EE502 - Linear Systems Theory II **Spring 2019**

Lecture 3

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3.1 State-Space Representation to Frequency Domain

In this lecture we will cover the conversion from state-space representations to frequency domain representations ($s-$domain for CT systems and $z-$domain for DT systems) and analyze the connections between two representations.

3.1.1 CT State-Space to $s-$domain

Note that a SS representation of an n^{th} order CTI-LTI system has the from below.

$$\begin{aligned}
&\text{Let } x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R}^q \ , \ u(t) \in \mathbb{R}^p, \\
&\quad \dot{x}(t) = Ax(t) + Bu(t), \\
&\quad y(t) = Cx(t) + Du(t), \\
&\text{where } A \in \mathbb{R}^{n \times n} \ , \ B \in \mathbb{R}^{n \times p} \ , \ C \in \mathbb{R}^{q \times n} \ , \ D \in \mathbb{R}^{q}
\end{aligned}$$

In order to convert state-space to frequency domain, we start with taking the Laplace transform of the both sides of the state-equation

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
sX(s) - x_0 &= AX(s) + BU(s) \\
sX(s) - AX(s) &= x_0 + BU(s) \\
(sI - A)\, X(s) &= x_0 + BU(s) \\
X(s) &= (sI - A)^{-1} x_0 + (sI - A)^{-1} BU(s)
\end{aligned}$$

Now let's concentrate on the output equation

$$\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
Y(s) &= C\,(sI - A)^{-1} x_0 + \left[C\,(sI - A)^{-1} B + D\right] U(s)
\end{aligned}$$

where $C\,(sI - A)^{-1} x_0$ corresponds to the initial-condition response and when $u(t) = 0$ we have

$$\begin{aligned}
Y(s) &= \left[C\,(sI - A)^{-1} B + D\right] U(s) \\
G(s) &= C\,(sI - A)^{-1} B + D
\end{aligned}$$

where $G(s)$ is called the **transfer function matrix** which has the following form for a general $p-$input–$q-$output MIMO system

$$G(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1p}(s) \\ \vdots & & \vdots \\ G_{q1}(s) & \cdots & G_{qp}(s) \end{bmatrix}$$

Definiton: $G_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)}$ is classified as follows

- $G_{ij}(s)$ is *proper* $\Leftrightarrow \deg(n_{ij}(s)) \leq \deg(d_{ij}(s)) \Leftrightarrow G_{ij}(\infty) = \lim_{s\to\infty} G_{ij}(s) = C$ where $|C| < \infty$
- $G_{ij}(s)$ is *strictly proper* $\Leftrightarrow \deg(n_{ij}(s)) < \deg(d_{ij}(s)) \Leftrightarrow G_{ij}(\infty) = \lim_{s\to\infty} G_{ij}(s) = 0$
- $G_{ij}(s)$ is *bi-proper* $\Leftrightarrow \deg(n_{ij}(s)) = \deg(d_{ij}(s)) \Leftrightarrow G_{ij}(\infty) = C$ where $|C| < \infty$ & $C \neq 0$
- $G_{ij}(s)$ is *improper* $\Leftrightarrow \deg(n_{ij}(s)) > \deg(d_{ij}(s)) \Leftrightarrow |G_{ij}(\infty)| \to \infty$

Remark: $G_{ij}(s)$ is strictly propper $\forall(i,j)$ iff $D = \mathbf{0}$

$$
\begin{aligned}
&G(s) = C\,(sI - A)^{-1}\,B = \frac{C\text{Adj}\,(sI-A)\,B}{\text{Det}\,(sI-A)}\ \text{, where}\\
&\det\left(\text{Det}\,(sI-A)\right) = n\\
&\text{Adj}\,(sI-A) = \left[\text{Cofactor}\,(sI-A)\right]^T
\end{aligned}
$$

Let Cofactor $(sI - A) = Co$ then $Co_{ij} = (-1)^{i+j}\text{Det}\,(M_{ij})$, where $\text{Det}\,(M_{ij})$ is called the minor of $(sI-A)_{ij}$ and is the the determinant of the submatrix formed by deleting the ith row and jth column. Note that $\deg(Co_{ij}) \leq (n-1)\ \forall(i,j)$ which imples that $G_{ij}(s)$ is strictly propper.

Definition:

- A scalar $\lambda \in \mathbb{C}$ is called a pole of $G_{ij}(s)$ if $|G_{ij}(\lambda)| \to \infty$
- A scalar $\gamma \in \mathbb{C}$ is called a zero of $G_{ij}(s)$ if $|G_{ij}(\gamma)| = 0$

Definition: Two polynomials are said to be coprime of they have no common root.

Remark:

- $\lambda \in \mathbb{C}$ is a pole of $G_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)}$ if $d_{ij}(s)$ and $n_{ij}(s)$ are coprime and $d_{ij}(\lambda) = 0$
- $\lambda \in \mathbb{C}$ is a zero of $G_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)}$ if $d_{ij}(s)$ and $n_{ij}(s)$ are coprime and $n_{ij}(\lambda) = 0$

3.1.2 DT State-Space to $z-$domain

Note that a SS representation of an n^{th} order DTI-LTI system has the from below.

$$
\begin{aligned}
&\text{Let } x[k] \in \mathbb{R}^n\ ,\ y[k] \in \mathbb{R}^q\ ,\ u[k] \in \mathbb{R}^p,\\
&x[k+1] = Ax[k] + Bu[k],\\
&\quad y[k] = Cx[k] + Du[k],\\
&\text{where } A \in \mathbb{R}^{n\times n}\ ,\ B \in \mathbb{R}^{n\times p}\ ,\ C \in \mathbb{R}^{q\times n}\ ,\ D \in \mathbb{R}^{q}
\end{aligned}
$$

In order to convert state-space to frequency domain, we start with taking the $Z-$transform of the both sides of the state-equation, where Z-transform of a unilateral (causal) discrete time signal $w[k]$ is given by

$$
W(z) = \mathcal{Z}\{w[k]\} = \sum_{k=0}^{\infty} w[k]z^{-k}
$$

$$
\begin{aligned}
x[k+1] &= Ax[k] + Bu[k] \\
zX(z) - zx[0] &= AX(z) + BU(z) \\
zX(z) - AX(z) &= zx[0] + BU(z) \\
(zI - A)\, X(Z) &= zx[0] + BU(z) \\
X(z) &= z\,(zI - A)^{-1}\, x[0] + (zI - A)^{-1}\, BU(z)
\end{aligned}
$$

I recommend to those of you not familiar with Z-transform operation on difference equations to read *shifting theorem* in EE402 Lecture Notes (Lecture # 2), indeed going over the whole Lecture would be very helpful.

Now let's concentrate on the output equation

$$
\begin{aligned}
y[k] &= Cx[k] + Du[k] \\
Y(z) &= zC\,(zI - A)^{-1}\, x[0] + \left[C\,(zI - A)^{-1}\, B + D\right] U(z)
\end{aligned}
$$

where $zC\,(sI - A)^{-1}\, x[0]$ corresponds to the initial-condition response and when $u[k] = 0$ we have

$$
\begin{aligned}
Y(z) &= \left[C\,(zI - A)^{-1}\, B + D\right] U(z) \\
G(z) &= C\,(zI - A)^{-1}\, B + D
\end{aligned}
$$

similar to the CT case $G(z)$ is called the **transfer function matrix**. Note that resultant frequency domain solution in DT systems is very simlar to the solution in CT systems (except the initial condition response). Without a big surprize state-space to transfer function related definitions, remarks, proofs, correlations etc. in CT systems are generally holds for DT systems, Such as *properness, poles, zeros* etc.