Frequency Response Techniques in Control Systems

Let's assume $u(t)$, $y(t)$, and $G(t)$ represents the input, output, and transfer function representation of an input-output continuous time system.

In order to characterize frequency response of a dynamical system, the test signal is

$$u(t) = e^{j\omega t}$$

which is an artificial complex periodic signal with a frequency of ω. The Laplace transform of $u(t)$ takes the form

$$U(s) = \mathcal{L}\{e^{j\omega t}\} = \frac{1}{s - j\omega}$$

Response of the system in s-domain is given by

$$Y(s) = G(s)U(s) = G(s)\frac{1}{s - j\omega}$$

Assuming that $G(s)$ is a rational transfer function we can perform a partial fraction expansion

$$\begin{aligned} Y(s) &= \frac{a}{s - j\omega} + [\text{terms due to the poles of } G(s)] \\ a &= \lim_{s \to j\omega} [(s - j\omega)Y(s)] = G(j\omega) \\ Y(s) &= \frac{G(j\omega)}{s - j\omega} + [\text{terms due to the poles of } G(s)] \end{aligned}$$

Taking the inverse Laplace transform yields

$$y(t) = G(j\omega)e^{j\omega t} + \mathcal{L}^{-1}\left[\text{terms due to the poles of } G(s)\right]$$

If we assume that the system is "stable" or system is a part of closed loop system and closed loop behavior is stable then at steady state we have

$$\begin{aligned} y_{ss}(t) &= G(j\omega)e^{j\omega t} \\ &= |G(j\omega)|e^{i\omega t + \angle[G(j\omega)]} \\ &= Me^{i\omega t + \theta} \end{aligned}$$

In other words complex periodic signal is scaled and phase shifted based on the following operators

$$\begin{aligned} M &= |G(j\omega)| \\ \theta &= \angle G(j\omega) \end{aligned}$$

It is very easy to show that for a general real time domain signal $u(t) = \sin(\omega t + \phi)$, the output $y(t)$ at steady state is computed via

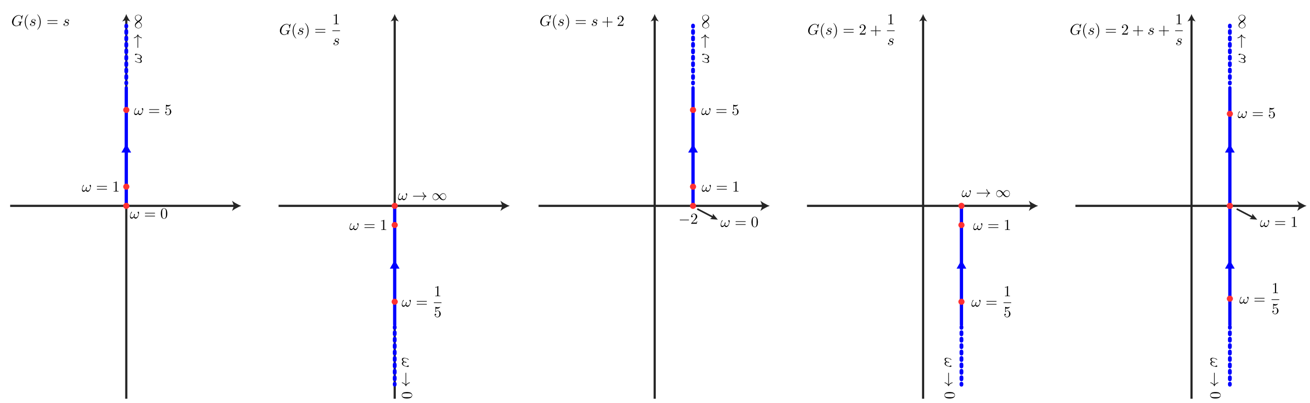
$$y_{ss}(t) = M\sin(\omega t + \phi + \theta)$$

2. Plotting Frequency Response: Polar Plot

We can consider the frequency response function $G(j\omega)$ as a mapping from positive $j\omega$ axis to a curve in the complex plane. In polar plot, we draw the frequency response function starting from $\omega = 0$ (or $\omega \to 0^+$) to $\omega \to \infty$.

Let's draw the polar plots of

$$G_1(s) = s \quad , \quad G_2(s) = \frac{1}{s} \quad , \quad G_3(s) = s + 2$$
$$G_4(s) = 2 + \frac{1}{s} \quad , \quad G_5(s) = 2 + s + \frac{1}{s}$$



Ex: Draw the polar plots of

$$G_1(s) = \frac{1}{s+1} \quad , \quad G_2(s) = \frac{s}{s+1}$$

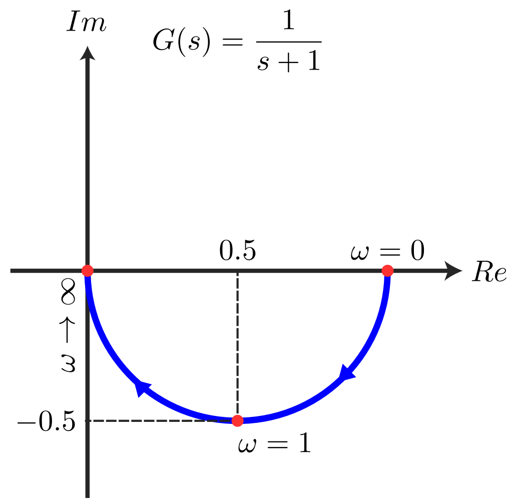
Let's analyze $G_1(j\omega)$ for $\omega \in [0, \infty)$

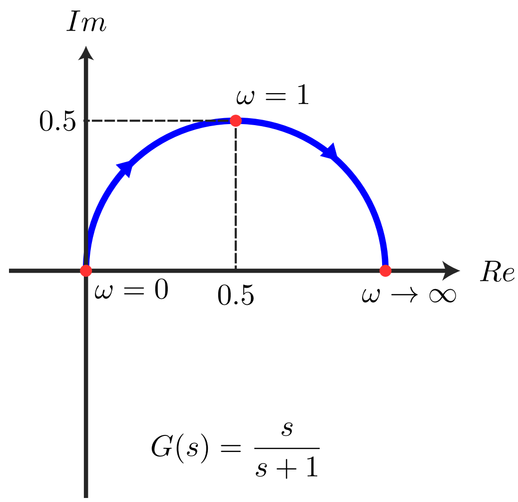
$$G_1(j\omega) = \frac{1}{j\omega + 1} = \frac{1 - j\omega}{\omega^2 + 1} = \frac{1}{\omega^2 + 1} - \frac{\omega}{\omega^2 + 1} j$$
$$|G_1(j\omega)| = \frac{1}{\sqrt{1 + \omega^2}}$$
$$\angle[G_1(j\omega)] = \arctan(-\omega)$$

Now let's analyze $G_2(j\omega)$ for $\omega \in [0, \infty)$

$$G_2(j\omega) = \frac{j\omega}{j\omega + 1} = \frac{j\omega + \omega^2}{\omega^2 + 1} = \frac{\omega^2}{\omega^2 + 1} + \frac{\omega}{\omega^2 + 1} j$$
$$|G_2(j\omega)| = \sqrt{\frac{\omega^2}{1 + \omega^2}}$$
$$\angle[G_2(j\omega)] = \arctan(1/\omega)$$

Polar plots of $G_1(s)$ and $G_2(s)$ are illustrated below.





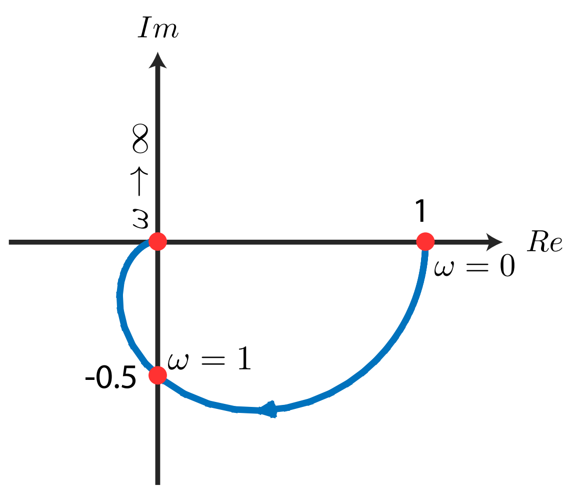
Ex: Draw the polar plot of $G(s) = \frac{1}{(s+1)^2}$

$$G(j\omega) = \frac{1}{(j\omega+1)^2} = \frac{(-j\omega+1)^2}{(\omega^2+1)^2}$$
$$= \left[\left(1-\omega^2\right) + j(-2\omega)\right] \frac{1}{(\omega^2+1)^2}$$

Some important points and associated features on the polar plot can be computed as

$$\omega \to 0 \ \Rightarrow G(j\omega) = 1$$
$$\omega \to 1 \ \Rightarrow G(j\omega) = -0.5j$$
$$\omega \to \infty \ \Rightarrow |G(j\omega)| \to 0 \quad \& \quad \angle[G(j\omega)] \to -\pi$$

Resultant polar plot is illustrated below



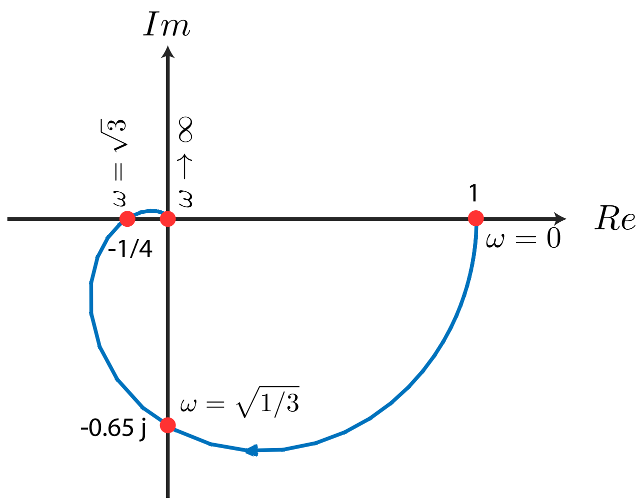
Ex: Draw the polar plot of $G(s) = \frac{1}{(s+1)^3}$

$$G(j\omega) = \frac{1}{(j\omega+1)^3} = \frac{(-j\omega+1)^3}{(\omega^2+1)^3}$$
$$= \left[\left(1-3\omega^2\right) + j(\omega^3 - 3\omega)\right] \frac{1}{(\omega^2+1)^3}$$

Some important points and associated features on the polar plot can be computed as

$$
\begin{aligned}
&\omega \to 0 \;\Rightarrow G(j\omega) = 1 \\
&\omega \to \sqrt{1/3} \;\Rightarrow G(j\omega) = -0.65j \\
&\omega \to \sqrt{3} \;\Rightarrow G(j\omega) = -1/4 \\
&\omega \to \infty \;\Rightarrow |G(j\omega)| \to 0 \quad \& \quad \angle[G(j\omega)] \to \pi/2
\end{aligned}
$$

Resultant polar plot is illustrated below

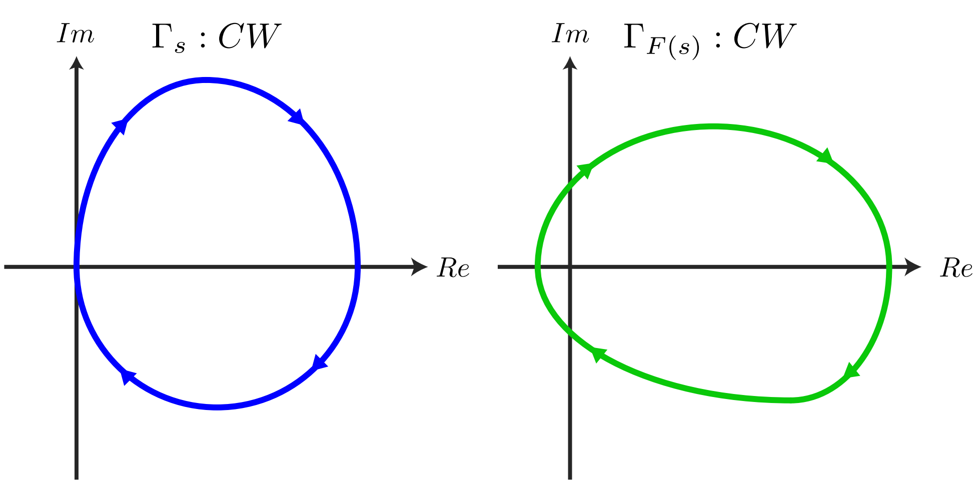


2. Nyquist Contour & Nyquist Plot

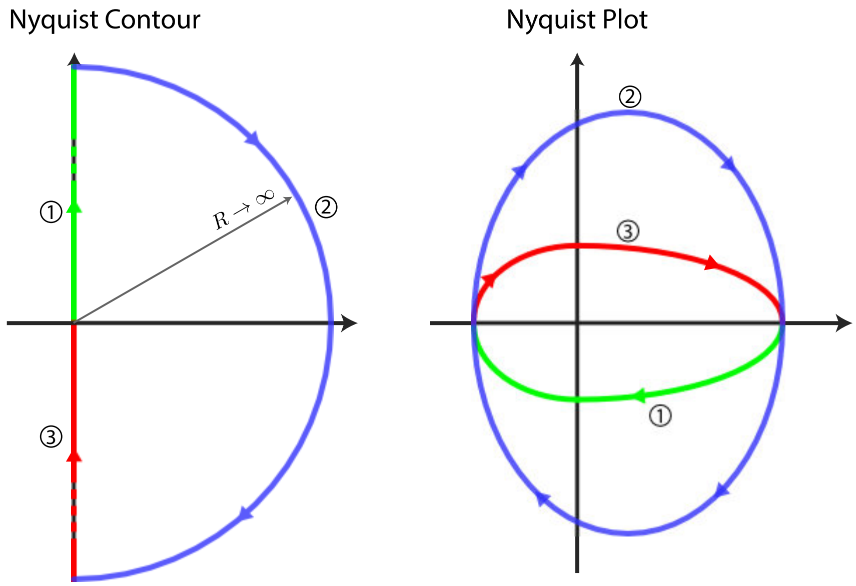
Nyquist plot is another tool that we use to to investigate the stability and robustness of a feedback system. The technique utilizes the frequency response characteristics of a system.

Definition: A contour Γ_s is a closed path with a direction in a complex plane.

Remark: A continuous function $F(s)$ maps a contour Γ_s in $s-$plane to another contour $\Gamma_{F(s)}$ in $F(s)$ plane. The figure below illustrates a clock-wise contour Γ_s and its map $\Gamma_{F(s)}$ which is also clock-wise in this example.



Let's consider an LTI transfer function $G(s) = \frac{N(s)}{D(s)}$ that has no zeros or poles on the imaginary axis. Nyquist contour/path, Γ_s, is defined in a way that it covers the whole open-right half plane. As illustrated in the Figure below, Nyquist contour is technically a half-circle for which the radius, $R \to \infty$. After that, one can draw the Nyquist plot, which is the mapped contour $\Gamma_{G(s)}$. Figure below illustrates a Nyquist contour and associated Nyquist plot.



Ex: Let's draw the Nyquist plot of $G(s) = \frac{1}{s+1}$

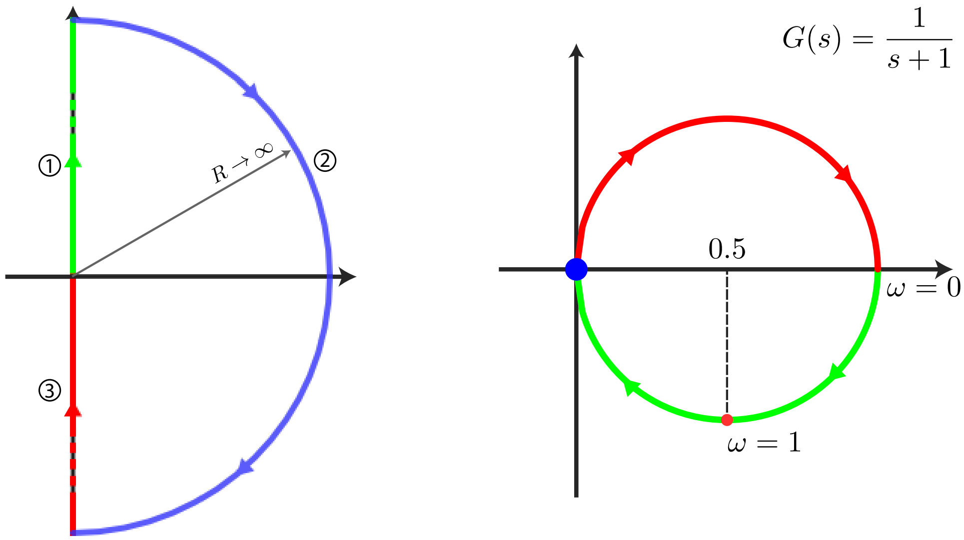
Solution: Based on the Nyquist contour we have three major paths. Now let's analyze the Nyquist paths

1. This part corresponds to the polar plot that we covered in the previously, We already plotted $G(j\omega)$, where $\omega : 0 \to \infty$.
2. This is the mapping of the infinite radius circular path on Nyquist contour. In this case if we write s in polar form, we get $s = Re^{j\theta}$ where $\theta : \pi/2 \to -\pi/2$. Then we can derive that

$$G\left(Re^{j\theta}\right) \approx \frac{1}{Re^{j\theta}} = \frac{e^{j(-\theta)}}{R}$$
$$\Rightarrow |G\left(Re^{j\theta}\right)| \approx 0$$

3. Last path (mapping of negative imaginary axis) is simply the conjugate of polar plot with reverse direction.

If we follow the procedure, we obtain the following Nyquist plot.



Ex: Let's draw the Nyquist plot of $G(s) = \frac{1}{(s+1)^2}$

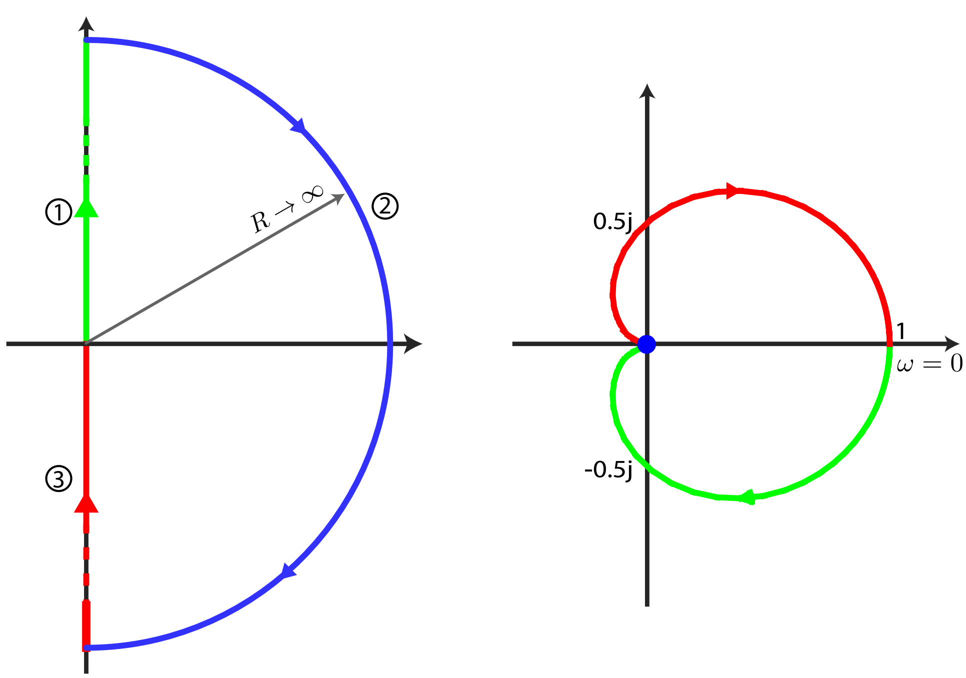
Solution: Let's analyze the Nyquist paths

1. Mapping of first path corresponds to the polar plot that we covered in the previously. We plot $G(j\omega)$, where $\omega : 0 \to \infty$.
2. This is the infinite radius circular path. In this case if we write s in polar form, we get $s = Re^{j\theta}$ where $\theta : \pi/2 \to -\pi/2$. Then we can derive that

$$G\left(Re^{j\theta}\right) =\approx \frac{1}{R^2 e^{j2\theta}} = \frac{e^{j(-2\theta)}}{R^2}$$
$$\Rightarrow |G\left(Re^{j\theta}\right)| \approx 0$$

3. Last path (mapping of negative imaginaty axis) is again the conjugate of polar plot with reverse direction.

If we follow the procedure, we obtain the following Nyquist plot.



Ex: Draw the Nyquist plot of $G(s) = \frac{1}{(s+1)^3}$

Solution: Firs let's analyze the Nyquist paths

1. This is the polar plot that we have covered in the previous lecture. We plotted $G(j\omega)$, where $\omega : 0 \rightarrow \infty$.
2. This is the mapping of the infinite radius circular path on Nyquist contour. In this case if we write s in polar form, we get $s = Re^{j\theta}$ and $\theta : \pi/2 \rightarrow -\pi/2$. Then we can derive that

$$G\left(Re^{j\theta}\right) \approx \frac{1}{R^3 e^{j3\theta}} = \frac{e^{j(-3\theta)}}{R^3}$$
$$\Rightarrow |G\left(Re^{j\theta}\right)| \approx 0$$

3. Last path (mapping of negative imaginaty axis) is again the conjugate of polar plot with reverse direction.

If we follow the procedure, we obtain the following Nyquist plot.