EE302 - Feedback Systems **Spring 2019**

Lecture 3

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3.1 Modeling of Mechanical Systems and Their Electrical Analogy

We use Kirchhoff's Current and Voltage laws to derive the dynamical (and static) relationships in Electrical Circuits. Similarly, we utilize Newton's laws of motion to derive equations of motion in (rigid body) mechanical dynamical systems.

3.1.1 Mechanical vs. Electrical Analogy Between Dependent Variables

There exist two different analogies that we can construct between electrical and mechanical systems. Mathematically, there is no difference between the two approaches. In this lecture, we will learn one of these analogies.

In electrical circuits, the core variables are Voltage, V, and current, I, whereas in translational mechanical systems, core variables are translational velocity, ν, and force, f. Similarly, in rotational mechanical systems, the core variables are angular velocity, ωn, and torque τ.

Voltage, $V \iff$ **Velocity,** $\nu \iff$ **Angular Velocity,** ω

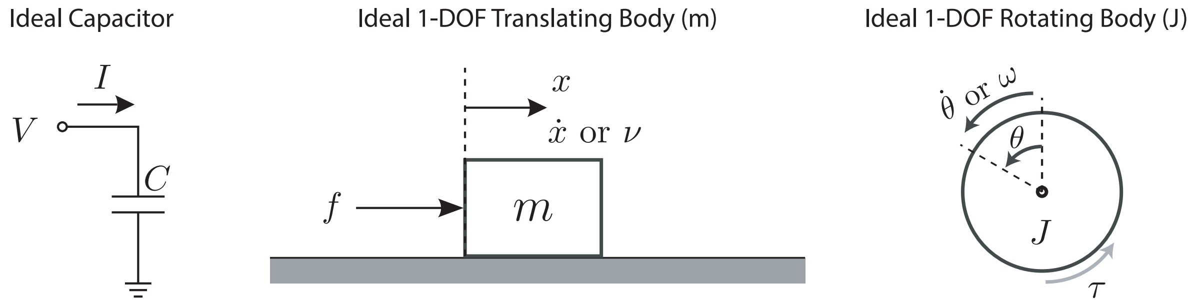
In electrical systems, voltage, also called electric potential difference, accounts for the difference in electric potential between two points. When we refer to the voltage of a node/point, we always measure it with respect to a reference point, e.g. *ground*. In mechanical systems, we measure the velocity either between two points in space, or (which is more general) with respect to an inertial reference frame, e.g. *ground* or *earth* in general. The analogy is similar with angular velocity. For this reason, we say that voltage, linear velocity, and angular velocity are the analog variables.

Current, $I \iff$ **force,** $f \iff$**Torque,** τ

In electrical systems, the current is the flow (or rate of change of) of electric charge and carried by electrons in motion. Roughly speaking, based on Newton's second law, the force acting upon a (rigid) body is equal to the rate of change of momentum related to this specific force component. Momentum can be considered as an analog of the electrical charge in this case. A similar analogy can also be constructed using Torque and Angular Momentum. For this reason, we say that Current, Force, and Torque are the analog variables.

3.1.2 Capacitor C, 1-DOF Translating Body with Mass m, and 1-DOF Rotating Body with Inertia J

If we follow the analogs between the variables, we can see that ideal capacitor for which one end is connected to the ground, 1-DOF translating body with a mass of m, and 1-DOF rotating body with an inertia of J analogs of each other. These are all ideal energy storage elements in their modeling domains and they are illustrated in the figure below.



The ODEs that govern the dynamics of these elements are provided below

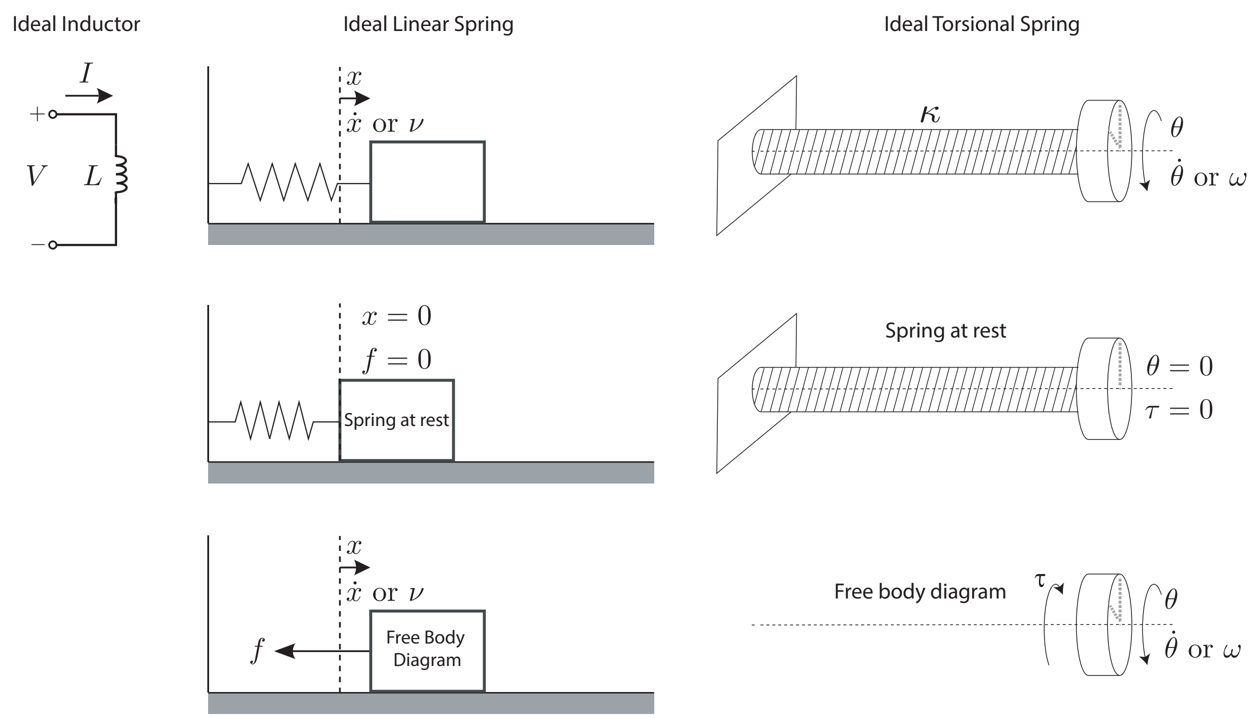
$$\begin{aligned} \text{Capacitor}: &\quad C\dot{V}(t) = I(t) \\ \text{Mass}: &\quad m\dot{\nu} = f(t) \ (\text{or } m\ddot{x} = f(t)) \\ \text{Inertia}: &\quad J\dot{\omega} = \tau(t) \ (\text{or } J\ddot{\theta} = \tau(t)) \end{aligned}$$

Based on these equations we can reach the following (system) parameter analogy as

$$C \equiv m \equiv J$$

3.1.3 Inductor L, Translational Spring k, and Torsional Spring κ

If we follow the analogs between the variables we can see that Ideal inductor (L), linear translational spring (k), and linear torsional spring (κ) are analogs of each other. These are also ideal energy storage elements in their modeling domains and they are illustrated in the figure below.



The ODEs that govern the dynamics of these elements are provided below

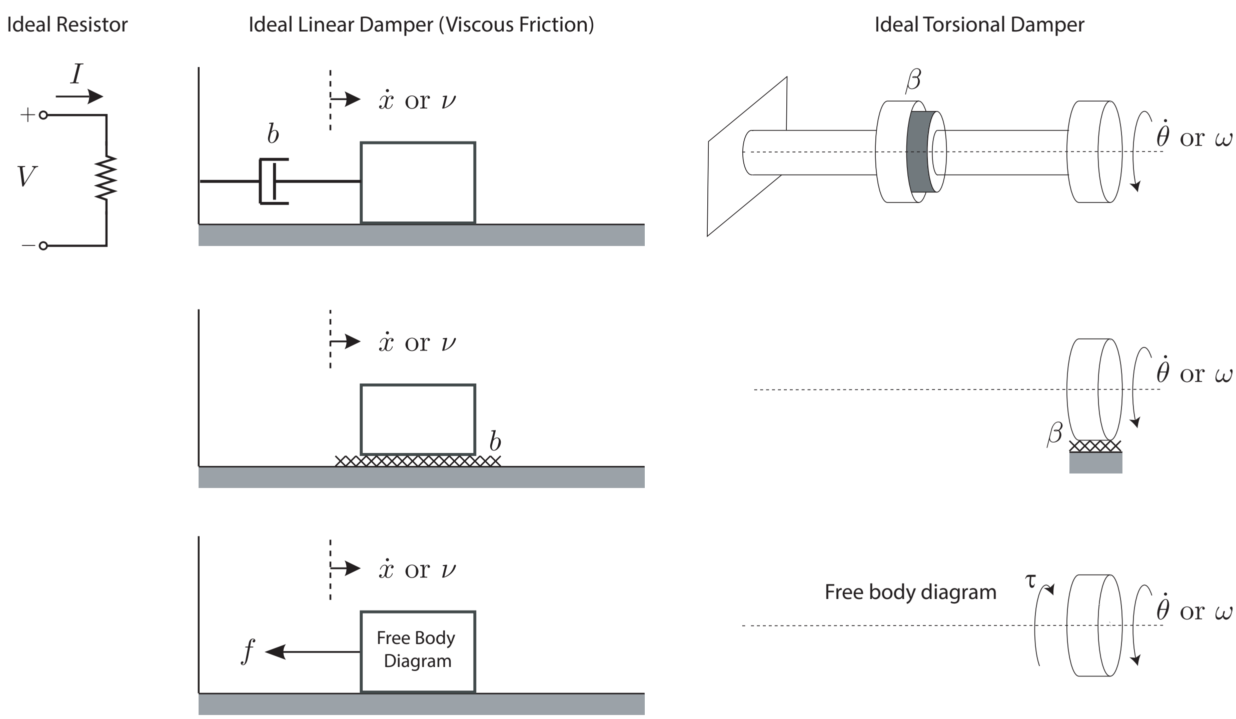
$$\begin{aligned} \text{Induction}: \quad & L\dot{I}(t) = V(t) \\ \text{TranslationalSpring}: \quad & f(t) = kx(t) \ \rightarrow \ \frac{1}{k}\dot{f}(t) = \nu(t) \\ \text{TorsionalSpring}: \quad & tau(t) = \kappa\theta(t) \ \rightarrow \ \frac{1}{\kappa}\dot{\tau}(t) = \omega(t) \end{aligned}$$

Based on these equations we can reach the following (system) parameter analogy as

$$L \equiv \frac{1}{k} \equiv \frac{1}{\kappa}$$

3.1.4 Resistor R, Damper (Viscous Friction) b, and Torsional Damper β

If we follow the analogs between the variables we can see that Ideal resistor (R), linear translational damper (k), and linear torsional damper (κ) are analogs of each other. These elements are ideal fully passive dissipative elements. Thus, these are memoryless (static) components as opposed to the previous elements. These elements are illustrated in the figure below.



The algebraic equations that govern the statics of these elements are provided below

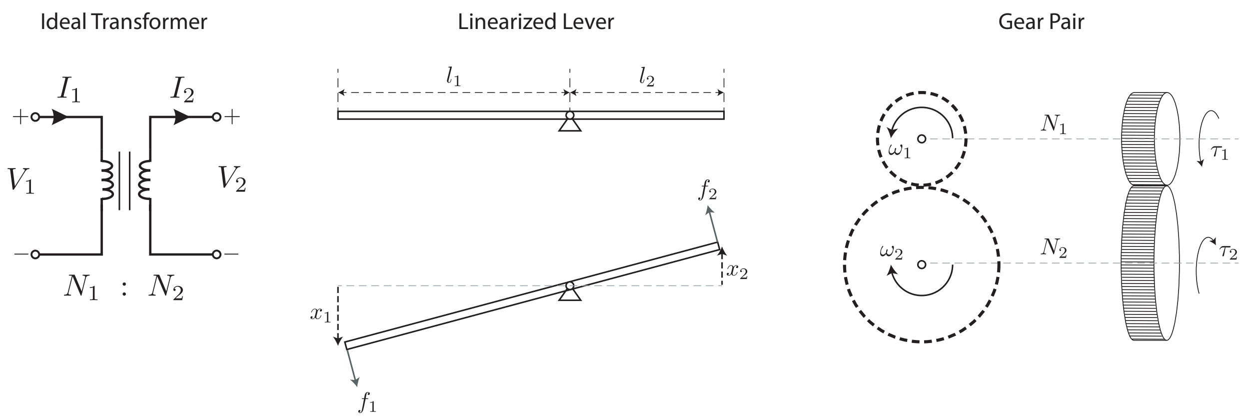
$$\begin{aligned} \text{Resistor}: \quad & V(t) = RI(t) \\ \text{Translational Damper}: \quad & f(t) = b\dot{x}(t) \text{ or } \frac{1}{b}f(t) = \nu(t) \\ \text{Torsional Damper}: \quad & \tau(t) = \beta\dot{\theta}(t) \ \rightarrow \ \frac{1}{\beta}\tau(t) = \omega(t) \end{aligned}$$

Based on these equations we can reach the following (system) parameter analogy as

$$R \equiv \frac{1}{b} \equiv \frac{1}{\beta}$$

3.1.5 Ideal Transformer, Linearized Lever, and Gear Pair

In both electrical and mechanical systems, we have transmission elements. In their ideal form, they conserve the energy after the transformation. In electrical systems, transformer is the component that achieves the transmission. In translational mechanical systems a linearized lever can achieve this under the assumption of small movements, where as for rotational systems a gear pair is one of the many solutions for mechanical transmission. These components are illustrated in the figure below.



The algebraic equations that govern the statics of these elements are provided below

$$\text{Electrical Transformer :} \quad \begin{array}{c} \frac{V_1}{N_1} = \frac{V_2}{N_2} \\ I_1 N_1 = I_2 N_2 \end{array}$$

$$\text{Lever :} \quad \begin{array}{c} \frac{\nu_1}{l_1} = \frac{\nu_2}{l_2} \\ f_1 l_1 = f_2 l_2 \end{array}$$

$$\text{Gear} - \text{Pair :} \quad \begin{array}{c} \frac{\omega_1}{r_2} = \frac{\omega_2}{r_1} \\ \tau_1 r_2 = \tau_2 r_1 \end{array}$$

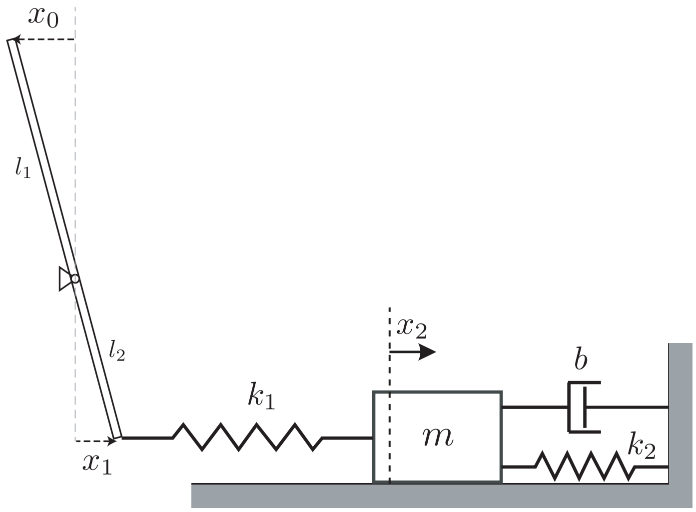
Based on these equations, it is to see that following (system) parameter analogy as

$$\frac{N_1}{N_2} \equiv \frac{l_1}{l_2} \equiv \frac{r_2}{r_1}$$

3.2 Examples

Ex 1: Let's consider the following translational mechanical system. It is given that when the lever is in vertical position, $[x_0\ x_1\ x_2] = 0$ and springs are at their rest length positions.

1. Given that $u(t) = x_0(t)$ and $y(t) = x_2(t)$, find the ODE of the system dynamics.



$$m\ddot{x}_2 = -b\dot{x}_2 - k_2x_2 + k_1(x_1 - x_2)$$
$$x_1 = x_0\frac{l_2}{l_1}$$
$$m\ddot{x}_2 + b\dot{x}_2 + (k_1 + k_2)x_2 = k_1\frac{l_2}{l_1}x_0$$
$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k_1 + k_2}{m}y = \frac{k_1}{m}\frac{l_2}{l_1}u$$

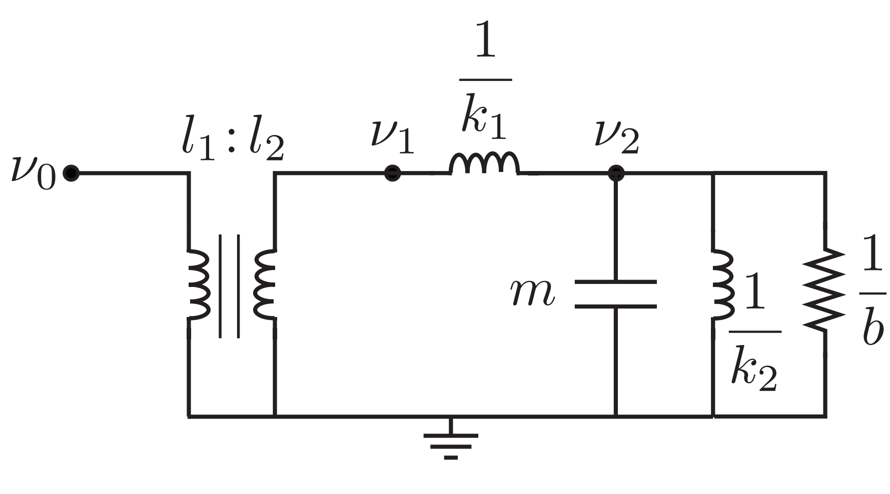
2. Find the transfer function for same input-output pair.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\frac{k_1}{m}\frac{l_2}{l_1}}{s^2 + \frac{b}{m}s + \frac{k_1+k_2}{m}}$$

3. Now, let's construct an electrical circuit analog of the mechanical system.
   Let

$$V_i \equiv \dot{x}_i$$
$$I_i \equiv f_i$$

then we can build the circuit analog as in the illustration below.



Let's also compute $\frac{\mathcal{V}_2(s)}{\mathcal{V}_l(s)}$ using node voltage analysis in impedance domain.

$$\begin{aligned}
\frac{\mathcal{V}_2(s)-\mathcal{V}_1(s)}{\frac{s}{k_1}}+\frac{\mathcal{V}_2(s)}{\frac{1}{ms}}+\frac{\mathcal{V}_2(s)}{\frac{s}{k_2}}+\frac{\mathcal{V}_2(s)}{\frac{1}{b}}&=0\\
\mathcal{V}_2(s)\left[ms+b+\frac{k_1+k_2}{s}\right]&=V_1(s)\left[\frac{k_1}{s}\right]\\
\mathcal{V}_2(s)\left[ms^2+bs+(k_1+k_2)\right]&=V_1(s)\left[k_1\right]
\end{aligned}$$

Since ideal transformer has the following relation, $\mathcal{V}_1(s)=\frac{l_2}{l_1}\mathcal{V}_0(s)$, we have the following transfer function between $\mathcal{V}_0(s)$ and $\mathcal{V}_2(s)$

$$\frac{\mathcal{V}_2(s)}{\mathcal{V}_0(s)}=\frac{\frac{k_1}{m}\frac{l_2}{l_1}}{s^2+\frac{b}{m}s+\frac{k_1+k_2}{m}}$$

Obviously this transfer function is equal to $G(s)$ computed from directly mechanical system and considering positional variables.

4. Convert the derived ODE into a state-space form

   We will solve the problem using a different approach. First let's integrate the ODE twice

$$y=\int\left[-\frac{b}{m}y+\int\left\{-\frac{k_1+k_2}{m}y+\frac{k_1}{m}\frac{l_2}{l_1}u\right\}dt\right]dt$$

   Then let the stat variable definitions be

$$\begin{aligned}
x_1=y&=\int\left[-\frac{b}{m}y+\int\left\{-\frac{k_1+k_2}{m}y+\frac{k_1}{m}\frac{l_2}{l_1}u\right\}dt\right]dt\\
x_2&=\int\left\{-\frac{k_1+k_2}{m}y+\frac{k_1}{m}\frac{l_2}{l_1}u\right\}dt
\end{aligned}$$

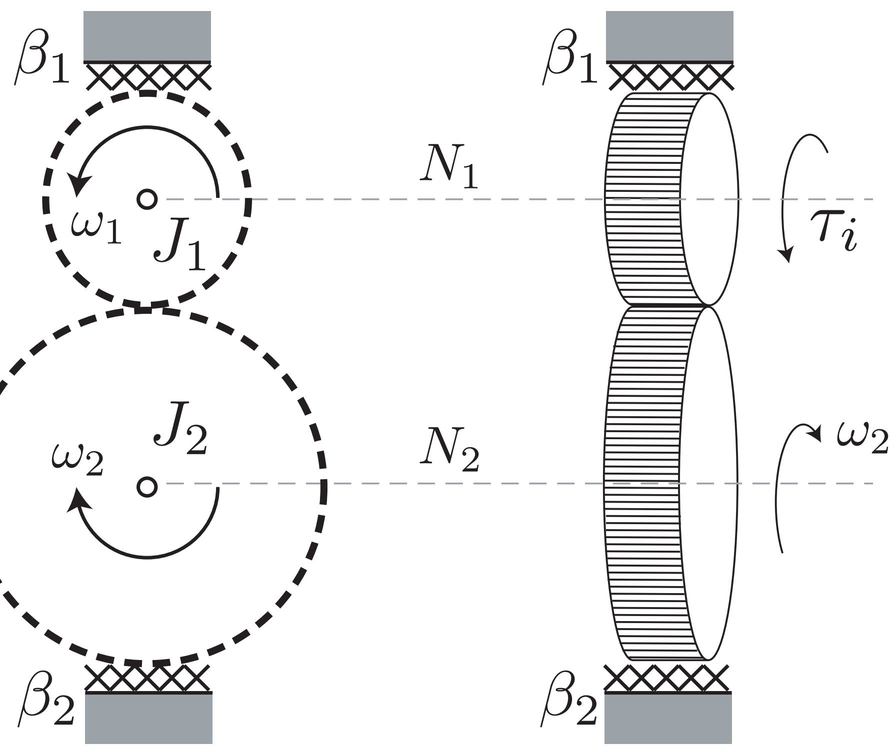
   Then state-equations take the form

$$\begin{aligned}
\dot{x}_1&=\left[-\frac{b}{m}y+\int\left\{-\frac{k_1+k_2}{m}y+\frac{k_1}{m}\frac{l_2}{l_1}u\right\}dt\right]\\
&=-\frac{b}{m}x_1+x_2\\
\dot{x}_2&=\left\{-\frac{k_1+k_2}{m}y+\frac{k_1}{m}\frac{l_2}{l_1}u\right\}\\
&=-\frac{k_1+k_2}{m}x_1+\frac{k_1}{m}\frac{l_2}{l_1}u
\end{aligned}$$

   If we gather the obtained state-equations in matrix form we obtain the following state-space representation

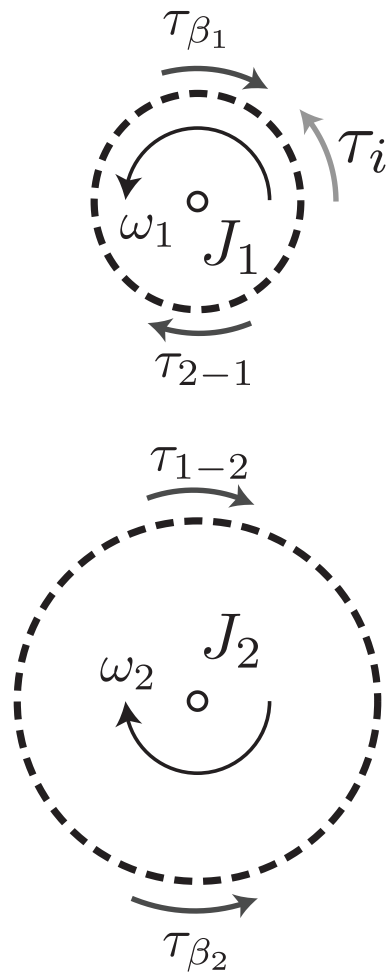
$$\begin{aligned}
\dot{x}&=\begin{bmatrix}-\frac{b}{m} & 1\\ -\frac{k_1+k_2}{m} & 0\end{bmatrix}x+\begin{bmatrix}0\\ \frac{k_1}{m}\frac{l_2}{l_1}\end{bmatrix}u\\
y&=\begin{bmatrix}1 & 0\end{bmatrix}x
\end{aligned}$$

Ex 2: Let's consider the following gear system. Unlike ideal gear pair case, now each gear has its own inertia, J_1 and J_2, as well as both gears are affected by viscous friction, β_1 and β_2 , due to mechanical contact with the environment.



1. Given that there is an external torque, τ_i, acting on the first gear is the input of the system, and the rotational speed of the second gear, ω_2, is the output of the system, find the ODE of the gear-box dynamics.

First let's draw the free-body diagrams of both gears separately and then write the equations of motion for each body.



$$
\begin{aligned}
J_1\dot{\omega}_1 &= \tau_i - \tau_{\beta_1} - \tau_{2-1}\\
&= \tau_i - \beta_1\omega_1 - \tau_{2-1}\\
J_2\dot{\omega}_2 &= -\tau_{\beta_2} + \tau_{1-2}\\
&= -\beta_2\omega_2 + \tau_{1-2}
\end{aligned}
$$

Based on the gear kinematics we know that

$$
\begin{aligned}
\omega_1 &= \frac{N_2}{N_1}\omega_2\\
\tau_{1-2} &= \frac{N_2}{N_1}\tau_{2-1}
\end{aligned}
$$

Thus we have the following derivations

$$
\begin{aligned}
\tau_{2-1} &= \tau_i - J_1\frac{N_2}{N_1}\dot{\omega}_2 - \beta_1\frac{N_2}{N_1}\omega_2\\
\tau_{1-2} &= \frac{N_2}{N_1}\tau_i - J_1\left(\frac{N_2}{N_1}\right)^2\dot{\omega}_2 - \beta_1\left(\frac{N_2}{N_1}\right)^2\omega_2
\end{aligned}
$$

Hence the ODE governing the dynamics can be formed as

$$
\left[J_2 + J_1\left(\frac{N_2}{N_1}\right)^2\right]\dot{y} + \left[\beta_2 + \beta_1\left(\frac{N_2}{N_1}\right)^2\right]y = \frac{N_2}{N_1}u
$$

It can be seen that the resultant ODE is a first order ODE. Ee can also consider the whole system as a single rotating body with an effective total inertia of $J_T = \left[J_2 + J_1\left(\frac{N_2}{N_1}\right)^2\right]$ and effective total viscous friction of $\beta_T = \left[\beta_2 + \beta_1\left(\frac{N_2}{N_1}\right)^2\right]$.

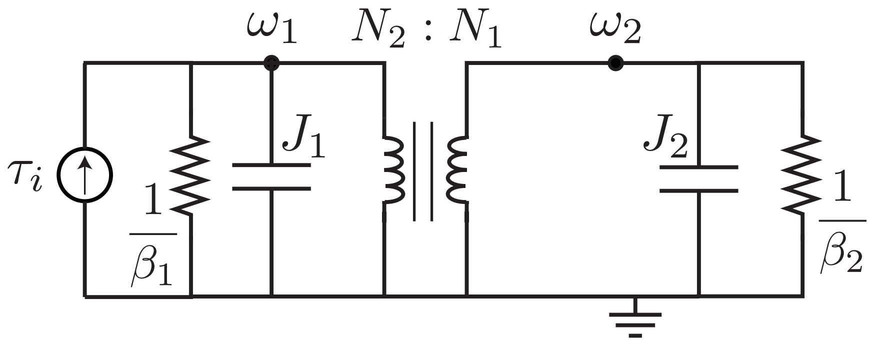
Take Home Problem: Now let's assume that output is $y(t) = \omega_1(t)$, and govern the ODE and re-compute the new effective inertia and viscous friction.

2. Compute the transfer function

$$
\begin{aligned}
G(s) = \frac{Y(s)}{U(s)} &= \frac{\frac{N_2}{N_1}}{\left[J_2 + J_1\left(\frac{N_2}{N_1}\right)^2\right]s + \left[\beta_2 + \beta_1\left(\frac{N_2}{N_1}\right)^2\right]}\\
&= \frac{\frac{N_2}{N_1}}{J_T s + \beta_T} = \frac{\frac{N_2}{N_1}\frac{1}{J_T}}{s + \frac{\beta_T}{J_T}}
\end{aligned}
$$

3. Now, build the electrical circuit analog of the gear-box system

The circuit diagram is given below



Take Home Problem: Solve the electrical circuit and compute the transfer function $G(s)$