ROB501 - Fundamentals & Emerging Topics in Robotics - Digital Control Systems

Lecture 5

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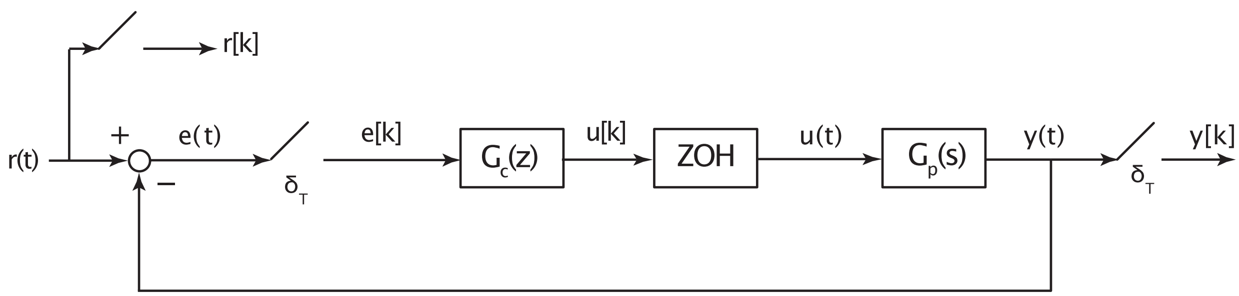
5.1 Steady-Sate (DC) Response Analysis

Let's remember the final value theorem. Given a discrete time signal $x[k]$ and its z-transform $X(z)$,if $X(z)$ has no poles outside the unit circle, then, final value theorem states that

$$\lim_{k\to\infty} x[k] = \lim_{z\to 1} \left[\left(1 - z^{-1}\right) X(z) \right]$$
$$x_{ss} = \lim_{z\to 1} \left[\frac{z-1}{z} X(z) \right]$$



Now let's find the pulse transfer function from the reference signal $r[k]$ to the error signal $e[k]$, to further analyze the steady-state error response.

$$E(z) = R(z) - E(z) \left(G_c(z) G(z)\right), \quad \text{where } G(z) = \mathcal{Z}\{G(s)\}$$
$$\frac{E(z)}{R(z)} = \frac{1}{1 + G_c(z)G(z)}$$

Note that $G_c(z)G(z)$ is the pulse transfer function from the error signal $E(z)$ to the signal which is fed to the negative terminal of the main difference operator, i.e. $F(z)$. This transfer function is called feed-forward or open-loop pulse transfer function of the closed-loop digital control system. For this system,

$$\frac{F(z)}{E(z)} = G_{OL} = G_c(z)G(z)$$

Then $E(z)$ can be written as

$$E(z) = R(z) \frac{1}{1 + G_{OL}(z)}$$

It is obvious that first requirement on m steady-state error performance is that closed-loop system have to be stable. Now let's analyze specific but fundamental input scenarios.

Unit-Step Input

We know that $r[k] = u[k]$ and $R(z) = \frac{1}{1-z^{-1}}$ then we have

$$\begin{aligned} e_{ss} &= \lim_{z\to 1}\left[\left(1-z^{-1}\right)R(z)\frac{1}{1+G_{OL}(z)}\right] \\ &= \lim_{z\to 1}\left[\left(1-z^{-1}\right)\frac{1}{1-z^{-1}}\frac{1}{1+G_{OL}(z)}\right] \\ e_{ss} &= \frac{1}{1+\lim_{z\to 1} G_{OL}(z)} \end{aligned}$$

If the DC gain of the system (also called static error constant) is constant, i.e. $G_{OL}(1) = K_{DC}$ then the steady state error can be computed as

$$e_{ss} = \frac{1}{1+K_{DC}}$$

It is obvious that

$$\begin{aligned} e_{ss} \neq 0 \quad &\text{if} \quad |K_{DC}| < \infty \\ e_{ss} \to 0 \quad &\text{if} \quad K_{DC} \to \infty \end{aligned}$$

Based on these results, we can have the following conclusions

- If $G_{OL}(1) = K_{DC}$, $0 < |K_{DC}| < \infty$, then $e_{ss} = 1/(1+K_{DC})$. These are **type 0** systems. We observe a bounded steady-state error and it is possible to reduce the by increasing the static gain constant K_P.
- If $G_{OL}(1) = \infty$, then $e_{ss} = 0$. These are **type *positive*** systems. The steady-state error is perfectly zero for such systems.

Now let's generalize the *type* of systems. An N *type* closed loop system has the following form of open-loop pulse transfer function

$$\begin{aligned} G_{OL}(z) &= \frac{1}{(z-1)^N} G_{DC}(z) \\ |G_{DC}(1)| &= K_{DC} \quad \text{where } 0 < |K_{DC}| < \infty \end{aligned}$$

It is easy to see that for unit-step response

- Type $N = 0$: $e_{ss} = 1/(1+K_{DC})$
- Type $N > 0$: $e_{ss} = 0$

Unit-Ramp Input

We know that $r[k] = ku[k]$ and $R(z) = \frac{z^{-1}}{(1-z^{-1})^2}$ then we have

$$\begin{aligned}
e_{ss} &= \lim_{z\to 1}\left[\left(1-z^{-1}\right)R(z)\frac{1}{1+G_{OL}(z)}\right] \\
&= \lim_{z\to 1}\left[\left(1-z^{-1}\right)\frac{z^{-1}}{(1-z^{-1})^2}\frac{1}{1+\frac{1}{(z-1)^N}G_{DC}(z)}\right] \\
&= \lim_{z\to 1}\left[\frac{1}{z-1}\frac{1}{1+\frac{1}{(z-1)^N}G_{DC}(z)}\right] \\
&= \lim_{z\to 1}\left[\frac{1}{(z-1)+\frac{1}{(z-1)^{N-1}}G_{DC}(z)}\right] \\
e_{ss} &= \frac{1}{\lim_{z\to 1}\left[\frac{1}{(z-1)^{N-1}}G_{DC}(z)\right]}
\end{aligned}$$

Based on this result we can have the following steady-state error conditions for the unit-ramp input based on the type condition of the system

- Type $N < 1$: $e_{ss} \to \infty$
- Type $N = 1$: $e_{ss} = \frac{1}{K_{DC}}$
- Type $N > 1$: $e_{ss} = 0$

Example 1: $G_{OL}(z) = K\frac{z}{z-0.5}$. Compute the steady-state error to unit-step, and unit-ramp inputs.

$$\begin{aligned}
G_{OL}(z) &= \frac{Kz}{z-0.5} \\
G_{DC}(1) &= 2K \quad \text{, Type 0}
\end{aligned}$$

Then the steady-state errors are computed as

- Unit-step: $e_{ss} = \frac{1}{1+2K}$
- Unit-ramp: $e_{ss} = \infty$

Example 2: $G_{OL}(z) = K\frac{z^2}{(z-1)(z-0.5)}$. Compute the steady-state erro to unit-step, unit-ramp, a and unit-quadratic inputs.

$$\begin{aligned}
G_{OL}(z) &= \frac{Kz^2}{(z-1)(z-0.5)} = \frac{1}{z-1}\frac{Kz^2}{z-0.5} \\
G_{DC}(1) &= 2K \quad \text{, Type 1}
\end{aligned}$$

Then the steady-state errors are computed as

- Unit-step: $e_{ss} = 0$
- Unit-ramp: $e_{ss} = \frac{1}{2K}$