EE502 - Linear Systems Theory **Spring 2013**

Lecture 1

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1.1 Continuous- & Discrete-Time Signals

A continuous time bilateral signal is a mapping defined by $f : \mathbb{R} \mapsto \mathbb{R}^n$ (or for unilateral case $f : \mathbb{R}^+ \mapsto \mathbb{R}^n$). Examples

$$f(t) = \sin(t) \ , \ f(t) = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix} \ , \ f(t) = u(t) \ , \ f(t) = \delta(t) \ , \ \text{where } t \in \mathbb{R}$$

A discrete time bilateral signal is a mapping defined by $g : \mathbb{Z} \mapsto \mathbb{R}$ (or for unilateral case $g : \mathbb{Z}^+ \mapsto \mathbb{R}$). Examples

$$f[n] = \sin[n] \ , \ f[n] = \begin{bmatrix} 5^n \\ 5^{-n} \end{bmatrix} \ , \ f[n] = u[n] \ , \ f[n] = \delta[n] \ , \ \text{where } n \in \mathbb{Z}$$

Graphical Examples

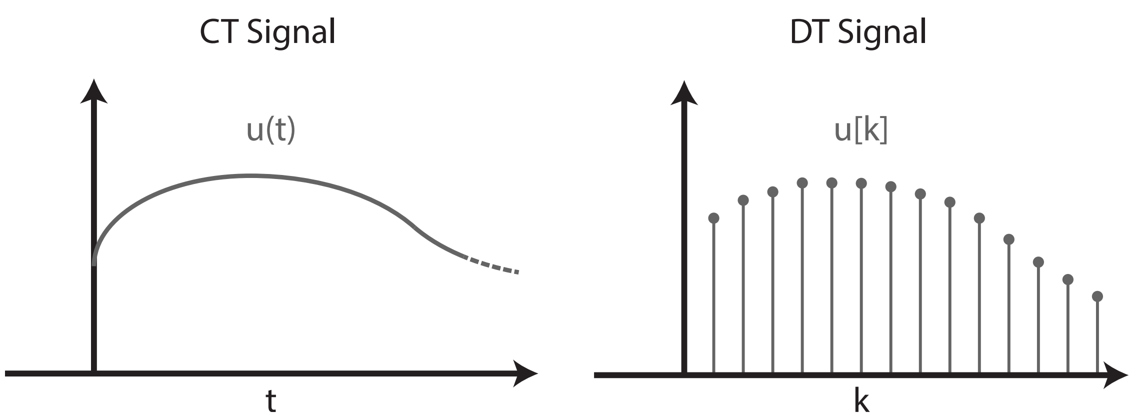


Figure 1.1: CT vs DT Signal

1.2 Continuous- & Discrete-Time Dynamical Systems

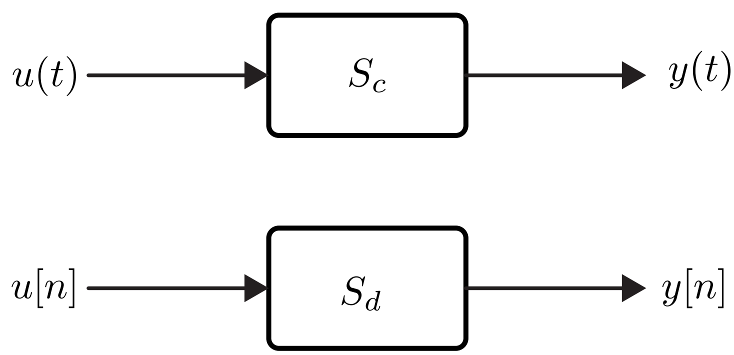
The system is modeled as a *mapping* from a set of input signals, $u(t)$ or $u[n]$, to a set of output signals, $y(t)$ or $y[n]$. We may represent continuous time and discrete time maps as

$$\begin{aligned} \text{Continuous} - \text{Time}: \quad & y(t) = (S_c\ u)(t) \\ \text{Discrete} - \text{Time}: \quad & y[n] = (S_d\ u)[n] \end{aligned}$$

The system, S, operates on the entire input signal $u(.)$ (or $u[.]$) and yields the entire output signal, $y(.)$ (or $y[.]$). A system can also be defined as a collection of constraints defined on the designated signals. For example we can define the system representations above as constraints in the designated signal spaces as

$$\begin{aligned} \text{Continuous} - \text{Time}: \quad & y(t) - (S_c\ u)(t) = 0 \\ \text{Discrete} - \text{Time}: \quad & y[n] - (S_d\ u)[n] = 0 \end{aligned}$$

Operation is performed on the entire input signal, $u(\cdot)$ or $u[\cdot]$ where the mappings S_c and S_d yield the signals $y(\cdot)$ and $y[\cdot]$.



1.2.1 Properties of Input–Output Systems

- **Linearity**

  A continuous time system is **linear** if and only if

$$\begin{gathered} (S_c\ (\alpha u_1 + \beta u_2))(t) = \alpha(S_c\ u_1)(t) + \beta(S_c\ u_2)(t) \\ \forall \alpha,\ \beta,\ u_1(\cdot),\ \&\ u_2(\cdot) \end{gathered}$$

  A discrete time system is **linear** if and only if

$$(S_d\ (\alpha u_1 + \beta u_2))[n] = \alpha(S_d\ u_1)[n] + \beta(S_d\ u_2)[n]$$
$$\forall \alpha,\ \beta,\ u_1[\cdot],\ \&\ u_2[\cdot]$$

- **Time Invariance**

  Let σ_T be the time-shift operator as

  $$(\sigma_T\ u)(t) = u(t-T)$$

  Then a continuous time system is time-invariant if and only if

  $$(S_c\ \sigma_T\ u)(t) = (\sigma_T\ y)(t) = y(t-T)\ \forall T \in \mathbb{R},\ \text{where } (S_c\ u)(t) = y(t)\ \forall T \in \mathbb{R}$$

  Similarly for discrete time systems

  $$(S_d\ \sigma_k\ u)[n] = (\sigma_k\ y)[n] = y[n-k]\ \forall k \in \mathbb{Z},\ \text{where } (S_d\ u))[n] = y[n]$$

- **Memoryless Systems:**

  A continuous time system is memoryless if and only if $y(\bar{t})$ only depends on $u(\bar{t})$, $\forall\ \bar{t} \in \mathbb{R}$.

  $$y(\bar{t}) = (S_c\ u)(\bar{t}) = f(u(\bar{t}))\ \forall \bar{t} \in \mathbb{R},$$

  A discrete time system is memoryless if and only if $y[\bar{n}]$ only depends on $u[\bar{n}]$, $\forall\ \bar{n} \in \mathbb{Z}$.

  $$y[\bar{n}] = (S_d\ u)[\bar{n}] = f(u[\bar{n}])\ \forall \bar{n} \in \mathbb{Z},$$

- **Causality:**

  We say the system is causal if the output does not depend on future values of the input. Mathematically we can show causality using the *truncation* operator, P_T. For continuous systems *truncation* is defined as

  $$(P_T\ u)(t) = \begin{cases} u(t) & \text{for } t \leq T \\ 0 & \text{otherwise} \end{cases}$$

  for discrete systems

  $$(P_k\ u)[n] = \begin{cases} u[n] & \text{for } n \leq k \\ 0 & \text{otherwise} \end{cases}$$

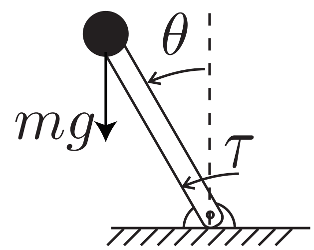
  then the system, S (continuous or discrete), is said to be causal if $P_T S = P_T S P_T\ , \forall T$

- **Finite & Infinite Dimensional Systems:**

  A continuous time dynamical system, S_c, is finite dimensional if there exist an ODE in u, y that models S_c.

  A discrete time dynamical system, S_d, is finite dimensional if there exist an Ordinary Difference Equation in u, y that models S_d.

1.2.2 Examples



1. $u(t) = \tau(t),\ y(t) = \theta(t)$

   Non-linear, Time-invariant, causal, system has memory, and finite dimensional

2. Now let's assume that $g = 0$, what happens?

   Linear, Time-invariant, causal, system has memory, and finite dimensional

3. $y(t) = \int_0^t (t-s)^3 u(s) ds$

   Linear, causal, system has memory, finite dimensional,

   Since the system is causal it may be OK to assume that the set of inputs are limited to causal signals and the convolution is unilateral. In this case the system is *time-invariant.*

   However if non-causal signals are allowed then the system becomes *time-varying.*

1.3 Representations of Dynamical Systems

1.3.1 Differential & Difference Equations

- **Continuous Time Systems - ODEs**

  Linear Time Invariant System (LTI)

  $$a_n y^{(n)} + ... + a_1 y' + a_0 y = b_n u^{(n)} + ... + b_1 u' + b_0 u$$

  Linear Time Varying System (LTV)

  $$a_n(t) y^{(n)} + ... + a_1(t) y' + a_0(t) y = b_n(t) u^{(n)} + ... + b_1(t) u' + b_0(t) u$$

  Non-linear Time Invariant System

  $$y^{(n)} = f(y^{(n-1)}, ..., y', y, u^{(n)}, ..., u', u)$$

  Non-linear Time Varying System

  $$y^{(n)} = f(y^{(n-1)}, ..., y', y, u^{(n)}, ..., u', u, t)$$

- **Discrete Time Systems - Difference Equations**

  Discrete-Time Linear Time Invariant System (LTI)

  $$a_n y[k] + a_{n-1} y[k-1] + ... + a_0 y[k-n] = b_n u[k] + ... + b_0 u[k-n]$$

  Discrete-Time Linear Time Varying System (LTV)

  $$a_n[k] y[k] + a_{n-1}[k] y[k-1] + ... + a_0[k] y[k-n] = b_n[k] u[k] + ... + b_0[k] u[k-n]$$

  Non-linear Time Invariant System

  $$y[k] = f(y[k-1], ..., y[k-n], u[k], ..., u[k-n])$$

  Non-linear Time Varying System

  $$y[k] = f(y[k-1], ..., y[k-n], u[k], ..., u[k-n], k)$$

Discussion

- When an ODE representation becomes *memoryless*?
- When a difference equation representation becomes *memoryless*?
- What about infinite dimensional systems?
- What about *causality*?

1.3.2 State-Space Representation of Dynamical Systems

- **Continuous-Time Dynamical Systems**

  Linear Time Invariant Systems

$$\begin{aligned} &\text{Let } x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R}^q \ , \ u(t) \in \mathbb{R}^p, \\ &\dot{x}(t) = Ax(t) + Bu(t), \\ &y(t) = Cx(t) + Du(t), \\ &\text{where } A \in \mathbb{R}^{n \times n} \ , \ B \in \mathbb{R}^{n \times p} \ , \ C \in \mathbb{R}^{q \times n} \ , \ D \in \mathbb{R}^q \end{aligned}$$

  Linear Time Varying Systems

$$\begin{aligned} &\text{Let } x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R}^q \ , \ u(t) \in \mathbb{R}^p, \\ &\dot{x}(t) = A(t)x(t) + B(t)u(t), \\ &y(t) = C(t)x(t) + D(t)u(t), \\ &\text{where } A(t) \in \mathbb{R}^{n \times n} \ , \ B(t) \in \mathbb{R}^{n \times p} \ , \ C(t) \in \mathbb{R}^{q \times n} \ , \ D(t) \in \mathbb{R}^q \end{aligned}$$

  Non-Linear Time Invariant Systems

$$\begin{aligned} &\text{Let } x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R}^q \ , \ u(t) \in \mathbb{R}^p, \\ &\dot{x}(t) = F(x(t), u(t)), \\ &y(t) = H(x(t), u(t)), \end{aligned}$$

  Non-Linear Time Varying Systems

$$\begin{aligned} &\text{Let } x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R}^q \ , \ u(t) \in \mathbb{R}^p, \\ &\dot{x}(t) = F(x(t), u(t), t), \\ &y(t) = H(x(t), u(t), t), \end{aligned}$$

- **Discrete-Time Dynamical Systems**

  Linear Time Invariant Systems

$$\begin{aligned} &\text{Let } x[n] \in \mathbb{R}^n \ , \ y[n] \in \mathbb{R}^q \ , \ u[n] \in \mathbb{R}^p, \\ &x[n+1] = Ax[n] + Bu[n], \\ &y[n] = Cx[n] + Du[n], \\ &\text{where } A \in \mathbb{R}^{n \times n} \ , \ B \in \mathbb{R}^{n \times p} \ , \ C \in \mathbb{R}^{q \times n} \ , \ D \in \mathbb{R}^q \end{aligned}$$

Linear Time Varying Systems

$$\begin{aligned}
\text{Let } x[n] \in \mathbb{R}^n \ , \ y[n] &\in \mathbb{R}^q \ , \ u[n] \in \mathbb{R}^p, \\
x[n+1] &= A[n]x[n] + B[n]u[n], \\
y[n] &= C[n]x[n] + D[n]u[n], \\
\text{where } A[n] \in \mathbb{R}^{n\times n} \ , \ B[n] &\in \mathbb{R}^{n\times p} \ , \ C[n] \in \mathbb{R}^{q\times n} \ , \ D[n] \in \mathbb{R}^{q}
\end{aligned}$$

Non-Linear Time Invariant Systems

$$\begin{aligned}
\text{Let } x[n] \in \mathbb{R}^n \ , \ y[n] &\in \mathbb{R}^q \ , \ u[n] in \mathbb{R}^p, \\
x[n+1] &= F(x[n], u[n]), \\
y[n] &= H(x[n], u[n]),
\end{aligned}$$

Non-Linear Time Varying Systems

$$\begin{aligned}
\text{Let } x[n] \in \mathbb{R}^n \ , \ y[n] &\in \mathbb{R}^q \ , \ u[n] \in \mathbb{R}^p, \\
x[n+1] &= F(x[n], u[n], n), \\
y[n] &= H(x[n], u[n], n),
\end{aligned}$$

Discussion

- When a state-space representation becomes *memoryless*?
- What about infinite dimensional systems?
- What about *causality*?

1.3.3 Impulse-Response Representation of Dynamical Systems

- **Continuous-Time Dynamical Systems**

  Linear Time Invariant Systems

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau$$

  Linear Time Varying Systems

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)u(\tau)d\tau$$

  where $h(t,\tau)$ is called time-varying impulse response function.

- **Discrete-Time Dynamical Systems**
  Linear Time Invariant Systems

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]u[k]$$

  Linear Time Varying Systems

$$y[n] = \sum_{k=-\infty}^{\infty} h[n,k]u[k]$$

Discussion

- Under what condition(s) an impulse response representation becomes *memoryless*?
- Under what condition(s) an impulse response representation becomes *causal*?
- What about finite and infinite dimensional systems?
- What are the differences between continuous time and discrete time impulse response?

1.3.4 Transfer Functions Representation of Dynamical Systems

- **Continuous-Time Dynamical Systems**
  Linear Time Invariant Systems

$$\begin{aligned} &Y(s) = G(s)U(s), \text{ where,} \\ &Y(s) = \mathcal{L}\{y(t)\} \text{ ,\& } U(s) = \mathcal{L}\{u(t)\} \end{aligned}$$

- **Discrete-Time Dynamical Systems**
  Linear Time Invariant Systems

$$\begin{aligned} &Y(z) = G(z)U(z), \text{ where,} \\ &Y(z) = \mathcal{Z}\{y[n]\} \text{ ,\& } U(z) = \mathcal{Z}\{u[n]\} \end{aligned}$$

  Discussion

  - Can we model/represent non-linear systems using transfer functions?
  - Can we model/represent linear time-varying systems using transfer functions?
  - What about finite and infinite dimensional systems?
  - What about *causality*?