EE502 - Linear Systems Theory **Spring 2013**

Lecture 9

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9.1 Externall Input-Output Stability

9.1.1 Signal Norms

A continuous time bilateral signal is a mapping defined by $f : \mathbb{R} \mapsto \mathbb{R}^n$ (or for unilateral case $f : \mathbb{R}^{\geq 0} \mapsto \mathbb{R}^n$), whereas discrete time bilateral signal is a mapping defined by $g : \mathbb{Z} \mapsto \mathbb{R}$ (or for unilateral case $g : \mathbb{Z}^{\geq 0} \mapsto \mathbb{R}$). Graphical Examples

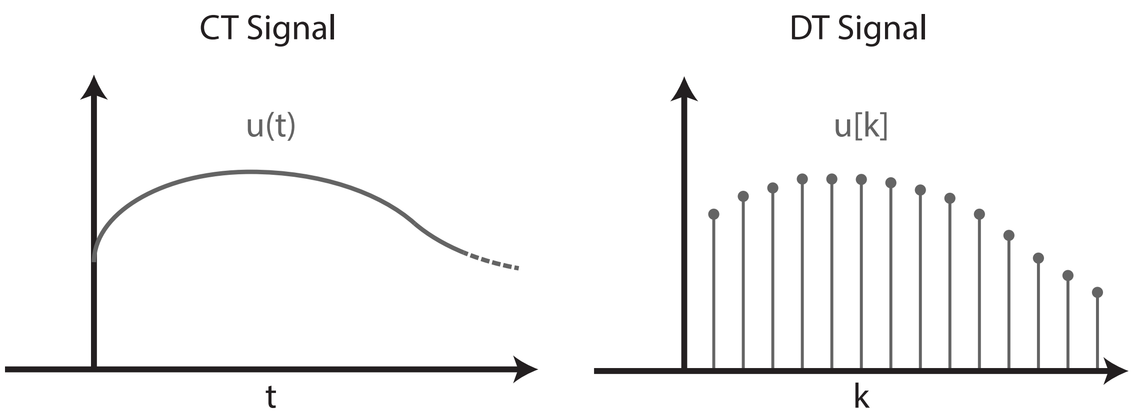

Figure 9.1: CT vs DT Signal

∞-norm

In the chacarterization and analysis of input-output stability of linear dynamical systems, most commonly used norm concept is the ∞-norm which is technically e measure of peak magnitude over time. For scalar signals ∞-norm is defined as

$$\begin{aligned} ||f||_\infty &\triangleq \sup_k |f(k)| \quad \text{(DT)} \\ &\triangleq \sup_t |f(t)| \quad \text{(CT)} \end{aligned}$$

The "sup" denotes the *supremum* or *least upper bound*, the value that is approached arbitrarily closely but never (i.e., at any finite time) exceeded. Note that this is the natural standard ∞-norm definition for finite-dimensional vectors to the infinite dimensional case, i.e. DT and CtT signals. Let's remember the ∞-norm of an n-dimensional vector,

$$||v||_\infty \triangleq \max_{i \in [1,n]} |v_i| \ , \ \text{where}\, v \in \mathbb{R}^n,$$

A scalar signal, $f(.)$ is called *bounded* if $||f||_\infty = M < \infty$ and that is the fundamental signal measure adopted in BIBO stability.

For multi-variate signals, we add a ned "dimension" in addition to the time dimension, thus in such a case we define ∞-norm as

$$\begin{aligned} ||f||_\infty &\triangleq \sup_k ||f(k)||_\infty \quad \text{(DT)} \\ &\triangleq \sup_t ||f(t)||_\infty \quad \text{(CT)} \end{aligned}$$

The space of all signals with finite ∞-norm are generally denoted by ℓ_∞ and $\mathcal{L}_\infty$ for DT and CT signals respectively. For multi-variate case, the dimension of the vector may be explicitly added as ℓ_∞^n and $\mathcal{L}_\infty^n$.

∞-norms of some example CT and DT uni-lateral signals (i.e. $t \geq 0$ and $k \geq 0$)

$$\begin{aligned} f(t) = 1\,,\ ||f||_\infty = 1 &\quad - \quad g[k] = 1\,,\ ||g||_\infty = 1 \\ f(t) = t\,,\ ||f||_\infty = \infty &\quad - \quad g[k] = k\,,\ ||g||_\infty = \infty \\ f(t) = e^t\,,\ ||f||_\infty = \infty &\quad - \quad g[k] = 2^k\,,\ ||g||_\infty = \infty \\ f(t) = 1 - e^{-t}\,,\ ||f||_\infty = 1 &\quad - \quad g[k] = 1 - 0.5^k\,,\ ||g||_\infty = 1 \\ f(t) = \delta(t)\,,\ ||f||_\infty = \infty &\quad - \quad g[k] = \delta[k]\,,\ ||g||_\infty = 1 \end{aligned}$$

2-norm

2-norm of a signal is the most fundamental measure of signal in optimal control theory and it can be considered as the square root of the "energy" of the signal. For scalar signals 2-norm is defined as

$$\begin{aligned} ||f||_2 &\triangleq \left[\sum_k (f[k])^2\right]^{\frac{1}{2}} \quad \text{(DT)} \\ &\triangleq \left[\int (f(t))^2 dt\right]^{\frac{1}{2}} \quad \text{(CT)} \end{aligned}$$

The space of all signals with finite 2-norm are generally denoted by ℓ_2 and $\mathcal{L}_2$ for DT and CT signals respectively. For multivariate signals, we adopt the inner product and obtain

$$\begin{aligned} ||f||_2 &\triangleq \left[\sum_k (f[k])^T f[k]\right]^{\frac{1}{2}} = \left[\sum_k ||f[k]||_2^2\right]^{\frac{1}{2}} \quad \text{(DT)} \\ &\triangleq \left[\int (f(t))^T f(t) dt\right] = \left[\int ||f(t)||_2^2\right]^{\frac{1}{2}} \quad \text{(CT)} \end{aligned}$$

2-norms of some example CT and DT uni-lateral signals (i.e. $t \geq 0$ and $k \geq 0$)

$$\begin{aligned} f(t) = 1\,,\ ||f||_2 = \infty &\quad - \quad g[k] = 1\,,\ ||g||_2 = \infty \\ f(t) = t\,,\ ||f||_2 = \infty &\quad - \quad g[k] = k\,,\ ||g||_2 = \infty \\ f(t) = e^t\,,\ ||f||_2 = \infty &\quad - \quad g[k] = 2^k\,,\ ||g||_2 = \infty \\ f(t) = e^{-t}\,,\ ||f||_2 = 1/\sqrt{2} &\quad - \quad g[k] = 0.5^k\,,\ ||g||_2 = 2/\sqrt{3} \\ f(t) = \delta(t)\,,\ ||f||_2 = 1 &\quad - \quad g[k] = \delta[k]\,,\ ||g||_2 = 1 \end{aligned}$$

1-norm

1-norm of a signal is refered as the "action" of the signal and for scalar signals 1-norm is defined as

$$\begin{aligned} ||f||_1 &\triangleq \left[\sum_k |f[k]|\right] \quad \text{(DT)} \\ &\triangleq \left[\int |f(t)|dt\right] \quad \text{(CT)} \end{aligned}$$

The space of all signals with finite 1-norm are generally denoted by ℓ_1 and $\mathcal{L}_1$ for DT and CT signals respectively. In order to generalize the 1-norm for multi-variate signals, we adopt the 1-norm definition of the vectors

$$\begin{aligned} ||f||_1 &\triangleq \left[\sum_k ||f[k]||_1\right] \quad \text{(DT)} \\ &\triangleq \left[\int ||f(t)||_1 dt\right] = \quad \text{(CT)} \\ ||v||_1 &\triangleq \sum_{i=1}^{n} |v_i| \ , \ \text{where}\, v \in \mathbb{R}^n, \end{aligned}$$

1-norms of some example CT and DT uni-lateral signals (i.e. $t \geq 0$ and $k \geq 0$)

$$\begin{aligned} f(t) = 1\,,\ ||f||_1 = \infty &\quad - \quad g[k] = 1\,,\ ||g||_1 = \infty \\ f(t) = t\,,\ ||f||_1 = \infty &\quad - \quad g[k] = k\,,\ ||g||_1 = \infty \\ f(t) = e^t\,,\ ||f||_1 = \infty &\quad - \quad g[k] = 2^k\,,\ ||g||_1 = \infty \\ f(t) = e^{-t}\,,\ ||f||_1 = 1 &\quad - \quad g[k] = 0.5^k\,,\ ||g||_1 = 2 \\ f(t) = \delta(t)\,,\ ||f||_1 = 1 &\quad - \quad g[k] = \delta[k]\,,\ ||g||_1 = 1 \end{aligned}$$

***p*-norm**

p-norm is technically generalization of the previous norms define in the Lecture Notes. Let $p > 0$, p-norm for vector valued signals are defined as

$$\begin{aligned} ||f||_p &\triangleq \left[\sum_k ||f[k]||_p^p\right]^{1/p} \quad \text{(DT)} \\ &\triangleq \left[\int ||f(t)||_p^p dt\right]^{1/p} = \quad \text{(CT)} \\ ||v||_p &\triangleq \left[\sum_{i=1}^{n} |v_i|^p\right]^{1/p} \ , \ \text{where}\, v \in \mathbb{R}^n, \end{aligned}$$

9.1.2 Input-Output Stability

The most important notion of input-output stability in the analysis of dynamical systems is termed ℓ_p (or $\mathcal{L}_p$) stability or p-stability.

Definition: A system with input signal, u, and output signal, y, is ℓ_p stable (or p-stable) if $\exists M \in \mathbb{R}$ such that

$$||y||_p \leq M||u||_p \, , \forall u \in \ell_p \, (\text{or}\, \mathcal{L}_p)$$

In other words a p-stable system is characterized by the requirement that for every input, u, with finite p-norm, p-norm of the output has also has to be finite.

In this context we can also define a induced system norm definition.

$$||S||_{p-ind} = \sup_{u \neq 0} \frac{||y||_p}{||u||_p}$$

BIBO stability & $\mathcal{L}_\infty$-stability for LTI Systems

If we directly follow the definition of ℓ_p stability or p-stability and signal norm definitions, we can see that BIBO stability definition is indeed a special case of p-stability where $p = \infty$. Lets first remember BIBO stability or ∞-stability for SISO LTI systems.

Theorem: Let, S_h, be a CT-LTI SISO system where $h(t)$ is its impulse response, i.e.

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

S_h is BIBO stable (or ∞-stable) if and only if $\exists M < \infty$ such that

$$||h||_1 = \int_{-\infty}^{\infty} |h(t)|dt = M$$

Proof: *Sufficiency* - Let $||h||_1 = M < \infty$ and $||u||_\infty = C < \infty$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau \right| \leq \int_{-\infty}^{\infty} |h(\tau)u(t-\tau)|d\tau \leq \int_{-\infty}^{\infty} |h(\tau)||u(t-\tau)|d\tau \leq \left[\int_{-\infty}^{\infty} |h(\tau)|d\tau \right] ||u||_\infty \, , \forall t$$

$$||y||_\infty \leq ||h||_1 ||u||_\infty \leq MC < \infty$$

Neccesity - Let $||h||_1 = \infty$, and for any t we can choose $u(t-\tau) = \text{sign}(h(\tau)$, then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)\text{sign}(h(\tau)d\tau = \int_{-\infty}^{\infty} |h(\tau)|d\tau = \infty$$

Thus a SISO CT-LTI system is BIBO stable (or ∞-stable) $\iff$ $||h||_1 = \int\limits_{-\infty}^{\infty} |h(t)|dt < \infty$. Furthermore, the least upper bound of the infinity norm of the output given that $||u||_\infty$ simply given by

$$\sup_{||u||_\infty=1} ||y||_\infty = ||h||_1 = \int\limits_{-\infty}^{\infty} |h(t)|dt$$

and this constant is is called the $\mathcal{L}_1$-norm (or $\mathcal{L}_{\infty-\mathrm{ind}}$-norm) of the system and showed as

$$||S_h||_1 = ||S_h||_{\infty-ind} = \sup_{||u||_\infty=1} ||y||_\infty = ||h||_1$$

Theorem: Let, S_g, be a DT-LTI SISO system where $g[k]$ is its impulse response, i.e.

$$y[k] = g[k] * u[k] = \sum_{n=-\infty}^{\infty} g[k-n]u[n] = \sum_{n=-\infty}^{\infty} g[n]u[k-n]$$

S_g is BIBO stable (or ∞-stable) if and only if $\exists M < \infty$ such that

$$||g||_1 = \sum_{n=-\infty}^{\infty} |g[n]| = M$$

Proof: Steps of the proof is very similar to the CT case.

Now let's generalize the BIBO stability for MIMO systems.

Theorem: Let, $S_{\mathcal{H}}$, be a CT-LTI MIMO system where $\mathcal{H}(t)$ is its impulse response matrix with m inputs and p outputs, i.e.

$$y(t) = \mathcal{H}(t) * u(t) = \int\limits_{-\infty}^{\infty} \mathcal{H}(t-\tau)u(\tau)d\tau = \int\limits_{-\infty}^{\infty} \mathcal{H}(\tau)u(t-\tau)d\tau$$

$S_{\mathcal{H}}$ is BIBO stable (or ∞-stable) if and only if $\exists M < \infty$ such that

$$\max_{i\in[1,p]} \sum ||h_{ij}||_1 = M$$

we can see that for multivariate BIBO stability all possible SISO input–output terminals has to be BIBO stability. It is enough to the sufficiency for the multi-variate case, since proof in SISO case covers necessity condition. Let $y_i(t)$ is the i^{th} output terminal then

$$\begin{aligned}
y_i(t) &= \sum_{j=1}^{m} \left[\int h_{ij}(\tau)u_j(t-\tau)d\tau \right] \\
|y_i(t)| &= \left| \sum_{j=1}^{m} \left[\int h_{ij}(\tau)u_j(t-\tau)d\tau \right] \right| \leq \sum_{j=1}^{m} \left[\int |h_{ij}(\tau)u_j(t-\tau)|\, d\tau \right] \leq \sum_{j=1}^{m} \left[\int |h_{ij}(\tau)|\, |u_j(t-\tau)|\, d\tau \right] \\
&\leq \sum_{j=1}^{m} \left[\int |h_{ij}(\tau)|\, ||u(t-\tau)||_\infty d\tau \right] \leq \sum_{j=1}^{m} \left[\int |h_{ij}(\tau)|\, d\tau \right] ||u||_\infty \leq \left[\sum_{j=1}^{m} ||h_{ij}||_1 \right] ||u||_\infty
\end{aligned}$$

Noe let's focus on all output variables

$$||y||_\infty = \sup_t ||y(t)||_\infty = \sup_t \max_{i\in[1,p]} |y_i(t)| \leq \max_{i\in[1,p]} \left[\sum_{j=1}^{m} ||h_{ij}||_1 \right] ||u||_\infty$$

thus if $\max_{i\in[1,p]}\left[\sum_{j=1}^{m}||h_{ij}||_1\right] = M < \infty$, then $||y||_\infty$ is finite for all finite $||u||_\infty$. The constant M derived in this proof is called the called the $\mathcal{L}_1$-norm (or $\mathcal{L}_{\infty-\text{ind}}$-norm) of the system and showed as

$$||S_\mathcal{H}||_1 = ||S_\mathcal{H}||_{\infty-ind} = \max_{i\in[1,p]}\left[\sum_{j=1}^{m}||h_{ij}||_1\right]$$

Theorem: Let, $S_\mathcal{G}$, be a DT-LTI MIMO system where $\mathcal{G}[k]$ is its impulse response matrix with m inputs and p outputs, i.e.

$$y[k] = \mathcal{G}[k] * u[k] = \sum_{n=-\infty}^{\infty}\mathcal{G}[k-n]u[k]$$

$S_\mathcal{G}$ is BIBO stable (or ∞-stable) if and only if $\exists M < \infty$ such that

$$||S_\mathcal{G}||_1 = ||S_\mathcal{G}||_{\infty-ind} = \max_{i\in[1,p]}\sum||h_{ij}||_1 = M$$

Steps of the proof is very similar to the CT case.

Theorem A finite-dimensional CT-LTI (or DT-LTI) system is BIBO or ∞-stable $\iff$

- poles of $\mathcal{H}(s)$ (or $\mathcal{G}(z)$ for DT case) are located in the O.L.P (or unit circle for DT systems) $\iff$
- eigenvalues associated with reachable and observable modes are located in the O.L.P (or unit circle for DT systems)

$\mathcal{L}_1$-stability of SISO LTI Systems

Let's focus on only CT SISO systems. In this part, we will attempt to find a conservative bound, $C < \infty$, such that

$$||y||_1 \leq C||u||_1\,, \forall u \in \mathcal{L}_1$$

From the ∞-stability derivation we know that

$$|y(t)| = |h(t) * u(t)| \leq \int_{-\infty}^{\infty}|h(t-\tau)||u(\tau)|d\tau$$

Mow let's analyze the $||y||_1$

$$\begin{aligned}||y||_1 &= \int_{-\infty}^{\infty}|y(t)|dt = \int_{-\infty}^{\infty}|h(t)*u(t)|dt \leq \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|h(t-\tau)||u(\tau)|d\tau\right)dt\\ &\leq \int_{-\infty}^{\infty}|u(\tau)|\left(\int_{-\infty}^{\infty}|h(t-\tau)|dt\right)d\tau = \int_{-\infty}^{\infty}|u(\tau)|\left(\int_{-\infty}^{\infty}|h(\gamma)|d\gamma\right)d\tau = ||h||_1\int_{-\infty}^{\infty}|u(\tau)d\tau\end{aligned}$$

$$||y||_1 \leq ||h||_1||u||_1$$

This implies that if $||h||_1 = M < \infty$ then system is p-stable for $p = 1$ and $C \leq M$. Indeed this is a conservative upper bound for general LTI systems, but inequality becomes equality (i.e. $||h||_1 = M = C$) for finite dimensional LTI systems. Same derivation can be easily adopted for DT-LTI SISO systems.

Indeed, we can generalize this derivation for any $p > 0$ such that

Theorem: If $||h_1|| = M < \infty$ and $||u||_p < \infty$, then $||y||_p < \infty$.

Proof of this theorem requires the adoption of Minkowski's inequalities, thus it is omitted from these lecture notes.