

## Lecture 9

*Lecturer: Asst. Prof. M. Mert Ankarali*

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## 9.1 Steady-State Response Analysis

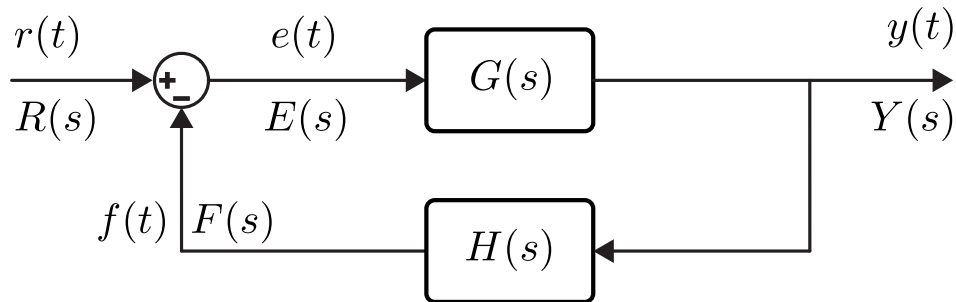
Fundamental concept that we need to perform steady-state response analysis of a control system is the final value theorem. Given a continuous time signal  $x(t)$  and its Laplace transform  $X(s)$ , if  $x(t)$  is convergent signal, final value theorem states that

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} [sX(s)]$$

$$x_{ss} = \lim_{s \rightarrow 0} [sX(s)]$$

### 9.1.1 Tracking Performance

The most important steady-state performance condition for a control system is the tracking performance under steady-state conditions. Let's consider the following fundamental feedback topology.



In order to achieve a good tracking performance, obviously the error signal  $e(t)$  need to be small. Accordingly, steady-state tracking performance is determined by the steady-state error of the closed-loop system, that we can compute using final value theorem as

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} [se(s)]$$

Let's compute  $E(s)/R(s)$ , i.e. transfer function from the reference input to the error signal,

$$E(s) = R(s) - E(s)G(s)H(s),$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)}$$

Note that  $G(s)H(s)$  is the transfer function from the error signal  $E(s)$  to the signal which is fed to the negative terminal of the main difference operator, i.e.  $F(s)$ . This transfer function is called the feed-forward or open-loop pulse transfer function of the closed-loop control system. For this system,

$$\frac{F(s)}{E(s)} = G_{OL} = G(s)H(s)$$

Then  $E(s)$  can be written as

$$E(s) = R(s) \frac{1}{1 + G_{OL}(s)}$$

It is obvious that first requirement on steady-state error performance is that closed-loop system have to be stable. Now let's analyze specific but fundamental input scenarios.

## Unit-Step Input

We know that  $r(t) = h(t)$  and  $R(s) = \frac{1}{s}$  then we have

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} \left[ sR(s) \frac{1}{1 + G_{OL}(s)} \right] \\ &= \lim_{s \rightarrow 0} \left[ s \frac{1}{s} \frac{1}{1 + G_{OL}(s)} \right] \\ e_{ss} &= \frac{1}{1 + \lim_{s \rightarrow 0} G_{OL}(s)} \end{aligned}$$

If the DC gain of the system (also called static error constant) is constant, i.e.  $\lim_{s \rightarrow 0} G_{OL}(s) = K_{DC}$  then the steady state error can be computed as

$$e_{ss} = \frac{1}{1 + K_{DC}}$$

It is obvious that

$$\begin{aligned} e_{ss} &\neq 0 \quad \text{if} \quad |K_{DC}| < \infty \\ e_{ss} &\rightarrow 0 \quad \text{if} \quad K_{DC} \rightarrow \infty \end{aligned}$$

At this point, it could be helpful to introduce the concept of system *type*, to generalize the steady-state error analysis.

**Definition:** Let's write the open-loop transfer function of a closed-loop system in the following standard form

$$G_{OL}(s) = \frac{K}{s^N} \frac{b_0 s^m + \dots + b_{m-1} s + 1}{a_0 s^n + \dots + a_{n-1} s + 1}$$

The closed-loop system is called as **Type N** system, where  $N$  is the # of integrators in the open-loop transfer function (OLTF).

Based on these results, we can have the following conclusions regarding steady-state error for unit-step input

- If  $G_{OL}(0) = K_P$ ,  $|K_P| < \infty$ , then

$$e_{ss} = 1/(1 + K_P)$$

These are **Type 0** (or **Type N**  $N \leq 0$ ) systems. We observe a bounded steady-state error and it is possible to reduce the error by increasing the static gain constant  $K_P$ .

- If  $G_{OL}(0) = \infty$ , then  $e_{ss} = 0$ . In other words, for **Type**  $N > 0$  systems, the steady-state error is perfectly zero .

Now let's summarize the steady-state error conditions

- Type  $N \leq 0$ :  $e_{ss} = \frac{1}{1+K_P}$
- Type  $N > 0$ :  $e_{ss} = 0$

## Unit-Ramp Input

We know that  $r(t) = th(t)$  and  $R(s) = \frac{1}{s^2}$  then we have

$$\begin{aligned}
 e_{ss} &= \lim_{s \rightarrow 0} \left[ sR(s) \frac{1}{1 + G_{OL}(s)} \right] \\
 &= \lim_{s \rightarrow 0} \left[ s \frac{1}{s^2} \frac{1}{1 + G_{OL}(s)} \right] \\
 &= \frac{1}{\lim_{s \rightarrow 0} s^{-1} G_{OL}(s)} \\
 e_{ss} &= \frac{1}{\lim_{s \rightarrow 0} \frac{K}{s^{N-1}} \frac{b_0 s^m + \dots + b_{m-1} s + 1}{a_0 s^n + \dots + a_{n-1} s + 1}}
 \end{aligned}$$

Based on this result we can have the following steady-state error conditions for the unit-ramp input based on the type condition of the system

- Type  $N < 1$ :  $e_{ss} \rightarrow \infty$
- Type  $N = 1$ :  $e_{ss} = \frac{1}{K_v}$
- Type  $N > 1$ :  $e_{ss} = 0$

where  $K_v$  is called the velocity error constant.

## Unit-Quadratic (Acceleration) Input

We know that  $r(t) = \frac{1}{2}t^2h(t)$  and  $R(s) = \frac{1}{s^3}$  then we have

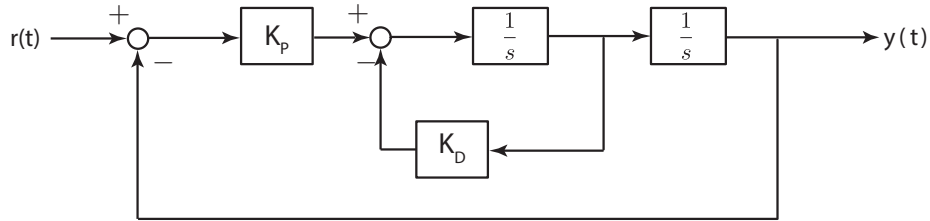
$$\begin{aligned}
 e_{ss} &= \lim_{s \rightarrow 0} \left[ sR(s) \frac{1}{1 + G_{OL}(s)} \right] \\
 &= \lim_{s \rightarrow 0} \left[ s \frac{1}{s^3} \frac{1}{1 + G_{OL}(s)} \right] \\
 &= \frac{1}{\lim_{s \rightarrow 0} s^{-2} G_{OL}(s)} \\
 e_{ss} &= \frac{1}{\lim_{s \rightarrow 0} \frac{K}{s^{N-2}} \frac{b_0 s^m + \dots + b_{m-1} s + 1}{a_0 s^n + \dots + a_{n-1} s + 1}}
 \end{aligned}$$

Based on this result we can have the following steady-state error conditions for the unit-ramp input based on the type condition of the system

- Type  $N < 2$ :  $e_{ss} \rightarrow \infty$
- Type  $N = 2$ :  $e_{ss} = \frac{1}{K_a}$
- Type  $N > 2$ :  $e_{ss} = 0$

where  $K_a$  is called the acceleration (parabolic) error constant.

**Example 1:** Compute the  $G_{OL}(s)$  for the following closed-loop system and define its **Type**. After that, compute the steady-state errors to unit-step, unit-ramp, a and unit-quadratic inputs.



**Solution:**

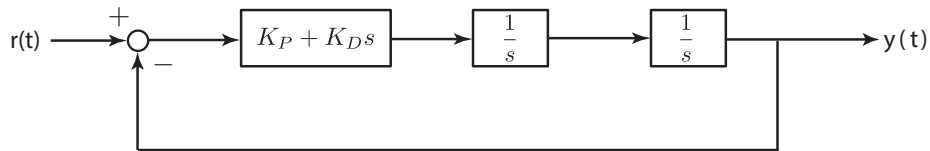
$$G_{OL}(s) = \frac{K_P}{s(s + K_D)}$$

$$\text{Type 1} \quad , \quad K_v = \frac{K_P}{K_D}$$

Then the steady-state errors are computed as

- Unit-step:  $e_{ss} = 0$
- Unit-ramp:  $e_{ss} = \frac{K_D}{K_P}$
- Unit-acceleration:  $e_{ss} = \infty$

**Example 2:** Compute the  $G_{OL}(s)$  for the following closed-loop system and define its **Type**. After that, compute the steady-state errors to unit-step, unit-ramp, a and unit-quadratic inputs.



**Solution:**

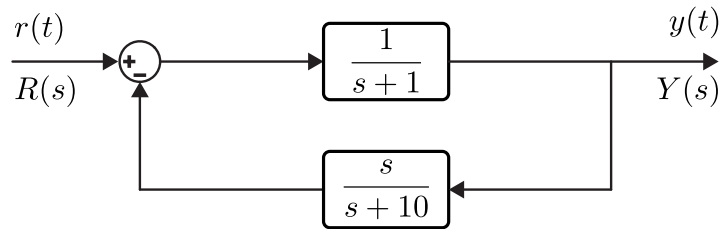
$$G_{OL}(s) = \frac{K_P + K_D s}{s^2}$$

$$\text{Type 2} \quad , \quad K_a = K_P$$

Then the steady-state errors are computed as

- Unit-step:  $e_{ss} = 0$
- Unit-ramp:  $e_{ss} = 0$
- Unit-acceleration:  $e_{ss} = \frac{1}{K_a}$

**Example 3:** Compute the  $G_{OL}(s)$  for the following closed-loop system and define its **Type**. After that, compute the steady-state errors to unit-step, unit-ramp, a and unit-quadratic inputs.



**Solution:**

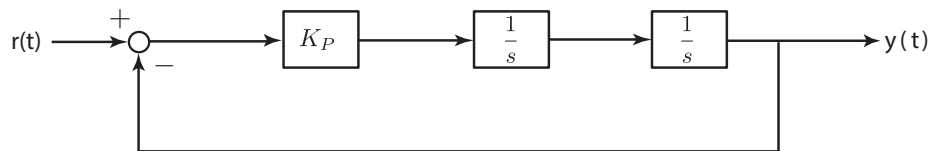
$$G_{OL}(s) = \frac{s}{(s+1)(s+10)}$$

Type **-1** ,  $K_P = 0$

Then the steady-state errors are computed as

- Unit-step:  $e_{ss} = 1$
- Unit-ramp:  $e_{ss} = \infty$
- Unit-acceleration:  $e_{ss} = \infty$

**Example 4:** Compute the steady-state error to unit-step input for the following system.



**Bad Solution:**

$$G_{OL}(s) = \frac{K_p}{s^2}$$

Type **2**

$$e_{ss} = 0 \quad \text{?????}$$

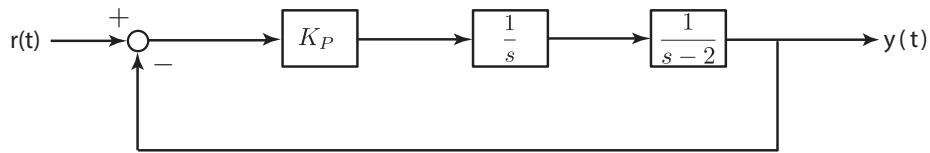
**Good Solution:** Let's compute  $Y(s)$  and then  $y(t)$ ,

$$Y(s) = \frac{\frac{K_p}{s^2}}{1 + \frac{K_p}{s^2}} R(s) = \frac{K_p}{s(s^2 + K_p)}$$

$$y(t) = 1 - \cos(Kt) \quad t > 0$$

Error function takes the form  $e(t) = \cos(Kt)$  which does not have a limit, i.e., there is no  $e_{ss}$ . If closed-loop transfer function has poles on imaginary axis then, we can not apply final value theorem.

**Example 5:** Compute the steady-state error to unit-step input for the following system when  $K_P = 1$ .



**Good Solution :)**  Let's check if  $y(t)$  is a convergent signal

$$Y(s) = \frac{\frac{1}{s(s-2)}}{1 + \frac{1}{s(s-2)}} R(s) = \frac{1}{s(s^2 - 2s + 1)}$$

$$y(t) = te^t - e^t + 1 \quad t > 0$$

Error function takes the form  $e(t) = e^t - te^t$ , thus

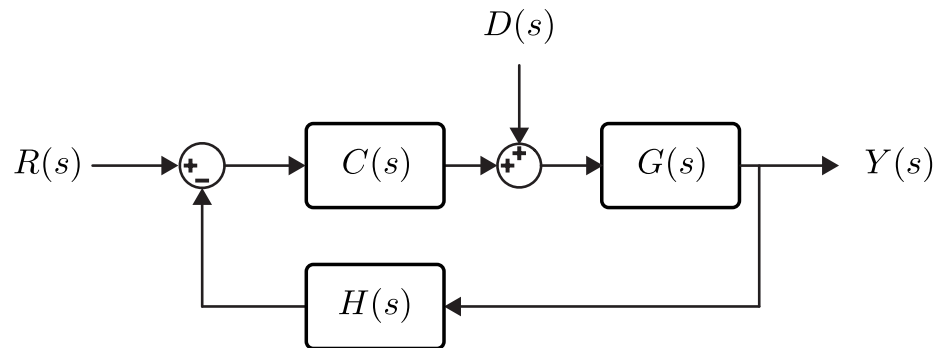
$$e_{ss} = \left| \lim_{t \rightarrow \infty} e(t) \right| = \infty$$

In conclusion, If closed-loop transfer function has poles on imaginary axis or open right half-plane then, we can not apply final value theorem.

### 9.1.2 Stead-State Response to Disturbances

When analyzing the steady-state response of a system in addition to the desired response to the reference input, it is also important to analyze the response to unwanted disturbances and noises.

Let's analyze the steady-state performance of the following topology which is perturbed by a disturbance input,  $d(t)$ .

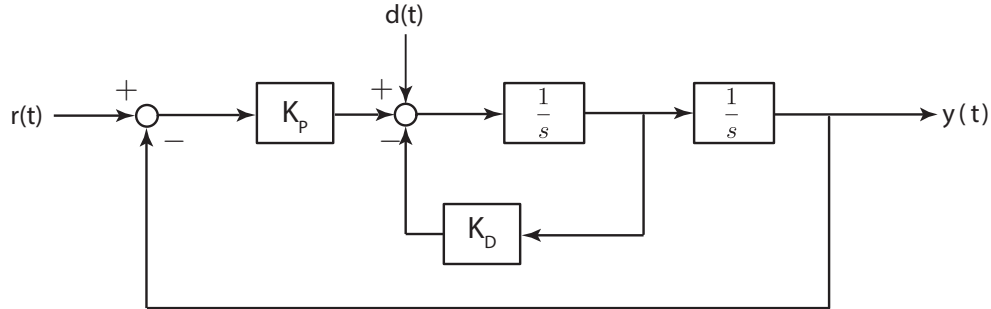


In order to analyze the response to the disturbance  $d(t)$ , we assume  $r(t) = 0$  (which is just fine due to the linearity). Let's first find the pulse transfer function from  $D(s)$  to  $Y(s)$ .

$$\begin{aligned} T_D(s) &= \frac{Y(s)}{D(s)} = \frac{G(s)}{1 + C(s)G(s)H(s)} \\ &= \frac{G(s)}{1 + G_{OL}(s)} \end{aligned}$$

Note that  $Y(s)$  depends on both  $G_{OL}(s)$  (OLTF) and  $G(s)$  (Plant TF). If one wants to generalize the steady-state disturbance rejection performance, he/she needs to analyze the conditions for both  $G_{OL}(s)$  and  $G(s)$ . Moreover, for a different topology and type of disturbance, we can have very different conditions. For this reason, in order to analyze steady-state disturbance/noise rejection performance, it is better to use fundamentals and apply final value theorem.

**Example 6:** The following closed-loop system is affected by a disturbance input  $d(t)$ . Compute the steady-state performance/response to a unit step disturbance input.



**Solution:** Lets compute  $Y(s)/D(s)$

$$Y(s) = (D(s) - Y(s)K_P) \frac{1}{s(s + K_D)}$$

$$Y(s) \left[ 1 + \frac{K_P}{s(s + K_D)} \right] = D(s) \frac{1}{s(s + K_D)}$$

$$\frac{Y(s)}{D(s)} = \frac{\frac{1}{s(s + K_D)}}{\frac{s^2 + K_D s + K_P}{s(s + K_D)}} = \frac{1}{s^2 + K_D s + K_P}$$

Now let's compute  $y_{ss}$ ,

$$y_{ss} = \lim_{s \rightarrow 0} [sY(s)] = \lim_{s \rightarrow 0} \left[ sD(s) \frac{1}{s^2 + K_D s + K_P} \right]$$

$$= \lim_{s \rightarrow 0} \left[ s \frac{1}{s} \frac{1}{s^2 + K_D s + K_P} \right]$$

$$= \frac{1}{K_P}$$

We can see that even if same system has 0 steady-state error when the reference signal is step-like input, the error under unit-step disturbance is not zero, i.e.,  $y_{ss} = 1/K_P$ . One can improve the disturbance rejection performance by increasing the  $K_P$  gain.