EE402 - Discrete Time Systems **Spring 2018**

Lecture 11

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11.1 Observability in CT-LTI Systems

In the context of observability of dynamical systems, it turns out that it is more convenient to think in terms of "un-observabile states" and then connect it to the concept of observability and fully observable systems. as reflected in the following definitions.

Definition: For LTI a contınıous-time state-space representation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

A state x_u is said to be unobservable over $t \in [0, T)$, if with $x(0) = x_u$ and $\forall\, u(t)$ we get the same $y(t)$ as we would with $x(0) = 0$.

The set, $\bar{\mathcal{O}}_T$, of all unobservale states over $t \in [0, T)$ forms a vecotor space, $\bar{\mathcal{O}}_T \subset \mathbb{R}^n$, and the system is called fully observable over $t \in [0, T)$, if $\dim[\bar{\mathcal{O}}_T] = 0$.

Note for linear dynamical systems observability of a state and system is independent from $u(t)$, in that respect we will only analuze zero-input response of the system in our derivations.

Theorem: $x_u \in \bar{\mathcal{O}}_T \iff x_u \in \bar{\mathcal{O}}_\tau,\, \forall \tau > 0 \iff x_u \in \mathcal{N}[\mathbf{O}]$, where

$$\mathbf{O} = \begin{bmatrix} C \\ \hline CA \\ \hline CA^2 \\ \hline \vdots \\ \hline CA^{n-1} \end{bmatrix}$$

Let's first show that $x_u \in \mathcal{N}[\mathbf{O}] \iff x_u \in \bar{\mathcal{O}}_\tau,\, \forall \tau > 0$. Let $x_u \in \mathcal{N}[\mathbf{O}]$, then

$$\mathbf{O}x_u = 0 \rightarrow \begin{array}{c} Cx_u = 0 \\ \hline CAx_u = 0 \\ \hline CA^2x_u = 0 \\ \hline \vdots \\ \hline CA^{n-1}x_u = 0 \end{array}$$

Moreover, by Cayley-Hamilton theorem, we can also conclude that $CA^l x_u = 0,\ \forall l \in \mathbf{Z}^+$. Now let's analyze the zero-input response of the system with $x(0) = x_u$

$$x(\tau) = Ce^{A\tau}x_u = 0 \Rightarrow x_u \in \bar{\mathcal{O}}_\tau$$

and indeed it is true for all $\tau \in \mathbb{R}$. Now let's show that $x_u \in \bar{\mathcal{O}}_T \Rightarrow x_u \in \mathcal{N}[\mathbf{O}]$

$$x(t) = Ce^{At}x_u = 0,\, \forall t \in [0, \tau],\, \forall \tau \in \mathbb{R} \Rightarrow x_u \in \bar{\mathcal{O}}_\tau$$

Now let's show that $x_u \in \bar{\mathcal{O}_T} \Rightarrow x_u \in \mathcal{N}[\mathbf{O}]$. If $x_u \in \bar{\mathcal{O}_T}$, then

$$x(t) = Ce^{At}x_u = 0,\, \forall t \in [0,T] \Rightarrow \begin{array}{c} x(0) = 0 \Rightarrow Cx_u = 0 \\ \left[\frac{d}{dt}x(t)\right]_{t=0} = 0 \Rightarrow CAx_u = 0 \\ \left[\frac{d^2}{dt^2}x(t)\right]_{t=0} = 0 \Rightarrow CA^2x_u = 0 \\ \vdots \\ \left[\frac{d^{n-1}}{dt^{n-1}}x(t)\right]_{t=0} = 0 \Rightarrow CA^{n-1}x_u = 0 \end{array} \Rightarrow \mathbf{O}x_u = 0 \iff x_u \in \mathcal{N}[\mathbf{O}]$$

Similar to the reachability, we show that for CT-LTI systems observability and unobservable (and observable) subspace are independent of time.

11.1.1 Observability Grammian

For a CT-LTI system, observability Grammian is defined as

$$\mathbf{Q}(t) = \int_0^t e^{A^T(t-\tau)}C^TCe^{A(t-\tau)}d\tau$$

Theorem: $\mathcal{N}[\mathbf{Q}(t)] = \mathcal{N}[\mathbf{O}]\ \forall t > 0$.

Proof: Let's first show that $\mathcal{N}[\mathbf{O}] \subset \mathcal{N}[\mathbf{Q}(t)]\ \forall t > 0$. If $x_u \in \mathcal{N}[\mathbf{O}]$, then we know that $CA^lx_u = 0,\ \forall l \in \mathbb{Z}^{\geq 0}$. Let's anlayze if x_u is in the null-space of $\mathbf{Q}(t)$

$$\mathbf{Q}(t)x_u = \int_0^t e^{A^T(t-\tau)}C^TCe^{A(t-\tau)}x_u d\tau = 0 \Rightarrow x_u \in \mathcal{N}[\mathbf{Q}(t)]\ \forall t > 0$$

Now let's show that $\mathcal{N}[\mathbf{Q}(t)] \subset \mathcal{N}[\mathbf{O}],\ \forall t > 0$. Let $x_u \in \mathcal{N}[\mathbf{Q}(t)]$, then

$$\mathbf{Q}(t)x_u = 0 \Rightarrow x_u^T\mathbf{Q}(t)x_u \iff \int_0^t x_u^Te^{A^T(t-\tau)}C^TCe^{A(t-\tau)}x_u d\tau = 0 \iff Ce^{A(t-\tau)}x_u = 0\,\forall \tau \in [0,t]$$

Then we know that

$$\begin{aligned} [Ce^{A\eta}x_u]_{\eta=0} = 0 &\Rightarrow Cx_u = 0 \\ \frac{d}{d\eta}[Ce^{A\eta}x_u]_{\eta=0} = 0 &\rightarrow CAx_u = 0 \\ \frac{d^2}{d\eta^2}[Ce^{A\eta}x_u]_{\eta=0} = 0 &\Rightarrow CA^2x_u = 0 \\ &\vdots \\ \frac{d^{n-1}}{d\eta^{n-1}}[Ce^{A\eta}x_u]_{\eta=0} = 0 &\Rightarrow CA^{n-1}x_u = 0 \\ &\Rightarrow \mathbf{O}x_u = 0 \Rightarrow x_u \in \mathcal{N}[\mathbf{O}] \Rightarrow \mathcal{N}[\mathbf{Q}(t)] \subset \mathcal{N}[\mathbf{O}]\,\forall t > 0 \end{aligned}$$