EE502 - Linear Systems Theory II **Spring 2023**

Lecture 13

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13.1 State Feedback

The state-feedback based control-policies for LTI systems start with the assumption that we have "access" to all of the states of the systems either via direct measurement or through some observer/estimator/tracker. In that context a family of state-feedback controllers for CT- and DT-LTI systems can be constructed as

$$u(t) = \gamma r(t) - Kx(t) \ \& \ u[k] = \gamma r[k] - Kx[k]$$

where $r(t)$ can be considered as the reference signal (most of the time it is), γ is a feed-forward scaling factor, and K is the state-feedback gain. Now let's find a state-space representation for dynamics of the closed-loop system for both CT- and DT-LTI systems under state-feedback rule proposed above

$$\dot{x} = Ax + B\left(\gamma r(t) - Kx(t)\right) \Rightarrow \dot{x} = (A - BK)\,x + \gamma Br$$
$$x[k+1] = Ax + B\left(\gamma r[k] - Kx[k]\right) \Rightarrow x[k+1] = (A - BK)\,x[k] + \gamma Br[k]$$

In both cases the closed loop system and input matrices take the following form.

$$A_c = A - BK \ , \ B_c = \gamma Br$$

A key question in this domain is whether we can find a K such that eigenvalues of A_c is located at arbitrary desired locations.

Theorem: (Eigenvalue/Pole Placement) Given (A, B), $\exists K$ s.t.

$$\det\left[\lambda I - (A - BK)\right] = \lambda^n + a_{n-1}^*\lambda^{n-1} + \cdots + a_1^*\lambda + a_0^*$$
$$\forall \mathcal{A} = \{a_0^*,\, a_1^* \cdots a_{n-1}^*\},\, a_i^* \in \mathbb{R}$$

if and only if (A, B) is reachable.

Proof: For a general complete proof, we need to show that reachability of (A, B) is necessary and sufficient.

Proof of necessity: Let's assume that (A, B) not reachable and $\exists(\lambda_u, w_u^T)$ pair such that $w_u^T A = w_u^T \lambda_u$ and $w_u^T B = 0$. Now check weather w_u^T is a left eigenvector of A_c

$$w_u^T A_c = w_u^T (A - BK) = w_u^T A - w_u^T BK = w_u^T \lambda_u - 0 = w_u^T \lambda_u$$
$$w_u^T B_c = w_u^T B\gamma = 0$$

Here not only we showed that λ_u could not be moved hence it is not possible to locate the poles arbitrary locations, we also showed that state-feedback rules do not affect reachability.

Proof of sufficiency: We will only show the sufficiency for a multi-input case, i.e. $B \in \mathbb{R}^{n\times 1}$, however, the reader should note that a complete proof multi-input case must be analyzed.

Let's assume that (A, B) is reachable, and we know that reachability is invariant under similarity transformations, i.e.

$$z = T^{-1}x\,,\ \det(T) \neq 0 \ \Rightarrow\ \dot{z} = \bar{A}z + \bar{B}u$$
$$\bar{A} = T^{-1}AT\ ,\ T^{-1}B$$

and $(\bar{A}, \bar{B})$ is reachable. Noe let's choose T such that

$$T = \mathbf{R} = \left[\ A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B\ \right]$$

then $\bar{B}$ can be derived as

$$B = T\bar{B} = \left[\ A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B\ \right]\bar{B} \ \Rightarrow\ B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and similarly $\bar{A}$ can be expressed as

$$A\mathbf{R} = \mathbf{R}\bar{A}$$
$$A\left[\ A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B\ \right] = \left[\ A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B\ \right]\bar{A}$$
$$\left[\ A^{n}B \mid A^{n-1}B \mid \cdots \mid AB \mid B\ \right] = \left[\ A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B\ \right] \left[\begin{array}{c|c|c|c|c|c} \bar{a}_{11} & 1 & 0 & 0 & \cdots & 0 \\ \bar{a}_{12} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \bar{a}_{1(n-2)} & 0 & 0 & \cdots & 1 & 0 \\ \bar{a}_{1(n-1)} & 0 & 0 & \cdots & 0 & 1 \\ \bar{a}_{1(n)} & 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

We can find a_{1i}'s using Cayley-Hamilton theorem

$$A^nB = \bar{a}_{11}A^{n-1}B + \bar{a}_{12}A^{n-2}B + \cdots + \bar{a}_{1(n-1)}AB + \bar{a}_{1(n)}B$$
$$A^n = -\left(a_{n-1}A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1A + a_I\right) \text{ where}$$
$$\det[\lambda I - A] = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

then $\bar{A}$ takes the form

$$\bar{A} = \left[\begin{array}{c|c|c|c|c|c} -a_{n-1} & 1 & 0 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ -a_2 & 0 & 0 & \cdots & 1 & 0 \\ -a_1 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

Note that this similarity transformation is in a companion form however we will transform this into (more useful) reachable canonical form

$$\tilde{A} = \left[\begin{array}{c|c|c|c|c|c} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{array}\right]\ ,\ \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Let's $\tilde{\mathbf{R}}$ be the reachability matrix of $(\tilde{A},\tilde{B})$ then we know that

$$\bar{A}=\tilde{\mathbf{R}}^{-1}A\tilde{\mathbf{R}}\ ,\ \bar{B}=\tilde{\mathbf{R}}^{-1}B$$

where $(\bar{A},\bar{B})$ are the matrices of the companion form derived above. Thus if we let $T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}=\mathbf{R}\tilde{\mathbf{R}}^{-1}$, where $\mathbf{R}$ is the reachability matrix of the original representation and $\tilde{\mathbf{R}}$ is the reachability matrix of the reachable canonical form and adopt T for similarity transformation we obtain

$$\begin{aligned} q &= \left(T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}\right)^{-1}x \Rightarrow \dot{q}=\tilde{A}q+\tilde{B}u \\ \tilde{A} &= \left(T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}\right)^{-1}A\,T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}\ ,\ \tilde{B}=\left(T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}\right)^{-1}B \end{aligned}$$

Now let's apply the state feedback rule in the transformed coordinates.

$$\begin{aligned} \dot{q} &= \tilde{A}q+\tilde{B}u\ ,\ u=\alpha r-\tilde{K}q \\ \dot{q} &= (\tilde{A}-\tilde{B}\tilde{K})q+\alpha\tilde{B}u \end{aligned}$$

$$\begin{aligned}
(\tilde{A}-\tilde{B}\tilde{K}) &= \left[\begin{array}{c|c|c|c|c|c} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{array}\right] - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix}\tilde{k}_0 & \tilde{k}_1 & \cdots & \tilde{k}_{n-1}\end{bmatrix} \\
&= \left[\begin{array}{c|c|c|c|c|c} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0-\tilde{k}_0 & -a_1-\tilde{k}_1 & -a_2-\tilde{k}_2 & \cdots & -a_{n-2}-\tilde{k}_{n-2} & -a_{n-1}-\tilde{k}_{n-1} \end{array}\right]
\end{aligned}$$

$$\det\left(\tilde{A}-\tilde{B}\tilde{K}\right)=\lambda^n+(a_{n-1}+\tilde{k}_{n-1}\lambda^{n-1}+\cdots+(a_1+\tilde{k}_1)\lambda+(a_0+\tilde{k}_0)$$

It is evident that $\exists\tilde{K}$ s.t.

$$\begin{gathered} \det\left(\tilde{A}-\tilde{B}\tilde{K}\right)=\lambda^n+a^*_{n-1}\lambda^{n-1}+\cdots+a^*_1\lambda+a^*_0 \\ \forall\mathcal{A}=\{a^*_0,\,a^*_1\,\cdots\,a^*_{n-1}\},\,a^*_i\in\mathbb{R} \end{gathered}$$

Now we need to find the state-feedback gain and rule in the original coordinate frame

$$\begin{aligned} u &= \alpha r-\tilde{K}q=\alpha r-\tilde{K}\left(T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}\right)^{-1}x \\ \dot{x} &= \left(A-B\tilde{K}\left(T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}\right)^{-1}\right)+\alpha Br\ ,\ K=\tilde{K}\left(T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}\right)^{-1} \end{aligned}$$

Now let's verify that $(A-BK)$ and $(\tilde{A}-\tilde{B}\tilde{K})$ has the same characteristic equations

$$\begin{aligned} \det(\lambda I-[A-BK]) &= \det\left(\lambda I-\left[A-B\tilde{K}\left(T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}\right)^{-1}\right]\right) \\ &= \det\left(\lambda I-\left(T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}\right)^{-1}\left[A-B\tilde{K}\left(T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}\right)^{-1}\right]T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}\right) \\ &= \det\left(\lambda I-\left[\left(T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}\right)^{-1}AT_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}-\left(T_{\mathbf{R}\tilde{\mathbf{R}}^{-1}}\right)^{-1}B\tilde{K}\right]\right) \\ &= \det\left(\lambda I-\left[\tilde{A}-\tilde{B}\tilde{K}\right]\right) \end{aligned}$$

this completes the proof for SISO case.

13.2 Stabilizability

Definition: An LTI system is called stabilizable, if $\forall x_0 \in \mathbb{R}^n$, $\exists\, u(t)$ (or $u[k]$) such that $\lim_{t\to\infty} x(t) = 0$ (or $\lim_{k\to\infty} x[k] = 0$).

An obvious **sufficient** condition on stabilizability for CT-LTI and DT-LTI systems is that, if (A, B) is reachable then (A, B) stabilizable, since in such a case $\forall x_0 \in \mathbb{R}^n$, $\exists\, u(t)$ or $u[k]$ such that $x(T) = 0\, \forall T < 0$ for CT systems, and $x[k] = 0\, \forall k > n$.

Another **sufficient** (but weaker) condition on stabilizability for DT-LTI systems is that if (A, B) is controllable then (A, B) stabilizable, since in such a case $\forall x_0 \in \mathbb{R}^n$, $\exists\, u[k]$ such that $x[k] = 0\, \forall k > n$.

Theorem The following statements are equal

- (A, B) is stabilizable $\iff$
- if unreachable modes of (A, B) are asymptotically stable $\iff$
- $\forall(\lambda_i, w_i)$ s.t. $w_i^T A = \lambda_i w_i^T \iff$ and $\mathrm{Re}\{\lambda_i\} \geq 0$ (or $|\lambda_i| \geq 0$), $w_i \notin \mathcal{N}(B) \iff$
- $\operatorname{rank}\,[A - \lambda I \mid B] = n$, $\forall \lambda \in \mathbb{C}$ s.t. $\mathrm{Re}\{\lambda_i\} \geq 0$ for CT systems (or $\forall \lambda \in \mathbb{C}$ s.t. $|\lambda| \geq 1$ for DT systems)

Proof: Given (A, B) that is not necessarily reachable, we can transform it into the standard unreachable form

$$\dot{x} = \begin{bmatrix} \dot{x}_r \\ \dot{x}_{\bar{r}} \end{bmatrix} = \left[\begin{array}{c|c} A_r & A_{12} \\ \hline 0 & A_{\bar{r}} \end{array}\right] \begin{bmatrix} x_r \\ x_{\bar{r}} \end{bmatrix} + \begin{bmatrix} B_r \\ 0 \end{bmatrix} u$$

Let $\dot{x}_{\bar{r}} = A_{\bar{r}} x_{\bar{r}}$ is not asymptotically stable then obviously the whole system is unstable; thus, the system is not stabilizable . Now let $\dot{x}_{\bar{r}} = A\bar{r} x_{\bar{r}}$ is asymptotically stable, then let $u = -K_r x_r$ then the closed-loop system takes the form

$$\dot{x} = \begin{bmatrix} \dot{x}_r \\ \dot{x}_{\bar{r}} \end{bmatrix} = \left[\begin{array}{c|c} A_r - B_r K_r & A_{12} \\ \hline 0 & A_{\bar{r}} \end{array}\right] \begin{bmatrix} x_r \\ x_{\bar{r}} \end{bmatrix}$$

Since (A_r, B_r) pair is reachable we can find K such that eigenvalues of $(A_r - B_r K_r)$ are located in the open-left-half of the complex plane, thus the system is stabilizable. The same logic can be adopted for DT systems.