EE502 - Linear Systems Theory II **Spring 2023**

Lecture 16

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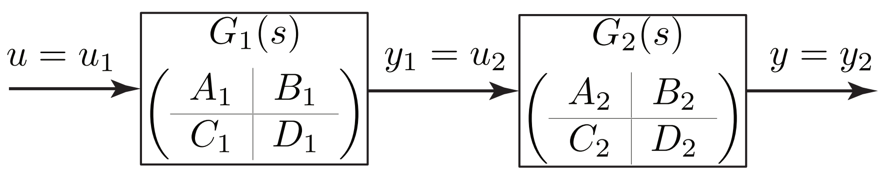
16.1 Minimality of Interconnected Systems

In this section we shall examine the conditions under which minimality is lost when minimal subsystems are interconnected in various configurations,

16.1.1 Series - Cascade Connection

Consider the following system structure where two sub systems, with transfer functions $G_1(s)$ and $G_2(s)$ and associated minimal representations $\left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right)$ and $\left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right)$, connected in series/cascade configuration.



The transfer function of the connection is simply equal to $G(s) = G_2(s)G_1(s)$. Let x_1 and x_2 state-variables of the sub-systems, then natural choice of the state variable for the series connection is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Under this definition the state-space representation for the whole system can be found as

$$A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline B_2C_1 & A_2 \end{array}\right] , \ B = \left[\begin{array}{c} B_1 \\ \hline B_2D_1 \end{array}\right] , \ C = \left[\begin{array}{c|c} D_2C_1 & C_2 \end{array}\right] , \ D = \left[\begin{array}{c} D_2D_2 \end{array}\right]$$

Clearly eigenvalues of A are the combination of the eigenvalues of A_1 and A_2 and the poles of the system Let's analyze the observability of the connection via PBH test.

$$\left[\begin{array}{c} \lambda I - A \\ \hline C \end{array}\right] = \left[\begin{array}{c|c} \lambda I - A_1 & 0 \\ \hline -B_2C_1 & \lambda I - A_2 \\ \hline D_2C_1 & C_2 \end{array}\right]$$

If we remember from the observability lecture that (A, C) (whole state-space model) pair is unobservable if and only if $\left[\begin{array}{c} \lambda I - A \\ \hline C \end{array}\right]$ losses rank for some λ, which can only happen if λ is an eigenvalue of A. Let's assume that λ_2 is an eigenvalue of A_2 but not eigenvalue of A_1. Then $\left[\begin{array}{c} \lambda I - A \\ \hline C \end{array}\right]_{\lambda=\lambda_2}$ looses rank, if $\exists v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

such that

$$\left[\begin{array}{c}\lambda_2 I - A \\ \hline C\end{array}\right] v = 0 \Rightarrow \left[\begin{array}{c|c}\lambda_2 I - A_1 & 0 \\ \hline -B_2C_1 & \lambda_2 I - A_2 \\ \hline D_2C_1 & C_2\end{array}\right]\begin{bmatrix}v_1 \\ v_2\end{bmatrix} = 0 \Rightarrow v_1 = 0 \,\&\, \left[\begin{array}{c}\lambda_2 I - A \\ \hline C_2\end{array}\right] v_2 = 0$$

which contradicts with the fact that (A_2, C_2) is observable since both individual sub-system representations are minimal. In that respect $\left[\begin{array}{c}\lambda I - A \\ \hline C\end{array}\right]$ can loose rank only at an eigenvalue of A_1 (i.e. a pole of A_2). Let's λ_1 is an eigenvalue of A_1. Then $\left[\begin{array}{c}\lambda I - A \\ \hline C\end{array}\right]_{\lambda=\lambda_1}$ looses rank, if $\exists v = \begin{bmatrix}v_1 \\ v_2\end{bmatrix}$ such that

$$\left[\begin{array}{c}\lambda_1 I - A \\ \hline C\end{array}\right] v = 0 \Rightarrow \left[\begin{array}{c|c}\lambda_1 I - A_1 & 0 \\ \hline -B_2C_1 & \lambda_1 I - A_2 \\ \hline D_2C_1 & C_2\end{array}\right]\begin{bmatrix}v_1 \\ v_2\end{bmatrix} = 0 \Rightarrow v_1 \neq 0 \,\&\, Av_1 = \lambda_1 v_1 \text{ and}$$

$$\left[\begin{array}{c|c}\lambda_1 I - A_2 & -B_2C_1 \\ \hline C_2 & D_2C_1\end{array}\right]\begin{bmatrix}v_2 \\ v_1\end{bmatrix} = 0 \Rightarrow \left[\begin{array}{c|c}\lambda_1 I - A_2 & -B_2 \\ \hline C_2 & D_2\end{array}\right]\begin{bmatrix}v_2 \\ C_1v_1\end{bmatrix} = 0$$

Note that $\left[\begin{array}{c|c}\lambda_1 I - A_2 & -B_2 \\ \hline C_2 & D_2\end{array}\right]\begin{bmatrix}v_2 \\ C_1v_1\end{bmatrix} = 0$ implies that λ_1 is a **right** zero of $G_2(s)$ where v_2 and C_1v_1 is the associated state-zero-direction and *input-zero-direction* respectively.

If we summarize the results, cascaded system is unobservable if $\exists(\lambda_1, v_1)$ and $v_2 \neq 0$ such that (λ_1, v_1) is an eigenvalue-eigenvector pair of A_1 and λ_1 is a **right** zero of $G_2(s)$ with v_2 and C_1v_1 as the state-zero-direction and *input-zero-direction* respectively.

Now let's analyze the reachability of the connection via PBH test.

$$\left[\begin{array}{c|c}\lambda I - A & B\end{array}\right] = \left[\begin{array}{c|c|c}\lambda I - A_1 & 0 & B_1 \\ \hline -B_2C_1 & \lambda I - A_2 & B_2D_1\end{array}\right]$$

If we remember from the reachability lecture that (A, B) (whole state-space model) pair is unreachable if and only if $\left[\begin{array}{c|c}\lambda I - A & B\end{array}\right]$ losses rank for some λ, which can only happen if λ is an eigenvalue of A. Let's assume that λ_1 is an eigenvalue of A_1 but not eigenvalue of A_2. Then $\left[\begin{array}{c|c}\lambda I - A & B\end{array}\right]_{\lambda=\lambda_1}$ looses rank, if $\exists w = \begin{bmatrix}w_1 \\ w_2\end{bmatrix}$ such that

$$\begin{aligned}&\begin{bmatrix}w_1^T & w_2^T\end{bmatrix}\left[\begin{array}{c|c}\lambda_1 I - A & B\end{array}\right] = 0 \Rightarrow \begin{bmatrix}w_1^T & w_2^T\end{bmatrix}\left[\begin{array}{c|c|c}\lambda_1 I - A_1 & 0 & B_1 \\ \hline -B_2C_1 & \lambda_1 I - A_2 & B_2D_1\end{array}\right] = 0 \\ &\Rightarrow w_2^T = 0 \,\&\, w_1^T\left[\begin{array}{c|c}\lambda_1 I - A_1 & B_1\end{array}\right] = 0\end{aligned}$$

which contradicts with the fact that (A_1, B_1) is reachable since both individual sub-system representations are minimal. Now let λ_2 is an eigenvalue of A_2. Then $\left[\begin{array}{c|c}\lambda I - A & B\end{array}\right]_{\lambda=\lambda_2}$ looses rank, if $\exists w = \begin{bmatrix}w_1 \\ w_2\end{bmatrix}$ such that

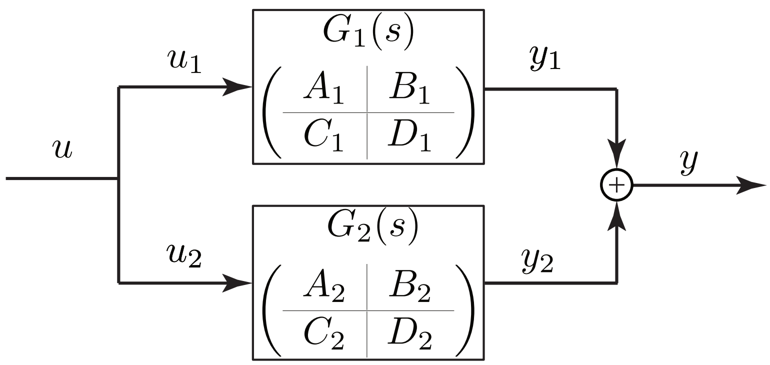
$$\begin{aligned}&\begin{bmatrix}w_1^T & w_2^T\end{bmatrix}\left[\begin{array}{c|c}\lambda_2 I - A & B\end{array}\right] = 0 \Rightarrow \begin{bmatrix}w_1^T & w_2^T\end{bmatrix}\left[\begin{array}{c|c|c}\lambda_2 I - A_1 & 0 & B_1 \\ \hline -B_2C_1 & \lambda_2 I - A_2 & B_2D_1\end{array}\right] = 0 \\ &\Rightarrow w_2 \neq 0 \,\&\, w_2^T A_2 = w_2^T \lambda_2 \text{ and } \begin{bmatrix}w_1^T & w_2^T\end{bmatrix}\left[\begin{array}{c|c}\lambda_2 I - A_1 & B_1 \\ \hline -B_2C_1 & B_2D_1\end{array}\right] = 0 \Rightarrow \\ &\begin{bmatrix}w_1^T & -(B_2^T w_2)^T\end{bmatrix}\left[\begin{array}{c|c}\lambda_2 I - A_1 & B_1 \\ \hline C_1 & -D_1\end{array}\right] = 0 \Rightarrow \begin{bmatrix}w_1^T & -(B_2^T w_2)^T\end{bmatrix}\left[\begin{array}{c|c}\lambda_2 I - A_1 & -B_1 \\ \hline C_1 & D_1\end{array}\right] = 0\end{aligned}$$

Note that $\begin{bmatrix} w_1^T & (-B_2^T w_2)^T \end{bmatrix} \left[\begin{array}{c|c} \lambda_2 I - A_1 & -B_1 \\ \hline C_1 & D_1 \end{array}\right] = 0$ implies that λ_2 is a **left** zero of $G_1(s)$ where w_1 and $(-B_2^T w_2)$ are the associated state-zero-direction and *input-zero-direction* respectively.

If we summarize the results, cascaded system is unreachable if $\exists(\lambda_2, w_2)$ and $w_1 \neq 0$ such that (λ_2, v_2) is a left eigenvalue-eigenvector pair of A_2 and λ_2 is a **left** zero of $G_1(s)$ with w_1 and $(-B_2^T w_2)$ as the state-zero-direction and *input-zero-direction* respectively.

16.1.2 Parallel Connection

Consider the following system structure where two sub systems, with transfer functions $G_1(s)$ and $G_2(s)$ and associated minimal representations $\left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right)$ and $\left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right)$, connected in parallel configuration.



The transfer function of the connection is simply equal to $G(s) = G_1(s) + G_2(s)$. Let x_1 and x_2 state-variables of the sub-systems, then natural choice of the state variable for the series connection is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Under this definition the state-space representation for the whole system can be found as

$$A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right], \; B = \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right], \; C = \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right], \; D = \left[\begin{array}{c} D_1 + D_2 \end{array}\right]$$

Clearly eigenvalues of A are the combination of the eigenvalues of A_1 and A_2 and the poles of the system Let's analyze the observability of the connection via one of the modal reachability tests.

$$\left[\begin{array}{c} \lambda I - A \\ \hline C \end{array}\right] = \left[\begin{array}{c|c} \lambda I - A_1 & 0 \\ \hline 0 & \lambda I - A_2 \\ \hline C_1 & C_2 \end{array}\right]$$

It is easy to observe that combined system is always observable if A_o1 and A_2 does not share any common eigenvalue. Thus a necessary (but not sufficient) condition such that parallel connection loses observability is that A_1 and A_2 will have at least 1 common eigenvalue. Let's assume that λ_c is a common eigenvalue/pole of both sub-systems Then $\left[\begin{array}{c} \lambda I - A \\ \hline C \end{array}\right]_{\lambda = \lambda_c}$ looses rank, if $\exists v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that

$$\left[\begin{array}{c} \lambda I - A \\ \hline C \end{array}\right] v = 0 \iff \left[\begin{array}{c|c} \lambda I - A_1 & 0 \\ \hline 0 & \lambda I - A_2 \\ \hline C_1 & C_2 \end{array}\right] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \iff$$
$$A_1 v_1 = \lambda_c v_1 \,,\; A_2 v_2 = \lambda_c v_2 \,, \& \, C_1 v_1 + C_2 v_2 = 0$$

In other words combined system looses observability if and only if $\exists\,\lambda_c$, v_1 , & v_2 such that λ_c is a common eigenvalue of both systems, and there exists an (v_1, v_2) eigenvector pair (v_i is an eigenvector of A_i associated with λ_c) such that $C_1 v_1 + C_2 v_2 = 0$.

Now analyze the reachability of the connection via one of the modal reachability tests.

$$\left[\begin{array}{c|c} \lambda I - A & B \end{array}\right] = \left[\begin{array}{c|c|c} \lambda I - A_1 & 0 & B_1 \\ \hline 0 & \lambda I - A_2 & B_2 \end{array}\right]$$

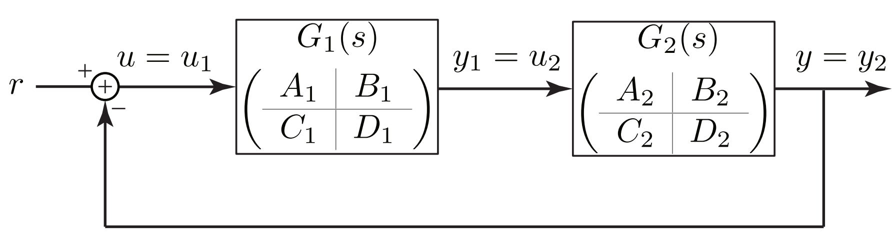
It is easy to observe that combined system is always reachable if A_1 and A_2 does not share any common eigenvalue. Thus a necessary (but not sufficient) condition such that parallel connection loses reahcability is that A_1 and A_2 will have at least 1 common eigenvalue. Let's assume that λ_c is a common eigenvalue/pole of both sub-systems Then $\left[\begin{array}{c|c} \lambda I - A & B \end{array}\right]_{\lambda=\lambda_c}$ looses rank, if $\exists w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ such that

$$\begin{aligned} & w^T \left[\begin{array}{c|c} \lambda_c I - A & B \end{array}\right] = 0 \iff \begin{bmatrix} w_1^T & w_2^T \end{bmatrix} \left[\begin{array}{c|c|c} \lambda_c I - A_1 & 0 & B_1 \\ \hline 0 & \lambda_c I - A_2 & B_2 \end{array}\right] = 0 \iff \\ & w_1^T A_1 = \lambda_c w_1^T \,,\; w_2^T A_2 = \lambda_c w_2^T \,, \& \; w_1^T B_1 + w_2^T B_2 = 0 \end{aligned}$$

In other words combined system looses reachability if and only if $\exists\,\lambda_c$, w_1 , & w_2 such that λ_c is a common eigenvalue of both systems, and there exists an (w_1, w_2) left-eigenvector pair (w_i is a left eigenvector of A_i associated with λ_c) such that $w_1^T B_1 + w_2^T B_2 = 0$.

16.1.3 Feedback Connection (Unity)

Consider the following system structure where two sub systems, with transfer functions $G_1(s)$ and $G_2(s)$ and associated minimal representations $\left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right)$ and $\left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right)$, connected in a unity feedback configuration.



The transfer function of the connection from the reference signal to the output signal is equal to $G(s) = (I + G_1(s)G_2(s))^{-1}\, G_1(s)G_2(s)$. Let x_1 and x_2 state-variables of the sub-systems, then natural choice of the state variable for the series connection is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Let's also assume that $D_1 = 0$ and $D_2 = 0$, under this assumption the state-space representation for the whole system can be found as

$$A_{CL} = \left[\begin{array}{c|c} A_1 & -B_1 C_2 \\ \hline B_2 C_1 & A_2 \end{array}\right] ,\; B_{CL} = \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right] ,\; C_{CL} = \left[\begin{array}{c|c} 0 & C_2 \end{array}\right] ,\; D = \left[\begin{array}{c} 0 \end{array}\right]$$

Unfortunately it is not straightforward to inspect the eigenvalues of the closed-loop system, since output feedback potentially moves the closed-loop eigenvalue locations. Note that in this topology, feedforward system, is the cascade configuration that we analyzed in the first part. In that respect, we can re-write the closed-loop matrices as

$$\begin{aligned}
A_{CL} &= \left[\begin{array}{c|c} A_1 & -B_1C_2 \\ \hline B_2C_1 & A_2 \end{array}\right] = \left[\begin{array}{c|c} A_1 & 0 \\ \hline B_2C_1 & A_2 \end{array}\right] - \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right] \left[\begin{array}{c|c} 0 & C_2 \end{array}\right] = A_{OL} - B_{OL}C_{OL} \\
B_{CL} &= \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right] = B_{OL} \\
C_{CL} &= \left[\begin{array}{c|c} 0 & C_2 \end{array}\right] = C_{OL}
\end{aligned}$$

Note that (A_{OL}, B_{OL}, C_{OL}) is the state-space representation of the cascade configuration. Let's assume that (A_{CL}, B_{CL}) pair is unreachable, then $\exists w$ such that

$$\begin{aligned}
w^T B_{CL} = 0 \iff w^T B_{OL} &= 0 \\
\lambda w^T &= w^T A_{CL} = w^T \left[A_{OL} - B_{OL}C_{OL}\right] = w^T A_{OL} \\
\lambda w^T &= w^T A_{OL}
\end{aligned}$$

This proves that (A_{CL}, B_{CL}) is unreachable if and only if (A_{OL}, B_{OL}) is unreachable. In that respect the feedback configuration looses reachability if and only if cascade configuration in the feed-forward path looses reachability.

Now let's analyze the observability of the feedback connection. Let's assume that (A_{CL}, C_{CL}) pair is unobservable, then $\exists v$ such that

$$\begin{aligned}
C_{CL}v = 0 \iff C_{OL} &= 0 \\
\lambda v &= A_{CL}v = \left[A_{OL} - B_{OL}C_{OL}\right] v = A_{OL}v \\
\lambda v &= A_{OL}v
\end{aligned}$$

This proves that (A_{CL}, C_{CL}) is unobservable if and only if (A_{OL}, C_{OL}) is unobservable. In that respect the feedback configuration looses observability if and only if cascade configuration in the feed-forward path looses observability.