EE302 - Discrete Time Systems **Spring 2019**

Lecture 21

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21.1 Reachability/Controllability

For a LTI continuous time state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- A state x_d is said to be **reachable** if there exist a finite time interval $t \in [0, t_f]$ and an input signal defined on this interval, $u(t)$, that transfers the state vector $x(t)$ from the origin (i.e. $x(0) = 0$) to the state x_d within this time interval, i.e. $x(t_f) = x_d$.
- A state x_d is said ti be **controllable** if there exist a finite time interval $t \in [0, t_f]$ and an input signal defined on this interval, $u(t)$, that transfers the state vector $x(t)$ from the initial state x_d (i.e. $x(0) = x_d$) to the origin this time interval, i.e. $x(t_f) = 0$.

- The set $\mathcal{R}$ of all reachable states is a linear (sub)space: $\mathcal{R} \subset \mathbb{R}^n$
- The set $\mathcal{C}$ of all controllable states is a linear (sub)space: $\mathcal{C} \subset \mathbb{R}^n$

For CT systems $x_d \in \mathcal{R}$ if and only if $x_d \in \mathcal{C}$, the Reachability and Controllability conditions are equivalent.

- If the reachable (or controllable) set is the entire state space, i.e., if $\mathcal{R} = \mathbb{R}^n$, then the system is called fully reachable/controllable.

One way of testing reachability/controllability is checking the rank (or the range space) the of reachability/controllability matrix

$$Q = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

A CT system is reachable/controllable if and only of

$$\text{rank}(Q) = n$$

or equivalently

$$\text{Ra}(\mathbf{M}) = \mathbb{R}^n$$

Example: Consider the following system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + Bu$$

Analyze the controllability for two different input, B, matrices

$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution:

Let's start with B_1 and derive the controllability matrix

$$Q_1 = \begin{bmatrix} B_1 & AB_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\det(Q_1) = 0$$

Thus the system is not fully controllable. Now let's get some intuition by writing the ODEs for individual states x_1 and x_2, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$\dot{x_1} = x_2 + u$$
$$\dot{x_2} = 0$$

Let's go with reachability definition, which is based on starting from zero initial conditions and going to a desired state, we can derive the following relations

$$x_1(T) = \int_0^T u(t)dt$$
$$x_2(T) = 0$$

Neither the input nor the first state has an affect on the second state, so $x_2 = \alpha$ for $\alpha \neq 0$ is not controllable (or reachable). However it is also easy to see that we can always find $u(t)$ that will drive the x_1 to a desired state, $x_1^*(T)$.

Now let's analyze the case with B_2

$$Q_2 = \begin{bmatrix} B_2 & AB_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\det(Q_2) = -1 \neq 0$$

Now the system is fully controllable.

Remark: A state-space representation that is in controllable canonical form is always fully controllable.

21.2 State-Feedback & Pole Placement

Given a continuous-time state-evolution equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

If direct measurements of all of the states of the system (e.g. $y(t) = x(t)$) are available, one of the most popular and powerful control method is the linear state feedback control,

$$u(t) = -Kx(t)$$

or when there is a reference input

$$u(t) = r(t) - Kx(t)$$

which can be thought as a generalization of P controller to the vector form. Under this control law, state-equations for the closed-loop system takes the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\left(r(t) - Kx(t)\right) \\ \dot{x}(t) &= \left(A - BK\right)x(t) + Br(t)\end{aligned}$$

The system matrix of this autonomous system is $\hat{A} = A - BK$. Important questions is how to choose K. Note that

$$K \in \mathbb{R}^n \quad \text{Single} - \text{Input}$$

As in all of the control design techniques, the most critical criterion is stability, thus we want all of the eigenvalues to be in the open-left-half s-plane. However, we know that there could be different requirements on the poles/eigenvalues of the system.

The fundamental principle of "pole-placement" design is that we first define a desired closed-loop eigenvalue set $\mathcal{E}^* = \{\lambda_1^*, \ \cdots \ , \ \lambda_1^*\}$, and then if possible we choose K^* such that the closed-loop eigenvalues match the desired ones.

The necessary and sufficient condition on arbitrary pole-placement is that the system should be fully Controllable/Reachable.

In Pole-Placement, first step is computing the desired characteristic polynomial.

$$\begin{aligned}\mathcal{E}^* &= \{\lambda_1^*, \ \cdots \ , \ \lambda_n^*\} \\ p^*(s) &= (s - \lambda_1^*)\cdots(s - \lambda_n^*) \\ &= s^n + a_1^* s^{n-1} + \cdots + a_{n-1}^* s + a_n^*\end{aligned}$$

Then we tune K such that

$$\det\left(sI - (A - BK)\right) = p^*(s)$$

21.2.1 Direct Design of State-Feedback Gain

If n is small, the most efficient method could be the direct design.

Example: Consider the following system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

Design a state-feedback rule such that poles are located at $\lambda_{1,2} = -1$

Solution: Desired characteristic equation can be computed as

$$p^*(s) = s^2 + 2s + 1$$

Let $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, then the characteristic equation of $\hat{A}$ can be computed as

$$\begin{aligned}
\det(sI - (A - BK)) &= \det\left(\begin{bmatrix} s-1+k_1 & k_1 \\ k_2 & s-2+k_2 \end{bmatrix}\right) \\
&= s^2 + s(k_1 + k_2 - 3) + (2 - 2k_1 - k_2)
\end{aligned}$$

If we match the equations

$$\begin{aligned}
s^2 + s(k_1 + k_2 - 3) + (2 - 2k_1 - k_2) &= s^2 + 2s + 1 \\
k_1 + k_2 &= 5 \\
2k_1 + k_2 &= 1 \\
k_1 &= -4 \\
k_2 &= 9
\end{aligned}$$

Thus $K = \begin{bmatrix} -4 & 9 \end{bmatrix}$.

21.2.2 Design of State-Feedback Gain Using Controllable Canonical Form

Let's assume that the state-space representation is in controllable canonical form and we have access to the all states of this form

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

Let $K = \begin{bmatrix} k_n & \cdots & k_1 \end{bmatrix}$, then closed-loop system takes the form

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \left(r - \begin{bmatrix} k_n & \cdots & k_1 \end{bmatrix} x\right)$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -(a_n + k_n) & -(a_{n-1} + k_{n-1}) & -(a_{n-2} + k_{n-2}) & \cdots & -(a_1 + k_1) \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} r$$

Let $p^*(s) = s^2 + a_1^* s + \cdots + a_{n-1}^* s + a_n^*$, then computation of K is super straight-forward

$$K = \begin{bmatrix} (a_n^* - a_n) & \cdots & (a_1^* - a_1) \end{bmatrix}$$

However, what if the system is not in controllable canonical form. We can find a transformation which finds the controllable canonical representation.

The controllability matrix of a state-space representation is given as

$$Q = \left[\begin{array}{c|c|c|c} B & AB & \cdots & A^{n-1}B \end{array}\right]$$

Let's define a transformation matrix T as follows:

$$\begin{aligned} T = QW \quad &, \quad x(t) = T\hat{x}(t) \\ \dot{\hat{x}} &= \left[T^{-1}AT\right]\hat{x} + T^{-1}Bu \end{aligned}$$

where

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & & \vdots \\ a_1 & 1 & & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix}$$

Then it is given that

$$T^{-1}AT = \hat{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$$T^{-1}B = \hat{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

We know how to design a state-feedback gain $\hat{K}$ for the controllable canonical form. Given $\hat{K}$ $u(t)$ is given as

$$\begin{aligned} u(t) &= r(t) - \hat{K}\hat{x}(t) \\ &= r(t) - \hat{K}T^{-1}\hat{x}(t) \\ K &= \hat{K}T^{-1} \end{aligned}$$

Example 2: Consider the following system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

Design a state-feedback rule using the controllable canonical form approach, such that poles are located at $\lambda_{1,2} = -1$

Solution: Characteristic equation of A can be derived as

$$\det\left(\begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix}\right) = s^2 - 3s + 2$$

The controllability matrix can be computed as

$$Q = \left[\begin{array}{c|c} B & AB \end{array}\right] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

The matrix W can be computed as

$$W = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$$

Transformation matrix, T and its inverse T^{-1} can be computed as

$$\begin{aligned} T = QW &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \\ T^{-1} &= \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} \end{aligned}$$

Given that desired characteristic polynomial is $p^*(s) = s^2 + 2s + 1$, $\hat{K}$ of controllable canonical from can be computed as

$$\begin{aligned} \hat{K} &= \begin{bmatrix} -a_2 & -a_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - 2 & 2 - (-3) \end{bmatrix} = \begin{bmatrix} -1 & 5 \end{bmatrix} \end{aligned}$$

Finally K can be computed as

$$\begin{aligned} K = \hat{K}T^{-1} &= \begin{bmatrix} -1 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 9 \end{bmatrix} \end{aligned}$$

As expected this is the same result with the one found with Direct-Method.

21.3 Observability

It turns out that it is more natural to think in terms of "un-observability" as reflected in the following definitions.

- A state x_o of a finite dimensional CT linear dynamical system is said to be unobservable, if with $x(0) = x_o$ and for every $u(t)$ we get the same $y(t)$ as we would with $x(0) = 0$.

The set $\bar{\mathcal{O}}$ of all unobservable states is a linear (sub)space: $\bar{\mathcal{O}} \subset \mathbb{R}^n$

- If the unobservable set only contains the origin, i.e., if $\bar{\mathcal{O}} = \{0\}$,
- If the dimension of unobservable subspace is equal to 0, $\dim = \left(\bar{\mathcal{O}}\right) = 0$,
- If any initial condition, $x(0)$ or $x[0]$, can be uniquely determined from input-output measurement,

then the system is called Observable.

21.3.0.1 Test of Observability on CT Systems

One way of testing Observability of CT systems is checking the rank (or the range space, or null space) the of the Observability matrix

$$\mathbf{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

A CT system is Observable if and only of

$$\operatorname{rank}(\mathbf{O}) = n$$

or equivalently

$$\operatorname{Ra}(\mathbf{O}) = \mathbb{R}^n$$

or equivalently

$$\dim\left(\mathcal{N}(\mathbf{O})\right) = 0$$

Remark: A state-space representation that is in observable canonical form is always fully Observable

Remark: A state-space representation is called minimal if it is both fully Controllable and Observable.

21.4 Luenberger Observer

In general the state, $x(t)$, of a system is not accessible and *observers, estimators, filters*) have to be used to extract this information. The output, $y(t)$, represents the measurements which is a function of $x(t)$ and $u(t)$.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

A Luenberger observers is built using a "simulated" model of the system and the errors caused by the mismatched initial conditions $x_0 \neq \hat{x}_0$ (or other types of perturbations) are reduced by introducing output error feedback.

Let's assume that the state vector of the simulated system is $\hat{x}$, then the state space equation of this synthetic system takes the form

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Hu \\ \hat{y} &= C\hat{x} + Du \end{aligned}$$

Note that since u is the input that is supplied by the controller, we assume that it is known apriori. If $x(0) = \hat{x}[= (0)$ and when there is no model mismatch or uncertainty in the system then we expect that $x(t) = \hat{x}(t)$ and $y(t) = \hat{y}(t)$ for all $t \in \mathbb{R}^+$. When $x(0) \neq \hat{x}(0)$, then we should observe a difference between the measured and predicted output $y(t) \neq \hat{y}(t)$ (if the initial condition is not in the unobservable sub-space). The core idea in Luenberger observer is feeding the error in the output prediction $y(t) - \hat{y}(t)$ to the simulated system via a linear feedback gain.

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Hu + L\left(y - \hat{y}\right) \\ \hat{y} &= C\hat{x} + Du \end{aligned}$$

In order to understand how a Luenberger observer works and to choose a proper observer gain L, we define an error signal $e = x - \hat{x}$. The dynamics w.r.t e can be derived as

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= (Ax + Hu) - \left(A\hat{x} + Hu + L\left(y - \hat{y}\right)\right) \\ \dot{e} &= (A - LC)\, e \end{aligned}$$

where $e(0) = x(0) - \hat{x}(0)$ denotes the error in the initial condition.

If the matrix $(A - LC)$ is stable then the errors in initial condition will diminish eventually. Moreover, in order to have a good observer/estimator performance the observer convergence should be sufficiently fast.

Observer Gain & Pole Placement

Similar to the state-feedback gain design, the fundamental principle of "pole-placement" Observer design is that we first define a desired closed-loop eigenvalue set and compute the associated desired characteristic polynomial.

$$\begin{aligned}\mathcal{E}^* &= \{\lambda_1^*,\ \cdots,\ \lambda_n^*\}\\ p^*(s) &= (s-\lambda_1^*)\cdots(s-\lambda_n^*)\\ &= s^n + a_0^* z^{n-1} + \cdots + a_{n-2}^* z + a_{n-1}^*\end{aligned}$$

The necessary and sufficient condition on arbitrary observer pole-placement is that the system should be fully Observable. Then, we ca tune L such that

$$\det\left(sI-(A-LC)\right) = p^*(s)$$

Direct Design of Observer Gain

If n is small, the most efficient method could be the direct design.

Example: Consider the following DT system

$$\begin{aligned}\dot{x} &= \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} x + \begin{bmatrix}1\\ 1\end{bmatrix} u\\ y &= \begin{bmatrix}1 & -1\end{bmatrix} x\end{aligned}$$

Design an observer such that estimater poles are located at $\lambda_{1,2} = -5$ (Dead-beat Observer)

Solution: Desired characteristic equation can be computed as

$$p^*(s) = s^2 + 10s + 25$$

Let $L = \begin{bmatrix} l_2\\ l_1\end{bmatrix}$, then the characteristic equation of $(G-LC)$ can be computed as

$$\begin{aligned}\det\left(sI-(A-LC)\right) &= \det\left(\begin{bmatrix} s-1+l_2 & -l_2\\ l_1 & s-2-l_1\end{bmatrix}\right)\\ &= s^2 + s(l_2-l_1-3) + (l_1-2l_2+2)\end{aligned}$$

If we match the equations

$$\begin{aligned}l_2 - l_1 &= 13\\ l_1 - 2l_2 &= 23\\ l_2 &= -49\\ l_1 &= -36\end{aligned}$$

Thus $L = \begin{bmatrix}-49\\ -36\end{bmatrix}$