EE402 - Discrete Time Systems **Spring 2018**

Lecture 9

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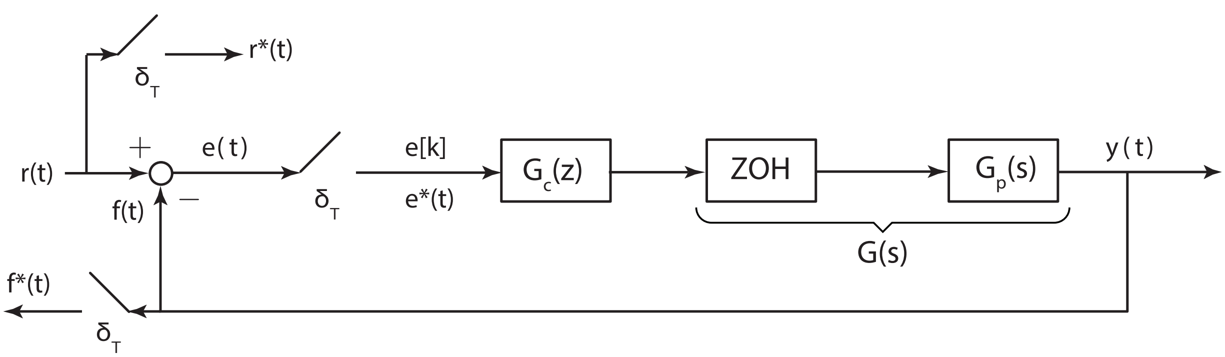
Steady-Sate (DC) Response Analysis

Let's remember the final value theorem. Given a discrete time signal $x[k]$ and its z-transform $X(z)$, if $x[k]$ is convergent sequence final value theorem states that

$$\lim_{k\to\infty} x[k] = \lim_{z\to 1}\left[\left(1-z^{-1}\right)X(z)\right]$$
$$x_{ss} = \lim_{z\to 1}\left[\frac{z-1}{z}X(z)\right]$$



Now let's find the pulse transfer function from the reference signal $r[k]$ to the error signal $e[k]$, to further analyze the steady-state error response.

$$E(z) = R(z) - E(z)\left(G_c(z)G(z)\right), \quad \text{where } G(z) = \mathcal{Z}\{G(s)\}$$
$$\frac{E(z)}{R(z)} = \frac{1}{1+G_c(z)G(z)}$$

Note that $G_c(z)G(z)$ is the pulse transfer function from the error signal $E(z)$ to the signal which is fed to the negative terminal of the main difference operator, i.e. $F(z)$. This transfer function is called feed-forward or open-loop pulse transfer function of the closed-loop digital control system. For this system,

$$\frac{F(z)}{E(z)} = G_{OL} = G_c(z)G(z)$$

Then $E(z)$ can be written as

$$E(z) = R(z)\frac{1}{1+G_{OL}(z)}$$

It is obvious that first requirement on m steady-state error performance is that closed-loop system have to be stable. Now let's analyze specific but fundamental input scenarios.

Unit-Step Input

We know that $r[k] = u[k]$ and $R(z) = \frac{1}{1-z^{-1}}$ then we have

$$\begin{aligned} e_{ss} &= \lim_{z\to 1}\left[\left(1-z^{-1}\right)R(z)\frac{1}{1+G_{OL}(z)}\right] \\ &= \lim_{z\to 1}\left[\left(1-z^{-1}\right)\frac{1}{1-z^{-1}}\frac{1}{1+G_{OL}(z)}\right] \\ e_{ss} &= \frac{1}{1+\lim_{z\to 1} G_{OL}(z)} \end{aligned}$$

If the DC gain of the system (also called static error constant) is constant, i.e. $G_{OL}(1) = K_{DC}$ then the steady state error can be computed as

$$e_{ss} = \frac{1}{1+K_{DC}}$$

It is obvious that

$$\begin{aligned} e_{ss} \neq 0 \quad &\text{if} \quad |K_{DC}| < \infty \\ e_{ss} \to 0 \quad &\text{if} \quad K_{DC} \to \infty \end{aligned}$$

Based on these results, we can have the following conclusions

- If $G_{OL}(1) = 0$, then $e_{ss} = 1$. These are **type *negative*** systems, and the steady-state error of step response type signals are always 100%.
- If $G_{OL}(1) = K_{DC}$, $0 < |K_{DC}| < \infty$, then $e_{ss} = 1/(1+K_{DC})$. These are **type 0** systems. We observe a bounded steady-state error and it is possible to reduce the by increasing the static gain constant K_P.
- If $G_{OL}(1) = \infty$, then $e_{ss} = 0$. These are **type *positive*** systems. The steady-state error is perfectly zero for such systems.

Now let's generalize the *type* of systems. An *N type* closed loop system has the following form of open-loop pulse transfer function

$$\begin{aligned} G_{OL}(z) &= \frac{1}{(z-1)^N}G_{DC}(z) \\ |G_{DC}(1)| &= K_{DC} \quad \text{where } 0 < |K_{DC}| < \infty \end{aligned}$$

It is easy to see that for unit-step response

- Type $N < 0$: $e_{ss} = 1$ (or $e_{ss} = 100\%$)
- Type $N = 0$: $e_{ss} = 1/(1+K_{DC})$
- Type $N > 0$: $e_{ss} = 0$

Unit-Ramp Input

We know that $r[k] = ku[k]$ and $R(z) = \frac{z^{-1}}{(1-z^{-1})^2}$ then we have

$$
\begin{aligned}
e_{ss} &= \lim_{z\to 1}\left[\left(1-z^{-1}\right)R(z)\frac{1}{1+G_{OL}(z)}\right]\\
&= \lim_{z\to 1}\left[\left(1-z^{-1}\right)\frac{z^{-1}}{(1-z^{-1})^2}\frac{1}{1+\frac{1}{(z-1)^N}G_{DC}(z)}\right]\\
&= \lim_{z\to 1}\left[\frac{1}{z-1}\frac{1}{1+\frac{1}{(z-1)^N}G_{DC}(z)}\right]\\
&= \lim_{z\to 1}\left[\frac{1}{(z-1)+\frac{1}{(z-1)^{N-1}}G_{DC}(z)}\right]\\
e_{ss} &= \frac{1}{\lim_{z\to 1}\left[\frac{1}{(z-1)^{N-1}}G_{DC}(z)\right]}
\end{aligned}
$$

Based on this result we can have the following steady-state error conditions for the unit-ramp input based on the type condition of the system

- Type $N<1$: $e_{ss}\to\infty$
- Type $N=1$: $e_{ss}=\frac{1}{K_{DC}}$
- Type $N>1$: $e_{ss}=0$

Unit-Quadratic (Acceleration) Input

We know that $r[k]=\frac{1}{2}k^2u[k]$ and $R(z)=\frac{z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}$ then we have

$$
\begin{aligned}
e_{ss} &= \lim_{z\to 1}\left[\left(1-z^{-1}\right)R(z)\frac{1}{1+G_{OL}(z)}\right]\\
&= \lim_{z\to 1}\left[\left(1-z^{-1}\right)\frac{z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}\frac{1}{1+\frac{1}{(z-1)^N}G_{DC}(z)}\right]\\
&= \lim_{z\to 1}\left[\frac{(z+1)}{2(z-1)^2}\frac{1}{1+\frac{1}{(z-1)^N}G_{DC}(z)}\right]\\
&= \lim_{z\to 1}\left[\frac{(z+1)/2}{(z-1)^2+\frac{1}{(z-1)^{N-2}}G_{DC}(z)}\right]\\
e_{ss} &= \frac{1}{\lim_{z\to 1}\left[\frac{1}{(z-1)^{N-2}}G_{DC}(z)\right]}
\end{aligned}
$$

- Type $N<2$: $e_{ss}\to\infty$
- Type $N=2$: $e_{ss}=\frac{1}{K_{DC}}$
- Type $N>2$: $e_{ss}=0$

Example 1: $G(z)=\frac{z-1}{z-0.5}$ and $G_C(z)=K$. Compute the steady-state erro to unit-step, unit-ramp, a and unit-quadratic inputs.

Solution:

$$G_{OL}(z) = K\frac{z-1}{z-0.5} = \frac{1}{(z-1)^{-1}}\frac{K}{z-0.5}$$
$$G_{DC}(1) = 2K \quad , \text{Type } -1$$

Then the steady-state errors are computed as

- Unit-step: $e_{ss} = 1$
- Unit-ramp: $e_{ss} = \infty$
- Unit-acceleration: $e_{ss} = \infty$

Example 2: $G(z) = \frac{z-1}{z-0.5}$ and $G_C(z) = K\frac{z}{z-1}$. Compute the steady-state error to unit-step, unit-ramp, a and unit-quadratic inputs.

Solution:

$$G_{OL}(z) = \frac{Kz}{z-0.5}$$
$$G_{DC}(1) = 2K \quad , \text{Type } 0$$

Then the steady-state errors are computed as

- Unit-step: $e_{ss} = \frac{1}{1+2K}$
- Unit-ramp: $e_{ss} = \infty$
- Unit-acceleration: $e_{ss} = \infty$

Example 3: $G(z) = \frac{z-1}{z-0.5}$ and $G_C(z) = K\frac{z^2}{(z-1)^2}$. Compute the steady-state erro to unit-step, unit-ramp, a and unit-quadratic inputs.

Solution:

$$G_{OL}(z) = \frac{Kz^2}{(z-1)(z-0.5)} = \frac{1}{z-1}\frac{Kz^2}{z-0.5}$$
$$G_{DC}(1) = 2K \quad , \text{Type } 1$$

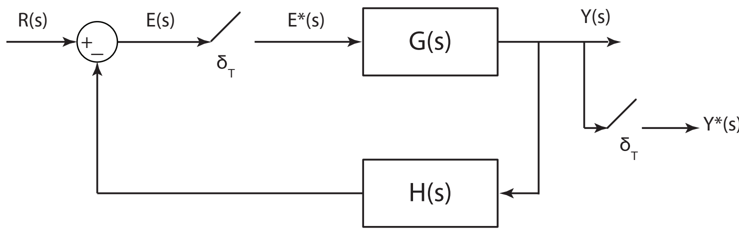
Then the steady-state errors are computed as

- Unit-step: $e_{ss} = 0$
- Unit-ramp: $e_{ss} = \frac{1}{2K}$
- Unit-acceleration: $e_{ss} = \infty$

Open-Loop Transfer Function for Different Topologies

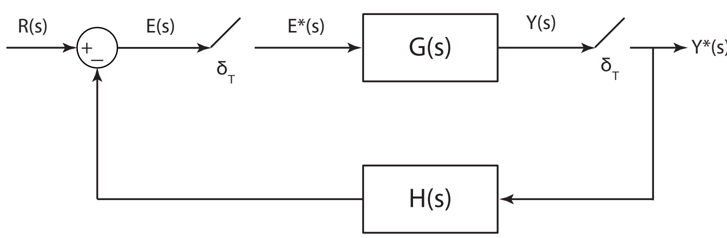
When computing the steady-state error it is important to carefully analyze the topology of the control system.

Compute the $G_{OL}(z)$ for the following system



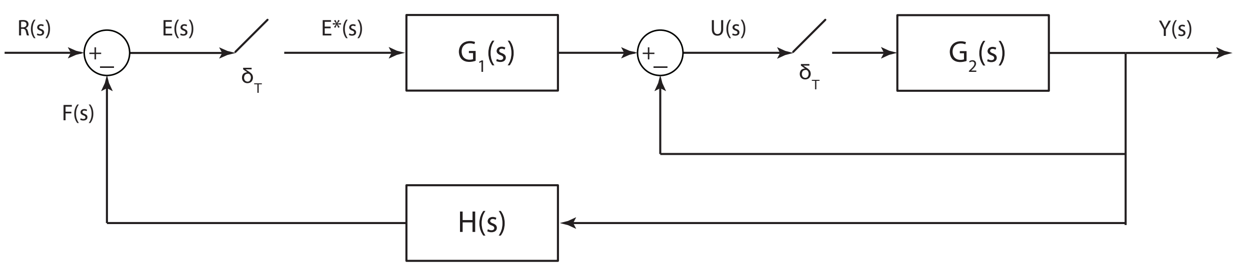
$$\begin{aligned} F(s) &= E^*(s)G(s)H(s) \\ F^*(s) &= E^*(s)[G(s)H(s)]^* = E^*(s)GH^*(s) \\ G_{OL}(z) &= GH(z) \end{aligned}$$

Now let's compte the $G_{OL}(z)$ for the following system



$$\begin{aligned} F(s) &= [E^*(s)G(s)] * H(s) = E^*(s)G^*(s)H(s) \\ F^*(s) &= E^*(s)G^*(s)H^*(s) \\ G_{OL}(z) &= G(z)H(z) \end{aligned}$$

Now let's compte the $G_{OL}(z)$ for the following system



From last week we know that

$$U^*(s) = \frac{G_1^*(s)}{1 + G_2^*(s) + G_1^*(s)GH^*(s)} R^*(s)$$

Then we can

$$U(s) = E^*(s)G_1(s) - U^*(s)G_2(s) \quad \rightarrow \quad U^*(s) = E^*(s)G_1^*(s) - U^*(s)G_2^*(s)$$

$$U^*(s) = \frac{G_1^*(s)}{1 + G_2^*(s)} E^*(s)$$

$$E^*(s) = \frac{1 + G_2^*(s)}{1 + G_2^*(s) + G_1^*(s)GH^*(s)} R^*(s)$$

$$\frac{E(z)}{R(z)} = \frac{1 + G_2(z)}{1 + G_2(z) + G_1(z)GH(z)}$$

This transfer function form does not (directly) fit to the form we analyzed, i.e. $\frac{E(z)}{R(z)} = \frac{1}{1+G_{OL}(z)}$, so we can not directly used the conditions and formulaes for this form. One way of comuting the steady-state errors is directly applying the final-value theorem.

The other way is we can simply convert the computed pulse transer function $E(z)/R(z)$ such that it fits the form $\frac{E(z)}{R(z)} = \frac{1}{1+G_{OL}(z)}$. If we carefully analyze the transer function we can obtain

$$\frac{E(z)}{R(z)} = \frac{1}{1 + \frac{G_1(z)G_2H(z)}{1+G_2(z)}}$$

$$G_{OL}(z) = \frac{G_1(z)G_2H(z)}{1 + G_2(z)}$$

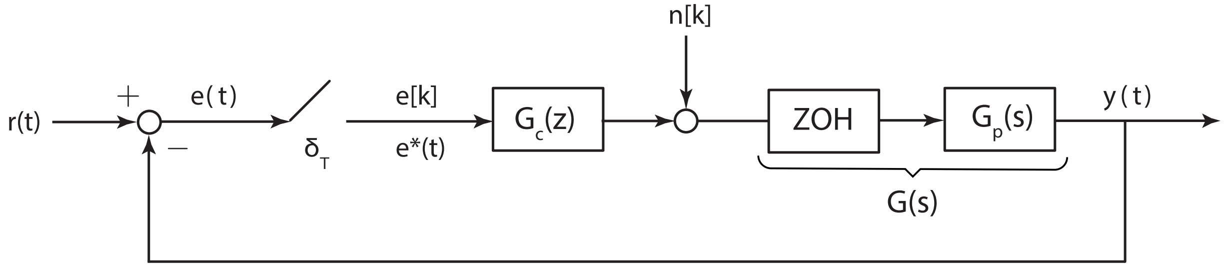
It is also possible to derive $G_{OL}(z)$ via direct computation of $F(z)/E(z)$.

Response to Disturbances

When analyzing response of a system in addition to the desired response to the reference input, it is also important to analyze the response (both steady-state, transient, and frequency) to unwanted disturbances and noises.

Process Disturbance/Uncertainty/Noise

Let's analyze a type of important disturbance on the fundamental discrete-time block diagram topology.



In order to analyze the response to the disturbance $n[k]$, we assume $r[k] = 0$ (which is just fine due to the linearity). Let's first find the pulse transfer function from $N(z)$ to $Y(z)$.

$$\begin{aligned} Y(z) &= (-Y(z)G_c(z) + N(z))G(z) \\ \frac{Y(z)}{N(z)} &= \frac{G(z)}{1 + G_c(z)G(z)} \end{aligned}$$

Technically, we want $\frac{Y(z)}{N(z)} = 0$, while also tracking the reference signal. Since it is not perfectly possible to achieve $\frac{Y(z)}{N(z)} = 0$ while satisfying other constraints, we want $\frac{Y(z)}{N(z)}$ to be "small". If $|G_C(z)G(z)| \gg 1$ then we have

$$\frac{Y(z)}{N(z)} \approx \frac{1}{G_c(z)}$$

Now let's consider a specific type of disturbance. An important class of process disturbance/uncertainty is in the form of DC bias, i.e. $n(t) = Nu(t)$ and $N(z) = \frac{N}{1-z^{-1}}$. Let's analyze DC steady state response using final value theorem.

$$\begin{aligned} y_{ss} &= \lim_{z\to 1} \left[\left(1 - z^{-1}\right) Y(z)\right] = \lim_{z\to 1} \left[\left(1 - z^{-1}\right) N(z) \frac{G(z)}{1 + G_c(z)G(z)}\right] \\ &= \lim_{z\to 1} \left[\left(1 - z^{-1}\right) \frac{N}{1 - z^{-1}} \frac{G(z)}{1 + G_c(z)G(z)}\right] = \lim_{z\to 1} \left[N \frac{G(z)}{1 + G_c(z)G(z)}\right] \\ &= N \frac{\lim_{z\to 1} G(z)}{1 + \lim_{z\to 1} G_C(z)G(z)} \end{aligned}$$

Let's analyze the steady-state disturbance response

- If plant is a type < 0 system (high pass filter plant) then $G(1) = 0$ and $y_{ss} = 0$.
- If plant is a type 0 system, then

  $$y_{ss} = \frac{NG(1)}{1 + G(1)\lim_{z\to 1} G_C(z)}$$

  Now let's analyze the response based on the type of $G_c(z)$

  - Type < 0, then

    $$y_{ss} = NG(1)$$

    In this case, controller has no control on the steady-state disturbance rejection performance.
  - Type 0, then

    $$y_{ss} = \frac{NG(1)}{1 + G(1)G_C(1)}$$

Obviously in order to "filter" the disturbance we should select a $G_C(z)$ such that $|G_C(1)G(1)| \gg 1$ then

$$y_{ss} = \frac{N}{G_C(1)}$$

Large gain $G_C(z)$ can effectively filter the disturbance (but not completely).

– Type > 0, then

$$\begin{aligned} y_{ss} &= \frac{NG(1)}{1 + G(1)\lim_{z\to 1} G_C(z)} \\ &= 0 \end{aligned}$$

Integral action on $G_C(z)$ can perfectly reject the DC disturbance on steady state.

- If plant is a type $m > 0$ then

$$\begin{aligned} y_{ss} &= \frac{N \lim_{z\to 1} G(z)}{1 + \lim_{z\to 1} G(z)G_C(z)} \\ \lim_{z\to 1} G(z) &= \infty \end{aligned}$$

Depending on the type of $G_C(z)$, we can conclude that

  – Type < 0, $y_{ss} = \infty$
  – Type 0, $y_{ss} = C$, where $0 < C < \infty$
  – Type > 0, $y_{ss} = 0$

Example 4: $G(z) = \frac{z-1}{z-0.5}$ and $G_C(z) = K$. Compute the steady-state response to a unit step process disturbance/noise.

Solution:

$$y_{ss} = \frac{\lim_{z\to 1} \frac{z-1}{z-0.5}}{1 + \lim_{z\to 1} K\frac{z-1}{z-0.5}} = 0$$

Plant perfectly rejects disturbance.

Example 5: $G(z) = \frac{z}{z-0.5}$ and $G_C(z) = K$. Compute the steady-state response to a unit step process disturbance/noise.

Solution:

$$y_{ss} = \frac{\lim_{z\to 1} \frac{z}{z-0.5}}{1 + \lim_{z\to 1} K\frac{z}{z-0.5}} = \frac{2}{1+2K}$$

Large gain K can be effective solution to reject disturbance

Example 6: $G(z) = \frac{z}{z-0.5}$ and $G_C(z) = K_P + K_I\frac{z}{z-1}$. Compute the steady-state response to a unit step process disturbance/noise.

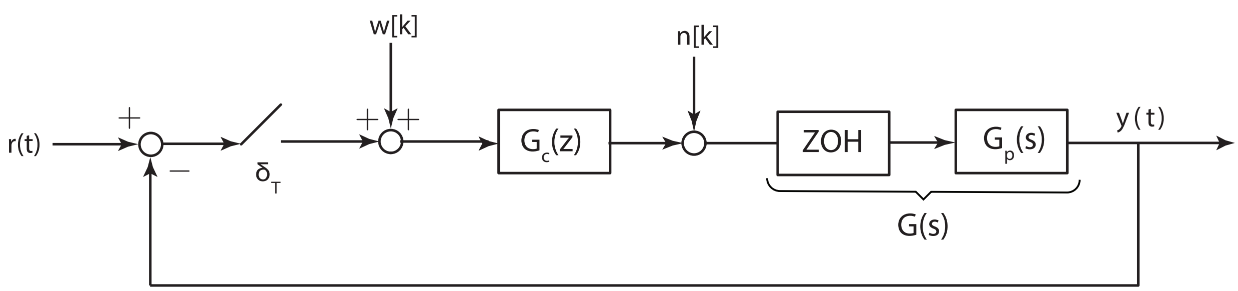
Solution:

$$y_{ss} = \frac{2}{1 + \lim_{z\to 1} 2\left(K_P + K_I \frac{z}{z-1}\right)} = 0$$

A PI controller can perfectly reject the DC process disturbance.

Measurement Disturbance/Uncertainty/Noise

Let's analyze a different type of important disturbance on the fundamental discrete-time block diagram topology.



In order to analyze the response to the disturbance $w[k]$, we assume $r[k] = 0$ and $n[k] = 0$

$$\begin{aligned} Y(z) &= (W(z) - Y(z))G_c(z)G(z) \\ \frac{Y(z)}{W(z)} &= \frac{G(z)G_C(z)}{1 + G_c(z)G(z)} \\ &= \frac{G_{OL}(z)}{1 + G_{OL}(z)} \end{aligned}$$

Technically, we want $\frac{Y(z)}{N(z)} = 0$, while also tracking the reference signal. Thus, practically we should design $G_C(z)$ such that $|G_C(z)G(z)| \ll 1$, to eliminate measurement noises/disturbances. (??????)

$$\frac{Y(z)}{N(z)} \approx G_{OL}(z)$$

This "requirement" obviously contradicts with requirements on stead-state tracking error performance and process noise/disturbance rejection performance. Most well known limitation of feedback control systems.

Now let's consider a specific type of measurement noise, i.e. DC measurement bias. $w(t) = Wu(t)$ and $W(z) = \frac{W}{1-z^{-1}}$. Let's analyze DC steady state response using final value theorem.

$$\begin{aligned} y_{ss} &= \lim_{z\to 1}\left[(1 - z^{-1})R(z)\frac{G_{OL}(z)}{1 + G_{OL}(z)}\right] = \lim_{z\to 1}\left[(1 - z^{-1})\frac{W}{1 - z^{-1}}\frac{G_{OL}(z)}{1 + G_{OL}(z)}\right] \\ &= \lim_{z\to 1}\left[\frac{WG_{OL}(z)}{1 + G_{OL}(z)}\right] \end{aligned}$$

Using the form $G_{OL}(z) = \frac{1}{(z-1)^N} G_{DC}(z)$, y_{ss} takes the form

$$y_{ss} = \lim_{z \to 1} \left[\frac{W G_{DC}(1) \frac{1}{(z-1)^N}}{1 + G_{DC}(1) \frac{1}{(z-1)^N}} \right]$$

Based on the type of the open-loop transfer function, $G_{OL}(z)$, we can conclude

- Type $N < 0$: $y_{ss} = 0$. Perfect rejection of measurement bias, but we know that this is unacceptable from reference tracking point of view.
- Type $N = 0$

  $$y_{ss} = \frac{W G_{DC}(1)}{1 + G_{DC}(1)}$$

  It seems that in order to "filter" the measurement bias $G_{DC}(1)$ should be selected very small.
- Type $N > 0$

  $$y_{ss} = W$$

  The disturbance is directly transfered to the output.

Example 7: $G(z) = \frac{z-1}{z-0.5}$ and $G_C(z) = K$. Compute the steady-state response to a unit step measurement disturbance/noise.

Solution:

$$G_{OL}(z) = K \frac{z-1}{z-0.5} \quad \text{Type } -1$$
$$y_{ss} = 0$$

Plant perfectly rejects measurement bias.

Example 8: $G(z) = \frac{z}{z-0.5}$ and $G_C(z) = K$. Compute the steady-state response to a unit step process disturbance/noise.

Solution:

$$G_{OL}(z) = K \frac{z}{z-0.5} \quad \text{Type } 0$$
$$y_{ss} = \frac{2K}{1+2K}$$

Small gain K can be effective solution to reject measurement bias.