EE502 - Linear Systems Theory II **Spring 2032**

Lecture 7

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7.1 Discrete-Time Linear Time Varying State Space Models

State-space representation of a (causal & finite dimensional) LTV DT system is given by

$$\begin{aligned}
\text{Let } x[k] \in \mathbb{R}^n \ , \ y[k] \in \mathbb{R}^m \ , \ u[k] \in \mathbb{R}^r, \\
x[k+1] = A[k]x[k] + B[k]u[k], \\
y[k] = C[l]x[k] + Du[k], \\
\text{where } G[k] \in \mathbb{R}^{n\times n} \ , \ B[k] \in \mathbb{R}^{n\times r} \ , \ C[k] \in \mathbb{R}^{m\times n} \ , \ D[d] \in \mathbb{R}^{m\times r}
\end{aligned}$$

Let's first assume that $u[k] = 0$, and find un-driven response.

$$\begin{aligned}
x[k+1] &= A[k]x[k] \\
y[k] &= C[k]x[k]
\end{aligned}$$

Unlike LTV-CT systems we easily can compute the response iteratively

$$\begin{aligned}
&x[0] = Ix[0] \quad , \quad y[0] = C[0]x[0] \\
&x[1] = A[0]x[0] \quad , \quad y[1] = C[1]x[1] \\
&x[2] = A[0]x[1] = A[1]A[0]x[0] \quad , \quad y[2] = C[2]x[2] \\
&x[3] = A[2]x[2] = A[3]A[1]A[0]x[0] \quad , \quad y[3] = C[3]x[3] \\
&\quad \vdots \\
&x[k] = A[k-1]x[k-1] = A[k-1]A[k-2]\cdots A[1]A[0]x[0] \quad , \quad y[k] = \quad , \quad y[k] = C[k]x[k] \\
&x[k] = \prod_{i=0}^{k-1} A[k-1-i]
\end{aligned}$$

Motivated by the LTI case, we define the **state transition matrix**, which relates the state of the undriven system at time k to the state at an earlier time m

$$x[k] = \Phi[k,m]x[m] \ , \ k \geq m \ , \ \text{where}$$
$$\Phi[k,m] = \begin{cases} \prod_{i=0}^{k-1} A[k-1-i] \ , & k > m \\ I & , \ k = m \end{cases}$$

Note that state-transition matrix satisfies following important properties undriven system at time k to the state at an earlier time m

$$\begin{aligned}
\Phi[k,k] &= I \\
x[k] &= \Phi[k,0]x[0] \\
\Phi[k+1,m] &= A[k]\Phi[k,m]
\end{aligned}$$

as you can see, the state-transition matrix satisfies the discrete dynamical state equations.

Now let's consider input-only state response (i.e. $x[0] = 0$).

$$x[k+1] = Gx[k] + Hu[k]$$

$$\begin{aligned}
x[1] &= Hu[0] \\
x[2] &= Gx[1] + Hu[1] = GHu[0] + Hu[1] \\
x[3] &= Gx[2] + Hu[2] = G^2Hu[0] + GHu[1] + Hu[2] \\
x[4] &= Gx[3] + Hu[3] = G^3Hu[0] + G^2Hu[1] + GHu[2] + Hu[3] \\
&\vdots \\
x[k] &= Gx[k-1] + Hu[k-1] \\
&= G^{k-1}Hu[0] + G^{k-2}Hu[1] + \cdots + GHu[k-2] + Hu[k-1] \\
&= \left[\begin{array}{c|c|c|c|c} G^{k-1}H & G^{k-2}H & \cdots & GH & H \end{array}\right] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix} \\
&= \sum_{j=0}^{k-1} G^{k-j-1}Hu[j] \\
&= \sum_{j=0}^{k-1} G^{j}Hu[k-j-1]
\end{aligned}$$

Given that $\Psi[k] = G^k$

$$\begin{aligned}
x[k] &= \sum_{j=0}^{k-1} \Psi[k-j-1]Hu[j] \\
&= \sum_{j=0}^{k-1} \Psi[j]Hu[k-j-1]
\end{aligned}$$

If we combine homegeneous and driven responses we can simply obtain

$$\begin{aligned}
x[k] &= \Psi[k]x[0] + \sum_{j=0}^{k-1} \Psi[k-j-1]Hu[j] \\
&= \Psi[k]x[0] + \sum_{j=0}^{k-1} \Psi[j]Hu[k-j-1]
\end{aligned}$$

whereas output at time k has the form

$$\begin{aligned}
y[k] &= C\Psi[k]x[0] + C\left(\sum_{j=0}^{k-1} \Psi[k-j-1]Hu[j]\right) + Du[k] \\
&= C\Psi[k]x[0] + C\left(\sum_{j=0}^{k-1} \Psi[j]Hu[k-j-1]\right) + Du[k]
\end{aligned}$$

Z-domain Solution of State-Space Equations

We already computed the transfer function under zero initial conditions.

$$Y(z) = \Big[C\,(zI - G)^{-1}\,H + D\Big]\,U(z)$$

$$\begin{aligned} z\,(zI - G)^{-1} &= I + z^{-1}G + z^{-2}G^2 + z^{-3}G^3 + \cdots \\ \mathcal{Z}^{-1}\Big[z\,(zI - G)^{-1}\Big] &= I\,\delta[k] + G\,\delta[k-1] + G^2\,\delta[k-2] + G^3\,\delta[k-3] + \cdots \end{aligned}$$

Example: Consider the following state-space representation

$$x[k+1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} x[k]$$

- Compute the closed form expression $\Psi[k]$ using the time expression

  Solution: The state-space representation is in Diagonal canonical form

$$\Psi[k] = G^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix}^k = \begin{bmatrix} 1^k & 0 & 0 \\ 0 & (1/2)^k & 0 \\ 0 & 0 & (-1)^k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^k & 0 \\ 0 & 0 & (-1)^k \end{bmatrix}$$

- Compute the closed form expression $\Psi[k]$ using the z-domain solution method

  Solution:

$$\begin{aligned}
\Psi[k] &= \mathcal{Z}^{-1}\left[z\left(zI-G\right)^{-1}\right] \\
&= \mathcal{Z}^{-1}\left[z\left(\begin{bmatrix} z-1 & 0 & 0 \\ 0 & (z-1/2) & 0 \\ 0 & 0 & z+1 \end{bmatrix}\right)^{-1}\right] \\
&= \mathcal{Z}^{-1}\left[\begin{bmatrix} \frac{z}{z-1} & 0 & 0 \\ 0 & \frac{z}{z-1/2} & 0 \\ 0 & 0 & \frac{z}{z+1} \end{bmatrix}\right] \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^k & 0 \\ 0 & 0 & (-1)^k \end{bmatrix} \quad \text{for} k \geq 0
\end{aligned}$$

- Compute the impulse response of the system from the time domain solution

  Solution:

$$\begin{aligned}
x[k] &= G^{k-1}Hu[0] \quad \text{for } k>0 \\
y[k] &= CG^{k-1}H \quad \text{for } k>0 \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^{k-1} & 0 \\ 0 & 0 & (-1)^{k-1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ (1/2)^{k-1} \\ (-1)^{k-1} \end{bmatrix} \\
y[k] &= 1 + 2(1/2)^{k-1} + 3(-1)^{k-1} \quad \text{for } k>0
\end{aligned}$$

- Compute the transfer function $\frac{Y(z)}{U(z)}$

Solution:

$$
\begin{aligned}
T(z) &= C\,(zI - G)^{-1}\,H \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} z-1 & 0 & 0 \\ 0 & z-1/2 & 0 \\ 0 & 0 & z+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{z-1} & 0 & 0 \\ 0 & \frac{1}{z-1/2} & 0 \\ 0 & 0 & \frac{1}{z+1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{z-1} \\ \frac{1}{z-1/2} \\ \frac{1}{z+1} \end{bmatrix} \\
T(z) &= \frac{1}{z-1} + \frac{2}{z-1/2} + \frac{3}{z+1}
\end{aligned}
$$

- Compute the inverse Z-transform of the transfer function

  Solution:

$$
\begin{aligned}
t[k] &= \mathcal{Z}^{-1}\left[\frac{1}{z-1} + \frac{2}{z-1/2} + \frac{3}{z+1}\right] \\
&= \left(1 + 2(1/2)^{k-1} + 3(-1)^{k-1}\right) h[k-1]
\end{aligned}
$$

where $h[k]$ is the unit step function