EE302 - Feedback Systems **Spring 2019**

Lecture 2

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Conversion Between Different LTI Representations

In this lecture we will cover the conversion between different LTI representations.

2.1 ODE to TF & TF to ODE

Conversion between ODE and TF representations is trivial in both directions

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = b_n u^{(n)} + ... + b_1 u' + b_0 u$$
$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

2.2 State-Space to TF

Note that a SS representation of an n^{th} order LTI system has the from below.

$$\begin{aligned} \text{Let } x(t) &\in \mathbb{R}^n \ , \ y(t) \in \mathbb{R} \ , \ u(t) \in \mathbb{R}, \\ \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ \text{where } A &\in \mathbb{R}^{n \times n} \ , \ B \in \mathbb{R}^{n \times 1} \ , \ C \in \mathbb{R}^{1 \times n} \ , \ D \in \mathbb{R} \end{aligned}$$

In order to convert state-space to transfer function, we start with taking the Laplace transform of the both sides of the state-equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ sX(s) &= AX(s) + BU(s) \\ sX(s) - AX(s) &= BU(s) \\ (sI - A)X(s)) &= BU(s) \\ X(s) &= (sI - A)^{-1} BU(s) \end{aligned}$$

Now let's concentrate on the output equation

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ Y(s) &= \left[C\,(sI - A)^{-1} B + D\right] U(s) \\ G(s) &= C\,(sI - A)^{-1} B + D \end{aligned}$$

2.3 ODE/TF to State-Space

Note that for a given LTI system, there exist infinitely many different SS representations. In this part, we learn two different ways converting a TF/ODE into State-Space form. For the sake of clarity, we will derive the realization for a general 3^{rd} order LTI system.

2.3.1 Canonical Realization I

In this method of realization, we will use the fact the system is LTI. Let's consider the transfer function of the system and let's perform some LTI operations.

$$\begin{aligned}
Y(s) &= \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} U(s) \\
&= \left(b_3 s^3 + b_2 s^2 + b_1 s + b_0\right) \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} U(s) \\
&= G_2(s) G_1(s) U(s) \text{ where} \\
G_1(s) &= \frac{H(s)}{U(s)} = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \\
G_2(s) &= \frac{Y(z)}{H(z)} = b_3 s^3 + b_2 s^2 + b_1 s + b_0
\end{aligned}$$

As you can see we introduced an intermediate variable $h(t)$ or with a Laplace transform of $H(s)$. First transfer function has static input dynamics, operates on $x(t)$, and produces an output, i.e. $h(t)$. Second transfer function is a non-causal system and operates on $h(t)$ and produces output $x(t)$. If we write the ODEs of both systems we obtain

$$\begin{aligned}
\dddot{h} &= -a_2 \ddot{h} - a_1 \dot{h} - a_0 h + u \\
y &= b_3 \dddot{h} + b_2 \ddot{h} + b_1 \dot{h} + b_0 h
\end{aligned}$$

Now let the state-variables be $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} h \\ x_2 \\ x_3 \end{bmatrix}$. Then, individual state equations take the form

$$\begin{aligned}
\dot{x_1} &= x_2 \\
\dot{x_2} &= x_3 \\
\dot{x_3} &= -a_2 x_3 - a_1 x_2 - a_0 x_1 + u
\end{aligned}$$

and the output equation take the form

$$\begin{aligned}
y &= b_3 \left(-a_2 x_3 - a_1 x_2 - a_0 x_1 + u\right) + b_2 x_3 + b_1 x_2 + b_0 x_1 \\
&= (b_0 - b_3 a_0) x_1 + (b_1 - b_3 a_1) x_2 + (b_2 - b_3 a_2) x_3 + b_3 u
\end{aligned}$$

If we re-write the equations in matrix form we obtain the state-space representation as

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} (b_0 - b_3 a_0) & (b_1 - b_3 a_1) & (b_2 - b_3 a_2) \end{bmatrix} x + [b_3]\, u
\end{aligned}$$

If we obtain a state-space model from this approach, the form will be in *controllable canonical form*. We will cover this later in the semester. Thus we can call this representation also as *controllable canonical realization*.

For a general n^{th} order system controllable canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} (b_0 - b_n a_0) & (b_1 - b_n a_1) & \cdots & (b_{n-1} - b_n a_{n-1}) \end{bmatrix} \quad , \quad D = b_n$$

2.3.2 Canonical Realization II

In this method will obtain a different minimal state-space realization. The process will be different and state-space structure will have a different topology. Let's start with the transfer function and perform some grouping based on the s elements.

$$\begin{aligned}
&Y(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} U(s) \\
&Y(s)\left(s^3 + a_2 s^2 + a_1 s + a_0\right) = \left(b_3 s^3 + b_2 s^2 + b_1 s + b_0\right) U(s) \\
&s^3 Y(s) = b_3 s^3 U(s) + s^2\left(-a_2 Y(s) + b_2 U(s)\right) + s\left(-a_1 Y(s) + b_1 U(s)\right) + \left(-a_0 Y(s) + b_0 U(s)\right)
\end{aligned}$$

Let's multiply both sides with $\frac{1}{s^3}$ and perform further grouping

$$\begin{aligned}
&Y(s) = b_3 U(s) + \frac{1}{s}\left(-a_2 Y(s) + b_2 U(s)\right) + \frac{1}{s^2}\left(-a_1 Y(s) + b_1 U(s)\right) + \frac{1}{s^3}\left(-a_0 Y(s) + b_0 U(s)\right) \\
&Y(s) = b_3 U(s) + \frac{1}{s}\left[\left(-a_2 Y(s) + b_2 U(s)\right) + \frac{1}{s}\left\{\left(-a_1 Y(s) + b_1 U(s)\right) + \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right)\right\}\right]
\end{aligned}$$

Let the Laplace domain representations of state variables $X(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix}$ defined as

$$\begin{aligned}
X_1(s) &= \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right) \\
X_2(s) &= \frac{1}{s}\left\{\left(-a_1 Y(s) + b_1 U(s)\right) + \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right)\right\} \\
X_3(s) &= \frac{1}{s}\left[\left(-a_2 Y(s) + b_2 U(s)\right) + \frac{1}{s}\left\{\left(-a_1 Y(s) + b_1 U(s)\right) + \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right)\right\}\right]
\end{aligned}$$

In this context output equation in s and time domains simply takes the form

$$Y(s) = X_3(s) + b_3 U(S) \quad \rightarrow \quad y(t) = x_3(t) + b_3 u(t)$$

Dependently the state equations (in s and time domains) take the form

$$\begin{aligned}
sX_1(s) &= -a_0 X_3(s) + (b_0 - a_0 b_3) U(s) & \rightarrow \quad \dot{x}_1 &= -a_0 x_3 + (b_0 - a_0 b_3) u \\
sX_2(s) &= X_1(s) - a_1 X_3(s) + (b_1 - a_1 b_3) U(s) & \rightarrow \quad \dot{x}_2 &= x_1 - a_1 x_3 + (b_1 - a_1 b_3) u \\
sX_3(s) &= X_2(s) - a_2 X_3(s) + (b_2 - a_2 b_3) U(s) & \rightarrow \quad \dot{x}_3 &= x_2 - a_2 x_3 + (b_2 - a_2 b_3) u
\end{aligned}$$

If we re-write all equations in matrix form, we obtain the state-space representation as

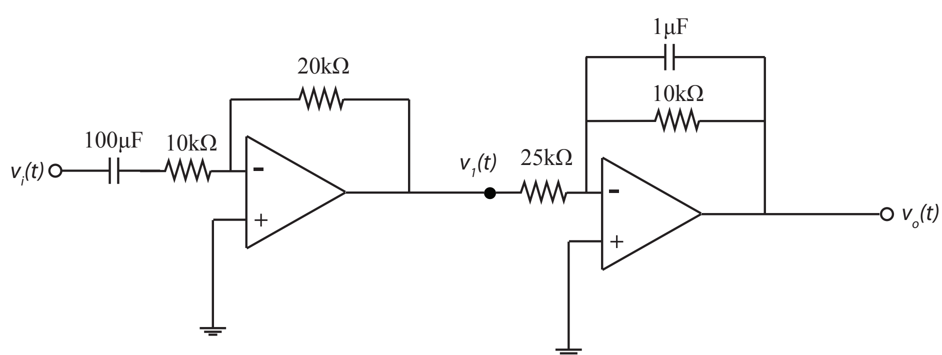
$$\dot{x} = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} x + \begin{bmatrix} (b_0 - b_3 a_0) \\ (b_1 - b_3 a_1) \\ (b_2 - b_3 a_2) \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x + [b_3]\, u$$

If we obtain a state-space model from this method, the form will be in *observable canonical form.* Thus we can call this representation also as *observable canonical realization.* This form and representation is the dual of the previous representation.

For a general n^{th} order system controllable canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \quad , \quad B = \begin{bmatrix} (b_0 - b_n a_0) \\ (b_1 - b_n a_1) \\ \vdots \\ (b_{n-2} - b_n a_{n-2}) \\ (b_{n-1} - b_n a_{n-1}) \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad , \quad D = b_n$$

Example:



1. Given that $u(t) = v_i(t)$ and $y(t) = v_o(t)$, compute the transferfunction for the given (ideal) OPAMP circuit

   Solution:

   First let's compute $V_1(s)/U(s)$

$$\frac{U(S)}{\frac{10^4}{s} + 10^4} = \frac{-V_1(S)}{2\,10^4}$$
$$\frac{V_1(s)}{U(s)} = \frac{-2s}{s+1}$$

Now let's compute $Y(s)/V_1(s)$

$$\begin{aligned}
\frac{V_1(s)}{2.510^4} &= -Y(s)\left(s10^{-6}+10^{-4}\right)\\
\frac{V_1(s)}{2.510^4} &= -Y(s)\left(s+100\right)10^{-6}\\
\frac{Y(s)}{V_1(s)} &= \frac{-40}{s+100}
\end{aligned}$$

Hence, the transfer function of the whole system/circuit can be found as

$$\begin{aligned}
G(s) = \frac{Y(s)}{U(s)} &= \frac{Y(s)}{V_1(s)}\frac{V_1(s)}{U(s)}\\
&= \frac{80s}{(s+100)(s+1)}\\
&= \frac{80s}{s^2+101s+100}
\end{aligned}$$

2. Now find a state-space representation from the given TF.

   Solution:

   If we follow the derivation of controllable canonical form for a second order system we obtain the following structure

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\\
y &= \begin{bmatrix} (b_0 - b_2 a_0) & (b_1 - b_2 a_1) \end{bmatrix} x + [b_2]\, u
\end{aligned}$$

   where

$$a_0 = 100\ ,\ a_1 = 101\ ,\ b_0 = 0\ ,\ b_1 = 80\ ,\&\ b_2 = 0$$

   Thus, the state-space representation takes the form

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -100 & -101 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\\
y &= \begin{bmatrix} 0 & 80 \end{bmatrix} x
\end{aligned}$$

3. Now re-compute the TF from the given state-space representation

   Solution:

$$\begin{aligned}
G(s) &= C\left(sI - A\right)^{-1} B + D\\
&= \begin{bmatrix} 0 & 80 \end{bmatrix} \begin{bmatrix} s & -1 \\ 100 & s+101 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\\
&= \begin{bmatrix} 0 & 80 \end{bmatrix} \frac{1}{s^2+101s+100} \begin{bmatrix} s+101 & 1 \\ -100 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\\
&= \begin{bmatrix} 0 & 80 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} \frac{1}{s^2+101s+100}\\
&= \frac{80s}{s^2+101s+100}
\end{aligned}$$

Take Home Problem:

1. First find state space representations of the sub-system transfer functions, i.e. $\frac{V_1(s)}{U(s)}$ and $\frac{Y(s)}{V_1(s)}$, separately.
2. Then combine the state-space representations of the sub-systems to find a state-space representation for the whole system.
3. Compute the TF from the computed state-space representation and compare it to the previous results.