EE402 - Discrete Time Systems **Spring 2018**

Lecture 13

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Matrix Exponential, e^{At}

Let's first review the matrix exponential, e^{At}. Let $t \in \mathbb{R}$ and $A \in \mathbb{R}^{nxn}$, then e^{At} defined as

$$\begin{aligned} e^{At} &:= I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \\ e^{At} &:= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \end{aligned}$$

which converges for all $t \in \mathbb{R}$ and $A \in \mathbb{R}^{nxn}$.

Now let's review some properties

- **Claim:**

$$\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$$

Proof:

$$\begin{aligned} \frac{d}{dt} e^{At} &= \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right) = \sum_{k=0}^{\infty} k \frac{t^{k-1}}{k!} A^k = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^k \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} A^{n+1} = A \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \right) A \\ \frac{d}{dt} e^{At} &= A e^{At} = e^{At} A \end{aligned}$$

- **Claim:** Let $t_1, t_2 \in$ R then

$$e^{At_1} e^{At_2} = e^{At_2} e^{At_1} = e^{A(t_1+t_2)}$$

Proof:

$$e^{At_1} e^{At_2} = \left(\sum_{k=0}^{\infty} \frac{t_1^k}{k!} A^k \right) \left(\sum_{j=0}^{\infty} \frac{t_2^j}{j!} A^j \right) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{t_1^k}{k!} \frac{t_2^j}{j!} A^{k+j}$$

Let $n = k + j$ and $j = n - k$, then

$$e^{At_1} e^{At_2} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{t_1^k \ t_2^{n-k}}{k!(n-k)!} A^n = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{t_1^k \ t_2^{n-k}}{n!} \frac{n!}{k!(n-k)!} A^n \quad = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{t_1^k \ t_2^{n-k}}{n!} \begin{pmatrix} n \\ k \end{pmatrix} A^n$$

Since for $n < k$, $\begin{pmatrix} n \\ k \end{pmatrix} = 0$,

$$e^{At_1}e^{At_2} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{t_1^k\ t_2^{n-k}}{n!}\begin{pmatrix} n \\ k \end{pmatrix}A^n = \sum_{n=0}^{\infty}\frac{A^n}{n!}\sum_{k=0}^{\infty}t_1^k\ t_2^{n-k}\begin{pmatrix} n \\ k \end{pmatrix}$$

Using *binomial* theorem we find

$$\begin{aligned} e^{At_1}e^{At_2} &= \sum_{n=0}^{\infty}\frac{A^n}{n!}(t_1+t_2)^n \\ e^{At_1}e^{At_2} &= e^{A(t_1+t_2)} \end{aligned}$$

Now let $t_1 = t$ and $t_2 = -t$, then we have

$$e^{At}e^{-At} = e^{A(t-t)} = I \quad \rightarrow \quad \left(e^{At}\right)^{-1} = e^{-At}$$

- **Claim:** Let $A, B \in \mathrm{R}^{n\times n}$ and $AB = BA$, then

$$e^{At}e^{Bt} = e^{Bt}e^{At} = e^{(A+B)t}$$

Proof: Possible mini project question

Note that if $AB \neq BA$ then

$$e^{At}e^{Bt} \neq e^{(A+B)t}$$

- **Claim:** Let $P \in \mathrm{R}^{n\times n}$ and $\det(P) \neq 0$, then

$$e^{\left(P^{-1}AP\right)t} = P^{-1}e^{At}P$$

Proof: Possible mini project question

Solution of CT State-Space Equations

CT state-space representation has the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ \dot{y}(t) &= Cx(t) + Du(t) \end{aligned}$$

where Let $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^q$

First consider the homogeneous solution, i.e. $u(t) = 0$ and $x(0) = x_0$.

$$\begin{aligned} \dot{x}(t) &= Ax(t) \quad , \quad x(0) = x_0 \\ \dot{y}(t) &= Cx(t) \end{aligned}$$

Let's test if $x(t) = e^{At}x_0$ is a solution of the homogeneous equation

$$\begin{aligned} x(0) &= e^{A0}x_0 = x_0 \\ \dot{x}(t) - Ax(t) &= \left(Ae^{At}\right)x_0 - Ae^{At}x_0 = 0 \end{aligned}$$

Now let's compute the forced response. First let's analyze the following derivative

$$\frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}\left[\dot{x}(t) - Ax(t)\right]$$

Now using this relation let's solve the state-space equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ \dot{x}(t) - Ax(t) &= Bu(t) \\ e^{-At}\left[\dot{x}(t) - Ax(t)\right] &= e^{-At}Bu(t) \\ \frac{d}{dt}\left[e^{-At}x(t)\right] &= e^{-At}Bu(t) \\ e^{-At}x(t) &= x(0) + \int_0^t e^{-A\tau}Bu(\tau)d\tau \\ x(t) &= e^{-At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \end{aligned}$$

Thus the solution of a system in state-space form can be written as

$$\begin{aligned} x(t) &= e^{-At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) &= Ce^{-At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{aligned}$$

The function $\Psi(t) = e^{At}$ is called the state-transition matrix of the system.

Example: Let's assume that system is a SISO system and $u(t) = \delta(t)$ (unit-impulse function) and $x_0 = 0$, compute the impulse response of the system, i.e. $y(t) = h(t)$,

$$\begin{aligned} h(t) &= \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau + D\delta(t) \\ &= Ce^{At}B + D\delta(t) \end{aligned}$$

S-Domain Solution of CT State-Space Equations

First take the Laplace transform of state evaluation equation

$$\begin{aligned}
\mathcal{L}\left[\dot{x}(t)\right] &= \mathcal{L}\left[Ax(t) + Bu(t)\right] \\
sX(s) - x(0) &= AX(s) + BU(s) \\
[sI - A]X(s) &= x(0) + BU(s) \\
X(s) &= [sI - A]^{-1}x_0 + [sI - A]^{-1}BU(s) \\
Y(s) &= C[sI - A]^{-1}x_0 + \left[C[sI - A]^{-1}B + D\right]U(s)
\end{aligned}$$

If we relate time and s-domain solutions we obtain

$$\begin{aligned}
e^{At} &= \mathcal{L}^{-1}\left[[sI - A]^{-1}\right] \\
h(t) &= \mathcal{L}^{-1}\left[C[sI - A]^{-1}B + D\right]
\end{aligned}$$

Discretization of CT State-Space Equations

Consider the CT system with the given state-space representation

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}$$

and suppose that the input is piece-wise constant over intervals of length T. That is

$$u(t) = u[k] \quad , \quad t \in (kT \;\; (k+1)T)$$

i.e. input of the system is the output of a ZOH operator. Let's derive the the DT state-space equations with respect to the sampled-state $x[k] = x(kT)$.

Let's start with the state evolution equation

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

It is obvious that due to time-invariant the initial time of the equation above can be generalized as

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau \quad , \quad t > t_0$$

Now let $t_0 = kT$ and $t = (k+1)T$,

$$x((k+1)T) = e^{AT}x(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau$$

Since the $u(t) = u[k]$ in this time interval

$$x[k+1] = e^{AT}x[k] + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bd\tau \; u[k]$$

Let $\lambda = (k+1)T - \tau$, then

$$\begin{aligned} x[k+1] &= e^{AT}x[k] - \int_T^0 e^{A\lambda}Bd\lambda \; u[k] \\ &= \left[e^{AT}\right]x[k] + \left[\left(\int_0^T e^{A\lambda}d\lambda\right)B\right]u[k] \end{aligned}$$

Given that

$$x[k+1] = Gx[k] + Hu[k]$$

G and H matrices can be extracted as

$$\begin{aligned} G &= e^{AT} \\ H &= \left(\int_0^T e^{A\lambda}d\lambda\right)B \end{aligned}$$

Claim: If A is invertable then we also have

$$H = A^{-1}\left(e^{AT} - I\right)B = \left(e^{AT} - I\right)A^{-1}B$$

Proof: Possible mini project question

Now let's consider the output equation

$$\begin{aligned} y(t) &= Cx(t) + Dx(t) \\ y(kT) &= Cx(kT) + Dx(kT) \\ y[k] &= Cx[k] + Dx[k] \end{aligned}$$

It can be seen that output equation matrices are not affected from the discretization.

Example: Consider the following CT-plant transfer function.

$$\frac{Y(s)}{U(s)} = \frac{1}{s} + \frac{1}{s + \ln(2)}$$

Find a CT state-space representation for this system

Solution:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 0 \\ 0 & -\ln(2) \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \end{aligned}$$

Compute the state-transition matrix

Solution:

$$\begin{aligned} e^{At} &= \begin{bmatrix} e^{0t} & 0 \\ 0 & e^{-\ln(2)t} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0.5^t \end{bmatrix} \end{aligned}$$

Discretize the CT State-Space equation under zero hold operation and ideal sampling of the defined state variables, with $T = 1s$.

Solution:

$$\begin{aligned} G &= e^{AT} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \\ H &= \left(\int_0^T e^{A\lambda} d\lambda \right) B = \left(\int_0^1 \begin{bmatrix} 1 & 0 \\ 0 & 0.5^\lambda \end{bmatrix} d\lambda \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \int_0^1 \begin{bmatrix} 1 \\ 0.5^\lambda \end{bmatrix} d\lambda \\ H &= \begin{bmatrix} 1 \\ 0.721 \end{bmatrix} \end{aligned}$$

Full DT state-space formulation takes the form

$$\begin{aligned} x[k+1] &= \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0.721 \end{bmatrix} u[k] \\ y[k] &= \begin{bmatrix} 1 & 1 \end{bmatrix} x[k] \end{aligned}$$

Compute the DT pulse transfer function $Y(z)/U(z)$ **Solution:**

$$\begin{aligned} \frac{Y(z)}{R(z)} &= C\,[zI - G]^{-1}\,H \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} [zI - G]^{-1} \begin{bmatrix} 1 \\ 0.721 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} z-1 & 0 \\ 0 & z-0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0.721 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z-0.5} \end{bmatrix} \begin{bmatrix} 1 \\ 0.721 \end{bmatrix} \\ &= \frac{1}{z-1} + \frac{0.721}{z-0.5} \end{aligned}$$

Now discretize $Y(s)/U(s)$ directly under ZOH operation

Solution:

$$\frac{Y(z)}{U(z)} = \mathcal{Z}\left[\frac{1-e^{-s}}{s}\frac{Y(s)}{U(s)}\right] = \frac{1}{z-1} + \frac{0.721}{z-0.5}$$