

## Lecture 14

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## 14.1 Frequency Response Techniques in Control Systems

Let's assume  $u(t)$ ,  $y(t)$ , and  $G(t)$  represents the input, output, and transfer function representation of an input-output continuous time system.

In order to characterize frequency response of a dynamical system, the test signal is

$$u(t) = e^{j\omega t}$$

which is an artificial complex periodic signal with a frequency of  $\omega$ . The Laplace transform of  $u(t)$  takes the form

$$U(s) = \mathcal{L}\{e^{j\omega t}\} = \frac{1}{s - j\omega}$$

Response of the system in s-domain is given by

$$Y(s) = G(s)U(s) = G(s)\frac{1}{s - j\omega}$$

Assuming that  $G(s)$  is a rational transfer function we can perform a partial fraction expansion

$$Y(s) = \frac{a}{s - j\omega} + [\text{terms due to the poles of } G(s)]$$

$$a = \lim_{s \rightarrow j\omega} [(s - j\omega)Y(s)] = G(j\omega)$$

$$Y(s) = \frac{G(j\omega)}{s - j\omega} + [\text{terms due to the poles of } G(s)]$$

Taking the inverse Laplace transform yields

$$y(t) = G(j\omega)e^{j\omega t} + \mathcal{L}^{-1}[\text{terms due to the poles of } G(s)]$$

If we assume that the system is “stable” or system is a part of closed loop system and closed loop behavior is stable then at steady state we have

$$\begin{aligned} y_{ss}(t) &= G(j\omega)e^{j\omega t} \\ &= |G(j\omega)|e^{i\omega t + \angle[G(j\omega)]} \\ &= Me^{i\omega t + \theta} \end{aligned}$$

In other words complex periodic signal is scaled and phase shifted based on the following operators

$$\begin{aligned} M &= |G(j\omega)| \\ \theta &= \angle G(j\omega) \end{aligned}$$

It is very easy to show that for a general real time domain signal  $u(t) = \sin(\omega t + \phi)$ , the output  $y(t)$  at steady state is computed via

$$y_{ss}(t) = M \sin(\omega t + \phi + \theta)$$

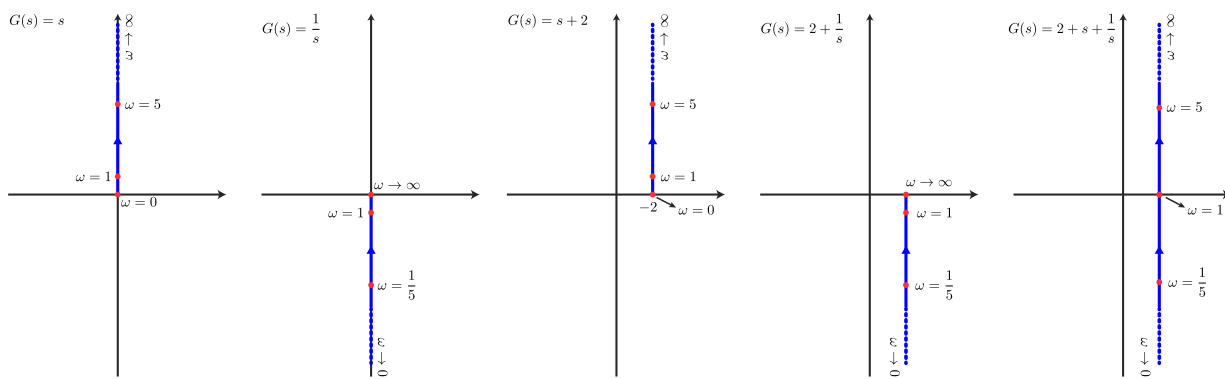
### 14.1.1 Plotting Frequency Response: Polar Plot

We can consider the frequency response function  $G(j\omega)$  as a mapping from positive  $j\omega$  axis to a curve in the complex plane. In polar plot, we draw the frequency response function starting from  $\omega = 0$  (or  $\omega \rightarrow 0^+$ ) to  $\omega \rightarrow \infty$ .

Let's draw the polar plots of

$$G_1(s) = s, \quad G_2(s) = \frac{1}{s}, \quad G_3(s) = s + 2$$

$$G_4(s) = 2 + \frac{1}{s}, \quad G_5(s) = 2 + s + \frac{1}{s}$$



Now let's draw the polar plots of

$$G_1(s) = \frac{1}{s+1}, \quad G_2(s) = \frac{s}{s+1}$$

Let's analyze  $G_1(j\omega)$  for  $\omega \in [0, \infty)$

$$G_1(j\omega) = \frac{1}{j\omega + 1} = \frac{1 - j\omega}{\omega^2 + 1} = \frac{1}{\omega^2 + 1} - \frac{\omega}{\omega^2 + 1}j$$

$$|G_1(j\omega)| = \frac{1}{\sqrt{1 + \omega^2}}$$

$$\angle[G_1(j\omega)] = \arctan(-\omega)$$

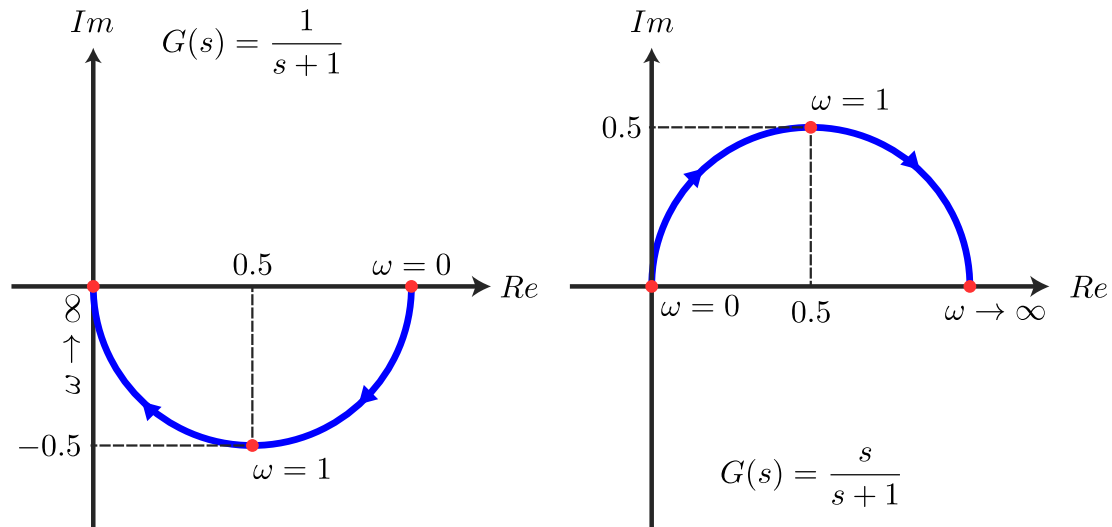
Now let's analyze  $G_2(j\omega)$  for  $\omega \in [0, \infty)$

$$G_2(j\omega) = \frac{j\omega}{j\omega + 1} = \frac{j\omega + \omega^2}{\omega^2 + 1} = \frac{\omega^2}{\omega^2 + 1} + \frac{\omega}{\omega^2 + 1}j$$

$$|G_2(j\omega)| = \sqrt{\frac{\omega^2}{1 + \omega^2}}$$

$$\angle[G_2(j\omega)] = \arctan(1/\omega)$$

Polar plots of  $G_1(s)$  and  $G_2(s)$  are illustrated below.

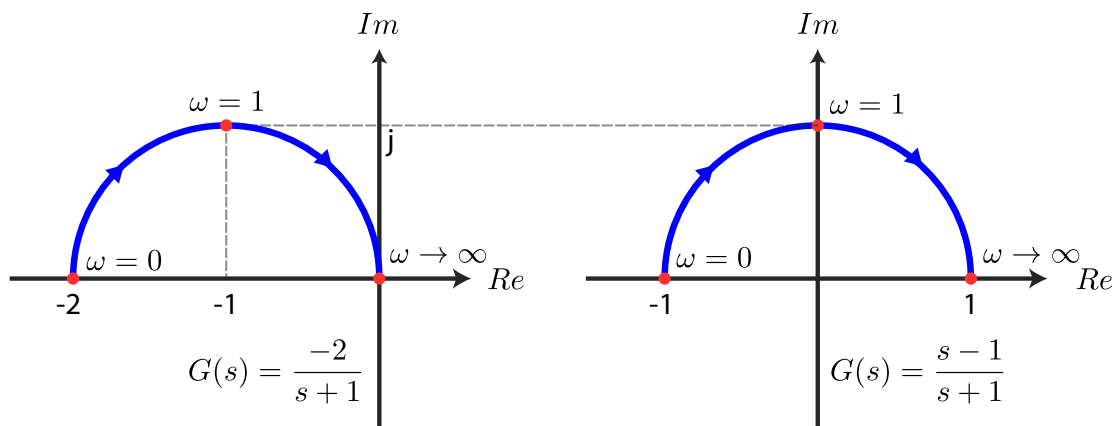


Now let's draw the polar plot of  $G(s) = \frac{s-1}{s+1}$  (note that there is a zero in open-right half plane). Note that

$$G(s) = 1 - \frac{2}{s+1}$$

$$G(j\omega) = 1 - 2 \left( \frac{1}{\omega^2 + 1} - \frac{\omega}{\omega^2 + 1} j \right)$$

Process of polar plot drawing of  $G(s) = \frac{s-1}{s+1}$  is illustrated below



Now let's draw the polar plot of  $G(s) = \frac{1}{(s+1)^2}$

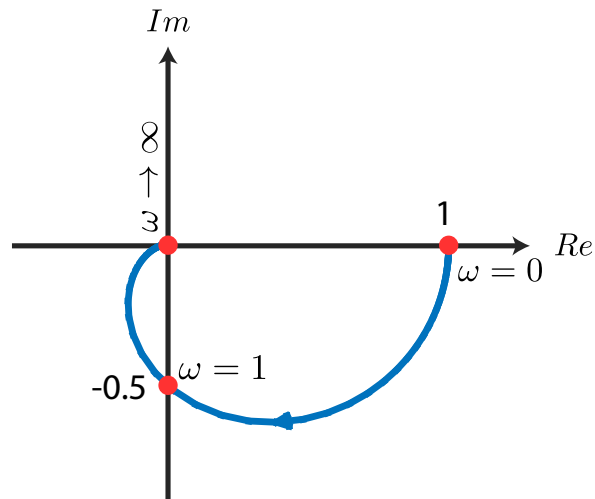
$$G(j\omega) = \frac{1}{(j\omega + 1)^2} = \frac{(-j\omega + 1)^2}{(\omega^2 + 1)^2}$$

$$= [(1 - \omega^2) + j(-2\omega)] \frac{1}{(\omega^2 + 1)^2}$$

Some important points and associated features on the polar plot can be computed as

$$\begin{aligned}\omega \rightarrow 0 &\Rightarrow G(j\omega) = 1 \\ \omega \rightarrow 1 &\Rightarrow G(j\omega) = -0.5j \\ \omega \rightarrow \infty &\Rightarrow |G(j\omega)| \rightarrow 0 \quad \& \quad \angle[G(j\omega)] \rightarrow -\pi\end{aligned}$$

Resultant polar plot is illustrated below



Now let's draw the polar plot of  $G(s) = \frac{1}{(s+1)^3}$

$$\begin{aligned}G(j\omega) &= \frac{1}{(j\omega + 1)^3} = \frac{(-j\omega + 1)^3}{(\omega^2 + 1)^3} \\ &= [(1 - 3\omega^2) + j(\omega^3 - 3\omega)] \frac{1}{(\omega^2 + 1)^3}\end{aligned}$$

Some important points and associated features on the polar plot can be computed as

$$\begin{aligned}\omega \rightarrow 0 &\Rightarrow G(j\omega) = 1 \\ \omega \rightarrow \sqrt{1/3} &\Rightarrow G(j\omega) = -0.65j \\ \omega \rightarrow \sqrt{3} &\Rightarrow G(j\omega) = -1/4 \\ \omega \rightarrow \infty &\Rightarrow |G(j\omega)| \rightarrow 0 \quad \& \quad \angle[G(j\omega)] \rightarrow \pi/2\end{aligned}$$

Resultant polar plot is illustrated below

