

11.1 State-Feedback & Pole Placement

Given a discrete-time state-evolution equation

$$x[k+1] = Gx[k] + Hu[k]$$

If direct measurements of all of the states of the system (e.g. $y[k] = x[k]$) are available, one of the most popular control methods is the linear state feedback control,

$$u[k] = -Kx[k]$$

which can be thought as a generalization of P controller to the vector form. Under this control law, without any reference signal, the system becomes an autonomous system

$$\begin{aligned} x[k+1] &= Gx[k] + H(-Kx[k]) \\ x[k+1] &= (G - HK)x[k] \end{aligned}$$

The system matrix of this new autonomous system is $\hat{G} = G - HK$. The important question is how we need to choose K ?. Note that

$$\begin{aligned} K &\in \mathbb{R}^n \quad \text{Single - Input} \\ K &\in \mathbb{R}^{n \times p} \quad \text{Multi - Input} \end{aligned}$$

As in all of the control design techniques, the most critical criterion is stability, thus we want all of the eigenvalues to be strictly inside the unit circle. However, we know that there could be different requirements on the poles/eigenvalues of the system.

The fundamental principle of “pole-placement” design is that we first define a desired closed-loop eigenvalue set $\mathcal{E}^* = \{\lambda_1^*, \dots, \lambda_n^*\}$, and then if possible we choose K^* such that the closed-loop eigenvalues match the desired ones.

The necessary and sufficient condition on arbitrary pole-placement is that the system should be fully reachable.

In Pole-Placement, the first step is computing the desired characteristic polynomial.

$$\begin{aligned} \mathcal{E}^* &= \{\lambda_1^*, \dots, \lambda_n^*\} \\ p^*(z) &= (z - \lambda_1^*) \cdots (z - \lambda_n^*) \\ &= z^n + a_1^* z^{n-1} + \cdots + a_{n-1}^* z + a_n^* \end{aligned}$$

Then we tune K such that

$$\det(zI - (G - HK)) = p^*(z)$$

Direct Design of State-Feedback Gain

If n is small, the most efficient method could be the direct design.

Example: Consider the following DT system

$$x[k+1] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]$$

Design a state-feedback rule such that poles are located at $\lambda_{1,2} = 0$ (Dead-beat gain)

Solution: Desired characteristic equation can be computed as

$$p^*(z) = z^2$$

Let $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, then the characteristic equation of \hat{G} can be computed as

$$\begin{aligned} \det(zI - (G - HK)) &= \det\left(\begin{bmatrix} z - 1 + k_1 & k_2 \\ k_1 & z - 2 + k_2 \end{bmatrix}\right) \\ &= z^2 + z(k_1 + k_2 - 3) + (2 - 2k_1 - k_2) \end{aligned}$$

If we match the equations

$$\begin{aligned} z^2 + z(k_1 + k_2 - 3) + (2 - 2k_1 - k_2) &= z^2 \\ k_1 + k_2 &= 3 \\ 2k_1 + k_2 &= 2 \\ k_1 &= -1 \\ k_2 &= 4 \end{aligned}$$

Thus $K = \begin{bmatrix} -1 & 4 \end{bmatrix}$. Now let's compute \hat{G}^k

$$\begin{aligned} \hat{G} &= \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \\ \hat{G}^2 &= \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &\vdots \end{aligned}$$

It can be seen that closed-loop system rejects all initial condition perturbations in 2 steps.