Constructive Approximation Theory
Assignment 10

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1 Files

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Cen_Diff2.cpp - Solution using central difference matrix,
error increases after a certain N.

Cen_Dif.cpp - Solution using central difference matrix,
 error keeps decreasing.

Greens_Trap.cpp - Solution using trapezoidal integration
and Greens function. O(N^2) complexity.

Greens_Fast.cpp - Fast algorithm to evaluate the Greens function integral.
 O(N) complexity.
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1.1 Compile and Run

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Sample:

g++ Cen_Diff2.cpp

./a.out >out.dat
```

Plot the data from out.dat using the file PlotCD.m

2 Central Difference

$$u_{xx} = -\pi^2 \sin^2(\pi x)$$

We divide the the interval $(0, 1)$ into $N + 1$ panels, with $h = \dfrac{1}{N + 1}$.

We get the central difference Linear system as,

$$\frac{1}{h^2}\begin{bmatrix} 1 & -2 & 1 & & & & 0 \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & \ddots & & \\ & & & \ddots & \ddots & 1 & \\ 0 & & & & 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix}$$

Let $x_i = ih$

Then,
$d_i = u_{xx}(x_i),\ c_i = 1,\ b_i = -2,\ a_i = 1$

and,

$y_i = u(x_i)$, the solution function.

We know that $y_0 = u(0) = 0$, and $y_{n+1} = u(1) = 0$,

Therefore we can rewrite the linear system as,

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix}.$$

2.1 Algorithm

We susbstitute every a_i with 0, b_i with 1, c_i with c_i', and d_i with d_i' given by the formulas:

$$c_i' = \begin{cases} -\dfrac{1}{2} & ; \quad i = 1 \\ -\dfrac{1}{2 + c_{i-1}'} & ; \quad i = 2, 3, \ldots, n-1 \end{cases}$$

$$d_i' = \begin{cases} -\dfrac{d_i}{2} & ; \quad i = 1 \\ -\dfrac{d_i - d_{i-1}'}{2 + c_{i-1}'} & ; \quad i = 2, 3, \ldots, n. \end{cases}$$

And our system becomes,

$$
\frac{1}{h^2}\begin{bmatrix} 1 & c'_1 & & & 0 \\ 0 & 1 & c'_2 & & \\ & 0 & 1 & \ddots & \\ & & \ddots & \ddots & c'_{n-1} \\ 0 & & & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} d'_1 \\ d'_2 \\ d'_3 \\ \vdots \\ d'_n \end{bmatrix}.
$$

We can solve this system easily in $O(N)$ by,

$$
\begin{aligned}
&y_n = d'_n \\
&y_i = d'_i - c'_i y_{i+1} \qquad ; \ i = n-1, n-2, \ldots, 1.
\end{aligned}
$$

2.2 Results

All the errors plotted in this report are global errors.

$$
e = \sum_{1}^{N} |y_i - u_i|
$$

where y_i = function value computed on the grid by solving the linear system. u_i = function value determined directly from the formula derived by integrating u_{xx}.

Figure 1: $\log - \log$ plot of global error vs N

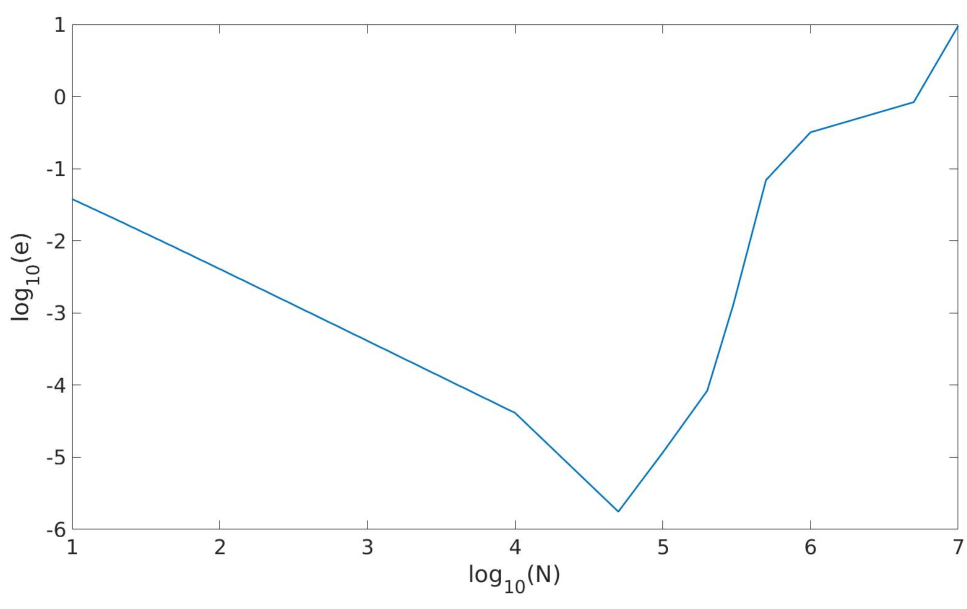
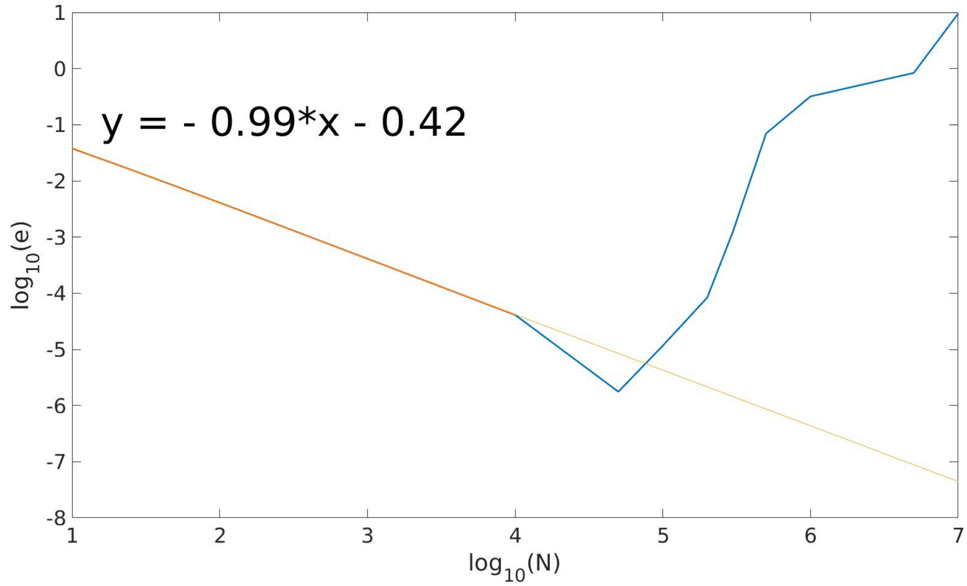


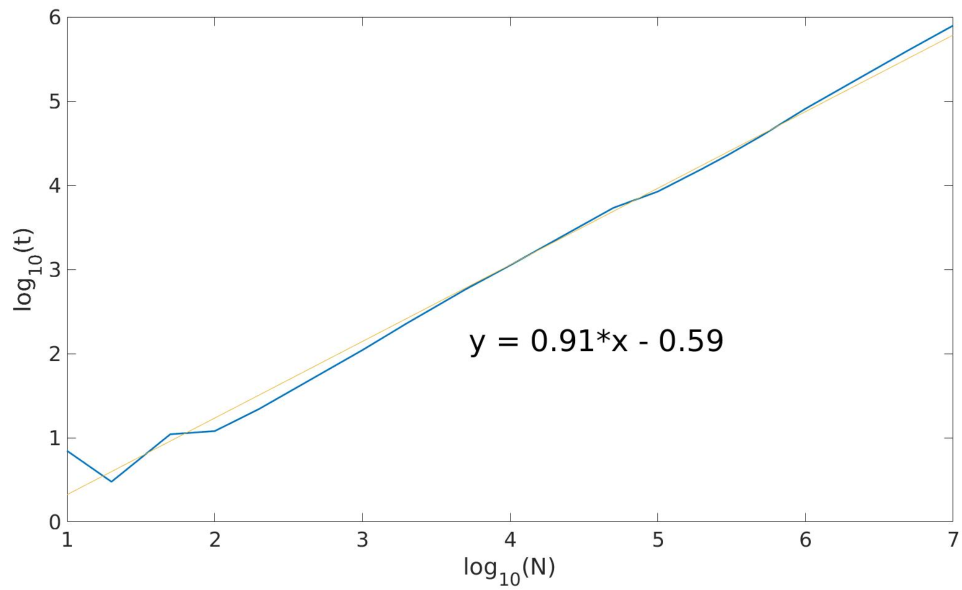
Figure 2: Slope of the linear decreasing part ≈ -1



Local truncation error for Central Difference is $O(h^2)$, therefore the global error is $O(h)$. This is confirmed from the graph.

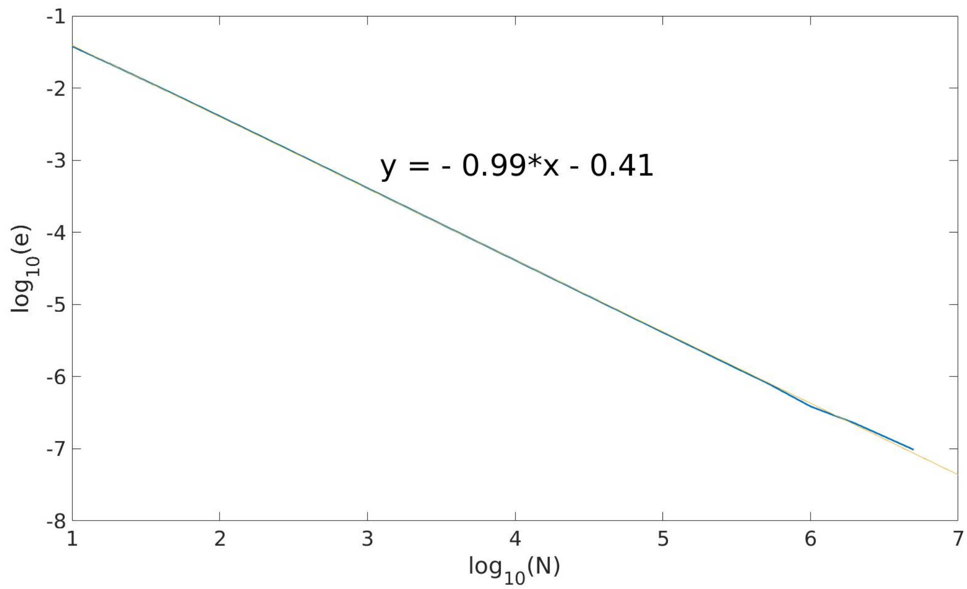
The time complexity is roughly $O(N)$. (Figure 3). We don't get the slope exactly equal to 1 probably because of noise in the time data.

Figure 3: Time Complexity of solution using Central Difference



Instead of computing c'_i recursively in a loop, if we just use the formula, $c'_i = -\frac{i}{i+1}$, the error keeps on decreasing for much longer. (Figure 4).

Figure 4: Error keeps on decreasing if we use $c'_i = -\dfrac{i}{i+1}$



3 Integration of product of Greens Function and u_{xx}

$$u(x) = \int_0^1 G(x-y)f(y)dy$$

3.1 Integration using Trapezoidal Rule

Figure 5: $\log - \log$ plot of global error vs N

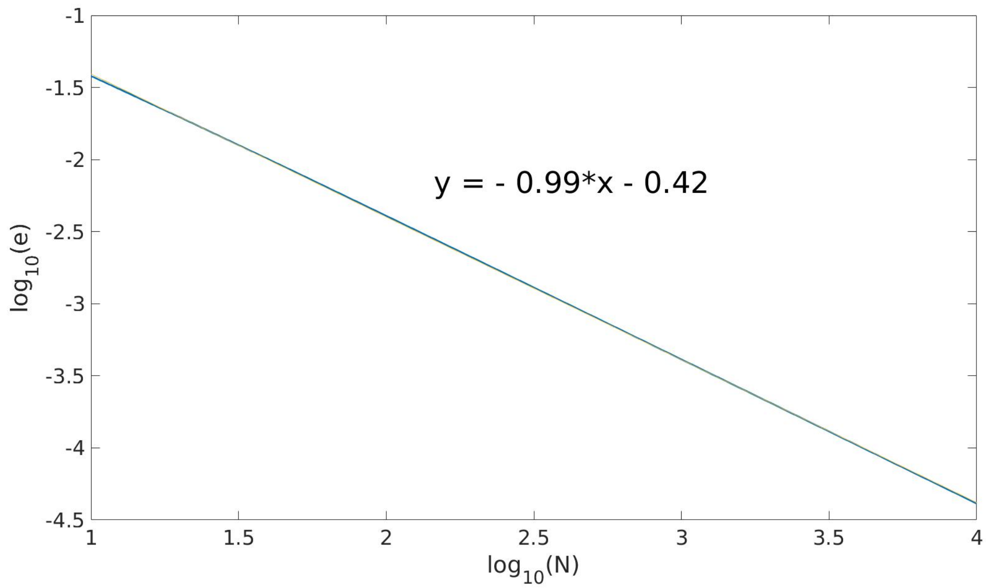
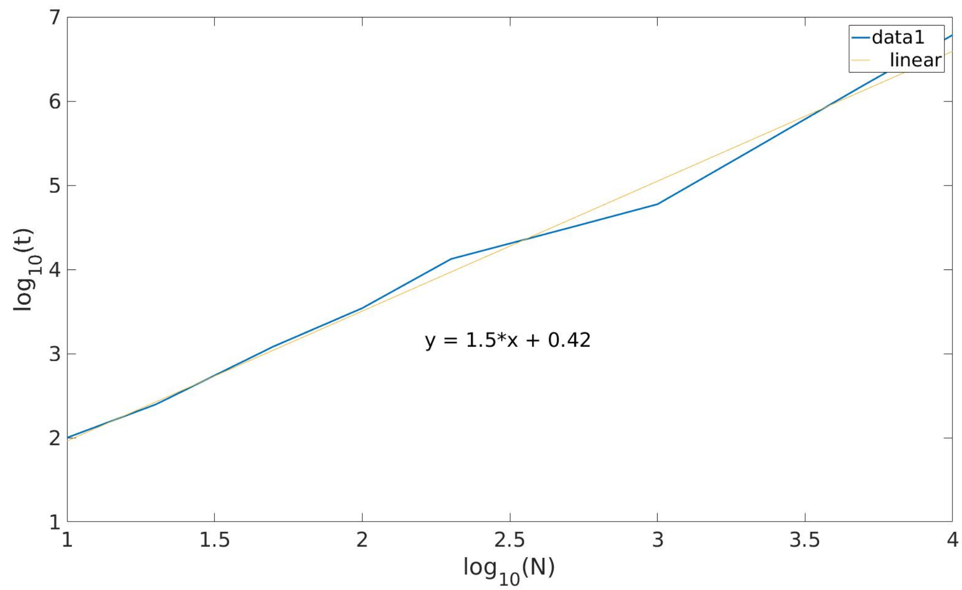


Figure 6: Time Complexity of solution using trapezoidal integration



3.2 Integration using Fast Algorithm

Figure 7: $\log - \log$ plot of global error vs N

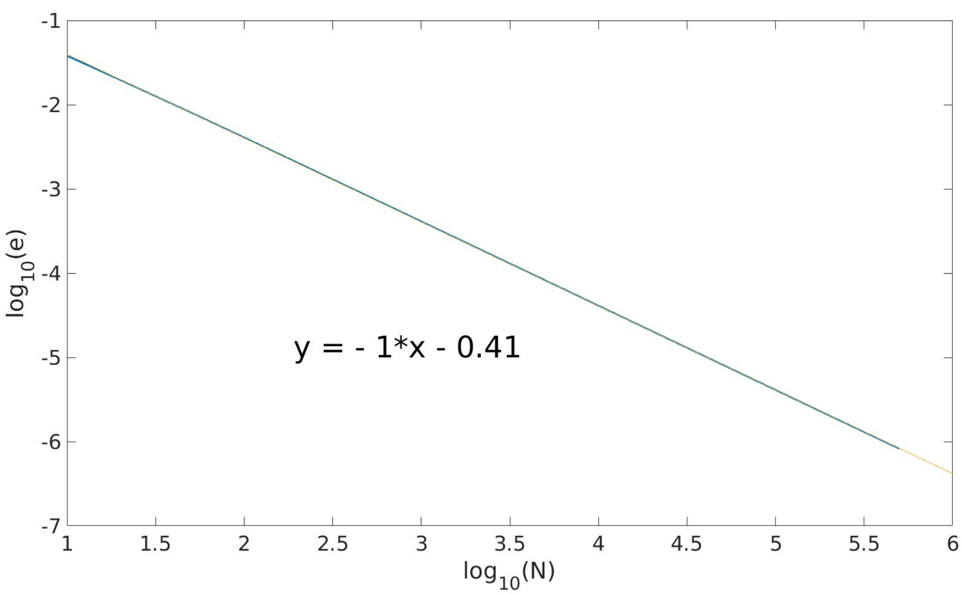


Figure 8: Time Complexity of solution using fast algorithm to evaluate the integral