13.1 State Feedback & Stabilizability

The state-feedback based control-policies for LTI systems starts with the assumption that we have "access" to the all of the states of the systems either via direct measurement or through some observer/estimator/tracker. In that context a family of state-feedback controllers for CT- and DT-LTI systems can be constructed as

$$u(t) = \gamma r(t) - Kx(t) \ \& \ u[k] = \gamma r[k] - Kx[k]$$

where $r(t)$ can be considered as the reference signal (most of the time it is), γ is a feed-forward scaling factor, and K is the state-feedback gain. Now let's find a state-space representation for dynamics of the closed-loop system for both CT- and DT-LTI systems under state-feedback rule proposed above

$$\dot{x} = Ax + B\left(\gamma r(t) - Kx(t)\right) \Rightarrow \dot{x} = (A - BK)\,x + \gamma Br$$
$$x[k+1] = Ax + B\left(\gamma r[k] - Kx[k]\right) \Rightarrow x[k+1] = (A - BK)\,x[k] + \gamma Br[k]$$

In both cases the closed loop system and input matrices takes the following form

$$A_c = A - BK \ , \ B_c = \gamma Br$$

A key question in this domain is that can I find a K such that eigenvalues of A_c is located at arbitrary desired locations.

Theorem: (Eigenvalue/Pole Placement) Given (A, B), $\exists K$ s.t.

$$\det\left[\lambda I - (A - BK)\right] = \lambda^n + a^*_{n-1}\lambda^{n-1} + \cdots + a^*_1\lambda + a^*_0$$
$$\forall \mathcal{A} = \{a^*_0,\, a^*_1 \cdots a^*_{n-1}\},\, a^*_i \in \mathbb{R}$$

if and only if (A, B) is reachable.

Proof: For a general complete proof we need to show that reachability of (A, B) is necessary and sufficient.

Proof of necessity: Let's assume that (A, B) not reachable and $\exists(\lambda_u, w_u^T)$ pair such that $w_u^T A = w_u^T \lambda_u$ and $w_u^T = 0$. Now check weather w_u^T is a left eigenvector of A_c

$$w_u^T A_c = w_u^T (A - BK) = w_u^T A - w_u^T BK = w_u^T \lambda_u - 0 = w_u^T \lambda_u$$
$$w_u^T B_c = w_u^T B\gamma = 0$$

Here not only we showed that λ_u can not be moved hence it is not possible to locate the poles arbitrary locations, we also showed that state-feedback rules does not affect the reachability.

Proof of sufficiency: We will only show the sufficiency for a multi-input case, i.e. $B \in \mathbb{R}^{n\times 1}$, however the reader should not the fact that for a complete proof multi-input case also needs to be analyzed. Let's assume that (A, B) is reachable and we know that reachability is invariant under similarity transformations, i.e.

$$z = T^{-1}x\,,\ \det(T) \neq 0 \Rightarrow \dot{z} = \bar{A}z + \bar{B}u$$
$$\bar{A} = T^{-1}AT \ , \ T^{-1}B$$

and $(\bar{A}, \bar{B})$ is reachable. Noe let's choose T such that

$$T = \mathbf{R} = \left[\ A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B\ \right]$$

then $\bar{B}$ can be derived as

$$B = T\bar{B} = \left[\ A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B\ \right]\bar{B} \Rightarrow B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and similarly $\bar{A}$ can be expressed as

$$A\mathbf{R} = \mathbf{R}\bar{A}$$

$$A\left[\ A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B\ \right] = \left[\ A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B\ \right]\bar{A}$$

$$\left[\ A^{n}B \mid A^{n-1}B \mid \cdots \mid AB \mid B\ \right] = \left[\ A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B\ \right] \left[\begin{array}{c|c|c|c|c|c} \bar{a}_{11} & 1 & 0 & 0 & \cdots & 0 \\ \bar{a}_{12} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \bar{a}_{1(n-2)} & 0 & 0 & \cdots & 1 & 0 \\ \bar{a}_{1(n-1)} & 0 & 0 & \cdots & 0 & 1 \\ \bar{a}_{1(n)} & 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

We can find a_{1i}'s using Cayley-Hamilton theorem

$$A^{n}B = \bar{a}_{11}A^{n-1}B + \bar{a}_{12}A^{n-2}B + \cdots + \bar{a}_{1(n-1)}AB + \bar{a}_{1(n)}B$$
$$A^{n} = -\left(a_{n-1}A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_{1}A + a_{I}\right) \text{ where}$$
$$\det[\lambda I - A] = \lambda^{n} + a_{n-1}\lambda^{n-1} + \cdots + a_{1}\lambda + a_{0}$$

then $\bar{A}$ takes the form

$$\bar{A} = \left[\begin{array}{c|c|c|c|c|c} -a_{n-1} & 1 & 0 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ -a_{2} & 0 & 0 & \cdots & 1 & 0 \\ -a_{1} & 0 & 0 & \cdots & 0 & 1 \\ -a_{0} & 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

Note that this similarity transformation is in a companion form however we will transform this into (more useful) reachable canonical form

$$\tilde{A} = \left[\begin{array}{c|c|c|c|c|c} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_{0} & -a_{1} & -a_{2} & \cdots & -a_{n-2} & -a_{n-1} \end{array}\right], \ \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Let's $\tilde{\mathbf{R}}$ be the reachability matrix of $(\tilde{A}, \tilde{B})$ then we know that

$$\bar{A} = \tilde{\mathbf{R}}^{-1}A\tilde{\mathbf{R}}\ ,\ \bar{B} = \tilde{\mathbf{R}}^{-1}B$$

where $(\bar{A}, \bar{B})$ are the matrices of the companion form derived above. Thus if we let $\hat{T} = \mathbf{R}\tilde{\mathbf{R}}^{-1}$, where $\mathbf{R}$ is the reachability matrix of the original representation and $\tilde{\mathbf{R}}$ is the reachability matrix of the reachable canonical form and adopt T for similarity transformation we obtain

$$q = \hat{T}^{-1}x \Rightarrow \dot{q} = \hat{A}q + \hat{B}u$$
$$\tilde{A} = \hat{T}^{-1}A\hat{T}\ , \ \tilde{B} = \hat{T}^{-1}B$$