EE402 - Discrete Time Systems **Spring 2018**

Lecture 10

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10.1 Reachability & Controllability of DT-LTI Systems

For LTI a discrete-time state-space representation

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

- A state x_r is said to be m-step **reachable**, if there exist an input sequence, $u[k], k \in \{0, 1, \cdots m-1\}$, that transfers the state vector $x[k]$ from the origin (i.e. $x[0] = 0$) to the state x_r in m number of steps, i.e. $x[m] = x_r$.
- A state x_d is said to be m-step **controllable**, if there exist an input sequence, $u[k], k \in \{0, 1, \cdots m-1\}$, that transfers the state vector $x[k]$ from the initial state x_c (i.e. $x[0] = x_c$) to the origin in m number of steps, i.e. $x[m] = 0$.

Note that

- the set $\mathcal{R}_m$ of all m-step reachable states is a linear (sub)space: $\mathcal{R}_m \subset \mathbb{R}^n$
- the set $\mathcal{C}_m$ of all m-step controllable states is a linear (sub)space: $\mathcal{C}_m \subset \mathbb{R}^n$

Let's characterize $\mathcal{R}_m$ and then try to generalize the reachability concept. When $x[0] = 0$, the solution of $x[m]$ is given by

$$x[m] = \left[\begin{array}{c|c|c|c|c} A^{m-1}B & A^{m-2}B & \cdots & AB & B \end{array} \right] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[m-2] \\ u[m-1] \end{bmatrix}$$

Let

$$\begin{aligned} \mathbf{R}_m &= \left[\begin{array}{c|c|c|c|c} A^{m-1}B & A^{m-2}B & \cdots & AB & B \end{array} \right] \\ \mathbf{U}_m &= \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[m-2] \\ u[m-1] \end{bmatrix} \end{aligned}$$

then if a state x_r is reachable at k steps, it should satisfy the following equation for some $\mathbf{U}_m$.

$$\mathbf{M}_m \mathbf{U}_m = x_m$$

In order this matrix equation to have a solution x_r should be in the range space of $\mathbf{M}_m$.

$$x_r \in \mathrm{Ra}(\mathbf{M}_m)$$

Thus m-step reachable sub-space is simply equal to range space of $\mathcal{R}_k$

$$\mathrm{Ra}(\mathbf{R}_m) = \mathcal{R}_m$$

Theorem: For $k < n < l$

$$\begin{aligned} \mathcal{R}_k \subset \mathcal{R}_n &= \mathcal{R}_l \\ \mathrm{Ra}(\mathbf{R}_k) \subset \mathrm{Ra}(\mathbf{R}_n) &= \mathrm{Ra}(\mathbf{R}_l) \end{aligned}$$

Proof: It is fairly easy to observe that

$$\begin{aligned} \mathcal{R}_i &\subset \mathcal{R}_{i+1} \\ \mathrm{Ra}(\mathbf{R}_i) &\subset \mathrm{Ra}(\mathbf{R}_{i+1}) \end{aligned}$$

since we add a new column (or columns for multi-input systems) to $\mathbf{R}_i$, thus it can only increase the dimension of the range-space. Thus we can conclude that

$$\begin{aligned} \mathcal{R}_k \subset \mathcal{R}_n &\subset \mathcal{R}_l \\ \mathrm{Ra}(\mathbf{R}_k) \subset \mathrm{Ra}(\mathbf{R}_n) &\subset \mathrm{Ra}(\mathbf{R}_l) \end{aligned}$$

In order prove $\mathcal{R}_n = \mathcal{R}_l$, we simply use the Cayley-Hamilton theorem. Based on Cayley-Hamilton theorem

$$\begin{aligned} A^n &= -a_1 A^{n-1} - \cdots - a_{n-1} A - a_n I \\ A^n B &= -a_1 A^{n-1} B - \cdots - a_{n-1} AB - a_n B \end{aligned}$$

which shows that $A^n B$ is linearly dependent to previous columns and thus

$$\begin{aligned} \mathcal{R}_n &= \mathcal{R}_l \\ \mathrm{Ra}(\mathbf{R}_n) &= \mathrm{Ra}(\mathbf{R}_l) \end{aligned}$$

This theorem shows that if x_r is reachable in n steps then it is reachable for $l > n$ steps, similarly, if it is not reachable in n steps then it is reachable for $l > n$ steps. In this context, the sub-space of states reachable in n-steps, $\mathcal{R}_n$ is referred as the reachable subspace of (A, N), and will be denoted simply by $\mathcal{R}$ and $\mathbf{R} = \mathbf{R}_k$ will be system wide the reachability matrix. The system is termed a (fully) reachable system if

$$\begin{aligned} \mathrm{rank}(\mathbf{R}) &= n \\ \mathrm{Ra}(\mathbf{R}) &= \mathcal{R} = \mathbb{R}^n \end{aligned}$$

Ex 10.1 *Solve the following problems regarding controllable sub-space*

- Show that $\mathcal{R} \subset \mathcal{C}$, $\forall (A, B)$, however $\mathcal{C} \subset \mathcal{R}$ not necessarily true $\forall (A, B)$.
- Similar to the reachable subspace, characterize the controllable subspace
- Derive conditions such that $\mathcal{R} = \mathcal{C}$

10.1.1 Reachability Gramian

An alternative characterization of $\mathbf{R}$ is using reachability Gramian (which is more critical for CT systems). m-step reachability Gramian, $\mathbf{P}_m$, is defined as

$$\mathbf{P}_m = \mathbf{R}_m {\mathbf{R}_m}^T = \sum_{i=0}^{k-1} A^i B B^T \left(A^T\right)^i \tag{10.1}$$

Note that $\mathcal{P}_m$ is a symmetric positive semi-definite matrix.

Lemma: $\mathcal{R}_m = \mathrm{Ra}(\mathbf{R}_m) = \mathrm{Ra}(\mathbf{P}_m)$

Proof: Let's fits show that $\mathrm{Ra}(\mathbf{P}_m) \subset \mathrm{Ra}(\mathbf{R}_m)$. If $x \in \mathrm{Ra}(\mathbf{P}_m)$, then $\exists v \in \mathbb{R}^n$ s.t. $x = \mathbf{P}_m v$ then

$$x = \mathbf{P}_m v = \mathbf{R}_m {\mathbf{R}_m}^T v = \mathbf{R}_m y \;\Rightarrow\; x \in \mathrm{Ra}(\mathbf{R}_m) \;\Rightarrow\; \mathrm{Ra}(\mathbf{P}_m) \subset \mathrm{Ra}(\mathbf{R}_m)$$

Now let's show that $\mathrm{Ra}(\mathbf{R}_m) \subset \mathrm{Ra}(\mathbf{P}_m)$. We know that

$$\mathrm{Ra}(\mathbf{R}_m) \subset \mathrm{Ra}(\mathbf{P}_m) \iff \mathrm{Ra}^{\perp}(\mathbf{P}_m) \subset \mathrm{Ra}^{\perp}(\mathbf{R}_m)$$

So we can equivalently show that $\mathrm{Ra}^{\perp}(\mathbf{P}_m) \subset \mathrm{Ra}^{\perp}(\mathbf{R}_m)$. Let $q \in \mathrm{Ra}^{\perp}(\mathbf{P}_m)$, then

$$\begin{aligned}
q^T \mathbf{P}_m = \mathbf{0} \;\Rightarrow\; q^T \mathbf{P}_m q = 0 \;&\iff\; q^T \mathbf{R}_m {\mathbf{R}_m}^T q = 0 \;\iff\; ({\mathbf{R}_m}^T q)^T ({\mathbf{R}_m}^T q) = 0 \\
&\iff\; {\mathbf{R}_m}^T q = \mathbf{0} \;\iff\; q^T \mathbf{R}_m = \mathbf{0}^T = \mathbf{0} \\
&\Rightarrow\; q \in \mathrm{Ra}^{\perp}(\mathbf{R}_m) \;\Rightarrow\; \mathrm{Ra}^{\perp}(\mathbf{P}_m) \subset \mathrm{Ra}^{\perp}(\mathbf{R}_m)
\end{aligned}$$

This completes the proof. As a result of this lemma, full reachable subspace $\mathcal{R} = \mathrm{Ra}(\mathbf{P}_l)$ for any $l \geq 0$. As a result we can make the following conclusions

- (A, B) pair is fully reachable $\iff$ $\dim\left[\mathrm{Ra}(\mathbf{P}_l)\right] = n$ for any $l \geq n$
- (A, B) pair is fully reachable $\iff$ $\det\left[\mathbf{P}_l\right] \neq 0$ for any $l \geq n$

If $x[k+1] = Ax[k]$ is asymptotically stable, then $\mathbf{P}_\infty = \lim_{k\to\infty} \sum_{i=0}^{k-1} A^i B B^T \left(A^T\right)^i \triangleq P$ is well defined and P satisfies the following Lyapunov equation

$$APA^T - P = -BB^T$$

To understand this derivation, refer to the Quadratic Lyapunov Functions for LTI systems section in Lecture Notes 8.

Ex 10.2 *Let $x[k+1] = Ax[k]$ be asymptotically stable. Then, show that*

$$APA^T - P = -BB^T$$

has a unique positive definite solution of P, if and only if, (A, B) pair is fully reachable.

10.1.2 Modal Aspects and Modal Reachability Tests

Lemma: The reachable sub-space, $\mathcal{R}$ is A invariant, i.e. $x \in \mathcal{R} \Rightarrow Ax \in \mathcal{R}$. We write this as $A\mathcal{R} \subset \mathcal{R}$.

Proof: Let $x \in \mathcal{R}$ then $\exists \mathbf{U}_n \in \mathbb{R}^{np}$, s.t. $x = \mathbf{R}\mathbf{U}_n$ where $B \in \mathbb{R}^{n\times p}$, then

$$x = \left[\begin{array}{c|c|c|c|c} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{array}\right]\mathbf{U}_n$$

Now let's expand Ax

$$Ax = \left[\begin{array}{c|c|c|c|c} A^{n}B & A^{n-1}B & \cdots & A^2B & AB \end{array}\right]\mathbf{U}_n$$

Using Cayley-Hamilton theorem, we reach that

$$\begin{aligned}
A^n &= -a_{n-1}A^{n-1} - \cdots - a_1 A - a_0 I \\
A^nB &= -a_{n-1}A^{n-1}B - \cdots - a_1 AB - a_0 B \\
Ax &= \left[\begin{array}{c|c|c|c|c} \sum\limits_{i=0}^{n-1} A^iB & A^{n-2}B & \cdots & A^2B & AB \end{array}\right]\mathbf{U}_n \\
Ax &\in \text{Span}\left\{\ A^{n-1}B, \quad A^{n-2}B, \quad \cdots \quad AB, \quad B\ \right\} = \mathcal{R}
\end{aligned}$$

Theorem: (A, B) pair is unreachable (or not fully reachable) if and only if $w^TB = 0$ for some left eigenvector of A, i.e. $w^TA = \lambda w^T$.

Proof: Let $w^TA = \lambda w^T$ and $w^TB = 0$, then

$$\begin{aligned}
w^T\mathbf{R} &= w^T \left[\begin{array}{c|c|c|c|c} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{array}\right] \\
&= \left[\begin{array}{c|c|c|c|c} w^TA^{n-1}B & w^TA^{n-2}B & \cdots & w^TAB & w^TB \end{array}\right] \\
&= \left[\begin{array}{c|c|c|c|c} \lambda^{n-1}w^TB & \lambda^{n-2}w^TB & \cdots & \lambda w^TB & w^TB \end{array}\right] = 0 \ \rightarrow\ (A, B) \text{ unreachable}
\end{aligned}$$

Now let's show that if (A, B) is not reachable, then $\exists w$ s.t. $w^TA = \lambda w^T$ and $w^TB = 0$. if (A, B) is not reachable, then there exist $q \in \mathbb{R}^n$ such that

$$q^T\mathbf{R} = 0 \ \Rightarrow\ \mathbf{R}^Tq = 0 \ \Rightarrow\ \begin{array}{c} B^T(A^T)^{n-1}q = 0 \\ B^T(A^T)^{n-2}q = 0 \\ \vdots \\ B^TA^Tq = 0 \\ B^Tq = 0 \end{array}$$

Define $\mathcal{S} = \text{Span}\left\{q\ ,\ A^Tq\ ,\ \cdots\ ,\ (A^T)^{n-1}q\right\}$, note that $\mathcal{S} \subset \mathcal{N}(B^T)$ and $\mathcal{S}$ is invariant under A^T (see the similar proof above), i.e. If $v \in \mathcal{S}$, then $A^Tv \in \mathcal{S}$. Let $\dim(\mathcal{S}) = j$ and $V \in \mathbb{R}^{n\times j}$ be a matrix whose columns form a basis for $\mathcal{S}$. Since $\mathcal{S}$ is invariant under A^T, we can find a transformation matrix, $\Gamma \in \mathbb{R}^{j\times j}$ such that

$$A^TV = V\Gamma$$

Let ν be an eigenvector of V, i.e. $\Gamma\nu = \alpha\nu$, $\alpha \in \mathcal{C}$, then

$$A^TV\nu = V\Gamma\nu = \alpha V\nu \ \rightarrow\ z = V\nu \in \mathbb{C}^n$$

where z is an eigenvector of A^T, moreover $z \in \mathcal{S} = \mathcal{N}(B^T)$. This completes the proof.

Theorem: *PBH Rechability Test* - (A, B) is reachable if and only if

$$\text{rank}\left[\begin{array}{c|c} \lambda I - A & B \end{array}\right] = n, \ \forall\lambda \in \mathcal{C}$$

Proof: Technically, it is the same argument as the eigenvector/modal reachability test proposed above. The same proof can be adapted for the PBH test.

Definition: If λ, w^T (eigenvalue, left-eigenvector) pair fails in one of the modal reachability tests, we call this pair an unreachable mode of the system.

10.1.3 Reachability & Similarity Transformation

Theorem: Reachability is invariant under state/similarity transformation $\bar{x} = P^{-1}x$ where $\det(P) \neq 0$. We know that system and input matrices under such a transformation take the form

$$\bar{A} = PAP^{-1} \ , \ \bar{B} = PB$$

Reachability matrix for the $(\bar{A}, \bar{B})$ can be written as

$$\begin{aligned}
\bar{\mathbf{R}} &= \left[\ \bar{A}^{n-1}\bar{B} \mid \bar{A}^{n-2}\bar{B} \mid \cdots \mid \bar{A}\bar{B} \mid \bar{B} \ \right] \\
&= \left[\ PA^{n-1}B \mid PA^{n-2}B \mid \cdots \mid PAP^{-1}PB \mid PB \ \right] \\
&= P \left[\ A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AP^{-1}B \mid B \ \right] = P\mathbf{R} \\
&\Rightarrow \text{rank}[\mathbf{R}] = \text{rank}[\bar{\mathbf{R}}]
\end{aligned}$$

Let $\dot{x} = Ax + Bu$ be an unreachable system; then it can be convenient and practical to choose coordinates (via similarity transformation) to highlight reachable and reachable "spaces". Let

$$\dim(\mathbf{R}) = r \ \& \ P^{-1} = T = \left[\ T_1 \mid T_2 \ \right] \ \text{where } T_1 \in \mathbb{R}^{n \times r} \ , T_2 \in \mathbb{R}^{n \times (n-r)} \ ,$$

Let's choose a T_1 such that $\text{Ra}(T_1) = \text{Ra}(\mathbf{R}) = \mathcal{R}$ and let's choose T_2 be a matrix such that columns of T are linearly independent. Let's analyze the similarity transformation.

$$\begin{aligned}
AT = A \left[\ T_1 \mid T_2 \ \right] = T\bar{A} = \left[\ T_1 \mid T_2 \ \right] \left[\begin{array}{c|c} A_{11} & A_{22} \\ \hline A_{21} & A_{22} \end{array} \right] \\
AT_1 = T_1 A_{11} + T_2 A_{22}
\end{aligned}$$

Note that the reachable sub-space is A invariant AT_1 must remain in $\text{Ra}(T_1) = \text{Ra}(\mathbf{R}) = \mathcal{R}$, since T_2 is composed of linearly independent columns, in order $T_1 A_{11} + T_2 A_{22}$ remain in $\text{Ra}(T_1)$, $A_{12} = \mathbf{0}$. Now let's analyze the effect of transformation on the input matrix.

$$\begin{aligned}
B &= T\bar{B} = \left[\ T_1 \mid T_2 \ \right] \left[\begin{array}{c} B_1 \\ B_2 \end{array} \right] \\
B &= T_1 B_1 + T_2 B_2
\end{aligned}$$

Similarly since $\text{Ra}(B) \subset \text{Ra}(T_1) = \text{Ra}(\mathbf{R}) = \mathcal{R} \ \Rightarrow \ B_2 = 0$, thus a standard unreachable form takes the form

$$\bar{x}[k+1] = \left[\begin{array}{c|c} A_{11} & A_{22} \\ \hline 0 & A_{22} \end{array} \right] \bar{x}[k] + \left[\begin{array}{c} B_1 \\ 0 \end{array} \right] u[k]$$

10.2 Reachability & Controllability of CT-LTI Systems

For LTI a continuous-time state-space representation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- A state x_r is said to be **reachable** in time $t_f > 0$, if there exists an input signal, $u(t), t \in [0, t_f]$, that transfers the state vector $x(t)$ from the origin (i.e. $x(0) = 0$) to the state x_r at the given time, i.e. $x(t) = x_r$.
- A state x_c is said to be **controllable** in time $t_f > 0$, if there exist an input signal, $u(t), t \in [0, t_f]$, that transfers the state vector $x(t)$ from the initial state x_c (i.e. $x(0) = x_c$) to the origin in given time frame, i.e. $x(t) = 0$.

Note that

- the set $\mathcal{R}_{t_f}$ of all states that reachable in time t_f is a linear (sub)space: $\mathcal{R}_{t_f} \subset \mathbb{R}^n$
- the set $\mathcal{R}_{t_f}$ of all states that controllable in time t_f is a linear (sub)space: $\mathcal{R}_{t_f} \subset \mathbb{R}^n$

Let's characterize $\mathcal{R}_{t_f}$. In CT derivation of the reachability matrix is not as intuitive as the case in DT systems. Instead, we construct the main concepts using the reachability Gramian for CT systems. Note that for a CT-LTI system $x(t_f)$ for a zero-state response written as

$$x(t_f) = \int_0^{t_f} e^{A(t_f - \tau)} B u(\tau) d\tau$$

Obviously if a $x_r \in \mathbb{R}$ is reachable in time t_f, then $\exists u(t),\ t \in [0, t_f]$ such that $x(t_f) = x_r$. In this context, the reachability Grammian is defined as

$$\mathbf{P}(t) = \int_0^{t} e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} d\tau \tag{10.2}$$

Theorem: $\mathcal{R}_{t_f} = \mathrm{Ra}[\mathbf{P}(t_f)]$, i.e. reachable in time t_f sub-space is characterized by the range-space of the reachability Gramian.

Proof: Let's first show that $\mathrm{Ra}[\mathbf{P}(t_f)] \subset \mathcal{R}_{t_f}$. Let $x^* \in \mathrm{Ra}[\mathbf{P}(t_f)]$, then $\exists v \in \mathbb{R}^n$ such that

$$x^* = \mathbf{P}(t_f) v \tag{10.3}$$

Let $u(t) = B^T e^{A(t_f - t)} v$, then

$$\begin{aligned} x(t_f) &= \int_0^{t_f} e^{A(t_f-\tau)} B u(\tau) d\tau \\ &= \int_0^{t_f} e^{A(t_f-\tau)} B B^T e^{A^T(t_f-\tau)} v d\tau \\ &= \mathbf{P}(t_f) v = x^* \ \Rightarrow \ x^* \in \mathcal{R}_{t_f} \ \Rightarrow \ \mathrm{Ra}[\mathbf{P}(t_f)] \subset \mathcal{R}_{t_f} \end{aligned}$$

Now let's show that $\mathcal{R}_{t_f} \subset \mathrm{Ra}[\mathbf{P}(t_f)]$. Equivalently we can show that $\mathrm{Ra}^\perp[\mathbf{P}(t_f)] \subset \mathcal{R}_{t_f}^\perp$. Let $q \in \mathrm{Ra}^\perp[\mathbf{P}(t_f)]$, then

$$q^T\mathbf{P}(t_f) = 0 \Rightarrow q^T\mathbf{P}(t_f)q = 0 \iff \int_0^{t_f} q^T e^{A(t_f-\tau)}BB^T e^{A^T(t_f-\tau)} q d\tau = 0 \iff \int_0^{t_f} ||B^T e^{A^T(t_f-\tau)}q||_2^2 d\tau = 0$$
$$\iff B^T e^{A^T(t_f-\tau)}q = 0 \,\forall\, \tau \in [0, t_f] \tag{10.4}$$

Let $x(t_f) \in \mathcal{R}_{t_f}$ (any vector in the reachable space), then we know that

$$x(t_f) = \int_0^{t_f} e^{A(t_f-\tau)}Bu(\tau)d\tau$$

Let's analyze $x(t_f)^T q$

$$x(t_f)^T q = \int_0^{t_f} u(\tau)^T \left[B^T e^{A^T(t_f-\tau)}q\right] d\tau = \int_0^{t_f} u(\tau)^T \,[0]\, d\tau = 0 \Rightarrow q \in \mathcal{R}_{t_f}^\perp \Rightarrow \mathrm{Ra}^\perp[\mathbf{P}(t_f)] \subset \mathcal{R}_{t_f}^\perp$$

This completes the proof.

Theorem: $\mathcal{R}_{t_f} = \mathrm{Ra}[\mathbf{P}(t_f)] = \mathrm{Ra}[\mathbf{R}] \;\forall t_f > 0$. This theorem claims that also, for CT-LTI systems, the reachability matrix characterizes the reachable in time t_f sub-space; moreover, reachable sub-space is indeed independent of t_f.

Let's concentrate the relation between the reachability grammian, $\mathbf{P}(t_f) = \int_0^{t_f} e^{A(t_f-\tau)}BB^T e^{A^T(t_f-\tau)}d\tau$ and the reachability matrix, $\mathbf{R} = \left[\begin{array}{c|c|c|c|c} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{array}\right]$. Note that

$$\mathrm{Ra}[\mathbf{P}(t_f)] = \mathrm{Ra}[\mathbf{R}] \iff \mathrm{Ra}^\perp[\mathbf{P}(t_f)] = \mathrm{Ra}^\perp[\mathbf{R}]$$

Firs show that if $\mathrm{Ra}^\perp[\mathbf{R}] \subset \mathrm{Ra}^\perp[\mathbf{P}(t_f)]$. Let $q \in \mathrm{Ra}^\perp[\mathbf{R}]$, then

$$q^T\mathbf{R} = 0 \iff q^T \left[\begin{array}{c|c|c|c|c} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{array}\right] = 0 \iff q^T A^i B = 0 \forall i \in \mathbb{Z}^+$$

Note that for $i \geq n$, we referred to the Cayley-Hamilton theorem. Now let's check if $q \in \mathrm{Ra}^\perp[\mathbf{P}_{t_f}]$

$$q^T\mathbf{P}_{t_f} = \int_0^{t_f} q^T e^{A(t_f-\tau)}BB^T e^{A^T(t_f-\tau)}d\tau$$
$$q^T A^i B = 0 \forall i \in \mathbb{Z}^+ \Rightarrow q^T e^{A(t_f-\tau)} = 0 \,\forall t_f \geq 0$$
$$\Rightarrow q^T\mathbf{P}_{t_f} = 0 \Rightarrow q \in \mathrm{Ra}^\perp[\mathbf{P}_{t_f}] \Rightarrow \mathrm{Ra}^\perp[\mathbf{R}] \subset \mathrm{Ra}^\perp[\mathbf{P}(t_f)]$$

Now let's show that if $\mathrm{Ra}^\perp[\mathbf{P}(t_f)] \subset \mathrm{Ra}^\perp[\mathbf{R}]$. Let $q \in \mathrm{Ra}^\perp[\mathbf{R}]$, from (**??**) we know that

$$qe^{A(t_f-\tau)}B = 0 \,\forall\, \tau \in [0, t_f]$$

Then we know that

$$
\begin{aligned}
[qe^{A\eta}B]_{\eta=0} = 0 &\Rightarrow q^T B = 0 \\
\frac{d}{dt}[qe^{A\eta}B]_{\eta=0} = 0 &\rightarrow q^T AB = 0 \\
\frac{d^2}{dt^2}[qe^{A\eta}B]_{\eta=0} = 0 &\Rightarrow q^T A^2 B = 0 \\
&\vdots \\
\frac{d^{n-1}}{dt^{n-1}}[qe^{A\eta}B]_{\eta=0} = 0 &\Rightarrow q^T A^{n-1} B = 0 \\
&\Rightarrow q^T \mathbf{R} = 0 \Rightarrow q \in \mathrm{Ra}^{\perp}[\mathbf{R}] \Rightarrow \mathrm{Ra}^{\perp}[\mathbf{P}(t_f)] \subset \mathrm{Ra}^{\perp}[\mathbf{R}]
\end{aligned}
$$

This completes the proof. This theorem indicates that similar to the DT systems range space of the reachability matrix, $\mathbf{R}$, characterizes the whole reachable space, and this space independent of the time interval. In that respect

Corollary: (A, B) is fully reachable if and only if $\dim(\mathrm{Ra}[\mathbf{R}]) = \dim(\mathrm{Ra}[\mathbf{P}(t_f)])$

Remark: This theorem shows that the condition for CT reachability is expressed in the same way as DT reachability. Hence all our DT results on standard forms for unreachable systems, modal tests, and so on; remain unchanged. We, therefore, do not repeat any of these DT results for the CT case but count on you to explicitly note the CT versions of our earlier DT results.

Ex 10.3 *Show that*

if $\dot{x} = Ax$ is asymptotically stable, then reachability Grammian at $t_f \to \infty$, $P := \mathbf{P}_\infty$, satisfies the following Lyapunov equation

$$AP + PA^T = -BB^T$$

, and this Lyapunov equation has a (unique) positive-definite solution for P, if and only if, (A, B) is fully reachable.

Ex 10.4 *Show that for a CT-LTI system $\mathcal{C} = \mathcal{R}$*

i.e., show that reachability is equivalent to controllability for CT systems.

Ex 10.5 *Let*

$$\dot{x} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} x + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} u$$

Analyze the reachability of the (A, B) pair and find the necessary and sufficient conditions such that (A, B) fully reachable.

Solution: Since A is already in diagonal Canonical form, the most convenient way to analyze reachability is the PBH's modal test

$$\left[\begin{array}{c|c} A - \lambda I & B \end{array}\right] = \left[\begin{array}{ccc|c} \lambda_1 & & 0 & b_1 \\ & \ddots & & \\ 0 & & \lambda_n & b_n \end{array}\right]$$

Necessary Condition: Let $\lambda = \lambda_i$, if $b_i = 0$, then $\left[\begin{array}{c|c} A - \lambda I & B \end{array}\right]$ drops rank. Thus, if $\exists \lambda_i$ such that $b_i = 0$, then the system is unreachable.

Sufficient Condition: Not that if $\lambda_i \neq \lambda_j$ for $i \neq j$, the condition above also becomes sufficient. Now analyze the general case, without loss of generality let's assume $\lambda_1 = \lambda_2 = \cdots = \lambda_p$ and all other eigenvalues are different. Let $\lambda = \lambda_1$ in the PBH test, then first p-rows of the PBH matrix takes the form

$$\left[\begin{array}{c|c} A - \lambda_1 I & B \end{array}\right] = \left[\begin{array}{c|c} 0_{1\times n} & b_1 \\ 0_{1\times n} & b_2 \\ & \\ \vdots & \\ 0_{1\times n} & b_p \end{array}\right]$$

PBH matrix does not drop rank, if and only if, $\{b_1^T, b_2^T, \cdots, b_p^T\}$ is a linearly independent set. We can see that number of inputs of the system has to be larger than or equal to p (indeed $\max(p)$ for system-wide reachability). Moreover, the rank of the following sub-matrix matrix has to be equal to the algebraic multiplicity of the eigenvalue, i.e.

$$B_{\lambda_1} = \begin{bmatrix} b_1 \\ b_2 \\ \\ \vdots \\ b_p \end{bmatrix}, \ \operatorname{rank}(B_{\lambda_1}) = p$$

Ex 10.6 *Let*

$$\dot{x} = \left[\begin{array}{ccc|cc} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right] x + \begin{bmatrix} b_{1,1} \\ b_{1,2} \\ b_{1,3} \\ b_{2,1} \\ b_{2,2} \end{bmatrix} u$$

Analyze the reachability of the (A, B) pair for $\lambda_1 neq \lambda_2$ and $\lambda_1 = \lambda_2$ and find the necessary and sufficient conditions such that (A, B) fully reachable.

Solution: Let's first assume that $\lambda_1 neq \lambda_2$, we can analyze each mode separately using the BPH test. Let's start with $\lambda = \lambda_1$

$$\left[\begin{array}{c|c} A - \lambda_1 I & B \end{array}\right] = \left[\begin{array}{ccc|cc|c} 0 & 1 & 0 & 0 & 0 & b_{1,1} \\ 0 & 0 & 1 & 0 & 0 & b_{1,2} \\ 0 & 0 & 0 & 0 & 0 & b_{1,3} \\ \hline 0 & 0 & 0 & \lambda_2 - \lambda_1 & 1 & b_{2,1} \\ 0 & 0 & 0 & 0 & \lambda_2 - \lambda_2 & b_{2,2} \end{array}\right]$$

The matrix above is full rank if and only if $b_{1,3} \neq 0$, this implies that $b_{1,1}$ and $b_{1,2}$ do not affect the reachability condition. Similarly for $\lambda = \lambda_2$ we can find that $b_{2,2} \neq 0$ is a necessary condition for reachability. So for $\lambda_1 neq \lambda_2$, (A, B) pair is reachable if and only if $b_{2,2} \neq 0$ and $b_{1,3} \neq 0$.

Now let's analyze the case $\lambda_2 = \lambda_1$. It is fairly easy to observe that $b_{2,2} \neq 0$ and $b_{1,3} \neq 0$ is a necessary but not sufficient condition for full reachability. Let's re-analyze the PBH matrix

$$\left[\begin{array}{c|c} A-\lambda_1 I & B \end{array}\right] = \left[\begin{array}{ccc|cc|c} 0 & 1 & 0 & 0 & 0 & b_{1,1} \\ 0 & 0 & 1 & 0 & 0 & b_{1,2} \\ 0 & 0 & 0 & 0 & 0 & b_{1,3} \\ \hline 0 & 0 & 0 & 0 & 1 & b_{2,1} \\ 0 & 0 & 0 & 0 & 0 & b_{2,2} \end{array}\right]$$

The matrix has full rank if and only if

$$\mathrm{Ra}\left\{\begin{bmatrix} b_{1,3} \\ b_{2,2} \end{bmatrix}\right\} = \mathbb{R}^2$$