

Lecture 9

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9.1 External Input-Output Stability

9.1.1 Signal Norms

A continuous time bilateral signal is a mapping defined by $f : \mathbb{R} \mapsto \mathbb{R}^n$ (or for unilateral case $f : \mathbb{R}^{\geq 0} \mapsto \mathbb{R}^n$), whereas discrete time bilateral signal is a mapping defined by $g : \mathbb{Z} \mapsto \mathbb{R}$ (or for unilateral case $g : \mathbb{Z}^{\geq 0} \mapsto \mathbb{R}$). Graphical Examples

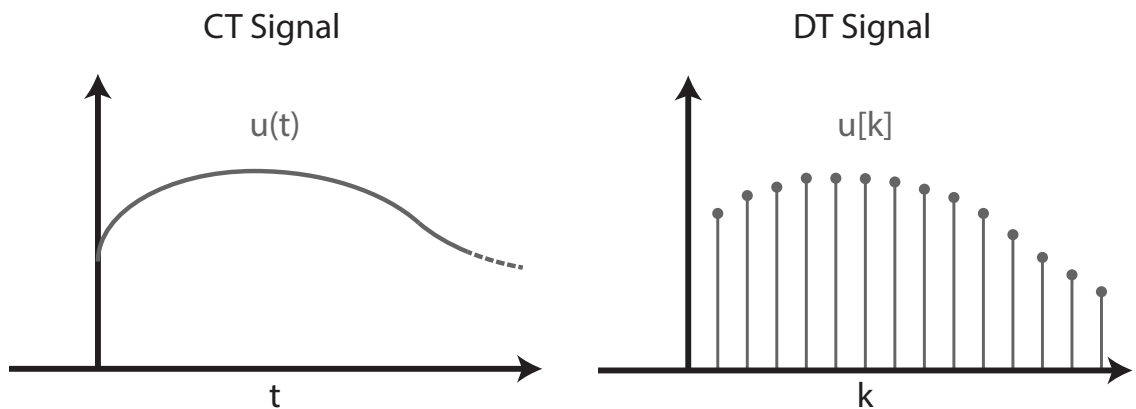


Figure 9.1: CT vs DT Signal

 ∞ -norm

In the characterization and analysis of input-output stability of linear dynamical systems, most commonly used norm concept is the ∞ -norm which is technically a measure of peak magnitude over time. For scalar signals ∞ -norm is defined as

$$\begin{aligned} \|f\|_{\infty} &\triangleq \sup_k |f(k)| \quad (\text{DT}) \\ &\triangleq \sup_t |f(t)| \quad (\text{CT}) \end{aligned}$$

The “sup” denotes the *supremum* or *least upper bound*, the value that is approached arbitrarily closely but never (i.e., at any finite time) exceeded. Note that this is the natural standard ∞ -norm definition for finite-dimensional vectors to the infinite dimensional case, i.e. DT and CtT signals. Let’s remember the ∞ -norm of an n -dimensional vector,

$$\|v\|_{\infty} \triangleq \max_{i \in [1, n]} |v_i|, \text{ where } v \in \mathbb{R}^n,$$

A scalar signal, $f(\cdot)$ is called *bounded* if $\|f\|_\infty = M < \infty$ and that is the fundamental signal measure adopted in BIBO stability.

For multi-variate signals, we add a new “dimension” in addition to the time dimension, thus in such a case we define ∞ -norm as

$$\begin{aligned}\|f\|_\infty &\triangleq \sup_k \|f(k)\|_\infty \quad (\text{DT}) \\ &\triangleq \sup_t \|f(t)\|_\infty \quad (\text{CT})\end{aligned}$$

The space of all signals with finite ∞ -norm are generally denoted by ℓ_∞ and \mathcal{L}_∞ for DT and CT signals respectively. For multi-variate case, the dimension of the vector may be explicitly added as ℓ_∞^n and \mathcal{L}_∞^n .

∞ -norms of some example CT and DT uni-lateral signals (i.e. $t \geq 0$ and $k \geq 0$)

$$\begin{aligned}f(t) = 1, \|f\|_\infty = 1 &\quad - \quad g[k] = 1, \|g\|_\infty = 1 \\ f(t) = t, \|f\|_\infty = \infty &\quad - \quad g[k] = k, \|g\|_\infty = \infty \\ f(t) = e^t, \|f\|_\infty = \infty &\quad - \quad g[k] = 2^k, \|g\|_\infty = \infty \\ f(t) = 1 - e^{-t}, \|f\|_\infty = 1 &\quad - \quad g[k] = 1 - 0.5^k, \|g\|_\infty = 1 \\ f(t) = \delta(t), \|f\|_\infty = \infty &\quad - \quad g[k] = \delta[k], \|g\|_\infty = 1\end{aligned}$$

2-norm

2-norm of a signal is the most fundamental measure of signal in optimal control theory and it can be considered as the square root of the “energy” of the signal. For scalar signals 2-norm is defined as

$$\begin{aligned}\|f\|_2 &\triangleq \left[\sum_k (f[k])^2 \right]^{\frac{1}{2}} \quad (\text{DT}) \\ &\triangleq \left[\int (f(t))^2 dt \right]^{\frac{1}{2}} \quad (\text{CT})\end{aligned}$$

The space of all signals with finite 2-norm are generally denoted by ℓ_2 and \mathcal{L}_2 for DT and CT signals respectively. For multivariate signals, we adopt the inner product and obtain

$$\begin{aligned}\|f\|_2 &\triangleq \left[\sum_k (f[k])^T f[k] \right]^{\frac{1}{2}} = \left[\sum_k \|f[k]\|_2^2 \right]^{\frac{1}{2}} \quad (\text{DT}) \\ &\triangleq \left[\int (f(t))^T f(t) dt \right]^{\frac{1}{2}} = \left[\int \|f(t)\|_2^2 \right]^{\frac{1}{2}} \quad (\text{CT})\end{aligned}$$

2-norms of some example CT and DT uni-lateral signals (i.e. $t \geq 0$ and $k \geq 0$)

$$\begin{aligned}f(t) = 1, \|f\|_2 = \infty &\quad - \quad g[k] = 1, \|g\|_2 = \infty \\ f(t) = t, \|f\|_2 = \infty &\quad - \quad g[k] = k, \|g\|_2 = \infty \\ f(t) = e^t, \|f\|_2 = \infty &\quad - \quad g[k] = 2^k, \|g\|_2 = \infty \\ f(t) = e^{-t}, \|f\|_2 = 1/\sqrt{2} &\quad - \quad g[k] = 0.5^k, \|g\|_2 = 2/\sqrt{3} \\ f(t) = \delta(t), \|f\|_2 = 1 &\quad - \quad g[k] = \delta[k], \|g\|_2 = 1\end{aligned}$$

1-norm

1-norm of a signal is referred as the “action” of the signal and for scalar signals 1-norm is defined as

$$\begin{aligned} \|f\|_1 &\triangleq \left[\sum_k |f[k]| \right] \quad (\text{DT}) \\ &\triangleq \left[\int |f(t)| dt \right] \quad (\text{CT}) \end{aligned}$$

The space of all signals with finite 1-norm are generally denoted by ℓ_1 and \mathcal{L}_1 for DT and CT signals respectively. In order to generalize the 1-norm for multi-variate signals, we adopt the 1-norm definition of the vectors

$$\begin{aligned} \|f\|_1 &\triangleq \left[\sum_k \|f[k]\|_1 \right] \quad (\text{DT}) \\ &\triangleq \left[\int \|f(t)\|_1 dt \right] = \quad (\text{CT}) \\ \|v\|_1 &\triangleq \sum_{i=1}^n |v_i|, \text{ where } v \in \mathbb{R}^n, \end{aligned}$$

1-norms of some example CT and DT uni-lateral signals (i.e. $t \geq 0$ and $k \geq 0$)

$$\begin{aligned} f(t) = 1, \|f\|_1 = \infty & \quad - \quad g[k] = 1, \|g\|_1 = \infty \\ f(t) = t, \|f\|_1 = \infty & \quad - \quad g[k] = k, \|g\|_1 = \infty \\ f(t) = e^t, \|f\|_1 = \infty & \quad - \quad g[k] = 2^k, \|g\|_1 = \infty \\ f(t) = e^{-t}, \|f\|_1 = 1 & \quad - \quad g[k] = 0.5^k, \|g\|_1 = 2 \\ f(t) = \delta(t), \|f\|_1 = 1 & \quad - \quad g[k] = \delta[k], \|g\|_1 = 1 \end{aligned}$$

p-norm

p -norm is technically generalization of the previous norms define in the Lecture Notes. Let $p > 0$, p -norm for vector valued signals are defined as

$$\begin{aligned} \|f\|_p &\triangleq \left[\sum_k \|f[k]\|_p^p \right]^{1/p} \quad (\text{DT}) \\ &\triangleq \left[\int \|f(t)\|_p^p dt \right]^{1/p} = \quad (\text{CT}) \\ \|v\|_p &\triangleq \left[\sum_{i=1}^n |v_i|^p \right]^{1/p}, \text{ where } v \in \mathbb{R}^n, \end{aligned}$$

9.1.2 Input-Output Stability

The most important notion of input-output stability in the analysis of dynamical systems is termed ℓ_p (or \mathcal{L}_p) stability or p -stability.

Definition: A system with input signal, u , and output signal, y , is ℓ_p stable (or p -stable) if $\exists M \in \mathbb{R}$ such that

$$\|y\|_p \leq M \|u\|_p, \forall u \in \ell_p \text{ (or } \mathcal{L}_p \text{)}$$

In other words a p -stable system is characterized by the requirement that for every input, u , with finite p -norm, p -norm of the output has also has to be finite.

BIBO Stability & ∞ -stability for LTI Systems

If we directly follow the definition of ℓ_p stability or p -stability and signal norm definitions, we can see that BIBO stability definition is indeed a special case of p -stability where $p = \infty$. Lets first remember BIBO stability or ∞ -stability for SISO LTI systems.

Theorem: Let, S_h , be a CT-LTI SISO system where $h(t)$ is its impulse response, i.e.

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(t - \tau)u(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)u(t - \tau)d\tau$$

S_h is BIBO stable (or ∞ -stable) if and only if $\exists M < \infty$ such that

$$\|h\|_1 = \int_{-\infty}^{\infty} |h(t)|dt = M$$

Proof: *Sufficiency* - Let $\|h\|_1 = M < \infty$ and $\|u\|_{\infty} = C < \infty$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)u(t - \tau)d\tau \\ |y(t)| &= \left| \int_{-\infty}^{\infty} h(\tau)u(t - \tau)d\tau \right| \leq \int_{-\infty}^{\infty} |h(\tau)u(t - \tau)|d\tau \leq \int_{-\infty}^{\infty} |h(\tau)||u(t - \tau)|d\tau \leq \left[\int_{-\infty}^{\infty} |h(\tau)|d\tau \right] \|u\|_{\infty}, \forall t \\ \|y\|_{\infty} &\leq \|h\|_1 \|u\|_{\infty} \leq MC < \infty \end{aligned}$$

Necessity - Let $\|h\|_1 = \infty$, and for any t we can choose $u(t - \tau) = \text{sign}(h(\tau))$, then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)\text{sign}(h(\tau))d\tau = \int_{-\infty}^{\infty} |h(\tau)|d\tau = \infty$$

Thus a SISO CT-LTI system is BIBO stable (or ∞ -stable) $\iff \|h\|_1 = \int_{-\infty}^{\infty} |h(t)|dt < \infty$. Furthermore,

the least upper bound of the infinity norm of the output given that $\|u\|_\infty$ simply given by

$$\sup_{\|u\|_\infty=1} \|y\|_\infty = \|h\|_1 = \int_{-\infty}^{\infty} |h(t)| dt$$

and this constant is called the \mathcal{L}_1 -norm (or \mathcal{L}_{∞} -ind-norm) of the system and showed as

$$\|S_h\|_1 = \sup_{\|u\|_\infty=1} \|y\|_\infty$$

Theorem: Let, S_g , be a DT-LTI SISO system where $g[k]$ is its impulse response, i.e.

$$y[k] = g[k] * u[k] = \sum_{n=-\infty}^{\infty} g[k-n]u[n] = \sum_{n=-\infty}^{\infty} g[n]u[k-n]$$

S_g is BIBO stable (or ∞ -stable) if and only if $\exists M < \infty$ such that

$$\|g\|_1 = \sum_{n=-\infty}^{\infty} |g[n]| = M$$

Proof: Steps of the proof is very similar to the CT case.