G_{OL}(s)}$ we shift $\Gamma_{F(s)}$ on real axis to the left (by **1**), similarly in order to obtain $\Gamma_{F(s)}$ we shift $\Gamma_{G_{OL}(s)}$ on real axis to the right ((by **1**)). From this fact we can derive the following equality which is critical for stability analysis

$$\text{N} \;=\; \#\text{ of CW enrichments of origin by } \Gamma_{F(s)} \;=\; \#\text{ of CW enrichments of } (-1+0j) \text{ by } \Gamma_{G_{OL}(s)}.$$

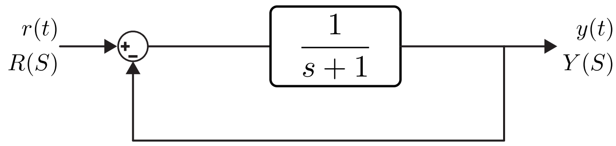
For example In this illustration above $N = 0$. If we apply Cauchy's Principle argument for $F(s)$, we can derive that

- $N = Z_F - P_F$
- P_F : # poles of $F(s)$ with positive real parts, which is indeed equal to the # poles of $G_{OL}(s)$ with positive real parts which is a "known" quantity.
- Z_F : # zeros of $F(s)$ with positive real parts, which is indeed equal to to the # unstable poles of $T(s)$ which is the desired output.

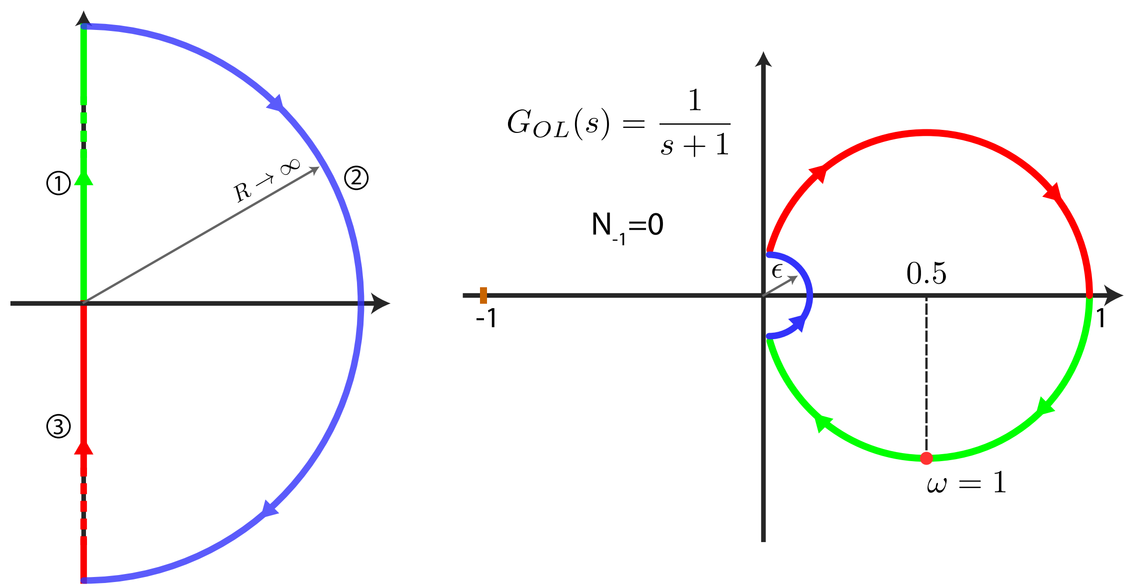
In conclusion, in order to analyze the closed-loop stability of a unity-feedback feed-back systems we apply the following procedure

- Draw the Nyquist plot of $G_{OL}(s)$
- Compute $N =$ # CW encirclements of $(-1 + 0j)$ by $\Gamma_{G_{OL}(s)}$
- Compute $P_F = P_{OL} =$ # open-loop poles with positive real parts
- Finally compute $\mathbf{P_{CL}} = Z_F = N + P_F = \mathbf{N} + \mathbf{P_{OL}} =$ # unstable closed-loop poles

Ex: Analyze the stability of the following feedback system using Nyquist plot.



Solution: For this given system $G_{OL}(s) = \frac{1}{s+1}$. In the previous lecture we already derived the Nyquist plot for $\frac{1}{s+1}$ which is illustrated below



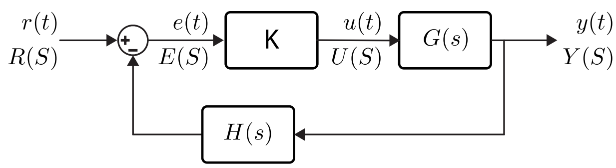
We can conclude from the derived Nyquist plot and open-loop transfer function

- $N = 0$
- $P_{OL} = 0$ (since there is no open-loop unstable pole)
- Finally compute $P_{CL} = 0 = \#$ unstable closed-loop poles.

Thus the closed-loop system is indeed stable.

16.1.2 Nyquist Stability for Feedback Systems with Varying DC Gains

Now let's consider the following feedback system topology for a SISO system where DC gain of the open-loop transfer function is adjusted with a gain parameter K (e.g. P controller).



We would like to test the stability of the closed-loop system for different values of K, moreover we would like to derive the range of K values that makes the closed-loop system stable. Indeed, we don't need to re-draw the Nyquist plot for each K that we want to test.

The analytic function, $F(s)$, that we adopt for analyzing stability can be written in the form

$$F(s) = 1 + G_{OL}(s) = 1 + K\frac{N(s)}{D(s)}$$

where

- Roots of $N(s)$ are open-loop zeros
- Roots of $D(s)$ are open-loop poles
- Varible gain parameter K

In order to analyze the stability of Nyquist plot, in the previous section we showed that we can draw the Nyqist plot $G_{OL}(s)$ and analyze the # encirclements of $(-1 + 0j)$.

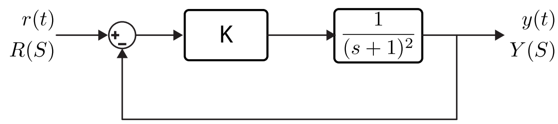
Now, in this case we will derive the Nyquist plot of $\frac{N(s)}{D(s)}$, i.e. $\Gamma_{\frac{N(s)}{D(s)}}$. Note that $\Gamma_{G_{OL}(s)} = K\Gamma_{\frac{N(s)}{D(s)}}$, i.e. we simply scale the Nyquist plot of $\frac{N(s)}{D(s)}$ with K (which can be either positive or negative) to find the Nyquist plot of $G_{OL}(s)$. It is very straightforward to conclude that

$$N \;=\; \#\text{ of CW enrichments of } (-1+0j) \text{ by } \Gamma_{G_{OL}(s)} \;=\; \#\text{ of CW enrichments of } \left(-\tfrac{1}{K}+0j\right) \text{ by } \tfrac{N(s)}{D(s)}.$$

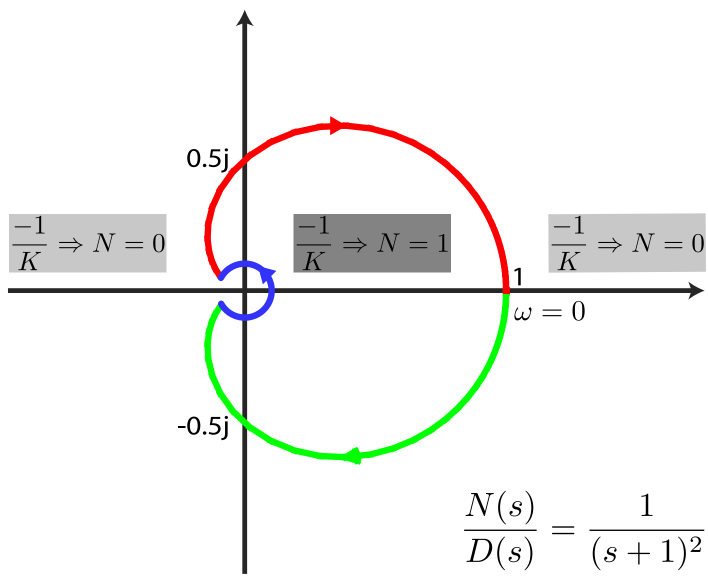
To sum-up, in order to analyze the closed-loop stability of a feedback systems for which we have a variable gain parameter K in the open-loop transfer function, we apply the following procedure

- Draw the Nyquist plot of $\frac{N(s)}{D(s)}$ where $G_{OL}(s) = K\frac{N(s)}{D(s)}$
- Compute $N =$ # CW encirclements of $\left(-\frac{1}{K} + 0j\right)$ by $\Gamma_{\frac{N(s)}{D(s)}}$
- Compute $P_{OL} =$ # open-loop poles with positive real parts, i.e. # roots of $D(s)$.
- Finally compute $P_{CL} = N + P_{OL} =$ # unstable closed-loop poles

Ex: Find the range of K values that makes the following closed-loop system stable.



Solution: For this given system $\frac{N(s)}{D(s)} = \frac{1}{(s+1)^2}$. In the previous lecture we already derived the Nyquist plot for $\frac{1}{(s+1)^2}$ which is illustrated below



We can see from the derived Nyquist plot that the closed path divides the real axis in three different parts. If we analyze these regions separately, we can then find a complete range of K values that makes the closed loop-system stable

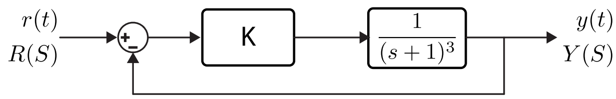
$$\frac{-1}{K} \in (-\infty, 0) \ \rightarrow \ K \in [0, \infty) \Rightarrow N = 0$$
$$\frac{-1}{K} \in (0, 1) \ \rightarrow \ K \in (-\infty, -1) \Rightarrow N = 1$$
$$\frac{-1}{K} \in (1, \infty) \rightarrow K \in (-1, 0] \Rightarrow N = 0$$

Note that since number of open-loop unstable poles is equal to 0, we have $P_{OL} = 0$. Thus, system is BIBO stable if and only if $N = 0$. Whereas, for the region when $N = 1$, there always exist one unstable pole.

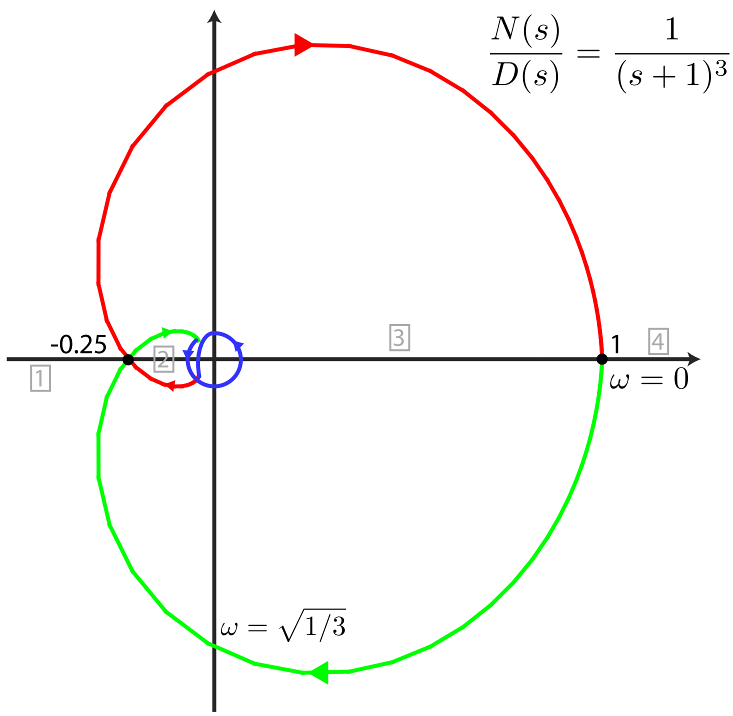
Note that for $K = -1$, we don't have a conclusion, indeed system is BIBO unstable since in this case there exist a pole on the imaginary axis. As a result

$$K \in (-1, \infty) \Leftrightarrow \text{BIBO} - \text{Stable}$$

Ex: Find the range of K values that makes the following closed-loop system stable. Also, for the K values that makes the closed-loop system unstable find the number of unstable poles.



Solution: For this given system $\frac{N(s)}{D(s)} = \frac{1}{(s+1)^3}$. In the previous lecture we already derived the Nyquist plot for $\frac{1}{(s+1)^2}$ which is illustrated below



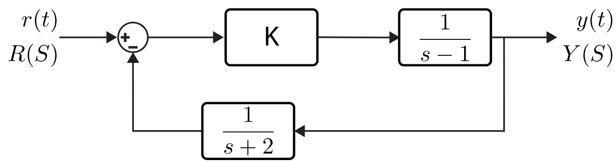
As we can see from the Nyquist plot pf $\frac{1}{(s+1)^3}$, it divides the region into four different parts. If we analyze these regions separately, we can then find a complete range of K values that makes the closed loop-system stable, as well as find the number of unstable poles for the unstable cases.

1. $\frac{-1}{K} \in \left(-\infty, \frac{-1}{4}\right) \rightarrow K \in [0, 4) \Rightarrow (N = 0, P_{CL} = 0)$, closed-loop system is stable
2. $\frac{-1}{K} \in \left(\frac{-1}{4}, 0\right) \rightarrow K \in (4, \infty) \Rightarrow (N = 2, P_{CL} = 2)$. Unstable CL system with 2 unstable poles
3. $\frac{-1}{K} \in (0, 1) \rightarrow K \in (-\infty, -1) \Rightarrow (N = 1, P_{CL} = 1)$. Unstable CL system with 1 unstable pole
4. $\frac{-1}{K} \in (1, \infty) \rightarrow K \in (-1, 0) \Rightarrow (N = 0, P_{CL} = 0)$, closed-loop system is stable

As a result

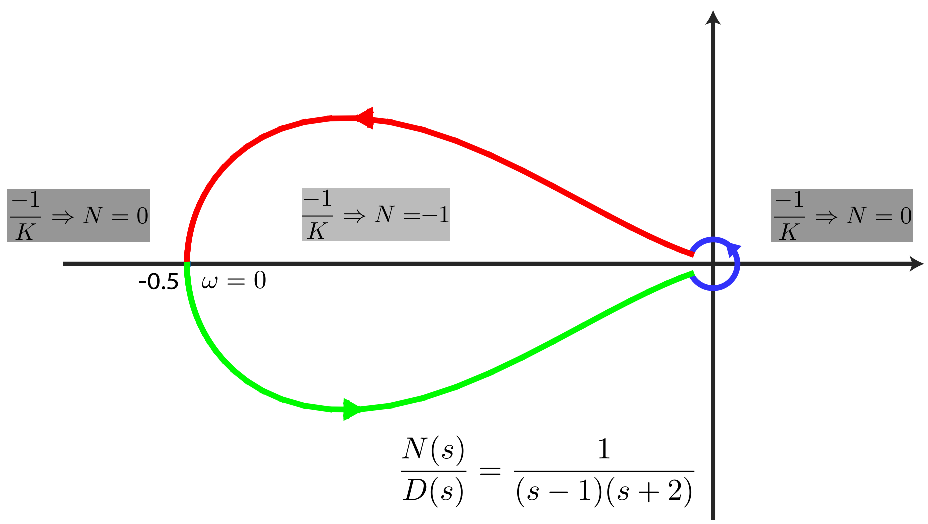
$$K \in (-1, 4) \Leftrightarrow \mathrm{BIBO-Stable}$$

Ex: Find the range of K values that makes the following closed-loop system stable. Also, for the K values that makes the closed-loop system unstable find the number of unstable poles.



Solution: For this given system $\frac{N(s)}{D(s)} = \frac{1}{(s-1)(s+2)}$. Note that for this system $P_{OL} = 1$.

In the previous lecture we already derived the Nyquist plot for $\frac{1}{(s-1)(s+2)}$ which is illustrated below



We can see that the Nyquist plot divides the real axis in three different parts. If we analyze these regions separately, we can then find a complete range of K values that makes the closed loop-system stable

1. $\frac{-1}{K} \in (-\infty, \frac{-1}{2}) \ \rightarrow K \in (0, 2) \ \Rightarrow N = 0 \rightarrow P_{CL} = 1$. Unstable CL-system with 1 unstable pole.
2. $\frac{-1}{K} \in \left(\frac{-1}{2}, 0\right) \rightarrow K \in (2, \infty) \Rightarrow \ N = -1 \rightarrow P = 1 - 1 = 0$. Stable CL-system
3. $\frac{-1}{K} \in (0, \infty) \rightarrow K in(-\infty, 0) \Rightarrow N = 0 \rightarrow P = 1$. Unstable CL-system with 1 unstable pole.

As a result

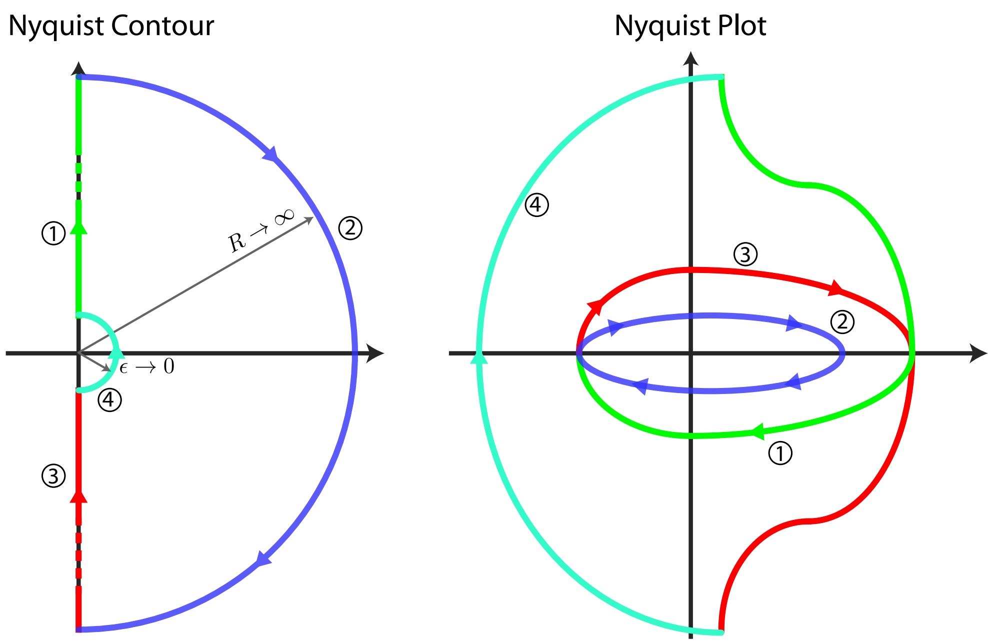
$$K \in (2, \infty) \Leftrightarrow \text{BIBO} - \text{Stable}$$

16.1.3 Nyquist Plot and Stability Test with Open-Loop Poles/Zeros on the Imaginary Axis

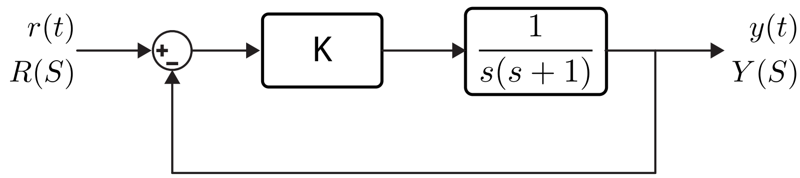
If you remember, when we introduced Nyquist contour and Nyquist stability test we assumed that there is no pole/zero on the imaginary axis. However, it is especially very common to have poles at the origin since it corresponds to simple integrator. In this course, we will explicitly cover the case when there exist a pole or zero at the origin.

Let's assume that $G_{OL}(s)$ has a pole (or zero, or multiple poles) at the origin. We simply modify the Nyquist contour by adding an infinitesimal notch at the origin to the original Nyquist contour.

The figure below illustrates this modified Nyquist contour and an illustrative Nyquist plot.



Ex: Find the range of K values that makes the following closed-loop system stable.



Solution: For this given system, $\frac{N(s)}{D(s)} = \frac{1}{s(s+1)}$. Note, there exist a pole at the origin, thus we need to utilize the modified Nyquist Contour. In the modified Nyquist contour there exist 4 major paths, and we need to draw the Nyquist plot based on these 4 paths.

1. This is the polar plot, where we need to plot $G(j\omega)$, where $\omega : 0 \to \infty$.

$$G(j\omega) = \frac{1}{j\omega(j\omega+1)} = \frac{-j-\omega}{\omega(\omega^2+1)} = \frac{-1}{\omega^2+1} + \frac{-1/\omega}{\omega^2+1}j$$

Some observations about polar plot

$$\begin{aligned} Re\{G(j\omega)\} < 0 \quad &\& \quad Im\{G(j\omega)\} < 0 \quad , \forall\omega > 0 \\ \lim_{\omega\to 0} Re\{G(j\omega)\} = -1 \quad &\& \quad \lim_{\omega\to 0} Re\{G(j\omega)\} = -\infty \\ \lim_{\omega\to\infty} |G(j\omega)| = 0 \quad &\& \quad \lim_{\omega\to\infty} \angle[G(j\omega)] = -\pi \end{aligned}$$

2. Note that since we are now only intersted in the circulation around $-1/K$. The origin in the Nyquist plot corresponds to $|K| \to \infty$, so that it is really not critical to correctly draw this detail for feedback systems. However, let's draw this detail for this example for practice. $s = Re^{j\theta}$ and $\theta : \pi/2 \to -\pi/2$. Then we can derive that

$$\begin{aligned} &G\left(Re^{j\theta}\right) \approx \frac{1}{R^2e^{j2\theta}} = \frac{e^{j(-2\theta)}}{R^2} \\ &\Rightarrow |G\left(Re^{j\theta}\right)| \approx \epsilon^2 \ll 1 \quad , \quad \angle[G\left(Re^{j\theta}\right)] \approx -2\theta \end{aligned}$$

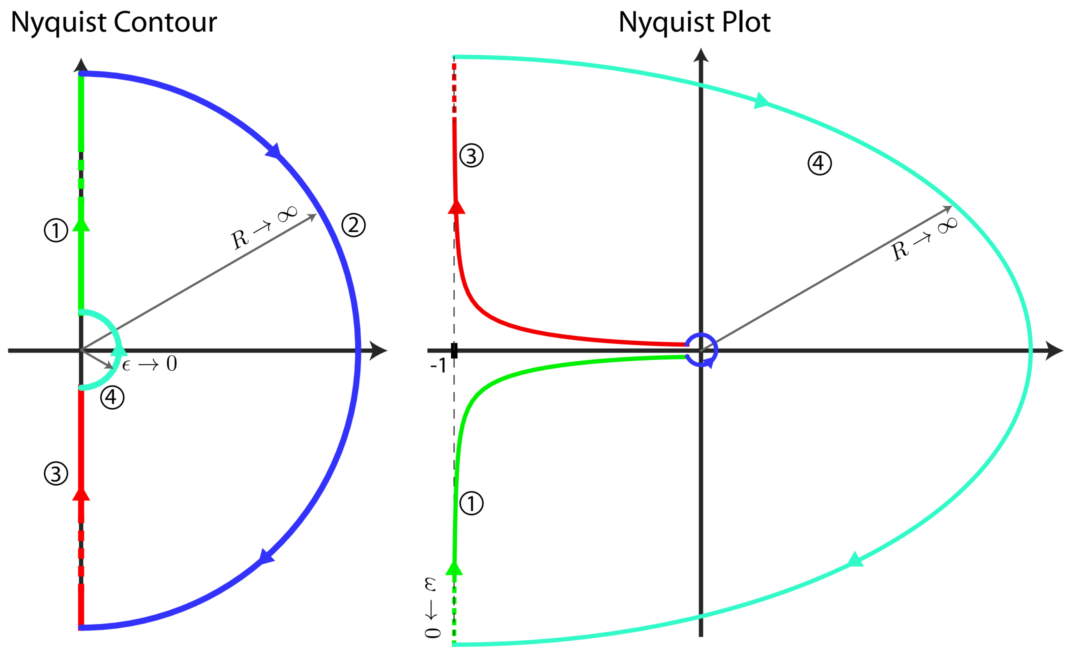
Note that when $\theta : \pi/2 \to -\pi/$, the infinite-small contour around origin rotates in CCW direction.

3. This part is simply the conjugate of polar plot with reverse direction.

4. Now this part is new, since we are dealing with this path since there is a pole at the origin. $s = \epsilon e^{j\phi}$, where $\epsilon \to 0$ and $\phi : -\pi/2 \to \pi/2$ (CCW direction). Then we can derive that

$$\begin{aligned} &G\left(\epsilon e^{j\phi}\right) \approx \frac{1}{\epsilon e^{j\phi}} = Re^{j(-\phi)} \\ &\Rightarrow |G\left(\epsilon e^{j\phi}\right)| \approx R \ll 1 \quad , \quad \angle[G\left(\epsilon e^{j\phi}\right)] \approx -\phi \end{aligned}$$

As a result, we obtain the following Nyquist plot for $\frac{N(s)}{D(s)} = \frac{1}{s(s+1)}$

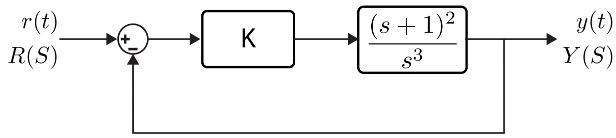


Note that the open-loop transfer function has no unstable poles, i.e. $P_{OL} = 0$. We can see from the Nyquist plot divides the real axis in two different parts. If we analyze these regions separately, we can then find a range of K values that makes the closed loop-system stable

1. $\frac{-1}{K} \in (-\infty, 0) \;\rightarrow\; K \in (0, \infty) \Rightarrow N = 0 \rightarrow P_{CL} = 0$. System is stable
2. $\frac{-1}{K} \in (0, \infty) \rightarrow K \in (-\infty, 0) \Rightarrow N = 1 \rightarrow P_{CL} = 1$. Unstable system with 1 unstable pole

As a result, $K \in (0, \infty) \Leftrightarrow$ BIBO-Stable

Ex: Find the range of K values that makes the following closed-loop system stable.



Solution: For this given system, $\frac{N(s)}{D(s)} = \frac{(s+1)^2}{s^3}$. Note, there exist three repeated pole at the origin, thus we need to utilize the modified Nyquist Contour.

In the modified Nyquist contour there exist 4 major paths, and this we need to draw Nyquist plot based on these 4 paths.

1. This is the polar plot, where we need to plot $G(j\omega)$, where $\omega : 0 \to \infty$.

$$G(j\omega) = \frac{(j\omega+1)^2}{(j\omega)^3} = \frac{(1-\omega^2)+2\omega j}{-j\omega^3} = \frac{(1-\omega^2)j - 2\omega}{\omega^3} = \frac{-2}{\omega^2} + \frac{1-\omega^2}{\omega^3} j$$

   Some observations about polar plot

$$\begin{aligned} Re\{G(j\omega)\} &< 0 \\ Im\{G(j\omega)\} > 0 \quad &, \ \forall\omega \ \in (0,1) \\ Im\{G(j\omega)\} < 0 \quad &, \ \forall\omega \ \in (1,\infty) \\ [G(j\omega)]_{\omega=1} &= -2 + 0j \\ \lim_{\omega\to 0} |G(j\omega)| = \infty \quad & \& \quad \lim_{\omega\to 0} \angle[G(j\omega)] = \pi/2 \\ \lim_{\omega\to \infty} |G(j\omega)| = 0 \quad & \& \quad \lim_{\omega\to \infty} \angle[G(j\omega)] = -\pi/2 \end{aligned}$$

2. Note that since we are intersted in the circulation around $-1/K$ and origin in the Nyquist plot corresponds to $|K| \to \infty$, it is really not critical to correctly draw this detail. Thus we will omit it for this problem.

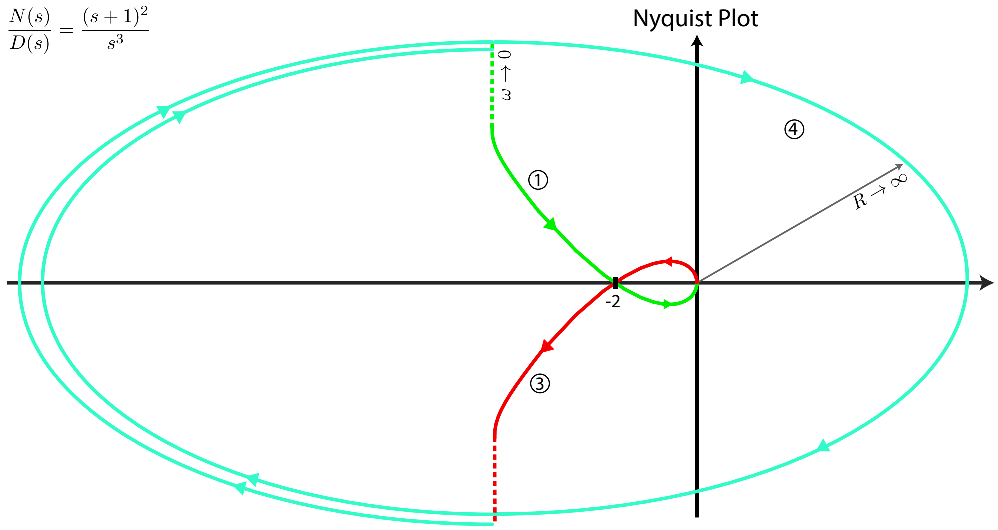
3. This part is simply the conjugate of polar plot with reverse direction.

4. This part is very critical for stability analysis $s = \epsilon e^{j\phi}$, where $\epsilon \to 0$ and $\phi : -\pi/2 \to \pi/2$ (CCW direction). Then we can derive that

$$\begin{aligned} &G\left(\epsilon e^{j\phi}\right) \approx \frac{1}{\epsilon^3 e^{j3\phi}} = R^3 e^{j(-3\phi)} \\ &\Rightarrow |G\left(\epsilon\ e^{j\phi}\right)| \approx R^3 \to \infty \quad , \quad \angle[G\left(\epsilon\ e^{j\phi}\right)] \approx -3\phi \end{aligned}$$

   Note that this path is a 1.5 circle that turns in CW direction

As a result, we obtain the following Nyquist plot for $\frac{N(s)}{D(s)} = \frac{(s+1)^2}{s^3}$



Note that the open-loop transfer function has no unstable poles, thus $P_{OL} = 0$. We can see that the derived Nyquist plot divides the real axis in three different parts. If we analyze these regions separately, we can then find a range of K values that makes the closed loop-system stable

1. $\frac{-1}{K} \in (-\infty, -2) \to K \in (0, 2) \Rightarrow N = 2 \to P_{CL} = 2.$ Unstable system with 2 unstable poles.
2. $\frac{-1}{K} \in (-2, 0) \to K \in (2, \infty) \Rightarrow N = 0 \to P_{CL} = 0.$ System is stable
3. $\frac{-1}{K} \in (0, \infty) \to K \in (-\infty, 0) \Rightarrow N = 1 \to P_{CL} = 1.$ Unstable system with 1 unstable pole.

As a result, $K \in (2, \infty) \Leftrightarrow$ BIBO-Stable