ROB501 - Fundamentals & Emerging Topics in Robotics - Digital Control Systems

Lecture 12

Lecturer: Asst. Prof. M. Mert Ankarali

12.1 Discrete-time Luenberger Observer

In general the state, $x[k]$, of a system is not accessible and *observers, estimators, filters*) have to be used to extract this information. The output, $y[k]$, represents the measurements which is a function of $x[k]$ and $u[k]$.

$$\begin{aligned} x[k+1] &= Gx[k] + Hu[k] \\ y[k] &= Cx[k] \end{aligned}$$

A Luenberger observers is built using a "simulated" model of the system and the errors caused by the mismatched initial conditions $x_0 \neq \hat{x}_0$ (or other types of perturbations) are reduced by introducing output error feedback.

Let's assume that the state vector of the simulated system is$\hat{x}[k]$, then the state space equation of this synthetic system takes the form

$$\begin{aligned} \hat{x}[k+1] &= G\hat{x}[k] + Hu[k] \\ \hat{y}[k] &= C\hat{x}[k] \end{aligned}$$

Note that since $u[k]$ is the input that is supplied by the controller, we assume that it is known apriori. If $x[0] = \hat{x}[0]$ and when there is no model mismatch or uncertainty in the system then we expect that $x[k] = \hat{x}[k]$ and $y[k] = \hat{y}[k]$ for all k. When $x[0] \neq \hat{x}[0]$, then we should observe a difference between the measured and predicted output $y[k] \neq \hat{y}[k]$. The core idea in Luenberger observer is feeding the error in the output prediction $y[k] - \hat{y}[k]$ to the simulated system via a linear feedback gain.

$$\begin{aligned} \hat{x}[k+1] &= G\hat{x}[k] + Hu[k] + L\left(y[k] - \hat{y}[k]\right) \\ \hat{y}[k] &= C\hat{x}[k] \end{aligned}$$

In order to understand how a Luenberger observer works and to choose a proper observer gain L, we define an error signal $e[k] = x[k] - \hat{x}[k]$. The dynamics w.r.t $e[k]$ can be derived as

$$\begin{aligned} e[k+1] &= x[k+1] - \hat{x}[k+1] \\ &= \left(Gx[k] + Hu[k]\right) - \left(G\hat{x}[k] + Hu[k] + L\left(y[k] - \hat{y}[k]\right)\right) \\ e[k+1] &= \left(G - LC\right) e[k] \end{aligned}$$

where $e[0] = x[0] - \hat{x}[0]$ denotes the error in the initial condition.

If the matrix $(G - LC)$ is stable then the errors in initial condition will diminish eventually. Moreover, in order to have a good observer/estimator performance the observer convergence should be sufficiently fast.

12.1.1 Observer Gain & Pole Placement

Similar to the state-feedback gain design, the fundamental principle of "pole-placement" Observer design is that we first define a desired closed-loop eigenvalue set and compute the associated desired characteristic polynomial.

$$\begin{aligned}
\mathcal{E}^* &= \{\lambda_1^*,\ \cdots,\ \lambda_n^*\} \\
p^*(z) &= (z-\lambda_1^*)\cdots(z-\lambda_n^*) \\
&= z^n + a_1^* z^{n-1} + \cdots + a_{n-1}^* z + a_n^*
\end{aligned}$$

The necessary and sufficient condition on arbitrary observer pole-placement is that the system should be fully Observable. Then, we tune L such that

$$\det\left(zI - (G - LC)\right) = p^*(z)$$

Direct Design of Observer Gain

If n is small, the most efficient method could be the direct design.

Example: Consider the following DT system

$$\begin{aligned}
x[k+1] &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k] \\
y[k] &= \begin{bmatrix} 1 & -1 \end{bmatrix} u[k]
\end{aligned}$$

Design an observer such that estimater poles are located at $\lambda_{1,2} = 0$ (Dead-beat Observer)

Solution: Desired characteristic equation can be computed as

$$p^*(z) = z^2$$

Let $L = \begin{bmatrix} l_2 \\ l_1 \end{bmatrix}$, then the characteristic equation of $(G - LC)$ can be computed as

$$\begin{aligned}
\det\left(zI - (G - LC)\right) &= \det\left(\begin{bmatrix} z - 1 + l_2 & -l_2 \\ l_1 & z - 2 - l_1 \end{bmatrix}\right) \\
&= z^2 + z(l_2 - l_1 - 3) + (l_1 - 2l_2 + 2)
\end{aligned}$$

If we match the equations

$$\begin{aligned}
l_2 - l_1 &= 3 \\
-l_1 + 2l_2 &= 2 \\
l_2 &= -1 \\
l_1 &= -4
\end{aligned}$$

Thus $L = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

12.2 Closed-Loop Observer & State-Feedback

In the state-feedback control policy the input is ideally defined by the following law

$$u[k] = -Kx[k]$$

However, as mentioned in Observer lecture, in general we don't have direct access to the all states of the system. In this case, we learnt how to design an Observer/Estimator of the states. In this respect, it is natural to assume that in a closed-loop system, the control policy that define the input should depend on the estimated states

$$u[k] = -K\hat{x}[k]$$

However the important question how this coupling affect the closed-loop behavior, and even deeper question can be even use such a policy. The advantage of LTI systems is that state-feedback gain, and observer gain can be seperatelly designed and we guarntee a stable closed-loop performance. In this section, we will analyze the coupled system Equations of motion for the closed-loop observer & state-feedback based control system is given below

$$\begin{aligned}
x[k+1] &= Gx[k] + Hu[k] \\
y[k] &= Cx[k] \\
\hat{x}[k+1] &= G\hat{x}[k] + Hu[k] + L\left(y[k] - \hat{y}[k]\right) \\
\hat{y}[k] &= C\hat{x}[k] \\
u[k] &= -K\hat{x}[k]
\end{aligned}$$

If we eliminate $u[k]$ and $\hat{y}[k]$ we obtain following dynamical representation

$$\begin{aligned}
x[k+1] &= Gx[k] - HK\hat{x}[k] \\
\hat{x}[k+1] &= G\hat{x}[k] - HK\hat{x}[k] + LC\left(x[k] - \hat{x}[k]\right) \\
y[k] &= Cx[k] - DK\hat{x}[k]
\end{aligned}$$

Now let's replace $\hat{x}[k]$ with $e[k] = x[k] - \hat{x}[k]$

$$\begin{aligned}
x[k+1] &= (G - HK)x[k] + HKe[k] \\
e[k+1] &= (G - LC)e[k] \\
y[k] &= (C - DK)x[k] + DKe[k]
\end{aligned}$$

Now let's defina a state for the whole system, $z[k] = \begin{bmatrix} x[k] \\ e[k] \end{bmatrix}$ then the state-space representation is given by

$$\begin{aligned}
z[k+1] &= \begin{bmatrix} (G - HK) & HK \\ 0_{n\times n} & (G - LC) \end{bmatrix} z[k] \\
y[k] &= \begin{bmatrix} (C - DK) & DK \end{bmatrix} z[k]
\end{aligned}$$

The system matrix is in block diagonal form and the eigenvalues of this new system matrix is find by taking the union of eigenvalues of $(G - HK)$ and eigenvalues of $(G - LC)$. Thus a seperate pole-placement can be performed for the state-feedback controller and the observer.