Constructive Approximation Theory
Assignment 2

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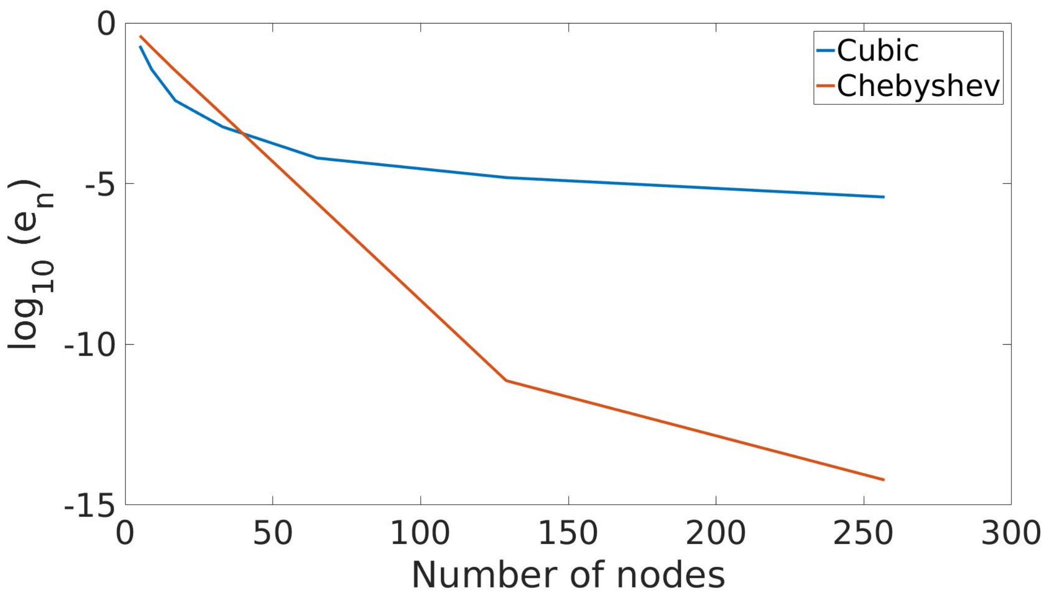
1 Approximation using MATLAB

1.1 Runge Function

Figure 1: Error comparisons for Chebyshev and Cubic Spline interpolation of Runge Function

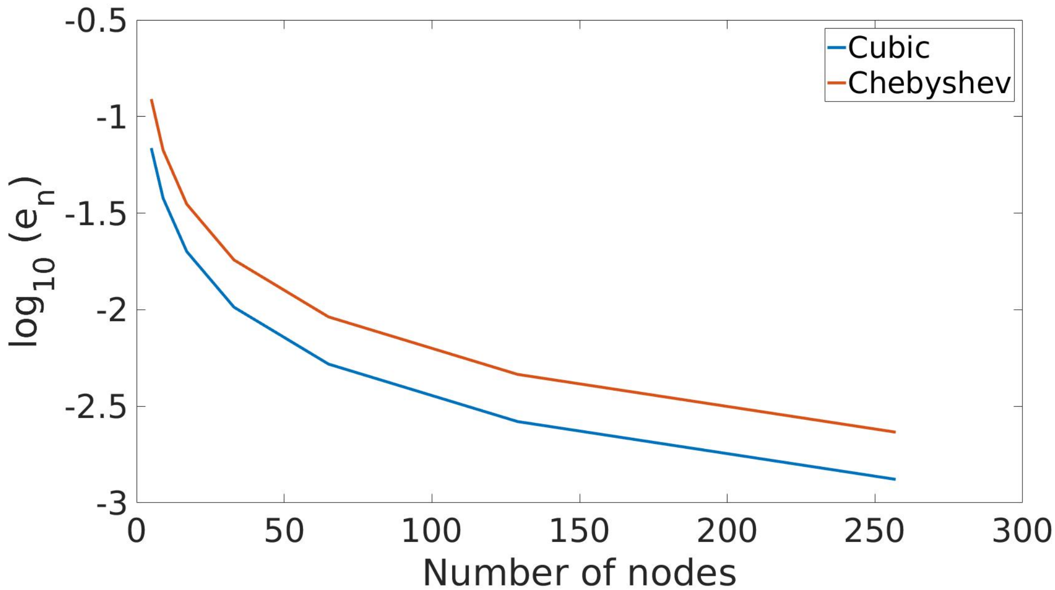


The log_{10} error plots for both the methods can be approximated to be linear, which implies that error decreases exponetially with increasing n.

While Cubic Spline gives better approximation at lesser number of nodes, Chebyshev interpolants give much better results after a certain point. Thus, for Runge function, Chebyshev interpolants converge faster.

1.2 $\mathbf{1 - |x|}$

Figure 2: Error comparisons for Chebyshev and Cubic Spline interpolation of $1 - |x|$



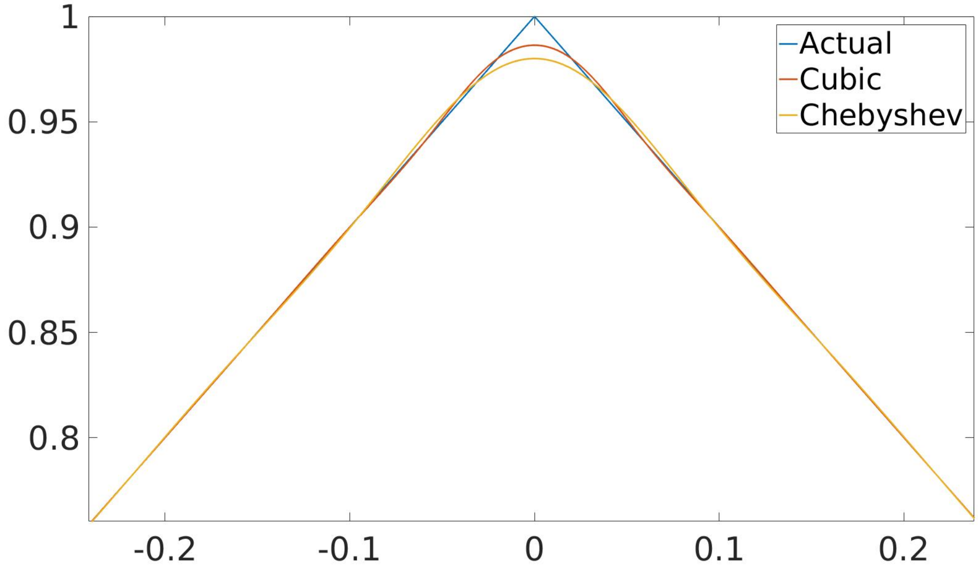
Again, the error can be considered to be exponentially decreasing.

The error behavior for both the methods seem identical, except for this function the Cubic spline consistently does better, and converges faster.

This is not surprising, considering that the function being interpolated here has non-differentiable point, a sharp turn, which gives the piecewise interpolant advantage over the global polynomial, as the former only takes local function values into consideration.

We can see in the figure below, that Chebyshev interpolant lags behind the cubic spline near the tip. It follows that, for interpolation, we need to take function properties like differentiability and various types of coninuity into account.

Figure 3: Interpolation for $n = 50$



2 Electric Potential:

- Location of charge: (Cartesian) $(0, 0, a)$. Location of point where potential is to be found: (Spherical) (r, θ, ϕ) = (Cartesian) $(rsin(\phi)cos(\theta).rsin(\phi)sin(\theta), rcos(\phi)$.
Distance between the points, $R = \sqrt{r^2 sin(\phi)^2 cos(\theta)^2 + r^2 sin(\phi)^2 sin(\theta)^2 + r^2((a/r) - cos(\phi))^2}$
$= r\sqrt{\frac{a^2}{r^2} - 2\frac{a}{r}cos(\phi) + 1}$ Potential at the given point $P = 1/R = \left(\frac{a^2}{r^2} - 2\frac{a}{r}cos(\phi) + 1\right)^{-1/2}/r$

- Let $f(x, y) = \left(1 - 2xy + x^2\right)^{-1/2}$

  Let us consider the following forumla for Binomial series expansion,
  $(1 - z)^{-1/2} = \sum_0^\infty \binom{1/2}{n} z^n$
  $= \sum_0^\infty \frac{\frac{1}{2}\frac{3}{2}\cdots\frac{2n-1}{2}}{n!} z^n$
  $= \sum_0^\infty \frac{1.3\cdots 2n-1}{2^n n!} \; \frac{2.4\cdots 2n}{2.4\cdots 2n} z^n$
  $= \sum_0^\infty \frac{2n!}{2^{2n}(n!)^2} z^n$

  Now, taking $z = 2xy - x^2$, we can write,

  $f(x, y) = \sum_0^\infty \frac{2n!}{2^{2n}(n!)^2}\left(2xy - x^2\right)^n = \sum_0^\infty \frac{2n!}{2^{2n}(n!)^2} x^n (2y - x)^n$

Now, taking the binomial series expansion of $(2y-x)^n$

$$f(x,y) = \sum_{n=0}^{\infty} \frac{2n!}{2^{2n}(n!)^2} x^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2y)^{n-k} x^k$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{2n!}{2^{n+k} n! k! (n-k)!} (y)^{n-k} x^{n+k}$$

Assuming $x < 1$ and $y \leq 1$, the series will be abolustely convergent, and we will be able to change the order of summation of the infinite series, and the total sum will be same, because on commutavity of addition.

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n (n-k)! k! (n-2k)!} (y)^{n-2k} x^n \ , \ ([n/2] = floor(n/2))$$

Now, taking $x^n, 1/2^n$ out of the inner series and multiplying $n!$ on both numerator and denominator, we get the expanded form of Rodrigue's formula for $Q_n(y)$. (Can be proven by taking binomial expansion and differentiating all terms)

$$\implies f(x,y) = \sum_{n=0}^{\infty} x^n \left(\frac{1}{2^n n!} \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{(n-2k)!} \binom{n}{k} (y)^{n-2k} \right)$$
$$= \sum_{n=0}^{\infty} x^n \left(\frac{1}{2^n n!} \frac{d^n (y^2-1)}{dx^n} \right)$$
$$= \sum_{n=0}^{\infty} x^n Q_n(y)$$

Now, taking $x = (a/r)$ and $y = cos(\phi)$,
$\implies f(a/r, cos(\phi)) = \sum_{n=0}^{\infty} (a/r)^n Q_n(cos(\phi))$

Our assumption from earlier holds as $x = a/r < 1$ and $y = cos(\phi) \leq 1$, and thus the result is valid.

The potential, $P = f(a/r, cos(\phi))/r = \frac{1}{r} \sum_{n=0}^{\infty} (a/r)^n Q_n(cos(\phi))$

- Let p_k be the expansion in finite k terms that gives us the required precision, and f be the infinite summation. So,
  $|f - p_k| = |\sum_{n=k+1}^{\infty} (a/r)^n Q_n(cos(\phi))| \approx |(a/r)^{k+1} Q_(k+1)(cos(\phi))|$, as the series is converging.

  $|f - p_k| \leq 2^{-53}$
  $\implies |(a/r)^{k+1} Q_(k+1)(cos(\phi))| \leq 2^{-53}$

  As $|Q_n(x)| \leq 1, |x| \leq 1$
  $\implies |Q_n(cos(\phi))| \leq 1$

Taking
$max\ |(a/r)^{k+1} Q_(k+1)(cos(\phi))| \leq 2^{-53}$
$\implies (1/2)^{k+1} \leq 2^{-53}$
$\implies k+1 = 53 \implies k = 52$

Thus we need 52 terms for the absolute error to be no greater than 2^{-53}

3 To find:

$p \in P_n$ that minimizes:

$$\int_{-1}^{1} \left(x^{n+1} - p(x)\right)^2 dx$$

Solution:

Let $p(x) = x^{n+1} - (q_{n+1}(x) + \Sigma_{r=0}^{n} a_r q_r(x))$

Where $q_r, 0 \leq r \leq n+1$ are Legendre polynomials scaled such that their leading coefficients are 1. We need to find coefficients a_r, for $0 \leq r \leq n$

We know that $q_{n+1}(x) = x^{n+1} + k_n(x)$, where $k_n(x) \in P_n$. Thus, $p(x) = k_n(x) - \Sigma_{r=0}^{n} a_r q_r(x) \implies p(x) \in P_n$

Now, let $E(a_0, a_1 ..., a_n) = \int_{-1}^{1} \left(x^{n+1} - p(x)\right)^2 dx = \int_{-1}^{1} \left(q_{n+1}(x) + \Sigma_{r=0}^{n} a_r q_r(x)\right)^2 dx$

Let, $\dfrac{\partial E}{\partial a_s} = 0,\ 0 \leq s \leq n$
$\implies \int_{-1}^{1} 2\left(q_{n+1}(x) + \Sigma_{r=0}^{n} a_r q_r(x)\right)(q_s(x))\, dx = 0$
$\implies \int_{-1}^{1} \left(q_{n+1}(x) q_s(x) + \Sigma_{r=0}^{n} a_r q_r(x) q_s(x)\right) dx = 0$
$\implies \cancel{\int_{-1}^{1} \left(q_{n+1(x)} q_s(x) dx\right)} + \int_{-1}^{1} \left(\Sigma_{r=0}^{n} a_r q_r(x) q_s(x)\right) dx = 0$
$\implies a_s \int_{-1}^{1} q_s(x)^2 dx = 0$
$q_s(x)^2$ will always be a non-zero even function, symmetric around $x = 0$, therefore,
$\int_{-1}^{1} q_s(x)^2 dx > 0$
$\implies a_s = 0$ for $0 \leq s \leq n$
$\implies p(x) = x^{n+1} - q_{n+1}(x)$ is the polynomial in P_n that will give least 2-norm error for approximating x^{n+1}.