Lecture 5

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5.1   Functions of a Matrix

In linear systems theory course, we are interested in matrix polynomials, specifically

- Matrix Exponential in CT Systems: e^{At}
- Matrix Power in DT Systems: A^k

which arise on the solution of state-space equations in their respective domains. Obviously A^k in DT systems is "easier" to analyze and understand compared to matrix exponential. Let's first review the matrix exponential, e^{At}. Let $t \in \mathbb{R}$ and $A \in \mathbb{R}^{nxn}$, then e^{At} defined as

$$e^{At} := I + At + \frac{1}{2!}A^2 t^2 + \frac{1}{3!}A^3 t^3 + \cdots$$

$$e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

which converges for all $t \in \mathbb{R}$ and $A \in \mathbb{R}^{nxn}$.

Now let's review some properties

- **Claim:** $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$

  Proof:

$$\frac{d}{dt}e^{At} = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k\right) = \sum_{k=0}^{\infty} k\frac{t^{k-1}}{k!}A^k = \sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}A^k$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!}A^{n+1} = A\sum_{n=0}^{\infty}\frac{t^n}{n!}A^n = \left(\sum_{n=0}^{\infty}\frac{t^n}{n!}A^n\right)A$$

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

- **Claim:** Let $A, B \in \mathrm{R}^{n \times n}$ and $AB = BA$, then

$$e^A e^B = e^B e^A = e^{(A+B)}$$

  Proof:

$$e^A e^B = \left(\sum_{k=0}^{\infty} \frac{1}{k!}A^k\right)\left(\sum_{j=0}^{\infty}\frac{1}{j!}B^j\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{A^k}{k!}\frac{B^j}{j!}$$

Let $n = k + j$ and $j = n - k$, then

$$e^A e^B = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{A^k \ B^{n-k}}{k!(n-k)!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^k \ B^{n-k}}{n!} \frac{n!}{k!(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} A^k \ B^{n-k} \begin{pmatrix} n \\ k \end{pmatrix}$$

Note that if $AB \neq BA$ we has to stop at this point. However, since $AB = BA$, we can adopt binomial theorem

$$e^A e^B = \sum_{n=0}^{\infty} \frac{1}{n!} (A + B)^n = e^{A+B} = e^B e^A$$

- **Claim:** Let $t_1, t_2 \in \mathR$ then

$$e^{At_1} e^{At_2} = e^{At_2} e^{At_1} = e^{A(t_1+t_2)}$$

Proof: Let $A := At_1$ and $B := At_2$, obviously $(At_1)(At_2) = (At_2)(At_1)$ hence we can use the previous property, i.e.

$$e^{At_1} e^{At_2} = e^{(At_1)+(At_2)} = e^{A(t_1+t_2)} = e^{At_2} e^{At_1}$$

Now let $t_1 = t$ and $t_2 = -t$, then we have

$$e^{At} e^{-At} = e^{A(t-t)} = I \quad \rightarrow \quad \left(e^{At}\right)^{-1} = e^{-At}$$

- **Claim:** Let $P \in \mathR^{n \times n}$ and $\det(P) \neq 0$, then

$$e^{\left(P^{-1}AP\right)t} = P^{-1} e^{At} P$$

Proof: Let's firs show that $\left(P^{-1}AP\right)^k = P^{-1}A^k P$

$$\begin{aligned}
\left(P^{-1}AP\right)^k &= \left(P^{-1}AP\right)\left(P^{-1}AP\right) \cdots \left(P^{-1}AP\right)\left(P^{-1}AP\right) \\
&= P^{-1}APP^{-1}APP^{-1} \cdots PP^{-1}APP^{-1}AP \\
&= P^{-1}AIAI \cdots IAIAP \\
&= A^k
\end{aligned}$$

Now let's expand

$$\begin{aligned}
e^{\left(P^{-1}AP\right)t} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(P^{-1}AP\right)^k \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} P^{-1}A^k P \\
&= P^{-1} \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right) P \\
&= P^{-1} e^{At} P
\end{aligned}$$

5.1.1 Computation of e^{At} and A^k

5.1.2 Computation via Solution of State-Space Equations and Frequency Domain Expressions

An LTI CT state-space representation of an autonomous system has the form

$$\dot{x}(t) = Ax(t) \text{ , where } x(t) \in \mathbb{R}^n$$

Let's test if $x(t) = e^{At}x_0$ is a solution of the homogeneous equation

$$x(0) = e^{A\,0}x_0 = x_0$$
$$\dot{x}(t) - Ax(t) = \left(Ae^{At}\right)x_0 - Ae^{At}x_0 = 0$$

Now let's remember Laplace domain solution of the same equation

$$\mathcal{L}\left[\dot{x}(t)\right] = \mathcal{L}\left[Ax(t)\right]$$
$$sX(s) - x(0) = AX(s)$$
$$[sI - A]X(s) = x(0)$$
$$X(s) = [sI - A]^{-1}x_0$$

If we connect time and s-domain solutions we obtain

$$e^{At} = \mathcal{L}^{-1}\left[[sI - A]^{-1}\right]$$

Now let's focus on A^k. An LTI DT state-space representation of an autonomous system has the form

$$x[k + 1] = Ax[k]$$

Unlike CT systems we can compute the response iteratively easily

$$x[1] = Gx[0]$$
$$x[2] = Gx[1] = G^2x[0]$$
$$x[3] = Gx[2] = G^3x[0]$$
$$\vdots$$
$$x[k] = Gx[k - 1] = G^kx[0]$$

Now let's remember form of the response in Z-domain.

$$\mathcal{Z}\left[x[k + 1]\right] = \mathcal{Z}\left[Gx[k]\right]$$
$$zX(z) - zx[0] = GX(z)$$
$$(zI - G)X(z) = zX(z)$$
$$X(z) = z\left(zI - G\right)^{-1}x[0]$$

If we connect time and s-domain solutions we obtain

$$G^k = \mathcal{Z}^{-1}\{z\left(zI - G\right)^{-1}\}$$

5.1.3 Computation via Diagonalization

Theorem: $A \in \mathbb{C}^{n \times n}$ is diagonalizable, if and only if there exists a (nonsingular) similarity transformation, $V \in \mathbb{C}^{n \times n}$, such that $A = V^{-1} \Lambda V$ where Λ is a diagonal matrix,

$$
\Lambda = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{bmatrix}
$$

where $\lambda_i \in \mathbb{C}$'s are the eigenvalues of A, which are the roots of the characteristic equation $d(\lambda) = \det(\lambda I - A)$.

Now let's compute A^k and e^{At} for a diagonalizable matrix

$$
A^k = \left(V^{-1} \Lambda V \right)^k = V^{-1} \Lambda^k V = V^{-1} \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} V
$$

$$
e^{At} = e^{V^{-1} \Lambda V t} = V^{-1} e^{\Lambda t} V = V^{-1} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} V
$$

A sufficient but not necessary condition that A will have n distinct eigenvalues in such a case characteristic equation will have the following form

$$
d(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n) \ , \ \text{ where } \lambda_i \neq \lambda_j \ , \ \text{ if } i \neq j
$$

In this case, we also have the following properties associated with A

- A has n linearly independent eigenvectors

- For each λ_i there exists an eigenvector, v_i, such that $Av_i = \lambda v_i$ and $\text{Span}\{v_1, \cdots, v_n\} = \mathbb{C}^n$

- $\forall \lambda_i$, geometric multiplicity is equal to algebraic multiplicity and they are both equal to 1, i.e. $\text{GM}(\lambda_i) = \text{AM}(\lambda_i) = 1$

- Minimal polynomial is equal to the characteristic equation, $m(\lambda) = d(\lambda)$

- For each λ_i, $\mathcal{N}(\lambda_i I - A) = \text{Span}\{v_i\}$ and $\dim[\mathcal{N}(\lambda_i I - A)] = 1$

Let's remember some concepts from EE501, to better understand and generalize *diagonalizable* and *non-diagonalizable* square matrices.

Definition: Given a matrix $A \in \mathbb{C}^{n \times n}$, the characteristic polynomial $d(\lambda)$ is defined as

$$
d(\lambda) = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_n)^k
$$

$$
k : \ \# \text{ distinct eigenvalues where} k \leq n
$$

$$
r_i = AM(\lambda_i) : \ \# \text{ algebraic multiplicty } \lambda_i \ , \ \text{ where } n = \sum_{i=1}^{k} r_i
$$

Theorem: Every $n \times n$ matrix satisfies its characteristic equation (Cayley-Hamilton Theorem)

$$d(\lambda) = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_n)^k = \lambda^n + d_{n-1}\lambda^{n-1} + \cdots + d_0$$
$$d(A) = A^n + d_{n-1}A^{n-1} + \cdots + d_1 A + d_0 I = 0_{n \times n}$$

Remark: Any power of a matrix $A \in \mathbb{C}^{n \times n}$ can be written as a liner combination of $\{I \ , \ A \ , \ A^2 \ , \ \cdots \ , \ A^{n-1}\}$