

## 10.1 Reachability/Controllability, & Observability

### 10.1.1 Reachability & Controllability of CT Systems

For an LTI continuous time state-space representation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

- A state  $x_d$  is said to be **reachable** if there exist a finite time interval  $t \in [0, t_f]$  and an input signal defined on this interval,  $u(t)$ , that transfers the state vector  $x(t)$  from the origin (i.e.  $x(0) = 0$ ) to the state  $x_d$  within this time interval, i.e.  $x(t_f) = x_d$ .
- A state  $x_d$  is said to be **controllable** if there exist a finite time interval  $t \in [0, t_f]$  and an input signal defined on this interval,  $u(t)$ , that transfers the state vector  $x(t)$  from the initial state  $x_d$  (i.e.  $x(0) = x_d$ ) to the origin within this time interval, i.e.  $x(t_f) = 0$ .
- The set  $\mathcal{R}$  of all reachable states is a linear (sub)space:  $\mathcal{R} \subset \mathbb{R}^n$
- The set  $\mathcal{C}$  of all controllable states is a linear (sub)space:  $\mathcal{C} \subset \mathbb{R}^n$

For CT systems  $x_d \in \mathcal{R}$  if and only if  $x_d \in \mathcal{C}$ , the Reachability and Controllability conditions are equivalent.

- If the reachable (or controllable) set is the entire state space, i.e., if  $\mathcal{R} = \mathbb{R}^n$ , then the system is called reachable (or controllable).

One way of testing reachability/controllability is checking the rank (or the range space) the of reachability/controllability matrix

$$\mathbf{M} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

A CT system is reachable/controllable if and only of

$$\text{rank}(\mathbf{M}) = n$$

or equivalently

$$\text{Ra}(\mathbf{M}) = \mathbb{R}^n$$

### 10.1.2 Reachability & Controllability of DT Systems

For LTI a discrete time state-space representation

$$\begin{aligned}x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k]\end{aligned}$$

- A state  $x_d$  is said to be **reachable**, if there exist an input sequence,  $u[k]$ , that transfers the state vector  $x[k]$  from the origin (i.e.  $x[0] = 0$ ) to the state  $x_d$  in finite number of steps, i.e.  $x[k] = x_d$  for some  $k \in \mathbb{Z}^+$ .
- A state  $x_d$  is said to be **controllable**, if there exist an input sequence,  $u[k]$ , that transfers the state vector  $x[k]$  from the initial state  $x_d$  (i.e.  $x[0] = x_d$ ) to the origin in finite number of steps, i.e.  $x[k] = 0$  for some  $k \in \mathbb{Z}^+$ .
- The set  $\mathcal{R}$  of all reachable states is a linear (sub)space:  $\mathcal{R} \subset \mathbb{R}^n$
- The set  $\mathcal{C}$  of all controllable states is a linear (sub)space:  $\mathcal{C} \subset \mathbb{R}^n$

Unlike from CT systems the Reachability and Controllability conditions are not equivalent.

- $x_d \in \mathcal{R} \Rightarrow x_d \in \mathcal{C}$
- $x_d \in \mathcal{C} \not\Rightarrow x_d \in \mathcal{R}$
- $\mathcal{R} \subset \mathcal{C}$

Thus Reachability implies Controllability but Controllability does not necessarily implies Reachability. For this reason, the term of Reachability is generally preferred for DT systems.

- If the reachable set is the entire state space, i.e., if  $\mathcal{R} = \mathbb{R}^n$ , then the system is called Reachable (and automatically Controllable).
- If the controllable set is the entire state space, i.e., if  $\mathcal{C} = \mathbb{R}^n$ , then the system is called Controllable. But there is no guarantee for Reachability.

**Example:** Consider the following autonomous system

$$x[k+1] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x[k]$$

What can we infer about the Reachability and Controllability of this system.

**Solution:** Since this is an autonomous system, obviously

$$B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus input has no affect on the states. If  $x[0] = 0$ , then  $x[k] = 0$ ,  $\forall k > 0$ . Thus the system is obviously NOT Reachable.

Now let's compute  $x[2]$  for a general  $x[0] = x_0$ ,

$$x[2] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 x_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Obviously  $\forall x_0 \in \mathbb{R}^n$   $x[2] = 0$ , thus all state-space is Controllable.

### Test of Reachability on DT Systems

When  $x[0] = 0$ , the solution of  $x[k]$  is given by

$$\begin{aligned} x[k] &= \sum_{j=0}^{k-1} G^{k-j-1} H u[j] \\ &= [ G^{k-1} H \mid G^{k-2} H \mid \cdots \mid G H \mid H ] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix} \end{aligned}$$

Let

$$\begin{aligned} \mathbf{M}_k &= [ G^{k-1} H \mid G^{k-2} H \mid \cdots \mid G H \mid H ] \\ \mathbf{U}_k &= \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix} \end{aligned}$$

then if a state  $x_d$  is reachable at  $k$  steps, it should satisfy the following equation for some  $\mathbf{U}_k$ .

$$\mathbf{M}_k \mathbf{U}_k = x_d$$

In order this matrix equation to have a solution  $x_d$  should be in the range space of  $\mathbf{M}_k$ .

$$x_d \in \text{Ra}(\mathbf{M}_k)$$

It is fairly easy to see that

$$\text{Ra}(\mathbf{M}_k) \subset \text{Ra}(\mathbf{M}_{k+1})$$

Thus increasing  $k$  increases the chance of  $x_d$  being in the reachable subset.

**Theorem:** For  $k < n < l$

$$\text{Ra}(\mathbf{M}_k) \subset \text{Ra}(\mathbf{M}_n) = \text{Ra}(\mathbf{M}_l)$$

**Proof:** In order to prove this Theorem, we need to use a different well-known theorem.

**Cayley-Hamilton Theorem** states that every square matrix satisfies its own characteristic equation. In other words, Let  $A \in \mathbb{R}^{n \times n}$ , and let  $p(\lambda)$  be the characteristic equation defined as

$$\begin{aligned} p(\lambda) &= \det(\lambda I - A) \\ &= \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n \end{aligned}$$

Then by Cayley-Hamilton theorem we conclude that

$$p(G) = G^n + a_1 G^{n-1} + \cdots + a_{n-1} G + a_n I = 0$$

Using this we can see easily that

$$G^n B = -a_1 G^{n-1} B - \dots - a_{n-1} G B - a_n I$$

Now lets observe  $M_{n+1}$

$$\mathbf{M}_{n+1} = [ G^n H \mid G^{n-1} H \mid \dots \mid G H \mid H ]$$

If we follow the Cayley-Hamilton theorem and associated derivations, we can see that the first column  $G^n H$  is a linear combination of other columns, thus it can not increase the rank of the matrix.

This the reachability matrix is defined as

$$\mathbf{M} = [ G^{n-1} H \mid G^{n-2} H \mid \dots \mid G H \mid H ]$$

where  $n$  is the dimension of the state-space.

The DT system is called reachable if

$$\text{rank}(\mathbf{M}) = n$$

or equivalently

$$\text{Ra}(\mathbf{M}) = \mathbb{R}^n$$

## 10.2 Observability

It turns out that it is more natural to think in terms of “un-observability” as reflected in the following definitions.

- For CT systems, a state  $x_o$  of a finite dimensional linear dynamical system is said to be unobservable, if with  $x(0) = x_o$  and for every  $u(t)$  we get the same  $y(t)$  as we would with  $x(0) = 0$ .
- For DT systems, a state  $x_o$  of a finite dimensional linear dynamical system is said to be unobservable, if with  $x[0] = x_o$  and for every  $u[k]$  we get the same  $y[k]$  as we would with  $x[0] = 0$ .

In other words, for both CT and DT systems an unobservable initial condition cannot be distinguished from the zero initial condition.

The set  $\bar{\mathcal{O}}$  of all unobservable states is a linear (sub)space:  $\bar{\mathcal{O}} \subset \mathbb{R}^n$

- If the unobservable set only contains the origin, i.e., if  $\bar{\mathcal{O}} = \{0\}$ ,
- If the dimension of unobservable subspace is equal to 0,  $\dim = (\bar{\mathcal{O}}) = 0$ ,
- If any initial condition,  $x(0)$  or  $x[0]$ , can be uniquely determined from input-output measurement,

then the system is called Observable.

### Test of Observability on CT Systems

One way of testing Observability of CT systems is checking the rank (or the range space, or null space) the of the Observability matrix

$$\mathbf{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

A CT system is Observable if and only of

$$\text{rank}(\mathbf{O}) = n$$

or equivalently

$$\text{Ra}(\mathbf{O}) = \mathbb{R}^n$$

or equivalently

$$\dim(\mathcal{N}(\mathbf{O})) = 0$$

### Test of Observability on DT Systems

Without loss of generality, let's assume that  $u[k] = 0$ . Under this assumption, we know that

$$y[k] = CG^k x_0$$

Based on this solution we can write

$$\begin{aligned} y[0] &= Cx_0 \\ y[1] &= CGx_0 \\ y[2] &= CG^2x_0 \\ &\vdots \\ y[k] &= CG^kx_0 \end{aligned}$$

If we combine these equations matrix form we obtain

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[k] \end{bmatrix} = \begin{bmatrix} C \\ CG \\ CG^2 \\ \vdots \\ CG^k \end{bmatrix} x_0$$

Let

$$\mathbf{Y}_k = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[k] \end{bmatrix}, \quad \mathbf{O}_k = \begin{bmatrix} C \\ CG \\ CG^2 \\ \vdots \\ CG^k \end{bmatrix}$$

Then the equation takes the simple form  $\mathbf{Y}_k = \mathbf{O}_k x_0$ . If  $x_0$  is an unobservable state, then for-all  $k$  we should have  $\mathbf{O}_k x_0 = 0$ , or equivalently  $x_0 \in \mathcal{N}(\mathbf{O}_k)$  (Null-space).

From this point, we can conclude that, the DT system is observable if and only if,

$$\forall k \in \mathbb{Z}, \dim(\mathcal{N}(\mathbf{O}_k)) = 0$$

However we don't need to test all  $k \in \mathbb{Z}$ . First of all it should be obvious that we should take  $k$  as large as possible to guarantee whether  $x_0$  is unobservable or not. Formally speaking,

$$\mathcal{N}(\mathbf{O}_{k+1}) \subset \mathcal{N}(\mathbf{O}_k)$$

However from Cayley-Hamilton theorem, we know that  $CA^n$  can be written as a linear combination of  $\{CA^{n-1}, CA^{n-2}, \dots, CA, C\}$ , thus we have

$$\mathcal{N}(\mathbf{O}_n) = \mathcal{N}(\mathbf{O}_{n-1})$$

For this reason it is necessary and sufficient to test  $\mathbf{O}_{n-1}$  for observability. In conclusion, observability matrix is defined as

$$\mathbf{O} = \begin{bmatrix} C \\ CG \\ CG^2 \\ \vdots \\ CG^{n-1} \end{bmatrix}$$

The DT system is called Observable if

$$\text{rank}(\mathbf{O}) = n$$

or equivalently

$$\text{Ra}(\mathbf{O}) = \mathbb{R}^n$$

$$\dim(\mathcal{N}(\mathbf{O})) = 0$$

**Example:** Consider the following state-space form of a DT system

$$\begin{aligned}x[k+1] &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k] \\ y[k] &= \begin{bmatrix} 1 & -1 \end{bmatrix} x(t)\end{aligned}$$

Is this system fully reachable and observable ?

**Solution:** Let's compute the reachability matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\text{rank}(\mathbf{M}) = 2$ , thus the state-space representation is fully reachable. Now, let's compute the observability matrix

$$\mathbf{O} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\text{rank}(\mathbf{M}) = 1$ , thus the state-space representation is not fully observable.

In order to gain some insight regarding the reason of the lack of observability, let's find the eigenvalues, as well as, closed-loop transfer function

$$\det \left( \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} \right) = \lambda^2 - 1 \rightarrow \lambda_{1,2} = \pm 1$$

$$\begin{aligned}H(z) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} z & -1 \\ -1 & z \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} z & 1 \\ 1 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{z^2 - 1} \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} z & 1 \\ 1 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{z^2 - 1} \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} \frac{1}{z^2 - 1} \\ &= \frac{-(z-1)}{z^2 - 1} \\ &= \frac{-1}{z+1}\end{aligned}$$

We can see that even if  $\lambda = 1$  is an eigenvalue of the state-space representation, it is not a pole of the transfer function. This implies that the mode of the system associated with  $\lambda = 1$  is reachable but not observable. For this reason original state-space representation is not a minimal state-space representation.