Constructive Approximation Theory
Assignment 5

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1 Equioscillation Theorem

- **Given:**
  Let $f \in C[-1, 1]$, and $p(x) \in P_n$ be a polynomial such that $f - p$ equioscillates at $n+2$ points.
- **To prove:**
  p minimizes $||f - p||_\infty$.
- **Proof by contradiction:**
  Assumption: p does not minimize $||f - p||_\infty$. Let $q(x) \in P_n$ be a polynomial such that $f - p - q$ minimizes $||f - p||_\infty$.

  Now, $f - p$ equisoscillates at $n + 2$ points, i.e. it has $n + 2$ peaks and on both sides of the vertical axis. If $f - p - q$ minimizes $||f - p||_\infty$, q must decrease each of these peaks (the maximum absolute value) of $f - p$, at $n + 2$ points.

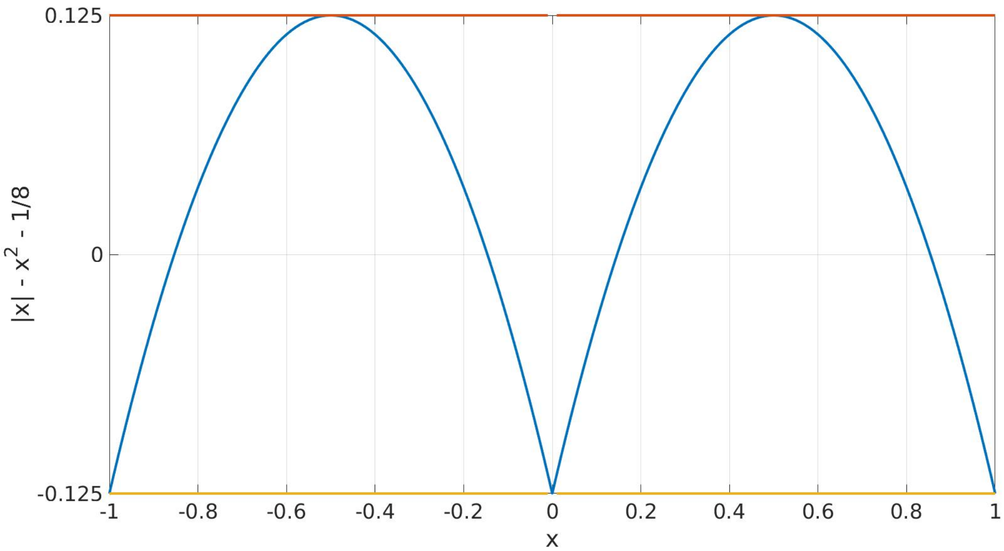
  As $f - p$ is changing signs between every peak, this must mean q must change its sign between these $n+2$ peaks, at least $n+1$ times. Therefore q must have at least $n+1$ roots. But $q \in P_n$ so this is not possible, therefore we have a contradiction, which follows directly from our assumption. Hence, our assumption is false, and p minimizes $||f - p||_\infty$.

2 $f(x) = |x|$

- $p(x) = x^2 + \dfrac{1}{8}$

  Plotting $f - p$, we see that, it equioscillates at 5 points, with peak absolute value of 0.125, at $-1, -1/2, 0, 1/2, 1$. Therefore by equioscillation theorem, p is the polynomial in P_3, that minimizes $||f - p||_\infty$.

Figure 1: $|f-p| = |x| - x^2 - \dfrac{1}{8}$ euioscillates at 5 points.



Therefore, we can conclude, $||f-p||_\infty = 0.125$

- - $p(x) = x^2 + \dfrac{1}{8}$
  - $p_L(x)$ = Polynomial obtained from Langrangian interpolation over 4 Legendre nodes.
  - $p_C(x)$ = Polynomial obtained from Langrangian interpolation over 4 Chebyshev nodes.
  - $\tilde{p}_L(x)$ = Polynomial obtained by truncating Legendre series upto fourth term.
  - $\tilde{p}_C(x)$ = Polynomial obtained by truncating Chebyshev series upto fourth term.
- For coefficient of the r^{th} Chebyshev polynomial in the Chebyshev series, we used the following formula

$$a_r = \frac{2}{\pi}\int_0^\pi f(\cos\theta)\cos(r\theta)d\theta$$

For the coefficients of the Legendre series, we used

$$a_r = \frac{2r+1}{2}\int_{-1}^1 f(x)Q_r(x)dx$$

We used trapezoidal numerical intergation method to evaluate the above integrals, with a step size of 0.001.

- We tabulate the errors for two different norms below.

Table 1: Error for different approximations for different norms

Approximation	$\|\|.\|\|_2$	$\|\|.\|\|_\infty$
$f(x) - p(x)$	0.1208	0.125
$f(x) - p_L(x)$	0.1152	0.2437
$f(x) - p_C(x)$	0.1304	0.2706
$f(x) - \tilde{p}_L(x)$	0.1021	0.1875
$f(x) - \tilde{p}_C(x)$	0.1090	0.2124

- From the table we can confirm that indeed p gives the lowest error in ∞-norm. The $||.||_\infty$ norm is always greater than the $||.||_2$ norm, which follows from theory. Series truncations give us better results than Langrangian interpolation on nodes.

- Legendre series trunction gives us the best $2-norm$ error and the $p(x) = x^2 + \dfrac{1}{8}$ gives us the best $\infty - norm$ error. Both of these follow from theory.

- Highest to lowest errors for $||.||_2$:

$$||f(x) - p_C(x)||_2 > ||f(x) - p(x)||_2 > ||f(x) - p_L(x)||_2 > ||f(x) - \tilde{p}_C(x)||_2 > ||f(x) - \tilde{p}_L(x)||_2$$

- Highest to lowest errors for $||.||_\infty$:

$$||f(x) - p_C(x)||_\infty > ||f(x) - p_L(x)||_\infty > ||f(x) - \tilde{p}_C(x)||_\infty > \tilde{p}_L(x)||_\infty > ||f(x) - p(x)||_\infty$$

3 Weistrass Approximation Theorem

3.1 Asymptotics of the Binomial Coefficient

- $I_k = \displaystyle\int_0^{\frac{\pi}{2}} \sin^k(x)dx$ Using integration by parts,

$$I_k = \cancel{-\sin^{k-1}(x)\cos(x)\Big|_0^{\frac{\pi}{2}}} - (k-1)\int_0^{\frac{\pi}{2}} \sin^{k-2}(x)\cos(x)(-\cos(x))dx$$
$$= (k-1)\int_0^{\frac{\pi}{2}} \sin^{k-2}(x)(1-\sin^2(x))dx$$
$$= (k-1)\int_0^{\frac{\pi}{2}} \sin^{k-2}(x)dx - (k-1)\int_0^{\frac{\pi}{2}} \sin^{k}(x)dx$$
$$\implies I_k = (k-1)I_{k-2} - (k-1)I_k$$
$$\implies I_k = \frac{k-1}{k} I_{k-2}$$

Now, if $k = 2m$

$$I_{2m} = \frac{2m-1}{2m} I_{2(m-1)}$$

Taking $I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$, we can solve the recursion as

$$I_{2m} = \frac{\pi}{2}\prod_{j=1}^{m}\frac{2j-1}{2j} = \frac{\pi}{2}\prod_{j=1}^{m}\frac{(2j-1)(2j)}{(2j)(2j)} = \frac{\pi}{2}\prod_{j=1}^{m}\frac{(2j-1)(2j)}{(2^2)(j)(j)}$$

$$\implies I_{2m} = \frac{\pi}{2}\frac{2m!}{2^{2m}m!m!} = \frac{\pi}{2^{2m+1}}\binom{2m}{m}$$

If $k = 2m+1$

$$I_{2m+1} = \frac{2m}{2m+1}I_{2m-1}$$

Taking $I_1 = \int_0^{\frac{\pi}{2}} sin(x)dx = 1$, we can solve the recursion as

$$I_{2m+1} = \prod_{j=1}^{m}\frac{2j}{2j+1} = \prod_{j=1}^{m}\frac{(2j)(2j)}{(2j)(2j+1)} = \frac{2^{2m}m!m!}{(2m+1)!}$$

$$= \frac{2^{2m}}{(2m+1)\binom{2m}{m}}$$

- $0 \leq \sin(x) \leq 1$ for $x \in [0, \pi/2]$

  Let a function $f(x) \in [0,1]$, for $x \in [0, \pi/2]$

  We can write,

  $\sin(x)f(x) \leq f(x)$ for all $x \in [0, \pi/2]$

  $\implies \sin(x)\sin^k(x) \leq \sin^k(x)$ for all $x \in [0, \pi/2]$

  $\implies \sin^{k+1}(x) \leq \sin^k(x)$ for all $x \in [0, \pi/2]$

  $\sin^{k+1}(x) = \sin^k(x)$ only at the endpoints of the closed interval, so we can write

  $\sin^{k+1}(x) < \sin^k(x)$ for all $x \in (0, \pi/2)$

  $$\implies \int_0^{\frac{\pi}{2}} \sin^{k+1}(x)dx < \int_0^{\frac{\pi}{2}} \sin^k(x)dx$$

  $\implies I_{k+1} < I_k$

  $\implies I_{2m+1} < I_{2m} < I_{2m-1}$

- Now, we know that, $I_{2m+1} = \frac{2m}{2m+1}I_{2m-1}$

$$\implies \lim_{m\to\infty} \frac{I_{2m-1}}{I_{2m+1}} = \lim_{m\to\infty} \frac{2m+1}{2m} = \lim_{m\to\infty} \frac{2+\cancel{\frac{1}{m}}}{2} = 1$$

- We know that, $I_{2m+1} < I_{2m} < I_{2m-1}$

$$\implies \lim_{m\to\infty} I_{2m+1} \leq I_{2m} \leq I_{2m-1}$$

$$\implies \lim_{m\to\infty} \frac{I_{2m+1}}{I_{2m+1}} \leq \frac{I_{2m}}{I_{2m+1}} \leq \frac{I_{2m-1}}{I_{2m+1}}$$

$$\implies \lim_{m\to\infty} 1 \leq \frac{I_{2m}}{I_{2m+1}} \leq 1$$

$$\implies \lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} = 1$$

- $\lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} = 1 \implies \lim_{m\to\infty} \frac{I_{2m}}{\cancel{I_{2m+1}}} \frac{\cancel{I_{2m+1}}}{I_{2m-1}} = 1$

$$\implies \lim_{m\to\infty} \frac{I_{2m}}{I_{2m-1}} = 1$$

$$\implies \lim_{m\to\infty} \frac{\frac{\pi}{2^{2m+1}}\binom{2m}{m}}{\frac{2^{2m}}{(2m)\binom{2m}{m}}} = 1$$

$$\implies \lim_{m\to\infty} \frac{\pi(\cancel{2}m)\binom{2m}{m}^2}{2^{4m+\cancel{1}}} = 1$$

$$\implies \lim_{m\to\infty} \binom{2m}{m} = \frac{4^m}{\sqrt{m\pi}}$$

$$\implies \binom{2m}{m} \sim \frac{4^m}{\sqrt{m\pi}}$$

3.2 Weistrass Proof

- Let $f \in C([-1,1])$. Uniform continuity implies that there is a number $\delta(\epsilon) > 0$ such that

  if $x_1, x_2 \in [-1,1]$, and $|x_1 - x_2| < \delta(\epsilon)$

  $\implies |f(x_1) - f(x_2)| < \epsilon$

  Now, let $m \in N$ such that, $h = \frac{2}{m} < \delta(\epsilon)$

  We divide the interval $[-1.1]$ into m disjoint intervals, each of size h.

Let $g(x)$ be our interpolant, then on the k^{th} interval, $g(x) = l_k(x)$ where l_k is the linear function joining the point

$$(-1 + (k-1)h, f(-1 + (k-1)h)) \quad \text{and} \quad (-1 + kh, f(-1 + kh))$$

As $l_k(-1 + kh) = l_{k+1}(-1 + kh)$, the linear functions will intersect at the endpoints of all intervals, and therefore $g(x)$ is continous and piecewise linear.

Let $x_k$and x_{k+1} be two endpoints of the k^{th} interval. We can write

$$l_k(x) = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}(x - x_k) + f(x_k)$$
$$= \frac{x_{k+1} - x}{x_{k+1} - x_k} f(x_k) + \frac{x - x_k}{x_{k+1} - x_k} f(x_{k+1})$$

So, in the k^{th} interval,
$$|f(x) - l_k(x)| = |f(x) - \frac{x_{k+1} - x}{x_{k+1} - x_k} f(x_k) - \frac{x - x_k}{x_{k+1} - x_k} f(x_{k+1})|$$
$$= |f(x)(\frac{x_{k+1} - x}{x_{k+1} - x_k} + \frac{x - x_k}{x_{k+1} - x_k}) - \frac{x_{k+1} - x}{x_{k+1} - x_k} f(x_k) - \frac{x - x_k}{x_{k+1} - x_k} f(x_{k+1})|$$
(Because $\frac{x_{k+1} - x}{x_{k+1} - x_k} + \frac{x - x_k}{x_{k+1} - x_k} = 1$)

$$\implies |f(x) - l_k(x)| = |(f(x) - f(x_k))\frac{x_{k+1} - x}{x_{k+1} - x_k} + (f(x) - f(x_{k+1}))\frac{x - x_k}{x_{k+1} - x_k}|$$

We know that $h < \delta(\epsilon)$

Thus, $|f(x) - l_k(x)| \le |\frac{x_{k+1} - x}{x_{k+1} - x_k} + \frac{x - x_k}{x_{k+1} - x_k}|\epsilon$
$\implies |f(x) - l_k(x)| \le \epsilon$
We can do this for every interval, and therefore conclude

$|f(x) - g(x)| \le \epsilon$
$\implies ||f(x) - g(x)||_\infty \le \epsilon$
Thus, we can approximate a uniformly continous function with a piecewise linear interpolant. **Moreover, for any $\epsilon > 0$ we can choose $h < \delta(\epsilon)$ such that, $||f(x) - g(x)||_\infty \le \epsilon$, and therefore our approximation will be uniform.**

- Let us consider the function, $r(x) = 0.5(x - d + |x - d|)$. This gives us a piecewise continous linear function that is,

$$r(x) = \begin{cases} x - d, & \text{if } x > d \\ 0, & \text{if } x \le d \end{cases} \tag{1}$$

To change the slope of $r(x)$ from 1 when $r(x) = x, x > a$ to u, we can simply multiply u with $r(x)$. We will construct our global piecewise linear function using this function as the basic component.

Consider: we add $(u_1x + d_1)$ to $u_2r(x)$

$(u_1x + d_1)) + u_2r(x) = (u_1x + d_1) + 0.5u_2(x - d + |x - d|)$

$$(u_1x + d_1) + u_2r(x) = \begin{cases} (u_1 + u_2)x - u_2d + d_1, & \text{if } x > d \\ u_1x + d_1, & \text{if } x \leq d \end{cases} \tag{2}$$

We now have a piecewise linear continous function, with two different slopes. As $x \to d$ the above function has single value $u_1d + d_1$, thus it is continous, its two linear "pieces" have slopes u_1 and $u_1 + u_2$. By taking suitable value for u_2 we can have any slope for the second linear piece.

Let's say, for the first interval, the linear function $l_1(x) = a_1 + b_1x$, and for second interval, $l_2(x) = a_2 + b_2x$. If x_1 is the right endpoint of the first interval, we need our global function to change its slope from b_1 to b_2.

We can get that if we take,

$$r_1(x) = \frac{b_2 - b_1}{2}(x - x_1 + |x - x_1|)$$

Therefore,

$$l_1(x) + r_1(x) = \begin{cases} a_1 + x_1(b_1 - b_2) + b_2x, & \text{if } x > x_1 \\ a_1 + b_1x, & \text{if } x \leq x_1 \end{cases} \tag{3}$$

Now we have added the linear functions for the first two intervals to our global function. We can similarly find $r_2(x), ...r_{m-1}(x)$ to add to $l_1(x)$ to get our global function, which, for m intervals, will look like

$$g(x) = a + bx + \sum_{k=1}^{m-1} c_k|x - x_k|$$

Where $c_k = \dfrac{b_{k+1} - b_k}{2}$, where b_k is the slope of l_k

And $b = b_1 + \sum_{k=1}^{m-1} \dfrac{b_{k+1} - b_k}{2}$

And $a = a_1 - \sum_{k=1}^{m-1} \dfrac{b_{k+1} - b_k}{2}x_k$

- By using Taylor Series expansion for $(1+y)^{1/2}$ at $y=0$,

  $(1+y)^{1/2} = 1 + \frac{\frac{1}{2}}{1!}y + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2!}y^2 + \cdots$
  $= \sum_0^\infty a_j y^j$

  The coefficient a_{j+1} is given by: $a_{j+1} = \frac{(\frac{1}{2})(\frac{1}{2}-1)\cdots(\frac{1}{2}-(j))}{(j+1)!}$
  $= \frac{(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2})(\frac{-5}{2})\cdots(\frac{-(2j-1)}{2})}{(j+1)!}$
  $= (-1)^j \frac{1 \times 3 \times 5 \cdots (2j-1) j! 2^j}{2^{j+1}(j+1)! j! 2^j}$
  $= (-1)^j \frac{\binom{2j}{j}}{2^{2j+1}(j+1)} = \binom{1/2}{j+1}$ (Let)

  Let $\binom{1/2}{0} = 1$

  Thus,

  $$(1+y)^{1/2} = \sum_0^\infty \binom{1/2}{j} y^j$$

  The result is true for $|y| \le 1$, for which the series on RHS converges.

- Let $y = x^2 - 1 = |x|^2 - 1$
  Therefore, $|x| = \sum_0^\infty \binom{1/2}{j}(x^2-1)^j$

  The result is true for $|x| \le \sqrt{2}$

- $f_s(n) = \sum_{m=n}^\infty \frac{1}{m^s}$

  $\lim_{n\to\infty} \frac{1}{n} = 0$
  $\implies \lim_{n\to\infty} \frac{1}{n^s} = 0 \quad , \quad s > 1$
  $\implies \lim_{n\to\infty} \sum_{m=n}^\infty \frac{1}{m^s} = 0 \quad , \quad s > 1$

  $\implies \lim_{n\to\infty} f_s(n) = 0$ (♠)

- $\binom{1/2}{j+1} = (-1)^j \frac{\binom{2j}{j}}{2^{2j+1}(j+1)}$
  (Substituting central bionomial coefficient using result derived previously)

  $= (-1)^j \frac{\frac{\cancel{4^j}}{\sqrt{\pi j}}}{2^{\cancel{2j}+1}(j+1)} = \frac{(-1)^j}{2\sqrt{\pi}(j^{1.5}+j^{0.5})}$
  $\lim_{n\to\infty} \sum_{j=n}^\infty \frac{1}{j^{1.5}} = 0$ Using (♠)

$\dfrac{1}{j^{1.5}} > \binom{1/2}{j+1}$
$\implies \lim_{n\to\infty} \sum_{j=n}^{\infty} \binom{1/2}{j+1} = 0$
$\binom{1/2}{j+1} > \binom{1/2}{j+1}(x^2-1)^j$ (Because $|x^2-1| \leq 1$ for $x \in [-1,1]$)

$\implies \lim_{n\to\infty} \sum_{j=n}^{\infty} \binom{1/2}{j+1}(x^2-1)^j = 0$

$\implies \lim_{n\to\infty} ||x| - \sum_{j=0}^{n} \binom{1/2}{j}(x^2-1)^j| = 0$

This means for any $\epsilon > 0$ we can find an n large enough such that
$||x| - \sum_{j=0}^{n} \binom{1/2}{j}(x^2-1)^j| < \epsilon$

Let the truncated series be S_n, therefore, uniform convergence implies, there exists n such that

$$|||x| - S_n||_\infty < \epsilon$$

for any $\epsilon > 0$.
Therefore we can approximate $|x|$ by truncating the series.

- Let $f(x)$ be our given function, and $g(x)$ be the piecewise linear approximation.

  For any given $\epsilon > 0$ we can find a $g(x)$ such that

$$|f(x) - g(x)| < \epsilon/2 \quad , \; x \in [-1,1]$$

We know from earlier the piecewise linear function is given by (for m intervals)

$$g(x) = a + bx + \sum_{k=1}^{m-1} c_k|x - x_k| \quad , \; x \in [-1,1]$$

We can find a truncated series $S_k(x)$ such that

$$|c_k|x - x_k| - S_k(x)| < \epsilon/2(m-1) \quad , \; x \in [-1,1]$$

$$\sum_{k=1}^{m-1} |c_k|x - x_k| - \sum_{k=1}^{m-1} S_k(x)| < \epsilon/2 \quad , \; x \in [-1,1]$$

Let $g_t(x) = a + bx + \sum_{k=1}^{m-1} S_k(x)$

$\implies |g(x) - g_t(x)| < \epsilon/2 \;$, $x \in [-1,1]$

$\implies |f(x) - g_t(x)| < \epsilon \;$, $x \in [-1,1]$

$$\implies ||f(x) - g_t(x)||_\infty < \epsilon \ \ , \ x \in [-1, 1]$$

Thus for any given $f(x)$ we can find a function $g_t(x)$ that approximates it uniformly. This concludes our proof of the Weierstrass theorem.