EE402 - Discrete Time Systems **Spring 2018**

Lecture 11

Lecturer: Asst. Prof. M. Mert Ankarali

11.1 Observability in CT-LTI Systems

In the context of observability of dynamical systems, it turns out that it is more convenient to think in terms of "un-observabile states" and then connect it to the concept of observability and fully observable systems. as reflected in the following definitions.

Definition: For LTI a contınıous-time state-space representation

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

A state x_u is said to be unobservable over $t \in [0, T)$, if with $x(0) = x_u$ and $\forall\, u(t)$ we get the same $y(t)$ as we would with $x(0) = 0$.

The set, $\bar{\mathcal{O}}_T$, of all unobservale states over $t \in [0, T)$ forms a vecotor space, $\bar{\mathcal{O}}_T \subset \mathbb{R}^n$, and the system is called fully observable over $t \in [0, T)$, if $\dim[\bar{\mathcal{O}}_T] = 0$.

Note for linear dynamical systems observability of a state and system is independent from $u(t)$, in that respect we will only analuze zero-input response of the system in our derivations.

Theorem: $x_u \in \bar{\mathcal{O}}_T \iff x_u \in \bar{\mathcal{O}}_\tau, \forall \tau > 0 \iff x_u \in \mathcal{N}[\mathbf{O}]$, where

$$\mathbf{O} = \begin{bmatrix} C \\ \hdashline CA \\ \hdashline CA^2 \\ \hdashline \vdots \\ \hdashline CA^{n-1} \end{bmatrix}$$

Let's first show that $x_u \in \mathcal{N}[\mathbf{O}] \iff x_u \in \bar{\mathcal{O}}_\tau, \forall \tau > 0$. Let $x_u \in \mathcal{N}[\mathbf{O}]$, then

$$\mathbf{O}x_u = 0 \rightarrow \begin{array}{c} Cx_u = 0 \\ \hdashline CAx_u = 0 \\ \hdashline CA^2x_u = 0 \\ \hdashline \vdots \\ \hdashline CA^{n-1}x_u = 0 \end{array}$$

Moreover, by Cayley-Hamilton theorem, we can also conclude that $CA^l x_u = 0, \ \forall l \in \mathbf{Z}^+$. Now let's analyze the zero-input response of the system with $x(0) = x_u$

$$x(\tau) = Ce^{A\tau}x_u = 0 \Rightarrow x_u \in \bar{\mathcal{O}}_\tau$$

and indeed it is true for all $\tau \in \mathbb{R}$. Now let's show that $x_u \in \bar{\mathcal{O}}_T \Rightarrow x_u \in \mathcal{N}[\mathbf{O}]$

$$x(t) = Ce^{At}x_u = 0, \forall t \in [0, \tau], \forall \tau \in \mathbb{R} \Rightarrow x_u \in \bar{\mathcal{O}}_\tau$$

Now let's show that $x_u \in \bar{\mathcal{O}_T} \Rightarrow x_u \in \mathcal{N}[\mathbf{O}]$. If $x_u \in \bar{\mathcal{O}_T}$, then

$$x(t) = Ce^{At}x_u = 0, \forall t \in [0,T] \Rightarrow \begin{array}{c} x(0) = 0 \Rightarrow Cx_u = 0 \\ \left[\frac{d}{dt}x(t)\right]_{t=0} = 0 \Rightarrow CAx_u = 0 \\ \left[\frac{d^2}{dt^2}x(t)\right]_{t=0} = 0 \Rightarrow CA^2x_u = 0 \\ \vdots \\ \left[\frac{d^{n-1}}{dt^{n-1}}x(t)\right]_{t=0} = 0 \Rightarrow CA^{n-1}x_u = 0 \end{array} \Rightarrow \mathbf{O}x_u = 0 \iff x_u \in \mathcal{N}[\mathbf{O}]$$

Similar to the reachability, we show that for CT-LTI systems observability and unobservable (and observable) subspace are independent of time.

11.1.1 Observability Grammian

For a CT-LTI system, observability Grammian is defined as

$$\mathbf{Q}(t) = \int_0^t e^{A^T(t-\tau)}C^TCe^{A(t-\tau)}d\tau$$

Theorem: $\mathcal{N}[\mathbf{Q}(t)] = \mathcal{N}[\mathbf{O}]\ \forall t > 0$.

Proof: Let's first show that $\mathcal{N}[\mathbf{O}] \subset \mathcal{N}[\mathbf{Q}(t)]\ \forall t > 0$. If $x_u \in \mathcal{N}[\mathbf{O}]$, then we know that $CA^lx_u = 0,\ \forall l \in \mathbb{Z}^{\geq 0}$. Let's anlayze if x_u is in the null-space of $\mathbf{Q}(t)$

$$\mathbf{Q}(t)x_u = \int_0^t e^{A^T(t-\tau)}C^TCe^{A(t-\tau)}x_u d\tau = 0 \Rightarrow x_u \in \mathcal{N}[\mathbf{Q}(t)]\ \forall t > 0$$

Now let's show that $\mathcal{N}[\mathbf{Q}(t)] \subset \mathcal{N}[\mathbf{O}],\ \forall t > 0$. Let $x_u \in \mathcal{N}[\mathbf{Q}(t)]$, then

$$\mathbf{Q}(t)x_u = 0 \Rightarrow x_u^T\mathbf{Q}(t)x_u \iff \int_0^t x_u^Te^{A^T(t-\tau)}C^TCe^{A(t-\tau)}x_u d\tau = 0 \iff Ce^{A(t-\tau)}x_u = 0\,\forall\tau \in [0,t]$$

Then we know that

$$\begin{aligned} [Ce^{A\eta}x_u]_{\eta=0} = 0 &\Rightarrow Cx_u = 0 \\ \frac{d}{d\eta}[Ce^{A\eta}x_u]_{\eta=0} = 0 &\rightarrow CAx_u = 0 \\ \frac{d^2}{d\eta^2}[Ce^{A\eta}x_u]_{\eta=0} = 0 &\Rightarrow CA^2x_u = 0 \\ &\vdots \\ \frac{d^{n-1}}{d\eta^{n-1}}[Ce^{A\eta}x_u]_{\eta=0} = 0 &\Rightarrow CA^{n-1}x_u = 0 \\ &\Rightarrow \mathbf{O}x_u = 0 \Rightarrow x_u \in \mathcal{N}[\mathbf{O}] \Rightarrow \mathcal{N}[\mathbf{Q}(t)] \subset \mathcal{N}[\mathbf{O}]\,\forall t > 0 \end{aligned}$$

Ex 11.1 *Show that*

if $\dot{x} = Ax$ is asymptotically stable, then observability Grammian at $t \to \infty$, $Q := \mathbf{Q}_\infty$, satisfies the following Lyapunov equation

$$AQ + QA^T = -C^T C$$

, and this Lyapunov equation has a (unique) positive-definite solution for Q, if and only if, (A, C) is fully observable.

11.1.2 Futher Results in CT-LTI Observability

In view of duality, we can use our reachability results to immediately derive various conclusions, tests, standard and canonical forms, etc., for observable and unobservable systems. The reader is highly encouraged to derive the following results, theorems, and claims by referring to the dual results in the reachability lecture.

Result 1: The unobservable sub-space, $\mathcal{N}[\mathbf{O}]$ is A invariant, i.e. $x \in \mathcal{N}[\mathbf{O}] \;\Rightarrow\; Ax \in \mathcal{N}[\mathbf{O}]$.

Result 2: (A, C) pair is unobservable $\iff$ $Cv = 0$ for some right eigenvector of A, i.e. $Av = \lambda v \iff$

$$\text{rank} \begin{bmatrix} \lambda I - A \\ \hline C \end{bmatrix} = n, \ \forall \lambda \in \mathcal{C}$$

Result 3: Observability is invariant under state/similarity transformation.