EE402 - Discrete Time Systems **Spring 2018**

Lecture 8

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Stability of Discrete Time Control Systems

For an LTI discrete time dynamical system which can be represented with a rational transfer function, closed loop poles determine the stability characteristics of the system.

- If all poles of the system are located strictly inside the unit-circle then the system is **(asymptotically) stable**. Asymptotically stability systems are also **BIBO stable**.
- If there exist some *simple* (non-repeated) poles on the unit circle and all remaining poles are located inside the unit circle, then the system is **critically/marginally stable**. Note that critically/marginally stable systems are **BIBO unstable**.
- If there exist at least one repeated pole on the the unit circle, then the system is **unstable**, of course also **BIBO unstable**.
- If there exist at least one pole outside of the unit circle, then the system is **unstable**, of course also **BIBO unstable**.

Jury Stability Test

Jury stability test similar to the Routh-Hurwitz in CT systems, can define the stability of a DT system given the characteristic equation which is in the form

$$D(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

without loss of generality we will assume that $a_0 > 0$.

First Order: When $n = 1$, $D(z)$ takes the form

$$D(z) = a_0 z + a_1$$

DT System is stable if

$$|a_1| < a_0$$

Second Order: When $n = 2$, $D(z)$ takes the form

$$D(z) = a_0 z^2 + a_1 z + a_2$$

DT System is stable if

$$\begin{aligned} |a_2| &< a_0 \\ D(1) &> 0 \\ D(-1) &> 0 \end{aligned}$$

Third Order: When $n = 3$, $D(z)$ takes the form

$$D(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3$$

We need to construct the Jury table

Row	z^0	z^1	z^2	z^3
1	a_3	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3
3	b_2	b_1	b_0	

where

$$b_0 = \begin{vmatrix} a_3 & a_2 \\ a_0 & a_1 \end{vmatrix} \quad , \quad b_1 = \begin{vmatrix} a_3 & a_1 \\ a_0 & a_2 \end{vmatrix} \quad , \quad b_2 = \begin{vmatrix} a_3 & a_0 \\ a_0 & a_3 \end{vmatrix}$$

Then DT system is stable if

$$\begin{aligned} |\mathbf{a_3}| &< a_0 \\ D(1) &> 0 \\ -D(-1) &> 0 \\ |b_2| &> |b_0| \end{aligned}$$

General Case: The jury table for systems with order n has $2n - 3$ rows and it has the from below

Row	z^0	z^1	z^2	$\cdots$	z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_2	a_1	a_0
2	a_0	a_1	a_2	$\cdots$	a_{n-2}	a_{n-1}	a_n
3	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_1	b_0	
4	b_0	b_1	b_2	$\cdots$	b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-3}	$\cdots$	c_0		
6	c_0	c_1	c_2	$\cdots$	c_{n-2}		
$\vdots$	$\vdots$						
2n - 3	q_2	q_1	q_0				

where

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix} \quad , \quad k \in \{0, 1, \cdots, n-1\}$$

$$c_k = \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix} \quad , \quad k \in \{0, 1, \cdots, n-2\}$$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix} \quad , \quad k \in \{0, 1, 3\}$$

Then DT system is stable if

$$\begin{aligned}|a_n| &< a_0\\ D(1) &> 0\\ (-1)^n D(-1) &> 0\\ |b_{n-1}| &> |b_0|\\ |c_{n-2}| &> |c_0|\\ &\dots\\ |q_2| &> |q_0|\end{aligned}$$

Example: Using Jury test, find if the following characteristic equation is stable or not

$$G(z) = \frac{0.02z^{-1} + 0.03z^{-2} + 0.02z^{-3}}{1 - 3z^{-1} + 4z^{-2} - 2z^{-3} + 0.5z^{-4}}$$

Solution: This is a 4^{th} order system for which the characteristic equation is

$$\begin{aligned}D(z) &= a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4\\ &= 1z^4 + -3z^3 + 4z^2 + -2z + 0.5\end{aligned}$$

Jury table for a $n = 4$ system has the form

Row	z^0	z^1	z^2	z^3	z^4
1	a_4	a_3	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	a_4
3	b_3	b_2	b_1	b_0	
4	b_0	b_1	b_2	b_3	
5	c_2	c_1	c_0		

Before computing the whole Jury table let's check conditions one-by-one

- Check if $|a_4| < a_0$

$$0.5 < 1 \quad \text{OK}$$

- Check if $D(1) > 0$

$$D(1) = 1 - 3 + 4 - 2 + 0.5 = 0.5 > 0 \quad \text{OK}$$

- Check if $(-1)^4 D(-1) > 0$

$$D(-1) = 1 + 3 + 4 + 2 + 0.5 = 10.5 > 0 \quad \text{OK}$$

- Let's compute b_0 and b_3 and check if $|b_3| > |b_0|$

$$\begin{aligned}b_0 &= \begin{vmatrix} a_4 & a_3 \\ a_0 & a_1 \end{vmatrix} = \begin{vmatrix} 0.5 & -2 \\ 1 & -3 \end{vmatrix} = 0.5\\ b_3 &= \begin{vmatrix} a_4 & a_0 \\ a_0 & a_4 \end{vmatrix} = \begin{vmatrix} 0.5 & 1 \\ 1 & 0.5 \end{vmatrix} = -0.75\\ |b_3| &= 0.75 > 0.5 = |b_0| \quad \text{OK}\end{aligned}$$

- Let's compute b_1 and b_2

$$b_1 = \begin{vmatrix} a_4 & a_2 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 0.5 & 4 \\ 1 & 4 \end{vmatrix} = -2$$
$$b_2 = \begin{vmatrix} a_4 & a_1 \\ a_0 & a_3 \end{vmatrix} = \begin{vmatrix} 0.5 & -3 \\ 1 & -2 \end{vmatrix} = 2$$

- Let's compute c_0 and c_2 and check if $|c_2| > |c_0|$

$$c_0 = \begin{vmatrix} b_3 & b_2 \\ b_0 & b_1 \end{vmatrix} = \begin{vmatrix} -0.75 & 2 \\ 0.5 & -2 \end{vmatrix} = 0.5$$
$$c_2 = \begin{vmatrix} b_3 & b_0 \\ b_0 & b_3 \end{vmatrix} = \begin{vmatrix} -0.75 & 0.5 \\ 0.5 & -0.75 \end{vmatrix} = 0.3125$$
$$|c_2| = 0.3125 \not> 0.5 = |c_0| \quad \text{NOT OK}$$

Final Jury Table is also given below

Row	z^0	z^1	z^2	z^3	z^4
1	$a_4 = 0.5$	$a_3 = -2$	$a_2 = 4$	$a_1 = -3$	$a_0 = 1$
2	$a_0 = 1$	$a_1 = -3$	$a_2 = 4$	$a_3 = -2$	$a_4 = 0.5$
3	$b_3 = -0.75$	$b_2 = 2$	$b_1 = -2$	$b_0 = 0.5$	
4	$b_0 = 0.5$	$b_1 = -2$	$b_2 = 2$	$b_3 = -0.75$	
5	$c_2 = 0.3125$	c_1	$c_0 = 0.5$		

Bilinear Transformation & Routh-Hurwitz Test

In Lecture 7, we showed that bilinear transformation has a 1-1 mapping between stable regions in z-plane and s-plane, as well as unstable regions in z-plane and s-plane. As a way of testing stability, we can transform the the characterstic polynomial using bilinear transformation, then we can apply Routh-Hurwitz test.

Routh-Hurwitz is simpler and easier then the Jury test, however amount of computation needed for transformation generally shadows the relative computational advantage of Routh-Hurwitz.

We know that the bilinear transformation has the form

$$z = \frac{1 + \frac{T}{2}\bar{s}}{1 - \frac{T}{2}\bar{s}}$$

Since we only consider the test of stability, for the sake of simplicity it is reasonable to assume that $T = 2$. Then, the transformation of a general $D(z)$ looks like

$$D(\bar{s}) = D(z)\big|_{z=\frac{1+\bar{s}}{1-\bar{s}}} = a_0 \left(\frac{1+\bar{s}}{1-\bar{s}}\right)^n + a_1 \left(\frac{1+\bar{s}}{1-\bar{s}}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{1+\bar{s}}{1-\bar{s}}\right) + a_n$$

Then clearing the fractions by multiplying both sides by $(1-\bar{s})^n$, we obtain

$$Q(\bar{s}) = b_0\bar{s}^n + b_1\bar{s}^{n-1} + \cdots + b_{n-1}\bar{s} + b_n$$

Testing the stability on $Q(s)$ using Routh-Hurwitz will yield the stability condition of the original DT system.

Example: Consider the following characterstic equation of a DT system

$$D(z) = (z-1) * (z-2) = z^2 - 3z + 2 \tag{8.1}$$

Test the stability (already known) using Bilinear Transformation and Routh-Hurwitz.

Solution:

$$\begin{aligned}
D(\bar{s}) &= D(z)|_{z=\frac{1+\bar{s}}{1-\bar{s}}} = \left(\frac{1+\bar{s}}{1-\bar{s}}\right)^2 - 3\left(\frac{1+\bar{s}}{1-\bar{s}}\right) + 2 \\
Q(\bar{s}) &= (1+\bar{s})^2 - 3(1+\bar{s})(1-\bar{s}) + 2\,(1-\bar{s})^2 \\
&= (1+2\bar{s}+\bar{s}^2) - 3(1-\bar{s}^2) + 2(1-2\bar{s}+\bar{s}^2) \\
&= 6\bar{s}^2 - 2\bar{s}
\end{aligned}$$

This artificial CT system is unstable since one coefficient is negative and one coefficient is equal to zero. It is clear from this example that just for testing stability Bilinear transformation is not very useful.