Lecture 14

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14.1 State Observer

Generally the full state measurment,$x(t)$ or $x[k]$, of a system is not accessible and *observers, estimators, filters*) have to be used to extract this information. The output, $y(t)$ or $y[k]$, represents the actual measurements which is a function of the input and the output. We will mainly focus on DT systems in this lecture however, all of the derivations and concepts are either same with the CT case or can be easily adopted with minor modifications. Let

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

A Luenberger observer is built using a "simulated" model of the system and the errors caused by the mismatched initial conditions $x_0 \neq \hat{x}_0$ (or other types of perturbations) are reduced by introducing output error feedback.

Let's assume that the state vector of the simulated system is $\hat{x}[k]$, then the state space equation of this synthetic system takes the form

$$\begin{aligned} \hat{x}[k+1] &= A\hat{x}[k] + Bu[k] \\ \hat{y}[k] &= C\hat{x}[k] + Du[k] \end{aligned}$$

Note that since $u[k]$ is the input that is supplied by the controller, we assume that it is known apriori. If $x[0] = \hat{x}[0]$ and when there is no model mismatch or uncertainty in the system then we expect that $x[k] = \hat{x}[k]$ and $y[k] = \hat{y}[k]$ for all k. When $x[0] \neq \hat{x}[0]$, then we should observe a difference between the measured and predicted output $y[k] \neq \hat{y}[k]$. The core idea in Luenberger observer is feeding the error in the output prediction $y[k] - \hat{y}[k]$ to the simulated system via a linear feedback gain.

$$\begin{aligned} \hat{x}[k+1] &= A\hat{x}[k] + Bu[k] + L\left(y[k] - \hat{y}[k]\right) \\ \hat{y}[k] &= C\hat{x}[k] + Du[k] \end{aligned}$$

In order to understand how a Luenberger observer works and to choose a proper observer gain L, we define an error signal $e[k] = x[k] - \hat{x}[k]$. The dynamics w.r.t $e[k]$ can be derived as

$$\begin{aligned} e[k+1] &= x[k+1] - \hat{x}[k+1] \\ &= \left(Ax[k] + Bu[k]\right) - \left(A\hat{x}[k] + Bu[k] + L\left(y[k] - \hat{y}[k]\right)\right) \\ e[k+1] &= \left(A - LC\right) e[k] \end{aligned}$$

where $e[0] = x[0] - \hat{x}[0]$ denotes the error in the initial condition.

If the matrix $(A - LC)$ is stable then the errors in initial condition will diminish eventually. Moreover, in order to have a good observer/estimator performance the observer convergence should be sufficiently fast.

Theorem: (Observer Eigenvalue Placement) Given (A, C), $\exists L$ s.t.

$$\det\left[\lambda I - (A - LC)\right] = \lambda^n + a_{n-1}^*\lambda^{n-1} + \cdots + a_1^*\lambda + a_0^*$$
$$\forall \mathcal{A} = \{a_0^*,\, a_1^* \cdots a_{n-1}^*\},\, a_i^* \in \mathbb{R}$$

if and only if (A, C) is observable.

Proof of necessity: Let's assume that (A, C) not observable and $\exists(\lambda_u, v_u)$ pair such that $Av_u = v_u\lambda_u$ and $Cv_u = 0$. Now check weather v_u is a right eigenvector of $(A - LC)$

$$(A - LC)v_u = Av_u - LCv_U = Av_u = \lambda_u v_U$$
$$Cv_u = 0$$

Here not only we showed that λ_u can not be moved hence it is not possible to locate the observer poles to arbitrary locations, we also showed that state-observer rule does not affect the observability.

Proof of sufficiency: Note that if (A, C) pair is observable then (A^T, C^T) pair is reachable. Then we know that $\exists L^T$ s.t.

$$\det\left[\lambda I - (A^T - C^T L^T)\right] = \lambda^n + a_{n-1}^*\lambda^{n-1} + \cdots + a_1^*\lambda + a_0^*$$
$$\forall \mathcal{A} = \{a_0^*,\, a_1^* \cdots a_{n-1}^*\},\, a_i^* \in \mathbb{R}$$

and obviously

$$\det\left[\lambda I - (A^T - C^T L^T)\right] = \det\left[\lambda I - (A - LC)\right]$$

which completes the proof.