EE502 - Linear Systems Theory II **Spring 2032**

Lecture 5

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5.1 Functions of a Matrix

In linear systems theory course, we are interested in matrix polynomials, specifically

- Matrix Exponential in CT Systems: e^{At}
- Matrix Power in DT Systems: A^k

which arise on the solution of state-space equations in their respective domains. Obviously A^k in DT systems is "easier" to analyze and understand compared to matrix exponential. Let's first review the matrix exponential, e^{At}. Let $t \in \mathbb{R}$ and $A \in \mathbb{R}^{nxn}$, then e^{At} defined as

$$
\begin{aligned}
e^{At} &:= I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \\
e^{At} &:= \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k
\end{aligned}
$$

which converges for all $t \in \mathbb{R}$ and $A \in \mathbb{R}^{nxn}$.

Now let's review some properties

- **Claim:** $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$

  Proof:

$$
\begin{aligned}
\frac{d}{dt}e^{At} &= \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}A^k\right) = \sum_{k=0}^{\infty} k\frac{t^{k-1}}{k!}A^k = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!}A^k \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!}A^{n+1} = A\sum_{n=0}^{\infty} \frac{t^n}{n!}A^n = \left(\sum_{n=0}^{\infty} \frac{t^n}{n!}A^n\right)A \\
\frac{d}{dt}e^{At} &= Ae^{At} = e^{At}A
\end{aligned}
$$

- **Claim:** Let $A, B \in \mathrm{R}^{n\times n}$ and $AB = BA$, then

$$
e^Ae^B = e^Be^A = e^{(A+B)}
$$

  Proof:

$$
e^Ae^B = \left(\sum_{k=0}^{\infty} \frac{1}{k!}A^k\right)\left(\sum_{j=0}^{\infty} \frac{1}{j!}B^j\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{A^k}{k!}\frac{B^j}{j!}
$$

Let $n = k + j$ and $j = n - k$, then

$$e^A e^B = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{A^k\ B^{n-k}}{k!(n-k)!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{A^k\ B^{n-k}}{n!}\frac{n!}{k!(n-k)!} = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n} A^k\ B^{n-k}\begin{pmatrix} n \\ k \end{pmatrix}$$

Note that if $AB \neq BA$ we has to stop at this point. However, since $AB = BA$, we can adopt binomial theorem

$$e^A e^B = \sum_{n=0}^{\infty}\frac{1}{n!}(A+B)^n = e^{A+B} = e^B e^A$$

- **Claim:** Let $t_1, t_2 \in \mathrm{R}$ then

$$e^{At_1}e^{At_2} = e^{At_2}e^{At_1} = e^{A(t_1+t_2)}$$

Proof: Let $A := At_1$ and $B := At_2$, obviously $(At_1)(At_2) = (At_2)(At_1)$ hence we can use the previous property, i.e.

$$e^{At_1}e^{At_2} = e^{(At_1)+(At_2)} = e^{A(t_1+t_2)} = e^{At_2}e^{At_1}$$

Now let $t_1 = t$ and $t_2 = -t$, then we have

$$e^{At}e^{-At} = e^{A(t-t)} = I \quad \rightarrow \quad \left(e^{At}\right)^{-1} = e^{-At}$$

- **Claim:** Let $P \in \mathrm{R}^{n\times n}$ and $\det(P) \neq 0$, then

$$e^{\left(P^{-1}AP\right)t} = P^{-1}e^{At}P$$

Proof: Let's firs show that $\left(P^{-1}AP\right)^k = P^{-1}A^kP$

$$\begin{aligned}\left(P^{-1}AP\right)^k &= \left(P^{-1}AP\right)\left(P^{-1}AP\right)\cdots\left(P^{-1}AP\right)\left(P^{-1}AP\right)\\ &= P^{-1}APP^{-1}APP^{-1}\cdots PP^{-1}APP^{-1}AP\\ &= P^{-1}AIAI\cdots IAIAP\\ &= A^k\end{aligned}$$

Now let's expand

$$\begin{aligned}e^{\left(P^{-1}AP\right)t} &= \sum_{k=0}^{\infty}\frac{t^k}{k!}\left(P^{-1}AP\right)^k\\ &= \sum_{k=0}^{\infty}\frac{t^k}{k!}P^{-1}A^kP\\ &= P^{-1}\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}A^k\right)P\\ &= P^{-1}e^{At}P\end{aligned}$$

5.1.1 Computation of e^{At} and A^k

5.1.2 Computation via Solution of State-Space Equations and Frequency Domain Expressions

An LTI CT state-space representation of an autonomous system has the form

$$\dot{x}(t) = Ax(t) \text{ , where } x(t) \in \mathbb{R}^n$$

Let's test if $x(t) = e^{At}x_0$ is a solution of the homogeneous equation

$$\begin{aligned} x(0) &= e^{A\,0}x_0 = x_0 \\ \dot{x}(t) - Ax(t) &= \left(Ae^{At}\right)x_0 - Ae^{At}x_0 = 0 \end{aligned}$$

Now let's remember Laplace domain solution of the same equation

$$\begin{aligned} \mathcal{L}\left[\dot{x}(t)\right] &= \mathcal{L}\left[Ax(t)\right] \\ sX(s) - x(0) &= AX(s) \\ [sI - A]X(s) &= x(0) \\ X(s) &= [sI - A]^{-1}x_0 \end{aligned}$$

If we connect time and s-domain solutions we obtain

$$e^{At} = \mathcal{L}^{-1}\left[[sI - A]^{-1}\right]$$

Now let's focus on A^k. An LTI DT state-space representation of an autonomous system has the form

$$x[k+1] = Ax[k]$$

Unlike CT systems we can compute the response iteratively easily

$$\begin{aligned} x[1] &= Gx[0] \\ x[2] &= Gx[1] = G^2x[0] \\ x[3] &= Gx[2] = G^3x[0] \\ &\vdots \\ x[k] &= Gx[k-1] = G^kx[0] \end{aligned}$$

Now let's remember form of the response in Z-domain.

$$\begin{aligned} \mathcal{Z}\left[x[k+1]\right] &= \mathcal{Z}\left[Gx[k]\right] \\ zX(z) - zx[0] &= GX(z) \\ (zI - G)\,X(z) &= zX(z) \\ X(z) &= z\,(zI - G)^{-1}\,x[0] \end{aligned}$$

If we connect time and s-domain solutions we obtain

$$G^k = \mathcal{Z}^{-1}\{z\,(zI - G)^{-1}\}$$

5.1.3 Computation via Diagonalization

Theorem: $A \in \mathbb{C}^{n\times n}$ is diagonalizable, if and only if there exists a (nonsingular) similarity transformation, $V \in \mathbb{C}^{n\times n}$, such that $A = V^{-1}\Lambda V$ where Λ is a diagonal matrix,

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{bmatrix}$$

where $\lambda_i \in \mathbb{C}$'s are the eigenvalues of A, which are the roots of the characteristic equation $d(\lambda) = \det(\lambda I - A)$.

Now let's compute A^k and e^{At} for a diagonalizable matrix

$$A^k = \left(V^{-1}\Lambda V\right)^k = V^{-1}\Lambda^k V = V^{-1}\begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} V$$

$$e^{At} = e^{V^{-1}\Lambda V t} = V^{-1}e^{\Lambda t}V = V^{-1}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} V$$

A sufficient but not necessary condition that A will have n distinct eigenvalues in such a case characteristic equation will have the following form

$$d(\lambda) = (\lambda - \lambda_1)\cdots(\lambda - \lambda_n)\ , \ \text{where } \lambda_i \neq \lambda_j\ , \ \text{if } i \neq j$$