EE502 - Linear Systems Theory II **Spring 2032**

Lecture 6

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6.1 Modal Decomposition of State-Space Models

6.1.1 Zero Input Response

Let's consider autonomous LTI CT and DT state-space models

In linear systems theory course, we are interested in matrix polynomials, specifically

$$\dot{x} = Ax$$
$$x[k+1] = Ax[k]$$

Let $x_0 = \alpha v_i$, where v_i is an eigenvector of A associated with eigenvalue λ_i, we can then find the solution for both systems

$$x(t) = e^{At}x_0 = \alpha e^{\lambda_i t} v_i$$
$$x[k] = A^k x_0 = \alpha \lambda_i^k v_i$$

Now let's assume that A is diagonalizable, then we now that there exist a set of n linearly independent eigenvectors $\mathcal{V} = \{v_i \ , \ \cdots \ , \ v_n\}$. Thus, we can write any initial condition, $x_0 \in \mathbb{R}^n$, as a linear combination of eigenvectors, i.e.

$$x_0 = \sum_{i=1}^{n} \alpha_i v_i$$

if we use this (modal) decomposition we can find the solutions of DT and CT equations as

$$x(t) = e^{At}x_0 = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} v_i$$
$$x[k] = A^k x_0 = \sum_{i=1}^{n} \alpha_i \lambda_i^k v_i$$

where $e^{\lambda_i t} v_i$ ($\lambda_i^k v_i$ in DT case) is called a "mode" of the system. Now let's try to find $\{\alpha_i \ , \ \cdots \ , \ \alpha_n\}$ via diagonalizition of A

$$A = V\Lambda V^{-1} = V \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{bmatrix} V^{-1} \text{ , where}$$

$$V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \text{ , where } Av_i = \lambda_i v_i$$

$$V^{-1} = \bar{V} = \begin{bmatrix} \bar{v}_1^T \\ \vdots \\ \bar{v}_n^T \end{bmatrix} \text{ , where } \bar{v}_i^T A = \lambda_i \bar{v}_i^T$$

$$\bar{V}V = V\bar{V} = I \text{ , } \bar{v}_i^T v_i = 1 \text{ , } \bar{v}_i^T v_j = 0 \text{ for } i \neq j$$

Now let's compute the zero-input responses for an arbitrary x_0

$$x(t) = e^{At}x_0 = Ve^{\Lambda t}V^{-1}x_0 = \begin{bmatrix} v_1 & \cdots v_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t}\bar{v}_1^T x_0 \\ \vdots \\ e^{\lambda_n t}\bar{v}_n^T x_0 \end{bmatrix} = \sum_{i=1}^{n} v_i e^{\lambda_i t} \left(\bar{v}_i^T x_0\right) \rightarrow \alpha_i = \bar{v}_i^T x_0$$

$$x[k] = V\Lambda^k V^{-1}x_0 = \begin{bmatrix} v_1 & \cdots v_n \end{bmatrix} \begin{bmatrix} \lambda_1^k \bar{v}_1^T x_0 \\ \vdots \\ \lambda_n^k \bar{v}_n^T x_0 \end{bmatrix} = \sum_{i=1}^{n} v_i \lambda_i^k \left(\bar{v}_i^T x_0\right) \rightarrow \alpha_i = \bar{v}_i^T x_0$$

Based on these results, we can see that in order to excite the i^{th} mode the system, we need $\bar{v}_i^T x_0 \neq 0$. If we treat initial conditions as inputs this generates a reachability/controllability argument. Let's also analyze the output response

$$y(t) = Cx(t) = Ce^{At}x_0 = \sum_{i=1}^{n} (Cv_i)e^{\lambda_i t} \left(\bar{v}_i^T x_0\right)$$

$$y[k] = Cx[k] = \sum_{i=1}^{n} (Cv_i)\lambda_i^k \left(\bar{v}_i^T x_0\right)$$

We can see that if $Cv_i = 0$, then we can not observe the i^{th} mode at the output $\forall x_0 \in \mathbb{R}^n$. Thus we can conclude that in order to have a fully observable system all modes needs to be observable, i.e. i.e. $Cv_i \neq 0 \; \forall i \in \{1, \cdots, n\}$.

Now let's try to find a similar solution for systems where matrix A can not be diagonalizable. For the sake of clarity let's focus on matrices that is composed of a single Jordan block

$$A = GJG^{-1} = G \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots \\ \vdots & & \ddots & & \\ & & & \ddots & 1 \\ 0 & & & 0 & \lambda \end{bmatrix} G^{-1}$$

where

$$
\begin{aligned}
G &= \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \\
Ag_1 &= \lambda g_1 \ \rightarrow \ (A-\lambda I)g_1 = 0 \\
Ag_2 &= \lambda g_2 + g_1 \ \rightarrow \ (A-\lambda I)g_2 = g_1 \ , \ \text{note } (A-\lambda I)^2 g_2 = 0 \ \& \ (A-\lambda I)g_2 \neq 0 \\
Ag_3 &= \lambda g_3 + g_2 \ \rightarrow \ (A-\lambda I)g_3 = g_2 \ , \ \text{note } (A-\lambda I)^3 g_3 = 0 \ \& \ (A-\lambda I)^2 g_3 \neq 0 \\
&\vdots \\
Ag_n &= \lambda g_n + g_{n-1} \ \rightarrow \ (A-\lambda I)g_n = g_{n-1} \ , \ \text{note } (A-\lambda I)^n g_n = 0 \ \& \ (A-\lambda I)^{n-1} g_n \neq 0
\end{aligned}
$$

and we also know that

$$
G^{-1} = \bar{G} = \begin{bmatrix} \bar{g}_1^T \\ \vdots \\ \bar{g}_n^T \end{bmatrix}
$$

$$
\bar{G}G = G\bar{G} = I \ , \ \bar{g}_i^T g_i = 1 \ , \ \bar{g}_i^T g_j = 0 \text{ for } i \neq j
$$

Let $x_0 = \alpha_1 g_1$, i.e. the eigenvector of A, then we can find the responses as

$$
\begin{aligned}
x(t) &= e^{At} g_1 = G e^{Jt} G^{-1} g_1 \alpha_1 \\
&= \begin{bmatrix} g_1 & \cdots g_n \end{bmatrix}
\begin{bmatrix}
e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda t} & \frac{t^{n-1}}{(n-1)!}e^{\lambda t} \\
0 & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda t} \\
\vdots & & \ddots & & & \vdots \\
 & & & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} \\
0 & & \cdots & & e^{\lambda t} & te^{\lambda t} \\
0 & & \cdots & & 0 & e^{\lambda t}
\end{bmatrix}
\begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \alpha_1 e^{\lambda t} g_1
\end{aligned}
$$

$$
x[k] = GJ^k G^{-1} x_0 = \alpha_1 \lambda^k g_1
$$

the format of the solution associated with g_1 seems to be exactly same with diagonal case (since g_1 is an eigenvector). This implies that, if we choose an initial condition inside the eigenvector space, the response stays in this space and acts like a "first-order" system. Now, let $x_0 = \alpha_2 g_2$, i.e. a first order generalized eigenvector of A, then we can find the responses as

$$
\begin{aligned}
x(t) &= G e^{Jt} G^{-1} g_2 \alpha_2 \\
&= \begin{bmatrix} g_1 & g_2 & \cdots g_n \end{bmatrix}
\begin{bmatrix}
e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda t} & \frac{t^{n-1}}{(n-1)!}e^{\lambda t} \\
0 & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda t} \\
\vdots & & \ddots & & & \vdots \\
 & & & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} \\
0 & & \cdots & & e^{\lambda t} & te^{\lambda t} \\
0 & & \cdots & & 0 & e^{\lambda t}
\end{bmatrix}
\begin{bmatrix} 0 \\ \alpha_2 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} g_1 & g_2 & \cdots g_n \end{bmatrix}
\begin{bmatrix} \alpha_2 te^{\lambda t} \\ \alpha_2 e^{\lambda t} \\ \vdots \\ 0 \end{bmatrix}
= \alpha_2 \left(te^{\lambda t} g_1 + e^{\lambda t} g_2 \right)
\end{aligned}
$$

$$
\begin{aligned}
x[k] &= GJ^kG^{-1}g_2\alpha_2 \\
&= \begin{bmatrix} g_1 & g_2 & \cdots g_n \end{bmatrix}
\begin{bmatrix}
\lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \binom{k}{n-2}\lambda^{k-n+2} & \binom{k}{n-1}\lambda^{k-n+1} \\
0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \binom{k}{n-2}\lambda^{k-n+2} \\
\vdots & & \ddots & & & \vdots \\
 & & & \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} \\
0 & & \cdots & & \lambda^k & \binom{k}{1}\lambda^{k-1} \\
0 & & \cdots & & 0 & \lambda^k
\end{bmatrix}
\begin{bmatrix} 0 \\ \alpha_2 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} g_1 & g_2 & \cdots g_n \end{bmatrix}
\begin{bmatrix} \alpha_2 k\lambda^{k-1} \\ \alpha_2\lambda^k \\ \vdots \\ 0 \end{bmatrix} = \alpha_2\left(k\lambda^{k-1}g_1 + \lambda^k g_2\right)
\end{aligned}
$$

We can observe that the response acts like a "second-order" (critically-damped) response. Moreover, the response does not stays inside the span of the generalized eigenvector, i.e. Span$\{g_2\}$, instead it navigates inside the span of the eigenvector and g_2, i.e. Span$\{g_1\ ,\ g_2\} = \mathcal{N}(A-\lambda I)^2$. Now, let $x_0 = \alpha_i g_i\ ,\ 0 \le i \le n$, i.e. order generalized eigenvector of order i, then we can find the responses as

$$
\begin{aligned}
x(t) &= Ge^{Jt}G^{-1}g_2\alpha_2 = \alpha_i e^{\lambda t}\sum_{j=1}^{i} g_j \frac{t^{i-j}}{(i-j)!} \\
x[k] &= GJ^kG^{-1}g_i\alpha_i = \alpha_i \sum_{j=1}^{i} g_j \binom{k}{i-j}\lambda^{k-i+j}
\end{aligned}
$$

Similar to the second-order case, we can see that response acts like an i^{th} order dynamical system, and trajectories stays inside, Span$\{g_1\ ,\ cdots\ g_i\} = \mathcal{N}(A-\lambda I)^i$. In this context, in order to excite all higher order modes of the system projection of the initial condition to the sub-space spanned highest order generalized eigenvector has to be non-zero. Now, let's try to find general solution for an arbitrary x_0. We can write any $x_0 \in \mathbb{R}^n$ as a linear combination of $\mathcal{G} = \{g_1\ ,\ g_2\ ,\ \cdots\ ,\ g_n\}$, thus we have

$$
\begin{aligned}
x_0 &= \sum_{i=1}^{n}\alpha_i g_i \\
x(t) &= \sum_{i=1}^{n}\alpha_i e^{\lambda t}\sum_{j=1}^{i} g_j \frac{t^{i-j}}{(i-j)!} \\
x[k] &= \sum_{i=1}^{n}\alpha_i \sum_{j=1}^{i} g_j \binom{k}{i-j}\lambda^{k-i+j}
\end{aligned}
$$

where

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = G^{-1}x_0 = \begin{bmatrix} \bar{g}_1^T \\ \bar{g}_2^T \\ \vdots \\ \bar{g}_n^T \end{bmatrix} x_0 \rightarrow \alpha_i = \bar{g}_i^T x_0$$

Based on these results, we can see that in order to excite all of the modes associated with a Jordan block of size n, we need $\alpha_n \bar{g}_n^T x_0 \neq 0$. If we treat initial conditions as inputs this generates a reachability/controllability argument. Thus in order for this Jordan block to be reachable/controllable, we need to excite highest order mode (generalized eigenvector).

Ex 6.1 *Let*

$$\dot{x} = Ax \text{ where } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

derive the solution of $x(t)$ using modal decomposition for an arbitrary $x_0 \in \mathbb{R}^3$

Solution: We know that Jordan canonical form of matrix A has the form

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the transformation matrices that leads to this Jordan form are

$$G = \begin{bmatrix} g_1 & g_2 & v \end{bmatrix}, \; G = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; G^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

where g_1 and v are eigenvectors and g_2 is the single generalized eigenvector associated with g_1. If we follow the derivation in this lecture note, we can first find all individual components of the responses as

$$\begin{aligned} x_{g_1}(t) &= \alpha_{g_1} e^t g_1 \\ x_{g_2}(t) &= \alpha_{g_2} \left(te^t g_1 + e^t g_2 \right) \\ x_v(t) &= \alpha_v e^t v \end{aligned}$$

where the combined solution and α_*'s can be derived using

$$x(t) = x_{g_1}(t) + x_{g_2}(t) + x_v(t) = e^t \left((\alpha_{g_1} + t\alpha_{g_2}) g_1 + \alpha_{g_2} g_2 + \alpha_v v \right)$$

$$\begin{bmatrix} \alpha_{g_1} \\ \alpha_{g_2} \\ \alpha_v \end{bmatrix} = G^{-1}x_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} x_0$$