EE502 - Linear Systems Theory II **Spring 2032**

Lecture 8

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8.1 Internal (Lyapunov) Stability

In internal stability, we are interested in un-driven (zero-input response) part of the dynamical system and solely focus on state evolution dynamics, i.e. autonomous part of the dynamical system. CT and DT non-linear autonomous systems can simply be expressed by

$$\dot{x} = F(x,t)$$
$$x[k+1] = F(x[k]], k)$$

For non-linear systems, in order to define and analyze the stability of a dynamical system, we need to define equilibrium points (or nominal solutions), since we will technically analyze the stability around such points. An equilibrium point for CT and DT non-linear systems are defined as

$$x_e \in \mathbb{R}^n$$
$$CT : 0 = F(x_e, t) \ \forall t > t_0$$
$$DT : x_e = F(x_e, k) \ \forall k > k_0$$

Obviously if a dynamical system at time t_0 (or k_0) starts from an an equilibrium point, $x(t_0) = x_e$ (or $x[k_o] = x_e$), it will remain on the equilibrium point $\forall \ t \geq t_0$ (or $\forall \ k \geq k_0$). A non-linear system can have a single equilibrium point, $x_e \in \mathcal{E}$, card$(\mathcal{E}) = 1$, have multiple finite number of equilibria, $x_e \in \mathcal{E}$, card$(\mathcal{E}) = n_e < \infty$, or infinite number of equilibrium points, $x_e \in \mathcal{E}$, card$(\mathcal{E}) = \infty$.

Ex 8.1 *Show that for an LTI dynamical system, set of equilibrium points define a vector space. Then characterize this vector space.*

Definition: Without loss of generality, let's assume that the equilibrium point that is point of interest is located at the origin $x_e = 0$.

1. The system is called *stable in the sense of Lyapunov (s.i.s.L)* around $x_e = 0$ if it satisfies

$$\forall \epsilon > 0, \ \exists \delta_L(\epsilon) \ s.t. \ if \ ||x(t_0)|| < \delta_L \ \rightarrow ||x(t)|| < \epsilon \ \forall t \geq t_0$$

2. The system is called *asymptotically stable* around around $x_e = 0$ if it is *stable in the sense of Lyapunov (s.i.s.L)* around $x_e = 0$ and *locally attractive*, i.e.

$$\exists \delta_a \ s.t. \ if \ ||x(t_0)|| < \delta_a \ \rightarrow \lim_{t \to \infty} ||x(t)|| = 0$$

3. The system is called *exponentially stable* around around $x_e = 0$ if it is *asymptotically stable* around $x_e = 0$ and satisfies

$$\exists \delta_e > 0, \, \alpha > 0, \, \sigma > 0 \ s.t. \ if \ ||x(t_0)|| < \delta_e \ \rightarrow ||x(t)|| \leq \alpha ||x(t_0)|| e^{-\sigma t} \ \forall t \geq t_0$$

Remark: If above stability conditions are satisfied $\forall\, t_0 \in \mathbb{R}$, then we call the system around the equilibrium *uniformly s.i.s.L*, *uniformly asymptotically stable*, and *uniformly exponentially stable* respectively. The difference between uniform and non-uniform stability is (slightly) important for only time-varying non-linear systems. Thus we will not use uniform stability definition in this course.

Remark: Note that as you can see the internal stability definitions, *s.i.s.L*, *asymptotic stability*, and *exponentially stability*, are all local stability definitions defined in the neighborhood of x_e. If a stability definition holds for all initial conditions, i.e. $x(t_0) \in \mathbb{R}^n$, then use the terms *globally s.i.s.L*, *globally asymptotically stable*, and *globally exponentially stable*.