EE302 - Feedback Systems **Spring 2019**

Lecture 8

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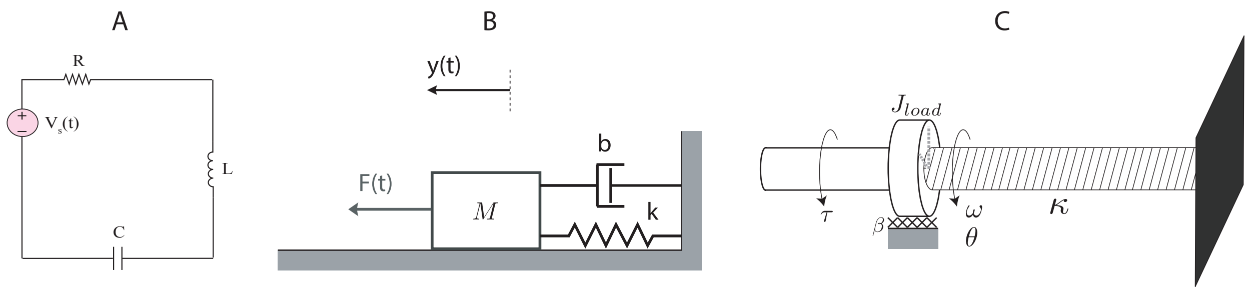
8.1 Second Order Systems

Let's derive the transfer functions for the following electrical and mechanical systems



$$G_A(s) = \frac{V_C(s)}{V_s(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$G_B(s) = \frac{Y(s)}{F(s)} = \frac{\frac{1}{M}}{s^2 + \frac{b}{M}s + \frac{k}{M}}$$

$$G_C(s) = \frac{\Theta(s)}{\mathcal{T}(s)} = \frac{\frac{1}{J}}{s^2 + \frac{\beta}{J}s + \frac{\kappa}{J}}$$

Most of the (passive) second order systems, can be put into the following the standard form

$$G(s) = K_{DC}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n}$$

where

$0 < \omega_n$	Undamped natural frequency
$0 < \zeta$	Damping ratio
K_{DC}	DC Gain

Accordingly for the systems that we analyzed previously we have the following relations

$$A: \quad \omega_n = \sqrt{\frac{1}{LC}} \ , \ \zeta = \frac{R}{2}\sqrt{\frac{C}{L}} \ , \ K_{DC} = 1$$

$$B: \quad \omega_n = \sqrt{\frac{k}{M}} \ , \ \zeta = \frac{b}{2}\sqrt{\frac{1}{Mk}} \ , \ K_{DC} = \frac{1}{k}$$

$$C: \quad \omega_n = \sqrt{\frac{\kappa}{J}} \ , \ \zeta = \frac{b}{2}\sqrt{\frac{1}{J\kappa}} \ , \ K_{DC} = \frac{1}{\kappa}$$

Definition: Given a transfer function, $G(s) = \frac{N(s)}{D(s)}$, where $N(s)$ and $D(s)$ are polynomials in s. Roots of $N(s)$ are called "zeros" of $G(s)$, and roots of $D(s)$ are called the "poles" of $G(s)$.

Behavior of the output $y(t)$ are majorly determined by the pole locations.

8.1.1 Step Response Types for the Second Order System in Standard Form

Given that $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, the poles can be computed as

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

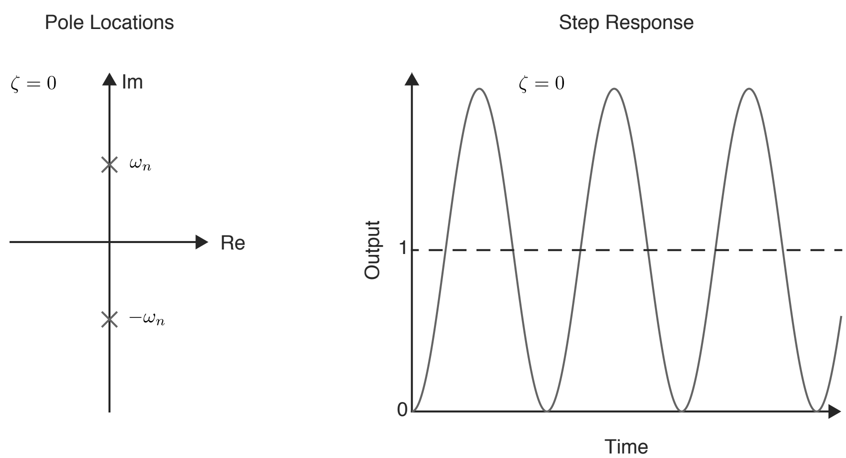
Case 1: When $\zeta = 0$, the system becomes undamped and $G(s)$ takes the form

$$G(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

We can compute the step-response as

$$\begin{aligned} Y(s) &= G(s)U(s) = \frac{1}{s}\frac{\omega_n^2}{s^2 + \omega_n^2} = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2} \\ y(t) &= 1 - \cos(\omega_n t) \quad \text{for } t > 0 \end{aligned}$$

Pole locations and step response when $\zeta = 0$ (undamped), is illustrated in the Figure below



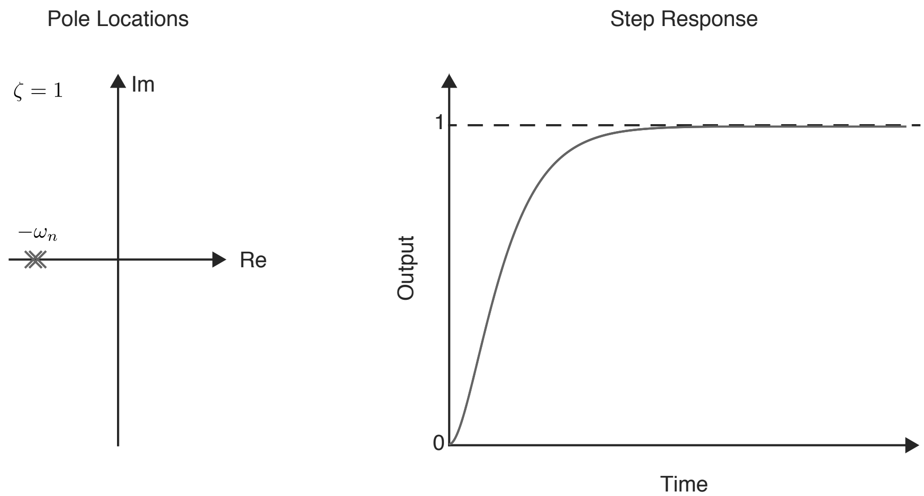
Case 2: When $\zeta = 1$, the system becomes "critically" damped and $G(s)$ takes the form

$$G(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n} = \frac{\omega_n^2}{(s + \omega_n)^2}$$

We can compute the step-response as

$$\begin{aligned} Y(s) &= G(s)U(s) = \frac{1}{s}\frac{\omega_n^2}{(s + \omega_n)^2} = \frac{1}{s} - \frac{1}{s + \omega_n} + \frac{\omega_n}{(s + \omega_n)^2} \\ y(t) &= 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t} \end{aligned}$$

Pole locations and step response when $\zeta = 1$ (critically damped), is illustrated in the Figure below



Case 3: When $\zeta > 1$, the system becomes over damped and there exist two real roots

$$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} > -\omega_n$$
$$p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} < -\omega_n$$

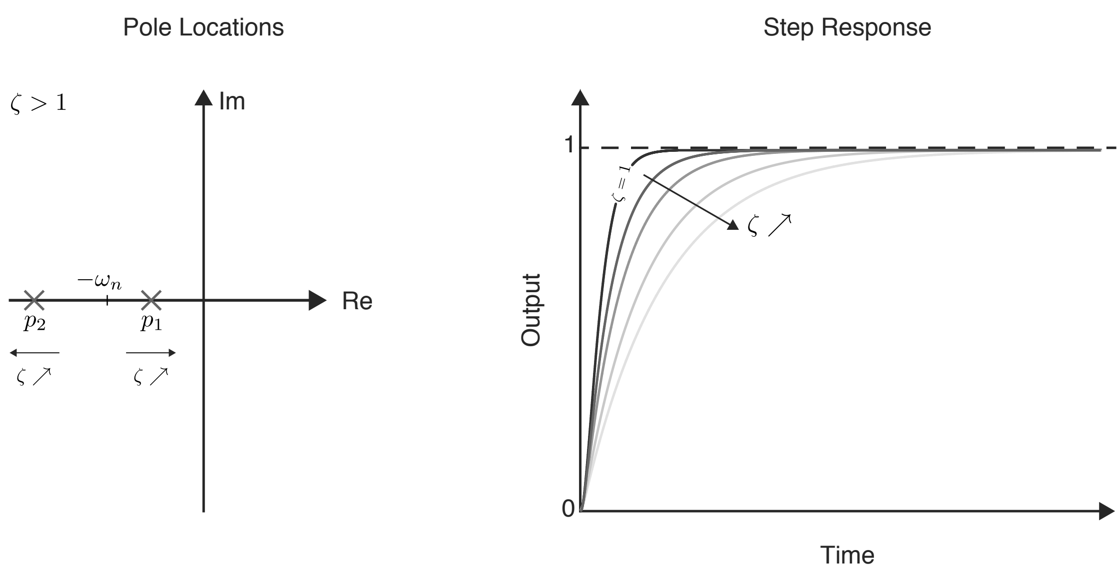
$G(s)$ can be written in terms of s_1 and s_2

$$G(s) = \frac{p_1 p_2}{(s + p_1)(s + p_2)}$$

where it is easy to see that $p_1 p_2 = \omega_n^2$. Finally, we can compute the step-response as

$$Y(s) = G(s)U(s) = \frac{p_1 p_2}{s(s - p_1)(s - p_2)}$$
$$y(t) = 1 + \frac{p_2}{p_1 - p_2} e^{p_1 t} - \frac{p_1}{p_1 - p_2} e^{p_2 t}$$

Pole locations and step response when $\zeta > 1$ (over damped), is illustrated in the Figure below



Case 3: When $0 < \zeta < 1$, the system becomes under damped and there exist two complex conjugate roots.

$$
\begin{aligned}
p_{1,2} &= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\
|p_{1,2}| &= \omega_n
\end{aligned}
$$

Let $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (which is called damped natural frequency), then we know that general solution of the ODE solution takes the form

$$
y(t) = y_p(t) + C_1 e^{-\sigma t}\cos(\omega_d t) + C_2 e^{-\sigma t}\sin(\omega_d t)
$$

Steady-state conditions leads that $y_p(t) = 1$. Then we can compute remaining coefficients from zero initial conditions constraints

$$
\begin{aligned}
y(0) = 0 \quad &\rightarrow \quad C_1 = -1 \\
\dot{y}(0) = 0 \quad &\rightarrow \quad \frac{d}{dt}\left[e^{-\sigma t}[-\cos(\omega_d t) + C_2\sin(\omega_d t)]\right]|_{t=0} = 0 \\
&\qquad \left[-\sigma e^{-\sigma t}[-\cos(\omega_d t) + C_2\sin(\omega_d t)] + e^{-\sigma t}[\omega_d\sin(\omega_d t) + C_2\omega_d\cos(\omega_d t)]\right]|_{t=0} = 0 \\
&\qquad [\sigma + C_2\omega_d] = 0 \\
&\qquad C_2 = -\frac{\sigma}{\omega_d} = -\frac{\zeta}{\sqrt{1-\zeta^2}}
\end{aligned}
$$

Finally output, $y(t)$, takes the form

$$
y(t) = 1 - e^{-\sigma t}\left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right]
$$

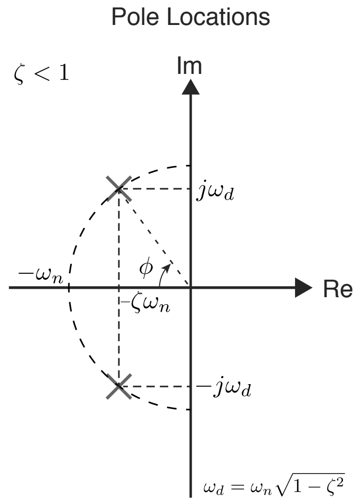
If we combine cos and sin terms into a single sin with phase shift we obtain

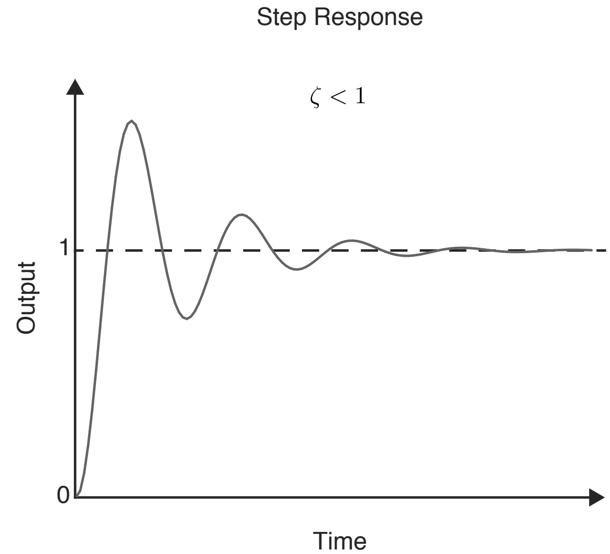
$$
\begin{aligned}
y(t) &= 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos(\omega_d t) + \zeta\sin(\omega_d t)\right] \\
&= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi) \quad t \geq 0
\end{aligned}
$$

where

$$
\begin{aligned}
\sin(\omega_d t + \phi) &= \sin(\phi)\cos(\omega_d t) + \cos(\phi)\sin(\omega_d t) \\
\cos(\phi) &= \zeta \\
\sin(\phi) &= \sqrt{1-\zeta^2} \\
\tan(\phi) &= \frac{\sqrt{1-\zeta^2}}{\zeta}
\end{aligned}
$$

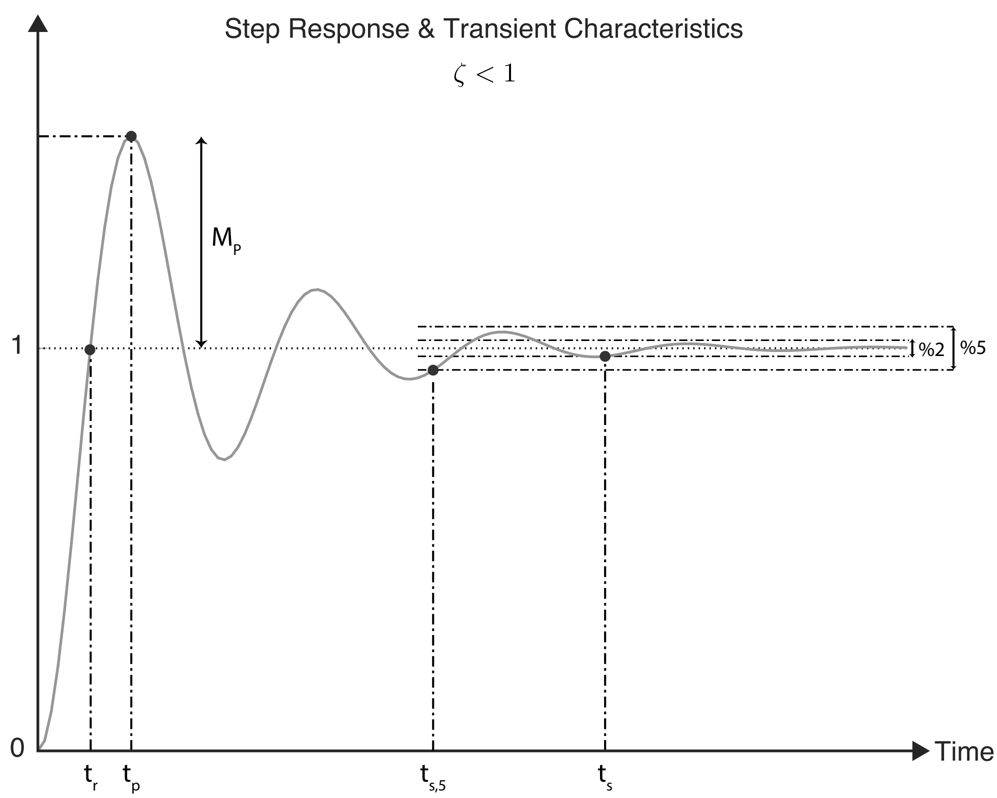
Pole locations and step response when $\zeta > 1$ (under damped), is illustrated in the Figure below





8.1.2 Transient Response Characteristics for Underdamped Second Order Systems in Standard Form

Important transient characteristics and performance metrics for 2^{nd} order underdamped systems are illustrated in the Figure below.



Rise Time (tr): The first time instant the response intersects the $y = 1$ line.

$$\begin{aligned}
y(t_r) &= 1 \\
1 &= 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \phi) \\
\pi &= \omega_d t_r + \phi \\
t_r &= \frac{\pi - \phi}{\omega_d}
\end{aligned}$$

Peak Time (t_p): The first time instant the response makes a peak

$$\begin{aligned}
&\left[\frac{dy}{dt}\right]_{t_p} = 0 \\
&\frac{d}{dt}\left[-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos(\omega_d t) + \zeta \sin(\omega_d t)\right]\right]_{t_p} = 0 \\
&\left[\zeta\omega_n e^{-\zeta\omega_n t}\left[\sqrt{1-\zeta^2}\cos(\omega_d t) + \zeta \sin(\omega_d t)\right] - e^{-\zeta\omega_n t}\left[-\omega_d\sqrt{1-\zeta^2}\sin(\omega_d t) + \zeta\omega_d \cos(\omega_d t)\right]\right]_{t_p} = 0 \\
&\left[\left[\zeta\sqrt{1-\zeta^2}\cos(\omega_d t) + \zeta^2 \sin(\omega_d t)\right] - \left[-(1-\zeta^2)\sin(\omega_d t) + \zeta\sqrt{1-\zeta^2}\cos(\omega_d t)\right]\right]_{t_p} = 0 \\
&\left[\left[\zeta^2 \sin(\omega_d t) + (1-\zeta^2)\sin(\omega_d t)\right]\right]_{t_p} = 0 \\
&\sin(\omega_d t_{t_p}) = 0 \\
&t_p = \frac{\pi}{\omega_d}
\end{aligned}$$

Maximum Overshoot (M_p): The maximum amount by which the response exceeds the value 1.

$$\begin{aligned}
M_p &= y(t_p) - 1 \\
&= \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos(\omega_d t) + \zeta \sin(\omega_d t)\right]\right]_{t_p} - 1 \\
&= \left[\frac{e^{-\zeta\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}}\sqrt{1-\zeta^2}(-1)\right] \\
M_p &= e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}} = e^{\frac{-\pi}{\tan\phi}}
\end{aligned}$$

Maximum Percentage Overshoot (MP_p) is simply calculated as $MP_p = M_p\,100$.

Settling Time (t_s): The earliest time instant such that $|y(t) - 1| \leq 0.02s$ or ($|y(t) - 1| \leq 0.05s$) for all $t \geq t_s$.Actual, settling time is very difficult to compute analytically (not so hard with numerical simulations). Thus we use following approximations.

$$\begin{aligned}
t_{s,5} &= \frac{3}{\zeta\omega_n} \quad \%5 \\
t_{s,2} &= \frac{4}{\zeta\omega_n} \quad \%2
\end{aligned}$$

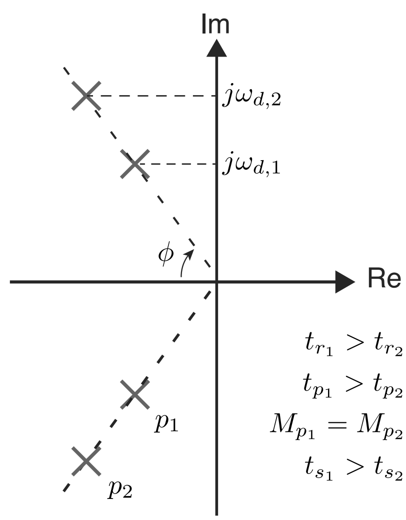
Example 1:

In this problem, we perform four different pole re-location cases. During re-locations we keep some parameters constant. Specifically

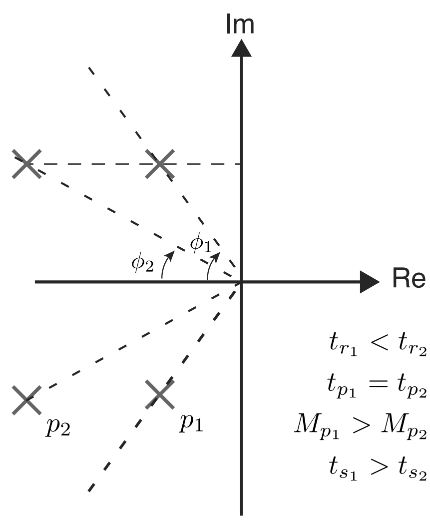
1. p_1 moved to a new location p_2 by keeping ζ (and ϕ) constant.
2. p_1 moved to a new location p_2 by keeping ω_d constant.
3. p_1 moved to a new location p_2 by keeping ω_n constant.
4. p_1 moved to a new location p_2 by keeping $\sigma = \zeta\omega_n$ constant.

For each four cases, explain what happens rise time, peak time, maximum overshoot, and settling time.

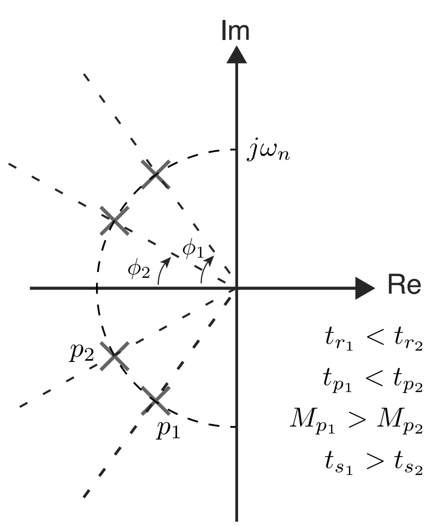
Constant ϕ, ζ



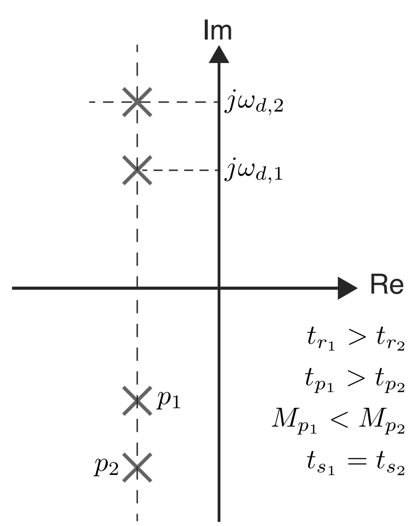
Constant ω_d



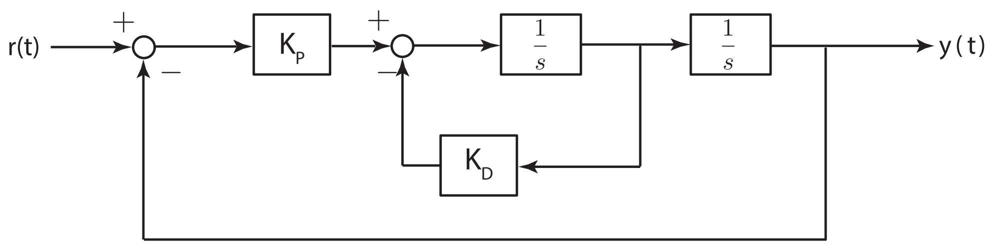
Constant ω_n



Constant $\sigma = \zeta\omega_n$



Example 2: Consider the following closed-loop system



Design K_P and K_D gains such that, maximum percent overshoot is less than %4.32, and settling time (%2) is less than 1 s.

Solution: Lets compute the closed-loop transfer function. In order to do that, first derive the transfer function from $E(s)$ to $Y(s)$ which is called feed-forward transfer function.

$$\frac{Y(s)}{E(s)} = K_P\left[\frac{1/s}{1+K_D/s}\right]\frac{1}{s} = \frac{K_P}{s(s+K_D)}$$

we can derive the $G(s)$ as

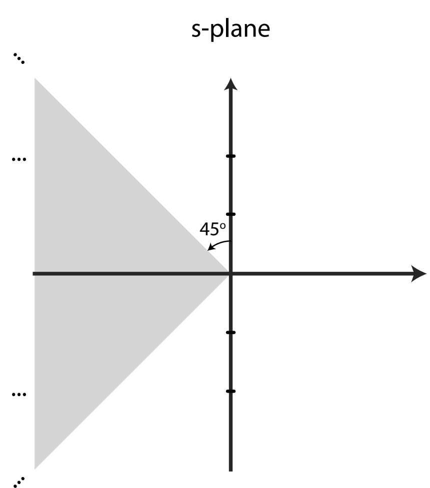
$$G(s) = \frac{\frac{K_P}{s(s+K_D)}}{1+\frac{K_P}{s(s+K_D)}} = \frac{K_P}{s^2+K_D s+K_P}$$

We can see that with K_P and K_D gains we have total control on the characteristic equation. Now let's analyze the performance requirements.

First requirement state that Maximum percent overshoot is less than %4.32, which means that $M_P < 0.0432$. Let's find a condition on ζ or ϕ,

$$\begin{aligned} M_P = e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}} = e^{-\pi/\tan\phi} &< 0.0432 \\ -\pi/\tan\phi &< -\pi \\ \tan\phi &< 1 \end{aligned}$$

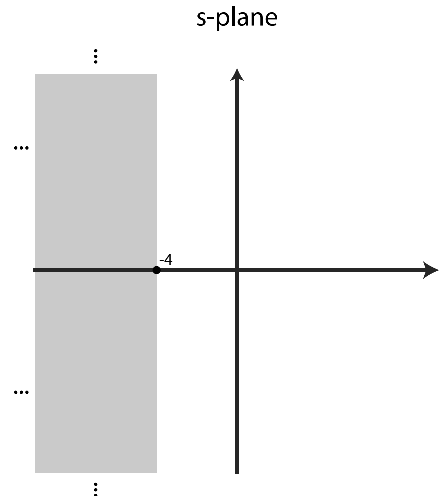
The region on s-plane that satisfy the M_P requirement is illustrated below.



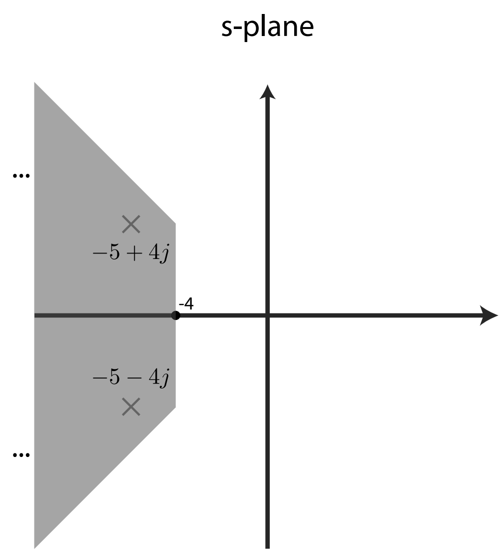
Second requirement state that settling time (%2) is less than 1 s, which implies

$$t_{s,2} = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} < 1$$
$$\sigma > 4$$

The region on s-plane that satisfy the settling time requirement is illustrated below.



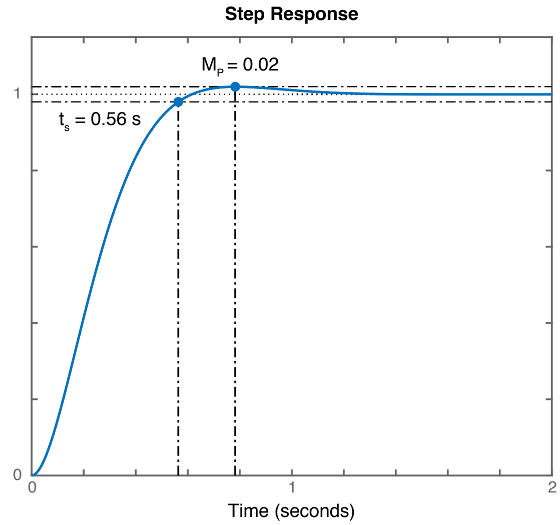
If we combine the requirements, we obtain the following region of possible pole locations.



Based on these requirements let's choose $p_{1,2} = -5 \pm 4j$ as desired pole locations. We can then compute the desired characteristic equation and then find the associated controller gains as

$$d^*(s) = (s + 5 + 4j)(s + 5 - 4j) = s^2 + 10s + 41$$
$$K_P = 41$$
$$K_D = 10$$

If we plot the step-response, we can illustrate the performance and check if we can meet the requirements.



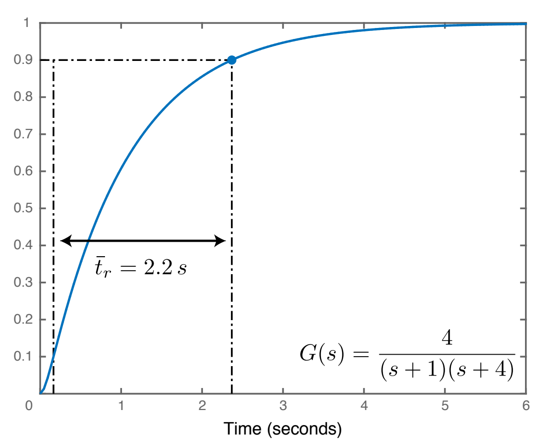
Transient Specifications for Over-damped Case

- Obviously, there is no over-shoot in over-damp case, thus $M_P = 0$.
- For settling time we can use the same approximate formula by considering the dominant/slowest pole, i.e.

$$
\begin{aligned}
t_{s,5} &= \frac{3}{\sigma_{min}} \quad \%5 \\
t_{s,2} &= \frac{4}{\sigma_{min}} \quad \%2
\end{aligned}
$$

- Since $y(t)$ only crosses the $y = 1$ as $t \to \infty$, t_r definition is not applicable for over-damped case. Instead, a different rise time definition can be used (which is applicable for both over-damped, under-damped, critically-damped systems, as well as first order systems). $\bar{t}_r$ is the time for $y(t)$ to go from 0.1 to 0.9. It is pretty hard to compute this time analytically, thus in general numerical and/or graphical methods are used.

  Rise-time concept is illustrated in the figure below, for an example system.



8.2 Higher Order Systems

In general, the poles closer to the $j\omega$ axis determine the behavior of the system. If the "distance" between the poles that are close to the $j\omega$ axis and other poles is high, then they dominate the behavior and we call them dominant poles. For example, figure given below illustrates a third order system, where we have two complex conjugate roots and one real root. In this case, the magnitude of the real pole is more than three times of the magnitude of the real part of the complex conjugate poles, thus the systems acts like a second order system.