EE402 - Discrete Time Systems | **Spring 2018**

Lecture 10

Lecturer: Asst. Prof. M. Mert Ankarali

10.1 Reachability & Controllability of DT-LTI Systems

For LTI a discrete time state-space representation

$$
\begin{aligned}
x[k+1] &= Ax[k] + Bu[k] \\
y[k] &= Cx[k] + Du[k]
\end{aligned}
$$

- A state x_r is said to be m-step **reachable**, if there exist an input sequence, $u[k], k \in \{0, 1, \cdots m-1\}$, that transfers the state vector $x[k]$ from the origin (i.e. $x[0] = 0$) to the state x_r in m number of steps, i.e. $x[m] = x_r$.
- A state x_d is said to be m-step **controllable**, if there exist an input sequence, $u[k], k \in \{0, 1, \cdots m-1\}$, that transfers the state vector $x[k]$ from the initial state x_c (i.e. $x[0] = x_c$) to the origin in m number of steps, i.e. $x[m] = 0$.

Note that

- the set $\mathcal{R}_m$ of all m-step reachable states is a linear (sub)space: $\mathcal{R}_m \subset \mathbb{R}^n$
- the set $\mathcal{C}_m$ of all m-step controllable states is a linear (sub)space: $\mathcal{C}_m \subset \mathbb{R}^n$

Let's characterize $\mathcal{R}_m$ and then try to generalize the reachability concept. When $x[0] = 0$, the solution of $x[m]$ is given by

$$
x[m] = \left[\begin{array}{c|c|c|c|c} A^{m-1}B & A^{m-2}B & \cdots & AB & B \end{array}\right] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[m-2] \\ u[m-1] \end{bmatrix}
$$

Let

$$
\begin{aligned}
\mathbf{R}_m &= \left[\begin{array}{c|c|c|c|c} A^{m-1}B & A^{m-2}B & \cdots & AB & B \end{array}\right] \\
\mathbf{U}_m &= \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[m-2] \\ u[m-1] \end{bmatrix}
\end{aligned}
$$

then if a state x_r is reachable at k steps, it should satisfy the following equation for some $\mathbf{U}_m$.

$$\mathbf{M}_m \mathbf{U}_m = x_m$$

In order this matrix equation to have a solution x_r should be in the range space of $\mathbf{M}_m$.

$$x_r \in \text{Ra}(\mathbf{M}_m)$$

Thus m-step reachable sub-space is simply equal to range space of $\mathcal{R}_k$

$$\text{Ra}(\mathbf{R}_m) = \mathcal{R}_m$$

Theorem: For $k < n < l$

$$\begin{aligned} \mathcal{R}_k \subset \mathcal{R}_n &= \mathcal{R}_l \\ \text{Ra}(\mathbf{R}_k) \subset \text{Ra}(\mathbf{R}_n) &= \text{Ra}(\mathbf{R}_l) \end{aligned}$$

Proof: It is fairly easy to observe that

$$\begin{aligned} \mathcal{R}_i &\subset \mathcal{R}_{i+1} \\ \text{Ra}(\mathbf{R}_i) &\subset \text{Ra}(\mathbf{R}_{i+1}) \end{aligned}$$

since we add a new column (or columns for multi-input systems) to $\mathbf{R}_i$, thus it can only increase the dimension of the range-space. Thus we can conclude that

$$\begin{aligned} \mathcal{R}_k \subset \mathcal{R}_n &\subset \mathcal{R}_l \\ \text{Ra}(\mathbf{R}_k) \subset \text{Ra}(\mathbf{R}_n) &\subset \text{Ra}(\mathbf{R}_l) \end{aligned}$$

In order prove $\mathcal{R}_n = \mathcal{R}_l$, we simply use the Cayley-Hamilton theorem. Based on Cayley-Hamilton theorem

$$\begin{aligned} A^n &= -a_1 A^{n-1} - \cdots - a_{n-1} A - a_n I \\ A^n B &= -a_1 A^{n-1} B - \cdots - a_{n-1} AB - a_n B \end{aligned}$$

which shows that $A^n B$ is linearly dependent to previous columns and thus

$$\begin{aligned} \mathcal{R}_n &= \mathcal{R}_l \\ \text{Ra}(\mathbf{R}_n) &= \text{Ra}(\mathbf{R}_l) \end{aligned}$$

This theorem shows that if x_r is reachable in n steps then it is reachable for $l > n$ steps, similarly, if it is not reachable in n steps then it is reachable for $l > n$ steps. In this context, the sub-space of states reachable in n-steps, $\mathcal{R}_n$ is referred as the reachable subspace of (A, N), and will be denoted simply by $\mathcal{R}$ and $\mathbf{R} = \mathbf{R}_k$ will be system wide the reachability matrix. The system is termed a (fully) reachable system if

$$\begin{aligned} \text{rank}(\mathbf{R}) &= n \\ \text{Ra}(\mathbf{R}) &= \mathcal{R} = \mathbb{R}^n \end{aligned}$$

Ex 10.1 *Solve the following problems regarding controllable sub-space*

- Show that $\mathcal{R} \subset \mathcal{C}$, $\forall (A, B)$, however $\mathcal{C} \subset \mathcal{R}$ not necessarily true $\forall (A, B)$.
- Similar to the reachable subspace, characterize the controllable subspace
- Derive conditions such that $\mathcal{R} = \mathcal{C}$

10.1.1 Reachability Gramian

An alternative characterization of $\mathbf{R}$ is using reachability Gramian (which is more critical for CT systems). m-step reachability Gramian, $\mathbf{P}_m$, is defined as

$$\mathbf{P}_m = \mathbf{R}_m {\mathbf{R}_m}^T = \sum_{i=0}^{k-1} A^i B B^T \left(A^T\right)^i \tag{10.1}$$

Note that $\mathcal{P}_m$ is a symmetric positive semi-definite matrix.

Lemma: $\mathcal{R}_m = \mathrm{Ra}(\mathbf{R}_m) = \mathrm{Ra}(\mathbf{P}_m)$

Proof: Let's fits show that $\mathrm{Ra}(\mathbf{P}_m) \subset \mathrm{Ra}(\mathbf{R}_m)$. If $x \in \mathrm{Ra}(\mathbf{P}_m)$, then $\exists v \in \mathbb{R}^n$ s.t. $x = \mathbf{P}_m v$ then

$$x = \mathbf{P}_m v = \mathbf{R}_m {\mathbf{R}_m}^T v = \mathbf{R}_m y \ \Rightarrow \ x \in \mathrm{Ra}(\mathbf{R}_m) \ \Rightarrow \ \mathrm{Ra}(\mathbf{P}_m) \subset \mathrm{Ra}(\mathbf{R}_m)$$

Now let's show that $\mathrm{Ra}(\mathbf{R}_m) \subset \mathrm{Ra}(\mathbf{P}_m)$. We know that

$$\mathrm{Ra}(\mathbf{R}_m) \subset \mathrm{Ra}(\mathbf{P}_m) \iff \mathrm{Ra}^{\perp}(\mathbf{P}_m) \subset \mathrm{Ra}^{\perp}(\mathbf{R}_m)$$

So we can equivalently show that $\mathrm{Ra}^{\perp}(\mathbf{P}_m) \subset \mathrm{Ra}^{\perp}(\mathbf{R}_m)$. Let $q \in \mathrm{Ra}^{\perp}(\mathbf{P}_m)$, then

$$\begin{aligned} q^T\mathbf{P}_m = \mathbf{0} \ \Rightarrow \ q^T\mathbf{P}_m q = 0 \ &\iff \ q^T \mathbf{R}_m {\mathbf{R}_m}^T q = 0 \ \iff \ ({\mathbf{R}_m}^T q)^T ({\mathbf{R}_m}^T q) = 0 \\ &\iff \ {\mathbf{R}_m}^T q = \mathbf{0} \ \iff \ q^T \mathbf{R}_m = \mathbf{0}^T = \mathbf{0} \\ &\Rightarrow \ q \in \mathrm{Ra}^{\perp}(\mathbf{R}_m) \ \Rightarrow \ \mathrm{Ra}^{\perp}(\mathbf{P}_m) \subset \mathrm{Ra}^{\perp}(\mathbf{R}_m) \end{aligned}$$

This completes the proof. As a result of this lemma, full reachable subspace $\mathcal{R} = \mathrm{Ra}(\mathbf{P}_l)$ for any $l \geq 0$. As a result we can make the following conclusions

- (A, B) pair is fully reachable $\iff$ $\dim\left[\mathrm{Ra}(\mathbf{P}_l)\right] = n$ for any $l \geq n$
- (A, B) pair is fully reachable $\iff$ $\det\left[\mathbf{P}_l\right] \neq 0$ for any $l \geq n$

If $x[k+1] = Ax[k]$ is asymptotically stable, then $\mathbf{P}_\infty = \lim_{k\to\infty} \sum_{i=0}^{k-1} A^i B B^T \left(A^T\right)^i \triangleq P$ is well defined and P satisfies the following Lyapunov equation

$$APA^T - P = -BB^T$$

To understand this derivation, refer to the Quadratic Lyapunov Functions for LTI systems section in Lecture Notes 8.

Ex 10.2 *Let $x[k+1] = Ax[k]$ be asymptotically stable. Then, show that*

$$APA^T - P = -BB^T$$

has a unique positive definite solution of P, if and only if, (A, B) pair is fully reachable.

10.1.2 Modal Aspects and Modal Reachability Tests

Lemma: The reachable sub-space, $\mathcal{R}$ is A invariant, i.e. $x \in \mathcal{R} \Rightarrow Ax \in \mathcal{R}$. We write this as $A\mathcal{R} \subset \mathcal{R}$.

Proof: Let $x \in \mathcal{R}$ then $\exists \mathbf{U}_n \in \mathbb{R}^{np}$, s.t. $x = \mathbf{R}\mathbf{U}_n$ where $B \in \mathbb{R}^{n \times p}$, then

$$x = \left[\; A^{n-1}B \;\middle|\; A^{n-2}B \;\middle|\; \cdots \;\middle|\; AB \;\middle|\; B \;\right] \mathbf{U}_n$$

Now let's expand Ax

$$Ax = \left[\; A^{n}B \;\middle|\; A^{n-1}B \;\middle|\; \cdots \;\middle|\; A^2B \;\middle|\; AB \;\right] \mathbf{U}_n$$

Using Cayley-Hamilton theorem we reach that

$$\begin{aligned}
A^n &= -a_{n-1}A^{n-1} - \cdots - a_1 A - a_0 I \\
A^n B &= -a_{n-1}A^{n-1}B - \cdots - a_1 AB - a_0 B \\
Ax &= \left[\; \sum_{i=0}^{n-1} A^i B \;\middle|\; A^{n-2}B \;\middle|\; \cdots \;\middle|\; A^2B \;\middle|\; AB \;\right] \mathbf{U}_n \\
Ax &\in \operatorname{Span}\left\{\; A^{n-1}B, \quad A^{n-2}B, \quad \cdots \quad AB, \quad B \;\right\} = \mathcal{R}
\end{aligned}$$