Lecture 2

Lecturer: Asst. Prof. M. Mert Ankarali

*Big Picture of EE402

In this course, the main focus will be on continuous-time systems that are controlled (sampled and actuated) by a digital computer interface. Such a discrete-time control system consists of four major parts as illustrated in Fig. 2.2,

1. *The plant* is a continuous-time dynamical system

2. The Analog-to-Digital Converter (ADC)

3. The Controller (μP), a microprocessor/microcontroller with a "real-time" OS
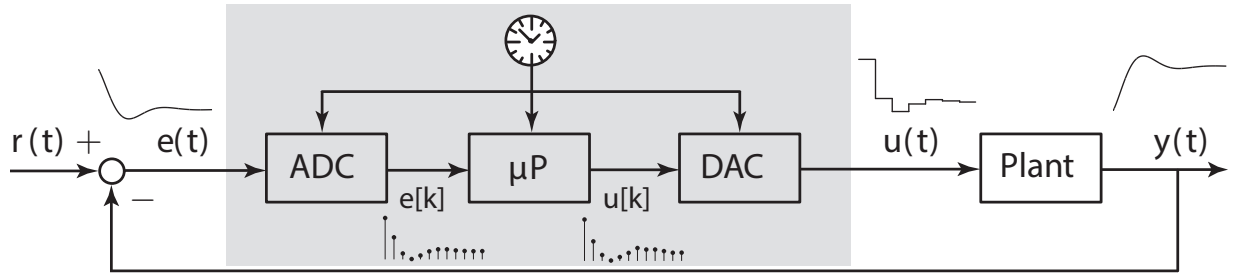
4. The Digital-to-Analog Converter (DAC)

Figure 2.1: Block diagram of a digital control system

Most of the time the plant is modeled as a "smooth" continuous dynamical system. In this course our focus is LTI systems thus, we will assume that (unless otherwise is given) that the plant is a continuous LTI plant model with a transfer function of $G_c(s)$ for which both the input and output are continuous time signals.

The "digital" blocks of the closed-loop block diagram structure are ADC, The Controller, and DAC. It is generally assumed (design preference) that all blocks shares a common "hard real time" clock.

A general ADC is a device that converts an analog signal to a digital signal. In this course we will model the ADC block as an *ideal sampler* for which the input is a continious-time signal, $e(t)$ and the output is a discrete-time signal, $e[k]$, where the relating between the continuous- and discrete-time signals are given as

$$e(kT) = e[k], \; k \in \mathbb{Z}^+,$$

where constant T is the *sampling time*.

The microcontoller/microprocessor processes some set of digital input signals to produce some set of digital output signals. The outputs are defined at only certain instances defined by the real-time clock. In this course, we will model the μP block as an ideal discrete-time LTI system for which the both the input and output are discrete-time signals, with a transfer function of $G_c(z)$.

The DAC is a device that converts a digital signal to an analog signal. In this course we assume that it is an ideal *Hold* element for which the input signal is a discrete-time signal, where as output is a continuous-time signal. The most commonly used *Hold* system is ZOH (Zero-Order-Hold) which is a mapping defined by the following relation

$$u(t) = u[k], \ \text{for} t \in [kT, (k+1)T)$$

Higher-order holds available but seldom used.

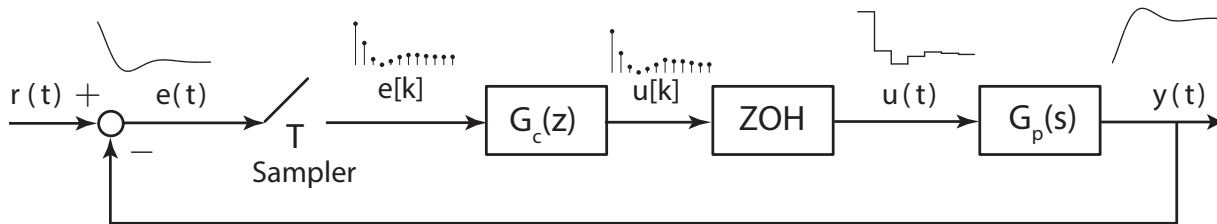The idealized and simplified block-diagram structure is given in Fig.

Figure 2.2: Block diagram of an LTI discrete-time control system

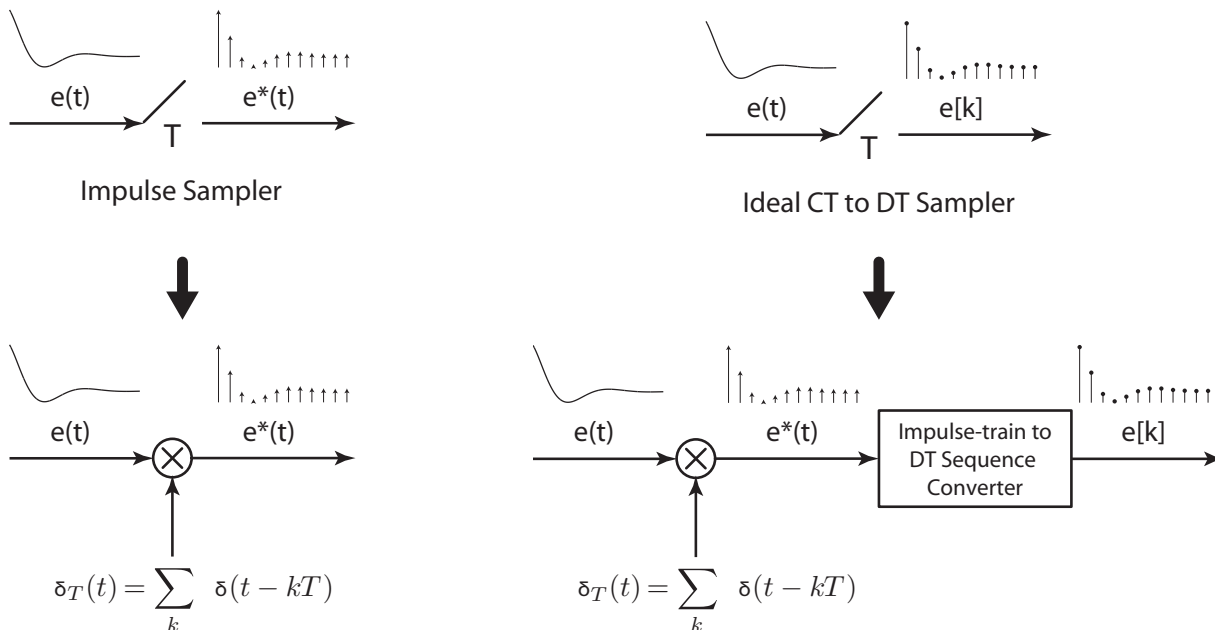**Major challenge:** Loop contains both continuous-time and discrete-time parts.

Sampling

Figure 2.3: Two different ideal samplers

Fig. 2.3 illustrates two different ideal sampler (both of them will be covered in this course. Firs column is an *impulse sampler* for which the output is a continuous signal but it is composed of trains of impulses

(impulse train). Second one is an ideal complete CT-to-DT sampler which convertes the impulse train into DT sequence.

The output of the impulse sampler, $x * (t)$, can be represented with the following infinte summations

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t - kT) = \sum_{k=0}^{\infty} x[k]\delta(t - kT)$$

or

$$x^*(t) = x(0)\delta(t) + x(T)\delta(t - T) + \cdots + x(kT)\delta(t - kT) + \cdots$$
$$= x[0]\delta(t) + x[1]\delta(t - T) + \cdots + x[k]\delta(t - kT) + \cdots$$

Now let's consider the Laplace transform of $x^*(t)$

$$X^*(s) = \mathcal{L}\{x^*(t)\} = \mathcal{L}\left\{\sum_{k=0}^{\infty} x(kT)\delta(t - kT)\right\} = \sum_{k=0}^{\infty} x(kT)\mathcal{L}\left\{\delta(t - kT)\right\}$$

$$= \sum_{k=0}^{\infty} x(kT) \int_{t=0}^{\infty} \delta(t - kT)e^{-st}dt = \sum_{k=0}^{\infty} x(kT)e^{-skT} = \sum_{k=0}^{\infty} x[k]e^{-skT}$$

Now let's define a map in complex domain such that

$$z = e^{Ts} \text{ or } s = \frac{1}{T}\ln z$$

Then we have

$$X^*(s)|_{s=(1/T)lnz} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

where

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

Z-transform

Z-transform of a (causal) discrete time signal $x[k]$ is given by

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

If $x[k]$ is a sampled signal from a continuous time signal $x(t)$ with a sampling time of T, we (abuse of notation) also use the following notation

$$X(z) = \mathcal{Z}\{x(kT)\} = \mathcal{Z}\{x^*(t)\}$$

Z-transforms of elementary functions

We assume that all signals are causal thus $t \in \mathbb{R}^+$ and $k \in \mathbb{Z}^+$

Unit-step function $x(t) = 1$ and thus $x(kT) = x[k] = 1$, the Z-transform is given by

$$X(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

Unit-ramp function $x(t) = t$ and thus $x(kT) = x[k] = kT$, the Z-transform is given by

$$X(z) = \sum_{k=0}^{\infty} kTz^{-k} = T(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots) = Tz(z^{-2} + 2z^{-3} + 3z^{-4} + \cdots)$$

$$= Tz\frac{d}{dz}\left(\int (z^{-2} + 2z^{-3} + 3z^{-4} + \cdots)dz\right) = Tz\frac{d}{dz}\left(-(z^{-1} + z^{-2} + z^{-3} + \cdots)\right)$$

$$= Tz\frac{d}{dz}\left(\frac{-1}{z-1}\right) = \frac{Tz}{(z-1)^2} = \frac{Tz^{-1}}{(1-z^{-1})^2}$$

Exponential sequence $x[k] = a^k$

$$X(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} \left(\frac{z}{a}\right)^{-k} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

Exponential function $x(t) = e^{bt}$ and thus $x(kT) = x(k) = e^{bTk}$

$$X(z) = \sum_{k=0}^{\infty} e^{bTk} z^{-k} = \sum_{k=0}^{\infty}(e^{bT})^k z^{-k} = \frac{1}{1 - e^{bT}z^{-1}} = \frac{z}{z - e^{bT}}$$

Cosine function $x(t) = \cos(\omega t)$, and thus $x(kT) = x(k) = \cos(\omega Tk)$

$$\cos(\omega Tk) = \frac{1}{2}\left(e^{j\omega Tk} + e^{-j\omega Tk}\right) X(z) = \frac{1}{2}\left(\frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}}\right) = \frac{1}{2}\frac{z(z - e^{-j\omega T}) + z(z - e^{j\omega T})}{(z - e^{-j\omega T})(z - e^{j\omega T})}$$

$$= \frac{1}{2}\frac{2z^2 - z(e^{-j\omega T} + e^{j\omega T})}{z^2 - z(e^{-j\omega T} + e^{j\omega T}) + 1} = \frac{z^2 - z\cos(\omega T)}{z^2 - z2\cos(\omega T) + 1}$$

$$= \frac{1 - z^{-1}\cos(\omega T)}{1 - z^{-1}2\cos(\omega T) + z^{-2}}$$

Properties and Theorems of the Z-transform

Linearity

$$x(k) = \alpha f(k) + \beta g(k) \;\rightarrow\; X(z) = \alpha F(z) + \beta G(z) \;, \forall\; \alpha,\; \beta,\; f(k),\; \&\; g(k)$$

Multiplication by a^k

$$\mathcal{Z}\{a^k x[k]\} = \sum_{k=0}^{\infty} a^k x[k]z^{-k} = \sum_{k=0}^{\infty} x[k](z/a)^{-k}$$

$$\mathcal{Z}\{a^k x[k]\} = X(z/a)$$

Complex translation theorem Let $y(t) = e^{-at}x(t)$ and $X(z) = \ddagger\{x(kT)\}$, then

$$\mathcal{Z}\{y(kT)\} = \mathcal{Z}\{e^{-aTk}x(kT)\} = X(e^{aT}z)$$

Shifting theorem Let $x(t)$ be a causal CT signal, thus we have $x(t) = 0$ for $t < 0$. Similarly sampled DT signal has the property $x[nk] = 0$ for $k < 0$. For the sake of simplicity lets work on the sampled (i.e. DT) signal. Let

$$\mathcal{Z}\{x^*(t)\} = \mathcal{Z}\{x[k]\} = X(z)$$

Shifting right by N (Causal shifting): Let $y[k] = x[k - N]$, then

$$\mathcal{Z}\{y[k]\} = \sum_{k=0}^{\infty} y[k]z^{-k} = \sum_{k=0}^{\infty} x[k-N]z^{-k} = \sum_{k=N}^{\infty} x[k-N]z^{-k}$$

Let $k = m + N$ then

$$\mathcal{Z}\{y[k]\} = \sum_{m=0}^{\infty} x[m]z^{-(m+N)} = z^{-N} \sum_{m=0}^{\infty} x[m]z^{-m}$$
$$\mathcal{Z}\{x[k-N]\} = z^{-N}X(z)$$

Shifting left by N (Non-causal shifting) & Bilateral Z transform: Let $y[k] = x[k + N]$,

$$\mathcal{Z}\{x[k+N]\} = \sum_{k=-\infty}^{\infty} x[k+N]z^{-k} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m-N)} = z^{N} \sum_{m=-\infty}^{\infty} x[m]z^{-m}$$
$$\mathcal{Z}\{x[k+N]\} = z^{N}X(z)$$

Shifting left by N (Non-causal shifting) & Unilateral Z transform: Let $y[k] = x[k + N]$,

$$\mathcal{Z}\{x[k+N]\} = \sum_{k=0}^{\infty} x[k+N]z^{-k}$$

Let $k = m - N$ then

$$\mathcal{Z}\{x[k+N]\} = \sum_{m=N}^{\infty} x[m]z^{-(m-N)} = z^{N} \sum_{m=N}^{\infty} x[m]z^{-m} = z^{N} \left(\sum_{k=0}^{\infty} x[k]z^{-k} - \sum_{k=0}^{N-1} x[k]z^{-k} \right)$$
$$\mathcal{Z}\{x[k+N]\} = z^{N} \left(X(z) - \sum_{k=0}^{N-1} x[k]z^{-k} \right)$$

From this equation we can obtain

$$\mathcal{Z}\{x[k+1]\} = zX(z) - zx[0]$$
$$\mathcal{Z}\{x[k+2]\} = z^{2}X(z) - z^{2}x[0] - zx[1]$$
$$\vdots$$

Example 1. Let $u[k]$ be the unit-step function. Compute $\mathcal{Z}\{u[k-1]\}$ both directly and using the shifting property.

$$\mathcal{Z}\{u[k-1]\} = \frac{z^{-1}}{1 - z^{-1}}$$

Example 2. Let $y[k] = \sum\limits_{n=0}^{k} x[k]$ where $k \in \mathbb{Z}^+$. Compute $Y(z)$ in terms of $X(z)$ using the shifting theorem.

$$Y(z) = \frac{1}{1 - z^{-1}} X(z)$$

Initial Value Theorem Let $X(z) = \mathcal{Z}\{x[n]\}$ and if the following limit exists, then the initial value of $x[0]$ or $x(0)$ is given by

$$x[0] = \lim_{z \to \infty} X(z)$$

Indeed the proof is very easy

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} \left[\sum_{k=0}^{\infty} x(k) z^{-k} \right] = \lim_{z \to \infty} \left[x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots \right] = x(0)$$

Final Value Theorem

Let's assume that $x(kT)$ or $x[k]$ is a convergent sequence (DT signal). Then the final value theorem states that

$$\lim_{k \to \infty} x[k] = \lim_{z \to 1} (1 - z^{-1}) X(z)$$

Proof: Let's take the Z transform of $x[k] - x[k-1]$

$$\mathcal{Z}\{x[k] - x[k-1]\} = \sum_{k=0}^{\infty} (x[k] - x[k-1]) z^{-k}$$

$$X(z) - X(z) z^{-1} = \left(x[0] \left(1 - z^{-1} \right) + x[1] \left(z^{-1} - z^{-2} \right) + x[2] \left(z^{-2} - z^{-3} \right) + x[3] \left(z^{-3} - z^{-4} \right) + \cdots \right) + \lim_{k \to \infty} x[k] z^{-k}$$

$$\lim_{z \to 1} X(z) \left(1 - z^{-1} \right) = (0 + 0 + \cdots) + \lim_{z \to 1} \lim_{k \to \infty} x[k] z^{-k}$$

$$\lim_{z \to 1} X(z) \left(1 - z^{-1} \right) = \lim_{k \to \infty} x[k]$$

Complex Differentiation Theorem

Consider

$$\frac{d}{dz} X(z) = \frac{d}{dz} \left[\sum_{k=0}^{\infty} x[k] z^{-k} \right] = \sum_{k=0}^{\infty} x[k] \frac{d}{dz} z^{-k} = \sum_{k=0}^{\infty} (-k) x[k] z^{-k-1}$$

$$-z \frac{d}{dz} X(z) = \sum_{k=0}^{\infty} k x[k] z^{-k}$$

$$-z \frac{d}{dz} X(z) = \mathcal{Z}\{k x[k]\}$$

In general

$$(-z)^m \frac{d}{dz^m} X(z) = \mathcal{Z}\{k^m x[k]\}$$

Example 3. Find the Z-transform of the unit ramp function, $r[k] = k$, $k \in \mathbb{Z}^+$ by applying the Complex Differentiation Theorem to the Z-transform of the unit step function.

Real Convolution Theorem Let $f[k]$ and $g[k]$ are causal signals and associated Z transforms are $F(z)$ and $G(z)$ respectively. The DT convolution operator is defined as

$$f[n] * g[n] = \sum_{k=0}^{n} f[n-k]g[k]$$

Real Convolution Theorem states that

$$\mathcal{Z}\{f[n] * g[n]\} = F(z)G(z)$$

Proof

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} f[n-k]g[k] \right] z^{-n}$$

Since we know that $f[m] = 0$ for $m < 0$, we can stretch the upper limit of the sum as

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} f[n-k]g[k] \right] z^{-n} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f[n-k]g[k]z^{-n}$$

Let $n = m + k$ then

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{k=0}^{\infty} \sum_{m=-k}^{\infty} f[m]g[k]z^{-m}z^{-k} = \sum_{k=0}^{\infty} g[k]z^{-k} \sum_{m=0}^{\infty} f[m]z^{-m}$$

$$\mathcal{Z}\{f[n] * g[n]\} = F(z)G(z)$$

The Inverse Z-transform

1. Direct division method

2. Z-tranform tables & partial-fraction expansion

3. "Simulation" method

4. Inversion integral method

Direct division

Direct division (or long division) method uses the fact that $X(z)$ can be expressed as

$$X(z) = \sum_{k=0}^{\infty} x[k]z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$$

The goal is finding the power series expansion of $X(z)$ using the long division approach. Here we assume that $X(z)$ cane be represnted as a ratio of two polynomials in z (or z^{-1})

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n} = \frac{b_0 z^{-n+m} + b_1 z^{-n+m-1} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

For the direct division method it is easer to work when the polynomials are written interms of powers of z^{-1}.

Example 4. Find the inverse Z-transform of $X(z) = \frac{z^{-1}}{1-2z^{-1}+z^{-2}}$.

$$
\begin{array}{r|l}
\phantom{z^{-1} - 2z^{-2} +} z^{-1} & 1 - 2\,z^{-1} + z^{-2} \\
\underline{z^{-1} - 2\,z^{-2} + z^{-3}} & z^{-1} + 2\,z^{-2} + 3\,z^{-3} + 4\,z^{-4} + \ldots \\
\end{array}
$$

$$2\,z^{-2} - z^{-3}$$
$$\underline{2\,z^{-2} - 4\,z^{-3} + 2\,z^{-4}}$$
$$3\,z^{-3} - 2\,z^{-4}$$
$$\underline{3\,z^{-3} - 6\,z^{-4} + 3\,z^{-5}}$$
$$4\,z^{-4} - 3\,z^{-5}$$
$$\vdots$$

Thus,

$$X(z) = 0 + 1z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \ldots$$
$$\downarrow$$
$$x[k] = 0\delta[k] + 1\delta[k-1] + 2\delta[k-2] + 3\delta[k-3] + 4\delta[k-4] + \ldots = k$$

Partial Fraction Expansion

In most applications $X(z)$ can be re-written in terms of poles and zeros as

$$X(z) = b_0 \frac{(z - z_1) \cdots (z - z_m)}{(z - p_1) \cdots (z - p_n)} \quad (m \leq n)$$

Specific (but extremely common) case

$$\frac{X(z)}{z} = \sum_{i=1}^{n} \frac{a_i}{(z - p_i)}$$

where all poles are distinct and simple order. We can compute each a_i using

$$a_i = \lim_{z \to p_i} \left[(z - p_i) \frac{X(z)}{z} \right]$$

Example 5. Find the inverse Z-transform of $X(z) = \frac{(1-b)z}{(z-1)(z-b)}$. Solution:

$$\frac{X(z)}{z} = \frac{(1-b)}{(z-1)(z-b)} = \frac{a_1}{z-1} + \frac{a_2}{z-b}$$

$$a_1 = \lim_{z \to 1} \left[(z-1)\frac{X(z)}{z} \right] = 1$$

$$a_2 = \lim_{z \to b} \left[(z-b)\frac{X(z)}{z} \right] = -1$$

$$X(z) = \frac{z}{z-1} - \frac{z}{z-b}$$

$$x[k] = 1 - b^k$$

Now let's assume that $\frac{X(z)}{z}$ has double pole at p_1 and all other poles are distinct

$$\frac{X(z)}{z} = \frac{c_1}{z - p_1} + \frac{c_2}{(z - p_1)^2} + \cdots$$

It is easy to show that

$$c_2 = \lim_{z \to p_1} \left[(z - p_1)^2 \frac{X(z)}{z} \right]$$

It is also possible to show that

$$c_1 = \lim_{z \to p_1} \left\{ \frac{d}{dz} \left[(z - p_1)^2 \frac{X(z)}{z} \right] \right\}$$

Example 6. Find the inverse Z-transform $X(z) = \frac{2z^2 - 3z}{(z-1)^2}$. Solution:

$$\frac{X(z)}{z} = \frac{c_1}{z-1} + \frac{c_2}{(z-1)^2}$$

$$c_1 = \lim_{z \to 1} \frac{d}{z} \left[(z-1)^2 \frac{X(z)}{z} \right] = 2$$

$$c_2 = \lim_{z \to 1} \left[(z-1)^2 \frac{X(z)}{z} \right] = -1$$

$$x[k] = 2 - k$$

Example 7. Find the inverse Z-transform $X(z) = \frac{(1-b)}{(z-1)(z-b)}$. Solution:

$$X(z) = \frac{(1-b)}{(z-1)(z-b)} = \frac{a_1}{z-1} + \frac{a_2}{z-b}$$

$$a_1 = \lim_{z \to 1} \left[(z-1)X(z) \right] = 1$$

$$a_2 = \lim_{z \to b} \left[(z-b)X(z) \right] = -1$$

$$X(z) = z^{-1} \left(\frac{z}{z-1} - \frac{z}{z-b} \right)$$

$$x[k] = [1 - b^{k-1}]u[k-1]$$

Example 8. Find the inverse Z-transform $X(z) = \frac{z^2 - 2}{(z-1)(z-2)}$. Solution:

$$X(z) = \frac{z^2 - 2}{z^2 - 3z + 2} = 1 + \frac{3z - 4}{z^2 - 3z + 2}$$

$$X(z) = 1 + \frac{a_1}{z - 1} + \frac{a_2}{z - 2}$$

$$a_1 = \lim_{z \to 1} [(z - 1)X(z)] = 1$$

$$a_2 = \lim_{z \to 2} [(z - 2)X(z)] = 2$$

$$X(z) = 1 + \frac{1}{z - 1} + \frac{2}{z - 2} = \frac{1}{z - 1} + \frac{z}{z - 2}$$

$$x[k] = 1 + 2^k - \delta[k]$$