

## Lecture 2

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## Big Picture of EE402

In this course, the main focus will be on continuous-time systems (plants) that are controlled (sampled and actuated) by a digital computer interface. Such a discrete-time control system consists of four major parts as illustrated in Fig. 2.2,

1. *The plant* is a continuous-time dynamical system
2. Analog-to-Digital Converter (ADC)
3. Controller ( $\mu P$ ), a microprocessor/microcontroller with a “real-time” OS
4. Digital-to-Analog Converter (DAC)

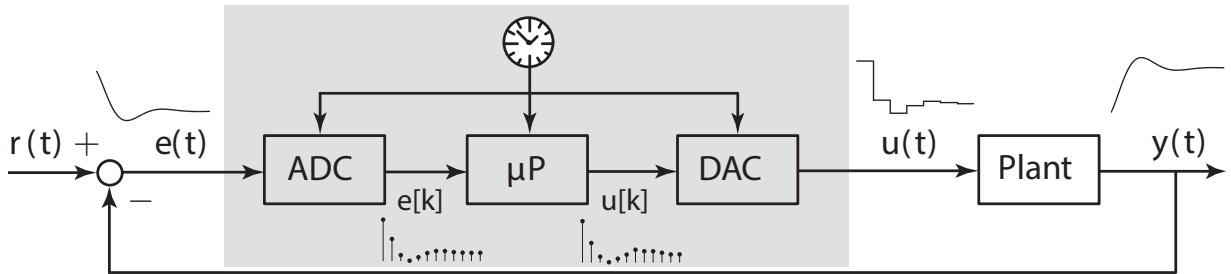


Figure 2.1: Block diagram of a digital control system

Most of the time, the plant is modeled as a “smooth” continuous dynamical system. In this course, we will cover only LTI systems. Thus, we will assume that (unless otherwise is given) the plant is a continuous LTI plant model with a transfer function of  $G_p(s)$  for which both the input and output are continuous-time signals.

The “digital” blocks inside the closed-loop block diagram structure are ADC, the Controller, and DAC. It is generally assumed (design requirement) that all blocks share a common “hard real-time” clock.

A general ADC is a device that converts an analog signal to a digital signal. In this course, we will model the ADC block as an *ideal sampler* for which the input is a continuous-time signal,  $e(t)$ , and the output is a discrete-time signal,  $e[k]$ , where the relation between the continuous- and discrete-time signals are given as

$$e(kT) = e[k], \quad k \in \mathbb{Z}^+,$$

where constant  $T$  is the *sampling time*.

The microcontroller/microprocessor processes some set of digital input signals to produce some set of digital output signals. The outputs are defined at only some specified instances determined by the real-time clock.

In this course, we will model the  $\mu P$  block as an ideal discrete-time LTI system for which both the input and output are discrete-time signals, with a transfer function of  $G_c(z)$ .

The DAC is a device that converts a digital signal to an analog signal. In this course, we assume that it is an ideal *Hold* element for which the input signal is a discrete-time signal, whereas the output is a continuous-time signal. The most commonly used *Hold* system is ZOH (Zero-Order-Hold) which is a mapping defined by the following relation

$$u(t) = u[k], \text{ for } t \in [kT, (k+1)T)$$

Higher-order hold operators exist, but they are extremely rarely used in practice.

The idealized and simplified block-diagram structure is given in Fig.

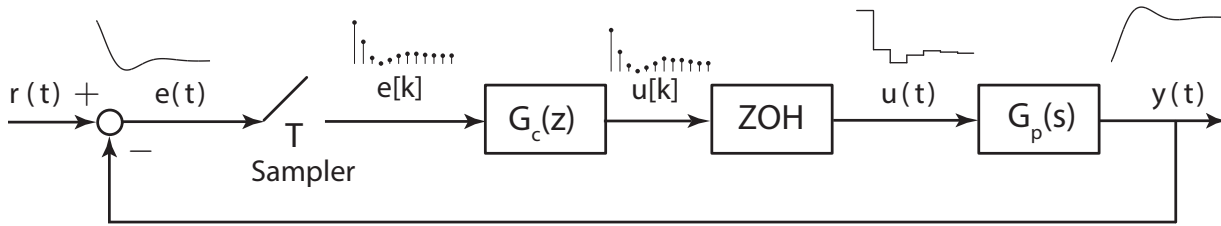


Figure 2.2: Block diagram of an LTI discrete-time control system

**Major challenge:** Loop contains both continuous-time and discrete-time parts.

## Sampling

Fig. 2.3 illustrates two different ideal samplers. Both of them will be covered in this course. First column is an *impulse sampler* for which the output is a continuous-time signal, but it is composed of trains of impulses (impulse train). Second one is an ideal complete CT-to-DT sampler which converts the impulse train into DT sequence.

The output of the impulse sampler,  $x^*(t)$ , can be represented with the following infinite summations

$$\begin{aligned} x^*(t) &= \sum_{k=0}^{\infty} x(kT)\delta(t - kT) = \sum_{k=0}^{\infty} x[k]\delta(t - kT) \\ \text{or} \\ x^*(t) &= x(0)\delta(t) + x(T)\delta(t - T) + \cdots + x(kT)\delta(t - kT) + \cdots \\ &= x[0]\delta(t) + x[1]\delta(t - T) + \cdots + x[k]\delta(t - kT) + \cdots \end{aligned}$$

Now let's consider the Laplace transform of  $x^*(t)$

$$\begin{aligned} X^*(s) &= \mathcal{L}\{x^*(t)\} = \mathcal{L}\left\{\sum_{k=0}^{\infty} x(kT)\delta(t - kT)\right\} = \sum_{k=0}^{\infty} x(kT)\mathcal{L}\{\delta(t - kT)\} \\ &= \sum_{k=0}^{\infty} x(kT) \int_{t=0}^{\infty} \delta(t - kT)e^{-st}dt = \sum_{k=0}^{\infty} x(kT)e^{-skT} = \sum_{k=0}^{\infty} x[k]e^{-skT} \end{aligned}$$

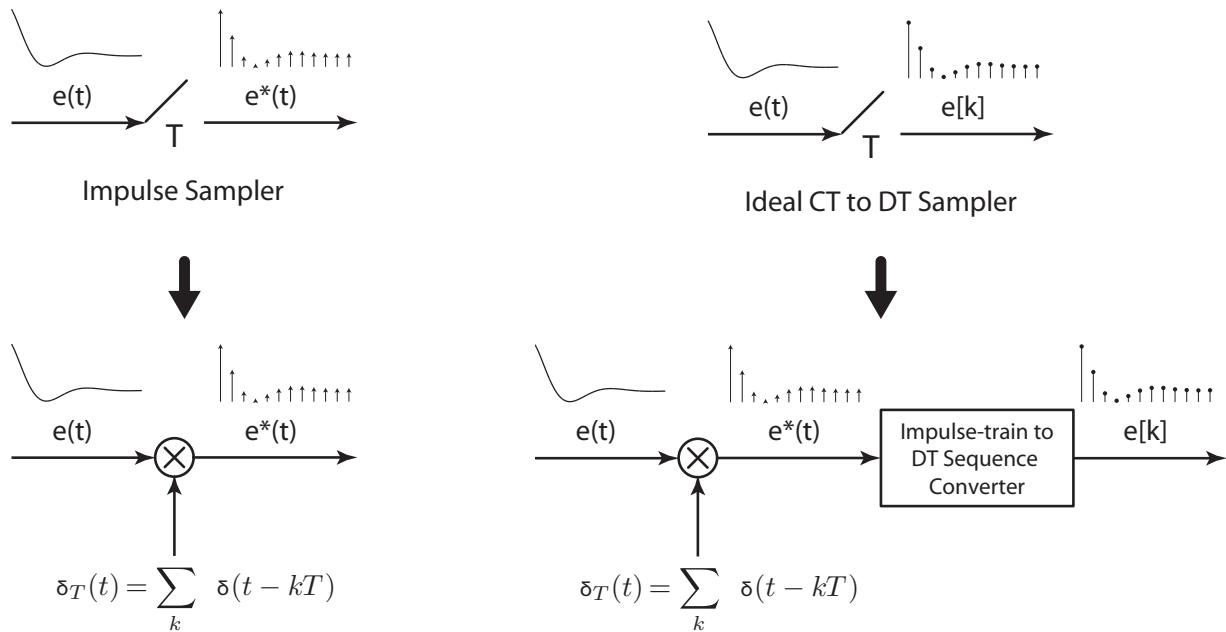


Figure 2.3: Two different ideal samplers

Now let's define a map in complex domain such that

$$z = e^{Ts} \text{ or } s = \frac{1}{T} \ln z$$

Then we have

$$X^*(s)|_{s=(1/T)\ln z} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

where

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

## Z-transform

Z-transform of a (causal) discrete time signal  $x[k]$  is given by

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

If  $x[k]$  is a sampled signal from a continuous time signal  $x(t)$  with a sampling time of  $T$ , we (abuse of notation) also use the following notation

$$X(z) = \mathcal{Z}\{x(kT)\} = \mathcal{Z}\{x^*(t)\}$$

## Z-transforms of elementary functions

We assume that all signals are causal thus  $t \in \mathbb{R}^+$  and  $k \in \mathbb{Z}^+$

Unit-step function  $x(t) = 1$  and thus  $x(kT) = x[k] = 1$ , the Z-transform is given by

$$X(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

Unit-ramp function  $x(t) = t$  and thus  $x(kT) = x[k] = kT$ , the Z-transform is given by

$$\begin{aligned} X(z) &= \sum_{k=0}^{\infty} kTz^{-k} = T(z^{-1} + 2z^{-2} + 3z^{-3} + \dots) = Tz(z^{-2} + 2z^{-3} + 3z^{-4} + \dots) \\ &= Tz \frac{d}{dz} \left( \int (z^{-2} + 2z^{-3} + 3z^{-4} + \dots) dz \right) = Tz \frac{d}{dz} (-(z^{-1} + z^{-2} + z^{-3} + \dots)) \\ &= Tz \frac{d}{dz} \left( \frac{-1}{z-1} \right) = \frac{Tz}{(z-1)^2} = \frac{Tz^{-1}}{(1-z^{-1})^2} \end{aligned}$$

Exponential sequence  $x[k] = a^k$

$$X(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} \left( \frac{z}{a} \right)^{-k} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

Exponential function  $x(t) = e^{bt}$  and thus  $x(kT) = x[k] = e^{bTk}$

$$X(z) = \sum_{k=0}^{\infty} e^{bTk} z^{-k} = \sum_{k=0}^{\infty} (e^{bT})^k z^{-k} = \frac{1}{1 - e^{bT}z^{-1}} = \frac{z}{z - e^{bT}}$$

Cosine function  $x(t) = \cos(\omega t)$ , and thus  $x(kT) = x[k] = \cos(\omega Tk)$

$$\begin{aligned} \cos(\omega Tk) &= \frac{1}{2} (e^{j\omega Tk} + e^{-j\omega Tk}) \quad X(z) = \frac{1}{2} \left( \frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}} \right) = \frac{1}{2} \frac{z(z - e^{-j\omega T}) + z(z - e^{j\omega T})}{(z - e^{-j\omega T})(z - e^{j\omega T})} \\ &= \frac{1}{2} \frac{2z^2 - z(e^{-j\omega T} + e^{j\omega T})}{z^2 - z(e^{-j\omega T} + e^{j\omega T}) + 1} = \frac{z^2 - z \cos(\omega T)}{z^2 - 2z \cos(\omega T) + 1} \\ &= \frac{1 - z^{-1} \cos(\omega T)}{1 - z^{-1} 2 \cos(\omega T) + z^{-2}} \end{aligned}$$

## Properties and Theorems of the Z-transform

### Linearity

$$x[k] = \alpha f[k] + \beta g[k] \rightarrow X(z) = \alpha F(z) + \beta G(z), \forall \alpha, \beta, f[k], \& g[k]$$

### Multiplication by $a^k$

$$\begin{aligned} \mathcal{Z}\{a^k x[k]\} &= \sum_{k=0}^{\infty} a^k x[k] z^{-k} = \sum_{k=0}^{\infty} x[k] (z/a)^{-k} \\ \mathcal{Z}\{a^k x[k]\} &= X(z/a) \end{aligned}$$

**Complex translation theorem**

Let  $y(t) = e^{-at}x(t)$  and  $X(z) = \mathcal{Z}\{x(kT)\}$ , then

$$\mathcal{Z}\{y(kT)\} = \mathcal{Z}\{e^{-aT^k}x(kT)\} = X(e^{aT}z)$$

**Shifting theorem**

Let  $x(t)$  be a causal CT signal, thus we have  $x(t) = 0$  for  $t < 0$ . Similarly, associated sampled DT signal has the property of  $x[k] = 0$  for  $k < 0$ . For the sake of simplicity let's work on the sampled (i.e. DT) signal. Let

$$\mathcal{Z}\{x^*(t)\} = \mathcal{Z}\{x[k]\} = X(z)$$

*Shifting right by N (Causal shifting):* Let  $y[k] = x[k - N]$ , then

$$\mathcal{Z}\{y[k]\} = \sum_{k=0}^{\infty} y[k]z^{-k} = \sum_{k=0}^{\infty} x[k - N]z^{-k} = \sum_{k=N}^{\infty} x[k - N]z^{-k}$$

Let  $k = m + N$  then

$$\begin{aligned}\mathcal{Z}\{y[k]\} &= \sum_{m=0}^{\infty} x[m]z^{-(m+N)} = z^{-N} \sum_{m=0}^{\infty} x[m]z^{-m} \\ \mathcal{Z}\{x[k - N]\} &= z^{-N} X(z)\end{aligned}$$

*Shifting left by N (Non-causal shifting) & Bilateral Z transform:* Let  $y[k] = x[k + N]$ ,

$$\begin{aligned}\mathcal{Z}\{x[k + N]\} &= \sum_{k=-\infty}^{\infty} x[k + N]z^{-k} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m-N)} = z^N \sum_{m=-\infty}^{\infty} x[m]z^{-m} \\ \mathcal{Z}\{x[k + N]\} &= z^N X(z)\end{aligned}$$

*Shifting left by N (Non-causal shifting) & Unilateral Z transform:* Let  $y[k] = x[k + N]$ ,

$$\mathcal{Z}\{x[k + N]\} = \sum_{k=0}^{\infty} x[k + N]z^{-k}$$

Let  $k = m - N$  then

$$\begin{aligned}\mathcal{Z}\{x[k + N]\} &= \sum_{m=N}^{\infty} x[m]z^{-(m-N)} = z^N \sum_{m=N}^{\infty} x[m]z^{-m} = z^N \left( \sum_{k=0}^{\infty} x[k]z^{-k} - \sum_{k=0}^{N-1} x[k]z^{-k} \right) \\ \mathcal{Z}\{x[k + N]\} &= z^N \left( X(z) - \sum_{k=0}^{N-1} x[k]z^{-k} \right)\end{aligned}$$

From this equation we can obtain

$$\begin{aligned}\mathcal{Z}\{x[k + 1]\} &= zX(z) - zx[0] \\ \mathcal{Z}\{x[k + 2]\} &= z^2X(z) - z^2x[0] - zx[1] \\ &\vdots\end{aligned}$$

**Example 1.** Let  $u[k]$  be the unit-step function. Compute  $\mathcal{Z}\{u[k-1]\}$  both directly and using the shifting property.

$$\mathcal{Z}\{u[k-1]\} = \frac{z^{-1}}{1-z^{-1}}$$

**Example 2.** Let  $y[k] = \sum_{n=0}^k x[k]$  where  $k \in \mathbb{Z}^+$ . Compute  $Y(z)$  in terms of  $X(z)$  using the shifting theorem.

$$Y(z) = \frac{1}{1-z^{-1}} X(z)$$

**Initial Value Theorem** Let  $X(z) = \mathcal{Z}\{x[n]\}$  and if the following limit exists, then the initial value of  $x[0]$  or  $x(0)$  is given by

$$x[0] = \lim_{z \rightarrow \infty} X(z)$$

Indeed the proof is very easy

$$\lim_{z \rightarrow \infty} X(z) = \lim_{z \rightarrow \infty} \left[ \sum_{k=0}^{\infty} x(k) z^{-k} \right] = \lim_{z \rightarrow \infty} [x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots] = x(0)$$

### Final Value Theorem

Let's assume that  $x(kT)$  or  $x[k]$  is a convergent sequence (DT signal). Then the final value theorem states that

$$\lim_{k \rightarrow \infty} x[k] = \lim_{z \rightarrow 1} (1-z^{-1})X(z)$$

**Proof:** Let's take the Z transform of  $x[k] - x[k-1]$

$$\begin{aligned} \mathcal{Z}\{x[k] - x[k-1]\} &= \sum_{k=0}^{\infty} (x[k] - x[k-1]) z^{-k} \\ X(z) - X(z)z^{-1} &= (x[0](1-z^{-1}) + x[1](z^{-1}-z^{-2}) + x[2](z^{-2}-z^{-3}) + x[3](z^{-3}-z^{-4}) + \dots) + \lim_{k \rightarrow \infty} x[k]z^{-k} \\ \lim_{z \rightarrow 1} X(z)(1-z^{-1}) &= (0+0+\dots) + \lim_{z \rightarrow 1} \lim_{k \rightarrow \infty} x[k]z^{-k} \\ \lim_{z \rightarrow 1} X(z)(1-z^{-1}) &= \lim_{k \rightarrow \infty} x[k] \end{aligned}$$

### Complex Differentiation Theorem

Consider

$$\begin{aligned} \frac{d}{dz} X(z) &= \frac{d}{dz} \left[ \sum_{k=0}^{\infty} x[k] z^{-k} \right] = \sum_{k=0}^{\infty} x[k] \frac{d}{dz} z^{-k} = \sum_{k=0}^{\infty} (-k) x[k] z^{-k-1} \\ -z \frac{d}{dz} X(z) &= \sum_{k=0}^{\infty} k x[k] z^{-k} \\ -z \frac{d}{dz} X(z) &= \mathcal{Z}\{kx[k]\} \end{aligned}$$

In general

$$(-z)^m \frac{d}{dz^m} X(z) = \mathcal{Z}\{k^m x[k]\}$$

**Example 3.** Find the Z-transform of the unit ramp function,  $r[k] = k, k \in \mathbb{Z}^+$  by applying the Complex Differentiation Theorem to the Z-transform of the unit step function.

**Solution:**

$$\mathcal{Z}\{r[k]\} = \mathcal{Z}\{ku[k]\}$$

$$\begin{aligned} R(z) &= (-z) \frac{d}{dz} U(z) = (-z) \frac{d}{dz} U(z) = (-z) \frac{d}{dz} \left( \frac{z}{z-1} \right) = (-z) \left( \frac{1}{z-1} - \frac{z}{(z-1)^2} \right) \\ &= \frac{z^2}{(z-1)^2} - \frac{z}{z-1} = \frac{z^2 - z(z-1)}{(z-1)^2} \\ R(z) &= \frac{z}{(z-1)^2} \end{aligned}$$

**Real Convolution Theorem** Let  $f[k]$  and  $g[k]$  are causal signals and associated Z transforms are  $F(z)$  and  $G(z)$  respectively. The DT convolution operator is defined as

$$f[n] * g[n] = \sum_{k=0}^n f[n-k]g[k]$$

Real Convolution Theorem states that

$$\mathcal{Z}\{f[n] * g[n]\} = F(z)G(z)$$

**Proof**

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^n f[n-k]g[k] \right] z^{-n}$$

Since we know that  $f[m] = 0$  for  $m < 0$ , we can stretch the upper limit of the sum as

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{\infty} f[n-k]g[k] \right] z^{-n} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f[n-k]g[k]z^{-n}$$

Let  $n = m + k$  then

$$\begin{aligned} \mathcal{Z}\{f[n] * g[n]\} &= \sum_{k=0}^{\infty} \sum_{m=-k}^{\infty} f[m]g[k]z^{-m}z^{-k} = \sum_{k=0}^{\infty} g[k]z^{-k} \sum_{m=0}^{\infty} f[m]z^{-m} \\ \mathcal{Z}\{f[n] * g[n]\} &= F(z)G(z) \end{aligned}$$

## The Inverse Z-transform

1. Direct division method
2. Z-transform tables & partial-fraction expansion
3. "Simulation" method
4. Inversion integral method

## Direct division

Direct division (or long division) method uses the fact that  $X(z)$  can be expressed as

$$X(z) = \sum_{k=0}^{\infty} x[k]z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \dots$$

The goal is finding the power series expansion of  $X(z)$  using the long division approach. Here we assume that  $X(z)$  can be represented as a ratio of two polynomials in  $z$  (or  $z^{-1}$ )

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \dots + b_m}{z^n + a_1 z^{n-1} + \dots + a_n} = \frac{b_0 z^{-n+m} + b_1 z^{-n+m-1} + \dots + b_m z^{-n}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}}$$

For the direct division method it is easier to work when the polynomials are written in terms of powers of  $z^{-1}$ .

**Example 4.** Find the inverse Z-transform of  $X(z) = \frac{z^{-1}}{1 - 2z^{-1} + z^{-2}}$ .

$$\begin{array}{r|l}
 \begin{array}{c} \textcolor{red}{z^{-1}} \\ z^{-1} - 2z^{-2} + z^{-3} \end{array} & \begin{array}{l} 1 - 2z^{-1} + z^{-2} \\ \hline \textcolor{red}{z^{-1}} + \textcolor{blue}{2z^{-2}} + \textcolor{green}{3z^{-3}} + \textcolor{magenta}{4z^{-4}} + \dots \end{array} \\
 \hline
 \begin{array}{c} \textcolor{blue}{2z^{-2}} - z^{-3} \\ 2z^{-2} - 4z^{-3} + 2z^{-4} \end{array} & \\
 \hline
 \begin{array}{c} \textcolor{green}{3z^{-3}} - 2z^{-4} \\ 3z^{-3} - 6z^{-4} + 3z^{-5} \end{array} & \\
 \hline
 \begin{array}{c} \textcolor{magenta}{4z^{-4}} - 3z^{-5} \\ \vdots \end{array} & 
 \end{array}$$

Thus,

$$\begin{aligned}
 X(z) &= 0 + 1z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \dots \\
 &\downarrow \\
 x[k] &= 0\delta[k] + 1\delta[k-1] + 2\delta[k-2] + 3\delta[k-3] + 4\delta[k-4] + \dots = k
 \end{aligned}$$

## Partial Fraction Expansion

In most applications  $X(z)$  can be re-written in terms of poles and zeros as

$$X(z) = b_0 \frac{(z - z_1) \cdots (z - z_m)}{(z - p_1) \cdots (z - p_n)} \quad (m \leq n)$$



Specific (but extremely common) case

$$\frac{X(z)}{z} = \sum_{i=1}^n \frac{a_i}{(z - p_i)}$$

where all poles are distinct and simple order. We can compute each  $a_i$  using

$$a_i = \lim_{z \rightarrow p_i} \left[ (z - p_i) \frac{X(z)}{z} \right]$$

**Example 5.** Find the inverse Z-transform of  $X(z) = \frac{(1-b)z}{(z-1)(z-b)}$ . Solution:

$$\begin{aligned} \frac{X(z)}{z} &= \frac{(1-b)}{(z-1)(z-b)} = \frac{a_1}{z-1} + \frac{a_2}{z-b} \\ a_1 &= \lim_{z \rightarrow 1} \left[ (z-1) \frac{X(z)}{z} \right] = 1 \\ a_2 &= \lim_{z \rightarrow b} \left[ (z-b) \frac{X(z)}{z} \right] = -1 \\ X(z) &= \frac{z}{z-1} - \frac{z}{z-b} \\ x[k] &= 1 - b^k \end{aligned}$$

Now let's assume that  $\frac{X(z)}{z}$  has double pole at  $p_1$  and all other poles are distinct

$$\frac{X(z)}{z} = \frac{c_1}{z - p_1} + \frac{c_2}{(z - p_1)^2} + \dots$$

It is easy to show that

$$c_2 = \lim_{z \rightarrow p_1} \left[ (z - p_1)^2 \frac{X(z)}{z} \right]$$

It is also possible to show that

$$c_1 = \lim_{z \rightarrow p_1} \left\{ \frac{d}{dz} \left[ (z - p_1)^2 \frac{X(z)}{z} \right] \right\}$$

**Example 6.** Find the inverse Z-transform  $X(z) = \frac{2z^2-3z}{(z-1)^2}$ . Solution:

$$\begin{aligned} \frac{X(z)}{z} &= \frac{c_1}{z-1} + \frac{c_2}{(z-1)^2} \\ c_1 &= \lim_{z \rightarrow 1} \frac{d}{dz} \left[ (z-1)^2 \frac{X(z)}{z} \right] = 2 \\ c_2 &= \lim_{z \rightarrow 1} \left[ (z-1)^2 \frac{X(z)}{z} \right] = -1 \\ x[k] &= 2 - k \end{aligned}$$

**Example 7.** Find the inverse Z-transform  $X(z) = \frac{(1-b)}{(z-1)(z-b)}$ . Solution:

$$\begin{aligned}
 X(z) &= \frac{(1-b)}{(z-1)(z-b)} = \frac{a_1}{z-1} + \frac{a_2}{z-b} \\
 a_1 &= \lim_{z \rightarrow 1} [(z-1)X(z)] = 1 \\
 a_2 &= \lim_{z \rightarrow b} [(z-b)X(z)] = -1 \\
 X(z) &= z^{-1} \left( \frac{z}{z-1} - \frac{z}{z-b} \right) \\
 x[k] &= [1 - b^{k-1}]u[k-1]
 \end{aligned}$$

**Example 8.** Find the inverse Z-transform  $X(z) = \frac{z^2-2}{(z-1)(z-2)}$ . Solution:

$$\begin{aligned}
 X(z) &= \frac{z^2-2}{z^2-3z+2} = 1 + \frac{3z-4}{z^2-3z+2} \\
 X(z) &= 1 + \frac{a_1}{z-1} + \frac{a_2}{z-2} \\
 a_1 &= \lim_{z \rightarrow 1} [(z-1)X(z)] = 1 \\
 a_2 &= \lim_{z \rightarrow 2} [(z-2)X(z)] = 2 \\
 X(z) &= 1 + \frac{1}{z-1} + \frac{2}{z-2} = \frac{1}{z-1} + \frac{z}{z-2} \\
 x[k] &= 1 + 2^k - \delta[k]
 \end{aligned}$$