Lecture 10

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10.0.1 Reachability & Controllability of DT-LTI Systems

For LTI a discrete time state-space representation

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

- A state x_r is said to be m-step **reachable**, if there exist an input sequence, $u[k], k \in \{0, 1, \cdots m-1\}$, that transfers the state vector $x[k]$ from the origin (i.e. $x[0] = 0$) to the state x_r in m number of steps, i.e. $x[m] = x_r$.
- A state x_d is said to be m-step **controllable**, if there exist an input sequence, $u[k], k \in \{0, 1, \cdots m-1\}$, that transfers the state vector $x[k]$ from the initial state x_c (i.e. $x[0] = x_c$) to the origin in m number of steps, i.e. $x[m] = 0$.

Note that

- the set $\mathcal{R}_m$ of all m-step reachable states is a linear (sub)space: $\mathcal{R}_m \subset \mathbb{R}^n$
- the set $\mathcal{C}_m$ of all m-step controllable states is a linear (sub)space: $\mathcal{C}_m \subset \mathbb{R}^n$

Let's characterize $\mathcal{R}_m$ and then try to generalize the reachability concept. When $x[0] = 0$, the solution of $x[m]$ is given by

$$x[m] = \left[\begin{array}{c|c|c|c|c} A^{m-1}B & A^{m-2}B & \cdots & AB & B \end{array}\right] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[m-2] \\ u[m-1] \end{bmatrix}$$

Let

$$\mathbf{R}_m = \left[\begin{array}{c|c|c|c|c} A^{m-1}B & A^{m-2}B & \cdots & AB & B \end{array}\right]$$
$$\mathbf{U}_m = \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[m-2] \\ u[m-1] \end{bmatrix}$$

then if a state x_r is reachable at k steps, it should satisfy the following equation for some $\mathbf{U}_m$.

$$\mathbf{M}_m\mathbf{U}_m = x_m$$

In order this matrix equation to have a solution x_r should be in the range space of $\mathbf{M}_m$.

$$x_r \in \mathrm{Ra}(\mathbf{M}_m)$$

Thus m-step reachable sub-space is simply equal to range space of $\mathcal{R}_k$

$$\mathrm{Ra}(\mathbf{R}_m) = \mathcal{R}_m$$

Theorem: For $k < n < l$

$$\mathcal{R}_k \subset \mathcal{R}_n = \mathcal{R}_l$$
$$\mathrm{Ra}(\mathbf{R}_k) \subset \mathrm{Ra}(\mathbf{R}_n) = \mathrm{Ra}(\mathbf{R}_l)$$

Proof: It is fairly easy to observe that

$$\mathcal{R}_i \subset \mathcal{R}_{i+1}$$
$$\mathrm{Ra}(\mathbf{R}_i) \subset \mathrm{Ra}(\mathbf{R}_{i+1})$$

since we add a new column (or columns for multi-input systems) to $\mathbf{R}_i$, thus it can only increase the dimension of the range-space. Thus we can conclude that

$$\mathcal{R}_k \subset \mathcal{R}_n \subset \mathcal{R}_l$$
$$\mathrm{Ra}(\mathbf{R}_k) \subset \mathrm{Ra}(\mathbf{R}_n) \subset \mathrm{Ra}(\mathbf{R}_l)$$

In order prove $\mathcal{R}_n = \mathcal{R}_l$, we simply use the Cayley-Hamilton theorem. Based on Cayley-Hamilton theorem

$$A^n = -a_1A^{n-1} - \cdots - a_{n-1}A - a_nI$$
$$A^nB = -a_1A^{n-1}B - \cdots - a_{n-1}AB - a_nB$$

which shows that A^nB is linearly dependent to previous columns and thus

$$\mathcal{R}_n = \mathcal{R}_l$$
$$\mathrm{Ra}(\mathbf{R}_n) = \mathrm{Ra}(\mathbf{R}_l)$$

This theorem shows that if x_r is reachable in n steps then it is reachable for $l > n$ steps, similarly if it is not reachable in n steps then it is reachable for $l > n$ steps. In this context, the sub-space of states reachable in n-steps, $\mathcal{R}_n$ is referred as the reachable subspace of (A, N), and will be denoted simply by $\mathcal{R}$ and $\mathbf{R} = \mathbf{R}_k$ will be system wide the reachability matrix. The system is termed a (fully) reachable system if

$$\mathrm{rank}(\mathbf{R}) = n$$
$$\mathrm{Ra}(\mathbf{R}) = \mathcal{R} = \mathbb{R}^n$$