EE302 - Feedback Systems **Spring 2019**

Lecture 17

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17.1 Stability Margins: Gain & Phase Margin

We already know that a binary stability metric is not enough to characterize the system performance and that we need metrics to evaluate how stable the system is and its robustness to perturbations. Using root-locus techniques we talked about some "good" pole regions which provides some specifications about stability and closed-loop performance.

Another common and powerful method is to use stability margins, specifically gain and phase margins, based on the frequency domain analysis of a feedback-system.

Phase and gain margins are derived from the Nyquists stability criterion and it is relatively easy to compute them only from the Polar Plot or Bode diagrams for a class of systems.

In this part of the course, we assume that

- Open-loop transfer function of the feedback system is a *minimum-phase* system, i.e.
  - No poles/zeros in the Open Right Half Plane
  - $\lim_{\omega \to \infty} [\frac{G_{OL}(s)}{s}]_{s=j\omega} = 0$
- The feed-back system is Type $0-2$ (i.e. no integrator of order larger than 3 in the open-loop transfer function).
- Polar plot of $G(j\omega)$ crosses the negative real-axis at most once.

Gain Margin

For a stable-system the *gain margin*, g_m, of a system is defined as the smallest amount that the open loop gain can be increased before the closed loop system goes unstable.

In terms of Nyquist & polar plot, we simply choose point, σ_{pc} where the polar plot crosses the negative-real axis and gain margin is simply equal to $g_m = \frac{1}{\sigma_p}$.

Alternatively, the gain margin can be computed based on the frequency where the phase of the loop transfer function $G_{OL}(j\omega)$ is -180^o. Let ω_p represent this frequency, called the phase crossover frequency. Then the gain margin for the system is given by

$$\angle[G_{OL}(j\omega_p)] = \pm - 180^0 \quad \Rightarrow \quad g_m = \frac{1}{|G_{OL}(j\omega_p)|} \quad \text{or} \quad G_m = -20\log_{10}|G_{OL}(j\omega_p)|$$

where G_m is the gain margin in dB scale. If the phase response never crosses the -180^o, i.e. $Re\{G(j\omega)\} \geq 0 \ \forall \ \omega \in [0, \infty]$, gain margin is simply ∞. Higher the gain margin is more robust and stable closed-loop system is.

Phase Margin

The *phase margin* is the amount of "phase lag" required to reach the (Nyquist) stability limit.

In terms of Nyquist & polar plot, we simply choose point, where the polar plot crosses the unit-circle, and phase margin is simply the "angular distance" between this point and the critical point $-1 + 0j$ in CW direction.

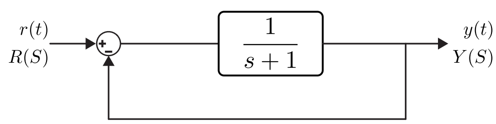
Alternatively, let ω_{gc} be the gain crossover frequency, the frequency where the loop transfer function satisfies $|G_{OL}(j\omega_g)| = 1$ (i.e. unit magnitude). The phase margin is given by

$$|G_{OL}(j\omega_g)| = 1 \quad \Rightarrow \quad \phi_m = \pi + \angle G_{OL}(j\omega_{gc})$$

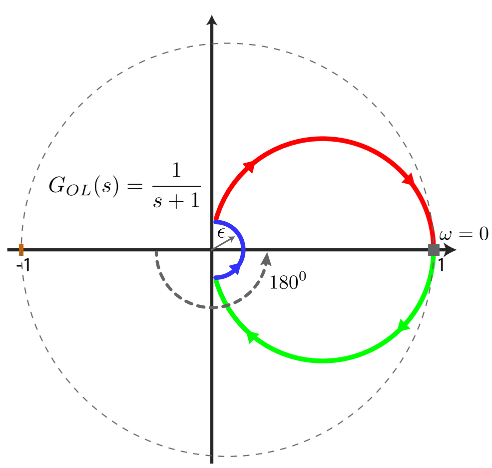
Higher the phase margin is more robust and stable closed-loop system is. Moreover, negative phase simply shows that the closed-loop system is indeed unstable.

Note that if the $G(j\omega)$ is strictly inside the unit-circle, then we can not compute the phase-crosover frequency which simply implies that $\phi_m = \infty$.

Ex: Compute the gain margin and phase margin for the following closed-loop system

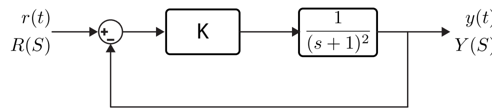


We already derived the Nyquist plot for this system



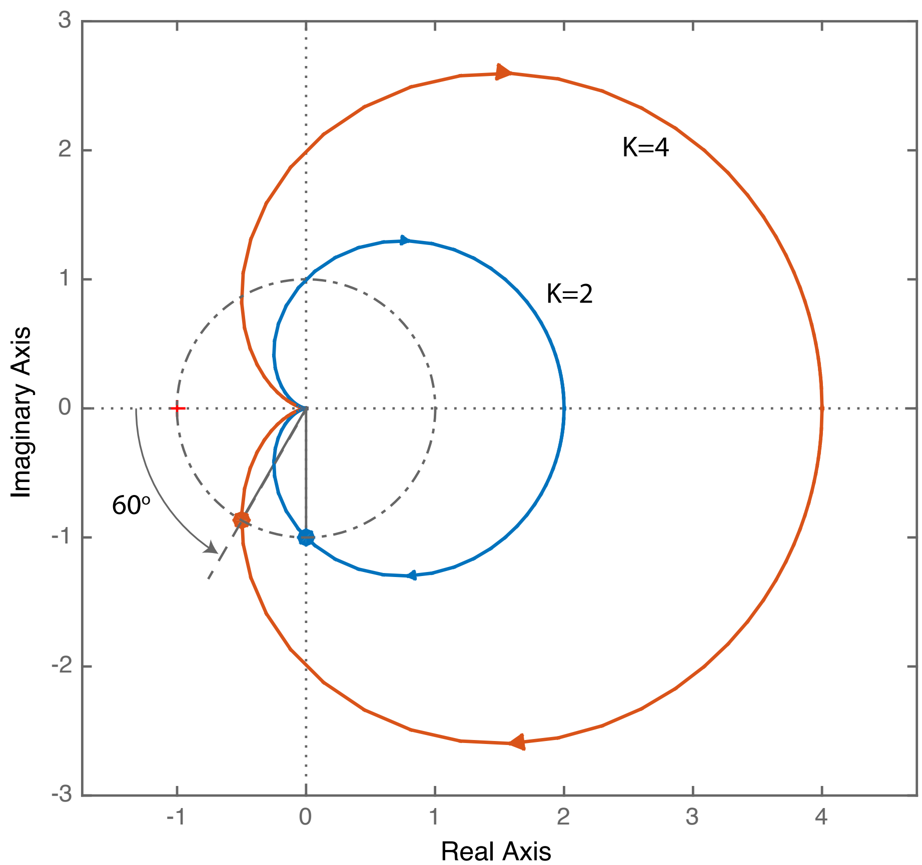
We can see that the Real part of the polar polat is always positive, thus $g_m = \infty$. Where as the polar plot crosses the unit circle only when $\omega = 0$, thus $\phi_m = 180^0$.

Ex: Compute the gain margin and phase margin for the following closed-loop system for $K = 2$ and $K = 4$.



Nyquist plots for both gain cases is illustrated in the Figure below. We can see from the illustration that

$$K = 2 \Rightarrow \phi_m = 90^o \ \& \ g_m = \infty$$
$$K = 4 \Rightarrow \phi_m = 60^o \ \& \ g_m = \infty$$



Now let's try to compute the phase margins analytically. Let's start with $K = 2$

$$|G(j\omega_g)| = 1 \ \rightarrow \ \frac{2}{\omega_g^2 + 1} = 1 \rightarrow \ \omega_g = 1$$
$$\angle[G(j)] = -2\angle[j + 1] = -90^o$$
$$\phi_m = 90^o$$

Now let's compute the phase margin for $K = 4$

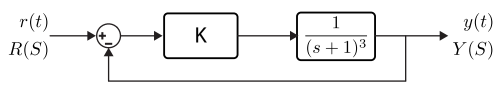
$$|G(j\omega_g)| = 1 \ \rightarrow \ \frac{4}{\omega_g^2 + 1} = 1 \rightarrow \ \omega_g = \sqrt{3}$$
$$\angle[G(j\sqrt{3})] = -2\angle[\sqrt{3}j + 1] = -120^o$$
$$\phi_m = 60^o$$

Now let's compute the closed-loop transfer function and compre the damping coefficients for both gain cases

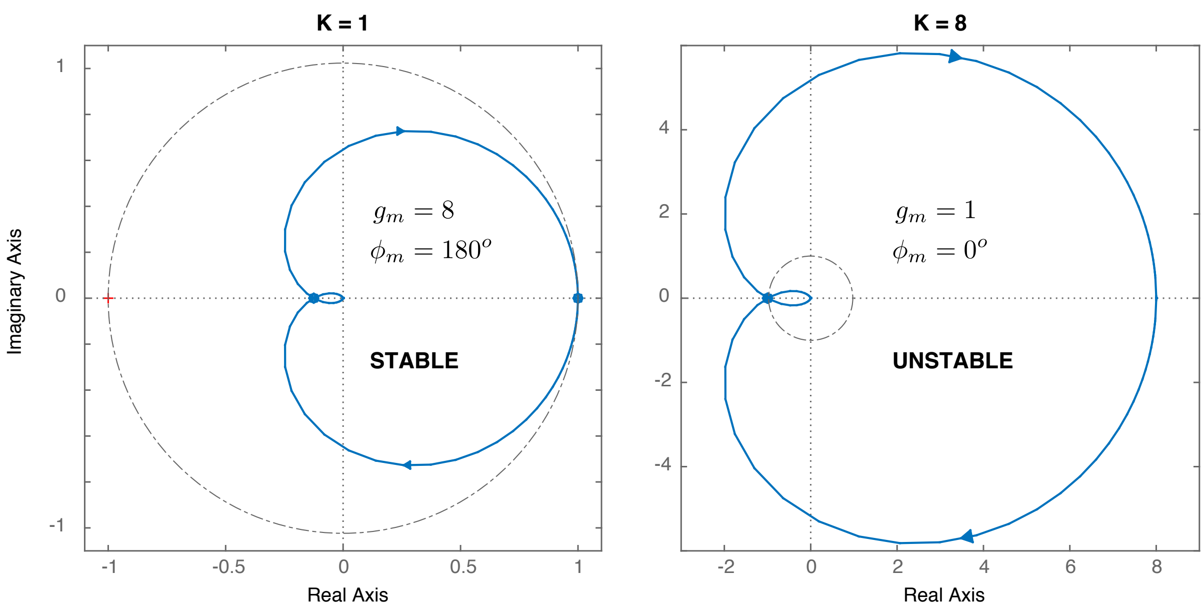
$$T_2 = \frac{2}{s^2 + 2s + 3} \ \rightarrow \ \zeta_2 = \frac{1}{\sqrt{3}}$$
$$T_4 = \frac{4}{s^2 + 2s + 5} \ \rightarrow \ \zeta_2 = \frac{1}{\sqrt{5}}$$

We can see that as we decrease the phase margin from 90^o to 60^o, we also decrease the damping ration which results in increased maximum-overshoot. In general good phase margin provides good transent performance in time domain.

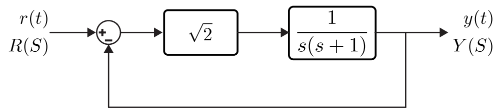
Ex: Compute the gain margin and phase margin for the following closed-loop system for $K = 1$ and $K = 8$ and comment on the stability of the system for both cases.



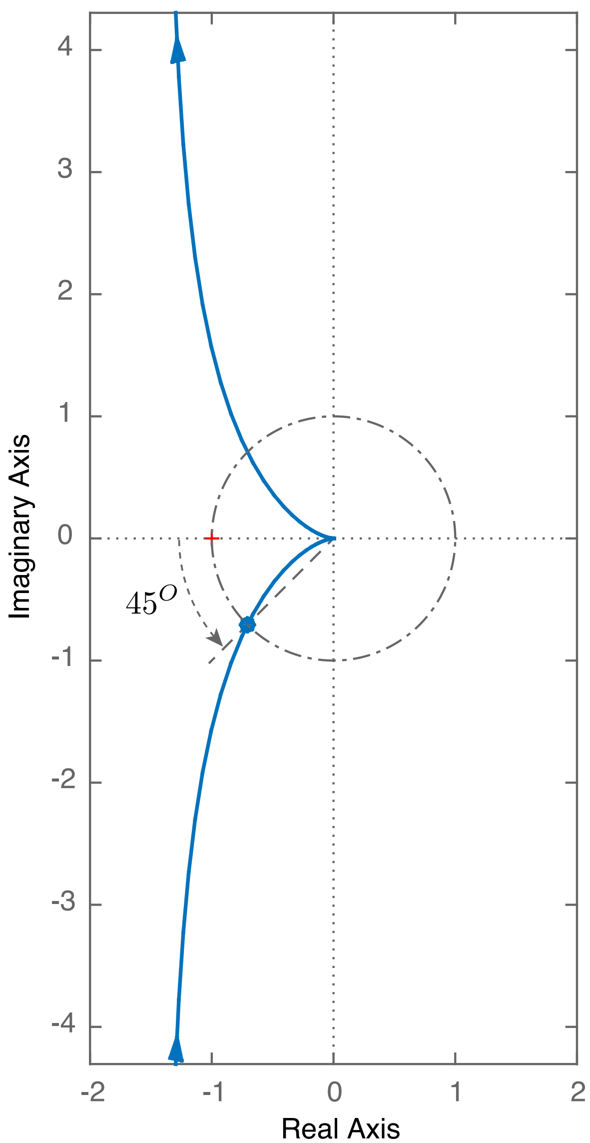
We already derived the Nyquist plot for the case $K = 1$, now let's illustrate both Nyquist plots side-by-side.



Ex: Compute the gain margin and phase margin for the following closed-loop system



We already derived the Nyquist plot for the case $K = 1$, now we have a different gain. Figure below illustrates the zoomed Nyquist plot (which is the important part for gain and phase margin computations).



$|G(j\omega_g)| = 1 \ \rightarrow |G(j\omega_g)|^2 = 1$

$$\frac{2}{\omega^2(\omega^2+1)} = 1$$

$\rightarrow \ \omega_g = 1$

$\angle[G(j)] = -(90^o + 45^0)$

$\phi_m = 45^0$

$g_m = \infty$