EE502 - Linear Systems Theory II **Spring 2032**

Lecture 5

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5.1 Functions of a Matrix

In linear systems theory course, we are interested in matrix polynomials, specifically

- Matrix Exponential in CT Systems: e^{At}
- Matrix Power in DT Systems: A^k

which arise on the solution of state-space equations in their respective domains. Obviously A^k in DT systems is "easier" to analyze and understand compared to matrix exponential. Let's first review the matrix exponential, e^{At}. Let $t \in \mathbb{R}$ and $A \in \mathbb{R}^{nxn}$, then e^{At} defined as

$$e^{At} := I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$
$$e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k$$

which converges for all $t \in \mathbb{R}$ and $A \in \mathbb{R}^{nxn}$.

Now let's review some properties

- **Claim:** $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$
  Proof:
$$\begin{aligned} \frac{d}{dt}e^{At} &= \frac{d}{dt}\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}A^k\right) = \sum_{k=0}^{\infty} k\frac{t^{k-1}}{k!}A^k = \sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}A^k \\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}A^{n+1} = A\sum_{n=0}^{\infty}\frac{t^n}{n!}A^n = \left(\sum_{n=0}^{\infty}\frac{t^n}{n!}A^n\right)A \\ \frac{d}{dt}e^{At} &= Ae^{At} = e^{At}A \end{aligned}$$

- **Claim:** Let $A, B \in \mathrm{R}^{n\times n}$ and $AB = BA$, then
$$e^Ae^B = e^Be^A = e^{(A+B)}$$

  Proof:

$$e^Ae^B = \left(\sum_{k=0}^{\infty}\frac{1}{k!}A^k\right)\left(\sum_{j=0}^{\infty}\frac{1}{j!}B^j\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{A^k}{k!}\frac{B^j}{j!}$$

Let $n = k + j$ and $j = n - k$, then

$$e^A e^B = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{A^k \ B^{n-k}}{k!(n-k)!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^k \ B^{n-k}}{n!} \frac{n!}{k!(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} A^k \ B^{n-k} \left(\begin{array}{c} n \\ k \end{array} \right)$$

Note that if $AB \neq BA$ we has to stop at this point. However, since $AB = BA$, we can adopt binomial theorem

$$e^A e^B = \sum_{n=0}^{\infty} \frac{1}{n!} (A + B)^n = e^{A+B} = e^B e^A$$

- **Claim:** Let $t_1, t_2 \in \mathrm{R}$ then

$$e^{At_1} e^{At_2} = e^{At_2} e^{At_1} = e^{A(t_1+t_2)}$$

Proof: Let $A := At_1$ and $B := At_2$, obviously $(At_1)(At_2) = (At_2)(At_1)$ hence we can use the previous property, i.e.

$$e^{At_1} e^{At_2} = e^{(At_1)+(At_2)} = e^{A(t_1+t_2)} = e^{At_2} e^{At_1}$$

Now let $t_1 = t$ and $t_2 = -t$, then we have

$$e^{At} e^{-At} = e^{A(t-t)} = I \quad \rightarrow \quad \left(e^{At}\right)^{-1} = e^{-At}$$

- **Claim:** Let $P \in \mathrm{R}^{n \times n}$ and $\det(P) \neq 0$, then

$$e^{\left(P^{-1}AP\right)t} = P^{-1} e^{At} P$$

Proof: Let's firs show that $\left(P^{-1}AP\right)^k = P^{-1}A^kP$

$$\begin{aligned} \left(P^{-1}AP\right)^k &= \left(P^{-1}AP\right)\left(P^{-1}AP\right) \cdots \left(P^{-1}AP\right)\left(P^{-1}AP\right) \\ &= P^{-1}APP^{-1}APP^{-1} \cdots PP^{-1}APP^{-1}AP \\ &= P^{-1}AIAI \cdots IAIAP \\ &= P^{-1}A^kP \end{aligned}$$

Now let's expand

$$\begin{aligned} e^{\left(P^{-1}AP\right)t} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(P^{-1}AP\right)^k \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} P^{-1}A^kP \\ &= P^{-1} \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right) P \\ &= P^{-1} e^{At} P \end{aligned}$$

5.1.1 Computation of e^{At} and A^k

5.1.2 Computation via Solution of State-Space Equations and Frequency Domain Expressions

An LTI CT state-space representation of an autonomous system has the form

$$\dot{x}(t) = Ax(t) \text{ , where } x(t) \in \mathbb{R}^n$$

Let's test if $x(t) = e^{At}x_0$ is a solution of the homogeneous equation

$$\begin{aligned} x(0) &= e^{A\,0}x_0 = x_0 \\ \dot{x}(t) - Ax(t) &= \left(Ae^{At}\right)x_0 - Ae^{At}x_0 = 0 \end{aligned}$$

Now let's remember Laplace domain solution of the same equation

$$\begin{aligned} \mathcal{L}\left[\dot{x}(t)\right] &= \mathcal{L}\left[Ax(t)\right] \\ sX(s) - x(0) &= AX(s) \\ [sI - A]X(s) &= x(0) \\ X(s) &= [sI - A]^{-1}x_0 \end{aligned}$$

If we connect time and s-domain solutions we obtain

$$e^{At} = \mathcal{L}^{-1}\left[[sI - A]^{-1}\right]$$

Now let's focus on A^k. An LTI DT state-space representation of an autonomous system has the form

$$x[k+1] = Ax[k]$$

Unlike CT systems we can compute the response iteratively easily

$$\begin{aligned} x[1] &= Gx[0] \\ x[2] &= Gx[1] = G^2x[0] \\ x[3] &= Gx[2] = G^3x[0] \\ &\vdots \\ x[k] &= Gx[k-1] = G^kx[0] \end{aligned}$$

Now let's remember form of the response in Z-domain.

$$\begin{aligned} \mathcal{Z}\left[x[k+1]\right] &= \mathcal{Z}\left[Gx[k]\right] \\ zX(z) - zx[0] &= GX(z) \\ (zI - G)\,X(z) &= zX(z) \\ X(z) &= z\,(zI - G)^{-1}\,x[0] \end{aligned}$$

If we connect time and s-domain solutions we obtain

$$G^k = \mathcal{Z}^{-1}\{z\,(zI - G)^{-1}\}$$

5.1.3 Computation via Diagonalization

Theorem: $A \in \mathbb{C}^{n\times n}$ is diagonalizable, if and only if there exists a (nonsingular) similarity transformation, $V \in \mathbb{C}^{n\times n}$, such that $A = V^{-1}\Lambda V$ where Λ is a diagonal matrix,

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{bmatrix}$$

where $\lambda_i \in \mathbb{C}$'s are the eigenvalues of A, which are the roots of the characteristic equation $d(\lambda) = \det(\lambda I - A)$.

Now let's compute A^k and e^{At} for a diagonalizable matrix

$$A^k = \left(V^{-1}\Lambda V\right)^k = V^{-1}\Lambda^k V = V^{-1}\begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} V$$

$$e^{At} = e^{V^{-1}\Lambda V t} = V^{-1}e^{\Lambda t}V = V^{-1}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} V$$

A sufficient but not necessary condition that A will have n distinct eigenvalues in such a case characteristic equation will have the following form

$$d(\lambda) = (\lambda - \lambda_1)\cdots(\lambda - \lambda_n)\ ,\ \text{where } \lambda_i \neq \lambda_j\ ,\ \text{if } i \neq j$$

In this case, we also have the following properties associated with A

- A has n linearly independent eigenvectors
- For each λ_i there exists an eigenvector, v_i, such that $Av_i = \lambda v_i$ and $\text{Span}\{v_1, \cdots, v_n\} = \mathbb{C}^n$
- $\forall\ \lambda_i$, geometric multiplicity is equal to algebraic multiplicity and they are both equal to 1, i.e. $\text{GM}(\lambda_i) = \text{AM}(\lambda_i) = 1$
- Minimal polynomial is equal to the characteristic equation, $m(\lambda) = d(\lambda)$
- For each λ_i, $\mathcal{N}(\lambda_i I - A) = \text{Span}\{v_i\}$ and $\dim[\mathcal{N}(\lambda_i I - A)] = 1$

Let's remember some concepts from EE501, to better understand and generalize *diagonalizable* and *non-diagonalizable* square matrices.

Definition: Given a matrix $A \in \mathbb{C}^{n\times n}$, the characteristic polynomial $d(\lambda)$ is defined as

$$d(\lambda) = (\lambda - \lambda_1)^{r_1}\cdots(\lambda - \lambda_k)^{r_k}$$

$$k:\ \#\text{ distinct eigenvalues where } k \leq n$$

$$r_i = AM(\lambda_i):\ \#\text{ algebraic multiplicty } \lambda_i\ ,\ \text{where } n = \sum_{i=1}^{k} r_i$$

Theorem: Every $n \times n$ matrix satisfies its characteristic equation (Cayley-Hamilton Theorem)

$$d(\lambda) = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_k)^{r_k} = \lambda^n + d_{n-1}\lambda^{n-1} + \cdots + d_0$$
$$d(A) = A^n + d_{n-1}A^{n-1} + \cdots + d_1 A + d_0 I = 0_{n\times n}$$

Remark: Any power of a matrix $A \in \mathbb{C}^{n\times n}$ can be written as a linear combination of $\mathcal{A}_n = \{I\ ,\ A\ ,\ A^2\ ,\ \cdots\ ,\ A^{n-1}\}$.

Note that Cayley-Hamilton theorem does not guarantee that $\{I\ ,\ A\ ,\ A^2\ ,\ \cdots\ ,\ A^{n-1}\}$ are linearly independent.

Definition: For an $A \in \mathbb{R}^{n\times n}$, the minimal polynomial $m(\lambda)$ is the monic polynomial with the smallest degree such that $m(A) = 0_{n\times n}$

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$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \ \rightarrow\ d_1(\lambda) = (\lambda+1)^2\,, \text{Let } m_1(\lambda) = (\lambda+1)\ \rightarrow\ m_1(A) = A + I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{n\times n}\ \checkmark$$

$$A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \ \rightarrow\ d_2(\lambda) = (\lambda+1)^2\,, \text{Let } m_2(\lambda) = (\lambda+1)\ \rightarrow\ m_1(A) = A + I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0_{n\times n}\ X$$

$$\text{Let } m_2(\lambda) = (\lambda+1)^2\ \rightarrow\ m_1(A) = (A+I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{n\times n}\ \checkmark$$

Theorem: Given $A \in \mathbb{R}^{n\times n}$, let $m(\lambda)$ be its minimal polynomial

1. $m(\lambda)$ is unique
2. $m(\lambda)$ divides $d(\lambda)$ without any reminder. $\exists q(\lambda)$ such that $d(\lambda) = q(\lambda)m(\lambda)$
3. Each root of $d(\lambda)$ is a root of $m(\lambda)$, then

$$m(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k} = \lambda^l + c_{l-1}\lambda^{l-1} + \cdots + c_0$$
$$\text{where } 1 \le m_i \le r_i\ ,\ \ k \le l \le n\ ,\ \ m_1 + \cdots + m_k = l$$
$$q(\lambda) = (\lambda - \lambda_1)^{r_1 - m_1} \cdots (\lambda - \lambda_k)^{r_k - m_k}$$

Proof: of (3)

Since m(λ) is the minimal polynomial, we know that $m(A) = 0_{n\times n}$. Let (λ_i, v_i) is an eigenvalue, eigenvector pair of Asuch that $Av_i = \lambda_v i$, then

$$m(A) = 0_{n\times n}\ \ \rightarrow m(A)v_i = 0_{n\times n}$$
$$\left(A^l + c_{l-1}A^{l-1} + \cdots + c_1 A + c_0 I\right) v_i = 0_{n\times n}$$
$$\left(\lambda_i^l v_i + c_{l-1}\lambda_i^{l-1} v_i + \cdots + c_1\lambda_i v_i + c_0 v_i\right) = 0_{n\times n}$$
$$\lambda_i^l + c_{l-1}\lambda_i^{l-1} + \cdots + c_1\lambda_i + c_0 = 0$$
$$m(\lambda_i) = 0\ \checkmark$$

Corollary: Let l be the order of the minimal polynomial, then the elements of $\mathcal{A}_l = \{I\ ,\ A\ ,\ A^2\ ,\ \cdots\ ,\ A^{l-1}\}$ are linearly independent and higher order A^i's can be written as a linear combination of $\{I\ ,\ A\ ,\ A^2\ ,\ \cdots\ ,\ A^{l-1}\}$

Theorem: $\mathbb{C}^n = \bigoplus_{i=1}^{k} \mathcal{N}\left((\lambda_i I - A)^{m_i}\right)$ where $\text{Dim}\left[\mathcal{N}\left((\lambda_i I - A)^{m_i}\right)\right] = r_i\ ,\ \ \forall i \in \{1, 2, \ldots k\}$