

## Lecture 13

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## 13.1 State Feedback

The state-feedback based control-policies for LTI systems start with the assumption that we have “access” to all of the states of the systems either via direct measurement or through some observer/estimator/tracker. In that context a family of state-feedback controllers for CT- and DT-LTI systems can be constructed as

$$u(t) = \gamma r(t) - Kx(t) \text{ \& } u[k] = \gamma r[k] - Kx[k]$$

where  $r(t)$  can be considered as the reference signal (most of the time it is),  $\gamma$  is a feed-forward scaling factor, and  $K$  is the state-feedback gain. Now let's find a state-space representation for dynamics of the closed-loop system for both CT- and DT-LTI systems under state-feedback rule proposed above

$$\begin{aligned} \dot{x} &= Ax + B(\gamma r(t) - Kx(t)) \Rightarrow \dot{x} = (A - BK)x + \gamma Br \\ x[k+1] &= Ax + B(\gamma r[k] - Kx[k]) \Rightarrow x[k+1] = (A - BK)x[k] + \gamma Br[k] \end{aligned}$$

In both cases the closed loop system and input matrices take the following form.

$$A_c = A - BK, \quad B_c = \gamma Br$$

A key question in this domain is whether we can find a  $K$  such that eigenvalues of  $A_c$  is located at arbitrary desired locations.

**Theorem: (Eigenvalue/Pole Placement)** Given  $(A, B)$ ,  $\exists K$  s.t.

$$\begin{aligned} \det[\lambda I - (A - BK)] &= \lambda^n + a_{n-1}^* \lambda^{n-1} + \dots + a_1^* \lambda + a_0^* \\ \forall \mathcal{A} &= \{a_0^*, a_1^* \dots a_{n-1}^*\}, a_i^* \in \mathbb{R} \end{aligned}$$

if and only if  $(A, B)$  is reachable.

**Proof:** For a general complete proof we need to show that reachability of  $(A, B)$  is necessary and sufficient.

**Proof of necessity:** Let's assume that  $(A, B)$  not reachable and  $\exists(\lambda_u, w_u^T)$  pair such that  $w_u^T A = w_u^T \lambda_u$  and  $w_u^T = 0$ . Now check whether  $w_u^T$  is a left eigenvector of  $A_c$

$$\begin{aligned} w_u^T A_c &= w_u^T (A - BK) = w_u^T A - w_u^T BK = w_u^T \lambda_u - 0 = w_u^T \lambda_u \\ w_u^T B_c &= w_u^T B \gamma = 0 \end{aligned}$$

Here not only we showed that  $\lambda_u$  can not be moved hence it is not possible to locate the poles arbitrary locations, we also showed that state-feedback rules does not affect the reachability.

**Proof of sufficiency:** We will only show the sufficiency for a multi-input case, i.e.  $B \in \mathbb{R}^{n \times 1}$ , however, the reader should note that a complete proof multi-input case must be analyzed.

Let's assume that  $(A, B)$  is reachable, and we know that reachability is invariant under similarity transformations, i.e.

$$z = T^{-1}x, \det(T) \neq 0 \Rightarrow \dot{z} = \bar{A}z + \bar{B}u$$

$$\bar{A} = T^{-1}AT, \quad T^{-1}B$$

and  $(\bar{A}, \bar{B})$  is reachable. Now let's choose  $T$  such that

$$T = \mathbf{R} = [A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B]$$

then  $\bar{B}$  can be derived as

$$B = T\bar{B} = [A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B] \bar{B} \Rightarrow B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and similarly  $\bar{A}$  can be expressed as

$$\mathbf{A}\mathbf{R} = \mathbf{R}\bar{\mathbf{A}}$$

$$A [A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B] = [A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B] \bar{A}$$

$$[A^n B \mid A^{n-1}B \mid \cdots \mid AB \mid B] = [A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B] \begin{bmatrix} \bar{a}_{11} & 1 & 0 & 0 & \cdots & 0 \\ \bar{a}_{12} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \bar{a}_{1(n-2)} & 0 & 0 & \cdots & 1 & 0 \\ \bar{a}_{1(n-1)} & 0 & 0 & \cdots & 0 & 1 \\ \bar{a}_{1(n)} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

We can find  $a_{1i}$ 's using Cayley-Hamilton theorem

$$A^n B = \bar{a}_{11}A^{n-1}B + \bar{a}_{12}A^{n-2}B + \cdots + \bar{a}_{1(n-1)}AB + \bar{a}_{1(n)}B$$

$$A^n = -(a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1A + a_0I) \text{ where}$$

$$\det[\lambda I - A] = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

then  $\bar{A}$  takes the form

$$\bar{A} = \begin{bmatrix} -a_{n-1} & 1 & 0 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ -a_2 & 0 & 0 & \cdots & 1 & 0 \\ -a_1 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Note that this similarity transformation is in a companion form however we will transform this into (more useful) reachable canonical form

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Let's  $\tilde{\mathbf{R}}$  be the reachability matrix of  $(\tilde{A}, \tilde{B})$  then we know that

$$\tilde{A} = \tilde{\mathbf{R}}^{-1} A \tilde{\mathbf{R}}, \quad \tilde{B} = \tilde{\mathbf{R}}^{-1} B$$

where  $(\tilde{A}, \tilde{B})$  are the matrices of the companion form derived above. Thus if we let  $T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}} = \mathbf{R}\tilde{\mathbf{R}}^{-1}$ , where  $\mathbf{R}$  is the reachability matrix of the original representation and  $\tilde{\mathbf{R}}$  is the reachability matrix of the reachable canonical form and adopt  $T$  for similarity transformation we obtain

$$q = (T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}})^{-1} x \Rightarrow \dot{q} = \tilde{A}q + \tilde{B}u$$

$$\tilde{A} = (T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}})^{-1} A T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}, \quad \tilde{B} = (T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}})^{-1} B$$

Now let's apply the state feedback rule in the transformed coordinates.

$$\dot{q} = \tilde{A}q + \tilde{B}u, \quad u = \alpha r - \tilde{K}q$$

$$\dot{q} = (\tilde{A} - \tilde{B}\tilde{K})q + \alpha\tilde{B}u$$

$$(\tilde{A} - \tilde{B}\tilde{K}) = \left[ \begin{array}{c|c|c|c|c|c|c} 0 & 1 & 0 & 0 & \cdots & 0 & \\ 0 & 0 & 1 & 0 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & \\ 0 & 0 & 0 & \cdots & 0 & 1 & \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} & \end{array} \right] - \left[ \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{array} \right] [\tilde{k}_0 \quad \tilde{k}_1 \quad \cdots \quad \tilde{k}_{n-1}]$$

$$= \left[ \begin{array}{c|c|c|c|c|c|c} 0 & 1 & 0 & 0 & \cdots & 0 & \\ 0 & 0 & 1 & 0 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & \\ 0 & 0 & 0 & \cdots & 0 & 1 & \\ -a_0 - \tilde{k}_0 & -a_1 - \tilde{k}_1 & -a_2 - \tilde{k}_2 & \cdots & -a_{n-2} - \tilde{k}_{n-2} & -a_{n-1} - \tilde{k}_{n-1} & \end{array} \right]$$

$$\det(\tilde{A} - \tilde{B}\tilde{K}) = \lambda^n + (a_{n-1} + \tilde{k}_{n-1})\lambda^{n-1} + \cdots + (a_1 + \tilde{k}_1)\lambda + (a_0 + \tilde{k}_0)$$

It is evident that  $\exists \tilde{K}$  s.t.

$$\det(\tilde{A} - \tilde{B}\tilde{K}) = \lambda^n + a_{n-1}^* \lambda^{n-1} + \cdots + a_1^* \lambda + a_0^*$$

$$\forall \mathcal{A} = \{a_0^*, a_1^* \cdots a_{n-1}^*\}, a_i^* \in \mathbb{R}$$

Now we need to find the state-feedback gain and rule in the original coordinate frame

$$u = \alpha r - \tilde{K}q = \alpha r - \tilde{K} (T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}})^{-1} x$$

$$\dot{x} = (A - B\tilde{K} (T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}})^{-1}) x + \alpha B r, \quad K = \tilde{K} (T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}})^{-1}$$

Now let's verify that  $(A - BK)$  and  $(\tilde{A} - \tilde{B}\tilde{K})$  has the same characteristic equations

$$\det(\lambda I - [A - BK]) = \det(\lambda I - [A - B\tilde{K} (T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}})^{-1}])$$

$$= \det(\lambda I - (T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}})^{-1} [A - B\tilde{K} (T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}})^{-1}] T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}})$$

$$= \det(\lambda I - [(T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}})^{-1} A T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}} - (T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}})^{-1} B \tilde{K}])$$

$$= \det(\lambda I - [\tilde{A} - \tilde{B}\tilde{K}])$$

this completes the proof for SISO case.

## 13.2 Stabilizability

**Definition:** An LTI system is called stabilizable, if  $\forall x_0 \in \mathbb{R}^n, \exists u(t)$  (or  $u[k]$ ) such that  $\lim_{t \rightarrow \infty} x(t) = 0$  (or  $\lim_{k \rightarrow \infty} x[k] = 0$ ).

An obvious **sufficient** condition on stabilizability for CT-LTI and DT-LTI systems is that, if  $(A, B)$  is reachable then  $(A, B)$  stabilizable, since in such a case  $\forall x_0 \in \mathbb{R}^n, \exists u(t)$  or  $u[k]$  such that  $x(T) = 0 \forall T < 0$  for CT systems, and  $x[k] = 0 \forall k > n$ .

Another **sufficient** (but weaker) condition on stabilizability for DT-LTI systems is that if  $(A, B)$  is controllable then  $(A, B)$  stabilizable, since in such a case  $\forall x_0 \in \mathbb{R}^n, \exists u[k]$  such that  $x[k] = 0 \forall k > n$ .

**Theorem** The following statements are equal

- $(A, B)$  is stabilizable  $\iff$
- if unreachable modes of  $(A, B)$  are asymptotically stable  $\iff$
- $\forall (\lambda_i, w_i)$  s.t.  $w_i^T A = \lambda_i w_i^T \iff$  and  $\text{Re}\{\lambda_i\} \geq 0$  (or  $|\lambda_i| \geq 0$ ),  $w_i \notin \mathcal{N}(B) \iff$
- $\text{rank}[A - \lambda I \mid B] = n, \forall \lambda \in \mathbb{C}$  s.t.  $\text{Re}\{\lambda_i\} \geq 0$  for CT systems (or  $\forall \lambda \in \mathbb{C}$  s.t.  $|\lambda| \geq 1$  for DT systems)

**Proof:** Given  $(A, B)$  that is not necessarily reachable, we can transform it into the standard unreachable form

$$\dot{x} = \begin{bmatrix} \dot{x}_r \\ \dot{x}_{\bar{r}} \end{bmatrix} = \left[ \begin{array}{c|c} A_r & A_{12} \\ \hline 0 & A_{\bar{r}} \end{array} \right] \begin{bmatrix} x_r \\ x_{\bar{r}} \end{bmatrix} + \begin{bmatrix} B_r \\ 0 \end{bmatrix} u$$

Let  $\dot{x}_{\bar{r}} = A_{\bar{r}} x_{\bar{r}}$  is not asymptotically stable then obviously the whole system is unstable; thus, the system is not stabilizable. Now let  $\dot{x}_{\bar{r}} = A_{\bar{r}} x_{\bar{r}}$  is asymptotically stable, then let  $u = -K_r x_r$  then the closed-loop system takes the form

$$\dot{x} = \begin{bmatrix} \dot{x}_r \\ \dot{x}_{\bar{r}} \end{bmatrix} = \left[ \begin{array}{c|c} A_r - B_r K_r & A_{12} \\ \hline 0 & A_{\bar{r}} \end{array} \right] \begin{bmatrix} x_r \\ x_{\bar{r}} \end{bmatrix}$$

Since  $(A_r, B_r)$  pair is reachable we can find  $K$  such that eigenvalues of  $(A_r - B_r K_r)$  are located in the open-left-half of the complex plane, thus the system is stabilizable. The same logic can be adopted for DT systems.