EE502 - Linear Systems Theory II **Spring 2032**

Lecture 7

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7.1 Discrete-Time Linear Time Varying State Space Models

State-space representation of a (causal & finite dimensional) LTV DT system is given by

$$\begin{aligned} &\text{Let } x[k] \in \mathbb{R}^n \ , \ y[k] \in \mathbb{R}^m \ , \ u[k] \in \mathbb{R}^r, \\ &x[k+1] = A[k]x[k] + B[k]u[k], \\ &y[k] = C[l]x[k] + Du[k], \\ &\text{where } A[k] \in \mathbb{R}^{n\times n} \ , \ B[k] \in \mathbb{R}^{n\times r} \ , \ C[k] \in \mathbb{R}^{m\times n} \ , \ D[d] \in \mathbb{R}^{m\times r} \end{aligned}$$

Let's first assume that $u[k] = 0$, and find un-driven response.

$$\begin{aligned} x[k+1] &= A[k]x[k] \\ y[k] &= C[k]x[k] \end{aligned}$$

Unlike LTV-CT systems we easily can compute the response iteratively

$$\begin{aligned} &x[0] = Ix[0] \quad , \quad y[0] = C[0]x[0] \\ &x[1] = A[0]x[0] \quad , \quad y[1] = C[1]x[1] \\ &x[2] = A[0]x[1] = A[1]A[0]x[0] \quad , \quad y[2] = C[2]x[2] \\ &x[3] = A[2]x[2] = A[3]A[1]A[0]x[0] \quad , \quad y[3] = C[3]x[3] \\ &\quad \vdots \\ &x[k] = A[k-1]x[k-1] = A[k-1]A[k-2]\cdots A[1]A[0]x[0] \quad , \quad y[k] = \quad , \quad y[k] = C[k]x[k] \\ &x[k] = \prod_{i=0}^{k-1} A[k-1-i] \end{aligned}$$

Motivated by the LTI case, we define the **state transition matrix**, which relates the state of the undriven system at time k to the state at an earlier time m

$$x[k] = \Phi[k,m]x[m] \ , \ k \geq m \ , \ \text{where}$$
$$\Phi[k,m] = \begin{cases} \prod_{i=0}^{k-1} A[k-1-i] \ , & k > m \\ I & , \ k = m \end{cases}$$

Note that state-transition matrix satisfies following important properties undriven system at time k to the state at an earlier time m

$$\begin{aligned} \Phi[k,k] &= I \\ x[k] &= \Phi[k,0]x[0] \\ \Phi[k+1,m] &= A[k]\Phi[k,m] \end{aligned}$$

as you can see, the state-transition matrix satisfies the discrete dynamical state equations. Now let's consider input-only state response (i.e. $x[0] = 0$).

$$x[k+1] = A[k]x[k] + B[k]u[k]$$

$$
\begin{aligned}
x[1] &= B[0]u[0] = \Phi[1,1]B[0]u[0] \\
x[2] &= A[1]x[1] + B[1]u[1] = A[1]B[0]u[0] + B[1]u[1] = \Phi[2,1]B[0]u[0] + \Phi[2,2]B[1]u[1] \\
x[3] &= A[2]x[2] + B[2]u[2] = A[2]A[1]B[0]u[0] + A[2]B[1]u[1] + B[2]u[2] \\
&= \Phi[3,1]B[0]u[0] + \Phi[3,2]B[1]u[1] + \Phi[3,3]B[2]u[2] \\
x[4] &= A[3]x[3] + B[3]u[3] = A[3]A[2]A[1]B[0]u[0] + A[3]A[2]B[1]u[1] + A[3]B[2]u[2] + B[3]u[3] \\
&= \Phi[4,1]B[0]u[0] + \Phi[4,2]B[1]u[1] + \Phi[4,3]B[2]u[2] + \Phi[4,4]B[3]u[3] \\
&\vdots \\
x[k] &= \Phi[k,1]B[0]u[0] + \Phi[k,2]B[1]u[1] + \cdots + \Phi[k,k-1]B[k-2]u[k-2] + \Phi[k,k]B[k-1]u[k-1] \\
&= \left[\begin{array}{c|c|c|c|c} \Phi[k,1]B[0] & \Phi[k,2]B[1] & \cdots & \Phi[k,k-1]B[k-2] & \Phi[k,k]B[k-1] \end{array}\right] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix} \\
&= \sum_{j=0}^{k-1} \Phi[k,j+1]B[j]u[j] \\
&= \Gamma[k,0]\mathcal{U}[k,0]
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma[k,0] &= \left[\begin{array}{c|c|c|c|c} \Phi[k,1]B[0] & \Phi[k,2]B[1] & \cdots & \Phi[k,k-1]B[k-2] & \Phi[k,k]B[k-1] \end{array}\right] \\
\mathcal{U}[k,0] &= \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix}
\end{aligned}
$$

Ex 7.1 *Let's consider following SISO LTP system*

$$
\begin{aligned}
x[k+1] = A[k]x[k] + B[k]u[k], \ , y[k] = C[k]x[k] \ , \text{ where} \\
A[k] = A[k+4] \ , \ B[k] = B[k+4] \ , \ C[k] = C[k+4]
\end{aligned}
$$

convert this SISO LTP system into a MISO LTI system

Solution: Let's derive $x[4]$ in terms of $x[0], u[0], u[1], u[2], u[3]$

$$x[4] = A[3]A[2]A[1]A[0]x[0] + A[3]A[2]A[1]B[0]u[0] + A[3]A[2]B[1]u[1] + A[3]B[2]u[2] + B[3]u[3]$$

$$= \begin{bmatrix} A[3]A[2]A[1]A[0] \end{bmatrix} x[0] + \begin{bmatrix} A[3]A[2]A[1]B[0] & A[3]A[2]B[1] & A[3]B[2] & B[3] \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix}$$

Now let's find $x[8]$ in terms of $x[4], u[4], u[5], u[6], u[7]$

$$x[8] = \begin{bmatrix} A[3]A[2]A[1]A[0] \end{bmatrix} x[4] + \begin{bmatrix} A[3]A[2]A[1]B[0] & A[3]A[2]B[1] & A[3]B[2] & B[3] \end{bmatrix} \begin{bmatrix} u[4] \\ u[5] \\ u[6] \\ u[7] \end{bmatrix}$$

Let $\mathcal{X}[m] = x[4k], \mathcal{U}[m] = \begin{bmatrix} u[4k] & u[4k+1] & u[4k+2] & u[4k+3] \end{bmatrix}^T$, and $\mathcal{Y}[m] = y[4k]$, then we have

$$\begin{aligned} \mathcal{X}[m+1] &= \begin{bmatrix} A[3]A[2]A[1]A[0] \end{bmatrix} \mathcal{X}[m] + \begin{bmatrix} A[3]A[2]A[1]B[0] & A[3]A[2]B[1] & A[3]B[2] & B[3] \end{bmatrix} \mathcal{U}[m] \\ &= \mathcal{A}\mathcal{X}[m] + \mathcal{B}\mathcal{U}[m] \\ \mathcal{Y}[m] &= [C[0]]\, \mathcal{X}[m] \\ &= \mathcal{C}\mathcal{X}[m] \end{aligned}$$

which is a multi input single output LTI state-space representation. Note that in this representation we technically perform periodic sampling and analyze the mapping between two sampling instants. However, in this representation we technically loose some information, e.g. measurements/outputs between to sampling instants at i.e. $y[4k+1]$, $y[4k+2]$ $y[4k+3]$ $k > 0$.

Ex 7.2 *Obtain a new state-space representation (still LTI) such that it also includes these measurements, i.e.* $y[4k+1]$, $y[4k+2]$ $y[4k+3]$ $k > 0$.