Properties and Theorems of the Z-transform

Properties and Theorems of the Z-transform

Linearity

$$x[k] = \alpha f[k] + \beta g[k] \quad \rightarrow X(z) = \ ? \ , \forall \ \alpha, \ \beta, \ f[k], \ \& \ \ g[k]$$

Properties and Theorems of the Z-transform

Linearity

$$x[k] = \alpha f[k] + \beta g[k] \;\; \rightarrow X(z) = \alpha F(z) + \beta G(z) \;\; , \forall \;\alpha,\; \beta,\; f[k],\; \& \;\; g[k]$$

Multiplication by a^k

Multiplication by a^k

Let $\mathscr{Z}\{x[k]\} = X(z)$ & $y[k] = a^k x[k]$ where $a \in \mathbb{C}$ $\rightarrow$ $Y(z) = ?$

Multiplication by a^k

$$\mathscr{Z}\{a^k x[k]\} = \sum_{k=0}^{\infty} a^k x[k] z^{-k} = \sum_{k=0}^{\infty} x[k](z/a)^{-k}$$

$$Y(z) = X(z/a)$$

Complex translation theorem

Complex translation theorem

Let $\mathscr{Z}\{x(kT)\} = X(z)$ & $y(t) = e^{-\alpha t}x(t)$ where $\alpha \in \mathbb{C}$ $\;\rightarrow\;$ $Y(z) = \mathscr{Z}\{y(kT)\} = ?$

Complex translation theorem

Let $\mathscr{Z}\{x(kT)\} = X(z)$ & $y(t) = e^{-\alpha t}x(t)$ where $\alpha \in \mathbb{C}$ $\rightarrow$ $Y(z) = \mathscr{Z}\{y(kT)\} = ?$

$$Y(z) = \mathscr{Z}\{y(kT)\} = \mathscr{Z}\{e^{-aTk}x(kT)\} = X(e^{aT}z)$$

Shifting theorem

Let $x(t) = 0$ for $t < 0$ & $x[k] = 0$ for $k < 0$

Shifting theorem

$$\text{Let } x(t) = 0 \text{ for } t < 0 \ \& \ x[k] = 0 \text{ for } k < 0$$

Shifting right by N (Causal shifting)

$$\text{Let } y[k] = x[k - N] \text{ where } N \in \mathbb{Z}^{+} \ \rightarrow \ Y(z) = ?$$

Shifting right by N (Causal shifting)

Let $x(t) = 0$ for $t < 0$ & $x[k] = 0$ for $k < 0$

Let $y[k] = x[k - N]$ where $N \in \mathbb{Z}^{+}$ $\rightarrow$ $Y(z) = ?$

Shifting right by N (Causal shifting)

Let $x(t) = 0$ for $t < 0$ & $x[k] = 0$ for $k < 0$

Let $y[k] = x[k - N]$ where $N \in \mathbb{Z}^+ \;\rightarrow\; Y(z) = ?$

$$\mathcal{Z}\{y[k]\} = \sum_{k=0}^{\infty} y[k]z^{-k} = \sum_{k=0}^{\infty} x[k-N]z^{-k} = \sum_{k=N}^{\infty} x[k-N]z^{-k}$$

Let $k = m + N$

$$Y(z) = \sum_{m=0}^{\infty} x[m]z^{-(m+N)} = z^{-N}\sum_{m=0}^{\infty} x[m]z^{-m} = z^{-N}X(z)$$

Shifting left by N (Non-causal shifting) & Bilateral Z transform

$$\text{Let } x(t) = 0 \text{ for } t < 0 \ \& \ x[k] = 0 \text{ for } k < 0$$

$$\text{Let } y[k] = x[k+N] \text{ where } N \in \mathbb{Z}^{+} \ \rightarrow \ Y(z) = \sum_{k=-\infty}^{\infty} y[k]z^{-k} = ?$$

Shifting left by N (Non-causal shifting) & Bilateral Z transform

Let $x(t) = 0$ for $t < 0$ & $x[k] = 0$ for $k < 0$

Let $y[k] = x[k + N]$ where $N \in \mathbb{Z}^+ \rightarrow Y(z) = \sum_{k=-\infty}^{\infty} y[k] z^{-k} = ?$

Shifting left by N (Non-causal shifting) & Bilateral Z transform

$$\text{Let } x(t) = 0 \text{ for } t < 0 \text{ \& } x[k] = 0 \text{ for } k < 0$$

$$\text{Let } y[k] = x[k+N] \text{ where } N \in \mathbb{Z}^{+} \;\rightarrow\; Y(z) = \sum_{k=-\infty}^{\infty} y[k]z^{-k} = ?$$

$$\mathscr{Z}\{x[k+N]\} = \sum_{k=-\infty}^{\infty} x[k+N]z^{-k} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m-N)} = z^{N} \sum_{m=-\infty}^{\infty} x[m]z^{-m}$$

$$Y(z) = \mathscr{Z}\{x[k+N]\} = z^{N}X(z)$$

Shifting left by N (Non-causal shifting) & Unilateral Z transform

$$\text{Let } x(t) = 0 \text{ for } t < 0 \text{ \& } x[k] = 0 \text{ for } k < 0$$

$$\text{Let } y[k] = x[k+N] \text{ where } N \in \mathbb{Z}^+ \;\; \rightarrow \;\; Y(z) = \sum_{k=0}^{\infty} y[k] z^{-k} = \,?$$

Shifting left by N (Non-causal shifting) & Unilateral Z transform

$$\text{Let } x(t) = 0 \text{ for } t < 0 \text{ \& } x[k] = 0 \text{ for } k < 0$$

$$\text{Let } y[k] = x[k + N] \text{ where } N \in \mathbb{Z}^{+} \;\rightarrow\; Y(z) = \sum_{k=0}^{\infty} y[k] z^{-k} = \,?$$

Shifting left by N (Non-causal shifting) & Unilateral Z transform

$$\text{Let } x(t) = 0 \text{ for } t < 0 \ \& \ x[k] = 0 \text{ for } k < 0$$

$$\text{Let } y[k] = x[k+N] \text{ where } N \in \mathbb{Z}^{+} \ \rightarrow \ Y(z) = \sum_{k=0}^{\infty} y[k]z^{-k} = ?$$

$$\mathscr{Z}\{x[k+N]\} = \sum_{k=0}^{\infty} x[k+N]z^{-k} \qquad \text{Let } k = m - N$$

$$Y(z) = \sum_{m=N}^{\infty} x[m]z^{-(m-N)} = z^{N}\sum_{m=N}^{\infty} x[m]z^{-m} = z^{N}\left(\sum_{k=0}^{\infty} x[k]z^{-k} - \sum_{k=0}^{N-1} x[k]z^{-k}\right)$$

$$Y(z) = z^{N}\left(X(z) - \sum_{k=0}^{N-1} x[k]z^{-k}\right)$$

Shifting left by N (Non-causal shifting) & Unilateral Z transform

$$Y(z) = z^N \left(X(z) - \sum_{k=0}^{N-1} x[k] z^{-k} \right)$$

$$\mathcal{Z}\{x[k+1]\} = zX(z) - zx[0]$$

$$\mathcal{Z}\{x[k+2]\} = z^2X(z) - z^2x[0] - zx[1]$$

$$\vdots$$

Example

Input is Unit-step function: $u[k]$

Output: $y[k] = u[k-1]$

Compute the Z-transform of the output both directly and using the shifting property.

Example

Input is Unit-step function: $u[k]$

Output: $y[k] = u[k-1]$

Compute the Z-transform of the output both directly and using the shifting property.

$$\mathscr{Z}\{u[k-1]\} = \frac{z^{-1}}{1 - z^{-1}}$$

Example

$$\text{Let } y[k] = \sum_{n=0}^{k} x[k] \text{ where } k \in \mathbb{Z}^{+}$$

Compute the Z-transform of the output using the shifting property

Example

$$\text{Let } y[k] = \sum_{n=0}^{k} x[k] \text{ where } k \in \mathbb{Z}^{+}$$

Compute the Z-transform of the output using the shifting property

$$Y(z) = \frac{1}{1 - z^{-1}} X(z)$$

Initial Value Theorem

Initial Value Theorem

$$X(z) = \mathscr{Z}\{x[n]\} \quad \rightarrow \quad x[0] = \lim_{z \to \infty} X(z) \qquad \text{(if limit exists)}$$

Initial Value Theorem

$$X(z) = \mathscr{Z}\{x[n]\} \quad \rightarrow \quad x[0] = \lim_{z\to\infty} X(z) \qquad \text{(if limit exists)}$$

Proof:

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \left[\sum_{k=0}^{\infty} x(k) z^{-k} \right] = \lim_{z\to\infty} \left[x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots \right] = x(0)$$

Final Value Theorem

Final Value Theorem

$$\lim_{k \to \infty} x[k] = \lim_{z \to 1} (1 - z^{-1}) X(z) \qquad \text{(if limit exists)}$$

Final Value Theorem

Proof:

$$\mathscr{Z}\{x[k] - x[k-1]\} = \sum_{k=0}^{\infty} \left(x[k] - x[k-1]\right) z^{-k}$$

$$X(z) - X(z)z^{-1} = \left(x[0]\left(1 - z^{-1}\right) + x[1]\left(z^{-1} - z^{-2}\right) + x[2]\left(z^{-2} - z^{-3}\right) + x[3]\left(z^{-3} - z^{-4}\right) + \cdots\right) + \lim_{k\to\infty} x[k]z^{-k}$$

$$\lim_{z\to 1} X(z)\left(1 - z^{-1}\right) = (0 + 0 + \cdots) + \lim_{z\to 1}\lim_{k\to\infty} x[k]z^{-k}$$

$$\lim_{z\to 1} X(z)\left(1 - z^{-1}\right) = \lim_{k\to\infty} x[k]$$

Complex Differentiation Theorem

$$\frac{d}{dz}X(z) = ?$$

Complex Differentiation Theorem

$$\frac{d}{dz}X(z) = \frac{d}{dz}\left[\sum_{k=0}^{\infty} x[k]z^{-k}\right] = \sum_{k=0}^{\infty} x[k]\frac{d}{dz}z^{-k} = \sum_{k=0}^{\infty}(-k)x[k]z^{-k-1}$$

$$-z\frac{d}{dz}X(z) = \sum_{k=0}^{\infty} kx[k]z^{-k}$$

$$-z\frac{d}{dz}X(z) = \mathscr{Z}\{kx[k]\}$$

Complex Differentiation Theorem

Higher Order(s)

$$(-z)^m \frac{d}{dz^m} X(z) = \mathscr{Z}\{k^m x[k]\}$$

Example

Find the Z-transform of the unit ramp function, by applying the Complex Differentiation Theorem to the Z-transform of the unit step function

Example

$$\mathcal{Z}\{r[k]\} = \mathcal{Z}\{ku[k]\}$$

$$R(z) = (-z)\frac{d}{dz}U(z) = (-z)\frac{d}{dz}U(z) = (-z)\frac{d}{dz}\left(\frac{z}{z-1}\right) = (-z)\left(\frac{1}{z-1} - \frac{z}{(z-1)^2}\right)$$

$$= \frac{z^2}{(z-1)^2} - \frac{z}{z-1} = \frac{z^2 - z(z-1)}{(z-1)^2}$$

$$R(z) = \frac{z}{(z-1)^2}$$

Real Convolution Theorem

$$f[n] * g[n] = \sum_{k=0}^{n} f[n-k]g[k]$$

$$\mathscr{Z}\{f[n] * g[n]\} = F(z)G(z)$$

Real Convolution Theorem

$$\mathcal{Z}\{f[n]*g[n]\} = F(z)G(z) \qquad f[n]*g[n] = \sum_{k=0}^{n} f[n-k]g[k]$$

Proof:

$$\mathcal{Z}\{f[n]*g[n]\} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} f[n-k]g[k]\right] z^{-n}$$

Since $f[m] = 0$ for $m < 0$

$$\mathcal{Z}\{f[n]*g[n]\} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty} f[n-k]g[k]\right] z^{-n} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} f[n-k]g[k]z^{-n}$$

Let $n = m + k$

$$\mathcal{Z}\{f[n]*g[n]\} = \sum_{k=0}^{\infty}\sum_{m=-k}^{\infty} f[m]g[k]z^{-m}z^{-k} = \sum_{k=0}^{\infty} g[k]z^{-k}\sum_{m=0}^{\infty} f[m]z^{-m}$$

$$\mathcal{Z}\{f[n]*g[n]\} = F(z)G(z)$$