# Constructive Approximation Theory
# Assignment 1

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## 1 VanderMonde Matrix

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^n \\ 1 & x_3 & x_3^2 & x_3^3 & \cdots & x_3^n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^n \end{bmatrix}$$

- **To Prove**: $\det(V)$ is a polynomial in the variables $x_0, x_1, \ldots, x_n$ with degree $\dfrac{n(n+1)}{2}$.
  **Proof:** We will use induction for our proof.

  - Basis: We can see that for n = 1,

    $$V = \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix}$$

  - Induction Hypothesis: For $n = k, det(V_k)$ is a polynomial with degree $\dfrac{k(k+1)}{2}$
  - Induction Step: For $n = k+1$ we can write V as

    $$V = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^k & x_0^{k+1} \\ 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^k & x_1^{k+1} \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^k & x_2^{k+1} \\ 1 & x_3 & x_3^2 & x_3^3 & \cdots & x_3^k & x_3^{k+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_k & x_k^2 & x_k^3 & \cdots & x_k^k & x_k^{k+1} \\ 1 & x_{k+1} & x_{k+1}^2 & x_{k+1}^3 & \cdots & x_{k+1}^k & x_{k+1}^{k+1} \end{bmatrix}$$

    Now, finding the determinant along the last column, $\det(V_{k+1}) = x_0^{k+1}\det(V_k^{(0)}) + x_1^{k+1}\det(V_k^{(1)}) \ldots + x_{k+1}^{k+1}\det(V_{k+1}^{(k+1)})$

where, $V_k^{(i)}$ is the matrix obtained from eliminating the $i^{th}$ row and last column from $V_{k+1}$. From our hypothesis, each $\det(V_k^{(i)})$ will be a polynomial of degree $\dfrac{k(k+1)}{2}$. Thus, degree of $\det(V_{k+1}) = (k+1) + \dfrac{k(k+1)}{2} = \dfrac{(k+1)(k+2)}{2} = \dfrac{(k+1)((k+1)+1)}{2}$. Hence, proved.

- If $x_i = x_j$ for $i \neq j$, then two rows of $V$ will be equal, $\implies$ $V$ will be singular $\implies$ $\det(V) = 0$.
- As $\det(V) = 0$, when $x_i = x_j$ for $i \neq j$, we can conclude that $x_i - x_j = (x_i - x_j)K$ (K is a constant), i.e. $x_i - x_j$ is a factor of $\det(V)$.
- Considering, $0 \leq j < i \leq n$, each $(x_i - x_j)$ will be a factor of $\det(V)$

  $\implies \det(V) = C\left(\prod_{0 \leq j < i \leq n} (x_i - x_j)\right)$, where C is a constant.

- $\det(V) = C\left(\prod_{0 \leq j < i \leq n} (x_i - x_j)\right) = C\left(\prod_{0 \leq j < n} (x_n - x_j)(\prod_{0 \leq j < n-1} (x_{n-1} - x_j) \cdots (x_1 - x_0)\right)$

  $\implies \det(V) = C\left((x_n^n + X_n)(x_{n-1}^{n-1} + X_{n-1} \cdots (x_1 - x_0)\right)$

  $\implies \det(V) = C\left(x_n^n x_{n-1}^{n-1} \cdots x_1 + F_n\right)$ (1)

  $X_n, X_{n-1}, F_n$ etc are the remainder terms of the polynomial products.

  Now, expanding $\det(V)$ along the first row, and the subsequent submatrices along their respective first rows,

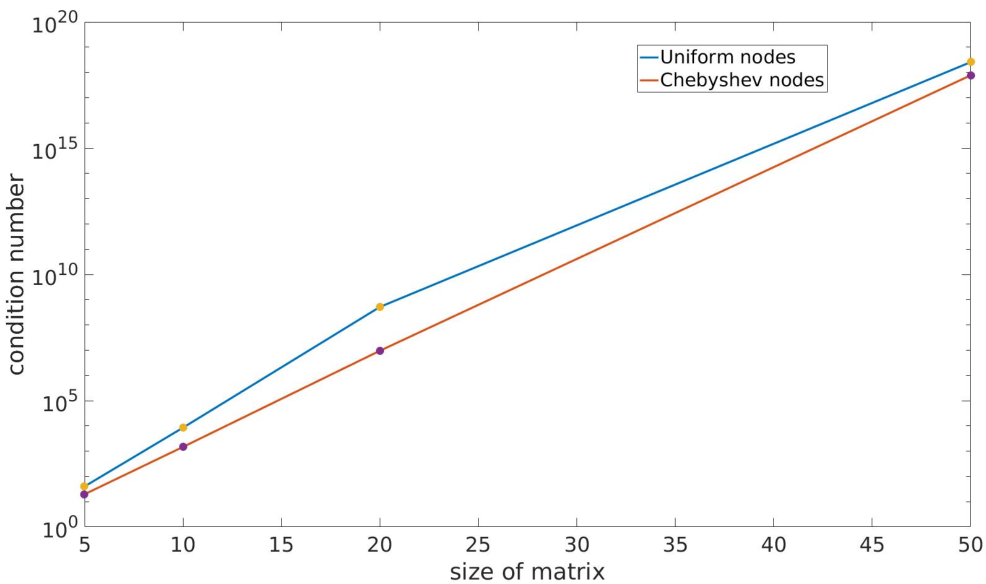
  $\det(V) = 1(x_1(x_2^2(\cdots) + Y_2) + Y_1) + Y_0$
  $\implies \det(V) = x_1 x_2^2 x_3^3 \cdots x_n^n + K_n$ (2)

  Comparing coefficients of $x_1 x_2^2 x_3^3 \cdots x_n^n$ in (1) and (2), we can conclude $C = 1$

# 2 Interpolation using MATLAB

- Plot of condition number for Vandermonde matrices of different sizes is given in Figure 1. The matrices are unstable for both set of nodes, exhibiting greater unstablity with uniform nodes. At $n = 5$ we the condition number is approximately to the order of $10^4$, and thus we lose 4 digits in accuracy. For 50 interpolant nodes, we lose almost 19 digits, which is more than the number of digits of a double precision computer, and would therefore give us completely garbage values.

Figure 1: Condition number of the Vandermonde Matrix for different sizes




- Interpolation by solving the Vandermonde matrix linear system. We have used MATLAB's $A\backslash b$ to find our interpolants. Figure 2 - 5 plot the interpolant along with the Runge Function. We observe that the interpolant gets closer to the real function as n increases, and Chebyshev nodes give us much more accurate results than uniform nodes, which diverge widely at the endpoints. The diversion increases with increasing the interpolant nodes.
  **Note:** We get extremely accurate results for Chebyshev nodes, n = 5. However, computing in MATLAB by finding the inverse, we don't get such accurate result for this size.

Figure 2: Vandermonde Interpolant for Runge Function, n = 5

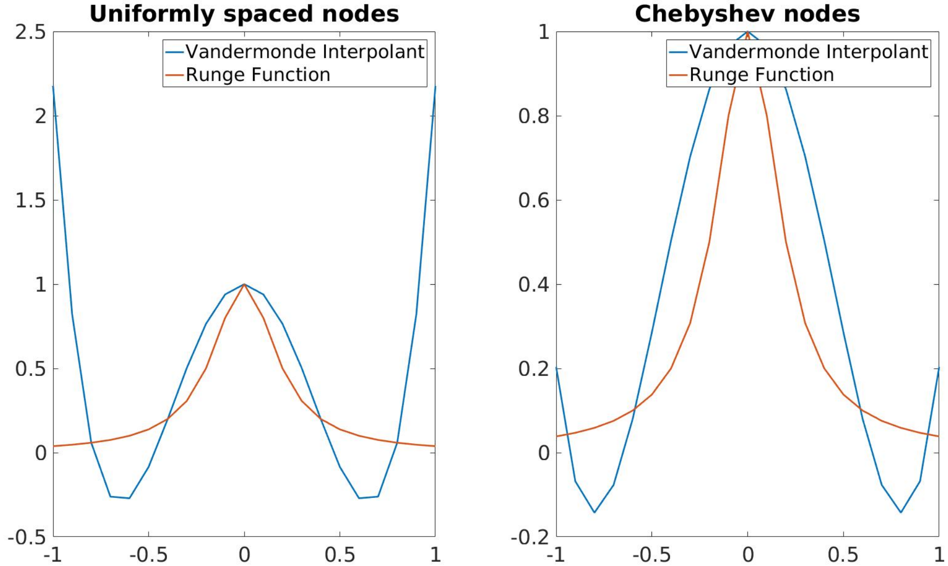



Figure 3: Vandermonde Interpolant for Runge Function, n = 10

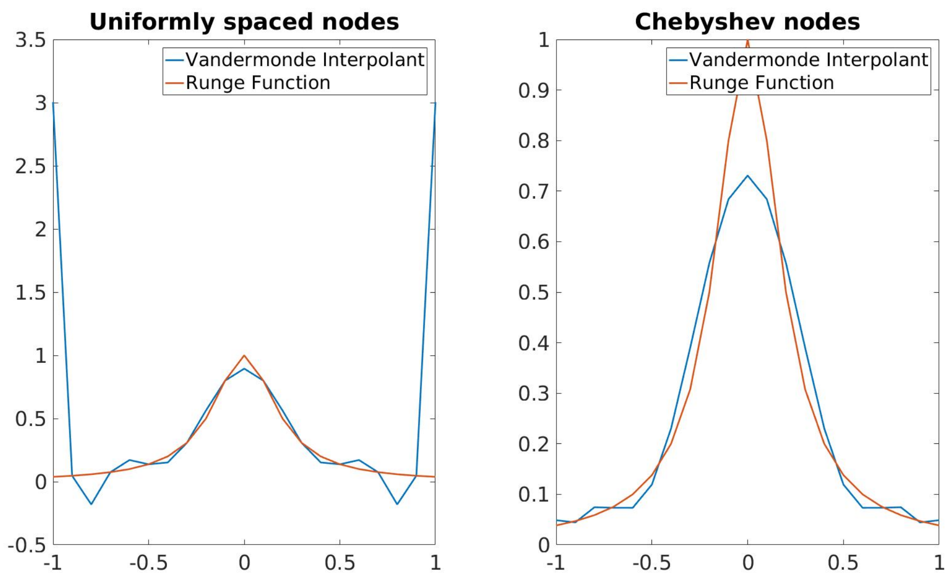


Figure 4: Vandermonde Interpolant for Runge Function, n = 20

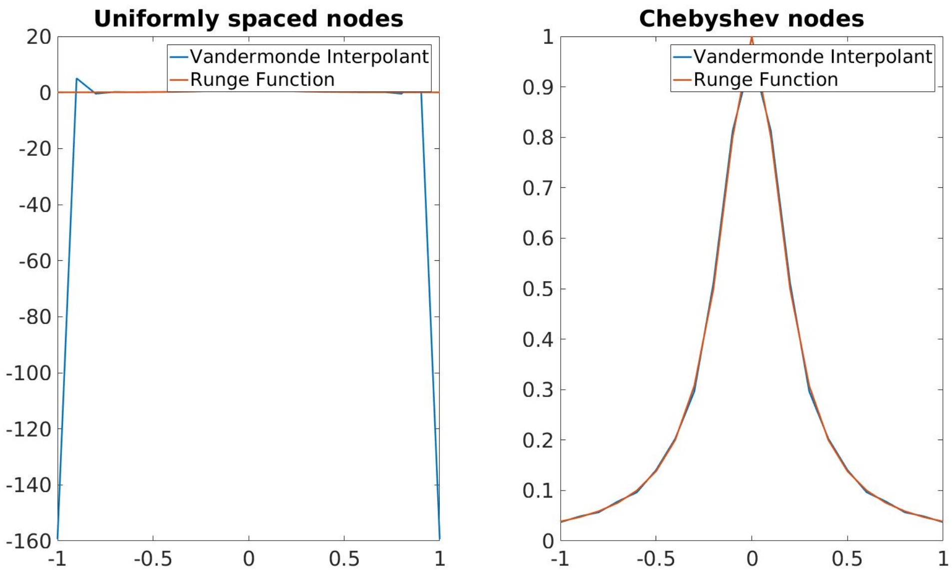
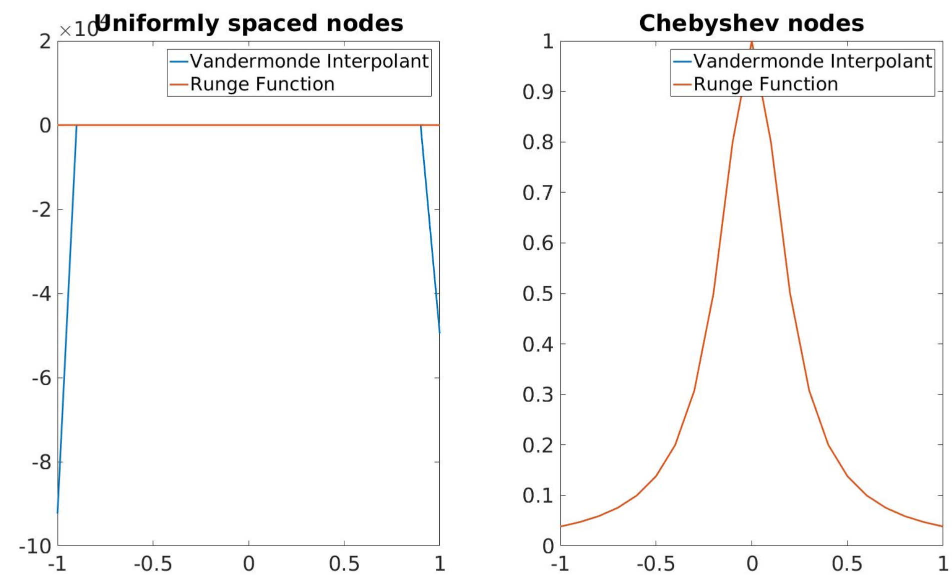



Figure 5: Vandermonde Interpolant for Runge Function, n = 50




- Interpolation by finding the Lagrangian polynomial. The interpolants are plotted in Figures

6 - 9. Similar to the Vandermonde interpolants, the Chebyshev nodes give us better results than the uniform nodes, which give us divergence at the endpoints. We notice that the Lagrangian plots are identical to the Vandermonde plots, and that is to be expected because both the methods find a $n^{th}$ degree polynomial to fit to $n+1$ nodes, and only once such unique polynomial exists. Therefore, both the methods give us the same interpolants, ignoring errors based on methods of computations.

Figure 6: Lagrangian Interpolant for Runge Function, n = 5

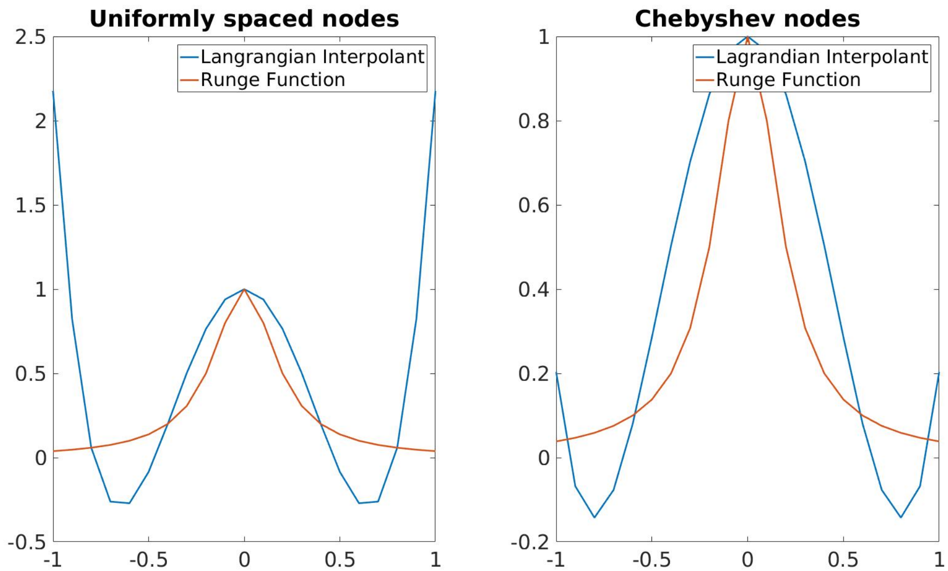


Figure 7: Lagrangian Interpolant for Runge Function, n = 10

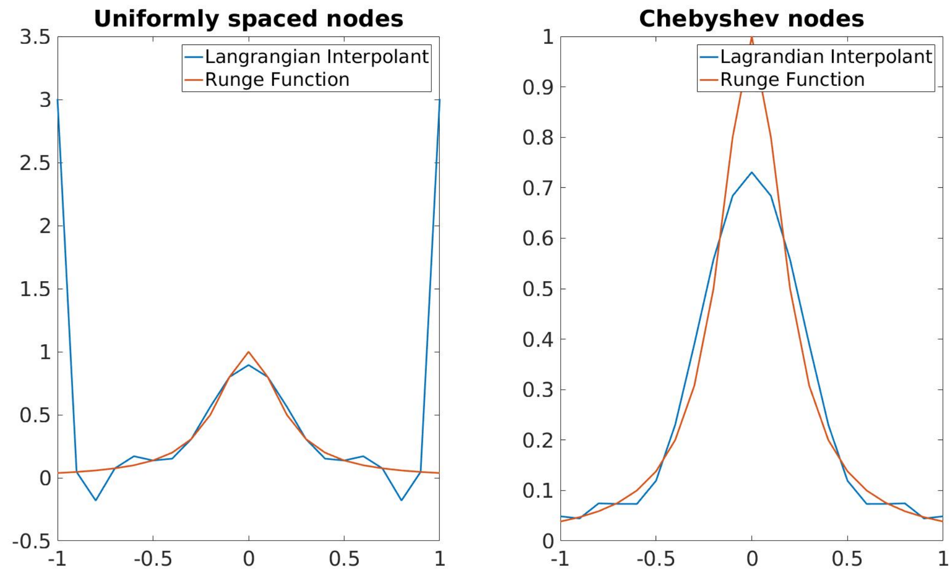



Figure 8: Lagrangian Interpolant for Runge Function, n = 20

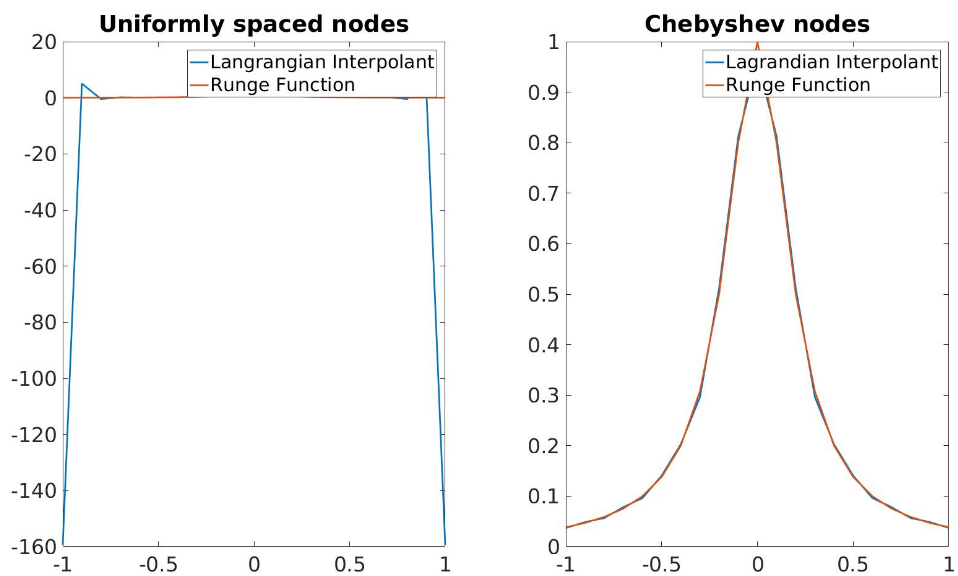
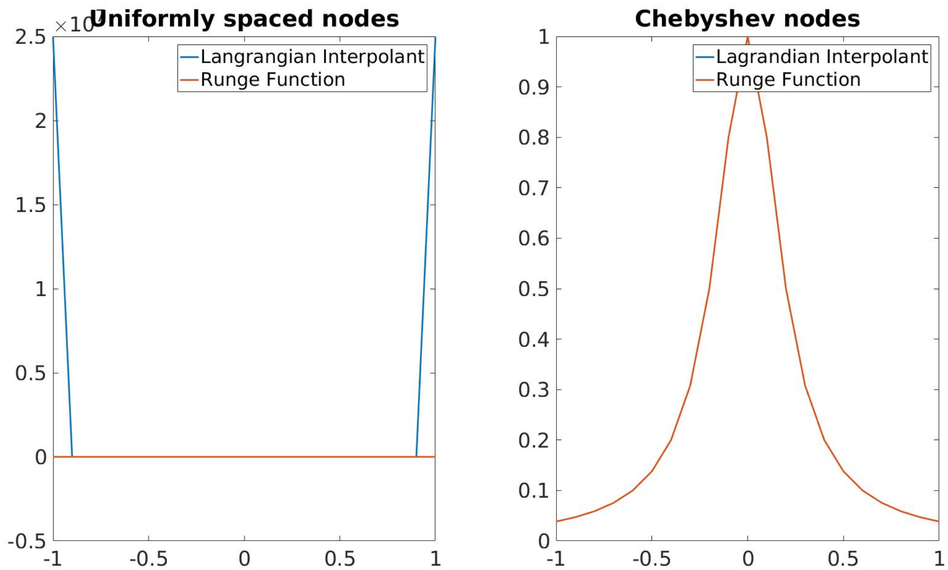


Figure 9: Lagrangian Interpolant for Runge Function, n = 50




- Number of Computations:
  - Vandermonde: Computing the Vandermonde matrix: $O(n^2)$
    Solving the system: $O(n^3)$
    Computing the value of the interpolant at a point: $O(n)$
    Total cost: $O(n^3)$
  - Lagrangian: Cost of computing each $\ell_j(x)$ : $O(n)$ Total cost: $O(n^2)$
- While both methods give us similar results, the accuracy of the Vandermonde method hinged heavily upon the method used to solve the linear system. With large number of nodes, the matrix can get very unstable because of large condition number, and thus this method is not as reliable as the Lagrangian method, which should be preferred between the two.

# 3 To prove:

$$\sum_{j=0}^{n} x_j^m \ell_j(x) = x^m$$

for any set of interpolation nodes, for all $m \in \{0, 1, 2, \ldots, n\}$.

**Proof:** Let $L(x)$ be the Lagrangian interpolant polynomial for the function $x^m$, using the interpolation nodes $x_j, j \in \{0, 1, 2, \ldots, n\}$.

We know that, $\sum_{j=0}^{n} x_j^m \ell_j(x_i) = x_i^m, \;\; m \in \{0, 1, 2, \ldots, n\}$,

As,

$$\ell_j(x_i) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Thus, the polynomial $L(x)$ fits the polynomial $x^m$ at all $n+1$ interpolation nodes. But, for $m \leq n$ there is only a unique polynomial that can fit through all $n+1$ points exactly. Thus, we can conclude that $L(x) = x^m \implies \sum_{j=0}^{n} x_j^m \ell_j(x) = x^m$

# 4 To prove:

$$H^\alpha([-1,1]) \supset H^\beta([-1,1])$$

, if $\alpha < \beta$

**Proof:** $H^\alpha([-1,1]) \implies |f(x) - f(y)| \leq C|x-y|^\alpha, C \in \Re^+$ Let $C_\alpha$ be the smallest $C$ for which the contuinity holds. Thus the continuity holds for $C \in [C_\alpha, \infty)$. Similarly, for $H^\beta$, for some $C_\beta$ the continuity holds for $C \in [C_\beta, \infty)$

We can write,
$|f(x_\alpha) - f(y_\alpha)| = C_\alpha |x-y|^\alpha$, and
$|f(x_\beta) - f(y_\beta)| = C_\beta |x-y|^\beta$
Now, as $\alpha < \beta$, for $x, y \in [-1, 1]$

$|x-y|^\alpha \geq |x-y|^\beta$,
$\implies \dfrac{|f(x_\alpha) - f(y_\alpha)|}{|x-y|^\alpha} \leq \dfrac{|f(x_\beta) - f(y_\beta)|}{|x-y|^\beta}$
$\implies C_\alpha \leq C_\beta$
$\implies [C_\alpha, \infty) \supset [C_\beta, \infty)$
$\implies H^\alpha([-1,1]) \supset H^\beta([-1,1])$

# 5 To Find Functions:

- Continous but not Holder continous for any $\alpha > 0$
  Let us consider the function:

$$f(x) = \begin{cases} 1/ln(0.25x + 0.25), & \text{if } x > -1 \\ 0, & \text{if } x \leq -1 \end{cases} \tag{2}$$

The function decreases monotonically for $-1 < x \leq 1$, and at $x = -1$, $f(-1^+) = f(-1^-) = 0$. Thus, the function is continous.
Let's assume that the function is Holder continous. Let $\alpha, C \in \Re^+$. At $x = -1$,

$|-1-1/ln(0.25x+0.25)| \leq C|-1-x|^{\alpha} \implies$
$C|1+x|^{\alpha}|1+1/ln(0.25x+0.25)| \geq 1$

But at $x \to -1^{+}, |1+1/ln(0.25x+0.25)| \to 1, |1+x| \to 1, \implies C|1+x|^{\alpha}|1+1/ln(0.25x+0.25)| \to 0$

Therefore the given function is not Holder continous for any $\alpha$.

- Lipschitz but not differentiable
  Let us consider the function $f(x) = |x|$.

  For $x < 0, f(x) = -x$, which is continous. For $x > 0, f(x) = x$, which is also continous. At $x = 0, f(0^{+}) = f(0^{-}) = 0$.
  Thus, $|x|$ is continous in $[-1, 1]$.

  For Lipschitz continuity, we need $|f(x) - f(y)| \leq C|x-y| \implies ||x| - |y|| \leq C|x-y| \implies C \geq \dfrac{||x|-|y||}{|x-y|}$

  Let $C_{\kappa} = \dfrac{||x|-|y||}{|x-y|}$

  $C_{\kappa} \in (0, 1]$ for $x, y \in [-1, 1]$

  Thus $f(x)$ is Lipschitz continous, as we can always find $C \geq C_{\kappa}$

- Differentiable but its derivative is not continuous
  Let us consider the function

$$f(x) = \begin{cases} x^2 sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \tag{3}$$

  The function is continous and oscillating, and can be diffentiated as,

$$f'(x) = \begin{cases} 2xsin(1/x) - cos(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \tag{4}$$

  At $x \to 0, 2xsin(1/x) \to 0$, but $cos(1/x)$ is undefined, therefore $x \to 0 f'(x)$ does not exist. $f(x)$ however is differentiable at $x = 0$, as

$$h \to 0 \ \frac{f(h) - f(0)}{h - 0} = h \to 0 \ \frac{f(h) - f(0)}{h} = h \to 0 \ hsin(1/h) = 0$$

  Thus, $f(x)$ is differentiable but $f'(x)$ is not continous.