

## Lecture 5

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## 5.1 Functions of a Matrix

In linear systems theory course, we are interested in matrix polynomials, specifically

- Matrix Exponential in CT Systems:  $e^{At}$
- Matrix Power in DT Systems:  $A^k$

which arise on the solution of state-space equations in their respective domains. Obviously  $A^k$  in DT systems is “easier” to analyze and understand compared to matrix exponential. Let’s first review the matrix exponential,  $e^{At}$ . Let  $t \in \mathbb{R}$  and  $A \in \mathbb{R}^{n \times n}$ , then  $e^{At}$  defined as

$$e^{At} := I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots$$

$$e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

which converges for all  $t \in \mathbb{R}$  and  $A \in \mathbb{R}^{n \times n}$ .

Now let’s review some properties

- **Claim:**  $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$

**Proof:**

$$\begin{aligned} \frac{d}{dt}e^{At} &= \frac{d}{dt} \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right) = \sum_{k=0}^{\infty} k \frac{t^{k-1}}{k!} A^k = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^k \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} A^{n+1} = A \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \left( \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \right) A \\ \frac{d}{dt}e^{At} &= Ae^{At} = e^{At}A \end{aligned}$$

- **Claim:** Let  $A, B \in \mathbb{R}^{n \times n}$  and  $AB = BA$ , then

$$e^A e^B = e^B e^A = e^{(A+B)}$$

**Proof:**

$$e^A e^B = \left( \sum_{k=0}^{\infty} \frac{1}{k!} A^k \right) \left( \sum_{j=0}^{\infty} \frac{1}{j!} B^j \right) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{A^k}{k!} \frac{B^j}{j!}$$

Let  $n = k + j$  and  $j = n - k$ , then

$$e^A e^B = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{A^k B^{n-k}}{k!(n-k)!} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{A^k B^{n-k}}{n!} \frac{n!}{k!(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n A^k B^{n-k} \binom{n}{k}$$

Note that if  $AB \neq BA$  we have to stop at this point. However, since  $AB = BA$ , we can adopt binomial theorem

$$e^A e^B = \sum_{n=0}^{\infty} \frac{1}{n!} (A + B)^n = e^{A+B} = e^B e^A$$

- **Claim:** Let  $t_1, t_2 \in \mathbb{R}$  then

$$e^{At_1} e^{At_2} = e^{At_2} e^{At_1} = e^{A(t_1+t_2)}$$

**Proof:** Let  $A := At_1$  and  $B := At_2$ , obviously  $(At_1)(At_2) = (At_2)(At_1)$  hence we can use the previous property, i.e.

$$e^{At_1} e^{At_2} = e^{(At_1)+(At_2)} = e^{A(t_1+t_2)} = e^{At_2} e^{At_1}$$

Now let  $t_1 = t$  and  $t_2 = -t$ , then we have

$$e^{At} e^{-At} = e^{A(t-t)} = I \quad \rightarrow \quad (e^{At})^{-1} = e^{-At}$$

- **Claim:** Let  $P \in \mathbb{R}^{n \times n}$  and  $\det(P) \neq 0$ , then

$$e^{(P^{-1}AP)t} = P^{-1} e^{At} P$$

**Proof:** Let's first show that  $(P^{-1}AP)^k = P^{-1}A^kP$

$$\begin{aligned} (P^{-1}AP)^k &= (P^{-1}AP)(P^{-1}AP) \cdots (P^{-1}AP)(P^{-1}AP) \\ &= P^{-1}APP^{-1}APP^{-1} \cdots PP^{-1}APP^{-1}AP \\ &= P^{-1}AIAI \cdots IAIAP \\ &= P^{-1}A^kP \end{aligned}$$

Now let's expand

$$\begin{aligned} e^{(P^{-1}AP)t} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} (P^{-1}AP)^k \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} P^{-1}A^kP \\ &= P^{-1} \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right) P \\ &= P^{-1} e^{At} P \end{aligned}$$

### 5.1.1 Computation of $e^{At}$ and $A^k$

### 5.1.2 Computation via Solution of State-Space Equations and Frequency Domain Expressions

An LTI CT state-space representation of an autonomous system has the form

$$\dot{x}(t) = Ax(t), \text{ where } x(t) \in \mathbb{R}^n$$

Let's test if  $x(t) = e^{At}x_0$  is a solution of the homogeneous equation

$$\begin{aligned} x(0) &= e^{A0}x_0 = x_0 \\ \dot{x}(t) - Ax(t) &= (Ae^{At})x_0 - Ae^{At}x_0 = 0 \end{aligned}$$

Now let's remember Laplace domain solution of the same equation

$$\begin{aligned} \mathcal{L}[\dot{x}(t)] &= \mathcal{L}[Ax(t)] \\ sX(s) - x(0) &= AX(s) \\ [sI - A]X(s) &= x(0) \\ X(s) &= [sI - A]^{-1}x_0 \end{aligned}$$

If we connect time and s-domain solutions we obtain

$$e^{At} = \mathcal{L}^{-1} [[sI - A]^{-1}]$$

Now let's focus on  $A^k$ . An LTI DT state-space representation of an autonomous system has the form

$$x[k+1] = Ax[k]$$

Unlike CT systems we can compute the response iteratively easily

$$\begin{aligned} x[1] &= Gx[0] \\ x[2] &= Gx[1] = G^2x[0] \\ x[3] &= Gx[2] = G^3x[0] \\ &\vdots \\ x[k] &= Gx[k-1] = G^kx[0] \end{aligned}$$

Now let's remember form of the response in Z-domain.

$$\begin{aligned} \mathcal{Z}[x[k+1]] &= \mathcal{Z}[Gx[k]] \\ zX(z) - zx[0] &= GX(z) \\ (zI - G)X(z) &= zX(z) \\ X(z) &= z(zI - G)^{-1}x[0] \end{aligned}$$

If we connect time and s-domain solutions we obtain

$$G^k = \mathcal{Z}^{-1}\{z(zI - G)^{-1}\}$$

### 5.1.3 Computation via Diagonalization

**Theorem:**  $A \in \mathbb{C}^{n \times n}$  is diagonalizable, if and only if there exists a (nonsingular) similarity transformation,  $V \in \mathbb{C}^{n \times n}$ , such that  $A = V^{-1}\Lambda V$  where  $\Lambda$  is a diagonal matrix,

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ & 0 & & \ddots \\ & & & & \lambda_n \end{bmatrix}$$

where  $\lambda_i \in \mathbb{C}$ 's are the eigenvalues of  $A$ , which are the roots of the characteristic equation  $d(\lambda) = \det(\lambda I - A)$ .

Now let's compute  $A^k$  and  $e^{At}$  for a diagonalizable matrix

$$\begin{aligned} A^k &= (V^{-1}\Lambda V)^k = V^{-1}\Lambda^k V = V^{-1} \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} V \\ e^{At} &= e^{V^{-1}\Lambda V t} = V^{-1}e^{\Lambda t}V = V^{-1} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} V \end{aligned}$$

A sufficient but not necessary condition that  $A$  will have  $n$  distinct eigenvalues in such a case characteristic equation will have the following form

$$d(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n), \text{ where } \lambda_i \neq \lambda_j, \text{ if } i \neq j$$

In this case, we also have the following properties associated with  $A$

- $A$  has  $n$  linearly independent eigenvectors
- For each  $\lambda_i$  there exists an eigenvector,  $v_i$ , such that  $Av_i = \lambda v_i$  and  $\text{Span}\{v_1, \dots, v_n\} = \mathbb{C}^n$
- $\forall \lambda_i$ , geometric multiplicity is equal to algebraic multiplicity and they are both equal to 1, i.e.  $\text{GM}(\lambda_i) = \text{AM}(\lambda_i) = 1$
- Minimal polynomial is equal to the characteristic equation,  $m(\lambda) = d(\lambda)$
- For each  $\lambda_i$ ,  $\mathcal{N}(\lambda_i I - A) = \text{Span}\{v_i\}$  and  $\dim[\mathcal{N}(\lambda_i I - A)] = 1$

Let's remember some concepts from EE501, to better understand and generalize *diagonalizable* and *non-diagonalizable* square matrices.

**Definition:** Given a matrix  $A \in \mathbb{C}^{n \times n}$ , the characteristic polynomial  $d(\lambda)$  is defined as

$$\begin{aligned} d(\lambda) &= (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_k)^{r_k} \\ k &: \# \text{ distinct eigenvalues where } k \leq n \\ r_i &= \text{AM}(\lambda_i) : \# \text{ algebraic multiplicity } \lambda_i, \text{ where } n = \sum_{i=1}^k r_i \end{aligned}$$

**Theorem:** Every  $n \times n$  matrix satisfies its characteristic equation (Cayley-Hamilton Theorem)

$$d(\lambda) = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_k)^{r_k} = \lambda^n + d_{n-1}\lambda^{n-1} + \cdots + d_0$$

$$d(A) = A^n + d_{n-1}A^{n-1} + \cdots + d_1A + d_0I = 0_{n \times n}$$

**Remark:** Any power of a matrix  $A \in \mathbb{C}^{n \times n}$  can be written as a linear combination of  $\mathcal{A}_n = \{I, A, A^2, \dots, A^{n-1}\}$ .

Note that Cayley-Hamilton theorem does not guarantee that  $\{I, A, A^2, \dots, A^{n-1}\}$  are linearly independent.

**Definition:** For an  $A \in \mathbb{R}^{n \times n}$ , the minimal polynomial  $m(\lambda)$  is the monic polynomial with the smallest degree such that  $m(A) = 0_{n \times n}$

**Ex 5.1**

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow d_1(\lambda) = (\lambda + 1)^2, \text{ Let } m_1(\lambda) = (\lambda + 1) \rightarrow m_1(A) = A + I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{n \times n} \checkmark$$

$$A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \rightarrow d_2(\lambda) = (\lambda + 1)^2, \text{ Let } m_2(\lambda) = (\lambda + 1) \rightarrow m_2(A) = A + I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0_{n \times n} \times$$

$$\text{Let } m_2(\lambda) = (\lambda + 1)^2 \rightarrow m_2(A) = (A + I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{n \times n} \checkmark$$

**Theorem:** Given  $A \in \mathbb{R}^{n \times n}$ , let  $m(\lambda)$  be its minimal polynomial

1.  $m(\lambda)$  is unique
2.  $m(\lambda)$  divides  $d(\lambda)$  without any remainder.  $\exists q(\lambda)$  such that  $d(\lambda) = q(\lambda)m(\lambda)$
3. Each root of  $d(\lambda)$  is a root of  $m(\lambda)$ , then

$$m(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k} = \lambda^l + c_{l-1}\lambda^{l-1} + \cdots + c_0$$

where  $1 \leq m_i \leq r_i, k \leq l \leq n, m_1 + \cdots + m_k = l$

$$q(\lambda) = (\lambda - \lambda_1)^{r_1 - m_1} \cdots (\lambda - \lambda_k)^{r_k - m_k}$$

**Proof:** of (3)

Since  $m(\lambda)$  is the minimal polynomial, we know that  $m(A) = 0_{n \times n}$ . Let  $(\lambda_i, v_i)$  is an eigenvalue, eigenvector pair of  $A$  such that  $Av_i = \lambda_i v_i$ , then

$$m(A) = 0_{n \times n} \rightarrow m(A)v_i = 0_{n \times n}$$

$$(A^l + c_{l-1}A^{l-1} + \cdots + c_1A + c_0I)v_i = 0_{n \times n}$$

$$(\lambda_i^l v_i + c_{l-1}\lambda_i^{l-1}v_i + \cdots + c_1\lambda_i v_i + c_0v_i) = 0_{n \times n}$$

$$\lambda_i^l + c_{l-1}\lambda_i^{l-1} + \cdots + c_1\lambda_i + c_0 = 0$$

$$m(\lambda_i) = 0 \checkmark$$

**Corollary:** Let  $l$  be the order of the minimal polynomial, then the elements of  $\mathcal{A}_l = \{I, A, A^2, \dots, A^{l-1}\}$  are linearly independent and higher order  $A^i$ 's can be written as a linear combination of  $\{I, A, A^2, \dots, A^{l-1}\}$

**Theorem:**  $\mathbb{C}^n = \mathcal{N}((\lambda_1 - A)^{m_1}) \oplus \cdots \oplus \mathcal{N}((\lambda_k I - A)^{m_k})$  where  $\text{Dim}[\mathcal{N}((\lambda_i I - A)^{m_i})] = r_i, \forall i$

Now let's state the necessary sufficient condition(s) such that a matrix can be diagonalizable

**Theorem:**  $A \in \mathbb{C}^{n \times n}$  is diagonalizable, if and only if

- Minimal polynomial has no repeated root. i.e.

$$m(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_n)^{m_k} \text{ where } m_i = 1 \ \forall i$$

$$m(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$$

- Geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity, i.e.

$$\forall i \text{ GM}(\lambda_i) = \dim(\mathcal{N}(\lambda_i - A)) = r_i = \text{AM}(\lambda_i)$$

- There exist  $n$  linearly independent eigenvectors of  $A$

$$\exists \mathcal{V} = \{v_1, \dots, v_n\}, \text{ where } Av_i = \lambda_j \text{ \& Span}(\mathcal{V}) = \mathbb{C}^n$$

### Ex 5.2

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow m(\lambda) = (\lambda + 1), \text{ GM}(-1) = \dim(\mathcal{N}((-1) - A)) = 2 = \text{AM}(-1), \mathcal{V} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \checkmark$$

$$A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \rightarrow m(\lambda) = (\lambda + 1)^2, \text{ GM}(-1) = \dim(\mathcal{N}((-1) - A)) = 1 \leq \text{AM}(-1), \mathcal{V} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \times$$

### 5.1.4 Computation via Jordan Form

When a matrix  $A$  is not diagonalizable, i.e. does not satisfy necessary and sufficient conditions listed above, there exists a similarity transformation,  $A = G^{-1}JG$ , such that  $J$  is Jordan canonical form (as close as possible to a diagonal form). Note that each  $A \in \mathbb{R}^{n \times n}$  is similar to only one such  $J$  (except for a reordering of the blocks):

$$J = \begin{bmatrix} J_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & J_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & \ddots & & \\ \mathbf{0} & & & \ddots & J_{N_J} \end{bmatrix} \text{ where } J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots \\ \vdots & & \ddots & & \\ & & & \ddots & 1 \\ 0 & & & 0 & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i \times n_i}$$

**Ex 5.3**

$$A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{array}{l} m(\lambda) = (\lambda + 1) \\ \text{GM}(-1) = \dim(\mathcal{N}((-1) - A)) = 4 \\ \mathcal{V} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \end{array}$$

$$A_2 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{array}{l} m(\lambda) = (\lambda + 1)^2 \\ \text{GM}(-1) = \dim(\mathcal{N}((-1) - A)) = 3 \\ \mathcal{V} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \end{array}$$

$$A_3 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{array}{l} m(\lambda) = (\lambda + 1)^2 \\ \text{GM}(-1) = \dim(\mathcal{N}((-1) - A)) = 2 \\ \mathcal{V} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \end{array}$$

$$A_4 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{array}{l} m(\lambda) = (\lambda + 1)^3 \\ \text{GM}(-1) = \dim(\mathcal{N}((-1) - A)) = 2 \\ \mathcal{V} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \end{array}$$

$$A_5 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{array}{l} m(\lambda) = (\lambda + 1)^4 \\ \text{GM}(-1) = \dim(\mathcal{N}((-1) - A)) = 1 \\ \mathcal{V} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \end{array}$$

**Proposition:** Jordan transformation of a matrix  $A \in \mathbb{C}^{n \times n}$  with  $d(\lambda) = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_n)^{r_k}$  and  $m(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_n)^{m_k}$ , for each  $\lambda_i$

- $m_i$  : Size of the largest Jordan block for  $\lambda_i$
- $\text{GM}(\lambda) = \dim(\mathcal{N}(\lambda_i - A))$  : # Jordan blocks for  $\lambda_i$
- $\dim(\mathcal{N}(\lambda_i - A)^k) - \dim(\mathcal{N}(\lambda_i - A)^{k-1})$  : # Jordan blocks for  $\lambda_i$  with size  $\geq k$