EE502 - Linear Systems Theory II **Spring 2032**

Lecture 12

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12.1 State-Space Models of Dynamical Systems

For a causal dynamical system, in order to compute output at a given time, t_0 (or k_o for DT systems), we need to know "only" the input signal over $(-\infty, t_o]$ (or $(-\infty, k_o]$ for DT systems). This requires a lot of data/information (indeed infinite amount of data), can we summarize it with something more manageable ? For example using some latent variables stored in a vector, a.k.a. *state-vector*.

State property of DT state-space models: Given the state vector $x[k_0]$ and input $u[k_0]$ at an arbitrary time k_0, we can compute the the present output, $y[k_0]$, and next state $x[k_0 + 1]$.

State property of CT state-space models: Given the initial time, t_0 and state $x(t_0)$ and input $u(t)$ for $t_0 \leq t \leq t_f$ (with t_0 & t_f arbitrary), we can compute the output $y(t)$ for $t_0 \leq t \leq t_f$ and the state $x(t)$ for $t_0 \leq t \leq t_f$.

In other words $x(t_0)$ (and $x[k_0]$ for DT systems) summarizes the whole input history, $t \in (-\infty, t_0)$ (or $k \in (-\infty, k_0)$) in a compact (most probably *finite-dimensional*) memory package, for the purpose of predicting the future output (and states)

Note that both definitions are not limited to LTI state-space models. Nonlinear and time-varying state-space models also are based on this definition. CT state-property is more and also applies for DT state-space models.

Note that the choice of state-variables is not unique (and there exist infinite possible of *realizations*), however, there are some options that are preferable to others (minimal representations, canonical forms, practical benefits etc.). When a state-space representation includes minimum possible of number of state variables, the representation is called minimal.

12.1.1 State-Space Representations of LTI, LTV, & Non-Linear Dynamical Systems

LTI Systems

State-space representation of a (causal & finite dimensional) LTI CT system is given by

$$\begin{aligned} \text{Let } x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R}^m \ , \ u(t) \in \mathbb{R}^r, \\ \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \\ \text{where } A \in \mathbb{R}^{n \times n} \ , \ B \in \mathbb{R}^{n \times r} \ , \ C \in \mathbb{R}^{m \times n} \ , \ D \in \mathbb{R}^{m \times r} \end{aligned}$$

State-space representation of a (causal & finite dimensional) LTI DT system is given by

$$\begin{aligned} \text{Let } x[k] \in \mathbb{R}^n \ , \ y[k] \in \mathbb{R}^m \ , \ u[k] \in \mathbb{R}^r, \\ x[k+1] = Gx[k] + Hu[k], \\ y[k] = Cx[k] + Du[k], \\ \text{where } G \in \mathbb{R}^{n \times n} \ , \ H \in \mathbb{R}^{n \times r} \ , \ C \in \mathbb{R}^{m \times n} \ , \ D \in \mathbb{R}^{m \times r} \end{aligned}$$

LTV Systems

State-space representation of a (causal & finite dimensional) LTV CT system is given by

$$\begin{aligned}\text{Let } x(t) &\in \mathbb{R}^n\ ,\ y(t) \in \mathbb{R}^q\ ,\ u(t) \in \mathbb{R}^p,\\ \dot{x}(t) &= A(t)x(t) + B(t)u(t),\\ y(t) &= C(t)x(t) + D(t)u(t),\\ \text{where } A(t) &\in \mathbb{R}^{n\times n}\ ,\ B(t) \in \mathbb{R}^{n\times p}\ ,\ C(t) \in \mathbb{R}^{q\times n}\ ,\ D(t) \in \mathbb{R}^{q}\end{aligned}$$

State-space representation of a (causal & finite dimensional) LTV DT system is given by

$$\begin{aligned}\text{Let } x[n] &\in \mathbb{R}^n\ ,\ y[n] \in \mathbb{R}^q\ ,\ u[n] \in \mathbb{R}^p,\\ x[n+1] &= A[n]x[n] + B[n]u[n],\\ y[n] &= C[n]x[n] + D[n]u[n],\\ \text{where } A[n] &\in \mathbb{R}^{n\times n}\ ,\ B[n] \in \mathbb{R}^{n\times p}\ ,\ C[n] \in \mathbb{R}^{q\times n}\ ,\ D[n] \in \mathbb{R}^{q}\end{aligned}$$

Non-Linear Systems

State-space representation of a (causal & finite dimensional) non-linear CT system is given by

$$\begin{aligned}\text{Let } x(t) &\in \mathbb{R}^n\ ,\ y(t) \in \mathbb{R}^q\ ,\ u(t) \in \mathbb{R}^p,\\ \dot{x}(t) &= F(x(t), u(t)),\\ y(t) &= H(x(t), u(t)),\end{aligned}$$

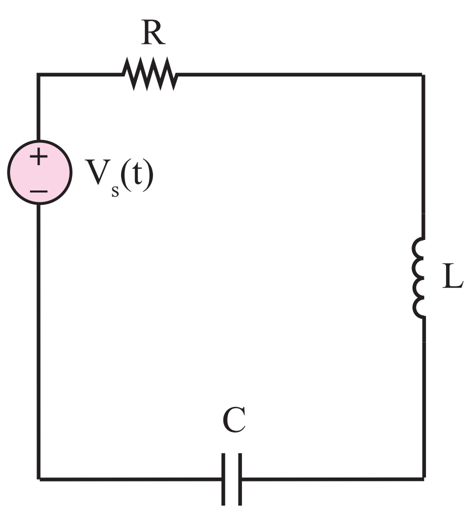
State-space representation of a (causal & finite dimensional) non-linear DT system is given by

$$\begin{aligned}\text{Let } x[n] &\in \mathbb{R}^n\ ,\ y[n] \in \mathbb{R}^q\ ,\ u[n] \in \mathbb{R}^p,\\ x[n+1] &= F(x[n], u[n]),\\ y[n] &= H(x[n], u[n]),\end{aligned}$$

12.1.2 Example CT and DT System Models

Ex 1: Series RLC Circuit

Given than input is $u(t) = V_s(t)$ and output os $y(t) = V_C(t)$, first find an ODE description of the given dynamical circuit.



$$\begin{aligned}V_L + V_R + V_C &= V_s(t)\\ L\frac{dI}{dt} + RI + V_C &= V_s(t)\\ L\frac{d}{dt}\left(C\frac{dV_C}{dt}\right) + R\left(C\frac{dV_C}{dt}\right) + V_C &= V_s(t)\\ LC\ \ddot{V}_C + RC\ \dot{V}_C + V_C &= V_s(t)\\ \ddot{y} + \frac{R}{L}\dot{y} + \frac{1}{LC}y &= \frac{1}{LC}u\end{aligned}$$

Now, find the transfer function representation of the system for the given input–output pair.

$$\mathcal{L}\left\{\ddot{y} + \frac{R}{L}\dot{y} + \frac{1}{LC}y\right\} = \mathcal{L}\left\{\frac{1}{LC}u\right\}$$
$$s^2Y(s) + s\frac{R}{L}Y(s) + \frac{1}{LC}Y(s) = \frac{1}{LC}U(s)$$
$$G(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Find a state-space representation of the system. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$, then

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -\frac{1}{LC}x_1 - \frac{R}{L}x_2 + \frac{1}{LC}u$$

If we put the equations in state-space form, we obtain

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \quad , \; B = \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} \quad , \; C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad , \; D = 0$$

Now let, $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} V_C \\ I \end{bmatrix}$, then

$$\dot{z_1} = \frac{1}{C}z_2$$
$$\dot{z_2} = -\frac{1}{L}z_1 - \frac{R}{L}z_2 + \frac{1}{L}u$$

If we put the equations in state-space form, we obtain

$$\dot{z} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} z + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} z$$

Home practice: Comment on the similarities and differences between two representations.

Ex 2: State-Space Realization of a Third Order FIR Systems

A third order FIR (Finite impulse response) filter has the following difference equation and transfer function

$$
\begin{aligned}
y[k] &= b_0 u[k] + b_1 u[k-1] + b_2 u[k-2] + b_3 u[k-3] \\
Y(z) &= \left(b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}\right) U(z)
\end{aligned}
$$

From inspection it is easy to see that we need at least three memory (unit delay) blocks to construct a the realization. Fig. 12.1 also provides the block-diagram realization of a third order FIR filter. Let

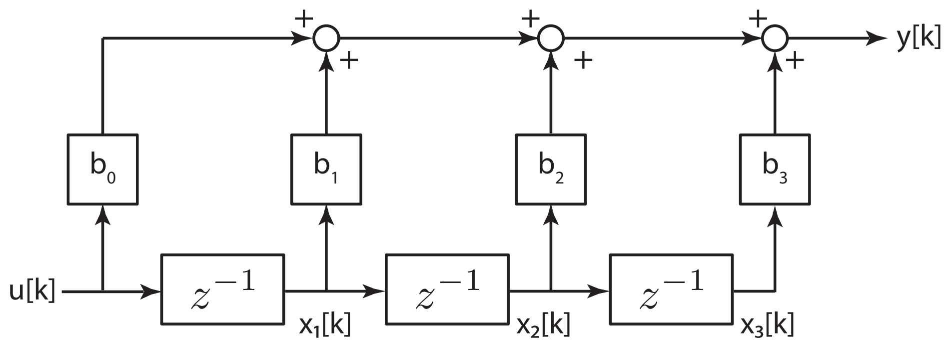


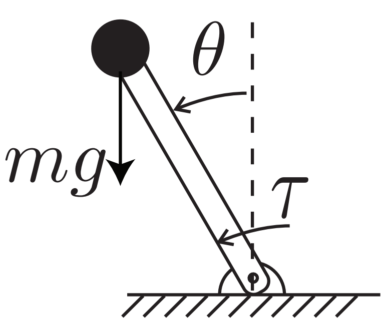
Figure 12.1: Block diagram realization of a third order FIR system

$x[k] = \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \end{bmatrix} = \begin{bmatrix} u[k-1] \\ u[k-2] \\ u[k-3] \end{bmatrix}$, then

$$
\begin{aligned}
x[k+1] &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u[k] \\
y &= \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} x[k] + [b_0] u[k]
\end{aligned}
$$

Ex 3: Pendulum

Given than input is $u(t) = \tau(t)$ and output os $y(t) = \theta(t)$, find a state-space model of the pendulum dynamics.



$$
\begin{aligned}
ml^2\ddot{\theta} &= \tau(t) + mgl\sin(\theta) \\
\text{Let } x &= \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \\
\dot{x} &= \begin{bmatrix} x_2 \\ \frac{g}{l}\sin(x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \\
f(x,u) &= \begin{bmatrix} x_2 \\ \frac{g}{l}\sin(x_1) + \frac{1}{ml^2} u \end{bmatrix}
\end{aligned}
$$

Ex 4: Predator-Prey Model Consider an island populated primaraly by goats and foxes. Goat's survive and breed by consuming the island's sources, while foxes survive and breed by consuming goats. To build a DT state-space model (based on behavioral obseervations) let's defined following state variables

$$x_1[k] : \#\text{goats}$$
$$x_2[k] : \#\text{foxes}$$

State-equation for the population of goats can be modeled as

$$x_1[k+1] = gx_1[k] - c_{fg}x_1[k]x_2[k]$$

where $g > 1$ (which models geometric growth rate of gots), and $c_{fg} > 0$. Note that $-c_{fg}x_1[k]x_2[k]$ models the negatif effect of fox population on goat population. On the other hand state-equation for the population of fox can be modeled as

$$x_2[k+1] = fx_2[k] - c_{gf}x_1[k]x_2[k]$$

where $0 < f < 1$ (which models geometric decay rate of foxes), and $c_{gf} > 0$. Note that $c_{gf}x_2[k]x_1[k]$ models the pozitif effect of goat population on fox population. If we combine both of the state equations we obtain

$$x[k+1] = \begin{bmatrix} gx_1[k+1] - c_{fg}x_1[k]x_2[k] \\ fx_2[k] + c_{gf}x_1[k]x_2[k] \end{bmatrix}$$

Note that as constructed this is an autonomous dynamical system (no external input)

12.1.3 Linearization of Non-linear Dynamical Systems

Consider a non-linear CT dynamical system represented in state-space form

$$\begin{aligned} &\text{Let } x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R}^q \ , \ u(t) \in \mathbb{R}^p, \\ &\dot{x}(t) = F(x(t), u(t)), \\ &y(t) = H(x(t), u(t)), \end{aligned}$$

Suppose that we are intersted in finding an approximate linear dynamical system model around a nominal point (equilibrium), $(x_o, u_o, y_o))$ that solves the equation constraints, i.e.

$$\begin{aligned} 0 &= F(x_0, u_o), \\ y(t) &= H(x_o, u_o), \end{aligned}$$

Since we are intersted in the dynamics around the nominal solution, we define set of "small" *perturbation* variables; $\delta x(t) = x(t) - x_o$, $\delta u(t) = u(t) - u_o$ & $\delta y(t) = y(t) - x_o$, . If we perform a (multivariate) first order Taylor series expansion, we can find the linerized state-space model as

$$\begin{aligned} \dot{\delta x} &\approx \left(\left[\frac{\partial F(x,u)}{\partial x} \right]_{(x_o,u_o)} \right) \delta x + \left(\left[\frac{\partial F(x,u)}{\partial u} \right]_{(x_o,u_o)} \right) \delta u, \\ \delta y &\approx \left(\left[\frac{\partial H(x,u)}{\partial x} \right]_{(x_o,u_o)} \right) \delta x + \left(\left[\frac{\partial H(x,u)}{\partial u} \right]_{(x_o,u_o)} \right) \delta u, \end{aligned}$$

where

$$\begin{aligned} A &= \left(\left[\frac{\partial F(x,u)}{\partial x} \right]_{(x_o,u_o)} \right) \ , \ B = \left(\left[\frac{\partial F(x,u)}{\partial u} \right]_{(x_o,u_o)} \right) \\ C &= \left(\left[\frac{\partial H(x,u)}{\partial x} \right]_{(x_o,u_o)} \right) \ , \ D = \left(\left[\frac{\partial H(x,u)}{\partial u} \right]_{(x_o,u_o)} \right) \end{aligned}$$

Ex 3-2: Linearization(s) of the Pendulum Model

Compute the approximate linear model of the pendulum around $(x,u) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0\right)$

$$A = \left(\left[\frac{\partial F(x,u)}{\partial x}\right]_{(x_o,u_o)}\right) = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix}, \; B = \left(\left[\frac{\partial F(x,u)}{\partial u}\right]_{(x_o,u_o)}\right) = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}, \; C \quad = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Compute the approximate linear model of the pendulum around $(x,u) = \left(\begin{bmatrix} \pi \\ 0 \end{bmatrix}, 0\right)$

$$A = \left(\left[\frac{\partial F(x,u)}{\partial x}\right]_{(x_o,u_o)}\right) = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix}, \; B = \left(\left[\frac{\partial F(x,u)}{\partial u}\right]_{(x_o,u_o)}\right) = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}, \; C \quad = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Compute the approximate linear model of the pendulum around $(x,u) = \left(\begin{bmatrix} -\pi/2 \\ 0 \end{bmatrix}, mgl\right)$

$$A = \left(\left[\frac{\partial F(x,u)}{\partial x}\right]_{(x_o,u_o)}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \left(\left[\frac{\partial F(x,u)}{\partial u}\right]_{(x_o,u_o)}\right) = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}, \; C \quad = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Now suppuse that we are intersted in finding an apprximate liner dynamical system model for a non-linear discrete-time dynamical system. Actually the process is exactly same (since we are still utilizing Taylor series expansion around a nominal point), except that the definition of nominal solution is different. Let's assume that a nominal point (equilibrium), $(x_o, u_o, y_o))$ solves the equation constraints of the non-linear discrete-time dynamical system, then we know that

$$\begin{aligned} x_o &= F(x_o, u_o), \\ y_o &= H(x_o, u_o), \end{aligned}$$

Cumputation of state-space matrices and definition of perturbation variables are completely same for the CT case.

Ex 4-2: Linearization of the Predator-Prey Model

Let $[g, f, c_{fg}, c_{gf}] = [2, 0.5, 0.1, 0.05]$, first compute the equilibrium point of the dynamical system

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 2\bar{x}_1 - 0.1\bar{x}_1\bar{x}_2 \\ 0.5\bar{x}_2 + 0.05\bar{x}_1\bar{x}_2 \end{bmatrix} \;\rightarrow\; x_0 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Now linarize the dynamics around the equilibrium and derive the approximate DT linear state-space representation

$$A = \left(\left[\frac{\partial F(x)}{\partial x}\right]_{(x_o)}\right) = \begin{bmatrix} 2-1 & -1 \\ 0.5 & 0.5+0.5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0.5 & 1 \end{bmatrix}$$

Let's go back to non-linear CT time dynamical models. Sometimes, we want to obtain a *Linear* dynamical system model around a nominal trajecotry (not a point) which still satisfies the constraints of the dynamical system. In such a case nominal solution takwa the form $(x_o(t), u_o(t), y_o(t)))$ ans still satisfies the equation constraints, i.e.

$$\dot{x_o}(t) = F(x_o(t), u_o(t)) \ , \ y_o(t) = H(x_o(t), u_o(t)), \ \forall t \in \mathbb{R}$$

We define set of "small" *perturbation* variables in a similar way; $\delta x(t) = x(t) - x_o(t)$, $\delta u(t) = u(t) - u_o(t)$ & $\delta y(t) = y(t) - y_o(t)$. Indeed formulation of the (multivariate) first order Taylor series expansion is (almost) exactly same, and we can find the linerized state-space model as

$$\begin{aligned}
\dot{\delta x}(t) &\approx \left(\left[\frac{\partial F(x,u)}{\partial x} \right]_{(x_o(t),u_o(t))} \right) \delta x + \left(\left[\frac{\partial F(x,u)}{\partial u} \right]_{(x_o(t),u_o(t))} \right) \delta u(t), \\
\delta y(t) &\approx \left(\left[\frac{\partial H(x,u)}{\partial x} \right]_{(x_o(t),u_o(t))} \right) \delta x(t) + \left(\left[\frac{\partial H(x,u)}{\partial u} \right]_{(x_o(t),u_o(t))} \right) \delta u(t), \\
&\text{where} \\
A(t) &= \left(\left[\frac{\partial F(x,u)}{\partial x} \right]_{(x_o(t),u_o(t))} \right) \ , \ B(t) = \left(\left[\frac{\partial F(x,u)}{\partial u} \right]_{(x_o(t),u_o(t))} \right) \\
C(t) &= \left(\left[\frac{\partial H(x,u)}{\partial x} \right]_{(x_o(t),u_o(t))} \right) \ , \ D(t) = \left(\left[\frac{\partial H(x,u)}{\partial u} \right]_{(x_o(t),u_o(t))} \right)
\end{aligned}$$

Note that in such a case state-space matrices (almost surely) potentially becomes time-dependent and hence linearitzation leads to a LTV (Linear-Time-Varying) state-space model.

Ex 3-3: Linearization(s) of the Pendulum Model Around a Trajectory

Let's go back to the simple pendulum model. We want to "control" the system such that it rotates with a constant angular velocity and analyze the dynamics around the nominal trajectory. In that respect nominal trajectory for the state variables can take the following form

$$x_o(t) = \begin{bmatrix} \theta_o(t) \\ \dot{\theta}_o(t) \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Compute the nominal solution for the input that satisfies the nominal state-trajectories

$$u_0(t) = mgl \sin(t)$$

Compute the approximate linear model of the pendulum around $(x_o(t), u_o(t))$

$$A(t) = \left(\left[\frac{\partial F(x,u)}{\partial x} \right]_{(x_o,u_o)} \right) = \begin{bmatrix} 0 & 1 \\ g/l \cos(t) & 0 \end{bmatrix} \ , \ B = \left(\left[\frac{\partial F(x,u)}{\partial u} \right]_{(x_o,u_o)} \right) = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \ , \ C \ = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

We can see that system matrix $A(t)$ is now time-dependent thus the approximate system is an LTV system. Indeed we can see that $A(t)$ is a periodic function, this this is a special class of LTV system and belongs to the group of LTP (Linear Time Periodic) systems.