EE502 - Linear Systems Theory II **Spring 2023**

Lecture 4

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4.1 Realization Theory

Definition: A transfer function (matrix) $G(s)$ is said to be realizable if there exist a (finite-dimensional) state-space realization of it.

Theorem: A Transfer-Function (matrix) $G(s)$ is realizable $\Leftrightarrow G_{ij}(s)$ is a proper rational transfer function $\forall(i,j)$

Proof:

Part I: Show that $G(s)$ is realizable $\Rightarrow G_{ij}(s)$ is proper $\forall(i,j)$

Assume $G(s)$ realizable $\exists(A,B,C,D)$ tuple such that $G(s) = C(sI-A)^{-1}B + D$

$$\lim_{s\to\infty} G(s) = \lim_{s\to\infty}\left[C\left(sI-A\right)^{-1}B\right] = \lim_{s\to\infty}\left[\frac{C\mathrm{Adj}\left(sI-A\right)B}{\mathrm{Det}\left(sI-A\right)} + D\right] = D \ , \ where$$
$$|D_{ij}| < \infty \ \forall(i,j) \Rightarrow G_{ij}(s) \text{ is proper } \forall(i,j)$$

Part II: Show that $G_{ij}(s)$ is proper $\forall(i,j) \Rightarrow G(s)$ is realizable

The task is to find (A,B,C,D) tuple such that $G(s) = C(sI-A)^{-1}B + D$

4.1.1 Canonical State-Space Realizations (SISO Systems)

For the sake of clarity, derivations are given for general 3^{rd} order LTI systems.

4.1.1.1 CT Reachable/Controllable Canonical Form

In this method of realization, we use the fact the system is LTI. Let's consider the transfer function of the system and let's perform some LTI operations.

$$\begin{aligned}
Y(s) &= \frac{b_3s^3 + b_2s^2 + b_1s + b_0}{s^3 + a_2s^2 + a_1s + a_0}U(s) \\
&= \left(b_3s^3 + b_2s^2 + b_1s + b_0\right)\frac{1}{s^3 + a_2s^2 + a_1s + a_0}U(s) \\
&= G_2(s)G_1(s)U(s) \text{ where} \\
G_1(s) &= \frac{H(s)}{U(s)} = \frac{1}{s^3 + a_2s^2 + a_1s + a_0} \\
G_2(s) &= \frac{Y(z)}{H(z)} = b_3s^3 + b_2s^2 + b_1s + b_0
\end{aligned}$$

As you can see we introduced an intermediate variable $h(t)$ or with a Laplace transform of $H(s)$. First transfer function has static input dynamics, operates on $u(t)$, and produces an output, i.e. $h(t)$. Second transfer function is a "non-causal" system and operates on $h(t)$ and produces output $y(t)$. If we write the ODEs of both systems we obtain

$$\begin{aligned}\dddot{h} &= -a_2\ddot{h} - a_1\dot{h} - a_0 h + u \\ y &= b_3\dddot{h} + b_2\ddot{h} + b_1\dot{h} + b_0 h\end{aligned}$$

Now let the state-variables be $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} h \\ \dot{h} \\ \ddot{h} \end{bmatrix}$. Then, individual state equations take the form

$$\begin{aligned}\dot{x_1} &= x_2 \\ \dot{x_2} &= x_3 \\ \dot{x_3} &= -a_2 x_3 - a_1 x_2 - a_0 x_1 + u\end{aligned}$$

and the output equation take the form

$$\begin{aligned}y &= b_3\left(-a_2 x_3 - a_1 x_2 - a_0 x_1 + u\right) + b_2 x_3 + b_1 x_2 + b_0 x_1 \\ &= (b_0 - b_3 a_0)x_1 + (b_1 - b_3 a_1)x_2 + (b_2 - b_3 a_2)x_3 + b_3 u\end{aligned}$$

If we re-write the equations in matrix form we obtain the state-space representation as

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} (b_0 - b_3 a_0) & (b_1 - b_3 a_1) & (b_2 - b_3 a_2) \end{bmatrix} x + [b_3]\, u\end{aligned}$$

If we obtain a state-space model from this approach, the form will be in *controllable canonical form.*

For a general n^{th} order transfer function controllable canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} (b_0 - b_n a_0) & (b_1 - b_n a_1) & \cdots & (b_{n-1} - b_n a_{n-1}) \end{bmatrix} \quad , \quad D = b_n$$

4.1.1.2 DT Reachable/Controllable Canonical Form

Similar to the CT case, we will use the fact the system is LTI. Let's consider the transfer function of the system and let's perform some LTI operations.

$$\begin{aligned}
Y(z) &= \frac{b_3 + b_2 z^{-1} + b_1 z^{-2} + b_0 z^{-3}}{1 + a_2 z^{-1} + a_1 z^{-2} + a_0 z^{-3}} U(z) \\
&= \left(b_3 + b_2 z^{-1} + b_1 z^{-2} + b_0 z^{-3}\right) \frac{1}{1 + a_2 z^{-1} + a_1 z^{-2} + a_0 z^{-3}} X(z) \\
&= G_2(z) G_1(z) U(z) \text{ where} \\
G_1(z) &= \frac{H(z)}{U(z)} = \frac{1}{1 + a_2 z^{-1} + a_1 z^{-2} + a_0 z^{-3}} \\
G_2(z) &= \frac{Y(z)}{H(z)} = b_3 + b_2 z^{-1} + b_1 z^{-2} + b_0 z^{-3}
\end{aligned}$$

As you can see we introduced an intermediate variable $h[k]$ with a Z-transform of $H(z)$, First transfer function, which is a system with static input dynamics operates on $u[n]$ and produces an output. Second transfer function operates on $h[n]$ and produces output $y[n]$. If we write the difference equations of both systems we obtain

$$\begin{aligned}
h[k] &= -a_2 h[k-1] - a_1 h[k-2] - a_0 h[k-3] + u[k] \\
y[k] &= b_3 h[k] + b_2 h[k-1] + b_1 h[k-2] + b_0 h[k-3]
\end{aligned}$$

As it can be seen that the delay/shifting operations are only performed on the signal $h[k]$. Now let the state-variables be $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} h[k-3] \\ h[k-2] \\ h[k-1] \end{bmatrix}$. Then, individual state equations take the form

$$\begin{aligned}
x_1[k+1] &= x_2[k] \\
x_2[k+1] &= x_3[k] \\
x_3[k+1] &= -a_2 x_1[k] - a_1 x_2[k] - a_0 x_3[k] + u
\end{aligned}$$

and the output equation take the form

$$\begin{aligned}
y[k] &= b_3\left(-a_2 x_3[k] - a_1 x_2[k] - a_0 x_1[k] + u[k]\right) + b_2 x_3[k] + b_1 x_2[k] + b_0 x_1[k] \\
&= (b_0 - b_3 a_0) x_1[k] + (b_1 - b_3 a_1) x_2[k] + (b_2 - b_3 a_2) x_3[k] + b_3 u[k]
\end{aligned}$$

If we re-write the equations in matrix form we obtain the state-space representation as

$$\begin{aligned}
x[k+1] &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[k] \\
y &= \begin{bmatrix} (b_0 - b_3 a_0) & (b_1 - b_3 a_1) & (b_2 - b_3 a_2) \end{bmatrix} x[k] + [b_3]\, u[k]
\end{aligned}$$

It can be seet that reachability/controllability canonical forms for DT and CT systems are exactly same. For a general n^{th} order transfer function controllable canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} (b_0 - b_n a_0) & (b_1 - b_n a_1) & \cdots & (b_{n-1} - b_n a_{n-1}) \end{bmatrix} \quad , \quad D = b_n$$

4.1.1.3 CT Observable Canonical Form

In this method will obtain a different minimal state-space realization, the form is called observable canonical form. The process is different and state-space structure will have a different topology. Let's start with a 3^{rd} transfer function and perform some grouping based on the s elements.

$$Y(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} U(s)$$
$$Y(s)\left(s^3 + a_2 s^2 + a_1 s + a_0\right) = \left(b_3 s^3 + b_2 s^2 + b_1 s + b_0\right) U(s)$$
$$s^3 Y(s) = b_3 s^3 U(s) + s^2 \left(-a_2 Y(s) + b_2 U(s)\right) + s\left(-a_1 Y(s) + b_1 U(s)\right) + \left(-a_0 Y(s) + b_0 U(s)\right)$$

Let's multiply both sides with $\frac{1}{s^3}$ and perform further grouping

$$Y(s) = b_3 U(s) + \frac{1}{s}\left(-a_2 Y(s) + b_2 U(s)\right) + \frac{1}{s^2}\left(-a_1 Y(s) + b_1 U(s)\right) + \frac{1}{s^3}\left(-a_0 Y(s) + b_0 U(s)\right)$$
$$Y(s) = b_3 U(s) + \frac{1}{s}\left[\left(-a_2 Y(s) + b_2 U(s)\right) + \frac{1}{s}\left\{\left(-a_1 Y(s) + b_1 U(s)\right) + \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right)\right\}\right]$$

Let the Laplace domain representations of state variables $X(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix}$ defined as

$$X_1(s) = \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right)$$
$$X_2(s) = \frac{1}{s}\left\{\left(-a_1 Y(s) + b_1 U(s)\right) + \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right)\right\}$$
$$X_3(s) = \frac{1}{s}\left[\left(-a_2 Y(s) + b_2 U(s)\right) + \frac{1}{s}\left\{\left(-a_1 Y(s) + b_1 U(s)\right) + \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right)\right\}\right]$$

In this context output equation in s and time domains simply takes the form

$$Y(s) = X_3(s) + b_3 U(S) \quad \rightarrow \quad y(t) = x_3(t) + b_3 u(t)$$

Dependently the state equations (in s and time domains) take the form

$$\begin{aligned} sX_1(s) &= -a_0 X_3(s) + (b_0 - a_0 b_3) U(s) & \rightarrow \quad \dot{x}_1 &= -a_0 x_3 + (b_0 - a_0 b_3) u \\ sX_2(s) &= X_1(s) - a_1 X_3(s) + (b_1 - a_1 b_3) U(s) & \rightarrow \quad \dot{x}_2 &= x_1 - a_1 x_3 + (b_1 - a_1 b_3) u \\ sX_3(s) &= X_2(s) - a_2 X_3(s) + (b_2 - a_2 b_3) U(s) & \rightarrow \quad \dot{x}_3 &= x_2 - a_2 x_3 + (b_2 - a_2 b_3) u \end{aligned}$$

If we re-write all equations in matrix form, we obtain the state-space representation as

$$\dot{x} = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} x + \begin{bmatrix} (b_0 - b_3 a_0) \\ (b_1 - b_3 a_1) \\ (b_2 - b_3 a_2) \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x + [b_3]\, u$$

If we obtain a state-space model from this method, the form will be in *observable canonical form.* Thus we can call this representation also as *observable canonical realization.* This form and representation is the dual of the previous representation.

For a general n^{th} order system controllable canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \quad , \quad B = \begin{bmatrix} (b_0 - b_n a_0) \\ (b_1 - b_n a_1) \\ \vdots \\ (b_{n-2} - b_n a_{n-2}) \\ (b_{n-1} - b_n a_{n-1}) \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad , \quad D = b_n$$

4.1.1.4 DT Observable Canonical Form

Let's start with the transfer function and perform some grouping based on the delay elements.

$$\begin{aligned} &Y(z)(1 + a_2 z^{-1} + a_1 z^{-2} + a_0 z^{-3}) = (b_3 + b_2 z^{-1} + b_1 z^{-2} + b_0 z^{-3})U(z) \\ &Y(z) = b_3 U(z) + z^{-1}\left(b_2 U(z) - a_2 Y(z)\right) + z^{-2}\left(b_1 U(z) - a_1 Y(z)\right) + z^{-3}\left(b_0 U(z) - a_0 Y(z)\right) \\ &Y(z) = b_3 U(z) + z^{-1}\left\{(b_2 U(z) - a_2 Y(z)) + z^{-1}\left[(b_1 U(z) - a_1 Y(z)) + z^{-1}\left(b_0 U(z) - a_0 Y(z)\right)\right]\right\} \end{aligned}$$

As you can see we have only z^{-1} terms in the representation there is a special toplogy embedded inside the expression. Let the Z-transform domain representations of state variables $X(z) = \begin{bmatrix} X_1(z) \\ X_2(z) \\ X_3(z) \end{bmatrix}$ defined as

$$\begin{aligned} X_1(z) &= \frac{1}{z}\left(-a_0 Y(z) + b_0 U(z)\right) \\ X_2(s) &= \frac{1}{z}\left\{(-a_1 Y(z) + b_1 U(z)) + \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right)\right\} \\ X_3(s) &= \frac{1}{z}\left[(-a_2 Y(z) + b_2 U(z)) + \frac{1}{z}\left\{(-a_1 Y(z) + b_1 U(z)) + \frac{1}{z}\left(-a_0 Y(z) + b_0 U(z)\right)\right\}\right] \end{aligned}$$

In this context output equation in s and time domains simply takes the form

$$Y(z) = X_3(z) + b_3 U(z) \quad \rightarrow \quad y[k] = x_3[k] + b_3 u[k]$$

Dependently the state equations (in z and time domains) take the form

$$\begin{aligned} zX_1(z) &= -a_0 X_3(s) + (b_0 - a_0 b_3)U(s) &\rightarrow\quad x_1[k+1] &= -a_0 x_3[k] + (b_0 - a_0 b_3)u[k] \\ zX_2(z) &= X_1(s) - a_1 X_3(s) + (b_1 - a_1 b_3)U(s) &\rightarrow\quad x_2[k+1] &= x_1[k] - a_1 x_3[k] + (b_1 - a_1 b_3)u[k] \\ zX_3(z) &= X_2(z) - a_2 X_3(z) + (b_2 - a_2 b_3)U(z) &\rightarrow\quad x_3[k+1] &= x_2[k] - a_2 x_3[k] + (b_2 - a_2 b_3)u[k] \end{aligned}$$

If we re-write all equations in matrix form, we obtain the state-space representation as

$$x[k+1] = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} x[k] + \begin{bmatrix} (b_0 - b_3 a_0) \\ (b_1 - b_3 a_1) \\ (b_2 - b_3 a_2) \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x[k] + [b_3]\, u[k]$$

If we obtain a state-space model from this method, the form will be in *observable canonical form.* Thus we can call this representation also as *observable canonical realization.* This form and representation is the dual of the controllable canonical form

For a general n^{th} order system observable canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \quad , \quad B = \begin{bmatrix} (b_0 - b_n a_0) \\ (b_1 - b_n a_1) \\ \vdots \\ (b_{n-2} - b_n a_{n-2}) \\ (b_{n-1} - b_n a_{n-1}) \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad , \quad D = b_n$$

4.1.1.5 CT Diagonal Canonical Form

If the transfer function of the CT-LTI system has distinct poles, we can expand it using partial fraction expansion

$$Y(s) = \left[b_3 + \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \frac{c_3}{s - p_3} \right] U(s)$$

Now let's concentrate on the candidate "state variables" and try to write state evaluation equations

$$\begin{aligned} X_1(s) &= \frac{1}{s - p_1} U(s) & \rightarrow \quad & \dot{x}_1 = p_1 x_1 + u \\ X_2(s) &= \frac{1}{s - p_2} U(s) & \rightarrow \quad & \dot{x}_2 = p_2 x_2 + u \\ X_3(s) &= \frac{1}{s - p_3} U(s) & \rightarrow \quad & \dot{x}_3 = p_3 x_3 + u \end{aligned}$$

where as output equation can be derived as

$$y(t) = b_3 u(t) + c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t)$$

If we combine the state and output equations, we can obtain the state space form as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \mathbf{x}(t) + b_3 u(t) \end{aligned}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad , \quad A = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \quad , \quad D = b_3$$

The form obtained with this approach is called diagonal canonical form. Obviously, this form is not applicable for "some" systems that has repeated roots.

For a general n^{th} order system with distinct roots diagonal canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} p_1 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & p_n \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix} \quad , \quad D = b_n$$

4.1.1.6 DT Diagonal Canonical Form

If the transfer function of the DT-LTI system has distinct poles, we can expand it using partial fraction expansion

$$Y(z) = \left[b_3 + \frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} + \frac{c_3}{z - p_3} \right] U(s)$$

Now let's concentrate on the candidate "state variables" and try to write state evaluation equations

$$\begin{aligned} X_1(z) &= \frac{1}{z - p_1} U(s) \quad \rightarrow \quad x_1[k+1] = p_1 x_1[k] + u \\ X_2(z) &= \frac{1}{z - p_2} U(s) \quad \rightarrow \quad x_2[k+1] = p_2 x_2[k] + u \\ X_3(z) &= \frac{1}{z - p_3} U(s) \quad \rightarrow \quad x_3[k+1] = p_3 x_3[k] + u \end{aligned}$$

where as output equation can be derived as

$$y[k] = b_3 u[k] + c_1 x_1[k] + c_2 x_2[k] + c_3 x_3[k]$$

If we combine the state and output equations, we can obtain the state space form as

$$x[k+1] = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$
$$y[k] = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} x[k] + b_3 u[k]$$

where

$$A = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \quad , \quad D = b_3$$

The form obtained with this approach is called diagonal canonical form. Obviously, this form is not applicable for "some" systems that has repeated roots.

For a general n^{th} order system with distinct roots diagonal canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} p_1 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & p_n \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix} \quad , \quad D = b_n$$

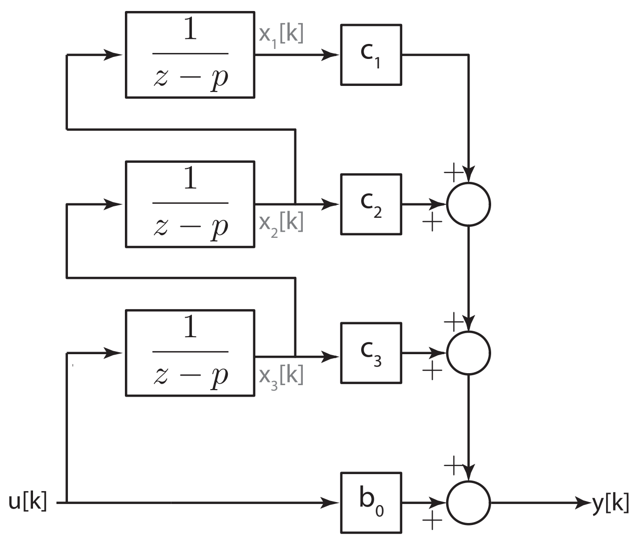
DT(CT) Jordan Canonical Form

Generalization of diagonal canonical farm is called Jordan canonical form which handles repeated roots.

In Jordan form the distinct roots has the same structure with Diagonal canonical form. Let's assume that the 3^{rd} order pulse transfer function has three repeated roots. In this case, we can expand it using partial fraction expansion

$$\begin{aligned} Y(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} X(z) \\ &= \left(b_0 + \frac{c_1}{(z-p)^3} + \frac{c_2}{(z-p)^2} + \frac{c_3}{z-p} \right) X(z) \end{aligned}$$

It is possible to realize this expanded form in block diagram form as given in the Figure below.



Now let's concentrate on the candidate "state variables" and try to write state evaluation equations

$$\begin{aligned} X_1(z) &= \frac{1}{z-p} X_2(z) \quad \rightarrow \quad x_1[k+1] = p\, x_1[k] + x_2[k] \\ X_2(z) &= \frac{1}{z-p} X_3(z) \quad \rightarrow \quad x_2[k+1] = p\, x_2[k] + x_3[k] \\ X_3(z) &= \frac{1}{z-p} U(z) \quad \rightarrow \quad x_3[k+1] = p\, x_3[k] + u[k] \end{aligned}$$

where as output equation can be derived as

$$y[k] = b_0 u[k] + c_1 x_1[k] + c_2 x_2[k] + c_3 x_3[k]$$

If we combine the state and output equations, we can obtain the state space form as

$$\mathbf{x}[k+1] = \begin{bmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} + b_0 u[k]$$

where

$$\mathbf{x} = \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_1[k] \end{bmatrix} \quad , \quad A = \begin{bmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \quad , \quad D = b_0$$

A, B, & C forms a Jordan block.

For a general n^{th} order system a Jordan block with m repeated roots inside a stat-space representation in Jordan canonical form looks like

$$A = \left[\begin{array}{c|ccccc|c} \ddots & & & & & & \\ \hline & \bar{p} & 1 & \cdots & 0 & 0 & \\ & 0 & \bar{p} & \cdots & 0 & 0 & \\ & & & \ddots & & & \\ & 0 & 0 & \cdots & \bar{p} & 1 & \\ & 0 & 0 & \cdots & 0 & \bar{p} & \\ \hline & & & & & & \ddots \end{array}\right] \quad , \quad B = \left[\begin{array}{c} \vdots \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \hline \vdots \end{array}\right]$$
$$C = \left[\begin{array}{c|ccccc|c} \cdots & c_1 & c_2 & \cdots & c_{n-1} & c_n & \cdots \end{array}\right]$$

Example: Given that

$$G(s) = \frac{s^2 + 8s + 10}{s^2 + 3s + 2}$$

find a controllable, observable, and diagonal canonical state-space representation of the given TF.

Solution:

If we follow the derivation of controllable canonical form for a second order system we obtain the following structure

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} (b_0 - b_2 a_0) & (b_1 - b_2 a_1) \end{bmatrix} x + [b_2]\, u$$

where

$$a_0 = 2 \ , \ a_1 = 3 \ , \ b_0 = 10 \ , \ b_1 = 8 \ , \& \ b_2 = 1$$

Thus, the state-space representation takes the form

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 8 & 5 \end{bmatrix} x + [1] u$$

Observable canonical form is the dual of the controllable canonical form thus for the given system, we know that

$$
\begin{aligned}
A_{OCF} &= A_{CCF}^T = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \\
B_{OCF} &= C_{CCF}^T = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \\
C_{OCF} &= B_{CCF}^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \\
D_{OCF} &= D_{CCF} = [1]
\end{aligned}
$$

In order to find the diagonal canonical form, we need to perform partial fraction expansion

$$G(s) = \frac{s^2 + 8s + 10}{s^2 + 3s + 2} = 1 + \frac{3}{s+1} + \frac{2}{s+2}$$

then SS matrices for the diagonal canonical form can be simply derived as

$$
\begin{aligned}
A_{DCF} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \\
B_{DCF} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
C_{DCF} &= \begin{bmatrix} 3 & 2 \end{bmatrix} \\
D_{DCF} &= [1]
\end{aligned}
$$

Example: Consider the following general state-space representation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Now let's consider the following state-space representation

$$\begin{aligned}\dot{\bar{x}}(t) &= A^T\bar{x}(t) + C^Tu(t),\\ y(t) &= B^T\bar{x}(t) + Du(t)\end{aligned}$$

Show that these two state-space representations results in same transfer function form

Solution: For the second representation we have

$$\begin{aligned}\bar{G}(s) &= \bar{C}\left(sI - \bar{A}\right)^{-1}\bar{B} + D\\ &= B^T\left(sI - A^T\right)^{-1}C^T + D\end{aligned}$$

Since $\bar{G}(s)$ is a scalar quantity we can take its transpose

$$\begin{aligned}\bar{G}(s) &= [\bar{G}(s)]^T = [B^T\left(sI - A^T\right)^{-1}C^T + D]^T\\ &= (C^T)^T\left(\left(sI - A^T\right)^{-1}\right)^T(B^T)^T + D\\ &= C\left(\left(sI - A^T\right)^T\right)^{-1}B + D\\ &= C\left(sI - A\right)^{-1}B + D\\ \bar{G}(s) &= G(s)\end{aligned}$$

This result also shows that controllable and observable canonical representations are similar.