Constructive Approximation Theory
Assignment 8

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1 Gaussian Quadrature

Let $f(x)$ be a polynomial of degree $\leq 2n-1$ and $\{p_k(x)\}_{k=0}^{\infty}$ be the series of orthogonal polynomials corresponding to the non-negative weight function $w(x)$ on $(-1,1)$.

We can write $f(x)$ as, $f(x) = p_n(x)q(x) + r(x)$ where, $q(x), r(x) \in P_{n-1}$

Also, we can write $q(x)$ as $q(x) = \sum_{k=0}^{n-1} c_k p_k(x)$

Now,

$$\int_{-1}^{1} w(x)f(x)dx = \int_{-1}^{1} p_n(x)q(x)w(x)dx + \int_{-1}^{1} r(x)w(x)dx$$

$$= \cancel{\sum_{k=0}^{n-1} c_k \int_{-1}^{1} p_n(x)p_k(x)w(x)dx} + \int_{-1}^{1} r(x)w(x)dx$$

We could cancel out the sum of integrals because p_n and p_k are orthogonal with respect to $w(x)$, and $k \neq n$

Thus,

$$\int_{-1}^{1} w(x)f(x)dx = \int_{-1}^{1} r(x)w(x)dx \qquad (1)$$

Now, interpolating $r(x)$ with a Langrangian polynomial of the same degree $(n-1)$, will give us the same polynomial, i.e. the interpolation will be exact. We interpolate $r(x)$ on the n roots of $p_n(x)$. Let $\left\{x_k^{(n)}\right\}_{k=1}^{n}$ be the zeroes of $p_n(x)$.

Let $l_i^{(n)}(x) = \prod_{j \neq i} \left(\frac{x - x_j^{(n)}}{x_i^{(n)} - x_j^{(n)}} \right)$

Then, $r(x) = \sum_{k=1}^{n} r(x_k^{(n)}) l_k^{(n)}(x)$

$\implies r(x)w(x) = \sum_{k=1}^{n} r(x_k^{(n)}) l_k^{(n)}(x)w(x)$

$$\implies \int_{-1}^{1} r(x)w(x)dx = \sum_{k=1}^{n} r(x_k^{(n)}) \int_{-1}^{1} l_k^{(n)}(x)w(x)dx = \sum_{k=1}^{n} r(x_k^{(n)})w_k^{(n)} = \int_{-1}^{1} w(x)f(x)dx$$

(Where, $w_k^{(n)} = \int_{-1}^{1} l_k^{(n)}(x)w(x)dx$)

$$\implies \int_{-1}^{1} w(x)f(x)dx = \sum_{k=1}^{n} \left(p_n(x_k^{(n)})q(x_k^{(n)}) + r(x_k^{(n)})\right) w_k^{(n)} \quad \text{(Because } p_n(x_k^{(n)})q(x_k^{(n)}) = 0 \text{)}$$

$= \sum_{k=1}^{n} f(x_k^{(n)})w_k^{(n)}$

2 Hermite Interpolation Bound

Let $w(x) = \prod_{j=0}^{n}(x - x_j)$

And, $g(t) = f(t) - p(t) - \frac{f(x) - p(x)}{w(x)^{m+1}} w(t)^{m+1}$

Now, $g(t)$ has $n+2$ zeroes, at $t = x_j, j \in 0, 1 \cdots n$ and $t = x$.

Therefore, by Rolle's Theorem, $g'(t)$ must have at least $n+1$ zeroes, between the $n+2$ zeroes of $g(t)$.

$g'(t) = f'(t) - p'(t) - \frac{f(x) - p(x)}{w(x)^{m+1}} ((m+1)w(t)^m w'(t))$

$f'(x_j) = p'(x_j)$

$w(x_j)^m = 0$

$\implies g'(x_j) = 0$

We have now shown that along with having $n+1$ zeroes between the zeroes of $g(t)$, $g'(t)$ also has $n+1$ more zeroes on $n+1$ zeroes of $g(t)$, at $x_j, j \in 0, 1, \cdots n$.

Thus, minimum zeroes of $g'(t) = 2n+2$

Similarly we can show, minimum zeroes of $g''(t) = 3n+2$

And, minimum zeroes of $g^{(m)}(t) = (m+1)n+2$

Now, if we keep on differentiating,

minimum zeroes of $g^{(m+1)}(t) = (m+1)n+1$

minimum zeroes of $g^{(m+2)}(t) = (m+1)n$

minimum zeroes of $g^{((m+1)(n+1))}(t) = 1$

Thus, there exists at least one number $\zeta(x) \in [x_0, x_n]$ for which, $g^{((m+1)(n+1))}(\zeta(x)) = 0$

$$\implies f^{((m+1)(n+1))}(\zeta(x)) - \cancel{p^{((m+1)(n+1))}(\zeta(x))} - \frac{f(x)-p(x)}{w(x)^{m+1}}((m+1)(n+1))! = 0$$

$$\implies f(x) - p(x) = \frac{f^{((m+1)(n+1))}(\zeta(x))}{((m+1)(n+1))!}\prod_{j=0}^{n}(x-x_j)^{m+1}$$

3 Gauss Quadrature Error Estimate

3.1 Weighted Mean Value Theorem:

Before moving on to the error estimate, we need to prove the following theorem:

If $f(x), w(x)$ are continous in $[a, b]$ and $w(x) \geq 0$

$$\int_a^b f(x)w(x)dx = f(c)\int_a^b w(x)dx$$

Proof:

Let m and M be the smallest and largest values of $f(x)$ in $[a, b]$.

Therefore for the interval $[a, b]$ we can write,

$$\implies mg(x) \leq f(x)g(x) \leq Mg(x)$$

$$\implies m\int_a^b g(x) \leq \int_a^b f(x)g(x) \leq M\int_a^b g(x)$$

$$\implies m \leq \frac{\int_a^b f(x)g(x)}{\int_a^b g(x)} \leq M$$

Let $d = \dfrac{\int_a^b f(x)g(x)}{\int_a^b g(x)}$

By Intermediate Value Theorem, for $m \leq d \leq M$ we can find a $c \in [a, b]$ such that $f(c) = d$

Therefore,

$$f(c) = \frac{\displaystyle\int_a^b f(x)g(x)}{\displaystyle\int_a^b g(x)}, \text{ for some } c \in [a, b]$$

$$\implies \int_a^b f(x)g(x) = f(c)\int_a^b g(x)$$

3.2 Error Estimate

Let $f(x)$ be interpolated by a Hermite interpolant, $h_{2n-1}(x)$ at n points, for first two dereivatives, i.e. $m = 1$. Thus, the degree of the interpolant $p(x)$ will e $2n - 1$.

Thus, $f(x_j) = h_{2n-1}(x_j)$ and $f'(x_j) = h'_{2n-1}(x_j)$ for $j \in 1, 2 \cdots n$

Hermite interpolation error:

$$f(x) - h_{2n-1}(x) = \frac{f^{(2n)}(\zeta(x))}{(2n)!}\prod_{j=1}^{n}(x - x_j)^2$$

Let $\displaystyle\prod_{j=1}^{n}(x - x_j) = p_n(x)$

$$\implies f(x) = h_{2n-1}(x) + \frac{f^{(2n)}(\zeta(x))}{(2n)!}p_n(x)^2$$

$$\implies \int_a^b f(x)w(x)dx = \int_a^b w(x)h_{2n-1}(x)dx + \int_a^b \frac{f^{(2n)}(\zeta(x))}{(2n)!}p_n(x)^2 w(x)dx \quad \spadesuit$$

Now,

$$\int_a^b w(x)h_{2n-1}(x_j)dx = \sum_{j=1}^{n} x_j h_{2n-1}(x_j)$$

(Using the result we proved in Section 1)

$$\implies \int_a^b w(x)h_{2n-1}(x_j)dx = \sum_{j=1}^{n} x_j f(x_j) \impliedby f(x_j) = h_{2n-1}(x_j)$$

As $f(x)$ is continous, and $p_n(x)^2 w(x) \geq 0$, for some value $\eta \in [a, b]$, we can apply the weighted mean value theorem for integrals to get

$$\int_a^b \frac{f^{(2n)}(\zeta(x))}{(2n)!}p_n(x)^2 w(x)dx = \frac{f^{(2n)}(\eta)}{(2n)!}\int_a^b p_n(x)^2 w(x)dx$$

Therefore, equation ♠ becomes

$$\int_a^b f(x)w(x)dx = \sum_{j=1}^{n} x_j f(x_j) + \frac{f^{(2n)}(\eta)}{(2n)!}\int_a^b p_n(x)^2 w(x)dx$$

$$\implies \int_a^b f(x)w(x)dx - \sum_{j=1}^{n} x_j f(x_j) = \frac{f^{(2n)}(\eta)}{(2n)!}||p_n(x)||_w^2$$

4 Trapezoidal integration of trigonometric function

For the given functiom

$$f(x) = a_0 + \sum_{k=1}^{n}(a_k\cos(2k\pi x) + b_k\sin(2k\pi x))$$

The integral will be

$$\int f(x)dx = a_0 x + \sum_{k=1}^{n}(2k\pi a_k\sin(2k\pi x) - 2k\pi b_k\cos(2k\pi x)) + C$$

$$I_e = \int_{-1}^{1} f(x)dx$$

We evaluate the integral using trapezoidal rule at K nodes in $[-1,1]$, with step size $h = \frac{2}{K-1}$.

$$I_t = \frac{h}{2}\left(f_1 + 2\sum_{j=2}^{K-1} f_j + f_K\right)$$

To Find: The value of K in relation to n for which the trapezoidal integration is exact, i.e $I_t = I_e$

We can see that,

$$\int_{-1}^{1} f(x)dx = a_0 x + \sum_{k=1}^{n}(2k\pi a_k\sin(2k\pi x) - 2k\pi b_k\cos(2k\pi x))\Big|_{-1}^{1} = 2a_0$$

Let us consider all the a_0 terms in I_t

$$\frac{h}{2}\left(a_0 + 2\sum_{j=2}^{K-1} a_0 + a_0\right) = \frac{h}{\cancel{2}}\cancel{2}(K-1)a_0 = 2a_0$$

Let $T(g(x))$ be the trapezoidal approximation of any function $g(x)$ in $[-1,1]$. Therefore,

$$T(f(x)) = T\left(a_0 + \sum_{k=1}^{n}(a_k\cos(2k\pi x) + b_k\sin(2k\pi x))\right)$$

$$\implies T(f(x)) = T(a_0) + \sum_{k=1}^{n} a_k T(\cos(2k\pi x)) + \sum_{k=1}^{n} b_k T(\sin(2k\pi x))$$

We know that $T(a_0) = 2a_0$. So, for $I_t = I_e$ we need to find the value of K for which

$$\sum_{k=1}^{n} a_k T(\cos(2k\pi x)) + \sum_{k=1}^{n} b_k T(\sin(2k\pi x)) = 0$$

Let us consider $T(\sin(2N\pi x))$, for any $N \geq 1$

$$T(\sin(2N\pi x)) = \frac{h}{2}\left(\cancel{\sin(-2N\pi)} + 2\sum_{j=1}^{K-2}\sin\left(2N\pi(-1+\frac{2j}{K-1})\right) + \cancel{\sin(2N\pi x)}\right)$$

$$\implies T(\sin(2N\pi x)) = \frac{h}{2}\left(2\sum_{j=1}^{K-2}\sin\left(4N\pi(\frac{j}{K-1})\right)\right)$$

Now,

$$\sin\left(4N\pi(\frac{j}{K-1})\right) = -\sin\left(4N\pi - 4N\pi(\frac{j}{K-1})\right) = -\sin\left(4N\pi(\frac{K-1-j}{K-1})\right)$$

If K is even, all the terms in the sum will cancel each other. We can rewrite the sum as,

$$T(\sin(2N\pi x)) = h\left(\sum_{j=1}^{(K-2)/2}\left(\sin\left(4N\pi(\frac{j}{K-1})\right) + \sin\left(4N\pi(\frac{K-1-j}{K-1})\right)\right)\right) = 0$$

In case K is odd, the function evaluation at the middle of the grid will be 0, and the rest of the terms will cancel each other.

$T(\sin(2N\pi x)) =$

$$h\left(\sum_{j=1}^{floor((K-2)/2)}\left(\sin\left(4N\pi(\frac{j}{K-1})\right) + \sin\left(4N\pi(\frac{K-1-j}{K-1})\right)\right) + \sin\left(\cancel{4}2N\pi(\frac{\cancel{(K-2+1)}/\cancel{2}}{\cancel{K-1}})\right)\right)$$

$= 0$

This result will be true for any $N \geq 1$ and $K \geq 1$.

Therefore, we can write,

$$\sum_{k=1}^{n} b_k T(\sin(2k\pi x)) = 0$$

Now, let us consider $T(\cos(2N\pi x))$ for any $N \geq 1$

$$T(\cos(2N\pi x)) = \frac{h}{2}\left(2 + 2\sum_{j=1}^{K-2} \cos\left(4N\pi(\frac{j}{K-1})\right)\right)$$

Let $K > 2N + 1$

$$T(\cos(2N\pi x)) = h\left(1 + \sum_{j=1}^{K-2} \cos\left(4N\pi(\frac{j}{K-1})\right)\right)$$

We can write,

$$1 = \cos\left(4N\pi(\frac{K-1}{K-1})\right)$$

$$\implies T(\cos(2N\pi x)) = h\left(\sum_{j=1}^{K-1} \cos\left(4N\pi(\frac{j}{K-1})\right)\right)$$

We know that,

$$\cos\left(4N\pi(\frac{j}{K-1})\right) = Re\left(\exp\left(i4N\pi(\frac{j}{K-1})\right)\right)$$

$$\implies \sum_{j=1}^{K-1} \cos\left(4N\pi(\frac{j}{K-1})\right) = Re\left(\sum_{j=1}^{K-1} \exp\left(i4N\pi(\frac{j}{K-1})\right)\right)$$

The sum on the right is a geometric series, and we can can evaluate it using the formula

$$S = \frac{r(1-r^n)}{1-r}$$

with

$$r = \exp\left(i4N\pi(\frac{1}{K-1})\right),\ n = K-1$$

$$\implies \sum_{j=1}^{K-1} \exp\left(i4N\pi(\frac{j}{K-1})\right) = \exp\left(i4N\pi(\frac{1}{K-1})\right) \frac{(1-\exp\left(i4N\pi(\frac{K-1}{K-1})\right))}{1-\exp\left(i4N\pi(\frac{1}{K-1})\right)}$$

The denominator is not 0 as $K > 2N+1, N > 1 \implies 0 < \dfrac{4N}{K-1} < 2$

We can see that the numerator is 0, therefore, for $K > 2N+1$

$$\sum_{j=1}^{K-1} \cos\left(4N\pi(\frac{j}{K-1})\right) = 0$$

$$\implies T(\cos(2N\pi x)) = 0$$

Let $K = 2N+1$. Then,

$$T(\cos(2N\pi x)) = h\left(1 + \sum_{j=1}^{K-2} \cos\left(2\cancel{4N}\pi(\frac{j}{\cancel{2N}})\right)\right) = h(1+K-2) = 2$$

So, we can conclude for $K = 2n+1$, $\displaystyle\sum_{k=1}^{n} a_k T(\cos(2k\pi x)) \neq 0 \implies I_t \neq I_e$

The last term will be 2 because $K = 2n+1$, and the rest of the terms will be 0 because $K > 2k+1, k = 1, 2 \cdots n-1$

Let $K > 2n+1$
Then,

$$\sum_{k=1}^{n} a_k T(\cos(2k\pi x)) = 0$$

We have already proven that for any $K \geq 1$

$$\sum_{k=1}^{n} b_k T(\sin(2k\pi x)) = 0$$

Therefore, if we take $K > 2n+1$

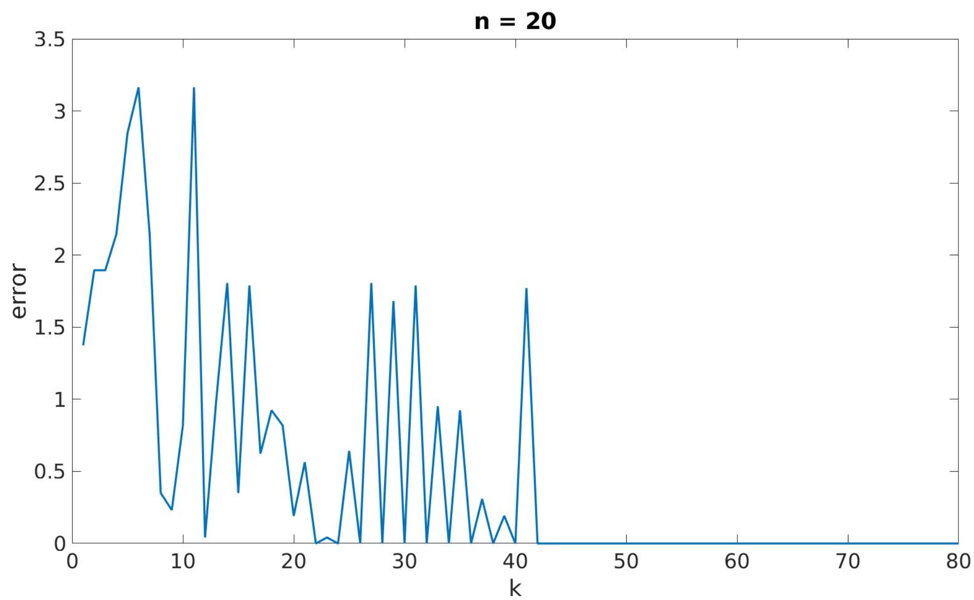
$$\sum_{k=1}^{n} a_k T(\cos(2k\pi x)) + \sum_{k=1}^{n} b_k T(\sin(2k\pi x)) = 0$$

$$\implies I_t = T(a_0) = 2a_0 = I_e$$

Therefore, for $K > 2n+1$ the given function $f(x)$ will integrate exactly on $[-1,1]$ by trapezoidal integration taking K nodes.

We have verified the result using MATLAB below.

Figure 1: Error = $|I_t - I_e|$



- Function to evalute the exact integral

```
function F = Fx(a_0,  a,b,x)

F = a_0.*x;

for i = 1:numel(a)
F = F + a(i)*2*i*pi*sin(2*i*pi.*x) - b(i)*2*i*pi*cos(2*i*pi.*x);
end

end
```

- Function to evalute the integral using trapezoidal approximation

```
function t = trapp(f,h)
t = 2*sum(f);
```

```
t = t - f(1) - f(end);

t = t*h/2;
end
```

- Function to compute $f(x)$ at chosen nodes

```
function f = fx(a_0,  a,b,x)

f = a_0;

for i = 1:numel(a)
f = f + a(i)*cos(2*i*pi.*x) + b(i)*sin(2*i*pi.*x);
end

end
```

- Main function

```
n = 20;

K = 1:80;

a = 2*rand(n,1) - 1; %uniformly distributed points in [-1,1]
b = 2*rand(n,1) - 1; %%uniformly distributed points in [-1,1]

a_0 = 2*rand(1,1) - 1; %pseudo-random number in [-1,1]

for k = 1:K(end) %k - number of points to approximate using trapezoidal rule
h = (2/(k-1));    %step size

x = -1:h:1;

f_x = fx(a_0,  a,b,x); %evaluating the function

Ie =  Fx(a_0,  a,b,1) - Fx(a_0,  a,b,-1) ; %Exact integral
if(k>1)
It = trapp(f_x,h); %trapezoidal approximation of integral
else
It = 0;
end

e(k) = abs(It - Ie); %error

end

plot(K,e) %Plotting the error
```