Constructive Approximation Theory
Assignment 7

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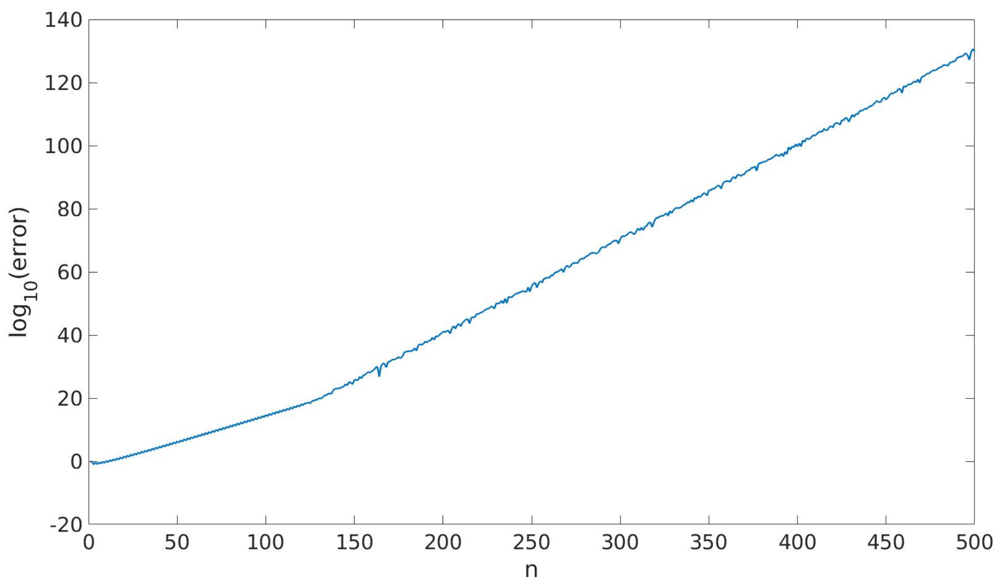
March 29, 2017

1 Uniform node interpolation

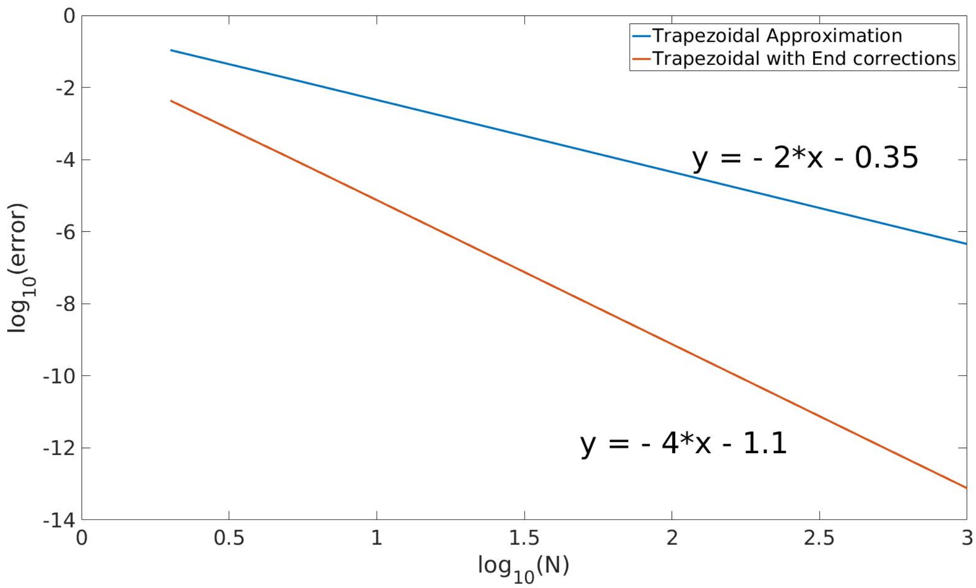
Figure 1: Error at $-1+\dfrac{1}{n}$ for uniform node Langrangian approximation of Runge function in $[-1,1]$



The plot with logarithmic scaling on y-axis is roughly linear, therefore, the error grows exponentially, which matches with out analysis of Lebesgue constant for uniform node interpolation.

2 Trapezoidal approximation of integrals

Figure 2: Error for Trapezoidal approximation with and without end corrections for different number of panels, N



The equations of the lines were found using MATLAB curve fitting tool.

The plot is scaled logarithmically on both x-axis and y-axis.

We know that, for Trapezoidal approximation, globally,

$$error = \mathcal{O}(h^2) = \mathcal{O}(N^{-2})$$

$$\implies \log(error) = -2N + C$$

Similarly, for Trapezoidal approximation using end corrections, globally

$$error = \mathcal{O}(h^4) \implies \log(error) = -4N + C$$

Thus, we can see from the slopes of the plots that the theory is validated.

3 Approximation of $\log(n!)$

3.1 Euler Maclaurin's formula

$$\sum_{k=a+1}^{b} f(k) = \int_a^b f(x)dx + \sum_{k=1}^{m} \frac{b_k}{k!} \left(f^{k-1}(b) - f^{k-1}(a)\right) + \frac{(-1)^{m+1}}{m!} \int_{a^+}^{b^-} B_m(\{t\}) f^{(m)}(t)dt$$

Where,

$b_k = k^{th}$ Bernoulli number, with $b_1 = \dfrac{1}{2}$

$B_m(y) =$ The m^{th} Bernoulli polynomial

$B_m(\{t\}) =$ Bernoulli periodic function with $\{t\} = t - floor(t)$

3.2 $\log(n!)$

$$\log(n!) = \log(2) + \log(3) \cdots \log(n) = \sum_{k=2}^{n} \log(k)$$

Applying the Euler-Maclaurin's formula, with $m = 2$, $a = 1$, and $b = n$, we have

$$\log(n!) = \int_1^n \log(x)dx + \frac{b_1}{1!}\left(\log(n) - \cancel{\log(1)}\right) + \frac{b_2}{2!}\left(\frac{1}{n} - 1\right) + \int_{1^+}^{n^-} \frac{B_2(\{t\})}{t^2}dt$$

$$\implies \log(n!) = n(\log(n) - 1) + \frac{\log(n)}{2} + \frac{1}{12n} - \frac{1}{12} + \int_{1^+}^{n^-} \frac{B_2(\{t\})}{t^2}dt$$

$$\implies \log(n!) = n\log(n) - n + \frac{\log(n)}{2} + \frac{1}{12n} - \frac{1}{12} + \int_{1^+}^{n^-} \frac{B_2(\{t\})}{t^2}dt$$

$$\implies \log(n!) = \log(n^n) + \log(e^{-n}) + \log(\sqrt{n}) + \frac{1}{12n} - \frac{1}{12} + \int_{1^+}^{n^-} \frac{B_2(\{t\})}{t^2}dt$$

$$\implies \log(n!) = \log\left(\left(\frac{n}{e}\right)^n \sqrt{n}\right) + \frac{1}{12n} - \frac{1}{12} + \int_{1^+}^{n^-} \frac{B_2(\{t\})}{t^2}dt$$

Now,

$$\int_{1^+}^{\infty} \frac{B_2(\{t\})}{t^2}dt$$

converges to some constant, because $B_2(\{t\})$ is periodic, and therefore bounded, and

$$\int_{1^+}^{\infty} \frac{1}{t^2}dt \text{ converges } \implies \int_{1^+}^{\infty} \frac{B_2(\{t\})}{t^2}dt < M\int_{1^+}^{\infty} \frac{1}{t^2}dt \text{ for } M = \max(B_2(\{t\}))$$

So, we can write,

$$\int_{1^+}^{n-} \frac{B_2(\{t\})}{t^2}dt = \int_{1^+}^{\infty} \frac{B_2(\{t\})}{t^2}dt - \int_{n}^{\infty} \frac{B_2(\{t\})}{t^2}dt$$

Therefore,

$$\log(n!) = \log\left(\left(\frac{n}{e}\right)^n \sqrt{n}\right) + \log(C) + \frac{1}{12n} - \int_n^\infty \frac{B_2(\{t\})}{t^2} dt$$

Where,

$$\log(C) = \int_{1^+}^\infty \frac{B_2(\{t\})}{t^2} dt - \frac{1}{12}$$

Now,

$$\frac{1}{12n} - \int_n^\infty \frac{B_2(\{t\})}{t^2} dt < \frac{1}{12n} - M\int_n^\infty \frac{1}{t^2} dt = \frac{1}{12n} + \frac{M}{n}$$

$$\implies \frac{1}{12n} - \int_n^\infty \frac{B_2(\{t\})}{t^2} dt = \mathcal{O}(\frac{1}{n})$$

Thus,

$$\log(n!) = \log\left(C\left(\frac{n}{e}\right)^n \sqrt{n}\right) + \mathcal{O}(\frac{1}{n})$$

4 $C = \sqrt{2\pi}$

We can write,

$$\lim_{n\to\infty} \log(n!) = \lim_{n\to\infty} \log\left(C\left(\frac{n}{e}\right)^n \sqrt{n}\right) + \cancel{\mathcal{O}(\frac{1}{n})}$$

$$\lim_{n\to\infty} n! = \lim_{n\to\infty} C\left(\frac{n}{e}\right)^n \sqrt{n}$$

We know that,

$$\lim_{n\to\infty} \binom{2n}{n} = \lim_{n\to\infty} \frac{4^n}{\sqrt{\pi n}}$$

$$\lim_{n\to\infty} \frac{(2n!)\sqrt{\pi n}}{n!n!4^n} = 1$$

Replacing the approximation of $n!$

$$\lim_{n\to\infty} \frac{\sqrt{2}\cancel{C}\cancel{2^{2n}}\cancel{\left(\frac{n}{e}\right)^{2n}}\cancel{\sqrt{n}}\sqrt{\pi\cancel{n}}}{\cancel{C}\cancel{\left(\frac{n}{e}\right)^n}\cancel{\sqrt{n}}C\cancel{\left(\frac{n}{e}\right)^n}\cancel{\sqrt{n}}\cancel{4^n}} = 1$$

$$\implies C = \sqrt{2\pi}$$

Thus, we can conclude Sterling's formula,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

5 $\mathcal{O}(\frac{1}{n^3})$ approximation of $log(n!)$

$$\log(n!) = \log(2) + \log(3)\cdots\log(n) = \sum_{k=2}^{n}\log(k)$$

Applying the Euler-Maclaurin's formula, with $m = 4$, $a = 1$, and $b = n$, we have

$$\log(n!) = \int_1^n \log(x)dx + \frac{b_1}{1!}\left(\log(n) - \cancel{\log(1)}\right) + \frac{b_2}{2!}\left(\frac{1}{n} - 1\right) +$$
$$\frac{b_3}{3!}\left(1 - \frac{1}{n^2}\right) + 2\frac{b_4}{4!}\left(\frac{1}{n^3} - 1\right) + 3\int_{1^+}^{n^-}\frac{B_4(\{t\})}{t^4}dt$$

$$b_1 = 1, b_2 = \frac{1}{6}, b_3 = 0, b_4 = -\frac{1}{30}$$

$$\implies \log(n!) = n\log(n) - n + \frac{\log(n)}{2} + \frac{1}{12n} - \frac{1}{12} + \frac{1}{420} - \frac{1}{420n^3} + 3\int_{1^+}^{n^-}\frac{B_4(\{t\})}{t^4}dt$$

Similar to how we proved in Section 3, we can show that

$$-\frac{1}{420n^3} + 3\int_n^{\infty}\frac{B_4(\{t\})}{t^4}dt = \mathcal{O}(\frac{1}{n^3})$$

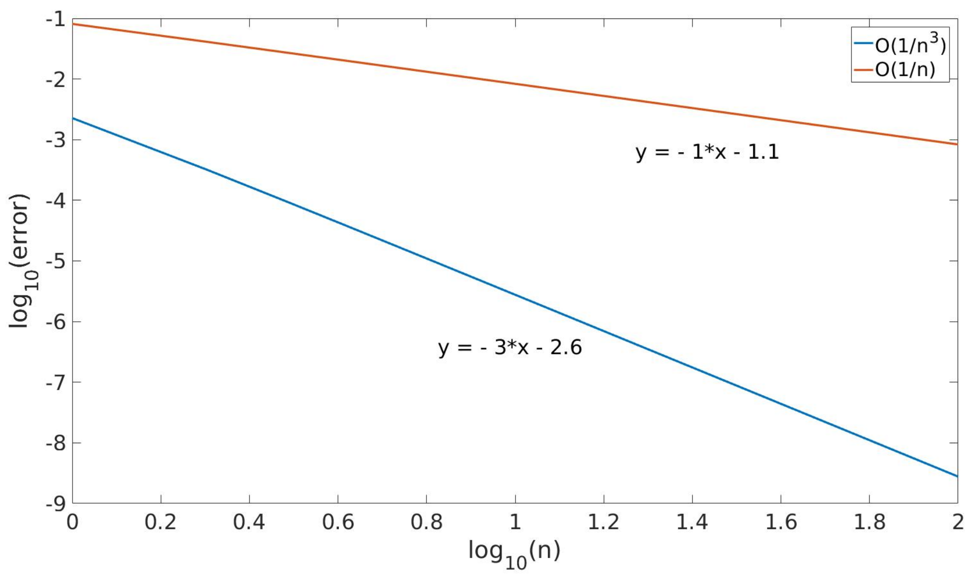
And,

$$\log(n!) = \log(n^n) + \log(e^{-n}) + \log(\sqrt{n}) + \log(e^{\frac{1}{12n}}) + \log(C) + \mathcal{O}(\frac{1}{n^3})$$

$$\implies \log(n!) = \log\left(C\left(n^{\left(n+\frac{1}{2}\right)}e^{\left(-n+\frac{1}{12n}\right)}\right)\right) + \mathcal{O}(\frac{1}{n^3})$$

Using MATLAB, we have plotted the error for both our approximations for $\log(n!)$. Similar to Section 2, the plots are log-log.

Figure 3: Error = $\log(\log(n!) - f(n))$ from both the approximations of $\log(n!)$ we have derived.



From the slopes of the plots, we have verified that indeed our appromixations are $\mathcal{O}(\frac{1}{n})$ and $\mathcal{O}(\frac{1}{n^3})$.