Lecture 6

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6.1 Modal Decomposition of State-Space Models

6.1.1 Zero Input Response

Let's consider autonomous LTI CT and DT state-space models

$$\dot{x} = Ax$$
$$x[k+1] = Ax[k]$$

Let $x_0 = \alpha v_i$, where v_i is an eigenvector of A associated with eigenvalue λ_i, we can then find the solution for both systems

$$x(t) = e^{At}x_0 = \alpha e^{\lambda_i t} v_i$$
$$x[k] = A^k x_0 = \alpha \lambda_i^k v_i$$

Now let's assume that A is diagonalizable, then we now that there exist a set of n linearly independent eigenvectors $\mathcal{V} = \{v_i \ , \ \cdots \ , \ v_n\}$. Thus, we can write any initial condition, $x_0 \in \mathbb{R}^n$, as a linear combination of eigenvectors, i.e.

$$x_0 = \sum_{i=1}^{n} \alpha_i v_i$$

if we use this (modal) decomposition we can find the solutions of DT and CT equations as

$$x(t) = e^{At}x_0 = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} v_i$$
$$x[k] = A^k x_0 = \sum_{i=1}^{n} \alpha_i \lambda_i^k v_i$$

where $e^{\lambda_i t} v_i$ ($\lambda_i^k v_i$ in DT case) is called a "mode" of the system. Now let's try to find $\{\alpha_i\ ,\ \cdots\ ,\ \alpha_n\}$ via diagonalizition of A

$$
\begin{aligned}
A = V\Lambda V^{-1} &= V \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{bmatrix} V^{-1} \text{ , where} \\
V &= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \text{ , where } Av_i = \lambda_i v_i \\
V^{-1} = \bar{V} &= \begin{bmatrix} \bar{v}_1^T \\ \vdots \\ \bar{v}_n^T \end{bmatrix} \text{ , where } \bar{v}_i^T A = \lambda_i \bar{v}_i^T \\
\bar{V}V &= V\bar{V} = I \ ,\ \bar{v}_i^T v_i = 1 \ ,\ \bar{v}_i^T v_j = 0 \text{ for } i \neq j
\end{aligned}
$$

Now let's compute the zero-input responses for an arbitrary x_0

$$
\begin{aligned}
x(t) &= e^{At}x_0 = Ve^{\Lambda t}V^{-1}x_0 = \begin{bmatrix} v_1 & \cdots v_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t}\bar{v}_1^T x_0 \\ \vdots \\ e^{\lambda_n t}\bar{v}_n^T x_0 \end{bmatrix} = \sum_{i=1}^{n} v_i e^{\lambda_i t} \left(\bar{v}_i^T x_0\right) \ \rightarrow\ \alpha_i = \bar{v}_i^T x_0 \\
x[k] &= V\Lambda^k V^{-1}x_0 = \begin{bmatrix} v_1 & \cdots v_n \end{bmatrix} \begin{bmatrix} \lambda_1^k \bar{v}_1^T x_0 \\ \vdots \\ \lambda_n^k \bar{v}_n^T x_0 \end{bmatrix} = \sum_{i=1}^{n} v_i \lambda_i^k \left(\bar{v}_i^T x_0\right) \ \rightarrow\ \alpha_i = \bar{v}_i^T x_0
\end{aligned}
$$

Based on these results, we can see that in order to excite the i^{th} mode the system, we need $\bar{v}_i^T x_0 \neq 0$. If we treat initial conditions as inputs this generates a reachability/controllability argument. Let's also analyze the output response

$$
\begin{aligned}
y(t) &= Cx(t) = Ce^{At}x_0 = \sum_{i=1}^{n} (Cv_i) e^{\lambda_i t} \left(\bar{v}_i^T x_0\right) \\
y[k] &= Cx[k] = \sum_{i=1}^{n} (Cv_i) \lambda_i^k \left(\bar{v}_i^T x_0\right)
\end{aligned}
$$

We can see that if $Cv_i = 0$, then we can not observe the i^{th} mode at the output $\forall x_0 \in \mathbb{R}^n$. Thus we can conclude that in order to have a fully observable system all modes needs to be observable, i.e. i.e. $Cv_i \neq 0\ \forall i \in \{1, \cdots, n\}$.

Now let's try to find a similar solution for systems where matrix A can not be diagonalizable. For the sake of clarity let's focus on matrices that is composed of a single Jordan block

$$
A = GJG^{-1} = G \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots \\ \vdots & & \ddots & & \\ & & & \ddots & 1 \\ 0 & & & 0 & \lambda \end{bmatrix} G^{-1}
$$

where

$$\begin{aligned}
G &= \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \\
Ag_1 &= \lambda g_1 \;\rightarrow\; (A-\lambda I)g_1 = 0 \\
Ag_2 &= \lambda g_2 + g_1 \;\rightarrow\; (A-\lambda I)g_2 = g_1 \;,\; \text{note } (A-\lambda I)^2 g_2 = 0 \;\&\; (A-\lambda I)g_2 \neq 0 \\
Ag_3 &= \lambda g_3 + g_2 \;\rightarrow\; (A-\lambda I)g_3 = g_2 \;,\; \text{note } (A-\lambda I)^3 g_3 = 0 \;\&\; (A-\lambda I)^2 g_3 \neq 0 \\
&\vdots \\
Ag_n &= \lambda g_n + g_{n-1} \;\rightarrow\; (A-\lambda I)g_n = g_{n-1} \;,\; \text{note } (A-\lambda I)^n g_n = 0 \;\&\; (A-\lambda I)^{n-1} g_n \neq 0
\end{aligned}$$

and we also know that

$$G^{-1} = \bar{G} = \begin{bmatrix} \bar{g}_1^T \\ \vdots \\ \bar{g}_n^T \end{bmatrix}$$

$$\bar{G}G = G\bar{G} = I \;,\; \bar{g}_i^T g_i = 1 \;,\; \bar{g}_i^T g_j = 0 \text{ for } i \neq j$$

Let $x_0 = \alpha_1 g_1$, i.e. the eigenvector of A, then we can find the responses as

$$\begin{aligned}
x(t) &= e^{At} g_1 = G e^{Jt} G^{-1} g_1 \alpha_1 \\
&= \begin{bmatrix} g_1 & \cdots g_n \end{bmatrix} \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda t} & \frac{t^{n-1}}{(n-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda t} \\ \vdots & & \ddots & & & \vdots \\ & & & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} \\ 0 & & \cdots & & e^{\lambda t} & te^{\lambda t} \\ 0 & & \cdots & & 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \alpha_1 e^{\lambda t} g_1
\end{aligned}$$

$$x[k] = GJ^k G^{-1} x_0 = \alpha_1 \lambda^k g_1$$

the format of the solution associated with g_1 seems to be exactly same with diagonal case (since g_1 is an eigenvector). This implies that, if we choose an initial condition inside the eigenvector space, the response stays in this space and acts like a "first-order" system. Now, let $x_0 = \alpha_2 g_2$, i.e. a first order generalized eigenvector of A, then we can find the responses as

$$\begin{aligned}
x(t) &= G e^{Jt} G^{-1} g_2 \alpha_2 \\
&= \begin{bmatrix} g_1 & g_2 & \cdots g_n \end{bmatrix} \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda t} & \frac{t^{n-1}}{(n-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda t} \\ \vdots & & \ddots & & & \vdots \\ & & & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} \\ 0 & & \cdots & & e^{\lambda t} & te^{\lambda t} \\ 0 & & \cdots & & 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_2 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} g_1 & g_2 & \cdots g_n \end{bmatrix} \begin{bmatrix} \alpha_2 te^{\lambda t} \\ \alpha_2 e^{\lambda t} \\ \vdots \\ 0 \end{bmatrix} = \alpha_2 \left(te^{\lambda t} g_1 + e^{\lambda t} g_2 \right)
\end{aligned}$$

$$
x[k] = GJ^kG^{-1}g_2\alpha_2
$$

$$
= \begin{bmatrix} g_1 & g_2 & \cdots g_n \end{bmatrix} \begin{bmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \binom{k}{n-2}\lambda^{k-n+2} & \binom{k}{n-1}\lambda^{k-n+1} \\ 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \binom{k}{n-2}\lambda^{k-n+2} \\ \vdots & & \ddots & & & \vdots \\ & & & \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} \\ 0 & & \cdots & & \lambda^k & \binom{k}{1}\lambda^{k-1} \\ 0 & & \cdots & & 0 & \lambda^k \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_2 \\ \vdots \\ 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} g_1 & g_2 & \cdots g_n \end{bmatrix} \begin{bmatrix} \alpha_2 k\lambda^{k-1} \\ \alpha_2\lambda^k \\ \vdots \\ 0 \end{bmatrix} = \alpha_2\left(k\lambda^{k-1}g_1 + \lambda^k g_2\right)
$$

We can observe that the response acts like a "second-order" (critically-damped) response. Moreover, the response does not stays inside the span of the generalized eigenvector, i.e. Span$\{g_2\}$, instead it navigates inside the span of the eigenvector and g_2, i.e. Span$\{g_1\ ,\ g_2\} = \mathcal{N}(A-\lambda I)^2$. Now, let $x_0 = \alpha_i g_i\ ,\ 0 \le i \le n$, i.e. order generalized eigenvector of order i, then we can find the responses as

$$
x(t) = Ge^{Jt}G^{-1}g_2\alpha_2 = \alpha_i e^{\lambda t}\sum_{j=1}^{i} g_j \frac{t^{i-j}}{(i-j)!}
$$

$$
x[k] = GJ^kG^{-1}g_i\alpha_i = \alpha_i \sum_{j=1}^{i} g_j \binom{k}{i-j}\lambda^{k-i+j}
$$

Similar to the second-order case, we can see that response acts like an i^{th} order dynamical system, and trajectories stays inside, Span$\{g_1\ ,\ cdots\ g_i\} = \mathcal{N}(A-\lambda I)^i$. In this context, in order to excite all higher order modes of the system projection of the initial condition to the sub-space spanned highest order generalized eigenvector has to be non-zero. Now, let's try to find general solution for an arbitrary x_0. We can write any $x_0 \in \mathbb{R}^n$ as a linear combination of $\mathcal{G} = \{g_1\ ,\ g_2\ ,\ \cdots\ ,\ g_n\}$, thus we have

$$
x_0 = \sum_{i=1}^{n} \alpha_i g_i
$$

$$
x(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda t}\sum_{j=1}^{i} g_j \frac{t^{i-j}}{(i-j)!}
$$

$$
x[k] = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{i} g_j \binom{k}{i-j}\lambda^{k-i+j}
$$

where

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = G^{-1}x_0 = \begin{bmatrix} \bar{g}_1^T \\ \bar{g}_2^T \\ \vdots \\ \bar{g}_n^T \end{bmatrix} x_0 \;\rightarrow\; \alpha_i = \bar{g}_i^T x_0$$

Based on these results, we can see that in order to excite all of the modes associated with a Jordan block of size n, we need $\alpha_n \bar{g}_n^T x_0 \neq 0$. If we treat initial conditions as inputs this generates a reachability/controllability argument. Thus in order for this Jordan block to be reachable/controllable, we need to excite highest order mode (generalized eigenvector).

Ex 6.1 *Let*

$$\dot{x} = Ax \text{ where } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

derive the solution of $x(t)$ using modal decomposition for an arbitrary $x_0 \in \mathbb{R}^3$

Solution: We know that Jordan canonical form of matrix A has the form

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the transformation matrices that leads to this Jordan form are

$$G = \begin{bmatrix} g_1 & g_2 & v \end{bmatrix},\; G = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\; G^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

where g_1 and v are eigenvectors and g_2 is the single generalized eigenvector associated with g_1. If we follow the derivation in this lecture note, we can first find all individual components of the responses as

$$\begin{aligned} x_{g_1}(t) &= \alpha_{g_1} e^t g_1 \\ x_{g_2}(t) &= \alpha_{g_2} \left(te^t g_1 + e^t g_2 \right) \\ x_v(t) &= \alpha_v e^t v \end{aligned}$$

where the combined solution and α_*'s can be derived using

$$\begin{aligned} x(t) = x_{g_1}(t) + x_{g_2}(t) + x_v(t) &= e^t \left((\alpha_{g_1} + t\alpha_{g_2}) g_1 + \alpha_{g_2} g_2 + \alpha_v v \right) \\ \begin{bmatrix} \alpha_{g_1} \\ \alpha_{g_2} \\ \alpha_v \end{bmatrix} = G^{-1}x_0 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} x_0 \end{aligned}$$

6.1.2 Zero State Response & Modal Decomposition

Let's consider input driven LTI CT and DT state-space models where $x_0 = 0$

$$x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$
$$y(t) = \int_0^t Ce^{A(t-\tau)} Bu(\tau) d\tau + Du(t)$$

$$x[k] = \left[\begin{array}{c|c|c|c|c} A^{k-1}B & A^{k-2}B & \cdots & AB & B \end{array}\right] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix} = \sum_{j=0}^{k-1} A^{k-j-1} Bu[j]$$
$$y[k] = \left[\begin{array}{c|c|c|c|c} CA^{k-1}B & CA^{k-2}B & \cdots & CAB & CB \end{array}\right] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix} = \sum_{j=0}^{k-1} CA^{k-j-1} Bu[j]$$

Now let's assume that A is diagonalizable,

$$A = V\Lambda V^{-1} = V \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{bmatrix} V^{-1} \text{ , where}$$
$$V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \text{ , where } Av_i = \lambda_i v_i$$
$$V^{-1} = \bar{V} = \begin{bmatrix} \bar{v}_1^T \\ \vdots \\ \bar{v}_n^T \end{bmatrix} \text{ , where } \bar{v}_i^T A = \lambda_i \bar{v}_i^T$$
$$\bar{V}V = V\bar{V} = I \text{ , } \bar{v}_i^T v_i = 1 \text{ , } \bar{v}_i^T v_j = 0 \text{ for } i \neq j$$

and derive the zero-state responses in modal coordinates (for CT systems first)

$$x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau = \begin{bmatrix} v_1 & \cdots v_n \end{bmatrix} \int_0^t \begin{bmatrix} e^{\lambda_1 (t-\tau)} \bar{v}_1^T \\ \vdots \\ e^{\lambda_n (t-\tau)} \bar{v}_n^T \end{bmatrix} Bu(\tau) d\tau$$

$$= \begin{bmatrix} v_1 & \cdots v_n \end{bmatrix} \begin{bmatrix} \int_0^t e^{\lambda_1 (t-\tau)} \bar{v}_1^T Bu(\tau) d\tau \\ \vdots \\ \int_0^t e^{\lambda_n (t-\tau)} \bar{v}_n^T Bu(\tau) d\tau \end{bmatrix} = \sum_{i=1}^n v_i \int_0^t e^{\lambda_i (t-\tau)} \beta_i u(\tau) d\tau \text{ where } \beta_i = \bar{v}_i^T B$$

$$= \sum_{i=1}^n v_i \beta_i \int_0^t e^{\lambda_i (t-\tau)} u(\tau) d\tau$$

$$y(t) = Cx(t) + Du(t) = \left[\sum_{i=1}^n C v_i \beta_i \int_0^t e^{\lambda_i (t-\tau)} u(\tau) d\tau \right] + Du(t)$$

$$= \left[\sum_{i=1}^n \gamma_i \beta_i \int_0^t e^{\lambda_i (t-\tau)} u(\tau) d\tau \right] + Du(t) \text{ where } \gamma_i = C v_i$$

We can see that in order to observe & excite a mode associated with λ_i, we need $\gamma_i = Cv_i \neq 0$ and $\beta_i = \bar{v}_i^T B \neq 0$ (only necessary conditions).

Ex 6.2 *Derive $x[k]$ and $y[k]$ using modal decomposition following the derivation details explained for CT systems. (Take-home example)*

Now let's try to find a similar solution for systems where matrix A can not be diagonalizable. For the sake of clarity let's focus on $A \in \mathbb{R}^{4\times 4}$ matrices that is composed of a single Jordan block

$$A = GJG^{-1} = G \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} G^{-1}$$

where

$$\begin{aligned} G &= \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \\ Ag_1 &= \lambda g_1 \;\rightarrow\; (A - \lambda I) g_1 = 0 \\ Ag_2 &= \lambda g_2 + g_1 \;\rightarrow\; (A - \lambda I) g_2 = g_1 \,,\; \text{note } (A-\lambda I)^2 g_2 = 0 \;\&\; (A-\lambda I) g_2 \neq 0 \\ Ag_3 &= \lambda g_3 + g_2 \;\rightarrow\; (A - \lambda I) g_3 = g_2 \,,\; \text{note } (A-\lambda I)^3 g_3 = 0 \;\&\; (A-\lambda I)^2 g_3 \neq 0 \\ Ag_4 &= \lambda g_4 + g_3 \;\rightarrow\; (A - \lambda I) g_4 = g_3 \,,\; \text{note } (A-\lambda I)^4 g_4 = 0 \;\&\; (A-\lambda I)^3 g_4 \neq 0 \end{aligned}$$

and we also know that

$$G^{-1} = \bar{G} = \begin{bmatrix} \bar{g}_1^T \\ \vdots \\ \bar{g}_n^T \end{bmatrix}$$

$$\bar{G}G = G\bar{G} = I \,,\; \bar{g}_i^T g_i = 1 \,,\; \bar{g}_i^T g_j = 0 \text{ for } i \neq j$$

$$
\begin{aligned}
x(t) &= \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \\
&= \begin{bmatrix} g_1 & \cdots g_n \end{bmatrix} \int_0^t \begin{bmatrix} e^{\lambda(t-\tau)} & (t-\tau)e^{\lambda(t-\tau)} & \frac{(t-\tau)^2}{2!}e^{\lambda(t-\tau)} & \frac{(t-\tau)^3}{3!}e^{\lambda(t-\tau)} \\ 0 & e^{\lambda(t-\tau)} & (t-\tau)e^{\lambda(t-\tau)} & \frac{(t-\tau)^2}{2!}e^{\lambda(t-\tau)} \\ 0 & 0 & e^{\lambda(t-\tau)} & (t-\tau)e^{\lambda(t-\tau)} \\ 0 & 0 & 0 & e^{\lambda(t-\tau)} \end{bmatrix} \begin{bmatrix} \bar{g}_1^T \\ \vdots \\ \bar{g}_4^T \end{bmatrix} B u(\tau) d\tau \\
&= \begin{bmatrix} g_1 & \cdots g_n \end{bmatrix} \int_0^t e^{\lambda(t-\tau)} \begin{bmatrix} 1 & (t-\tau) & \frac{(t-\tau)^2}{2!} & \frac{(t-\tau)^3}{3!} \\ 0 & 1 & (t-\tau) & \frac{(t-\tau)^2}{2!} \\ 0 & 0 & 1 & (t-\tau) \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_4 \end{bmatrix} u(\tau) d\tau \ , \ \beta_i = \bar{g}_i^T B \\
&= \begin{bmatrix} g_1 & \cdots g_n \end{bmatrix} \int_0^t \begin{bmatrix} \beta_1 + \beta_2(t-\tau) + \beta_3 \frac{(t-\tau)^2}{2!} + \beta_4 \frac{(t-\tau)^3}{3!} \\ \beta_2 + \beta_3(t-\tau) + \beta_4 \frac{(t-\tau)^2}{2!} \\ \beta_3 + \beta_4(t-\tau) \\ \beta_4 \end{bmatrix} e^{\lambda(t-\tau)} u(\tau) d\tau \\
&= \int_0^t g_1 \left(\beta_1 + \beta_2(t-\tau) + \beta_3 \frac{(t-\tau)^2}{2!} + \beta_4 \frac{(t-\tau)^3}{3!} \right) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&\quad + \int_0^t g_2 \left(\beta_2 + \beta_3(t-\tau) + \beta_4 \frac{(t-\tau)^2}{2!} \right) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&\quad + \int_0^t g_3 \left(\beta_3 + \beta_4(t-\tau) \right) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&\quad + \int_0^t g_4 \left(\beta_4 \right) e^{\lambda(t-\tau)} u(\tau) d\tau
\end{aligned}
$$

whereas the output equation takes the form

$$
\begin{aligned}
y(t) =& Cx(t) + Du(t) \\
=& \int_0^t (Cg_1) \left(\beta_1 + \beta_2(t-\tau) + \beta_3 \frac{(t-\tau)^2}{2!} + \beta_4 \frac{(t-\tau)^3}{3!} \right) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&+ \int_0^t (Cg_2) \left(\beta_2 + \beta_3(t-\tau) + \beta_4 \frac{(t-\tau)^2}{2!} \right) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&+ \int_0^t (Cg_3) \left(\beta_3 + \beta_4(t-\tau) \right) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&+ \int_0^t (Cg_4) \left(\beta_4 \right) e^{\lambda(t-\tau)} u(\tau) d\tau \\
&+ Du(t)
\end{aligned}
$$

We can see that in order to observe & excite all of the modes associated with λ, we need $\gamma_1 = Cg_1 \neq 0$ and $\beta_4 = \bar{g}_4^T B \neq 0$.

Ex 6.3 *Derive $x[k]$ and $y[k]$ using modal decomposition for a $A \in \mathbb{R}^{4\times 4}$ matrice that is composed of a single Jordan block following the derivation details explained for CT systems. (Take-home example)*

6.2 Zero-State Response to Fundamental Inputs & Steady-State Response

Let's first focus on single-input CT systems

$$\dot{x} = Ax + Bu \ , \ y = Cx + Du$$

Fundamental test signal for the analysis of LTI CT systems is $u(t) = e^{s_0 t}$ where $s_0 \in mathbbC$. Note that for a multi-input system, $u(t) \in mathbbC^q$ the test signal becomes $u(t) = u_0 s_0 t$ where $u_0 \in \mathbb{C}^q$ and $u_o \neq 0$. However as you may guess, in such a case we may need to explore more than one u_0.

$$\begin{aligned}
y(t) &= C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) = C\int_0^t e^{A(t-\tau)}Be^{s_o\tau}d\tau + De^{s_0 t} = C\int_0^t e^{A(t-\tau)}e^{Is_o\tau}Bd\tau + De^{s_0 t} \\
&= Ce^{At}\int_0^t e^{(s_oI-A)\tau}Bd\tau + De^{s_0 t}
\end{aligned}$$

Let's assume that s_0 is not an eigenvalue, then $\det(s_oI - A) \neq 0$, which implies that $(s_oI - A)^{-1}$ exists. Note that

$$\int e^{B\lambda}d\lambda = B^{-1}e^{B\lambda} = e^{B\lambda}B^{-1} \ , \ \text{if } \det(B) \neq 0$$

then we can write the output equation as

$$\begin{aligned}
y(t) &= Ce^{At}\left[e^{(s_oI-A)\tau}\right]_{\tau=0}^{\tau=t}(s_oI-A)^{-1}B + De^{s_0 t} = Ce^{At}\left[e^{(s_oI-A)t} - I\right](s_oI-A)^{-1}B + De^{s_0 t} \\
&= C\left[e^{Is_o t} - e^{At}\right](s_oI-A)^{-1}B + De^{s_0 t} \\
&= Ce^{At}\left[-(s_oI-A)^{-1}B\right] + \left[C(s_oI-A)^{-1}B + D\right]e^{s_0 t} \\
&= \qquad \underbrace{Ce^{At}\bar{x}_0}_{transient} \qquad + \qquad \underbrace{G(s_0)e^{s_0 t}}_{steady-state} \ , \ \text{where}
\end{aligned}$$

$$\begin{aligned}
\bar{x}_0 &= \left[-(s_oI-A)^{-1}B\right] \ , \ \text{quasi initial} - \text{condition} \\
G(s) &= C(sI-A)^{-1}B + D \ , \ \text{TF} - \text{matrix}
\end{aligned}$$

Ex 6.4 *Find $y(t)$ using this time using Laplace transform based solution*

Now let's focus on single-input DT systems

$$x[k+1] = Ax[k] + Bu[k] \ , \ y = Cx[k] + Du[k]$$

Fundamental test signal for the analysis of LTI DT systems is $u(t) = z_0^k$ where $z_0 \in \mathbb{C}$. Note that for a multi-input system, $u[k] \in \mathbb{C}^q$ the test signal becomes $u[k] = u_0 z_0^k$ where $u_0 \in \mathbb{C}^q$ and $u_o \neq 0$. However as you may guess, in such a case we may need to explore more than one u_0.

$$\begin{aligned}
y[k] &= \left[\sum_{j=0}^{k-1} CA^{k-j-1}Bu[j]\right] + Du[k] = \left[\sum_{j=0}^{k-1} CA^{j}Bu[k-j-1]\right] + Du[k] \\
&= C\left[\sum_{j=0}^{k-1} A^{j} z_0^{-j}\right] Bz^{k-1} + Dz_0^k = C\left[\sum_{j=0}^{k-1}\left(\frac{A}{z_0}\right)^j\right] z^{k-1}B + Dz_0^k
\end{aligned}$$

Let $M = \left(\frac{A}{z_0}\right)$, then

$$\begin{aligned}
\sum_{j=0}^{k-1} M^j &= I + M + M^2 + \cdots + M^{n-1} \\
M^k + \sum_{j=0}^{k-1} M^j &= \sum_{j=0}^{k} M^j = I + M\sum_{j=0}^{k-1} M^j \\
(I-M)\sum_{j=0}^{k-1} M^j &= I - M^k \\
\sum_{j=0}^{k-1} M^j &= (I-M)^{-1}\left(I-M^k\right) = \left(I-M^k\right)(I-M)^{-1}
\end{aligned}$$

we can then find $y[k]$ as

$$\begin{aligned}
y[k] &= C\left[\left(I - A^k z_0^{-k}\right)\left(I - Az_0^{-1}\right)^{-1}\right] z_0^{k-1}B + Dz_0^k = C\left[\left(z_0^k I - A^k\right)(z_0 I - A)^{-1}\right]B + Dz_0^k \\
&= CA^k\left(-(z_0I-A)^{-1}B\right) + \left(C(z_0I-A)^{-1}B + D\right)z_0^k \\
&= \underbrace{CA^k\bar{x}_0}_{transient} + \underbrace{G(z_0)z_0^k}_{steady-state}\ , \text{ where}
\end{aligned}$$

$$\begin{aligned}
\bar{x}_0 &= \left(-(z_0I-A)^{-1}B\right)\ , \text{ quasi initial} - \text{condition} \\
G(z) &= C(zI-A)^{-1}B + D\ , \text{ TF} - \text{matrix}
\end{aligned}$$

Ex 6.5 *Let*

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u\ ,\ y = \begin{bmatrix} -1 & 1 \end{bmatrix} x$$

Compute (zero-state) steady-state and transient responses for $u_1(t) = 1$ and $u_2(t) = e^{-t}$.

Solution: Let's start with SS response

$$\begin{aligned}
y_1^{ss}(t) &= \left(C(s_0I-A)^{-1}B + D\right)_{s_0=0} e^{0t} = C(-A)^{-1}B = \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 1/4 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = 1 \\
y_2^{ss}(t) &= C(-I-A)^{-1}Be^{-t} = \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \mathbf{0}\ ???
\end{aligned}$$

Now let's analyze transient response

$$\begin{aligned}
y_1^{tr}(t) &= Ce^{At}\bar{x}_0 \ , \ \text{where} \ , \ \bar{x}_0 = A^{-1}B = \begin{bmatrix} -1/2 & -1/4 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -e^{-2t} & -te^{-2t} + e^{-2t} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (2t-1)e^{-2t} \\
y_2^{tr}(t) &= Ce^{At}\bar{x}_0 \ , \ \text{where} \ , \ \bar{x}_0 = (I+A)^{-1}B = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -4 \\ -4 \end{bmatrix} = \begin{bmatrix} -e^{-2t} & -te^{-2t} + e^{-2t} \end{bmatrix} \begin{bmatrix} -4 \\ -4 \end{bmatrix} = 4te^{-2t}
\end{aligned}$$

Ex 6.6 *Now, let $u_3(t) = e^{jt}$ and compute (zero-state) steady-state responses*

Solution:

$$y_3^{ss}(t) = \Big(C\,(s_0 I - A)^{-1}\,B\Big)_{s_0=j} e^{jt} = G(j)e^{jt} = |G(j)|e^{jt+\angle[G(j)]}$$

$$G(j) = C(jI-A)^{-1}B = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} j+2 & -1 \\ 0 & j+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.4-0.2j & 0.12-0.16j \\ 0 & 0.4-0.2j \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.48-0.64j \\ 1.6-0.8j \end{bmatrix} = 1.12 - 0.16j = 1.1314\angle[-8.1301^o]$$

6.2.1 Response to Real Sinusoidal Inputs

We can write CT and DT cosine/sine signals as a sum of two complex exponential functions

$$\begin{aligned}
\cos(\omega t) &= \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t} \ , \ \sin(\omega t) = \frac{1}{2}e^{j\omega t} - \frac{1}{2}e^{-j\omega t} \ , \ \omega \in \mathbb{R} \\
\cos(\omega_d k) &= \frac{1}{2}e^{j\omega_d k} + \frac{1}{2}e^{-j\omega_d k} \ , \ \cos(\omega t) = \frac{1}{2}e^{j\omega_d k} - \frac{1}{2}e^{-j\omega t} \ , \ \omega_d \in (-\pi, \pi)
\end{aligned}$$

Using linearity (and assuming A, B, C, D are all matrices with real entires) we can find the CT response to $\cos(\omega t)$ signal as

$$\begin{aligned}
y_c(t) &= \frac{1}{2}\left[Ce^{At}\bar{x}_0^+ + G(j\omega)e^{j\omega t}\right] + \frac{1}{2}\left[Ce^{At}\bar{x}_0^- + G(-j\omega)e^{-j\omega t}\right] \ \text{where} \\
\bar{x}_0^+ &= \left[-(j\omega I - A)^{-1}B\right] \ , \ \bar{x}_0^- = \left[-(-j\omega I - A)^{-1}B\right] \ , \ \left(\bar{x}_0^+\right)^* = \bar{x}_0^- \\
G(j\omega) &= \left[C(j\omega I - A)^{-1}B + D\right] = (G(-j\omega))^*
\end{aligned}$$

$$\begin{aligned}
y_c(t) &= Ce^{At}\left[\frac{1}{2}\bar{x}_0^+ + \frac{1}{2}\bar{x}_0^-\right] + \frac{1}{2}\left[G(j\omega)e^{j\omega t}\right] + \frac{1}{2}\left[G(j\omega)e^{j\omega t}\right]^* \ \text{note} \\
G(j\omega)e^{j\omega t} &= |G(j\omega)|\left(\cos\phi + j\sin\phi\right) \ \ \phi = \angle[G(j\omega)]
\end{aligned}$$

$$y_c(t) = \underbrace{Ce^{At}\mathrm{Re}\left\{\bar{x}_0^+\right\}}_{transient} + \underbrace{|G(j\omega)|\cos\left(\omega t + \angle[G(j\omega)]\right)}_{sinusoidal\ steady-state} \ , \ \text{where}$$

$$y_s(t) = \underbrace{Ce^{At}\mathrm{Im}\left\{\bar{x}_0^+\right\}}_{transient} + \underbrace{|G(j\omega)|\sin\left(\omega t + \angle[G(j\omega)]\right)}_{sinusoidal\ steady-state}$$

Now let's apply the some process on DT systems with discrete-time cosine input signal

$$
\begin{aligned}
y_c[k] &= \frac{1}{2}\left[CA^k\bar{x}_0^+ + G(e^{j\omega_d})e^{j\omega_d k}\right] + \frac{1}{2}\left[CA^k\bar{x}_0^- + G(e^{-j\omega_d})e^{-j\omega_d k}\right] \text{ where} \\
\bar{x}_0^+ &= \left[-(e^{j\omega_d}I - A)^{-1}B\right] = \left(\bar{x}_0^-\right)^* \\
G(e^{j\omega_d}) &= \left[C(e^{j\omega_d}I - A)^{-1}B + D\right] = \left(G(e^{-j\omega_d})\right)^*
\end{aligned}
$$

$$
y_c[k] = CA^k\left[\frac{1}{2}\bar{x}_0^+ + \frac{1}{2}\bar{x}_0^-\right] + \frac{1}{2}\left[G(e^{j\omega_d k})e^{j\omega_d k}\right] + \frac{1}{2}\left[G(e^{j\omega_d k})e^{j\omega_d k}\right]^* \text{ note}
$$

$$
y_c[k] = \underbrace{CA^k\mathrm{Re}\left\{\bar{x}_0^+\right\}}_{transient} + \underbrace{|G(e^{j\omega_d k})\cos\left(\omega_d k + \angle[G(j\omega)]\right)}_{sinusoidal\ steady-state}, \text{ where}
$$

$$
y_s[k] = \underbrace{CA^k\mathrm{Im}\left\{\bar{x}_0^+\right\}}_{transient} + \underbrace{|G(e^{j\omega_d k})\sin\left(\omega_d k + \angle[G(j\omega)]\right)}_{sinusoidal\ steady-state}
$$

Ex 6.7 *Let*

$$
x[k+1] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\ ,\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x
$$

Compute (zero-state) the transient and steady-state responses for $u[k] = \cos(\omega_d k)$ for $\omega_d \in \{0, \pi/2, \pi\}$

Solution: Let's start with $\omega = 0$

$$
\begin{aligned}
\omega = 0 \ &\rightarrow\ u[k] = 1 \\
\bar{x}_0 = \bar{x}_0^+ &= \left[-(I - A)^{-1}B\right] = -\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\
y^{tr}[k] &= CA^k\bar{x}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix} A^k\begin{bmatrix} -1 \\ -1 \end{bmatrix} \ \text{,note } A^0 = I \ \&\ A^k = 0\ ,\ k > 1, \\
y^{tr}[k] &= -\delta[k] - \delta[k-1]
\end{aligned}
$$

$$
\begin{aligned}
G(e^{j0}) &= G(1) = \left[C(I - A)^{-1}B\right] = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \\
y^{ss}[k] &= 1
\end{aligned}
$$

$$
y[k] = 1 - \delta[k] - \delta[k-1]
$$

Now let's analyze the case $\omega = \pi/2$

$$\begin{aligned}
\omega = \pi/2 \;\rightarrow\; u[k] &= \cos\left(\frac{\pi}{2}k\right) \\
\bar{x}_0^+ &= \left[-(jI-A)^{-1}B\right] = -\begin{bmatrix} j & -1 \\ 0 & j \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} j & 1 \\ 0 & j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ j \end{bmatrix} \\
y^{tr}[k] &= CA^k \mathrm{Re}\left(\bar{x}_0^+\right) = \begin{bmatrix} 1 & 0 \end{bmatrix} A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ , note } A^0 = I \;\&\; A^k = 0\ ,\ k > 1, \\
y^{tr}[k] &= \delta[k-1]
\end{aligned}$$

$$\begin{aligned}
G(e^{j\pi/2}) = G(j) &= \left[C(jI-A)^{-1}B\right] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -j & -1 \\ 0 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1 \\
y^{ss}[k] &= \cos\left(\frac{\pi}{2}k + \pi\right) = -\cos\left(\frac{\pi}{2}k\right)
\end{aligned}$$

$$y[k] = -\cos\left(\frac{\pi}{2}k\right) + \delta[k-1]$$