Lecture 15

Lecturer: Asst. Prof. M. Mert Ankarali

Reachability/Controllability, & Observability

Reachability & Controllability of CT Systems

For an LTI continuous time state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- A state x_d is said to be **reachable** if there exist a finite time interval $t \in [0, t_f]$ and an input signal defined on this interval, $u(t)$, that transfers the state vector $x(t)$ from the origin (i.e. $x(0) = 0$) to the state x_d within this time interval, i.e. $x(t_f) = x_d$.
- A state x_d is said to be **controllable** if there exist a finite time interval $t \in [0, t_f]$ and an input signal defined on this interval, $u(t)$, that transfers the state vector $x(t)$ from the initial state x_d (i.e. $x(0) = x_d$) to the origin within this time interval, i.e. $x(t_f) = 0$.

- The set $\mathcal{R}$ of all reachable states is a linear (sub)space: $\mathcal{R} \subset \mathbb{R}^n$
- The set $\mathcal{C}$ of all controllable states is a linear (sub)space: $\mathcal{C} \subset \mathbb{R}^n$

For CT systems $x_d \in \mathcal{R}$ if and only if $x_d \in \mathcal{C}$, the Reachability and Controllability conditions are equivalent.

- If the reachable (or controllable) set is the entire state space, i.e., if $\mathcal{R} = \mathbb{R}^n$, then the system is called reachable (or controllable).

One way of testing reachability/controllability is checking the rank (or the range space) the of reachability/controllability matrix

$$\mathbf{M} = \left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array} \right]$$

A CT system is reachable/controllable if and only of

$$\text{rank}(\mathbf{M}) = n$$

or equivalently

$$\text{Ra}(\mathbf{M}) = \mathbb{R}^n$$

Reachability & Controllability of DT Systems

For LTI a discrete time state-space representation

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

- A state x_d is said to be **reachable**, if there exist an input sequence, $u[k]$, that transfers the state vector $x[k]$ from the origin (i.e. $x[0] = 0$) to the state x_d in finite number of steps, i.e. $x[k] = x_d$ for some $k \in \mathbb{Z}^+$.
- A state x_d is said ti be **controllable**, if there exist an input sequence, $u[k]$, that transfers the state vector $x[k]$ from the initial state x_d (i.e. $x[0] = x_d$) to the origin in finite number of steps, i.e. $x[k] = 0$ for some $k \in \mathbb{Z}^+$

- The set $\mathcal{R}$ of all reachable states is a linear (sub)space: $\mathcal{R} \subset \mathbb{R}^n$
- The set $\mathcal{C}$ of all controllable states is a linear (sub)space: $\mathcal{C} \subset \mathbb{R}^n$

Unlike from CT systems the Reachability and Controllability conditions are not equivalent.

- $x_d \in \mathcal{R} \Rightarrow x_d \in \mathcal{C}$
- $x_d \in \mathcal{C} \not\Rightarrow x_d \in \mathcal{R}$
- $\mathcal{R} \subset \mathcal{C}$

Thus Reachability implies Controllability but Controllability does not necessarily implies Reachability. For this reason, the term of Reachability is generally preferred for DT systems.

- If the reachable set is the entire state space, i.e., if $\mathcal{R} = \mathbb{R}^n$, then the system is called Reachable (and automatically Controllable).
- If the controllable set is the entire state space, i.e., if $\mathcal{C} = \mathbb{R}^n$, then the system is called Controllable. But there is no guarantee for Reachability.

Example: Consider the following autonomous system

$$x[k+1] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x[k]$$

What can we infer about the Reachability and Controllability of this system.

Solution: Since this is an autonomous system, obviously

$$B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus input has no affect on the states. If $x[0] = 0$, then $x[k] = 0,\ \forall k > 0$. Thus the system is obviously NOT Reachable.

Now let's compute $x[2]$ for a general $x[0] = x_0$,

$$x[2] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 x_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Obviously $\forall x_0 \in \mathbb{R}^n$ $x[2] = 0$, thus all state-space is Controllable.

Test of Reachability on DT Systems

When $x[0] = 0$, the solution of $x[k]$ is given by

$$\begin{aligned} x[k] &= \sum_{j=0}^{k-1} G^{k-j-1} H u[j] \\ &= \left[\begin{array}{c|c|c|c|c} G^{k-1}H & G^{k-2}H & \cdots & GH & H \end{array} \right] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix} \end{aligned}$$

Let

$$\mathbf{M}_k = \left[\begin{array}{c|c|c|c|c} G^{k-1}H & G^{k-2}H & \cdots & GH & H \end{array} \right]$$
$$\mathbf{U}_k = \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix}$$

then if a state x_d is reachable at k steps, it should satisfy the following equation for some $\mathbf{U}_k$.

$$\mathbf{M}_k \mathbf{U}_k = x_d$$

In order this matrix equation to have a solution x_d should be in the range space of $\mathbf{M}_k$.

$$x_d \in \mathrm{Ra}(\mathbf{M}_k)$$

It is fairly easy to see that

$$\mathrm{Ra}(\mathbf{M}_k) \subset \mathrm{Ra}(\mathbf{M}_{k+1})$$

Thus increasing k increases the chance of x_d being in the reachable subset.

Theorem: For $k < n < l$

$$\mathrm{Ra}(\mathbf{M}_k) \subset \mathrm{Ra}(\mathbf{M}_n) = \mathrm{Ra}(\mathbf{M}_l)$$

Proof: In order to prove this Theorem, we need to use a different well-known theorem.

Cayley-Hamilton Theorem states that every square matrix satisfies its own characteristic equation. In other words, Let $A \in \mathbb{R}^{n\times n}$, and let $p(\lambda)$ be the characteristic equation defined as

$$\begin{aligned} p(\lambda) &= \det(\lambda I - A) \\ &= \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n\lambda \end{aligned}$$

Then by Cayley-Hamilton theorem we conclude that

$$p(G) = G^n + a_1 G^{n-1} + \cdots + a_{n-1} G + a_n I = 0$$

Using this we can see easily that

$$G^n B = -a_1 G^{n-1} B - \cdots - a_{n-1} GB - a_n I$$

Now lets observe M_{n+1}

$$\mathbf{M}_{n+1} = \left[\begin{array}{c|c|c|c|c} G^n H & G^{n-1} H & \cdots & GH & H \end{array} \right]$$

If we follow the Cayley-Hamilton theorem and associated derivations, we can see that the first column $G^n H$ is a linear combination of other columns, thus it can not increase the rank of the matrix.

This the reachability matrix is defined as

$$\mathbf{M} = \left[\begin{array}{c|c|c|c|c} G^{n-1} H & G^{n-2} H & \cdots & GH & H \end{array} \right]$$

where n is the dimension of the state-space.

The DT system is called reachable if

$$\operatorname{rank}(\mathbf{M}) = n$$

or equivalently

$$\operatorname{Ra}(\mathbf{M}) = \mathbb{R}^n$$

Observability

It turns out that it is more natural to think in terms of "un-observability" as reflected in the following definitions.

- For CT systems, a state x_o of a finite dimensional linear dynamical system is said to be unobservable, if with $x(0) = x_o$ and for every $u(t)$ we get the same $y(t)$ as we would with $x(0) = 0$.
- For DT systems, a state x_o of a finite dimensional linear dynamical system is said to be unobservable, if with $x[0] = x_o$ and for every $u[k]$ we get the same $y[k]$ as we would with $x[0] = 0$.

In other words, for both CT and DT systems an unobservable initial condition cannot be distinguished from the zero initial condition.

The set $\bar{\mathcal{O}}$ of all unobservable states is a linear (sub)space: $\bar{\mathcal{O}} \subset \mathbb{R}^n$

- If the unobservable set only contains the origin, i.e., if $\bar{\mathcal{O}} = \{0\}$,
- If the dimension of unobservable subspace is equal to 0, $\dim = \left(\bar{\mathcal{O}}\right) = 0$,
- If any initial condition, $x(0)$ or $x[0]$, can be uniquely determined from input-output measurement,

then the system is called Observable.

Test of Observability on CT Systems

One way of testing Observability of CT systems is checking the rank (or the range space, or null space) the of the Observability matrix

$$\mathbf{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

A CT system is Observable if and only of

$$\text{rank}(\mathbf{O}) = n$$

or equivalently

$$\text{Ra}(\mathbf{O}) = \mathbb{R}^n$$

or equivalently

$$\dim(\mathcal{N}(\mathbf{O})) = 0$$

Test of Observability on DT Systems

Without loss of generality, let's assume that $u[k] = 0$. Under this assumption, we know that

$$y[k] = CG^k x_0$$

Based on this solution we can write

$$\begin{aligned} y[0] &= Cx_0 \\ y[1] &= CGx_0 \\ y[2] &= CG^2 x_0 \\ &\vdots \\ y[k] &= CG^k x_0 \end{aligned}$$

If we combine these equations matrix form we obtain

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[k] \end{bmatrix} = \begin{bmatrix} C \\ CG \\ CG^2 \\ \vdots \\ CG^k \end{bmatrix} x_0$$

Let

$$\mathbf{Y}_k = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[k] \end{bmatrix} \quad , \quad \mathbf{O}_k = \begin{bmatrix} C \\ CG \\ CG^2 \\ \vdots \\ CG^k \end{bmatrix}$$

Then the equation takes the simple form $\mathbf{Y}_k = \mathbf{O}_k x_0$. If x_0 is an unobservable state, then for-all k we should have $\mathbf{O}_k x_0 = 0$, or equivalently $x_0 \in \mathcal{N}(\mathbf{O}_k)$ (Null-space).

From this point, we can conclude that, the DT system is observable if and only if,

$$\forall k \in \mathbb{Z}, \dim\left(\mathcal{N}\left(\mathbf{O}_k\right)\right) = 0$$

However we don't need to test all $k \in \mathbb{Z}$. First of all it should be obvious that we should take k as large as possible to guarantee weather x_0 is unobservable ot not. Formally speaking,

$$\mathcal{N}\left(\mathbf{O}_{k+1}\right) \subset \mathcal{N}\left(\mathbf{O}_k\right)$$

However from Cayley-Hamilton theorem, we know that CA^n can be written as a linear combination of $\{CA^{n-1},\ CA^{n-2},\ \cdots, CA,\ C\}$, thus we have

$$\mathcal{N}\left(\mathbf{O}_n\right) = \mathcal{N}\left(\mathbf{O}_{n-1}\right)$$

For this reason it is necessary and sufficient to test $\mathbf{O}_{n-1}$ for observability. In conclusion, observability matrix is defined as

$$\mathbf{O} = \begin{bmatrix} C \\ CG \\ CG^2 \\ \vdots \\ CG^{n-1} \end{bmatrix}$$

The DT system is called Observable if

$$\operatorname{rank}\left(\mathbf{O}\right) = n$$

or equivalently

$$\operatorname{Ra}\left(\mathbf{O}\right) = \mathbb{R}^n$$

$$\dim\left(\mathcal{N}(\mathbf{O})\right) = 0$$

Example: Consider the following state-space form of a DT system

$$x[k+1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} 1 & -1 \end{bmatrix} x(t)$$

Is this system fully reachable and observable ?

Solution: Let's compute the reachability matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

rank($\mathbf{M}$) $= 2$, thus the state-space representation is fully reachable. Now, let's compute the observability matrix

$$\mathbf{O} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

rank($\mathbf{M}$) $= 1$, thus the state-space representation is not fully observable.

In order to gain some insight regarding the reason of the lack of observability, let's find the eigenvalues, as well as, closed-loop transfer function

$$\det\left(\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix}\right) = \lambda^2 - 1 \quad \rightarrow \quad \lambda_{1,2} = \pm 1$$

$$\begin{aligned} H(z) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} z & -1 \\ -1 & z \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} z & 1 \\ 1 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{z^2-1} \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} z & 1 \\ 1 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{z^2-1} \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} \frac{1}{z^2-1} \\ &= \frac{-(z-1)}{z^2-1} \\ &= \frac{-1}{z+1} \end{aligned}$$

We can see that even if $\lambda = 1$ is an eigenvalue of the state-space representation, it is not a pole of the transfer function. This implies that the mode of the system associated with $\lambda = 1$ is reachable bot not observable. For this reason original state-space representation is not a minimal state-space representation.