EE502 - Linear Systems Theory II **Spring 2032**

Lecture 8

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8.1 Internal (Lyapunov) Stability

In internal stability, we are interested in un-driven (zero-input response) part of the dynamical system and solely focus on state evolution dynamics, i.e. autonomous part of the dynamical system. CT and DT non-linear autonomous systems can simply be expressed by

$$x \in \mathbb{D} \subset \mathbb{R}^n$$
$$\dot{x} = F(x,t)$$
$$x[k+1] = F(x[k]], k)$$

For non-linear systems, in order to define and analyze the stability of a dynamical system, we need to define equilibrium points (or nominal solutions), since we will technically analyze the stability around such points. An equilibrium point for CT and DT non-linear systems are defined as

$$x_e \in \mathbb{D}$$
$$CT : 0 = F(x_e, t) \ \forall t > t_0$$
$$DT : x_e = F(x_e, k) \ \forall k > k_0$$

Obviously if a dynamical system at time t_0 (or k_0) starts from an an equilibrium point, $x(t_0) = x_e$ (or $x[k_o] = x_e$), it will remain on the equilibrium point $\forall \ t \geq t_0$ (or $\forall \ k \geq k_0$). A non-linear system can have a single equilibrium point, $x_e \in \mathcal{E}$, $\text{card}(\mathcal{E}) = 1$, have multiple finite number of equilibria, $x_e \in \mathcal{E}$, $\text{card}(\mathcal{E}) = n_e < \infty$, or infinite number of equilibrium points, $x_e \in \mathcal{E}$, $\text{card}(\mathcal{E}) = \infty$.

Ex 8.1 *Show that for an LTI dynamical system, set of equilibrium points define a vector space. Then characterize this vector space.*

Definition: Without loss of generality, let's assume that the equilibrium point that is point of interest is located at the origin $x_e = 0$.

1. The system is called *stable in the sense of Lyapunov (s.i.s.L)* around $x_e = 0$ if it satisfies

$$\forall \epsilon > 0, \ \exists \delta_L(\epsilon) \ s.t. \ if \ ||x(t_0)|| < \delta_L \ \rightarrow ||x(t)|| < \epsilon \ \forall t \geq t_0$$

2. The system is called *asymptotically stable* around around $x_e = 0$ if it is *stable in the sense of Lyapunov (s.i.s.L)* around $x_e = 0$ and *locally attractive*, i.e.

$$\exists \delta_a \ s.t. \ if \ ||x(t_0)|| < \delta_a \ \rightarrow \lim_{t \to \infty} ||x(t)|| = 0$$

3. The system is called *exponentially stable* around around $x_e = 0$ if it is *asymptotically stable* around $x_e = 0$ and satisfies

$$\exists \delta_e > 0, \, \alpha > 0, \, \sigma > 0 \ s.t. \ if \ ||x(t_0)|| < \delta_e \ \rightarrow ||x(t)|| \leq \alpha ||x(t_0)|| e^{-\sigma t} \ \forall t \geq t_0$$

Remark: If above stability conditions are satisfied $\forall\, t_0 \in \mathbb{R}$, then we call the system around the equilibrium *uniformly s.i.s.L*, *uniformly asymptotically stable*, and *uniformly exponentially stable* respectively. The difference between uniform and non-uniform stability is (slightly) important for only time-varying non-linear systems. Thus we will not use uniform stability definition in this course.

Remark: Note that as you can see the internal stability definitions, *s.i.s.L*, *asymptotic stability*, and *exponentially stability*, are all local stability definitions defined in the neighborhood of x_e. If asymptotic and exponential stability definition holds for all initial conditions, i.e. $x(t_0) \in \mathbb{D}$, then we use the terms *globally asymptotically stable*, and *globally exponentially stable*.

Ex 8.2 *Consider the pendulum dynamics*

$$\dot{x} = \begin{bmatrix} x_2 \\ -sin(x_1) \end{bmatrix} \ , \text{ where } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

Analyze the stability of the dynamics around $x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ using Lyapunov's stability definitions.

Solution: We know that a mechanical pendulum is an energy-conserving system since there is no dissipative or active element. In that respect, at any time instant, we can write total energy as

$$E(x) = \frac{1}{2}x_2^2 + 1 - \cos x_1 \ , \text{ note } E(0) = 0$$

Let $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$ and $||x(0)||_2 < \epsilon$, $\epsilon \in \mathbb{R}^+$, then we know that $(x_1(0)^2 + x_2(0)^2) < \epsilon^2$. Due to the conservation of energy, we also know that

$$\begin{aligned}
&E(x(t)) = E(x(0)) \\
&\frac{1}{2}x_2(t)^2 + 1 - \cos x_1(t) = \frac{1}{2}x_2(0)^2 + 1 - \cos x_1(0) < \frac{1}{2}x_2(0)^2 + \frac{1}{2}x_1(0)^2 < \frac{\epsilon^2}{2} \\
&\frac{1}{2}x_2(t)^2 + 1 - \cos x_1(t) < \frac{\epsilon^2}{2} \ \rightarrow \ x_2(t)^2 < \epsilon^2 + 4
\end{aligned}$$

We already know that $x_1 = \theta$ and it is bounded since $\theta \in \mathbb{S}$, so

$$x_1(t)^2 \leq 1 \ \rightarrow \ (x_1(t)^2 + x_2(t)^2) < \epsilon^2 + 5 \ \rightarrow \ ||x(t)||_2 < \sqrt{\epsilon^2 + 5} = \delta(\epsilon)$$

This shows that the dynamics around the equilibrium is *stable in the sense of Lyapunov*. Note that since $E(t) = E(0)\, \forall t > 0$, $||x(t)||_2 \neq 0 \forall t > 0$, thus the system around the equilibrium is not asymptotically stable (neither local nor global), and hence not exponentially stable.

8.1.1 Internal Stability of LTI Systems

Lyapunov's stability definitions may not be very useful for analyzing the stability of non-linear systems, however we can easily derive necessary and sufficient conditions for stability. One should also note the in LTI systems (and linear systems in general) we are interested in the stability of the origin $x_e = 0$. The following matrix differential and difference equations express autonomous CT and DT LTI systems

$$\begin{aligned}
\dot{x} &= Ax \\
x[k+1] &= Ax[k]
\end{aligned}$$

and we know the analytical solutions to the zero-input responses have the following forms

$$x(t) = e^{At}x_0$$
$$x[k] = A^k x_0$$

It is easier to analyze the internal stability using Jordan decomposition of the system matrix A,

$$x(t) = Ge^{Jt}G^{-1}x_0 \;\rightarrow\; \left[G^{-1}x(t)\right] = e^{Jt}\left[G^{-1}x_0\right] \;\rightarrow\; \alpha(t) = e^{Jt}\alpha_0$$
$$x[k] = GJ^kG^{-1}x_0 \;\rightarrow\; \alpha[k] = J^k\alpha_0$$

Note that since G and G^{-1} finite and invertible matrices, we know that

$$||x|| = 0 \iff ||G^{-1}x|| = 0 \iff x = 0$$
$$||x|| < M_1 < \infty \iff ||G^{-1}x|| < M_2 < \infty \text{ where} M_1, M_2 \in \mathbb{R}$$
$$||x|| \to \infty \iff ||G^{-1}x|| \to \infty$$

We know that e^{Jt} and J^k has the following block diagonal form

$$e^{Jt} = \begin{bmatrix} e^{J_1 t} & & & & \\ & e^{J_2 t} & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & e^{J_n t} \end{bmatrix}$$

$$J^k = \begin{bmatrix} J_1^k & & & & \\ & J_2^k & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & J_n^k \end{bmatrix}$$

where an individual block associated with a Jordan block of A has the following form

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda_i t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda_i t} & \frac{t^{n-1}}{(n-1)!}e^{\lambda_i t} \\ 0 & e^{\lambda_i t} & te^{\lambda_i t} & \frac{t^2}{2!}e^{\lambda_i t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda_i t} \\ \vdots & & \ddots & & & \vdots \\ & & & e^{\lambda_i t} & te^{\lambda_i t} & \frac{t^2}{2!}e^{\lambda_i t} \\ 0 & & \cdots & & e^{\lambda_i t} & te^{\lambda_i t} \\ 0 & & \cdots & & 0 & e^{\lambda_i t} \end{bmatrix}$$

$$J_i^k = \begin{bmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} & \cdots & \binom{k}{n-2}\lambda_i^{k-n+2} & \binom{k}{n-1}\lambda_i^{k-n+1} \\ 0 & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} & \cdots & \binom{k}{n-2}\lambda_i^{k-n+2} \\ \vdots & & \ddots & & & \vdots \\ & & & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} \\ 0 & & \cdots & & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} \\ 0 & & \cdots & & 0 & \lambda_i^k \end{bmatrix}$$

Based on this decomposition, we can derive the stability conditions.

1. - A CT LTI system is *asympotitically & exponentially stable*
     $\iff \lim_{t\to\infty} e^{J_i t} = 0 \ \forall i \iff \text{Re}\{\lambda_i\} < 0 \ \forall i$
   - A DT LTI system is *asympotitically & exponentially stable*
     $\iff \lim_{k\to\infty} J_i^k = 0 \ \forall i \iff |\lambda_i| < 1 \ \forall i$
2. - A CT LTI system is *s.i.s.L*
     $\iff \lim_{t\to\infty} e^{J_i t} = M_i < \infty \ \forall i \iff \{\text{Re}\{\lambda_i\} < 0 \text{ or } \{\text{Re}\{\lambda_i\} = 0 \text{ and } J_i \in \mathbb{R}\}\} \ \forall i$
   - A DT LTI system is *s.i.s.L*
     $\iff \lim_{k\to\infty} J_i^k = M_i < \infty \ \forall i \iff \{|\lambda_i| < 1 \text{ or } \{|\lambda_i| = 1 \text{ and } J_i \in \mathbb{R}\}\} \ \forall i$
3. - A CT LTI system is *unstable*
     $\iff \exists i \ \lim_{t\to\infty} e^{J_i t} = \infty \iff \{\text{Re}\{\lambda_i\} > 0 \text{ or } \{\text{Re}\{\lambda_i\} = 0 \text{ and } J_i \in \mathbb{R}^{n\times n}, n > 1\}\} \ \forall i$
   - A DT LTI system is *unstable*
     $\iff \exists i \ \lim_{k\to\infty} J_i^k = \infty \iff \exists i\{|\lambda_i| > 1 \text{ or } \{|\lambda_i| = 1 \text{ and } J_i \in \mathbb{R}^{n\times n}, n > 1\}\}$

8.1.2 Internal Stability of LTV Systems

Indeed, for LTI systems we derived the conditions of stability using the state-transition matrix, which leads us to some conditions based on the eigenvalues of the system matrix. For general LTV systems, if we follow Lyapunov's stability definitions, we can construct the stability conditions.

Theroem: The LTV system with the following state equation and state-transition matrix

$$\dot{x}(t) = A(t)x(t) \ , \ x(t) = \Phi(t, t_0)x_0$$

1. is *s.i.s.L.* $\iff \exists M(t_0) \ s.t. \ ||\Phi(t, t_0)|| \leq M(t_0) < \infty \ , \ \forall t \geq t_0$
2. is *asymptotically stable* $\iff$ if it is *s.i.s.L.* and $\lim_{t\to\infty} \Phi(t, t_0) = 0$
3. is *exponentially stable* $\iff$ if it is *asymptotically stable* and $\exists \sigma(t_0), C(t_0) > 0$ s.t. $||\Phi(t, t_0)|| \leq C(t_0)e^{-\sigma(t_0)(t-t_0} \ \forall t \geq t_0$

If these stability definitions hold $\forall t_0 \in \mathbb{R}$, then they become *uniformly stable* in their respective categories. Note that same conditions are valid for the stability of DT LTV systems if we replace time variables.

Ex 8.3 *Let*

$$x[k+1] = \begin{bmatrix} 0 & \rho^k \\ 0 & (1/2)^k \end{bmatrix}$$

Analyze the stability of the origin based on the value of ρ

Solution: In LTI systems, we know that eigenvalues of the system matrix provide necessary and sufficient conditions on Lyapunov, asymptotic, and exponential stability. We can see that for this LTV system, eigenvalues are independent of k, and they satisfy LTI stability conditions, i.e.

$$\lambda(A[k]) \in \{0 \ , \ 1/2\} \& |\lambda_{1,2}| < 1$$

Let's try to find a solution to the initial value problem for the given LTV system and see if LTI intuition is valid or not for this LTV system. Let $x_{0,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then

$$x[k]_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

now let $x_{0,2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$x[k]_2 = \begin{bmatrix} (\rho/2)^{k-1} \\ (1/2)^k \end{bmatrix} \quad k > 0$$

thus we can write $\Phi[k, 0]$ as

$$\Phi[k, 0] = \begin{bmatrix} x[k]_1 & x[k]_2 \end{bmatrix} = \begin{bmatrix} 0 & (\rho/2)^k \\ 0 & (1/2)^{k-1} \end{bmatrix} \quad k > 0$$

Based on this result, we can conclude that.

- LTV system around the origin is asymptotically & exponentially stable if $|\rho| < 2$
- LTV system around the origin is stable i.s.L if $|\rho| \leq 2$
- LTV system around the origin is unstable if $|\rho| > 2$

As you can see in this example (and in general) for LTV systems, eigenvalues of the system matrix $A[k]$ do not imply stability.

Ex 8.4 *Let*

$$\dot{x} = \begin{bmatrix} 0 & e^{-t} \\ 0 & 0 \end{bmatrix}$$

Analyze the stability of the origin

Solution: Since

$$A(t) = \begin{bmatrix} 0 & e^{-t} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} e^{-t} = \bar{A} f(t)$$

We can find the state transition matrix via

$$\Phi(t, t_0) = e^{\int\limits_{t_0}^{t} A(\tau) d\tau} = \exp\left(\int\limits_{t_0}^{t} \begin{bmatrix} 0 & e^{-\tau} \\ 0 & 0 \end{bmatrix} d\tau \right) = \exp\left(\begin{bmatrix} 0 & e^{-t_0} - e^{-t} \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & e^{-t_0} - e^{-t} \\ 0 & 1 \end{bmatrix}$$

We can see that $||\Phi(t, t_0)|| < M < \infty \ \forall \ (t, t_0)$ but $\lim_{t \to \infty} \neq 0$ thus system around the origin is (only) stable in the sense of Lyapunov. However, if we had only checked the eigenvalues, we may have (wrongly) concluded that the system was unstable.

8.1.3 Lyapunov's Direct Method

Consider an autonomous time-invariant CT or DT non-linear state-space mode;

$$\dot{x} = F(x(t))$$
$$x[k+1] = F(x[k])$$

with an equilibrium point x_e. Without loss of generality, let's assume that $x_e = 0$. Consider a continuous scalar function $V : \mathbb{R}^m \mapsto \mathbb{R}$ (or $V : D \mapsto F$ in general, where D is the domain and F is the field), where $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ inside some "ball" enclosing the equilibrium point. In that context, $V(x)$ can be considered as a function of "energy" (indeed, for a mechanical system, it can be a very good choice), or mathematically $V(x)$ can be a norm. Then $\dot{V}(x)$ for CT systems and $\triangle V(x) := V(x[k+1]) - V(x[k])$ DT systems can provide critical information regarding the internal stability of the dynamical system.

Lyapunov Functions, Lyapunov Theorem for Local & Global Stability

Definition: Let $V : \mathbb{R}^m \mapsto \mathbb{R}$.

- We call $V(x)$ a *locally positive definite (lpd)* function around $x = 0$ if
  1. $V(0) = 0$
  2. $V(x) > 0,\ ||x|| \in (0, r)$ for some $0 < r$
- We call $V(x)$ a *locally negative definite (lnd)* function around $x = 0$ if
  1. $V(0) = 0$
  2. $V(x) < 0,\ ||x|| \in (0, r)$ for some $0 < r$
- We call $V(x)$ a *locally positive semi definite (lpsd)* function around $x = 0$ if
  1. $V(0) = 0$
  2. $V(x) \geq 0,\ ||x|| \in (0, r)$ for some $0 < r$
- We call $V(x)$ a *locally negative semi definite (lnsd)* function around $x = 0$ if
  1. $V(0) = 0$
  2. $V(x) \leq 0,\ ||x|| \in (0, r)$ for some $0 < r$

Definition: Let $V(x)$ be a *lpd* function, then we call it a "candidate" Lyapunov function. If $\dot{V}(x)$ (or $\triangle V(x)$) for the dynamical system of interest is a *lnsd* function, then $V(x)$ is called a **Lyapunov function** of the system.

Theorem: If there exists a Lyapunov function of the system, $V(x)$, then

1. Then $x_e = 0$ is a (locally) stable equilibrium point *in the sense of Lyapunov*
2. and if $\dot{V}(x) < 0$ (or $\triangle V(x) < 0$) $||x|| \in (0, r_a)$ (i.e. *lsd*), then $x_e = 0$ is a (locally) asymptotically stable equilibrium point.

Theorem: If there exists a Lyapunov function of the system, $V(x)$, defined on the entire state-space (or domain in general), i.e.

$$\exists V : \mathbb{R}^n \to \mathbb{R} \ s.t. \ V(0) = 0 \ \ , \ V(x) > 0, \ \forall x \in \mathbb{R}^n \ (i.e.nd) \ (i.e. \ pd) \text{ and}$$
$$\dot{V}(x) \leq 0, \ \forall x \in \mathbb{R}^n \ (i.e. \ nd)$$

then

1. Then $x_e = 0$ is a stable equilibrium point *in the sense of Lyapunov* (in a more global sense)
2. if $x_e = 0$ is a globally stable equilibrium point *in the sense of Lyapunov* and if

$$\dot{V}(x) < 0, \ \forall x \in \mathbb{R}^n$$
$$||x|| \to \infty \implies V(x) \to \infty$$

   then $x_e = 0$ is a globally asymptotically stable equilibrium point.

Ex 8.5 *Let's go back to the pendulum dynamics*

$$\dot{x} = \begin{bmatrix} x_2 \\ -sin(x_1) \end{bmatrix} \ , \text{ where } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

Analyze the stability of the dynamics around $x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ using Lyapunov's direct method.

Solution: We first need to find a candidate Lyapunov function. Let's test if the total mechanical energy can be adopted as a Lyapunov candidate

$$V(x) = E(x) = \frac{1}{2}x_2^2 + 1 - \cos x_1$$
$$V(0) = 0$$
$$V(x) > 0 \ , \ \forall x \neq 0$$

Now let's test if $V(x)$ is a Lyapunov function.

$$\frac{d}{dt}V(x) = -x_2 \sin x_1 + \sin(x_1)x_2 = 0$$
$$\dot{V}(x) \geq 0 \ \forall x$$

Thus we can conclude that $E(x)$ is a Lyapunov function and $x_e = 0$ is a stable equilibrium point in the sense of Lyapunov (globally). We know that since $x_{e,k} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix} \forall k \in \mathbb{Z}$ is an equilibrium point, $x_e = 0$ is not globally asymptotically stable. Indeed, since $\dot{V}(x) = 0 \, \forall x$, we also know that $x_e = 0$ is not even locally asymptotically stable. However, one should know the fact that Lyapunov's direct method, in general, does not imply un-stability.

Ex 8.6 *Now let's add a linear damping term to the pendulum dynamics*

$$\dot{x} = \begin{bmatrix} x_2 \\ -\sin(x_1) - x_2 \end{bmatrix} \ , \text{ where } b > 0$$

Analyze the stability of the dynamics around $x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ using Lyapunov's direct method.

Solution: Total mechanical energy can again be a good candidate as a Lyapunov function

$$\begin{aligned} V(x) &= E(x) = \frac{1}{2}x_2^2 + 1 - \cos x_1 \\ V(0) &= 0 \ , \ V(x) > 0 \ , \ \forall x \neq 0 \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}V(x) &= -x_2^2 - x_2 \sin x_1 + \sin(x_1)x_2 = -x_2^2 \geq 0 \\ \dot{V}(x) &= 0, \ \forall x \in \{x | \begin{bmatrix} 0 & 1 \end{bmatrix} x = 0\} \\ \dot{V}(x) &> 0, \ \forall x \in \{x | \begin{bmatrix} 0 & 1 \end{bmatrix} x \neq 0\} \end{aligned}$$

In this case we can again conclude that $E(x)$ is a Lyapunov function and $x_e = 0$ is a stable equilibrium point in the sense of Lyapunov (globally). However, we can still not conclude that the system is asymptotically stable (even though intuitively, we know that it is at least locally asymptotically stable), since $\dot{V}(x)$ is a *nsd* function (locally and globally). Let's try another candidate $V(x)$

$$\begin{aligned} V(x) &= \frac{1}{2}x_2^2 + a(1 - \cos x_1) + \frac{b}{2}\left(x_1 + x_2\right)^2 , \ a > 0, \ b > 0 \\ V(0) &= 0 \ , \ V(x) > 0 \ , \ \forall x \neq 0 \ \rightarrow \ pd \ \checkmark \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}V(x) &= -x_2^2 - x_2\sin(x_1) + a\sin(x_1)x_2 + b\left(x_1 + x_2\right)\dot{x_1} + b\left(x_1 + x_2\right)\dot{x_2} \\ &= (b-1)x_2^2 + (a-1)x_2\sin(x_1) + bx_1x_2 + b\left(x_1 + x_2\right)\dot{x_2} \\ &= (b-1)x_2^2 + (a-1)\sin(x_1)x_2 + bx_1x_2 - b\sin(x_1)\left(x_1 + x_2\right) - bx_2\left(x_1 + x_2\right) \\ &= -x_2^2 + (a-b-1)\sin(x_1)x_2 - b\sin(x_1)x_1 \end{aligned}$$

Let $(a, b) = (2, 1)$, then

$$\begin{aligned} \frac{d}{dt}V(x) &= -\left(x_2^2 + \sin(x_1)x_1\right) \geq 0 \\ \frac{d}{dt}V(x) &= 0 \iff x_1 = k\pi \, k \in \mathbb{Z} \rightarrow \ l.n.d \end{aligned}$$

We can now see that when $x_1 \in (\pi, \pi)$, $\dot{V}(x)$ is a locally negative definite function, the system around $x_e = 0$ is also locally asymptotically stable.

8.1.4 Lyapunov's Indirect Method

Consider an autonomous time-invariant CT or DT non-linear state-space mode;

$$\begin{aligned} \dot{x} &= F(x(t) \\ x[k+1] &= F(x[k]) \end{aligned}$$

with an equilibrium point x_e. Without loss of generality, let's assume that $x_e = 0$. Let the linearization of dynamics around the equilibrium yields

$$\begin{aligned} \dot{\delta x} &= Ax(t) \\ \delta x[k+1] &= \delta x[k]) \end{aligned}$$

Theorem:

1. If the linearized system matrix, A, is asymptotically/exponentially stable, then the non-linear system is also *locally asymptotically stable* around the equilibrium point, x_e, inside a region locally defined around the equilibrium point
2. If the linearized system matrix, A, is asymptoticall/exponentially unstable, then the non-linear system is also *locally unstable* around the equilibrium point
3. If the linearized system matrix, A, is stable i.s.L., we can not comment on the stability of the non-linear system around the equilibrium

Ex 8.7 *Consider the following scalar DT dynamical system*

$$x[k+1] = \frac{x[k]}{1+(x[k])^2}$$

Analyze the stability of the dynamics around $x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ using both Lyapunov's direct and indirect methods.

Solution: Let's first start with the direct method and see if $V(x) = x^2$ is a Lyapunov function

$$\begin{aligned} V(x) &= x^2 \ , \ V(0) = 0 \ , \ V(x) > 0 \ , \ \forall x \neq 0 \ \rightarrow \ pd \ \checkmark \\ \triangle V(x) &= x^2 \left(\frac{1}{(1+x^2)^2} - 1 \right) \ , \ \triangle V(0) = 0 \ , \ \triangle V(x) < 0 \ , \ \forall x \neq 0 \ \rightarrow \ nd \ \checkmark \end{aligned}$$

Thus we can conclude that the system around the origin is globally asymptotically stable. Now let's adopt the indirect method

$$\begin{aligned} \delta x[k+1] &= \frac{\partial}{\partial} \left(\frac{x}{1+(x)^2} \right)_{x=0} \delta x[k] \\ \delta x[k+1] &= \delta x[k] \end{aligned}$$

We can see that linearized dynamics has a single eigenvalue at $z = 1$, which implies that the linear dynamical system is stable i.s.L. However, based on only this result, we can not comment on the stability of the non-linear system. However, we know from the direct method that the system around the origin is, indeed, asymptotically stable.

8.1.5 Quadratic Lyapunov Functions and Lyapunov Stability for LTI Systems

Consider the function

$$V(x) = x^T P x, \ x \in \mathbb{R}^n, P \in \mathbb{R}^{n \times n}, \& \ P^T = P$$

Proposition: $V(x)$ is a positive definite function if and only if all the eigenvalues of P are positive

Proof: Since P is a symmetric real matrix, we know that it can be diagonalized by an *orthagonal matirx*, i.e.

$$P = U^T \Lambda U, \ U^T U = I, \& \ \Lambda = \text{diag}\left[\lambda_1, \quad \cdots, \quad \lambda_n\right]$$

Let $y = Ux \;\rightarrow\; ||y||_2 = ||x||_2$, then

$$V(x) = x^T P x = y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2$$

Thus

$$V(x) > 0 \;\forall x \neq 0 \quad \Longleftrightarrow \quad \lambda_i > 0 \;\forall i$$

Definition: A symmetric matrix $P^T = P$

- is called *positive definite* if it satisfies

  $$x^T P x > 0 \;\forall x \neq 0$$

  and use the notation, $P > 0$ for such matrices

- is called *positive semi definite* if

  $$x^T P x \geq 0 \;\forall x \neq 0$$

  and use the notation, $P \geq 0$ for such matrices

Corollaries:

- For a symmetric positive definite or symmetric positive semi-definite matrix

  $$\lambda_{min}(P)||x||_2^2 \leq x^T P x \leq \lambda_{max}(P)||x||_2^2$$

- If $P \geq 0$ but $P \not> 0$, then $\lambda_{min} = 0$

8.1.6 Quadratic Lyapunov Functions for LTI Systems

Consider defining a Lyapunov function candidate of the form

$$V(x) = x^T P x \;\; P > 0$$

for an LTI CT or DT system. Then for a CT-LTI system, $\dot{x} = Ax$, we have

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P x = x^T A^T P x + x^T P A x \\ &= x^T (A^T P + PA) x = -x^T Q x \\ Q &= -(A^T P + PA) \end{aligned}$$

note that $Q^T = Q$. Based on Lyapunov's direct method, we conclude that

- $\dot{x} = Ax$ is stable i.s.L. if $Q = -(A^T P + PA) \geq 0$ or
- $\dot{x} = Ax$ is asymptotically stable if $Q = -(A^T P + PA) > 0$

Similarly for a DT-LTI system, $x[k+1] = Ax$, we have

$$\begin{aligned} \triangle V(x) &= x^T A^T P A x - x^T P x \\ &= x^T (A^T P A - P) x = -x^T Q x \\ Q &= -A^T (P - I) A \end{aligned}$$

note that again $Q^T = Q$. Based on Lyapunov's direct method, we conclude that

- $x[k+1] = Ax[k]$ is stable i.s.L. if $Q = -A^T(P-I)A \geq 0$ or
- $x[k+1] = Ax[k]$ is asymptotically stable if $Q = -A^T(P-I)A \geq 0$

Theorem: Given a CT-LTI dynamic system, $\dot{x} = Ax$, and any $Q > 0$, there exists a definite positive solution P of the Lyapunov equation, $Q = -(A^T P + PA)$, if and only if $\dot{x} = Ax$ is asymptotically stable. Solution P, in this case, is unique.

Proof:

1. For a given $Q > 0$, if $P > 0$ is a solution of $Q = -(A^T P + PA)$, then $V(x) = x^T P x$ is p.d. and $\dot{V}(x)$ is n.d., hence $\dot{x} = Ax$ globally asymptotically stable
2. Now let's try to find "a" solution of $P > 0$, given $Q > 0$. Let

$$P = \int_0^\infty e^{tA^T} Q e^{At} dt$$

P is well defined since, all eigenvalues of A is in O.L.P, and e^{At} exponentially converges to 0. Now

$$\begin{aligned} A^T P + PA &= \int_0^\infty A^T e^{tA^T} Q e^{At} dt + \int_0^\infty e^{tA^T} Q e^{At} A dt = \int_0^\infty A^T e^{tA^T} Q e^{At} + e^{tA^T} Q e^{At} A \; dt \\ &= \int_0^\infty \frac{d}{dt} \left[e^{tA^T} Q e^{At} \right] dt = \left[e^{tA^T} Q e^{At} \right]_0^\infty = 0 - Q \\ A^T P + PA &= -Q \end{aligned}$$

i.e., P satisfies the Lyapunov equation

3. Let's show P is symmetric

$$P^T = \int_0^\infty \left[e^{tA^T} Q e^{At} \right]^T dt = \int_0^\infty \left[e^{At} \right]^T Q \left[e^{tA^T} \right]^T dt = \int_0^\infty e^{tA^T} Q e^{At} dt = P$$

4. Let's show $P > 0$. Let $x \in \mathbb{R}^n$

$$x^T P x = \int_0^\infty x^T e^{tA^T} Q e^{At} x dt = \int_0^\infty z(t)^T Q z(t) \;\; z(t) = e^{At} x$$

Note that

$$z(t) = 0 \iff x = 0 \;, \; \forall t < \infty$$

Since $Q > 0$,

$$z(t)^T Qz(t) = 0 \iff z(t) = 0 \ , \ \forall t$$
$$z(t)^T Qz(t) > 0 \ \forall z(t) \neq 0$$

Thus

$$x^T Px = \int_0^\infty z(t)^T Qz(t) = 0 \iff z(t) \ \forall t \iff x = 0$$

$$x^T Px = \int_0^\infty z(t)^T Qz(t) > 0 \ , \ \forall x \neq 0$$

5. Let's show P is unique. Let's prove this by contradiction. Let's assume that $\exists \, \bar{P} \neq P$ and $\bar{P}^T = \bar{P}$ such that

$$A^T \bar{P} + \bar{P}A = -Q \ , \ A^T P + PA = -Q$$

Let $W(t) = e^{tA^T} \bar{P} e^{At}$ and let's analyze the following integral

$$\int_0^\infty -\dot{W}(t)dt = [W(t)]_\infty^0 = W(0) - W(\infty) = \bar{P}$$

$$\int_0^\infty -\dot{W}(t)dt = \int_0^\infty -\left(e^{tA^T} A^T \bar{P} e^{At} + e^{tA^T} \bar{P} A e^{At}\right) dt = \int_0^\infty e^{tA^T} (-\left(A^T \bar{P} + \bar{P}A\right)) e^{At} dt$$
$$= \int_0^\infty e^{tA^T} Q e^{At} dt = P = \bar{P}$$

which is the contradiction to the assumption that $\exists \, \bar{P} \neq P$. This completes the proof

Theorem: Given a DT-LTI dynamic system, $x[k+1] = Ax[k]$, and any $Q > 0$, there exists a definite positive solution P of the Lyapunov equation, $Q = P - A^T PA$, if and only if $x[k+1] = Ax[k]$ is asymptotically stable. Solution P, in this case, is unique.

Proof:

1. For a given $Q > 0$, if $P > 0$ is a solution of $Q = P - A^T PA$, then $V(x) = x^T Px$ is p.d. and $\dot{V}(x)$ is n.d., hence $x[k+1] = Ax[k]$ globally asymptotically stable

2. Now let's try to find "a" solution of $P > 0$, given $Q > 0$. Let

$$P = \sum_{k=0}^\infty A^{k^T} Q A^k$$

P is well defined since, all eigenvalues of A is in inside the unit circle, and A^k exponentially converges to 0. Now

$$P - Q = \sum_{k=1}^\infty A^{k^T} Q A^k = \sum_{n=0}^\infty A^{n+1^T} Q A^{n+1} = A^T \left(\sum_{n=0}^\infty A^{n^T} Q A^n \right) A$$
$$P - Q = A^T PA$$
$$-Q = A^T PA - P$$

i.e., P satisfies the Lyapunov equation

3. Remaining parts of the proof are left to the student as a take-home problem