EE402 - Discrete Time Systems **Spring 2018**

Lecture 2

Lecturer: Asst. Prof. M. Mert Ankarali

Big Picture of EE402

In this course, the main focus will be on continuous-time systems (plants) that are controlled (sampled and actuated) by a digital computer interface. Such a discrete-time control system consists of four major parts as illustrated in Fig. **??**,

1. *The plant* is a continuous-time dynamical system
2. Analog-to-Digital Converter (ADC)
3. Controller (μP), a microprocessor/microcontroller with a "real-time" OS
4. Digital-to-Analog Converter (DAC)

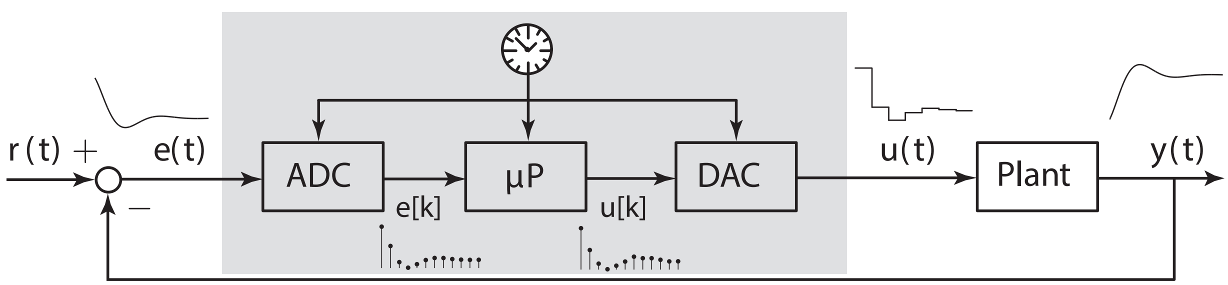

Figure 2.1: Block diagram of a digital control system

Most of the time, the plant is modeled as a "smooth" continuous dynamical system. In this course, we will cover only LTI systems. Thus, we will assume that (unless otherwise is given) the plant is a continuous LTI plant model with a transfer function of $G_p(s)$ for which both the input and output are continuous-time signals.

The "digital" blocks inside the closed-loop block diagram structure are ADC, the Controller, and DAC. It is generally assumed (design requirement) that all blocks share a common "hard real-time" clock.

A general ADC is a device that converts an analog signal to a digital signal. In this course, we will model the ADC block as an *ideal sampler* for which the input is a continuous-time signal, $e(t)$, and the output is a discrete-time signal, $e[k]$, where the relation between the continuous- and discrete-time signals are given as

$$e(kT) = e[k], \; k \in \mathbb{Z}^+,$$

where constant T is the *sampling time.*

The microcontroller/microprocessor processes some set of digital input signals to produce some set of digital output signals. The outputs are defined at only some specified instances determined by the real-time clock.

In this course, we will model the μP block as an ideal discrete-time LTI system for which both the input and output are discrete-time signals, with a transfer function of $G_c(z)$.

The DAC is a device that converts a digital signal to an analog signal. In this course, we assume that it is an ideal *Hold* element for which the input signal is a discrete-time signal, whereas the output is a continuous-time signal. The most commonly used *Hold* system is ZOH (Zero-Order-Hold) which is a mapping defined by the following relation

$$u(t) = u[k], \text{ for } t \in [kT, (k+1)T)$$

Higher-order hold operators exist, but they are extremely rarely used in practice.

The idealized and simplified block-diagram structure is given in Fig.

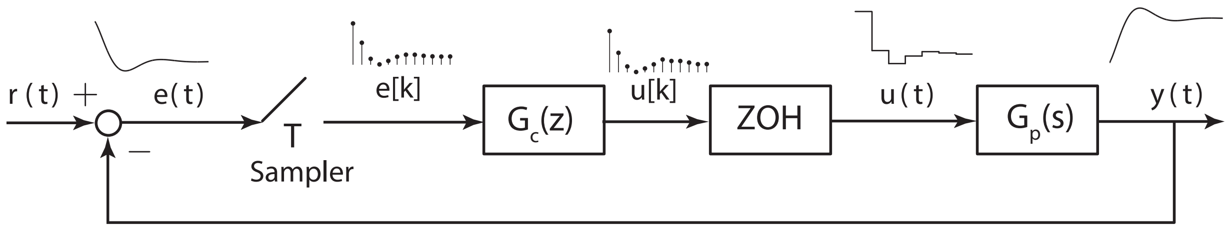

Figure 2.2: Block diagram of an LTI discrete-time control system

Major challenge: Loop contains both continuous-time and discrete-time parts.

Sampling

Fig. **??** illustrates two different ideal samplers. Both of them will be covered in this course. First column is an *impulse sampler* for which the output is a continuous-time signal, but it is composed of trains of impulses (impulse train). Second one is an ideal complete CT-to-DT sampler which convertes the impulse train into DT sequence.

The output of the impulse sampler, $x^*(t)$, can be represented with the following infinte summations

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT) = \sum_{k=0}^{\infty} x[k]\delta(t-kT)$$

or

$$\begin{aligned} x^*(t) &= x(0)\delta(t) + x(T)\delta(t-T) + \cdots + x(kT)\delta(t-kT) + \cdots \\ &= x[0]\delta(t) + x[1]\delta(t-T) + \cdots + x[k]\delta(t-kT) + \cdots \end{aligned}$$

Now let's consider the Laplace transform of $x^*(t)$

$$\begin{aligned} X^*(s) = \mathcal{L}\{x^*(t)\} &= \mathcal{L}\left\{\sum_{k=0}^{\infty} x(kT)\delta(t-kT)\right\} = \sum_{k=0}^{\infty} x(kT)\mathcal{L}\left\{\delta(t-kT)\right\} \\ &= \sum_{k=0}^{\infty} x(kT)\int_{t=0}^{\infty} \delta(t-kT)e^{-st}dt = \sum_{k=0}^{\infty} x(kT)e^{-skT} = \sum_{k=0}^{\infty} x[k]e^{-skT} \end{aligned}$$

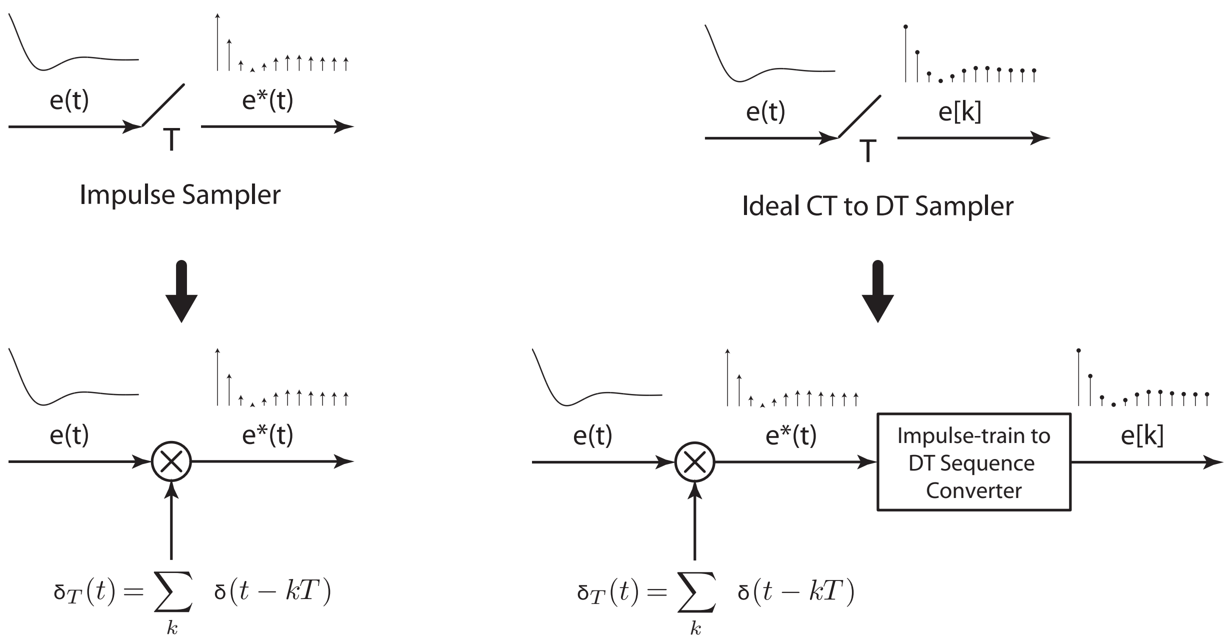

Figure 2.3: Two different ideal samplers

Now let's define a map in complex domain such that

$$z = e^{Ts} \text{ or } s = \frac{1}{T} \ln z$$

Then we have

$$X^*(s)|_{s=(1/T)lnz} = \sum_{k=0}^{\infty} x[k] z^{-k}$$

where

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k] z^{-k}$$

Z-transform

Z-transform of a (causal) discrete time signal $x[k]$ is given by

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k] z^{-k}$$

If $x[k]$ is a sampled signal from a continuous time signal $x(t)$ with a sampling time of T, we (abuse of notation) also use the following notation

$$X(z) = \mathcal{Z}\{x(kT)\} = \mathcal{Z}\{x^*(t)\}$$

Z-transforms of elementary functions

We assume that all signals are causal thus $t \in \mathbb{R}^+$ and $k \in \mathbb{Z}^+$

Unit-step function $x(t) = 1$ and thus $x(kT) = x[k] = 1$, the Z-transform is given by

$$X(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

Unit-ramp function $x(t) = t$ and thus $x(kT) = x[k] = kT$, the Z-transform is given by

$$\begin{aligned}
X(z) &= \sum_{k=0}^{\infty} kTz^{-k} = T(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots) = Tz(z^{-2} + 2z^{-3} + 3z^{-4} + \cdots) \\
&= Tz\frac{d}{dz}\left(\int (z^{-2} + 2z^{-3} + 3z^{-4} + \cdots)dz\right) = Tz\frac{d}{dz}\left(-(z^{-1} + z^{-2} + z^{-3} + \cdots)\right) \\
&= Tz\frac{d}{dz}\left(\frac{-1}{z-1}\right) = \frac{Tz}{(z-1)^2} = \frac{Tz^{-1}}{(1-z^{-1})^2}
\end{aligned}$$

Exponential sequence $x[k] = a^k$

$$X(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} \left(\frac{z}{a}\right)^{-k} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

Exponential function $x(t) = e^{bt}$ and thus $x(kT) = x[k] = e^{bTk}$

$$X(z) = \sum_{k=0}^{\infty} e^{bTk} z^{-k} = \sum_{k=0}^{\infty} (e^{bT})^k z^{-k} = \frac{1}{1-e^{bT}z^{-1}} = \frac{z}{z-e^{bT}}$$

Cosine function $x(t) = \cos(\omega t)$, and thus $x(kT) = x(k) = \cos(\omega Tk)$

$$\begin{aligned}
\cos(\omega Tk) = \frac{1}{2}\left(e^{j\omega Tk} + e^{-j\omega Tk}\right) X(z) &= \frac{1}{2}\left(\frac{z}{z-e^{j\omega T}} + \frac{z}{z-e^{-j\omega T}}\right) = \frac{1}{2}\frac{z(z-e^{-j\omega T}) + z(z-e^{j\omega T})}{(z-e^{-j\omega T})(z-e^{j\omega T})} \\
&= \frac{1}{2}\frac{2z^2 - z(e^{-j\omega T} + e^{j\omega T})}{z^2 - z(e^{-j\omega T} + e^{j\omega T}) + 1} = \frac{z^2 - z\cos(\omega T)}{z^2 - z2\cos(\omega T) + 1} \\
&= \frac{1 - z^{-1}\cos(\omega T)}{1 - z^{-1}2\cos(\omega T) + z^{-2}}
\end{aligned}$$

Properties and Theorems of the Z-transform

Linearity

$$x[k] = \alpha f[k] + \beta g[k] \ \rightarrow X(z) = \alpha F(z) + \beta G(z) \ , \forall\ \alpha,\ \beta,\ f[k],\ \&\ g[k]$$

Multiplication by a^k

$$\begin{aligned}
\mathcal{Z}\{a^k x[k]\} &= \sum_{k=0}^{\infty} a^k x[k] z^{-k} = \sum_{k=0}^{\infty} x[k](z/a)^{-k} \\
\mathcal{Z}\{a^k x[k]\} &= X(z/a)
\end{aligned}$$

Complex translation theorem

Let $y(t) = e^{-at}x(t)$ and $X(z) = \mathcal{Z}\{x(kT)\}$, then

$$\mathcal{Z}\{y(kT)\} = \mathcal{Z}\{e^{-aTk}x(kT)\} = X(e^{aT}z)$$

Shifting theorem

Let $x(t)$ be a causal CT signal, thus we have $x(t) = 0$ for $t < 0$. Similarly, associated sampled DT signal has the property of $x[k] = 0$ for $k < 0$. For the sake of simplicity lets work on the sampled (i.e. DT) signal. Let

$$\mathcal{Z}\{x^*(t)\} = \mathcal{Z}\{x[k]\} = X(z)$$

Shifting right by N (Causal shifting): Let $y[k] = x[k-N]$, then

$$\mathcal{Z}\{y[k]\} = \sum_{k=0}^{\infty} y[k]z^{-k} = \sum_{k=0}^{\infty} x[k-N]z^{-k} = \sum_{k=N}^{\infty} x[k-N]z^{-k}$$

Let $k = m + N$ then

$$\begin{aligned}
\mathcal{Z}\{y[k]\} &= \sum_{m=0}^{\infty} x[m]z^{-(m+N)} = z^{-N}\sum_{m=0}^{\infty} x[m]z^{-m} \\
\mathcal{Z}\{x[k-N]\} &= z^{-N}X(z)
\end{aligned}$$

Shifting left by N (Non-causal shifting) & Bilateral Z transform: Let $y[k] = x[k+N]$,

$$\begin{aligned}
\mathcal{Z}\{x[k+N]\} &= \sum_{k=-\infty}^{\infty} x[k+N]z^{-k} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m-N)} = z^{N}\sum_{m=-\infty}^{\infty} x[m]z^{-m} \\
\mathcal{Z}\{x[k+N]\} &= z^{N}X(z)
\end{aligned}$$

Shifting left by N (Non-causal shifting) & Unilateral Z transform: Let $y[k] = x[k+N]$,

$$\mathcal{Z}\{x[k+N]\} = \sum_{k=0}^{\infty} x[k+N]z^{-k}$$

Let $k = m - N$ then

$$\begin{aligned}
\mathcal{Z}\{x[k+N]\} &= \sum_{m=N}^{\infty} x[m]z^{-(m-N)} = z^{N}\sum_{m=N}^{\infty} x[m]z^{-m} = z^{N}\left(\sum_{k=0}^{\infty} x[k]z^{-k} - \sum_{k=0}^{N-1} x[k]z^{-k}\right) \\
\mathcal{Z}\{x[k+N]\} &= z^{N}\left(X(z) - \sum_{k=0}^{N-1} x[k]z^{-k}\right)
\end{aligned}$$

From this equation we can obtain

$$\begin{aligned}
\mathcal{Z}\{x[k+1]\} &= zX(z) - zx[0] \\
\mathcal{Z}\{x[k+2]\} &= z^2X(z) - z^2x[0] - zx[1] \\
&\vdots
\end{aligned}$$

Example 1. Let $u[k]$ be the unit-step function. Compute $\mathcal{Z}\{u[k-1]\}$ both directly and using the shifting property.

$$\mathcal{Z}\{u[k-1]\} = \frac{z^{-1}}{1-z^{-1}}$$

Example 2. Let $y[k] = \sum_{n=0}^{k} x[k]$ where $k \in \mathbb{Z}^+$. Compute $Y(z)$ in terms of $X(z)$ using the shifting theorem.

$$Y(z) = \frac{1}{1-z^{-1}} X(z)$$

Initial Value Theorem Let $X(z) = \mathcal{Z}\{x[n]\}$ and if the following limit exists, then the initial value of $x[0]$ or $x(0)$ is given by

$$x[0] = \lim_{z\to\infty} X(z)$$

Indeed the proof is very easy

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \left[\sum_{k=0}^{\infty} x(k)z^{-k}\right] = \lim_{z\to\infty} \left[x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots\right] = x(0)$$

Final Value Theorem

Let's assume that $x(kT)$ or $x[k]$ is a convergent sequence (DT signal). Then the final value theorem states that

$$\lim_{k\to\infty} x[k] = \lim_{z\to 1} (1-z^{-1})X(z)$$

Proof: Let's take the Z transform of $x[k] - x[k-1]$

$$\begin{aligned}
\mathcal{Z}\{x[k]-x[k-1]\} &= \sum_{k=0}^{\infty} \left(x[k]-x[k-1]\right) z^{-k} \\
X(z) - X(z)z^{-1} &= \left(x[0]\left(1-z^{-1}\right) + x[1]\left(z^{-1}-z^{-2}\right) + x[2]\left(z^{-2}-z^{-3}\right) + x[3]\left(z^{-3}-z^{-4}\right) + \cdots\right) + \lim_{k\to\infty} x[k]z^{-k} \\
\lim_{z\to 1} X(z)\left(1-z^{-1}\right) &= (0+0+\cdots) + \lim_{z\to 1}\lim_{k\to\infty} x[k]z^{-k} \\
\lim_{z\to 1} X(z)\left(1-z^{-1}\right) &= \lim_{k\to\infty} x[k]
\end{aligned}$$

Complex Differentiation Theorem

Consider

$$\begin{aligned}
\frac{d}{dz}X(z) &= \frac{d}{dz}\left[\sum_{k=0}^{\infty} x[k]z^{-k}\right] = \sum_{k=0}^{\infty} x[k]\frac{d}{dz}z^{-k} = \sum_{k=0}^{\infty}(-k)x[k]z^{-k-1} \\
-z\frac{d}{dz}X(z) &= \sum_{k=0}^{\infty} kx[k]z^{-k} \\
-z\frac{d}{dz}X(z) &= \mathcal{Z}\{kx[k]\}
\end{aligned}$$

In general

$$(-z)^m \frac{d}{dz^m} X(z) = \mathcal{Z}\{k^m x[k]\}$$

Example 3. Find the Z-transform of the unit ramp function, $r[k] = k$, $k \in \mathbb{Z}^+$ by applying the Complex Differentiation Theorem to the Z-transform of the unit step function.

Solution:

$$\begin{aligned}
\mathcal{Z}\{r[k]\} &= \mathcal{Z}\{ku[k]\} \\
R(z) &= (-z)\frac{d}{dz}U(z) = (-z)\frac{d}{dz}U(z) = (-z)\frac{d}{dz}\left(\frac{z}{z-1}\right) = (-z)\left(\frac{1}{z-1} - \frac{z}{(z-1)^2}\right) \\
&= \frac{z^2}{(z-1)^2} - \frac{z}{z-1} = \frac{z^2 - z(z-1)}{(z-1)^2} \\
R(z) &= \frac{z}{(z-1)^2}
\end{aligned}$$

Real Convolution Theorem Let $f[k]$ and $g[k]$ are causal signals and associated Z transforms are $F(z)$ and $G(z)$ respectively. The DT convolution operator is defined as

$$f[n] * g[n] = \sum_{k=0}^{n} f[n-k]g[k]$$

Real Convolution Theorem states that

$$\mathcal{Z}\{f[n] * g[n]\} = F(z)G(z)$$

Proof

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} f[n-k]g[k]\right] z^{-n}$$

Since we know that $f[m] = 0$ for $m < 0$, we can stretch the upper limit of the sum as

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} f[n-k]g[k]\right] z^{-n} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} f[n-k]g[k]z^{-n}$$

Let $n = m + k$ then

$$\begin{aligned}
\mathcal{Z}\{f[n] * g[n]\} &= \sum_{k=0}^{\infty}\sum_{m=-k}^{\infty} f[m]g[k]z^{-m}z^{-k} = \sum_{k=0}^{\infty} g[k]z^{-k} \sum_{m=0}^{\infty} f[m]z^{-m} \\
\mathcal{Z}\{f[n] * g[n]\} &= F(z)G(z)
\end{aligned}$$

The Inverse Z-transform

1. Direct division method
2. Z-tranform tables & partial-fraction expansion
3. "Simulation" method
4. Inversion integral method

Direct division

Direct division (or long division) method uses the fact that $X(z)$ can be expressed as

$$X(z) = \sum_{k=0}^{\infty} x[k]z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$$

The goal is finding the power series expansion of $X(z)$ using the long division approach. Here we assume that $X(z)$ cane be represnted as a ratio of two polynomials in z (or z^{-1})

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n} = \frac{b_0 z^{-n+m} + b_1 z^{-n+m-1} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

For the direct division method it is easer to work when the polynomials are written interms of powers of z^{-1}.

Example 4. Find the inverse Z-transform of $X(z) = \frac{z^{-1}}{1-2z^{-1}+z^{-2}}$.

$$\begin{array}{r|l}
z^{-1} & 1 - 2z^{-1} + z^{-2} \\ \cline{2-2}
z^{-1} - 2z^{-2} + z^{-3} & z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \ldots \\ \cline{1-1}
2z^{-2} - z^{-3} & \\
2z^{-2} - 4z^{-3} + 2z^{-4} & \\ \cline{1-1}
3z^{-3} - 2z^{-4} & \\
3z^{-3} - 6z^{-4} + 3z^{-5} & \\ \cline{1-1}
4z^{-4} - 3z^{-5} & \\
\vdots &
\end{array}$$

Thus,

$$X(z) = 0 + 1z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \ldots$$
$$\downarrow$$
$$x[k] = 0\delta[k] + 1\delta[k-1] + 2\delta[k-2] + 3\delta[k-3] + 4\delta[k-4] + \ldots = k$$

Partial Fraction Expansion

In most applications $X(z)$ can be re-written in terms of poles and zeros as

$$X(z) = b_0 \frac{(z-z_1)\cdots(z-z_m)}{(z-p_1)\cdots(z-p_n)} \quad (m \le n)$$

Specific (but extremely common) case

$$\frac{X(z)}{z} = \sum_{i=1}^{n} \frac{a_i}{(z - p_i)}$$

where all poles are distinct and simple order. We can compute each a_i using

$$a_i = \lim_{z \to p_i} \left[(z - p_i) \frac{X(z)}{z} \right]$$

Example 5. Find the inverse Z-transform of $X(z) = \frac{(1-b)z}{(z-1)(z-b)}$. Solution:

$$\begin{aligned}
\frac{X(z)}{z} &= \frac{(1-b)}{(z-1)(z-b)} = \frac{a_1}{z-1} + \frac{a_2}{z-b} \\
a_1 &= \lim_{z \to 1} \left[(z-1) \frac{X(z)}{z} \right] = 1 \\
a_2 &= \lim_{z \to b} \left[(z-b) \frac{X(z)}{z} \right] = -1 \\
X(z) &= \frac{z}{z-1} - \frac{z}{z-b} \\
x[k] &= 1 - b^k
\end{aligned}$$

Now let's assume that $\frac{X(z)}{z}$ has double pole at p_1 and all other poles are distinct

$$\frac{X(z)}{z} = \frac{c_1}{z - p_1} + \frac{c_2}{(z - p_1)^2} + \cdots$$

It is easy to show that

$$c_2 = \lim_{z \to p_1} \left[(z - p_1)^2 \frac{X(z)}{z} \right]$$

It is also possible to show that

$$c_1 = \lim_{z \to p_1} \left\{ \frac{d}{dz} \left[(z - p_1)^2 \frac{X(z)}{z} \right] \right\}$$

Example 6. Find the inverse Z-transform $X(z) = \frac{2z^2 - 3z}{(z-1)^2}$. Solution:

$$\begin{aligned}
\frac{X(z)}{z} &= \frac{c_1}{z-1} + \frac{c_2}{(z-1)^2} \\
c_1 &= \lim_{z \to 1} \frac{d}{z} \left[(z-1)^2 \frac{X(z)}{z} \right] = 2 \\
c_2 &= \lim_{z \to 1} \left[(z-1)^2 \frac{X(z)}{z} \right] = -1 \\
x[k] &= 2 - k
\end{aligned}$$

Example 7. Find the inverse Z-transform $X(z) = \frac{(1-b)}{(z-1)(z-b)}$. Solution:

$$
\begin{aligned}
X(z) &= \frac{(1-b)}{(z-1)(z-b)} = \frac{a_1}{z-1} + \frac{a_2}{z-b} \\
a_1 &= \lim_{z\to 1}\left[(z-1)X(z)\right] = 1 \\
a_2 &= \lim_{z\to b}\left[(z-b)X(z)\right] = -1 \\
X(z) &= z^{-1}\left(\frac{z}{z-1} - \frac{z}{z-b}\right) \\
x[k] &= [1-b^{k-1}]u[k-1]
\end{aligned}
$$

Example 8. Find the inverse Z-transform $X(z) = \frac{z^2-2}{(z-1)(z-2)}$. Solution:

$$
\begin{aligned}
X(z) &= \frac{z^2-2}{z^2-3z+2} = 1 + \frac{3z-4}{z^2-3z+2} \\
X(z) &= 1 + \frac{a_1}{z-1} + \frac{a_2}{z-2} \\
a_1 &= \lim_{z\to 1}\left[(z-1)X(z)\right] = 1 \\
a_2 &= \lim_{z\to 2}\left[(z-2)X(z)\right] = 2 \\
X(z) &= 1 + \frac{1}{z-1} + \frac{2}{z-2} = \frac{1}{z-1} + \frac{z}{z-2} \\
x[k] &= 1 + 2^k - \delta[k]
\end{aligned}
$$