EE302 - Feedback Systems	**Spring 2019**

Lecture 1

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Dynamical Systems

A dynamical system is a collection of "elements" for which there is a time-dependent cause and effect relationships between "variables". The behavior of a dynamical system changes with time, usually in response to external inputs.

The elements can range from atoms and molecules to oceans and planets. Similarly, time scales can range from pico seconds to years and even decades.

Modeling of Dynamical Systems

Goal: Obtaining a mathematical description of the dynamic relationship between the variables of the given dynamical system/behavior. ***Always***, the mathematical description is a simplification of the real phenomena. Always remember:

"All models are wrong but some are useful" George E. Box, 1978.

Representations of (Continious-Time) Dynamical Systems

1. Differential equations

$$a_n y^{(n)} + ... + a_1 y' + a_0 y = b_n u^{(n)} + ... + b_1 u' + b_0 u$$

   Differential equations can model both linear and non-linear systems, time-invariant and time-varying ones.

2. Impulse response representation

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(t-\tau) u(\tau) d\tau$$

   Impulse-response representation can be used to model the dynamic relation between input, $u(t)$, and output $y(t)$ signals. This representation is limited to Linear-Time-Invariant (LTI) systems.

   Time varying impulse response representation can be used to model Linear-Time-Varying (LTV) systems.

3. Transfer function representation

$$
\begin{aligned}
&Y(s) = G(s)U(s), \text{ where,}\\
&Y(s) = \mathcal{L}\{y(t)\}\ , \&\ U(s) = \mathcal{L}\{u(t)\}
\end{aligned}
$$

   Transfer functions are limited to LTI systems.

4. State-space representation

$$
\begin{aligned}
\text{Let } & x(t) \in \mathbb{R}^n\ ,\ y(t) \in \mathbb{R}\ ,\ u(t) \in \mathbb{R},\\
& \dot{x}(t) = Ax(t) + Bu(t),\\
& y(t) = Cx(t) + Du(t),\\
\text{where } & A \in \mathbb{R}^{n\times n}\ ,\ B \in \mathbb{R}^{n\times 1}\ ,\ C \in \mathbb{R}^{1\times n}\ ,\ D \in \mathbb{R}
\end{aligned}
$$

   Similar to differential equations, we can model both linear and non-linear systems, and time-invariant and time-varying systems using state-space representations.

Feedback Systems

The *feedback* terms constitute the case, where dynamical systems are connected in a way that the output of each system influences its own driving input (directly or indirectly), and dynamic relations between different sub-systems are tightly coupled.

Simple causal reasoning about a feedback system is difficult (if not possible). For example, if we observe the feedback system illustrated in Fig. 1.1(B), we see that the first system influences the second, and the second system influences the first, leading to a circular argument. We can conclude that formal methods are necessary to understand the behavior of feedback systems.

Fig. 1.1 illustrates the idea of feedforward vs. feedback in block diagram topology. Open-loop and closed-loop terms are also commonly used to refer to feedforward and feedback structures, respectively. When systems connect each other in a cyclic topology, we refer to the topolgy as a closed-loop system. If one cuts an interconnection such that the cyclic structure is broken, the system becomes an open-loop system.

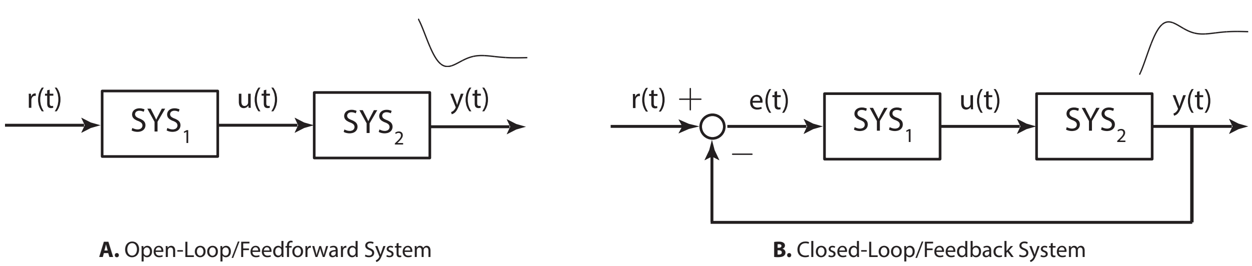


Figure 1.1: Open- and closed-loop systems

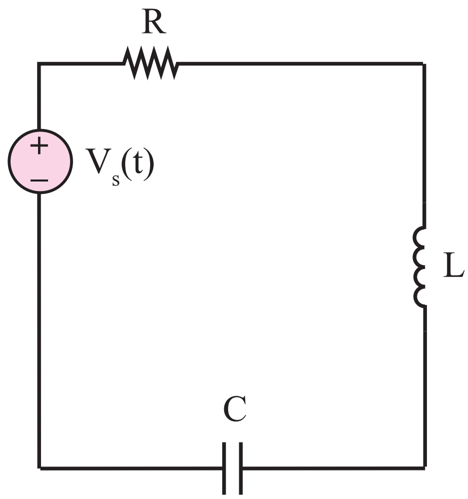
Feedback based control may not be mandatory for some control applications, yet a feedforward based control policy can be more advantageous than a feedback-based control policy in some cases. The core benefit of feedback in a control system is that feedback reduces uncertainties in systems and improves the robustness. Since uncertainties are unavoidable in real life, feedback control systems are ubiquitous in both synthetic and biological control systems.

Dynamic Modeling of Some Physical Systems

Electrical Circuits

Ex 1: Series RLC Circuit

Given than input is $u(t) = V_s(t)$ and output os $y(t) = V_C(t)$, find an ODE description of the given dynamical circuit.



$$\begin{aligned}
V_L + V_R + V_C &= V_s(t) \\
L\frac{dI}{dt} + RI + V_C &= V_s(t) \\
L\frac{d}{dt}\left(C\frac{dV_C}{dt}\right) + R\left(C\frac{dV_C}{dt}\right) + V_C &= V_s(t) \\
LC\ \ddot{V}_C + RC\ \dot{V}_C + V_C &= V_s(t) \\
\ddot{y} + \frac{R}{L}\dot{y} + \frac{1}{LC}y &= \frac{1}{LC}u
\end{aligned}$$

Find the transfer function representation of the system for the given input–output pair.

$$\begin{aligned}
\mathcal{L}\left\{\ddot{y} + \frac{R}{L}\dot{y} + \frac{1}{LC}y\right\} &= \mathcal{L}\left\{\frac{1}{LC}u\right\} \\
s^2Y(s) + s\frac{R}{L}Y(s) + \frac{1}{LC}Y(s) &= \frac{1}{LC}U(s) \\
G(s) = \frac{Y(s)}{U(s)} &= \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}
\end{aligned}$$

Find a state-space representation of the system.

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$, then

$$\begin{aligned}
\dot{x_1} &= x_2 \\
\dot{x_2} &= -\frac{1}{LC}x_1 - \frac{R}{L}x_2 + \frac{1}{LC}u
\end{aligned}$$

If we put the equations in state-space form, we obtain

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x
\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \quad , \ B = \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} \quad , \ C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad , \ D = 0$$

Now let, $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} V_C \\ I \end{bmatrix}$, then

$$\begin{aligned} \dot{z}_1 &= \frac{1}{C} z_2 \\ \dot{z}_2 &= -\frac{1}{L} z_1 - \frac{R}{L} z_2 + \frac{1}{L} u \end{aligned}$$

If we put the equations in state-space form, we obtain

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} z + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} z \end{aligned}$$

It can be seen that state-space representation of a dynamical system is not unique. Indeed there exist ifinitelly many state-space representations of the same system.