EE402 - Discrete Time Systems	Spring 2018

Lecture 5

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Let's remember the idealized and simplified block-diagram structure a discrete-time control system (See Fig. ??)

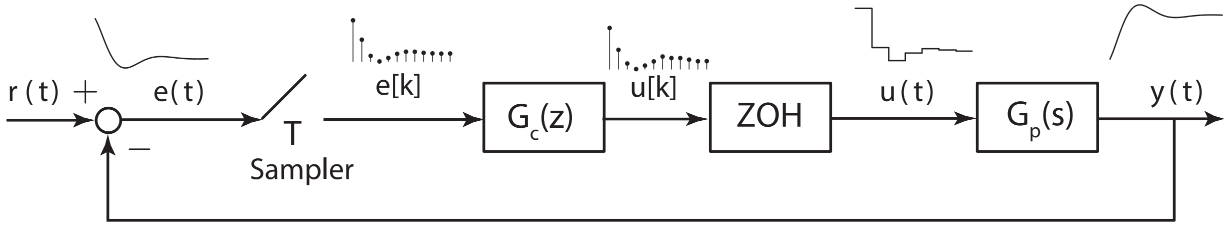


Figure 5.1: Block diagram of an LTI discrete-time control system

Loop contains both continuous-time and discrete-time signals and blocks.

- We can treat the system as a completely discrete-time system. We technically restrict ourselves into sampled time instants (which may be just fine)
- Alternatively, we can use continuous time signals (as much as possible) and deal with starred versions of signals and starred Laplace transform.

Sampling - Review

Fig. ?? illustrates an ideal *impulse sampler* and an ideal complete CT-to-DT sampler.

The output of the impulse sampler, $x^*(t)$, can be represented with the following infinte summations

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT) = \sum_{k=0}^{\infty} x[k]\delta(t-kT)$$

Now let's consider the Laplace transform of $x^*(t)$

$$X^*(s) = \mathcal{L}\{x^*(t)\} = \sum_{k=0}^{\infty} x(kT) \int_{t=0}^{\infty} \delta(t-kT)e^{-st}dt = \sum_{k=0}^{\infty} x(kT)e^{-skT} = \sum_{k=0}^{\infty} x[k]e^{-skT}$$

Now let's define a map in complex domain such that $z = e^{Ts}$ or $s = \frac{1}{T}\ln z$. Then we have

$$X^*(s)|_{s=(1/T)lnz} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

where

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

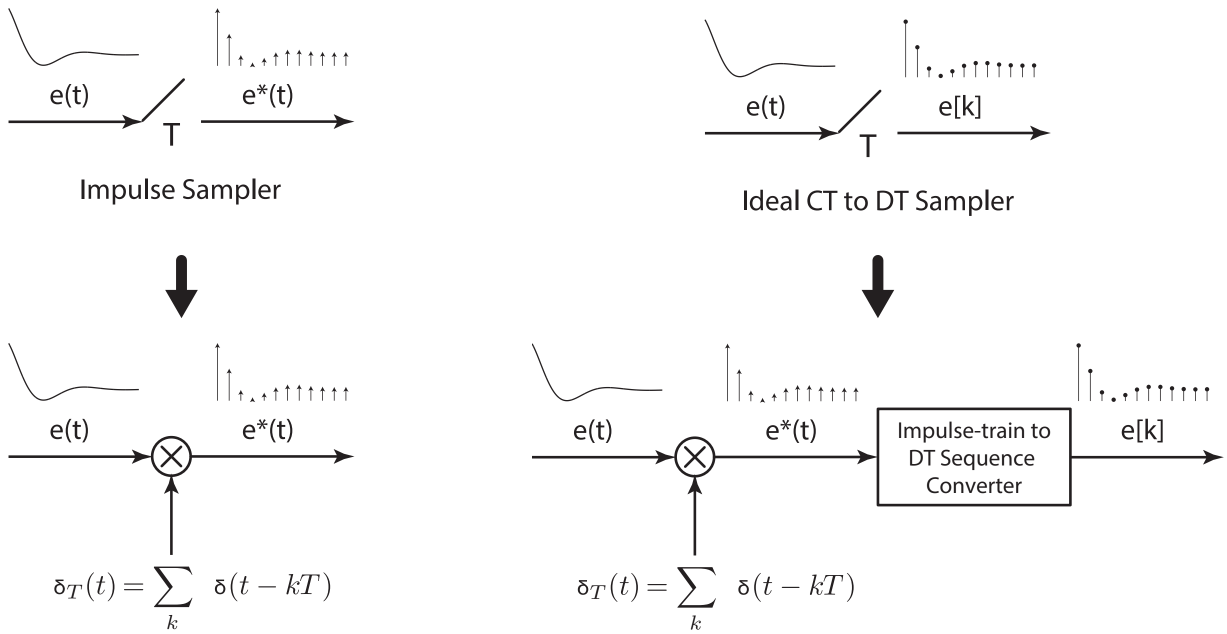
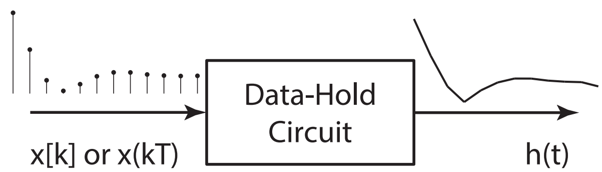


Figure 5.2: Two different ideal samplers

Data Hold Operation

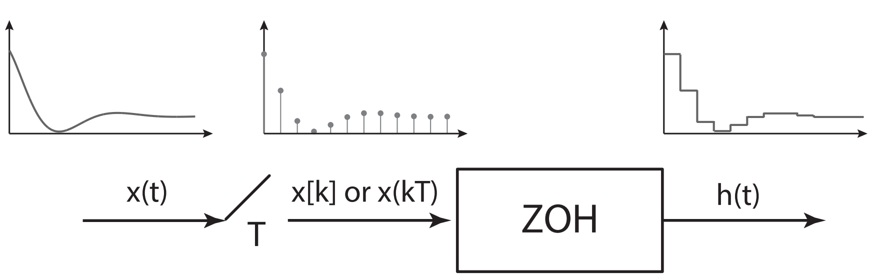
Data-Hold operation is an idealized model of a DAC device which converts a digital signal to an analog signal. In terms of the terminology used in this class, Data-Hold operation is the process of obtaining a CT signal $h(t)$ from a DT sequence. A general data-hold operation block circuit is shown below



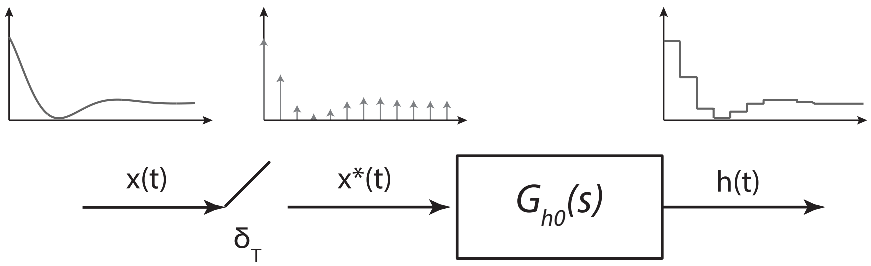
Simplest and most dominantly used (I have never seen a practical usage of other hold operations) hold circuit/operation is the zero-order-hold (ZOH). Basically, at each time instant kT ZOH "samples" the input $x[k]$ or $x(kT)$ and "holds" this value at the output until the next sampling event. Mathematically,

$$h(kT + t) = x(kT) = x[k], \text{ for } 0 \le t < T$$

The figure below illustrates a serios connection of an ideal CT-DT sampler and an ideal ZOH block.



If we model the sampler using an ideal impulse sampler (not CT-DT converter) then it becomes more convenient to model the ZOH with a CT transfer function as shown with the block diagram below



Let's assume that $x(t)$ is a strictly causal signal, then from the definition of ZOH we can express $h(t)$ in terms of $x(t)$ (or $x^*(t)$, $x[k]$, $x(kT)$) as

$$h(t) = x(0)\,[u(t) - u(t-T)] + x(T)\,[u(t-T) - u(t-2T)] + x(2T)\,[u(t-2T) - u(t-3T)] + \cdots$$

$$h(t) = \sum_{k=0}^{\infty} x(kT)\,[u(t-kT) - u(t-(k+1)T)]$$

If we take the Laplace transform of $h(t)$, we obtain

$$\begin{aligned}
\mathcal{L}\{h(t)\} &= \sum_{k=0}^{\infty} x(kT)\mathcal{L}\left\{[u(t-kT) - u(t-(k+1)T)]\right\} \\
&= \sum_{k=0}^{\infty} x(kT)\left[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}\right] \\
&= \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x[k]e^{-kTs} \\
H(s) &= \frac{1-e^{-Ts}}{s}X^*(s) = G_{h0}(s)X^*(s) \\
G_{h0}(s) &= \frac{1-e^{-Ts}}{s}
\end{aligned}$$

Z-transform & ZOH

When analyzing the discrete time control systems, we will (frequently) need to compute the Z-transform of sampled signals, for which the Laplace transform involves the term $\frac{1-e^{-Ts}}{s}$.

Let $\mathcal{L}\{x(t)\} = X(s) = \frac{1-e^{-Ts}}{s}G(s)$. Now let's analyze the z-transform of the sampled version of the signal, i.e. $X(z) = \mathcal{Z}\{x^*(t)\}$. First let's find $x(t)$ from $X(z)$

$$\begin{aligned}
x(t) = \mathcal{L}^{-1}\{X(s)\} &= \mathcal{L}^{-1}\left\{\frac{1-e^{-Ts}}{s}G(s)\right\} = \mathcal{L}^{-1}\left\{(1-e^{-Ts})\frac{G(s)}{s}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} - \mathcal{L}^{-1}\left\{e^{-Ts}\frac{G(s)}{s}\right\}
\end{aligned}$$

Let $\hat{g}(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}$ then

$$x(t) = \hat{g}(t) - \hat{g}(t-T)$$

$x(kT)$ and $x[k]$ takes the form

$$x(kT) = \hat{g}(kT) - \hat{g}(kT - T)$$
$$x[k] = \hat{g}[k] - \hat{g}[k-1]$$

Then $X(z)$ takes the form

$$X(z) = \left(1 - z^{-1}\right)\hat{G}(z)$$

where $\hat{G}(z) = \mathcal{Z}\left\{\left[\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right]^*\right\}$. In the textbook this notation is shortened to have $\hat{G}(z) = \mathcal{Z}\left\{\frac{G(s)}{s}\right\}$. After that we have

$$X(z) = \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

Example 1. Obtain the z transform of $x(kT)$ where $T = 1$ and $X(s)$ is given as

$$X(s) = \frac{1 - e^{-s}}{s}\frac{1}{s+1}$$

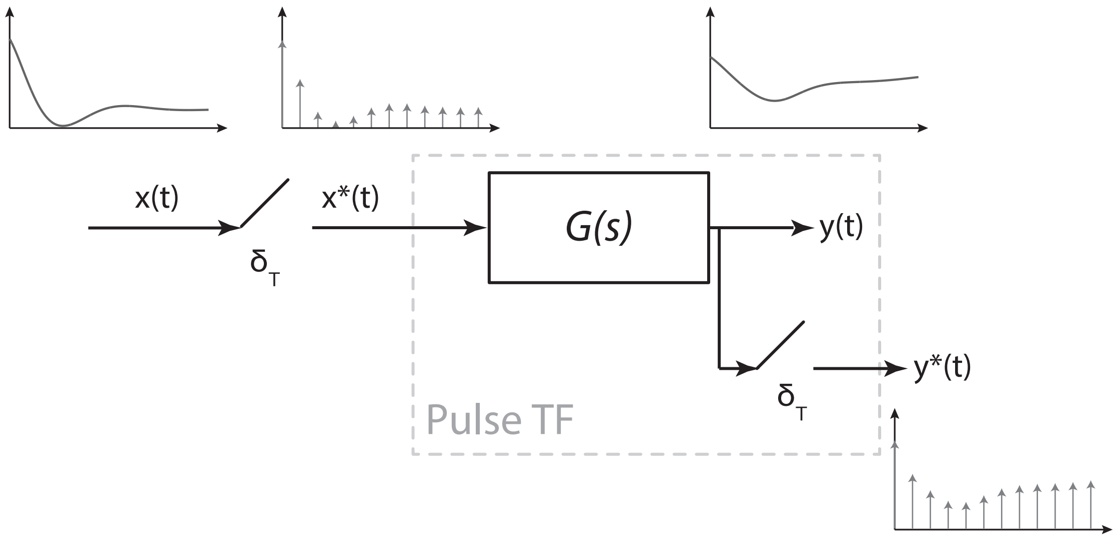
Solution:

$$\begin{aligned}
X(z) &= \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{1}{s(s+1)}\right\} = \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} \\
&= \frac{z-1}{z}\left(\frac{z}{z-1} - \frac{z}{z-e^{-1}}\right) = 1 - \frac{z-1}{z-e^{-1}} \\
X(z) &= \frac{1-e^{-1}}{z-e^{-1}}
\end{aligned}$$

Pulse Transfer Function

The CT transfer function relates the Laplace transform of the continious-time output, $y(t)$ and $Y(s)$, to that of the continious-time input $x(t)$ and $X(s)$.

The Pulse Transfer Function will relate the Z transform of the output, $y^*(t)$ (or $y(kT)$) and $Y(z)$, to that of the sampled input, $x^*(t)$ (or $x(kT)$) and $X(z)$. Note that without a feedback-loop the sampling at the output remains purely synthetic. The figure below illustrates the signals and associated transfer function blocks:



Let $g(t)$ be the impulse response of the transfer function $G(s)$, then we know that

$$y(t) = \int_0^t g(t-\tau)x^*(\tau)d\tau$$

$$y(t) = \int_0^t g(t-\tau)\sum_{k=0}^{\infty} x(kT)\delta(\tau - kT)d\tau$$

Let $t = nT + \hat{t}$ where $\hat{t} \in [0, T)$ then

$$y(nT+\hat{t}) = \int_0^{nT+\hat{t}} g(nT+\hat{t}-\tau)\sum_{k=0}^{\infty} x(kT)\delta(\tau - kT)d\tau$$

$$y(nT+\hat{t}) = \sum_{k=0}^{n} x(kT) \int_0^{nT+\hat{t}} g(nT+\hat{t}-\tau)\delta(\tau - kT)d\tau$$

$$y(nT+\hat{t}) = \sum_{k=0}^{n} x(kT)g((n-k)T+\hat{t})$$

Let $\hat{t} = 0$, then

$$y(nT) = \sum_{k=0}^{n} x(kT)g((n-k)T)$$

$$y[n] = \sum_{k=0}^{n} x[k]g[(n-k)]$$

In other words

$$y(nT) = x(nT) * g(nT) = g(nT) * x(nT)$$
$$y[n] = x[n] * g[n] = g[n] * x[n]$$

The result is pretty interesting: the impulse response of the "discretized" system is equal to the signal which is obtained by sampling the impulse response function of original the continuous time system.

If we take the Z transform of the equation given by the convolution (remember the properties of Z-transform) we obtain

$$Y(z) = G(z)X(z) \;\rightarrow\; G(z) = \frac{Y(z)}{X(z)}$$

where $G(z)$ is called the **Pulse Transfer Function of the DT System**. Note that

$$G(z) = \sum_{k=0}^{\infty} g(kT)z^{-k}$$

If we know $g(t)$ and $G(s)$, given the sampling time T, we can compute $g[k]$ and $G(z)$.

Can we say something extra regarding the starred Laplace transforms and Z transforms of y, x, and g?

$$\begin{aligned} &Y(s) = G(s)X^*(s) \;\rightarrow Y^*(s) = [G(s)X^*(s)]^* \\ &Y(z) = G(z)X(z) \end{aligned}$$

Given that starred Laplace transform is the z-transform where z is evaluated e^{TS} we can conclude that

$$Y^*(s) = [G(s)X^*(s)]^* = G^*(s)X^*(s)$$