ROB501 - Fundamentals & Emerging Topics in Robotics - Digital Control Systems

Lecture 3

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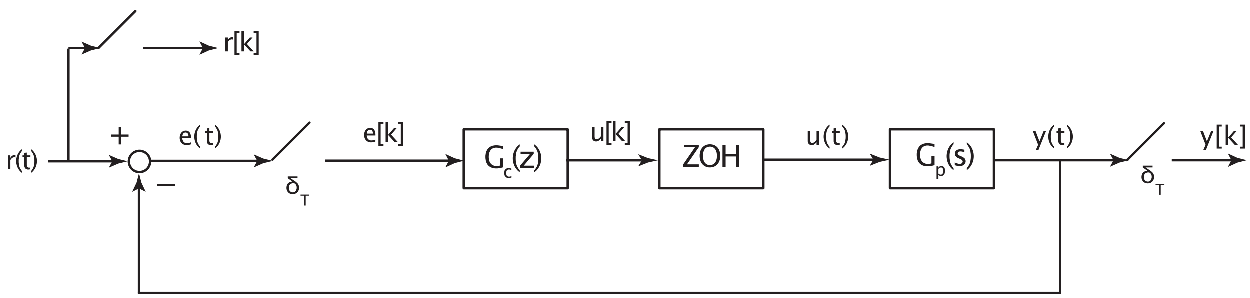
3.1 Closed Loop Digital Control Systems

Consider the fundamental DT-control system below



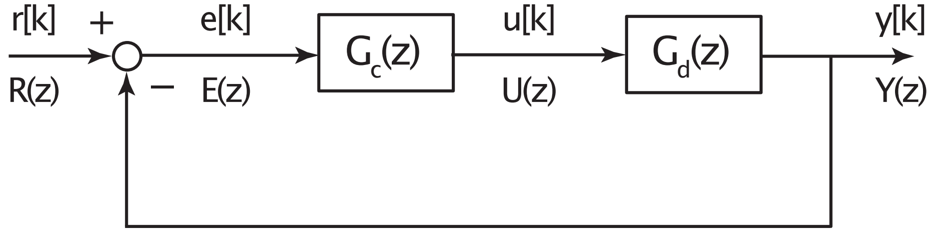
Previously, we learned how to derive the transfer function of a fundamental open-loop digital control system composed of a ZOH operator, CT plant, and uniform sampler in cascaded form. We can use this derivation to find the transfer function from $u[k]$ to $y[k]$ such that

$$\frac{Y(z)}{U(z)} = G_d(z) = \mathcal{Z}\left\{\frac{G_p(s)}{s}\right\}$$

Now let's derive the closed-loop transfer function from $r[k]$ to $y[k]$

$$\begin{aligned}
E(z) &= R(z) - Y(z) \quad , \quad Y(z) = G_c(z)G_d(z)E(z) \\
E(z) &= R(z) - G_c(z)G_d(z)E(z) \\
E(z) &= \frac{R(z)}{1 + G_c(z)G_d(z)} \\
\frac{Y(z)}{R(z)} &= \frac{G_c(z)G_d(z)}{1 + G_c(z)G_d(z)}
\end{aligned}$$

Note if we only care the signal flow in the sampled instants we can re-draw the block diagram such that all time domain signals are in DT and all systems are represented in Z-domain. The fundamental block diagram can be re-drawn as



Example: Let $G_p(s) = \frac{1}{s+1}$, $T = 1$, $G_c(z) = K$ (Discrete P Controller) . First find PTF (in z-domain).

Solution: First let's find $G_d(z)$

$$G_d(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G_p(s)}{s}\right\} = \frac{1 - e^{-1}}{z - e^{-1}}$$

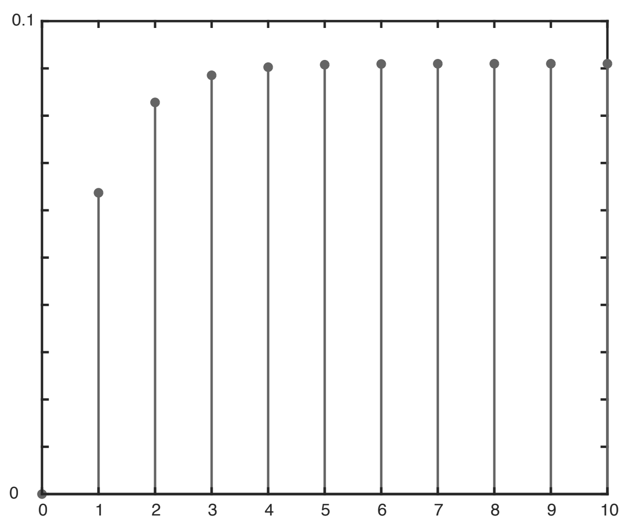
which we already knew from the Lecture Notes 4. Now let's compute the closed-loop PTF, $T(z)$.

$$T(z) = \frac{G_c(z)G_d(z)}{1 + G_c(z)G_d(z)} = \frac{K(1 - e^{-1})}{z + K - (K+1)e^{-1}}$$

Let $\mathbf{K} = \mathbf{0.1}$, then compute the step-response of the closed-loop PTF

$$Y(z) = R(z)T(z) = \frac{0.063\, z}{(z-1)(z-0.3)}$$
$$y[k] = \mathcal{Z}^{-1}[Y(z)] = \left[0.09 - 0.09(0.3)^k\right] u[k]$$

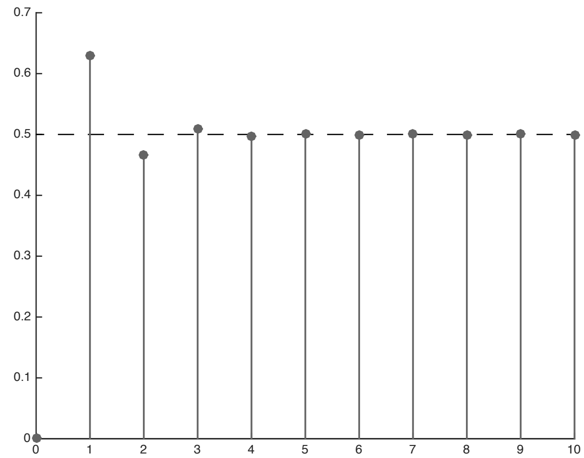
If we plot the step response we obtain the following plot



Now, Let $\mathbf{K} = \mathbf{1}$, then compute the step-response of the closed-loop PTF

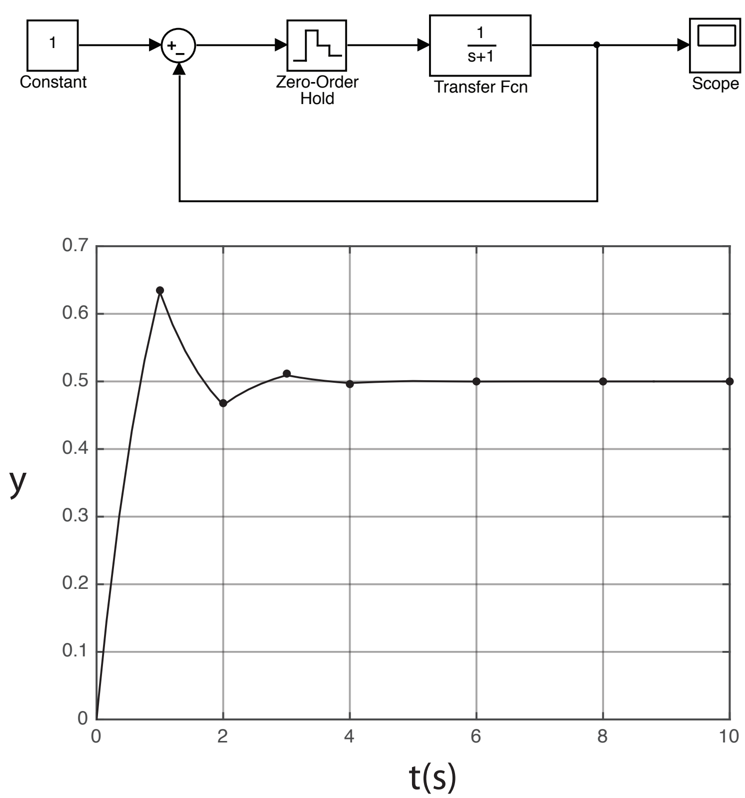
$$Y(z) = R(z)T(z) = \frac{z}{z-1}\,\frac{0.63}{z+0.26} = \frac{0.63\, z}{(z-1)(z+0.26)}$$
$$y[k] = \mathcal{Z}^{-1}[Y(z)] = \left[0.5 - 0.5(-0.26)^k\right] u[k]$$

If we plot the step response we obtain the following plot



What about inter-sample behavior?

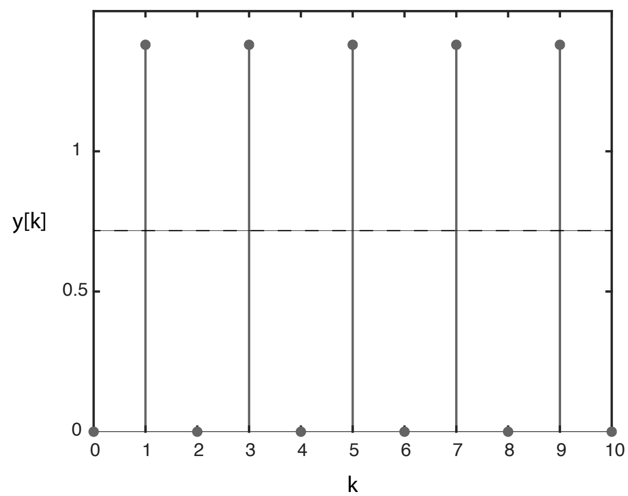
We can simulate the system and analyze the behavior. The figure below shows the Simulink model as well as the output.



Now let $K = \frac{1+e^{-1}}{1-e^{-1}}$, then $Y(z)$ and $y[k]$ takes the form

$$Y(z) = \frac{1.37z}{(z-1)(z+1)}$$
$$y[k] = 0.69 - 0.69(-1)^n$$

The graph of $y[k]$ is illustrated below. It can be seen that the output shows an oscillatory behavior.



Now let $K = \frac{e^{-1}}{1-e^{-1}}$, then $Y(z)$ and $y[k]$ takes the form

$$Y(z) = \frac{0.37}{z-1}$$
$$y[k] = 0.37u[k-1]$$

Output converges to its steady state value in "finite time" (dead-beat behavior/controller).

Take home message: It can be seen that even if the plant is a simple first order transfer function, depending on the value of K, we can observe very interesting behavior in the closed-loop DT system.

3.2 Digital PID Controller

In this section, we will try to obtain a form for the digital PID controller. The continuous transfer function of a PID is given as

$$G_{PID}(s) = K_P + K_D s + \frac{K_I}{s}$$

One idea is to start from continuous PID form and then "discretize" it. One way of deriving a discrete controller, $G_c(z)$, is sampling the impulse response of the CT PID controller (or any controller) to obtain the impulse response of the DT PID controller. Based in this approach if possible $G_c(z)$ simply commuted as

$$G_c(z) = \mathcal{Z}\{G_c(s)\}$$

Let's start with PI controller.

Digitization of PI Controller: It is a well known fact that the PI Controller is in the form

$$G_{PI}(s) = K_P + \frac{K_I}{s}$$

The discretization simply gives

$$\begin{aligned} G_{PI}(z) &= \mathcal{Z}\{G_{PI}(s)\} \\ &= K_P + K_I \frac{1}{1 - z^{-1}} \\ &= \frac{(K_I + K_P) - K_P z^{-1}}{1 - z^{-1}} \\ &= \frac{b_0 + b_1 z^{-1}}{1 - z^{-1}} \end{aligned}$$

Now let's discretize PID controller which has the following CT transfer function

$$G_{PID}(s) = K_P + K_D s + \frac{K_I}{s}$$

The problem is $K_D s$ term is non-causal. Let us approximate the effect of $K_D s$ in time domain and then perform a discretization. A causal approximate derivative can be find by computing the backward difference.

$$\frac{dx(t)}{dt} \approx \frac{x(t) - x(t - \Delta t)}{\Delta t}$$

Now let's compute the approximate derivative term at the sampling instants and let $\Delta t = T$ we have

$$\frac{dx(t)}{dt}|_{t=kT} \approx \frac{x(kT) - x((k-1)T)}{T}$$

If we take the z-transform we can simply obtain a transfer function for this FIR filter as

$$G_D(z) = \frac{K_D}{T}(1 - z^{-1})$$

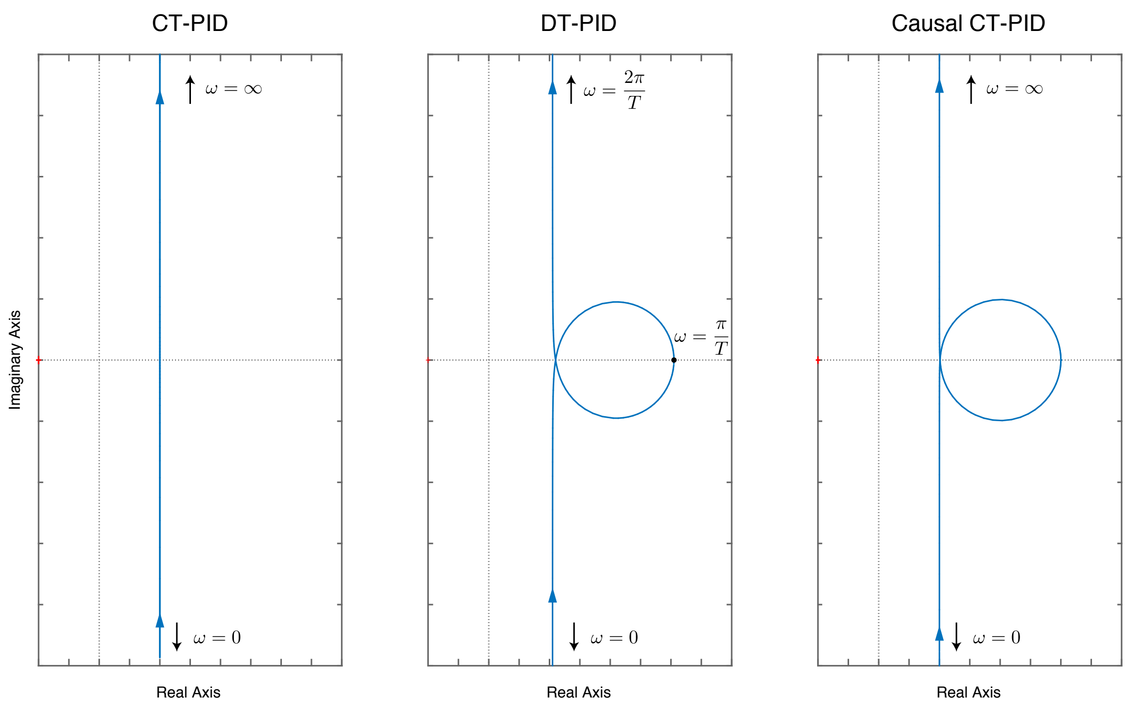
Note that instead of K_D/T, we can just use K_D for the gain. If we combine PI and D terms we obtain the following pulse transfer function for the digital PID controller.

$$\begin{aligned} G_{PID}(z) &= K_P + K_I \frac{1}{1 - z^{-1}} + K_D \left(1 - z^{-1}\right) \\ &= \frac{K_P + K_D + K_I - (K_P + 2K_D)z^{-1} + K_D z^{-2}}{1 - z^{-1}} \\ &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - z^{-1}} \end{aligned}$$

The Figure below illustrates the frequency response characteristics of an ideal CT-PID, a DT-PID, as well as an approximate causal CT-PID controllers. The transfer function of a causal CT-PID has the form below

$$\begin{aligned} G_{PID}(s) &\approx K_P + K_D \frac{s}{\gamma s + 1} + \frac{K_I}{s} \quad \text{where,} \\ &\gamma > 0 \ \& \ \gamma \approx 0 \end{aligned}$$

Qualitatively, at low and "high" frequencies all controllers behaves similar. However, for some intermediate range of frequencies there are significant differences between the CT-PID and DT-PID (as well as approximate causal CT-PID). Remarkably, for this frequency range DT-PID and approximate causal CT-PID controllers' the frequency response polar polts are qualitatively similar. This shows that if we choose the right parameters, digitization of derivative term has a similar effect as implementing an approximate analog derivative circuit.



Note that frequency response function for CT and DT systems are found by the Fourier (CT or DT) transforms of the impulse response functions, or simply they can be computed from the s-domain or z-domain transfer functions

$$\begin{aligned} \text{CT}: \quad & G_c(s)|_{s=j\omega} = G_c(j\omega) \\ \text{DT}: \quad & G_d(z)|_{z=e^{j\omega}} = G_d(e^{j\omega}) \end{aligned}$$

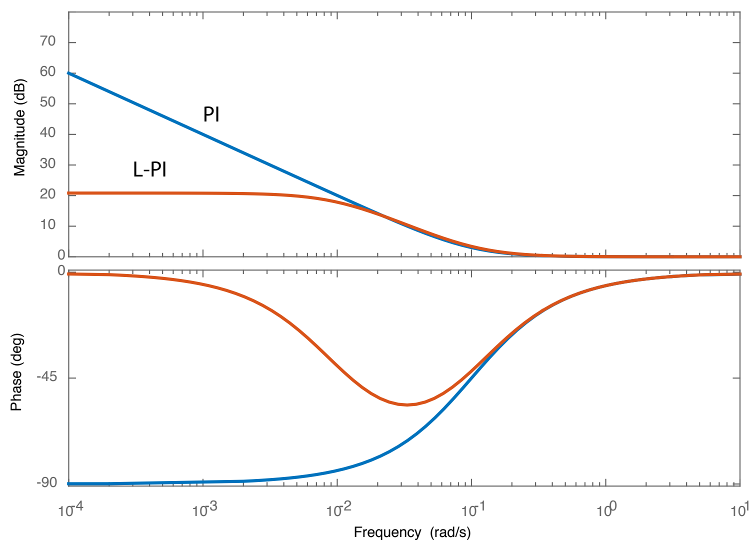
Note that in DT case ω stands for DT frequency. Sometimes $G_d(j\omega)$ or $G_d(\omega)$ used instead of $G_d(e^{j\omega})$. We will cover the Frequency respons later in the class.

PI & PID Controllers with Leaky Integrator

Some times for some practical and other considerations instead of a true integrator (or accumulator) a leaky version is used. A leaky PI controller is in the form

$$\begin{aligned} G_{L-PI}(s) &= K_P + K_I \frac{1}{s+\alpha} \\ &= K_P \frac{s + (\alpha + K_I/K_P)}{s+\alpha} \end{aligned}$$

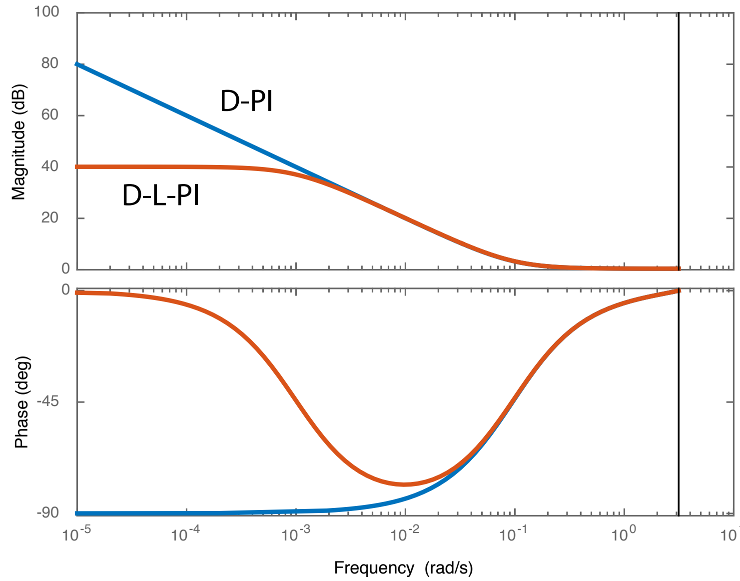
where $\alpha > 0$ and $\alpha \approx 0$ (considering the bandwith of the closed loop system). It can be seen that a leaky-PI controller has the same form with the compensator controller that we covered in 302. If we obscure the frequency response characteristics of both classical and leaky PI controllers, we observe that the behavior is quite different at low frequencies but similar at high frequencies.



If we discretize this CT controller using the emulation operation approach, we obtain

$$\begin{aligned} G_{L-PI}(z) = \mathcal{Z}\{G_{L-PI}(s)\} &= K_P + \frac{K_I}{1 - e^{-\alpha T} z^{-1}} \\ &= K_P + \frac{K_I}{1 - \beta z^{-1}} \\ &= \frac{(K_P + K_I) - K_P \beta z^{-1}}{1 - \beta z^{-1}} \\ &= \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} \end{aligned}$$

where similar to the CT case, $\beta < 1$ and $\beta \approx 1$. Similar to the CT case, this DT transfer function has one zero and one pole and it has the equivalent form with a DT-compensator controller. The bode plots of DT-PI and DT-Leaky-PI controllers are illustrated in the Figure below. It can be seen that at low frequencies the differences are significant, but at high frequencies the responses between classical and leaky PI controllers are very similar.



One interesting result is that both for classical and leaky cases, CT and DT frequency responses are qualitatively very similar.