ROB501 - Fundamentals & Emerging Topics in Robotics - Digital Control Systems

Lecture 4

Lecturer: Asst. Prof. M. Mert Ankarali

4.1 Stability of Discrete Time Control Systems

For an LTI discrete time dynamical system which can be represented with a rational transfer function, closed loop poles determine the stability characteristics of the system.

- If all poles of the system are located strictly inside the unit-circle then the system is **(asymptotically) stable**. Asymptotically stability systems are also **BIBO stable**.
- If there exist some *simple* (non-repeated) poles on the unit circle and all remaining poles are located inside the unit circle, then the system is **critically/marginally stable**. Note that critically/marginally stable systems are **BIBO unstable**.
- If there exist at least one repeated pole on the the unit circle, then the system is **unstable**, of course also **BIBO unstable**.
- If there exist at least one pole outside of the unit circle, then the system is **unstable**, of course also **BIBO unstable**.

4.1.1 Jury Stability Test

Jury stability test similar to the Routh-Hurwitz in CT systems, can define the stability of a DT system given the characteristic equation which is in the form

$$D(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

without loss of generality we will assume that $a_0 > 0$.

First Order: When $n = 1$, $D(z)$ takes the form

$$D(z) = a_0 z + a_1$$

DT System is stable if

$$|a_1| < a_0$$

Second Order: When $n = 2$, $D(z)$ takes the form

$$D(z) = a_0 z^2 + a_1 z + a_2$$

DT System is stable if

$$\begin{aligned} |a_2| &< a_0 \\ D(1) &> 0 \\ D(-1) &> 0 \end{aligned}$$

Third Order: When $n = 3$, $D(z)$ takes the form

$$D(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3$$

We need to construct the Jury table

Row	z^0	z^1	z^2	z^3
1	a_3	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3
3	b_2	b_1	b_0	

where

$$b_0 = \begin{vmatrix} a_3 & a_2 \\ a_0 & a_1 \end{vmatrix} \quad , \quad b_1 = \begin{vmatrix} a_3 & a_1 \\ a_0 & a_2 \end{vmatrix} \quad , \quad b_2 = \begin{vmatrix} a_3 & a_0 \\ a_0 & a_3 \end{vmatrix}$$

Then DT system is stable if

$$\begin{aligned} |\mathbf{a_3}| &< a_0 \\ D(1) &> 0 \\ \mathbf{-}D(-1) &> 0 \\ |b_2| &> |b_0| \end{aligned}$$

General Case: The jury table for systems with order n has $2n - 3$ rows and it has the form below

Row	z^0	z^1	z^2	$\cdots$	z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_2	a_1	a_0
2	a_0	a_1	a_2	$\cdots$	a_{n-2}	a_{n-1}	a_n
3	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_1	b_0	
4	b_0	b_1	b_2	$\cdots$	b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-3}	$\cdots$	c_0		
6	c_0	c_1	c_2	$\cdots$	c_{n-2}		
$\vdots$	$\vdots$						
2n - 3	q_2	q_1	q_0				

where

$$\begin{aligned} b_k &= \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix} \quad , \quad k \in \{0, 1, \cdots, n-1\} \\ c_k &= \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix} \quad , \quad k \in \{0, 1, \cdots, n-2\} \\ q_k &= \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix} \quad , \quad k \in \{0, 1, 3\} \end{aligned}$$

Then DT system is stable if

$$
\begin{aligned}
|a_n| &< a_0 \\
D(1) &> 0 \\
(-1)^n D(-1) &> 0 \\
|b_{n-1}| &> |b_0| \\
|c_{n-2}| &> |c_0| \\
&\cdots \\
|q_2| &> |q_0|
\end{aligned}
$$

Example: Using Jury test, find if the following characteristic equation is stable or not

$$G(z) = \frac{0.02z^{-1} + 0.03z^{-2} + 0.02z^{-3}}{1 - 3z^{-1} + 4z^{-2} - 2z^{-3} + 0.5z^{-4}}$$

Solution: This is a 4^{th} order system for which the characteristic equation is

$$
\begin{aligned}
D(z) &= a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 \\
&= 1z^4 + -3z^3 + 4z^2 + -2z + 0.5
\end{aligned}
$$

Jury table for a $n = 4$ system has the form

Row	z^0	z^1	z^2	z^3	z^4
1	a_4	a_3	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	a_4
3	b_3	b_2	b_1	b_0	
4	b_0	b_1	b_2	b_3	
5	c_2	c_1	c_0		

Before computing the whole Jury table let's check conditions one-by-one

- Check if $|a_4| < a_0$

$$0.5 < 1 \quad \text{OK}$$

- Check if $D(1) > 0$

$$D(1) = 1 - 3 + 4 - 2 + 0.5 = 0.5 > 0 \quad \text{OK}$$

- Check if $(-1)^4 D(-1) > 0$

$$D(-1) = 1 + 3 + 4 + 2 + 0.5 = 10.5 > 0 \quad \text{OK}$$

- Let's compute b_0 and b_3 and check if $|b_3| > |b_0|$

$$
\begin{aligned}
b_0 &= \begin{vmatrix} a_4 & a_3 \\ a_0 & a_1 \end{vmatrix} = \begin{vmatrix} 0.5 & -2 \\ 1 & -3 \end{vmatrix} = 0.5 \\
b_3 &= \begin{vmatrix} a_4 & a_0 \\ a_0 & a_4 \end{vmatrix} = \begin{vmatrix} 0.5 & 1 \\ 1 & 0.5 \end{vmatrix} = -0.75 \\
|b_3| &= 0.75 > 0.5 = |b_0| \quad \text{OK}
\end{aligned}
$$

- Let's compute b_1 and b_2

$$b_1 = \begin{vmatrix} a_4 & a_2 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 0.5 & 4 \\ 1 & 4 \end{vmatrix} = -2$$

$$b_2 = \begin{vmatrix} a_4 & a_1 \\ a_0 & a_3 \end{vmatrix} = \begin{vmatrix} 0.5 & -3 \\ 1 & -2 \end{vmatrix} = 2$$

- Let's compute c_0 and c_2 and check if $|c_2| > |c_0|$

$$c_0 = \begin{vmatrix} b_3 & b_2 \\ b_0 & b_1 \end{vmatrix} = \begin{vmatrix} -0.75 & 2 \\ 0.5 & -2 \end{vmatrix} = 0.5$$

$$c_2 = \begin{vmatrix} b_3 & b_0 \\ b_0 & b_3 \end{vmatrix} = \begin{vmatrix} -0.75 & 0.5 \\ 0.5 & -0.75 \end{vmatrix} = 0.3125$$

$$|c_2| = 0.3125 \not> 0.5 = |c_0| \quad \text{NOT OK}$$

Final Jury Table is also given below

Row	z^0	z^1	z^2	z^3	z^4
1	$a_4 = 0.5$	$a_3 = -2$	$a_2 = 4$	$a_1 = -3$	$a_0 = 1$
2	$a_0 = 1$	$a_1 = -3$	$a_2 = 4$	$a_3 = -2$	$a_4 = 0.5$
3	$b_3 = -0.75$	$b_2 = 2$	$b_1 = -2$	$b_0 = 0.5$	
4	$b_0 = 0.5$	$b_1 = -2$	$b_2 = 2$	$b_3 = -0.75$	
5	$c_2 = 0.3125$	c_1	$c_0 = 0.5$		