EE302 - Feedback Systems **Spring 2019**

Lecture 19

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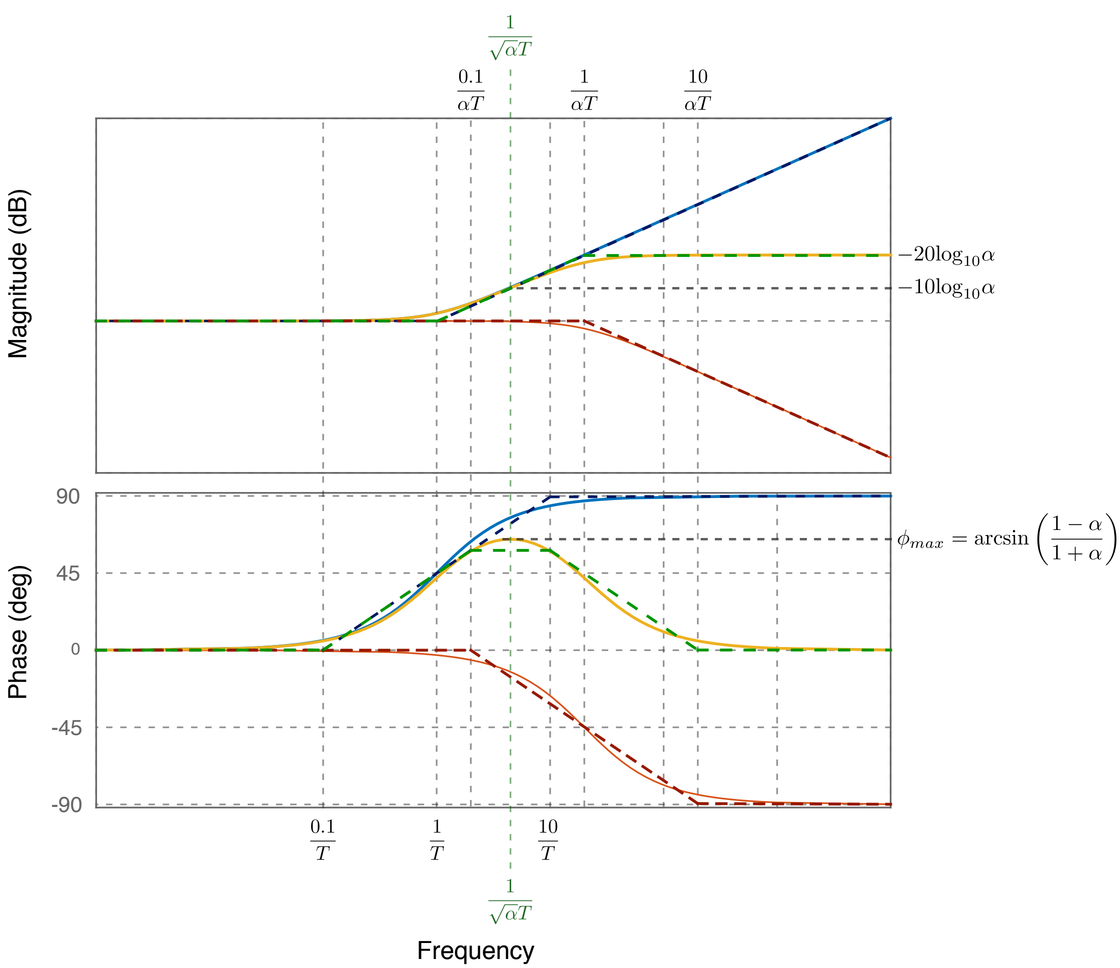
19.1 Lead Compensator Design

The lead-compensator is a controller which has the form of a first-order high-pass filter

$$G_c(s) = K_c \frac{Ts+1}{T\alpha s+1} \quad \alpha \in (0,1)$$

In general, we first design K_c based on the setady-state requirements of the system, and then design T and α based on the phase-margin requirement. First let's illustrate the bode-plots of a unity gain lead-compensator to understand how we can utilize its properties for the design process.



We can see that (both from the actual and approximate bode-plots), phase-lead compensator is a type of high-pass filter for which the cut-off (mid) frequency is $\omega_c = \frac{1}{\sqrt{\alpha}T}$. The low and high frequency gains are respectively 0 dB and $-20\log_{10}\alpha$ dB. The lead-compensator phase, ϕ_{max}, peaks at its cut-off frequency, and basically we will try to use this positive maximum phase shift to improve the phase margin of the feedback system.

Let's derive a formula for ϕ_{max}, which we will need during the design phase

$$\begin{aligned}
\phi_{max} &= \angle[G_c(j\omega_c)] = \angle\left[1 + T\omega_c j\right] - \angle\left[1 + \alpha T\omega_c j\right] = \arctan(T\omega_c) - \arctan(\alpha T\omega_c) \\
&= \arctan\left(\frac{1}{\sqrt{\alpha}}\right) - \arctan\left(\sqrt{\alpha}\right) \\
&= \pi/2 - 2\arctan\left(\sqrt{\alpha}\right) \\
\sin\phi_{max} &= \cos\left(2\arctan\left(\sqrt{\alpha}\right)\right) = 1 - 2\left[\sin\left(\arctan\left(\sqrt{\alpha}\right)\right)\right]^2 = 1 - 2\frac{\alpha}{1+\alpha} \\
&= \frac{1-\alpha}{1+\alpha} \\
\phi_{max} &= \arcsin\left(\frac{1-\alpha}{1+\alpha}\right) \;\Rightarrow\; \alpha = \frac{1-\sin\phi_{max}}{1+\sin\phi_{max}}
\end{aligned}$$

As expected when $\alpha = 1$, $\phi_{max} = 0$ since numerator and denominator time constants becomes equal in this case. Accordingly, we can see that

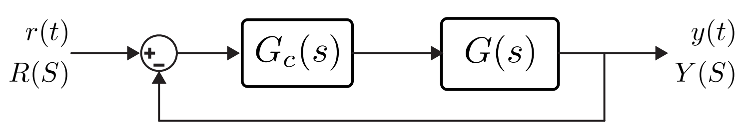
$$\alpha \searrow \Rightarrow \; \phi_{max} \nearrow$$

Theoretical maximum value for ϕ_{max} is 90^o, however practically $\phi_{max} < 75^o$ for analog lead compensator circuits. Another important factor that we will need to pay attention is the gain-shift of the lead-compensator at the cut-off frequency

$$M_{dB}(j\omega_c) = -10\log_{10}\alpha \quad \& \quad |G(j\omega_c)| = \frac{1}{\sqrt{\alpha}}$$

We will illustrate the design process on an example

Ex: Consider the following feedback system illustrated below. It is given that $G(s) = \frac{1}{s(s+1)}$ and we want to design a lead-compensator, $G_c(s) = K_c\frac{Ts+1}{T\alpha s+1}$, such that unit-ramp steady-state error satisfies, $e_{ss} = 0.1$ and phase-margin of the compensated system satisfies $\phi_m^* = \left[45^0, 55^0\right]$.



Solution:

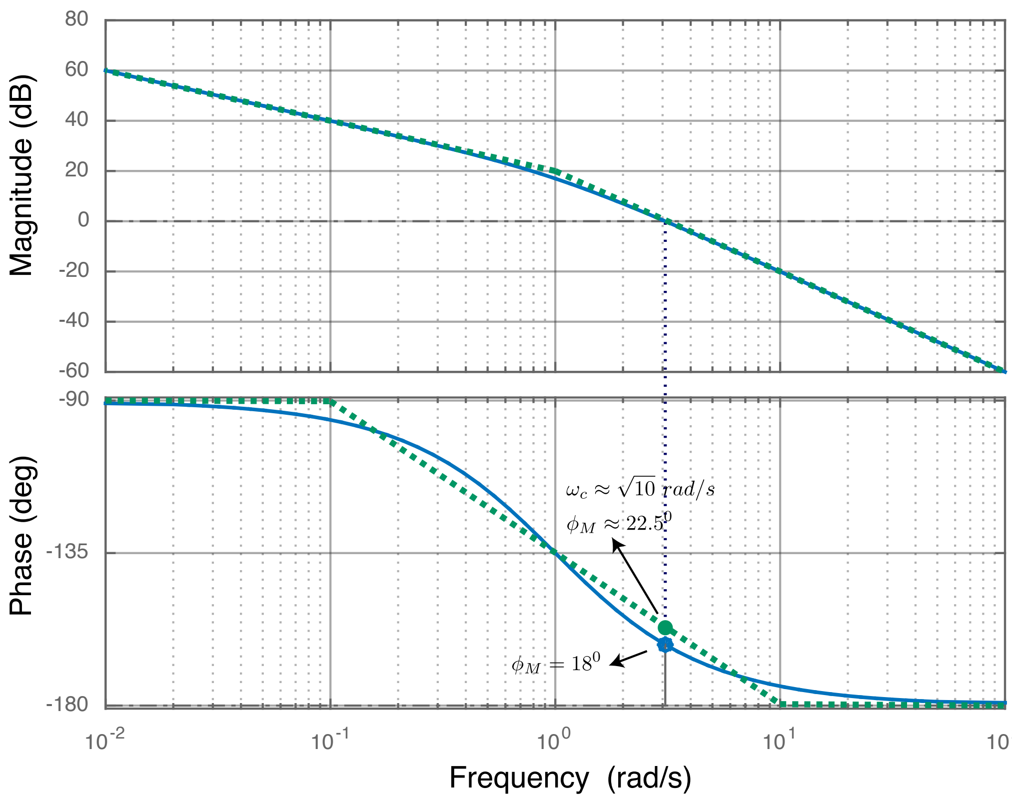
Step 1: We design/compute K_c based on the steady-state requirement on the unit-ramp error.

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K_c} = 1 \;\rightarrow\; K_c = 10 \tag{19.1}$$

Step 2: Embed K_c into $G(s)$, i.e.

$$\bar{G}(s) = K_c G(s) = \frac{10}{s(s+1)},$$

draw the bode-plot for $\bar{G}(s)$, and compute the gain-crosover frequency, ω_{gc}, and the phase margin, ϕ_M, for the uncompensated $\bar{G}(s)$. The bode plot of $\bar{G}(s)$ and its approximation is illustrated in the Figure below.



If we concentrate on the approximate bode-plots, we can estimate the gain-crosover frequency and the phase-margin as

$$\omega_{gc} \approx \sqrt{10}\ rad/s = 3.16\ rad/s$$
$$\phi_m \approx 22.5^o$$

If we compute the actual values from the actual Bode plot, we obtain

$$\omega_{gc} = 3.08\ rad/s$$
$$\phi_m \approx 18^o$$

We can see that approximation is pretty good for the ω_c, however a little crude for ϕ_m. So let's compute $\angle[G(j\omega_{gc})]$ for $\omega_{gc} \approx \sqrt{10}$ and estimate the phase-margin based on this frequency

$$\angle[G(j\omega_{gc})] = \angle[G(j\sqrt{10})] = -90^0 - \arctan\left(\sqrt{10}\right) = -162.5^o$$
$$\phi_M \approx 17.5^o$$

Step 3: Compute the required phase increment, $\Delta\phi$ to be added by the compensator and compute α

$$\phi_{max} \approx \Delta\phi = \phi_M^* - \phi_M + \epsilon$$
$$\epsilon = 5^0 - 10^0$$

So for the given problem, we can compute ϕ_{max} and α as

$$\phi_{max} \approx 47.5^o - 17.5^0 + 7^0 = 37^0$$
$$\alpha = \frac{1 - \sin\phi_{max}}{1 + \sin\phi_{max}} \approx \frac{1}{4}$$

Step 4: Estimate the "new" gain-crossover frequency, $\hat{\omega}_{gc}$, and place the peak point of the lead-compensator, at this estimated $\hat{\omega}_{gc}$.

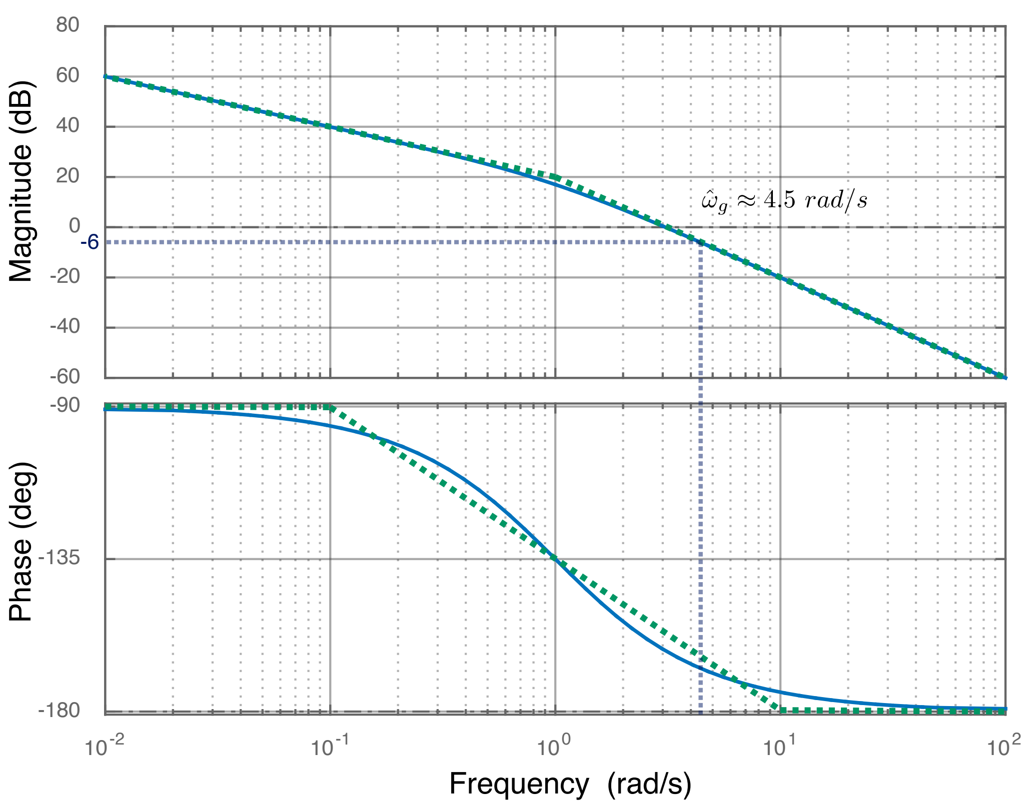
We already know that, lead-compensator at its center/cutoff frequency shifts the bode magnitude by $-10\log_{10}\alpha$ (or increases the gain by $\frac{1}{\sqrt{6}}$) and this causes a shift in gain-crossover frequency. For this reason, we can estimate the new gain-crossover frequency as the point where the bode magnitude of $\bar{G}(s)$ crosses $10\ \log_{10}\alpha$, i.e.

$$M_{dB}(j\hat{\omega}_{gc}) = 10\ \log_{10}\alpha \quad or \quad |G(j\hat{\omega}_{gc})| = \sqrt{\alpha}$$

In our problem,

$$M_{dB}(j\hat{\omega}_{gc}) \approx -6\ dB \quad or \quad |G(j\hat{\omega}_{gc})| = \frac{1}{2}$$

We can indeed estimate the new gain-crossover frequency graphically from the bode-plot. The figure below, illustrates how we can find the new-gain crossover frequency graphically



where $\hat{\omega}_{gc} \approx 4.5\ rad/s$. We can also estimate the new-gain crosover frequency numerically

$$\begin{aligned} |G(j\hat{\omega}_{gc})| &= \frac{1}{2} \\ \frac{100}{\hat{\omega}_{gc}^2\left(\hat{\omega}_{gc}^2+1\right)} &= \frac{1}{4} \\ \hat{\omega}_{gc}^4 + \hat{\omega}_{gc}^2 - 400 &= 0 \\ \hat{\omega}_{gc} \approx 400^{1/4} &= 4.47\ rad/s \end{aligned}$$

which is very close to the graphical estimation.

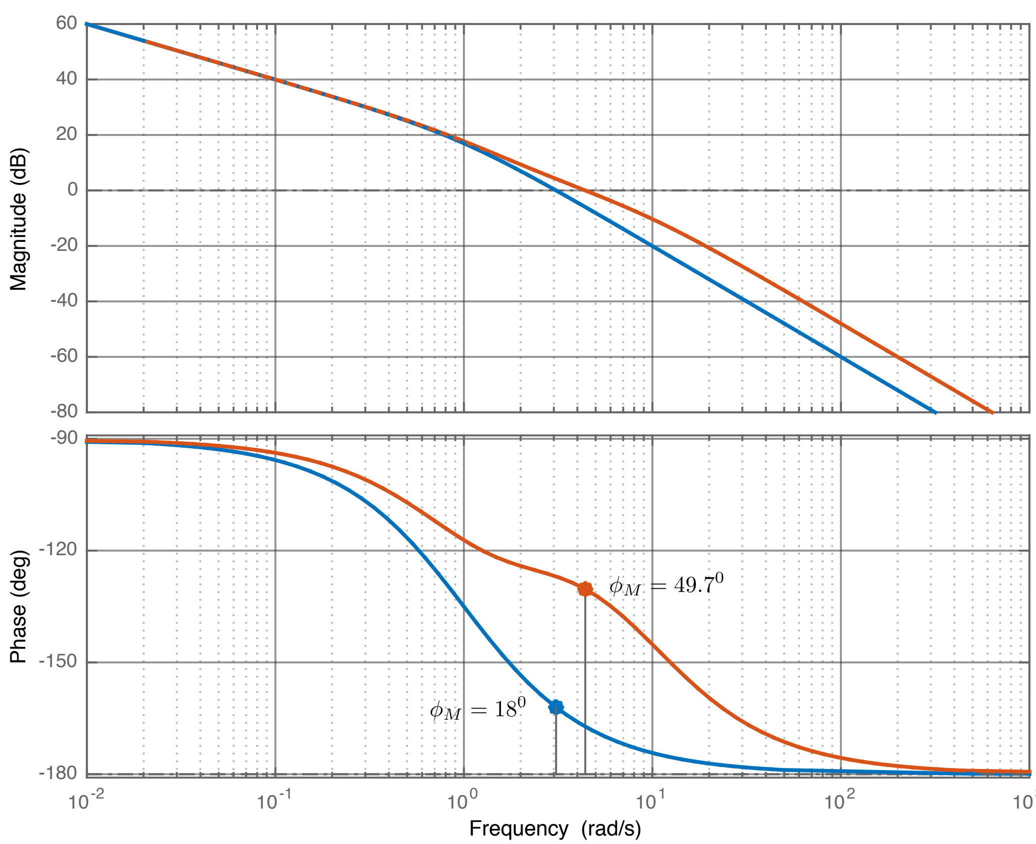
Step 5: Compute T using $\hat{\omega}_{gc}$ and check if the lead-compensator meets the phase-margin requirement. Otherwise, repeat the process with a higher $\Delta\phi$ angle.

$$\hat{\omega}_{gc} = \omega_c = \frac{1}{\sqrt{\alpha}T} \quad \Rightarrow \quad T = \frac{1}{\sqrt{\alpha}\hat{\omega}_{gc}}$$

In our example

$$T = \frac{1}{\frac{4.5}{2}} \approx 0.45 \quad \Rightarrow \quad G_c(s) = 10\frac{0.45s+1}{0.1125s+1} = 40\frac{s+2.25}{s+9}$$

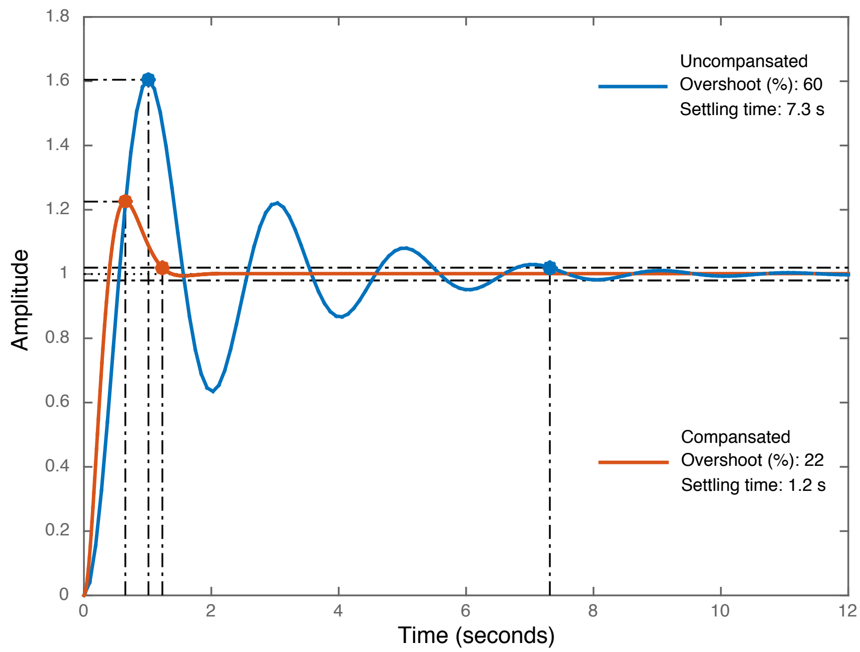
The Figure below illustrates the bode plots of both compensated and uncompensated ($\bar{G}(s) = K_cG(s)$) systems. Compensated systems has a phase margin of $\phi_m = 49.7^o$ which meets the requirements.



Now let's compute new phase margin directly using the new gain-crossover frequency

$$\begin{aligned}\phi_M &= -180^o + \angle[G_c(j4.45\ rad/s)\bar{G}(j4.45\ rad/s)] = -180^o + 37^o + \angle[\bar{G}(j4.45\ rad/s)]\\ &= -180^o + 37^o - 90^o - \angle[j4.45 + 1] = -180^o + 37^o - 90^o - 77^0\\ &= 50^o\end{aligned}$$

which is consistent with the phase-margin on the bode-plot. In Figure below, we compare the closed-loop step responses of both uncompensated and compensated closed-loop transfer functions. We can clearly see that, the lead-compensator substantially improves both the settling time and over-shoot performance.



Phase-Lead Design Guideline:

1. Design/compute K_c based on the steady-state requirements
2. Embed K_c into $G(s)$, i.e. $\bar{G(s)} = K_cG(s)$, then draw the bode-plots for $\bar{G}(s)$, and compute the gain-crossover frequency, ω_{gc}, and the phase margin, ϕ_M, for the uncompensated $\bar{G}(s)$.
3. Compute the required phase increment, $\Delta\phi$ to be added by the compensator and design/compute α
4. Estimate the "new" gain-crossover frequency, $\hat{\omega}_{gc}$, and place the peak point of the lead-compensator, at this estimated $\hat{\omega}_{gc}$.
5. Check the phase-margin if it does not meet the requirements increase $\Delta\phi$ and repeat the process.

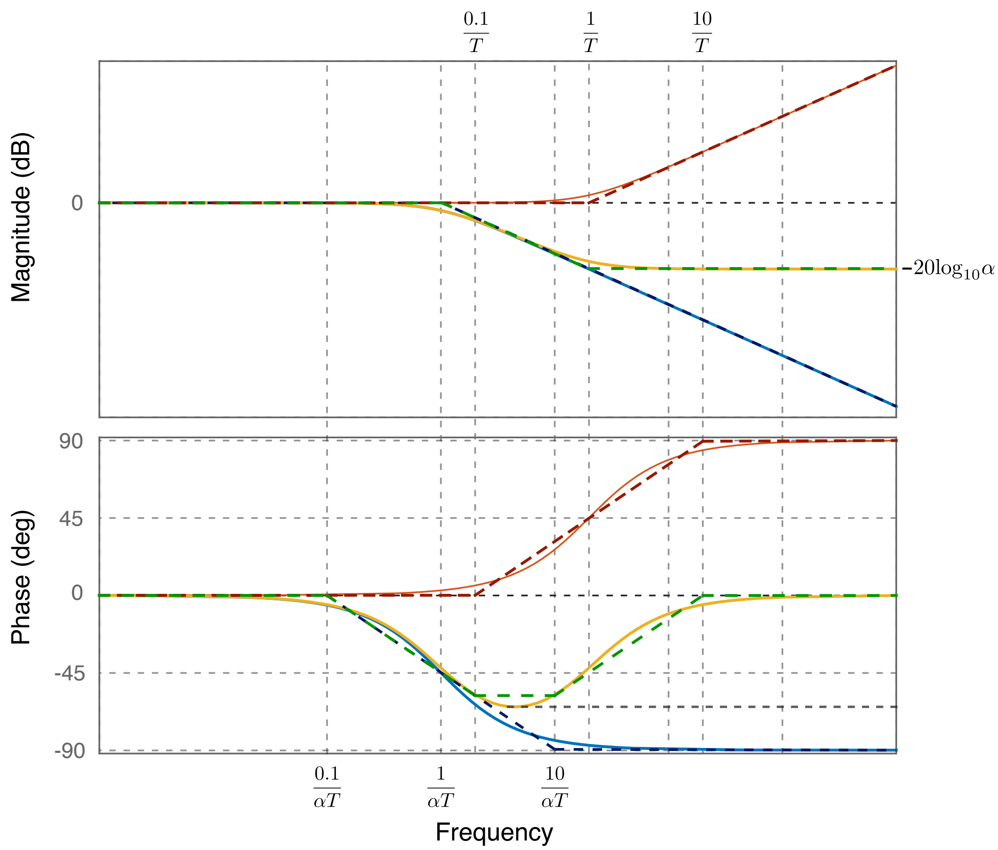
19.2 Lag Compensator Design

The lag-compensator is a controller which has the form of a first-order low-pass filter

$$G_c(s) = K_c \frac{Ts+1}{T\alpha s+1} \quad \alpha \in (1,\infty)$$

In general, we first design K_c based on the steady-state requirements of the system, then design α based on the phase-margin requirement, and finally choose a T such that phase-lag of the compensator does not interfere with the gain-crossover frequency.

First let's illustrate the bode-plots of a unity gain lag-compensator to understand how we can utilize its properties for the design process.



In lag-compensator design, we basically use the negative-gain shift of the compensator in the high-frequency region and we try to push the low and mid frequency region to the left (in frequency axis) such that they don't interfere with the gain-crossover frequency. For this reason, design process easier compared to the lead-compensator.

We will illustrate the lag-compensator design process on an the same example

Ex: Consider the feedback system that we analyzed previously in lead compensatory case. Plant is same, $G(s) = \frac{1}{s(s+1)}$. However now we want to design a lag-compensator, $G_c(s) = K_c \frac{Ts+1}{T\alpha s+1}$, $a \in (1,\infty)$), such that unit-ramp steady-state error satisfies, $e_{ss} 0.1$ and phase-margin of the compensated system satisfies $\phi_m^* > 40^o$.

Solution:

Step 1: Same as the lead-design, we design/compute K_c based on the steady-state requirement on the unit-ramp error.

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K_c} = 1 \;\rightarrow\; K_c = 10 \tag{19.2}$$

Step 2: Embed K_c into $G(s)$, i.e.

$$\bar{G(s)} = K_c G(s) = \frac{10}{s(s+1)},$$

draw the bode-plot for $\bar{G}(s)$.

Step 3: Compute/find the required gain-crossover frequency, ω^*_{gc}, based on the required phase-margin, ϕ^*_M, and compute/find the magnitude of $G(s)$ at ω^*_{gc}, i.e. $|\bar{G}(j\omega^*_{gc})|$ or $20\log_2 0|\bar{G}(j\omega^*_{gc})|$.

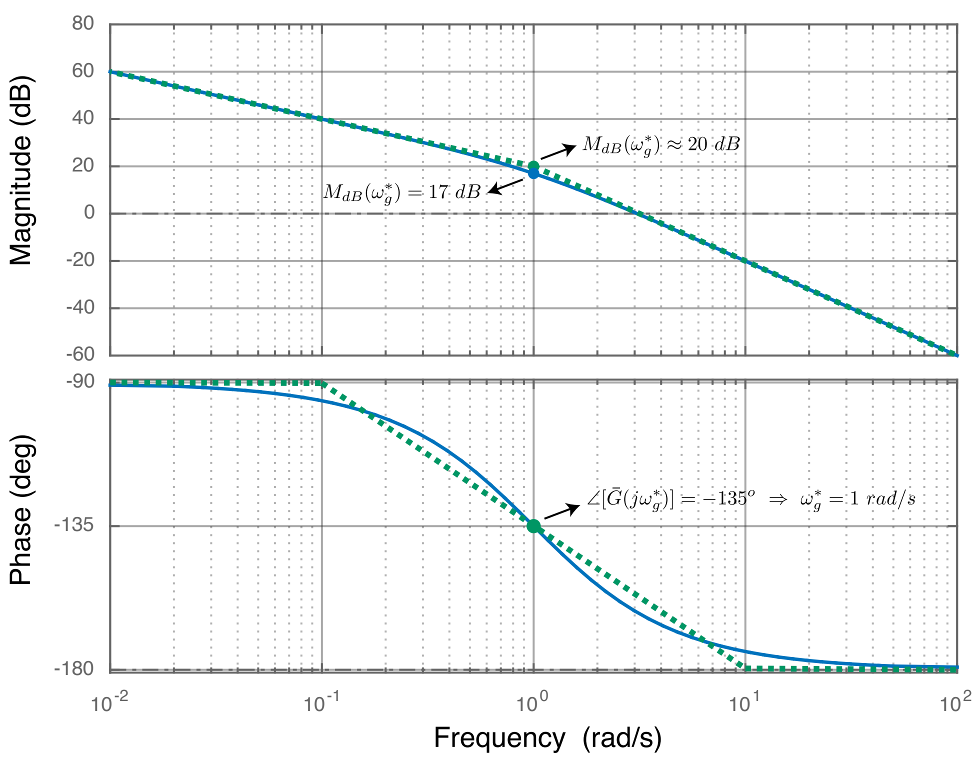
$$\angle[\bar{G}(j\omega^*_{gc})] = -180^o + \phi^*_M + \epsilon$$
$$\epsilon \approx 5^0$$

Note that at high-frequency region, lag-compensator acts like negative gain shift on magnitude plot while not affecting the phase response. Such a gain shift technically changes the gain-crossover frequency, which can be potentially used for changing the phase margin.

For our example, required phase-margin, ϕ^*_M can be computed as

$$\angle[\bar{G}(j\omega^*_{gc})] \approx 45^p \quad \Rightarrow \quad \omega^*_{gc} \approx 1\ rad/s$$

The bode plot of $\bar{G(s)}$, required gain-crossover frequency, and the magnitude of $\bar{G(s)}$ at the desired gain-crossover frequency are illustrated in the Figure below.



From the bode-plots we can observe that at the desired $\omega_{gc}^* \approx 1\ rad/s$, bode-plot approximation has a magnitude of 20 dB, whereas the magnitude in the actual bide plot is approximately 17 dB. In this example, let's use the magnitude of the approximation in the next Step.

Step 4: Compute α to compensate the the magnitude at the new-gain crossover frequency

$$20log_{10}\alpha = 20\log_{10}|\bar{G}(j\omega_{gc}^*)| \quad \text{or} \quad \alpha = |\bar{G}(j\omega_{gc}^*)|$$

In our example, α, can be computed as

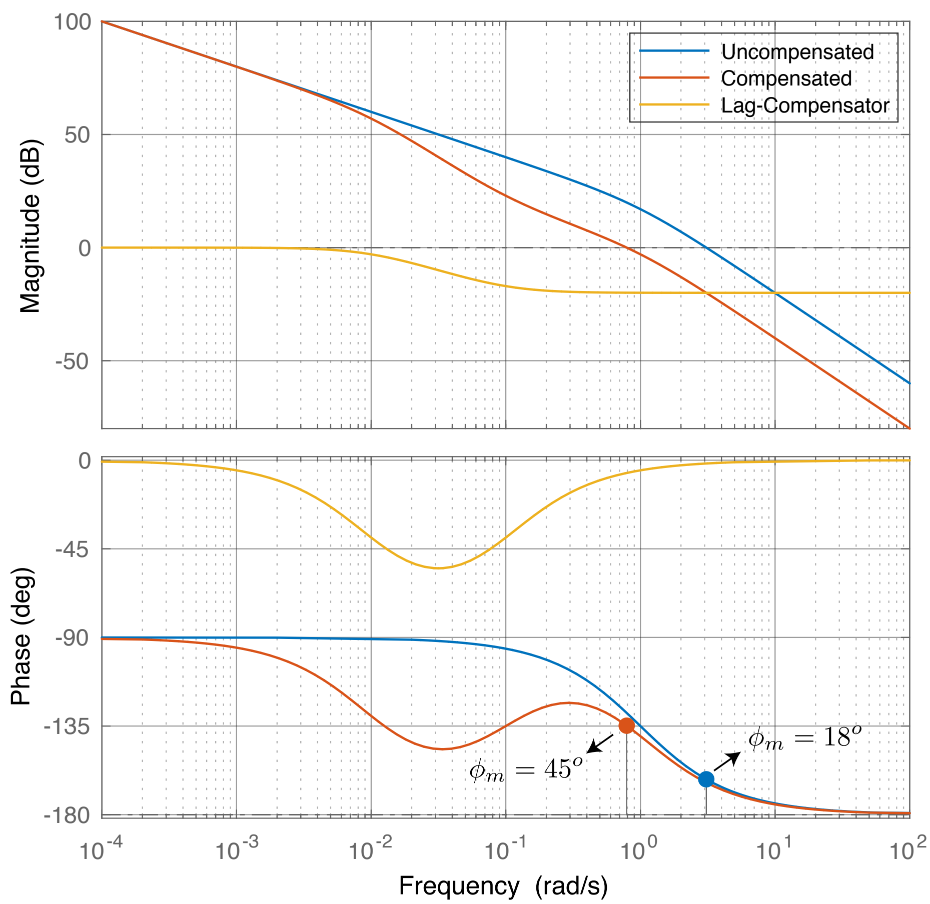
$$20log_{10}\alpha \approx 20\ dB \ \Rightarrow \ \alpha \approx 10$$

Step 4: Choose T such that "$10/T \leq \omega_{gc}^*$". Note that $10/T$ is the frequency where the phase of the compensator approximately re-approaches to zero. This is required in order the negative phase bump of the compensator does not affect the phase-margin.

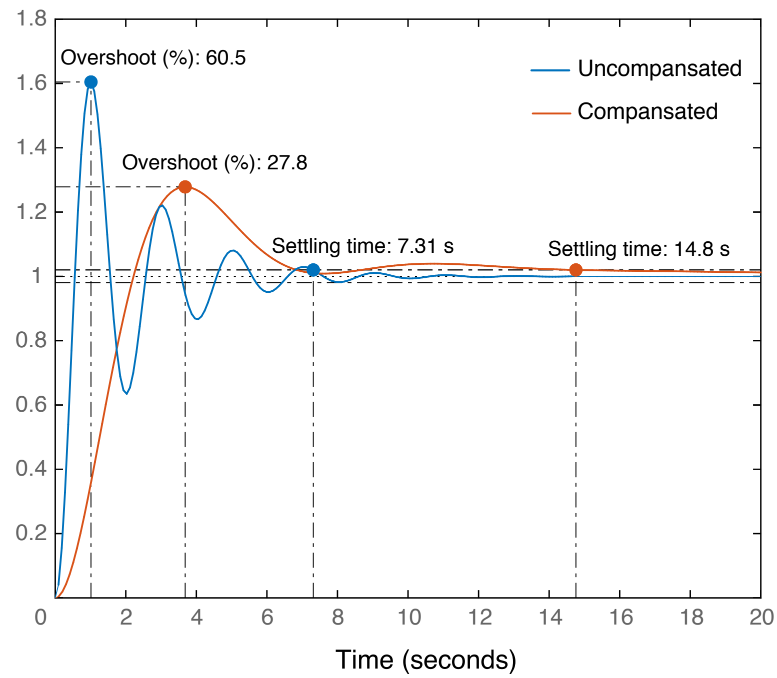
$$\frac{1}{T} \approx 0.1\omega_{gc}^* \ \Rightarrow \ T \approx 10$$

$$G_c(s) = 10\frac{10s+1}{100s+1} = \frac{s+0.1}{s+0.01}$$

The Figure below illustrates the bode plots of the uncompensated ($\bar{G}(s) = K_cG(s)$) system, designed lag-compensator, and the compensated system. Compensated systems has a phase margin of $\phi_m = 45^o$ which meets the requirements.



In Figure below, we compare the closed-loop step responses of both uncompensated and compensated closed-loop transfer functions. We can see that, similar to the the lead-compensator, lag-compensator improves the over-shoot performance. However, the settling-time of the compensated system is approximately two times of the original system. This is major the drawback of the lag-compensator.



The reason behind the reduced convergence speed performance is that new gain-crossover frequency is less than the original gain-crossover frequency. PM is closely related with over-shoot performance. On the other hand gain-crossover frequency determines the band-with of the system which is closely related with the rise and settling times.

Phase-Lag Design Guideline:

1. Design/compute K_c based on the steady-state requirements
2. Compute/find the required gain-crossover frequency, ω_{gc}^*, based on the required phase-margin, ϕ_M^*, and compute/find the magnitude of $G(s)$ at ω_{gc}^*
3. Compute α to compensate the the magnitude at the new-gain crossover frequency
4. Chose T at such that "$10/T \leq \omega_{gc}^*$". This is required in order the negative phase bump of the compensator does not affect the phase-margin. Note that $10/T$ is the frequency where the phase of the compensator approximately re-approaches to zero.
5. Check the phase-margin if it does not meet the requirements change $\Delta\phi$ and repeat the process.