ROB501 - Fundamentals & Emerging Topics in Robotics - Digital Control Systems

Lecture 2

Lecturer: Asst. Prof. M. Mert Ankarali

Let's remember the idealized and simplified block-diagram structure a discrete-time control system (See Fig. 2.1)

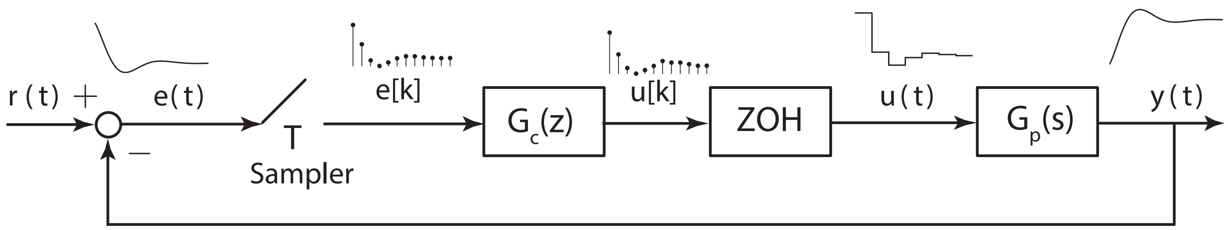


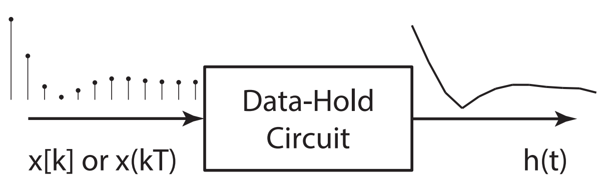
Figure 2.1: Block diagram of an LTI discrete-time control system

Loop contains both continuous-time and discrete-time signals and blocks.

- We can treat the system as a completely discrete-time system. We technically restrict ourselves into sampled time instants (in this course we will fundementally follow this path)
- Alternatively, we can use continuous time signals (as much as possible) and deal with starred versions of signals and starred Laplace transform (I sometimes follow this path in EE402 course).

2.1 Data Hold Operation

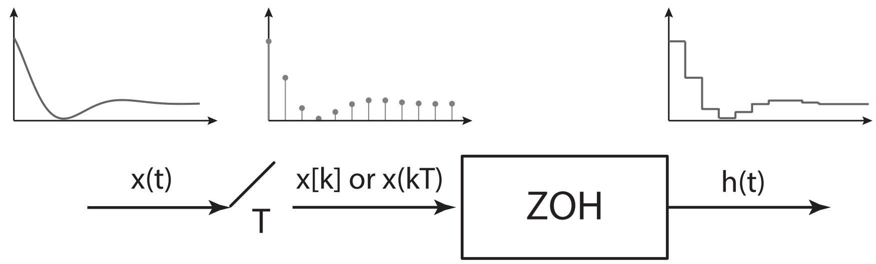
Data-Hold operation is an idealized model of a DAC device which converts a digital signal to an analog signal. In terms of the terminology used in this class, Data-Hold operation is the process of obtaining a CT signal $h(t)$ from a DT sequence. A general data-hold operation block circuit is shown below



Simplest and most dominantly used (I have never seen a practical usage of other hold operations) hold circuit/operation is the zero-order-hold (ZOH). Basically, at each time instant kT ZOH "samples" the input $x[k]$ or $x(kT)$ and "holds" this value at the output until the next sampling event. Mathematically,

$$h(kT + t) = x(kT) = x[k], \text{ for } 0 \leq t < T$$

The figure below illustrates a serios connection of an ideal CT-DT sampler and an ideal ZOH block.



Let's assume that $x(t)$ is a strictly causal signal, then from the definition of ZOH we can express $h(t)$ in terms of $x(t)$ (or $x[k]$, $x(kT)$) as

$$h(t) = x(0)\,[u(t) - u(t-T)] + x(T)\,[u(t-T) - u(t-2T)] + x(2T)\,[u(t-2T) - u(t-3T)] + \cdots$$

$$h(t) = \sum_{k=0}^{\infty} x(kT)\,[u(t-kT) - u(t-(k+1)T)]$$

where $u(t)$ is the unit-step function.

If we take the Laplace transform of $h(t)$, we obtain

$$\begin{aligned}
\mathcal{L}\{h(t)\} &= \sum_{k=0}^{\infty} x(kT)\mathcal{L}\left\{[u(t-kT) - u(t-(k+1)T)]\right\} \\
&= \sum_{k=0}^{\infty} x(kT)\left[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}\right] \\
H(s) &= \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x[k]e^{-kTs}
\end{aligned}$$

Z-transform & ZOH

When analyzing the discrete time control systems, we will (frequently) need to compute the Z-transform of sampled signals, for which the Laplace transform involves the term $\left(1-e^{-Ts}\right)$.

Let $\mathcal{L}\{x(t)\} = X(s) = \left(1-e^{-Ts}\right)G(s)$. Now let's analyze the z-transform of the sampled version of the signal, i.e. $X(z) = \mathcal{Z}\{x(kT)\}$. First let's find $x(t)$ from $X(z)$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\left(1-e^{-Ts}\right)G(s)\right\} = \mathcal{L}^{-1}\left\{G(s)\right\} - \mathcal{L}^{-1}\left\{e^{-Ts}G(s)\right\}$$

Let $g(t) = \mathcal{L}^{-1}\left\{G(s)\right\}$ then

$$x(t) = g(t) - g(t-T)$$

$x(kT)$ and $x[k]$ takes the form

$$\begin{aligned}
x(kT) &= g(kT) - g(kT-T) \\
x[k] &= \hat{g}[k] - \hat{g}[k-1]
\end{aligned}$$

Then $X(z)$ takes the form

$$X(z) = \left(1-z^{-1}\right)G(z)$$

where $G(z) = \mathcal{Z}\left\{\left[\mathcal{L}^{-1}\left\{G(s)\right\}\right]^*\right\}$ and $*$ is the sampling operation. In the textbook this notation is shortened to have $G(z) = \mathcal{Z}\left\{G(s)\right\}$. After that we have

$$X(z) = \left(1 - z^{-1}\right)\mathcal{Z}\left\{G(s)\right\}$$

Example 1. Obtain the z transform of $x(kT)$ where $T = 1$ and $X(s)$ is given as

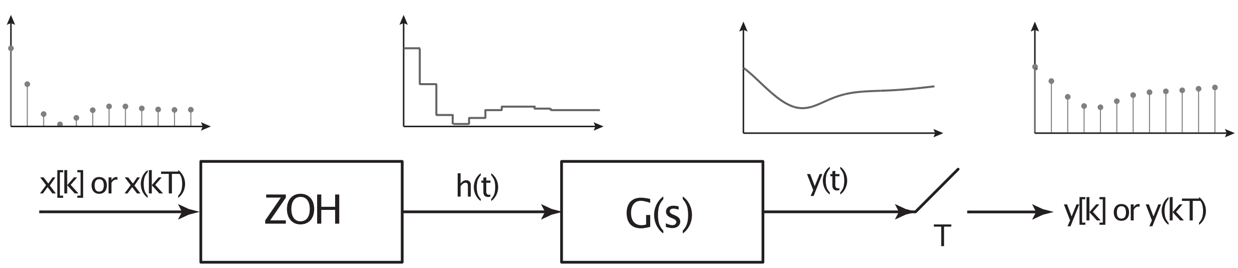
$$X(s) = \frac{1 - e^{-s}}{s}\frac{1}{s+1}$$

Solution:

$$\begin{aligned}
X(z) &= \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{1}{s(s+1)}\right\} = \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} \\
&= \frac{z-1}{z}\left(\frac{z}{z-1} - \frac{z}{z-e^{-1}}\right) = 1 - \frac{z-1}{z-e^{-1}} \\
X(z) &= \frac{1-e^{-1}}{z-e^{-1}}
\end{aligned}$$

2.2 Discretization of CT TF under ZOH and Ideal Sampling Operators

The figure below illustrates an open loop fundamental digital control control system that is composed of a CT plant, $G(s)$, a ZOH operator, and an ideal syncronous sampler. Our goal is to find a DT (z-domain) transfer function between the discrete-time input signal, $x[k]$ and the discrete-time output signal, $y[k]$, i.e. $fracY(z)X(z)$.



Let's first concentrate on input–output dynamics of $G(s)$

$$\begin{aligned}
Y(s) &= G(s)H(s) \\
&= G(s)\frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x[k]e^{-kTs} \\
&= \left(1-e^{-Ts}\right)\frac{G(s)}{s}\sum_{k=0}^{\infty} x[k]e^{-kTs} \\
&= \left(1-e^{-Ts}\right)\hat{G}(s)\sum_{k=0}^{\infty} x[k]e^{-kTs}
\end{aligned}$$

where $\hat{G}(s)$ is also the Laplace transform of the step-response of $G(s)$. Let $P(s) = \hat{G}(s)\sum_{k=0}^{\infty} x[k]e^{-kTs}$, and try to derive $P(z) = \mathcal{Z}\left\{P(s)\right\}$. First take the inverse Laplace transform of the expression

$$\begin{aligned} p(t) &= \mathcal{L}^{-1}\left\{\hat{G}(s)\sum_{k=0}^{\infty} x[k]e^{-kTs}\right\} = \sum_{k=0}^{\infty} x[k]\mathcal{L}^{-1}\left\{\hat{G}(s)e^{-kTs}\right\} \\ &= \sum_{k=0}^{\infty} x[k]\hat{g}(t-kT) \end{aligned}$$

If we limit ourselves to causal $g(t)$ case and sample $p(t)$, we will obtain

$$p(nT) = p[k] = \sum_{k=0}^{n} x[k]\hat{g}((n-k)\,T) = \sum_{k=0}^{n} x[k]\hat{g}[n-k]$$

Note that this is the expression of the discrete-time convolution, and thus we can infer the followings

$$\begin{aligned} p(nT) &= \hat{g}(nT) * x(nT) = x(nT) * \hat{g}(nT) \\ p[n] &= \hat{g}[n] * x[n] = x[n] * \hat{g}[n] \\ P(z) &= \hat{G}(z)X(z) \end{aligned}$$

If we use the derivation that we found previously regarding the Z-transform of sampled signals, for which the Laplace transform involves the term $\left(1 - e^{-Ts}\right)$, we can compute $Y(z)$ as

$$\begin{aligned} Y(z) &= \left(1 - z^{-1}\right)P(z) = \left(1 - z^{-1}\right)\hat{G}(z)X(z) \\ G_d(z) = \frac{Y(z)}{X(z)} &= \left(1 - z^{-1}\right)\hat{G}(z) \quad \text{where } \hat{G}(z) = \mathcal{Z}\left\{\frac{G(s)}{s}\right\} \end{aligned}$$

We call $G_d(z)$ as the discretized transfer function of $G(s)$ under ZOH and ideal sampling operators. The result is pretty interesting: the impulse response of the "discretized" system is obtained by sampling the step response function of original the continuous time system.