EE302 - Feedback Systems **Spring 2019**

Lecture 11

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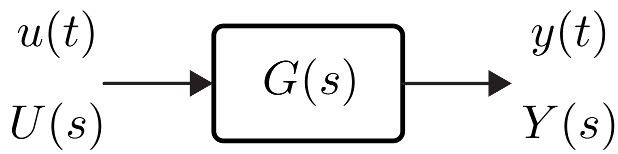
11.1 Stability

A SISO system is called BIBO (bounded-input–bounded-output) stable, if the output will be bounded for **every** input to the system that is bounded.



If we use the impulse response representation for an LTI SISO system, i.e.. $y(t) = \int_0^t g(t)u(t-\tau)d\tau$ (we assume that system is causal), the system is BIBO stable if and only if its impulse response is absolutely integrable.

$$\text{BIBO Stable} \;\Leftrightarrow\; \int_0^\infty |g(t)|dt < B < \infty$$

However, we genrally use transfer function representation in this course. Let

$$\frac{Y(s)}{U(s)} = G(s) = \frac{N(s)}{D(s)}$$

In this context, a rational transfer function representation is BIBO stable if and only if, the poles of $G(s)$ (or roots of $D(s)$) are strictly located in the open left half $s-$plane.

Ex: Show that $G(s) = \frac{1}{s}$ is not BIBO stable

Solution: Let's first check the impulse response condition

$$g(t) = 1,\; t \geq 0$$
$$\lim_{t\to\infty} \int_0^t |g(\tau)|d\tau = \lim_{t\to\infty} \int_0^t 1 d\tau = \infty$$

Thus the system is BIBO unstable. The system has a single pole at the origin, so we already know that it is BIBO unstable. Now, let's find a specific bounded input such that output is unbounded. Let $u(t)$ be the unit-step input then

$$Y(s) = \frac{1}{s^2} \;\Rightarrow\; y(t) = t\ , t \geq 0$$
$$\lim_{t\to\infty} |y(t)| = \infty$$

Obviously, we obtained an un-bounded output.

Ex: Let $G(s) = \frac{1}{s^2+1}$. Find a bounded input, such that output is unbounded. **Solution:** Let $u(t) = \cos t$, $t \geq 0$, then

$$Y(s) = G(s)U(s) = \frac{s}{(s^2+1)^2}$$
$$y(t) = t \sin t \ , t \geq> 0$$

Now we obtain an un-bounded output due to the resonance effect.

11.1.1 Stability of First & Second Order Systems

In order to gain some intuition about how to check stability of general rational LTI systems, we will analyze first and second order systems.

The transfer function of a first order system has the form

$$D(s) = a_0 s + a_1$$
$$a_0 > 0, \text{ w.l.g}$$

The single pole of the system and associated stability condition can be derived as

$$p = -\frac{a_1}{a_0}$$
$$\text{Stable} \Leftrightarrow a_1 > 0$$

Now lets analyze second order systems. Transfer function of a second order system has the following form

$$D(s) = a_0 s^2 + a_1 s + a_2$$
$$a_0 > 0, \text{ w.l.g}$$

If we carefully analyze Sign$[D(0]$, we can see that

- Sign$[D(0)] < 0$, one pole is located in the open-left half plane, where as the other one is located in the open- right half plane.
- Sign$[D(0)] = 0$, there exist at least one pole at the origin.

In this context, we can derive the first condition (necessary but not sufficient)

$$\text{Stable} \Rightarrow a_2 > 0$$

Under this condition, we can re-write the characteristic equation in a more standard form

$$D(s) = a_0 \left(s^2 + \frac{a_1}{a_0} s + \frac{a_2}{a_0} \right)$$
$$D(s) = a_0 \left(s^2 + 2\zeta\omega_n s + \omega_n^2 \right)$$
$$\text{where}$$
$$\omega_n = \sqrt{a_2/a_0} > 0$$
$$\zeta = \frac{a_1}{2a_0\omega_n}$$

From the analysis of second order system in standard form, we know that when

- $\zeta < 0$, system poles are located in the open right-half plane
- $\zeta = 0$, system poles are located on the imaginary axis
- $\zeta > 0$, system poles are located in the open left-half plane

It is also easy to see that $\zeta > 0 \Leftrightarrow a_1 > 0$. As a result, we can derive a necessary and sufficient condition on BIBO stability.

A second order system with $D(s) = a_o s^2 + a_1 s + a_2$ (with $a_0 > 0$), is BIBO stable if and only if $a_i > 0,\ \forall i \in \{1, 2\}$.

11.1.2 Routh's Stability Criterion

Routh's Stability Criterion (a.k.a RouthHurwitz stability criterion) is a mathematical test that is a necessary and sufficient condition for the stability of an LTI system. The Routh test is a very computationally efficient algorithm for the test of absolute LTI system stability.

In this test, one first constructs the routh table for a given

$$D(s) = a_o s^n + a_1 s^{n-1} + a_2 s^{n-3} + \cdots + a_{n-2} s^2 + a_{n-1} s^1 + a_n$$

which has $n+1$ rows.

s^n	a_0	a_2	a_4	a_6	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
s^{n-4}	d_1	d_2	d_3	d_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	e_1	e_2	0	$\cdots$	
s^1	f_1	0	0	$\cdots$	
s^0	g_0	0	0	$\cdots$	

where the coefficents b_i are computed with

$$\begin{aligned}
b_1 &= -\det\left(\begin{bmatrix} a_0 & a_2 \\ a_1 & a_3 \end{bmatrix}\right)/a_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \\
b_2 &= -\det\left(\begin{bmatrix} a_0 & a_4 \\ a_1 & a_5 \end{bmatrix}\right)/a_1 = \frac{a_1 a_4 - a_0 a_5}{a_1} \\
b_3 &= -\det\left(\begin{bmatrix} a_0 & a_6 \\ a_1 & a_7 \end{bmatrix}\right)/a_1 = \frac{a_1 a_6 - a_0 a_7}{a_1} \\
&\vdots
\end{aligned}$$

coefficents c_i are computed with

$$
\begin{aligned}
c_1 &= -\det\left(\begin{bmatrix} a_1 & b_1 \\ a_3 & b_2 \end{bmatrix}\right)/b_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \\
c_2 &= -\det\left(\begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}\right)/b_1 = \frac{b_1 a_5 - a_1 b_3}{b_1} \\
c_3 &= -\det\left(\begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}\right)/b_1 = \frac{b_1 a_7 - a_1 b_4}{b_1} \\
&\vdots
\end{aligned}
$$

Coefficents d_i are computed with

$$
\begin{aligned}
d_1 &= -\det\left(\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}\right)/c_1 = \frac{c_1 b_2 - b_1 b_2}{c_1} \\
d_2 &= -\det\left(\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix}\right)/c_1 = \frac{c_1 b_3 - b_1 c_3}{c_1} \\
d_3 &= -\det\left(\begin{bmatrix} b_1 & b_4 \\ c_1 & c_4 \end{bmatrix}\right)/c_1 = \frac{c_1 b_4 - b_1 c_4}{c_1} \\
&\vdots
\end{aligned}
$$

Other coefficients are computed with the same structure with the computation flow of d_i's.

Result 1: The system is NOT BIBO stable if $\exists i,\ s.t,\ a_i \leq 0$. In other words, since we assumed $a_0 > 0$, all other coefficients have to be strictly positive. This is a necessary, but not sufficient condition.

Result 2: The system is BIBO stable, if and only if all the coefficients in the first row strictly positive. Routh test provides a necessary and sufficient condition.

Result 3: The # of roots of D(s) with positive real parts is equal to the # of sign changes in the first column (shown in table below) of the Routh array.

a_0
a_1
b_1
c_1
d_1
$\vdots$
e_1
f_1
g_0

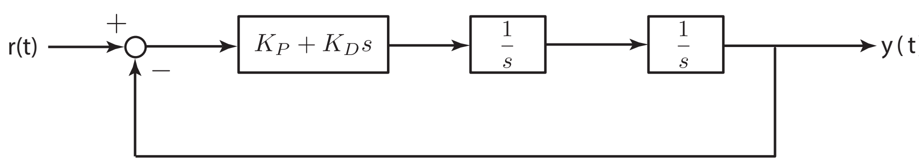
Ex: Let $D(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$, is this system is stable. If not what is the number of poles with positive real parts.

Solution: Let's build the Routh table

s^4	1	3	5	0
s^3	2	4	0	0
s^2	$\frac{2\cdot 3-1\cdot 4}{2}=1$	$\frac{2\cdot 5-1\cdot 0}{2}=5$	0	0
s^1	$\frac{1\cdot 4-2\cdot 5}{1}=-6$	0	0	0
s^0	5	0	0	0

sign changes is equal to 2, thus the system is unstable and 2 out of 4 poles are located in the open right-half plane. If we compute the poles numerically using a programming environment, we can find that $p_{1,2} = -1.29 \pm 0.86j$ and $p_{3,4} = 0.29 \pm 1.4j$, which indeed verifies the Routh table result.

Ex: Consider the following closed-loop system, find the set of K_P and K_D gains such that closed-loop system is stable

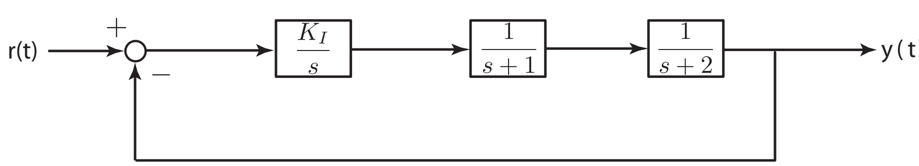


Solution: Denominator of the closed-loop transfer function of the system has the following form $D(s) = s^2 + K_D s + K_P$. Now, lets construct the Routh table for the $D(s)$

s^2	1	K_P
s^1	K_D	0
s^0	K_P	0

In order the closed-loop system to be stable, # sign changes in the first column must be equal to zero, thus the closed-loop system is BIBO stable if and only if $K_P > 0$ and $K_D > 0$.

Ex: Consider the following closed-loop system, find the set of K_I gains such that closed-loop system is stable



Solution: Denominator of the closed-loop transfer function of the system has the following form

$$D(s) = s^3 + 3s^2 + 2s + K_I$$

Now, lets construct the Routh table for the $D(s)$

s^3	1	2
s^2	3	K_I
s^1	$\frac{6-K_I}{3}$	0
s^0	K_I	0

In order the closed-loop system to be stable, # sign changes in the first column must be equal to zero, thus the closed-loop system is BIBO stable if and only if $K_I \in (0, 6)$.

11.1.2.1 Routh Hurwitz Stability Test: Special Cases

Case I: Let's analyze the stability of "$D(s) = s^3 - 3s + 2$". Since $a_1 = 0$ and $a_2 = -3$, we know that the system is not BIBO stable. Let's construct the Routh table to verify this result.

s^3	1	-3
s^2	0	2
s^1	?	
s^0	?	

We can see that one of the coefficients in the Routh array is zero, and we can not complete the Rout table. If a coefficient in the Routh array is zero, we know from the Routh Hurwitz test that the system is BIBO unstable. However, what can we do, if we want to compute the # poles with positive real parts.

Solution Type 1: Lets find a new $\bar{D}(s) = (s+\alpha)D(s),\ \alpha > 0$. We know that $D(s)$ and $\bar{D}(s)$ has the same # poles with positive real parts. Then we can construct a Routh table for $\bar{D}(s)$ to seek an answer.

Let $\bar{D}(s) = (s+2)(s^3 - 3s + 2) = s^4 + 2s^3 - 3s^2 - 4s + 4$, the Routh table takes the form

s^4	1	-3	4
s^3	2	-4	0
s^2	-1	4	0
s^1	4	0	
s^0	4	0	

sign changes in the Routh array of $\bar{D}(s)$ is equal to 2, so we can conclude that $D(s)$ has 2 poles with positive real parts. If we compute the poles of $D(s)$, we find that $p_1 = 2$, $p_{2,3} = 1$, which verifies our finding.

Solution Type 2: Replace 0 element with an infinitesimal but non-zero element $\epsilon > 0$.

s^3	1	-3
s^2	ϵ	2
s^1	$-\frac{3\epsilon+2}{\epsilon}$	0
s^0	2	

sign changes in the perturbed Routh array of $D(s)$ is equal to 2, so we can conclude that $D(s)$ has 2 poles with positive real parts.

Solution Type 3: Replace s with $1/q$

$$D(s)|_{s=1/q} = q^{-3} - 3q^{-1} + 2 = q^{-3}\left(2q^2 - 3q^2 + 1\right)$$

Now let's define $\bar{D}(q) = q^3 D(s)|_{s=1/q} = 2q^3 - 3q^2 + 1$. Note that we simply flip the coefficients of $D(s)$ to find the coefficients of $\bar{D}(q)$. It is easy to see that if p_i is a pole of $D(s)$, then $1/p_i$ is a pole of $\bar{D}(q)$. Let $p_i = \sigma + j\omega$, then

$$\begin{aligned}\frac{1}{p_i} &= \frac{1}{\sigma + j\omega} = \frac{\sigma - j\omega}{\sigma^2 + \omega^2} \\ &= \frac{\sigma}{\sigma^2 + \omega^2} - j\frac{\omega}{\sigma^2 + \omega^2} \\ \mathrm{Sign}(\mathrm{Re}\{p_i\}) &= \mathrm{Sign}\left(\mathrm{Re}\left\{\frac{1}{p_i}\right\}\right)\end{aligned}$$

We can see that $D(s)$ and $\bar{D}(q)$ have same number of stable and unstable poles. Thus, we can perform a Routh Hurwitz test on $\bar{D}(q)$

s^3	2	0
s^2	-3	1
s^1	2/3	0
s^0	1	

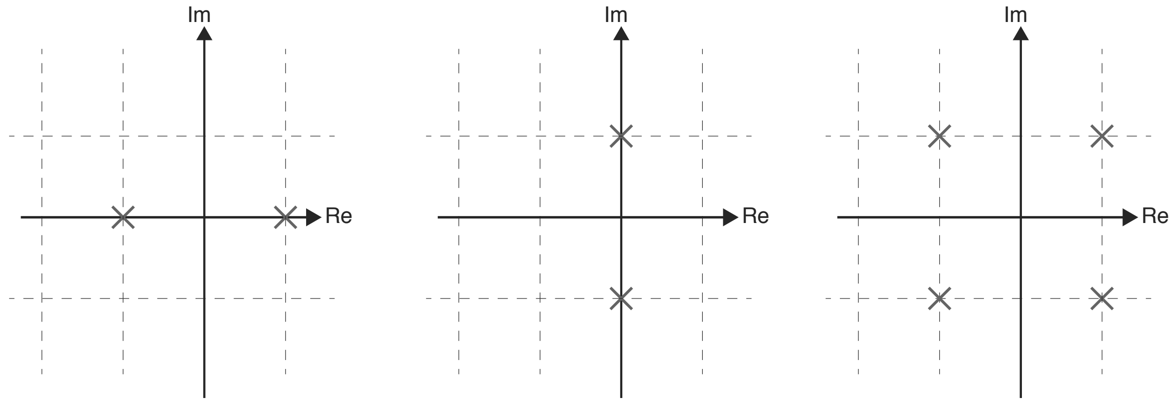
\# sign changes in the Routh array of $\bar{D}(q)$ is equal to 2, so we can conclude that $D(s)$ has 2 poles with positive real parts.

Case II:

Ex: Analyze the stability of $D(s) = s^4 + 2s^3 + 2s^2 + 2s + 1$ using Routh table

s^4	1	2	1
s^3	2	2	0
s^2	1	1	0
s^1	0	0	
s^0	?	?	

We can see that one row in Routh table is completely zero. Thus we know that system is indeed BIBO stable. However, what we can do in order to find number of unstable poles. This happens when there exists roots of equal magnitude located radially opposite in $s-$plane, i.e. symmetric w.r.t origin. These cases are illustrated in the figrue below.



This case happens always after an even row, in this example right after s^2 row. In such cases, we compute the Auxiliary polynomial, $A(s) = s^2 + 1$ in this example. After that, we compute its derivative, $A'(s) = 2s$ in this example, and use the coefficient of $A'(s)$ in replacement of the zero row and compute the Routh array accordingly. This process is illustrated below

s^4	1	2	1	
s^3	2	2	0	
s^2	1	1	0	$\rightarrow A(s) = s^2 + 1$
s^1	**2**	**0**		$\leftarrow A'(s) = 2$
s^0	1	0		

\# sign changes in the Routh array of $D(s)$ with Auxiliary polynomial is equal to 0, so we can conclude that $D(s)$ has 0 poles with positive real parts. This means that all unstable poles are located on the imaginary axis. If we compute the poles numerically we find that $p_{1,2} = -1$ and $p_{2,3} = -j$ which verifies our finding. Indeed, we can see that problematic roots are the roots of $A(s)$.

Ex: Compute # unstable poles of $D(s) = s^3 + 2s^2 - s - 2$ using Routh table

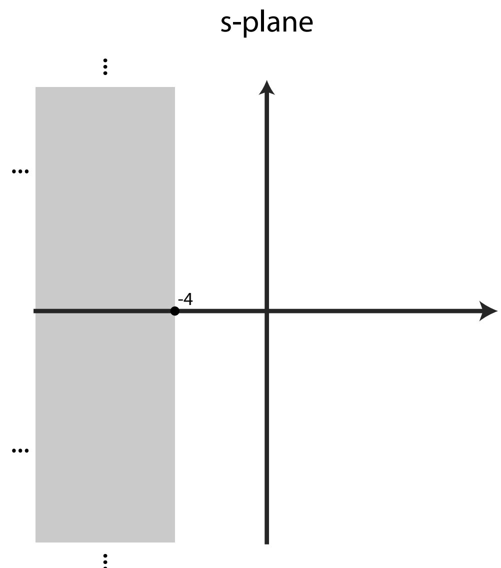
s^3	1	-1	
s^2	2	-2	$\rightarrow A(s) = 2s^2 - 2$
s^1	**2**	**0**	$\leftarrow A'(s) = 2s^2 - 2$
s^0	-2	0	

\# sign changes in the Routh array of $D(s)$ with Auxiliary polynomial is equal to 1, so we can conclude that $D(s)$ has 1 pole with positive real part. If we compute the poles numerically we find that $p_1 = -1$, $p_2 = -2$ and $p_3 = 1$ which verifies our finding. Indeed, we can see that problematic roots is one of the roots of $A(s)$.

This case occurs when there are roots of equal magnitude lying radially opposite in the s-plane.

11.1.2.2 Routh Hurwitz Stability Test: Relative Stability

Let's assume that we not only interested in the absolute stability of a system, but also we would like to test weather all poles are located in a region where the real part of the whole poles has an upper bound of -4. Similarly we can think that, we define a performance region based on a settling time requirement.



We can still use Routh Hurwitz test to compute the # poles for which real parts is lower than -4. However, we need to perform a change of variables.

Replace s with $z - 4$

$$D(s)|_{s=z-4} = D(z)$$

If we apply a Routh Hurwitz test on $D(z)$, we compute the # poles of $D(z)$ with positive real parts. Based on change of variables, we defined

$$\text{If } \mathrm{Re}\{z_i\} > 0 \Leftrightarrow \mathrm{Re}\{s_i\} > -4$$

where z_i and s_i are poles in assocated planes.

Ex: Let $D(s) = s^3 + 8s^2 + 19s + 12$, first test if the system is BIBO stable using Routh Hurwitz test. If the system is BIBO stable, then check if the real parts of all the poles are located in the region $\sigma \in (-\infty, -2)$.

Solution: First test the absolute stability using Routh test on $D(s)$

s^3	1	19
s^2	8	12
s^1	17.5	0
s^0	12	0

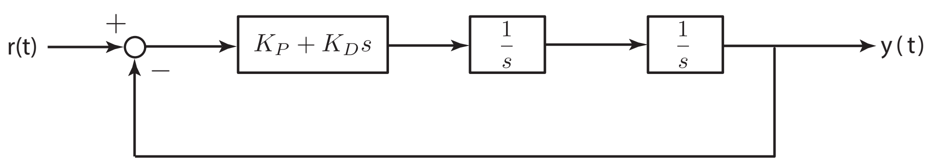
Since all of the coefficients of the Routh table are positive, the system is BIBO stable. Now check the relative stability by applying Routh test on "$D(z) = D(s)|_{s=z-2} = z^3 + 2z^2 - z - 2$",

s^3	1	-1	
s^2	2	-2	$\rightarrow A(z) = 2z^2 - 2$
s^1	4	0	$\leftarrow A'(z) = 4$
s^0	-2	0	

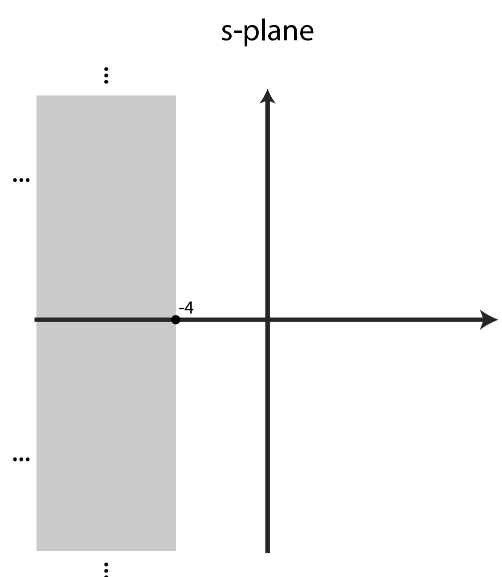
Since in the Routh table (with Auxiliary polynomial), there exist one sign change, $D(z)$ has one pole in the open right half z-plane. This means that $D(s)$ has a single pole where its real part is located in the $\sigma \in (-2, 0)$ region.

If we compute the roots of $D(s)$ numerically, we find that $p_1 = -4$, $p_2 = -3$, and $p_3 = -1$. This verifies that the system is BIBO stable but, only two of the poles are located in the $\sigma \in (-\infty, -2)$ region.

Ex: Consider the closed-loop system that we analyzed previously in terms of absolute stability.



Now let's fix $K_P = 20$, and find the set of K_D gains such that closed-loop poles are located in the desired (gray) region illustrated below.



Solution:

We can use Routh Hurwitz test on

$$\begin{aligned} D(z) = D(s)|_{s=z-4} &= (z-4)^2 + K_D(z-4) + 20 \\ &= z^2 + (K_D - 8)z + (36 - 4K_D) \end{aligned}$$

in order to solve this problem. In order to achieve the desired pole locations, $D(z)$ can not have any poles

s^2	1	$36-4K_D$
s^1	K_D-8	0
s^0	$36-4K_D$	0

with positive real parts, thus the coefficients of the Routh array has to be positive.

$$K_D > 8$$
$$K_D < 9$$

In conclusion, desired pole locations require that $K_D \in (8, 9)$.