

Lecture 6

Lecturer: Assoc. Prof. M. Mert Ankarali

6.1 Modal Decomposition of State-Space Models

6.1.1 Zero Input Response

Let's consider autonomous LTI CT and DT state-space models

$$\begin{aligned}\dot{x} &= Ax \\ x[k+1] &= Ax[k]\end{aligned}$$

Let $x_0 = \alpha v_i$, where v_i is an eigenvector of A associated with eigenvalue λ_i , we can then find the solution for both systems

$$\begin{aligned}x(t) &= e^{At}x_0 = \alpha e^{\lambda_i t}v_i \\ x[k] &= A^k x_0 = \alpha \lambda_i^k v_i\end{aligned}$$

Now let's assume that A is diagonalizable, then we know that there exist a set of n linearly independent eigenvectors $\mathcal{V} = \{v_1, \dots, v_n\}$. Thus, we can write any initial condition, $x_0 \in \mathbb{R}^n$, as a linear combination of eigenvectors, i.e.

$$x_0 = \sum_{i=1}^n \alpha_i v_i$$

if we use this (modal) decomposition we can find the solutions of DT and CT equations as

$$\begin{aligned}x(t) &= e^{At}x_0 = \sum_{i=1}^n \alpha_i e^{\lambda_i t} v_i \\ x[k] &= A^k x_0 = \sum_{i=1}^n \alpha_i \lambda_i^k v_i\end{aligned}$$

where $e^{\lambda_i t} v_i$ ($\lambda_i^k v_i$ in DT case) is called a “mode” of the system. Now let’s try to find $\{\alpha_i, \dots, \alpha_n\}$ via diagonalization of A

$$A = V \Lambda V^{-1} = V \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \ddots \\ & & & & \lambda_n \end{bmatrix} V^{-1}, \text{ where}$$

$$V = [v_1 \ \dots \ v_n], \text{ where } Av_i = \lambda_i v_i$$

$$V^{-1} = \bar{V} = \begin{bmatrix} \bar{v}_1^T \\ \vdots \\ \bar{v}_n^T \end{bmatrix}, \text{ where } \bar{v}_i^T A = \lambda_i \bar{v}_i^T$$

$$\bar{V}V = V\bar{V} = I, \ \bar{v}_i^T v_i = 1, \ \bar{v}_i^T v_j = 0 \text{ for } i \neq j$$

Now let’s compute the zero-input responses for an arbitrary x_0

$$x(t) = e^{At} x_0 = V e^{\Lambda t} V^{-1} x_0 = [v_1 \ \dots \ v_n] \begin{bmatrix} e^{\lambda_1 t} \bar{v}_1^T x_0 \\ \vdots \\ e^{\lambda_n t} \bar{v}_n^T x_0 \end{bmatrix} = \sum_{i=1}^n v_i e^{\lambda_i t} (\bar{v}_i^T x_0) \rightarrow \alpha_i = \bar{v}_i^T x_0$$

$$x[k] = V \Lambda^k V^{-1} x_0 = [v_1 \ \dots \ v_n] \begin{bmatrix} \lambda_1^k \bar{v}_1^T x_0 \\ \vdots \\ \lambda_n^k \bar{v}_n^T x_0 \end{bmatrix} = \sum_{i=1}^n v_i \lambda_i^k (\bar{v}_i^T x_0) \rightarrow \alpha_i = \bar{v}_i^T x_0$$

Based on these results, we can see that in order to excite the i^{th} mode the system, we need $\bar{v}_i^T x_0 \neq 0$. If we treat initial conditions as inputs this generates a reachability/controllability argument. Let’s also analyze the output response

$$y(t) = Cx(t) = C e^{At} x_0 = \sum_{i=1}^n (C v_i) e^{\lambda_i t} (\bar{v}_i^T x_0)$$

$$y[k] = Cx[k] = \sum_{i=1}^n (C v_i) \lambda_i^k (\bar{v}_i^T x_0)$$

We can see that if $C v_i = 0$, then we can not observe the i^{th} mode at the output $\forall x_0 \in \mathbb{R}^n$. Thus we can conclude that in order to have a fully observable system all modes needs to be observable, i.e. i.e. $C v_i \neq 0 \ \forall i \in \{1, \dots, n\}$.

Now let’s try to find a similar solution for systems where matrix A can not be diagonalizable. For the sake of clarity let’s focus on matrices that is composed of a single Jordan block

$$A = G J G^{-1} = G \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots \\ \vdots & & \ddots & & \\ & & & \ddots & 1 \\ 0 & & & 0 & \lambda \end{bmatrix} G^{-1}$$

where

$$\begin{aligned}
 G &= \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \\
 Ag_1 &= \lambda g_1 \rightarrow (A - \lambda I)g_1 = 0 \\
 Ag_2 &= \lambda g_2 + g_1 \rightarrow (A - \lambda I)g_2 = g_1, \text{ note } (A - \lambda I)^2 g_2 = 0 \text{ \& } (A - \lambda I)g_2 \neq 0 \\
 Ag_3 &= \lambda g_3 + g_2 \rightarrow (A - \lambda I)g_3 = g_2, \text{ note } (A - \lambda I)^3 g_3 = 0 \text{ \& } (A - \lambda I)^2 g_3 \neq 0 \\
 &\vdots \\
 Ag_n &= \lambda g_n + g_{n-1} \rightarrow (A - \lambda I)g_n = g_{n-1}, \text{ note } (A - \lambda I)^n g_n = 0 \text{ \& } (A - \lambda I)^{n-1} g_n \neq 0
 \end{aligned}$$

and we also know that

$$\begin{aligned}
 G^{-1} &= \bar{G} = \begin{bmatrix} \bar{g}_1^T \\ \vdots \\ \bar{g}_n^T \end{bmatrix} \\
 \bar{G}G &= G\bar{G} = I, \quad \bar{g}_i^T g_i = 1, \quad \bar{g}_i^T g_j = 0 \text{ for } i \neq j
 \end{aligned}$$

Let $x_0 = \alpha_1 g_1$, i.e. the eigenvector of A , then we can find the responses as

$$\begin{aligned}
 x(t) &= e^{At} g_1 = G e^{Jt} G^{-1} g_1 \alpha_1 \\
 &= \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!} e^{\lambda t} & \frac{t^{n-1}}{(n-1)!} e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!} e^{\lambda t} \\ \vdots & & \ddots & & & \vdots \\ 0 & & & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} \\ 0 & & \cdots & & e^{\lambda t} & te^{\lambda t} \\ 0 & & \cdots & & 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
 &= \alpha_1 e^{\lambda t} g_1
 \end{aligned}$$

$$x[k] = G J^k G^{-1} x_0 = \alpha_1 \lambda^k g_1$$

the format of the solution associated with g_1 seems to be exactly same with diagonal case (since g_1 is an eigenvector). This implies that, if we choose an initial condition inside the eigenvector space, the response stays in this space and acts like a “first-order” system. Now, let $x_0 = \alpha_2 g_2$, i.e. a first order generalized eigenvector of A , then we can find the responses as

$$\begin{aligned}
 x(t) &= G e^{Jt} G^{-1} g_2 \alpha_2 \\
 &= \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix} \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!} e^{\lambda t} & \frac{t^{n-1}}{(n-1)!} e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!} e^{\lambda t} \\ \vdots & & \ddots & & & \vdots \\ 0 & & & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} \\ 0 & & \cdots & & e^{\lambda t} & te^{\lambda t} \\ 0 & & \cdots & & 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_2 \\ \vdots \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix} \begin{bmatrix} \alpha_2 t e^{\lambda t} \\ \alpha_2 e^{\lambda t} \\ \vdots \\ 0 \end{bmatrix} = \alpha_2 (t e^{\lambda t} g_1 + e^{\lambda t} g_2)
 \end{aligned}$$

$$\begin{aligned}
x[k] &= GJ^k G^{-1} g_2 \alpha_2 \\
&= \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix} \begin{bmatrix} \lambda^k & \binom{k}{1} \lambda^{k-1} & \binom{k}{2} \lambda^{k-2} & \cdots & \binom{k}{n-2} \lambda^{k-n+2} & \binom{k}{n-1} \lambda^{k-n+1} \\ 0 & \lambda^k & \binom{k}{1} \lambda^{k-1} & \binom{k}{2} \lambda^{k-2} & \cdots & \binom{k}{n-2} \lambda^{k-n+2} \\ \vdots & & \ddots & & & \vdots \\ & & & \lambda^k & \binom{k}{1} \lambda^{k-1} & \binom{k}{2} \lambda^{k-2} \\ 0 & & \cdots & & \lambda^k & \binom{k}{1} \lambda^{k-1} \\ 0 & & \cdots & & 0 & \lambda^k \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_2 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix} \begin{bmatrix} \alpha_2 k \lambda^{k-1} \\ \alpha_2 \lambda^k \\ \vdots \\ 0 \end{bmatrix} = \alpha_2 (k \lambda^{k-1} g_1 + \lambda^k g_2)
\end{aligned}$$

We can observe that the response acts like a “second-order” (critically-damped) response. Moreover, the response does not stay inside the span of the generalized eigenvector, i.e. $\text{Span}\{g_2\}$, instead it navigates inside the span of the eigenvector and g_2 , i.e. $\text{Span}\{g_1, g_2\} = \mathcal{N}(A - \lambda I)^2$. Now, let $x_0 = \alpha_i g_i$, $0 \leq i \leq n$, i.e. order generalized eigenvector of order i , then we can find the responses as

$$\begin{aligned}
x(t) &= G e^{Jt} G^{-1} g_2 \alpha_2 = \alpha_i e^{\lambda t} \sum_{j=1}^i g_j \frac{t^{i-j}}{(i-j)!} \\
x[k] &= G J^k G^{-1} g_i \alpha_i = \alpha_i \sum_{j=1}^i g_j \binom{k}{i-j} \lambda^{k-i+j}
\end{aligned}$$

Similar to the second-order case, we can see that response acts like an i^{th} order dynamical system, and trajectories stay inside, $\text{Span}\{g_1, \dots, g_i\} = \mathcal{N}(A - \lambda I)^i$. In this context, in order to excite all higher order modes of the system projection of the initial condition to the sub-space spanned highest order generalized eigenvector has to be non-zero. Now, let's try to find general solution for an arbitrary x_0 . We can write any $x_0 \in \mathbb{R}^n$ as a linear combination of $\mathcal{G} = \{g_1, g_2, \dots, g_n\}$, thus we have

$$\begin{aligned}
x_0 &= \sum_{i=1}^n \alpha_i g_i \\
x(t) &= \sum_{i=1}^n \alpha_i e^{\lambda t} \sum_{j=1}^i g_j \frac{t^{i-j}}{(i-j)!} \\
x[k] &= \sum_{i=1}^n \alpha_i \sum_{j=1}^i g_j \binom{k}{i-j} \lambda^{k-i+j}
\end{aligned}$$

where

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = G^{-1} x_0 = \begin{bmatrix} \bar{g}_1^T \\ \bar{g}_2^T \\ \vdots \\ \bar{g}_n^T \end{bmatrix} x_0 \rightarrow \alpha_i = \bar{g}_i^T x_0$$

Based on these results, we can see that in order to excite all of the modes associated with a Jordan block of size n , we need $\alpha_n \bar{g}_n^T x_0 \neq 0$. If we treat initial conditions as inputs this generates a reachability/controllability argument. Thus in order for this Jordan block to be reachable/controllable, we need to excite highest order mode (generalized eigenvector).

Ex 6.1 Let

$$\dot{x} = Ax \text{ where } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

derive the solution of $x(t)$ using modal decomposition for an arbitrary $x_0 \in \mathbb{R}^3$

Solution: We know that Jordan canonical form of matrix A has the form

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the transformation matrices that leads to this Jordan form are

$$G = \begin{bmatrix} g_1 & g_2 & v \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

where g_1 and v are eigenvectors and g_2 is the single generalized eigenvector associated with g_1 . If we follow the derivation in this lecture note, we can first find all individual components of the responses as

$$\begin{aligned} x_{g_1}(t) &= \alpha_{g_1} e^t g_1 \\ x_{g_2}(t) &= \alpha_{g_2} (te^t g_1 + e^t g_2) \\ x_v(t) &= \alpha_v e^t v \end{aligned}$$

where the combined solution and α_* 's can be derived using

$$\begin{aligned} x(t) &= x_{g_1}(t) + x_{g_2}(t) + x_v(t) = e^t ((\alpha_{g_1} + t\alpha_{g_2})g_1 + \alpha_{g_2}g_2 + \alpha_v v) \\ \begin{bmatrix} \alpha_{g_1} \\ \alpha_{g_2} \\ \alpha_v \end{bmatrix} &= G^{-1} x_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} x_0 \end{aligned}$$

6.1.2 Zero State Response & Modal Decomposition

Let's consider input driven LTI CT and DT state-space models where $x_0 = 0$

$$x(t) = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

$$y(t) = \int_0^t C e^{A(t-\tau)} B u(\tau) d\tau + D u(t)$$

$$x[k] = \begin{bmatrix} A^{k-1}B & A^{k-2}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix} = \sum_{j=0}^{k-1} A^{k-j-1} B u[j]$$

$$y[k] = \begin{bmatrix} CA^{k-1}B & CA^{k-2}B & \cdots & CAB & CB \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix} = \sum_{j=0}^{k-1} CA^{k-j-1} B u[j]$$

Now let's assume that A is diagonalizable,

$$A = V \Lambda V^{-1} = V \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \ddots \\ & & & & \lambda_n \end{bmatrix} V^{-1}, \text{ where}$$

$$V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}, \text{ where } A v_i = \lambda_i v_i$$

$$V^{-1} = \bar{V} = \begin{bmatrix} \bar{v}_1^T \\ \vdots \\ \bar{v}_n^T \end{bmatrix}, \text{ where } \bar{v}_i^T A = \lambda_i \bar{v}_i^T$$

$$\bar{V} V = V \bar{V} = I, \bar{v}_i^T v_i = 1, \bar{v}_i^T v_j = 0 \text{ for } i \neq j$$

and derive the zero-state responses in modal coordinates (for CT systems first)

$$\begin{aligned}
 x(t) &= \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \int_0^t \begin{bmatrix} e^{\lambda_1(t-\tau)} \bar{v}_1^T \\ \vdots \\ e^{\lambda_n(t-\tau)} \bar{v}_n^T \end{bmatrix} Bu(\tau) d\tau \\
 &= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \int_0^t e^{\lambda_1(t-\tau)} \bar{v}_1^T Bu(\tau) d\tau \\ \vdots \\ \int_0^t e^{\lambda_n(t-\tau)} \bar{v}_n^T Bu(\tau) d\tau \end{bmatrix} = \sum_{i=1}^n v_i \int_0^t e^{\lambda_i(t-\tau)} \beta_i u(\tau) d\tau \text{ where } \beta_i = \bar{v}_i^T B \\
 &= \sum_{i=1}^n v_i \beta_i \int_0^t e^{\lambda_i(t-\tau)} u(\tau) d\tau \\
 y(t) &= Cx(t) + Du(t) = \left[\sum_{i=1}^n C v_i \beta_i \int_0^t e^{\lambda_i(t-\tau)} u(\tau) d\tau \right] + Du(t) \\
 &= \left[\sum_{i=1}^n \gamma_i \beta_i \int_0^t e^{\lambda_i(t-\tau)} u(\tau) d\tau \right] + Du(t) \text{ where } \gamma_i = C v_i
 \end{aligned}$$

We can see that in order to observe & excite a mode associated with λ_i , we need $\gamma_i = C v_i \neq 0$ and $\beta_i = \bar{v}_i^T B \neq 0$ (only necessary conditions).

Ex 6.2 Derive $x[k]$ and $y[k]$ using modal decomposition following the derivation details explained for CT systems. (Take-home example)

Now let's try to find a similar solution for systems where matrix A can not be diagonalizable. For the sake of clarity let's focus on $A \in \mathbb{R}^{4 \times 4}$ matrices that is composed of a single Jordan block

$$A = G J G^{-1} = G \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} G^{-1}$$

where

$$\begin{aligned}
 G &= \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \\
 A g_1 &= \lambda g_1 \rightarrow (A - \lambda I) g_1 = 0 \\
 A g_2 &= \lambda g_2 + g_1 \rightarrow (A - \lambda I) g_2 = g_1, \text{ note } (A - \lambda I)^2 g_2 = 0 \text{ \& } (A - \lambda I) g_2 \neq 0 \\
 A g_3 &= \lambda g_3 + g_2 \rightarrow (A - \lambda I) g_3 = g_2, \text{ note } (A - \lambda I)^3 g_3 = 0 \text{ \& } (A - \lambda I)^2 g_3 \neq 0 \\
 A g_4 &= \lambda g_4 + g_3 \rightarrow (A - \lambda I) g_4 = g_3, \text{ note } (A - \lambda I)^4 g_4 = 0 \text{ \& } (A - \lambda I)^3 g_4 \neq 0
 \end{aligned}$$

and we also know that

$$\begin{aligned}
 G^{-1} &= \bar{G} = \begin{bmatrix} \bar{g}_1^T \\ \vdots \\ \bar{g}_n^T \end{bmatrix} \\
 \bar{G} G &= G \bar{G} = I, \quad \bar{g}_i^T g_i = 1, \quad \bar{g}_i^T g_j = 0 \text{ for } i \neq j
 \end{aligned}$$

$$\begin{aligned}
x(t) &= \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \\
&= \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \int_0^t \begin{bmatrix} e^{\lambda(t-\tau)} & (t-\tau)e^{\lambda(t-\tau)} & \frac{(t-\tau)^2}{2!}e^{\lambda(t-\tau)} & \frac{(t-\tau)^3}{3!}e^{\lambda(t-\tau)} \\ 0 & e^{\lambda(t-\tau)} & (t-\tau)e^{\lambda(t-\tau)} & \frac{(t-\tau)^2}{2!}e^{\lambda(t-\tau)} \\ 0 & 0 & e^{\lambda(t-\tau)} & (t-\tau)e^{\lambda(t-\tau)} \\ 0 & 0 & 0 & e^{\lambda(t-\tau)} \end{bmatrix} \begin{bmatrix} \bar{g}_1^T \\ \vdots \\ \bar{g}_4^T \end{bmatrix} B u(\tau) d\tau \\
&= \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \int_0^t e^{\lambda(t-\tau)} \begin{bmatrix} 1 & (t-\tau) & \frac{(t-\tau)^2}{2!} & \frac{(t-\tau)^3}{3!} \\ 0 & 1 & (t-\tau) & \frac{(t-\tau)^2}{2!} \\ 0 & 0 & 1 & (t-\tau) \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_4 \end{bmatrix} u(\tau) d\tau, \quad \beta_i = \bar{g}_i^T B \\
&= \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \int_0^t \begin{bmatrix} \beta_1 + \beta_2(t-\tau) + \beta_3 \frac{(t-\tau)^2}{2!} + \beta_4 \frac{(t-\tau)^3}{3!} \\ \beta_2 + \beta_3(t-\tau) + \beta_4 \frac{(t-\tau)^2}{2!} \\ \beta_3 + \beta_4(t-\tau) \\ \beta_4 \end{bmatrix} e^{\lambda(t-\tau)} u(\tau) d\tau \\
&= \int_0^t g_1 \left(\beta_1 + \beta_2(t-\tau) + \beta_3 \frac{(t-\tau)^2}{2!} + \beta_4 \frac{(t-\tau)^3}{3!} \right) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&\quad + \int_0^t g_2 \left(\beta_2 + \beta_3(t-\tau) + \beta_4 \frac{(t-\tau)^2}{2!} \right) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&\quad + \int_0^t g_3 (\beta_3 + \beta_4(t-\tau)) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&\quad + \int_0^t g_4 (\beta_4) e^{\lambda(t-\tau)} u(\tau) d\tau
\end{aligned}$$

whereas the output equation takes the form

$$\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= \int_0^t (Cg_1) \left(\beta_1 + \beta_2(t-\tau) + \beta_3 \frac{(t-\tau)^2}{2!} + \beta_4 \frac{(t-\tau)^3}{3!} \right) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&\quad + \int_0^t (Cg_2) \left(\beta_2 + \beta_3(t-\tau) + \beta_4 \frac{(t-\tau)^2}{2!} \right) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&\quad + \int_0^t (Cg_3) (\beta_3 + \beta_4(t-\tau)) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&\quad + \int_0^t (Cg_4) (\beta_4) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&\quad + Du(t)
\end{aligned}$$

We can see that in order to observe & excite all of the modes associated with λ , we need $\gamma_1 = Cg_1 \neq 0$ and $\beta_4 = \bar{g}_4^T B \neq 0$.

Ex 6.3 Derive $x[k]$ and $y[k]$ using modal decomposition for a $A \in \mathbb{R}^{4 \times 4}$ matrix that is composed of a single Jordan block following the derivation details explained for CT systems. (Take-home example)

6.2 Zero-State Response to Fundamental Inputs & Steady-State Response

Let's first focus on single-input CT systems

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

Fundamental test signal for the analysis of LTI CT systems is $u(t) = e^{s_0 t}$ where $s_0 \in \mathbb{C}$. Note that for a multi-input system, $u(t) \in \mathbb{C}^q$ the test signal becomes $u(t) = u_0 e^{s_0 t}$ where $u_0 \in \mathbb{C}^q$ and $u_0 \neq 0$. However as you may guess, in such a case we may need to explore more than one u_0 .

$$\begin{aligned} y(t) &= C \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau + Du(t) = C \int_0^t e^{A(t-\tau)} B e^{s_0 \tau} d\tau + D e^{s_0 t} = C \int_0^t e^{A(t-\tau)} e^{I s_0 \tau} B d\tau + D e^{s_0 t} \\ &= C e^{At} \int_0^t e^{(s_0 I - A)\tau} B d\tau + D e^{s_0 t} \end{aligned}$$

Let's assume that s_0 is not an eigenvalue, then $\det(s_0 I - A) \neq 0$, which implies that $(s_0 I - A)^{-1}$ exists. Note that

$$\int e^{B\lambda} d\lambda = B^{-1} e^{B\lambda} = e^{B\lambda} B^{-1}, \quad \text{if } \det(B) \neq 0$$

then we can write the output equation as

$$\begin{aligned} y(t) &= C e^{At} \left[e^{(s_0 I - A)\tau} \right]_{\tau=0}^{\tau=t} (s_0 I - A)^{-1} B + D e^{s_0 t} = C e^{At} \left[e^{(s_0 I - A)t} - I \right] (s_0 I - A)^{-1} B + D e^{s_0 t} \\ &= C \left[e^{I s_0 t} - e^{At} \right] (s_0 I - A)^{-1} B + D e^{s_0 t} \\ &= C e^{At} \left[-(s_0 I - A)^{-1} B \right] + \left[C(s_0 I - A)^{-1} B + D \right] e^{s_0 t} \\ &= \underbrace{C e^{At} \bar{x}_0}_{\text{transient}} + \underbrace{G(s_0) e^{s_0 t}}_{\text{steady-state}}, \quad \text{where} \end{aligned}$$

$$\bar{x}_0 = \left[-(s_0 I - A)^{-1} B \right], \quad \text{quasi initial - condition}$$

$$G(s) = C(sI - A)^{-1} B + D, \quad \text{TF - matrix}$$

Ex 6.4 Find $y(t)$ using this time using Laplace transform based solution

Now let's focus on single-input DT systems

$$x[k+1] = Ax[k] + Bu[k], \quad y = Cx[k] + Du[k]$$

Fundamental test signal for the analysis of LTI DT systems is $u(t) = z_0^k$ where $z_0 \in \mathbb{C}$. Note that for a multi-input system, $u[k] \in \mathbb{C}^q$ the test signal becomes $u[k] = u_0 z_0^k$ where $u_0 \in \mathbb{C}^q$ and $u_0 \neq 0$. However as

you may guess, in such a case we may need to explore more than one u_0 .

$$\begin{aligned} y[k] &= \left[\sum_{j=0}^{k-1} C A^{k-j-1} B u[j] \right] + D u[k] = \left[\sum_{j=0}^{k-1} C A^j B u[k-j-1] \right] + D u[k] \\ &= C \left[\sum_{j=0}^{k-1} A^j z_0^{-j} \right] B z^{k-1} + D z_0^k = C \left[\sum_{j=0}^{k-1} \left(\frac{A}{z_0} \right)^j \right] z^{k-1} B + D z_0^k \end{aligned}$$

Let $M = \left(\frac{A}{z_0} \right)$, then

$$\begin{aligned} \sum_{j=0}^{k-1} M^j &= I + M + M^2 + \dots + M^{k-1} \\ M^k + \sum_{j=0}^{k-1} M^j &= \sum_{j=0}^k M^j = I + M \sum_{j=0}^{k-1} M^j \\ (I - M) \sum_{j=0}^{k-1} M^j &= I - M^k \\ \sum_{j=0}^{k-1} M^j &= (I - M)^{-1} (I - M^k) = (I - M^k) (I - M)^{-1} \end{aligned}$$

we can then find $y[k]$ as

$$\begin{aligned} y[k] &= C \left[(I - A^k z_0^{-k}) (I - A z_0^{-1})^{-1} \right] z_0^{k-1} B + D z_0^k = C \left[(z_0^k I - A^k) (z_0 I - A)^{-1} \right] B + D z_0^k \\ &= C A^k \left(- (z_0 I - A)^{-1} B \right) + \left(C (z_0 I - A)^{-1} B + D \right) z_0^k \\ &= \underbrace{C A^k \bar{x}_0}_{\text{transient}} + \underbrace{G(z_0) z_0^k}_{\text{steady-state}}, \text{ where} \end{aligned}$$

$$\bar{x}_0 = \left(- (z_0 I - A)^{-1} B \right), \text{ quasi initial - condition}$$

$$G(z) = C(zI - A)^{-1} B + D, \text{ TF - matrix}$$

Ex 6.5 Let

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u, \quad y = \begin{bmatrix} -1 & 1 \end{bmatrix} x$$

Compute (zero-state) steady-state and transient responses for $u_1(t) = 1$ and $u_2(t) = e^{-t}$.

Solution: Let's start with SS response

$$y_1^{ss}(t) = \left(C(s_0 I - A)^{-1} B + D \right)_{s_0=0} e^{0t} = C(-A)^{-1} B = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/4 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = 1$$

$$y_2^{ss}(t) = C(-I - A)^{-1} B e^{-t} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \mathbf{0} ???$$

Now let's analyze transient response

$$\begin{aligned}
 y_1^{tr}(t) &= Ce^{At}\bar{x}_0, \text{ where } \bar{x}_0 = A^{-1}B = \begin{bmatrix} -1/2 & -1/4 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\
 &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -e^{-2t} & -te^{-2t} + e^{-2t} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (2t-1)e^{-2t} \\
 y_2^{tr}(t) &= Ce^{At}\bar{x}_0, \text{ where } \bar{x}_0 = (I+A)^{-1}B = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix} \\
 &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -4 \\ -4 \end{bmatrix} = \begin{bmatrix} -e^{-2t} & -te^{-2t} + e^{-2t} \end{bmatrix} \begin{bmatrix} -4 \\ -4 \end{bmatrix} = 4te^{-2t}
 \end{aligned}$$

Ex 6.6 Now, let $u_3(t) = e^{jt}$ and compute (zero-state) steady-state responses

Solution:

$$\begin{aligned}
 y_3^{ss}(t) &= \left(C(s_0I - A)^{-1}B \right)_{s_0=j} e^{jt} = G(j)e^{jt} = |G(j)|e^{jt+\angle[G(j)]} \\
 G(j) &= C(jI - A)^{-1}B = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} j+2 & -1 \\ 0 & j+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.4-0.2j & 0.12-0.16j \\ 0 & 0.4-0.2j \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 4 \end{bmatrix} \\
 &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.48-0.64j \\ 1.6-0.8j \end{bmatrix} = 1.12-0.16j = 1.1314\angle[-8.1301^\circ]
 \end{aligned}$$

6.2.1 Response to Real Sinusoidal Inputs

We can write CT and DT cosine/sine signals as a sum of two complex exponential functions

$$\begin{aligned}
 \cos(\omega t) &= \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}, \quad \sin(\omega t) = \frac{1}{2}e^{j\omega t} - \frac{1}{2}e^{-j\omega t}, \quad \omega \in \mathbb{R} \\
 \cos(\omega_d k) &= \frac{1}{2}e^{j\omega_d k} + \frac{1}{2}e^{-j\omega_d k}, \quad \cos(\omega t) = \frac{1}{2}e^{j\omega_d k} - \frac{1}{2}e^{-j\omega t}, \quad \omega_d \in (-\pi, \pi)
 \end{aligned}$$

Using linearity (and assuming A, B, C, D are all matrices with real entries) we can find the CT response to $\cos(\omega t)$ signal as

$$\begin{aligned}
 y_c(t) &= \frac{1}{2} [Ce^{At}\bar{x}_0^+ + G(j\omega)e^{j\omega t}] + \frac{1}{2} [Ce^{At}\bar{x}_0^- + G(-j\omega)e^{-j\omega t}] \text{ where} \\
 \bar{x}_0^+ &= [-(j\omega I - A)^{-1}B], \quad \bar{x}_0^- = [-(-j\omega I - A)^{-1}B], \quad (\bar{x}_0^+)^* = \bar{x}_0^- \\
 G(j\omega) &= [C(j\omega I - A)^{-1}B + D] = (G(-j\omega))^* \\
 y_c(t) &= Ce^{At} \left[\frac{1}{2}\bar{x}_0^+ + \frac{1}{2}\bar{x}_0^- \right] + \frac{1}{2} [G(j\omega)e^{j\omega t}] + \frac{1}{2} [G(j\omega)e^{j\omega t}]^* \text{ note} \\
 G(j\omega)e^{j\omega t} &= |G(j\omega)| (\cos \phi + j \sin \phi) \quad \phi = \angle[G(j\omega)]
 \end{aligned}$$

$$y_c(t) = \underbrace{Ce^{At}\text{Re}\{\bar{x}_0^+\}}_{\text{transient}} + \underbrace{|G(j\omega)| \cos(\omega t + \angle[G(j\omega)])}_{\text{sinusoidal steady-state}}, \text{ where}$$

$$y_s(t) = \underbrace{Ce^{At}\text{Im}\{\bar{x}_0^+\}}_{\text{transient}} + \underbrace{|G(j\omega)| \sin(\omega t + \angle[G(j\omega)])}_{\text{sinusoidal steady-state}}$$

Now let's apply the some process on DT systems with discrete-time cosine input signal

$$y_c[k] = \frac{1}{2} [CA^k \bar{x}_0^+ + G(e^{j\omega_d})e^{j\omega_d k}] + \frac{1}{2} [CA^k \bar{x}_0^- + G(e^{-j\omega_d})e^{-j\omega_d k}] \text{ where}$$

$$\bar{x}_0^+ = [-(e^{j\omega_d}I - A)^{-1}B] = (\bar{x}_0^-)^*$$

$$G(e^{j\omega_d}) = [C(e^{j\omega_d}I - A)^{-1}B + D] = (G(e^{-j\omega_d}))^*$$

$$y_c[k] = CA^k \left[\frac{1}{2} \bar{x}_0^+ + \frac{1}{2} \bar{x}_0^- \right] + \frac{1}{2} [G(e^{j\omega_d k})e^{j\omega_d k}] + \frac{1}{2} [G(e^{j\omega_d k})e^{j\omega_d k}]^* \text{ note}$$

$$y_c[k] = \underbrace{CA^k \text{Re} \{ \bar{x}_0^+ \}}_{\text{transient}} + \underbrace{|G(e^{j\omega_d k}) \cos(\omega_d k + \angle[G(j\omega)])|}_{\text{sinusoidal steady-state}}, \text{ where}$$

$$y_s[k] = \underbrace{CA^k \text{Im} \{ \bar{x}_0^+ \}}_{\text{transient}} + \underbrace{|G(e^{j\omega_d k}) \sin(\omega_d k + \angle[G(j\omega)])|}_{\text{sinusoidal steady-state}}$$

Ex 6.7 Let

$$x[k+1] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Compute (zero-state) the transient and steady-state responses for $u[k] = \cos(\omega_d k)$ for $\omega_d \in \{0, \pi/2, \pi\}$

Solution: Let's start with $\omega = 0$

$$\omega = 0 \rightarrow u[k] = 1$$

$$\bar{x}_0 = \bar{x}_0^+ = [-(I - A)^{-1}B] = - \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$y^{tr}[k] = CA^k \bar{x}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix} A^k \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \text{ note } A^0 = I \text{ \& } A^k = 0, \quad k > 1,$$

$$y^{tr}[k] = -\delta[k] - \delta[k-1]$$

$$G(e^{j0}) = G(1) = [C(I - A)^{-1}B] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

$$y^{ss}[k] = 1$$

$$y[k] = 1 - \delta[k] - \delta[k-1]$$

Now let's analyze the case $\omega = \pi/2$

$$\begin{aligned}\omega = \pi/2 &\rightarrow u[k] = \cos\left(\frac{\pi}{2}k\right) \\ \bar{x}_0^+ &= [-(jI - A)^{-1}B] = -\begin{bmatrix} j & -1 \\ 0 & j \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} j & 1 \\ 0 & j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ j \end{bmatrix} \\ y^{tr}[k] &= CA^k \text{Re}(\bar{x}_0^+) = [1 \quad 0] A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ note } A^0 = I \text{ \& } A^k = 0, \quad k > 1, \\ y^{tr}[k] &= \delta[k]\end{aligned}$$

$$\begin{aligned}G(e^{j\pi/2}) &= G(j) = [C(jI - A)^{-1}B] = [1 \quad 0] \begin{bmatrix} -j & -1 \\ 0 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1 \\ y^{ss}[k] &= \cos\left(\frac{\pi}{2}k + \pi\right) = -\cos\left(\frac{\pi}{2}k\right)\end{aligned}$$

$$y[k] = -\cos\left(\frac{\pi}{2}k\right) + \delta[k]$$

Solution for $\omega = \pi$ is take-home problem.

6.2.2 Response to Inputs of the Form, $u(t) = t^i e^{s_0 t}$ and $u[k] = k^i z_0^{k-i}$

Let $y_0(t)$ and $y_0[k]$ be the inputs for $u(t) = t^i e^{s_0 t}$ and $u_0[k] = k^i z_0^k$

$$\begin{aligned}y_0(t) &= \int_0^t C e^{t-\tau} u(\tau) d\tau = \int_0^t C e^{t-\tau} e^{s_0 \tau} d\tau \\ y_0[k] &= \sum_{j=0}^{k-1} C A^{k-j-1} B z_0^j\end{aligned}$$

Let's take the derivative of the expressions with respect to s_0 and z_0 respectively

$$\begin{aligned}\frac{\partial}{\partial s_0} y_0(t) &= \frac{\partial}{\partial s_0} \int_0^t C e^{t-\tau} (\tau e^{s_0 \tau}) d\tau = \frac{\partial}{\partial s_0} \int_0^t C e^{t-\tau} u_1(\tau) d\tau = y_1(t) \\ \frac{\partial}{\partial z_0} y_0[k] &= \frac{\partial}{\partial z_0} \sum_{j=0}^{k-1} C A^{k-j-1} B (j z_0^{j-1}) = \frac{\partial}{\partial z_0} \sum_{j=0}^{k-1} C A^{k-j-1} B u_1[k] = y_1[k]\end{aligned}$$

It is straightforward to extend the result for higher order inputs of the form $u(t) = t^i e^{s_0 t}$ and $u[k] = k^i z_0^{k-i}$. Let's explore $y_1(t)$

$$\begin{aligned}y_1(t) &= \frac{\partial}{\partial s_0} y_0(t) = C e^{At} \left[-\left\{ \frac{\partial}{\partial s_0} (s_0 I - A)^{-1} \right\} B \right] + \left[C \left\{ \frac{\partial}{\partial s_0} (s_0 I - A)^{-1} \right\} B \right] e^{s_0 t} \\ &\quad + [C(s_0 I - A)^{-1} B + D] t e^{s_0 t}\end{aligned}$$

Let's derive $\left\{ \frac{\partial}{\partial s_0} (s_o I - A)^{-1} \right\}$

$$\begin{aligned} (s_o I - A)^{-1} (s_o I - A) &= I \\ \frac{\partial}{\partial s_0} [(s_o I - A)^{-1} (s_o I - A)] &= 0 \\ \frac{\partial}{\partial s_0} [(s_o I - A)^{-1}] (s_o I - A) + (s_o I - A)^{-1} &= 0 \\ \frac{\partial}{\partial s_0} [(s_o I - A)^{-1}] &= -(s_o I - A)^{-2} \end{aligned}$$

then we can express $y_1(t)$ as

$$y_1(t) = C e^{At} [(s_o I - A)^{-2} B] + [-C(s_o I - A)^{-2} B] e^{s_0 t} + [C(s_o I - A)^{-1} B + D] t e^{s_0 t}$$

$$\begin{aligned} \bar{x}_{0,1} &= [(s_o I - A)^{-2} B] \\ G'(s_0) &= -C(s_o I - A)^{-2} B \end{aligned}$$

$$y_1(t) = \underbrace{C e^{At} \bar{x}_{0,1}}_{\text{transient}} + \underbrace{G'(s_0) e^{s_0 t} + G(s_0) t e^{s_0 t}}_{\text{steady-state}}$$

It is relatively easy to derive a similar expression for $y_1[k]$.

Ex 6.8 Let

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u, \quad y = [-1 \quad 1] x$$

Compute (zero-state) steady-state and transient responses for $u(t) = t$

Solution: $s_0 = 0$. Let's first compute the transient part **Solution:** Let's start with SS response

$$\begin{aligned} y^{ss}(t) &= \left(C (-A)^{-1} B \right) t - \left(C (-A)^{-2} B \right) \\ &= [-1 \quad 1] \begin{bmatrix} 1/2 & 1/4 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} t + [-1 \quad 1] \begin{bmatrix} 1/2 & 1/4 \\ 0 & 1/2 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 4 \end{bmatrix} \\ &= t + [-1 \quad 1] \begin{bmatrix} 1/4 & 1/4 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = t - 0 \\ &= t \end{aligned}$$

Now let's analyze transient response

$$\begin{aligned} y^{tr}(t) &= C e^{At} \bar{x}_{0,1}, \quad \text{where } \bar{x}_0 = (-A)^{-2} B = \begin{bmatrix} -1/2 & -1/4 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\ &= [-1 \quad 1] \begin{bmatrix} e^{-2t} & t e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = [-e^{-2t} \quad -t e^{-2t} + e^{-2t}] \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (2t - 1) e^{-2t} \end{aligned}$$