Constructive Approximation Theory
Assignment 3

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1 MATLAB Approximation using Legendre nodes and Legendre polynomial

We use the Lagrangian polynomials to find the interpolant at the Legendre nodes. For the Legendre polynomial interpolant, given by the formula,

$$P_n(x) = \sum_{r=0}^{\infty} a_r Q_r(x)$$

we find the value of the Legendre polynonials, Q_r, at x by the following recursion

$$Q_r = ((2r-1)xQ_{r-1} - (r-1)Q_{r-2})/r$$

To find the Legendre cofficients, a_r, we use the following formula

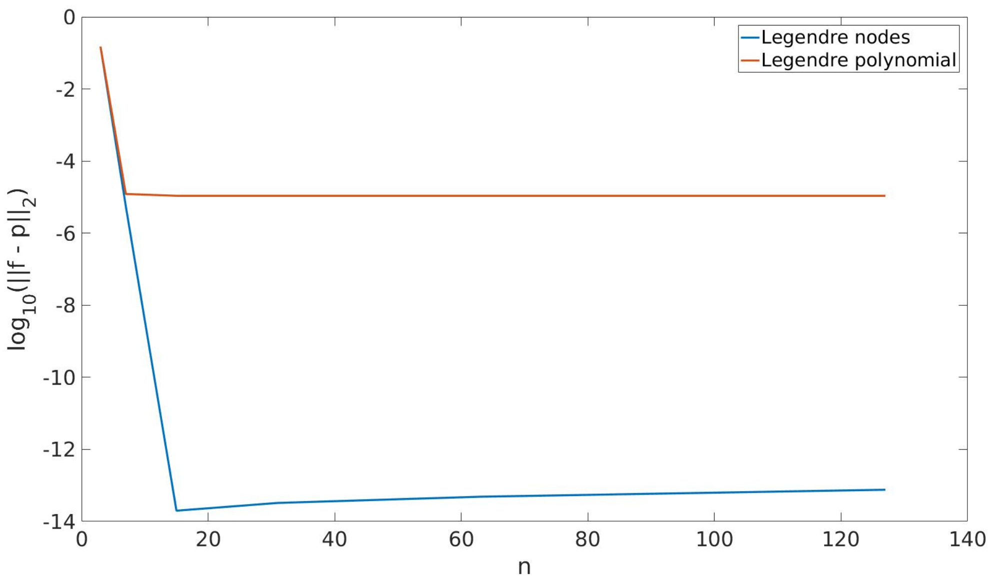
$$a_r = \frac{2r+1}{2^{r+1}r!} \int_{-1}^{1} f^{(r)}(x)(1-x^2)^r dx$$

A recurrence relation can be found for a_r, but it is numerically unstable because of accumulating finite precision error. We therefore evaluated the integral using trapezoidal rule (*trapz* function in MATLAB) with step size 0.001.

Table 1: 2-norm errors for respective interpolants for different n

n	Legendre Nodes	Legendre Polynomial
3.0	0.1503944276415	0.149638
7.0	0.000005534459	0.00001220
15.0	0.0000000000000199138	0.000010888
31.0	0.0000000000000326856	0.000010888
63.0	0.0000000000000485987	0.000010888
127.0	0.00000000000007595980	0.000010888

Figure 1: 2-norm errors for respective interpolants for different n



The interpolant found using Legendre nodes gives us lesser error than the one found using Legendre polynomials, this goes contrary to theory. This anamoly can be chalked up to errors in computations in finding the Legendre coefficients, and the error from the recursion relation used to find the Legendre polynomials.

Observations: Interpolation using Legendre polynomials is computationally expensive. Recurrence relations are not always reliable for computations, have to check for their stability. Using nodes to find the global polynomial is more preferrable than finding a series of polynomials and their coefficients.

2 Density of nodes

- Uniform nodes.
  Number of points in the interval $(x-\epsilon, x+\epsilon) = N\dfrac{2\epsilon}{2}$
  Size of the interval $= 2\epsilon$
  $\implies \rho_N(x) = \lim_{\epsilon\to 0} \dfrac{N\frac{\cancel{2\epsilon}}{2}}{\cancel{2\epsilon}} = \dfrac{N}{2}$

  $\implies \rho(x) = \lim_{N\to\infty} \dfrac{\frac{\cancel{N}}{2}}{\cancel{N}} = \dfrac{1}{2}$

- Chebyshev Nodes.
  Size of the interval $(x-\epsilon, x+\epsilon) = 2\epsilon$
  The length of a curve $y = f(x)$ from x_0 to x_1 is given by: $\int_{x_0}^{x_1} \sqrt{1+(y')^2}dx$. Now, equation of the unit circle in terms of y is given by $y = \sqrt{1-x^2}$.

  Therefore, length of the arc on the upper half of the unit circle from $x-\epsilon$ to $x+\epsilon$ is given by, $L = \displaystyle\int_{x-\epsilon}^{x+\epsilon} \sqrt{1+\frac{x^2}{1-x^2}}dx = \int_{x-\epsilon}^{x+\epsilon} \sqrt{\frac{1}{1-x^2}}dx$.

  Length of the unit half circle $= \pi$. Chebyshev nodes are projections onto the horizontal axis from unformly distributed points on a unit half circle. As the points on the half circle are uniformly distributed, therefore total number of points in the interval $(x-\epsilon, x+\epsilon) =$
  $N\dfrac{\displaystyle\int_{x-\epsilon}^{x+\epsilon} \sqrt{\frac{1}{1-x^2}}dx}{\pi}$

  So, $\rho_N(x) = \lim_{\epsilon\to 0} \dfrac{N}{2\epsilon\pi}\displaystyle\int_{x-\epsilon}^{x+\epsilon} \sqrt{\frac{1}{1-x^2}}dx$

  As the inerval is infinitesimally small when $\epsilon \to 0$, we can compute the integral using trapezoidal rule, which will give us,
  $\rho_N(x) = \lim_{\epsilon\to 0} \dfrac{N}{\cancel{2\epsilon}\pi}\dfrac{1}{2}\left(\sqrt{\dfrac{1}{1-(x+\epsilon)^2}} + \sqrt{\dfrac{1}{1-(x-\epsilon)^2}}\right)\cancel{2\epsilon}$
  $\implies \rho_N(x) = \dfrac{N}{\pi\sqrt{1-x^2}}$
  $\implies \rho(x) = \dfrac{1}{\pi\sqrt{1-x^2}}$

3 Chebyshev polynomials

$$T_n(x) = \cos(n \arccos(x))$$

- **Recurrence Relation**
  Consider the formula,

  $\cos(a) + \cos(b) = 2\cos\left(\dfrac{a+b}{2}\right)\cos\left(\dfrac{a-b}{2}\right)$
  Now, we can write,

  $\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos(n\theta)cos(\theta)$

  Taking $\theta = \arccos(x)$, we can write,

  $T_{n+1}(x) + T_{n-1}(x) = 2T_n(x)x$
  $\implies T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

- **To prove: Leading Coefficient of $T_n = 2^{n-1}$, and T_n is a polynomial of degree n**

  - **Proof by induction**

  - Basis: We can see that for $n = 1$, $T_1 = x = 2^0 x^1$.
  - Hypothesis: Let's assume T_n is a polynomial of degree n and its leading coefficient is 2^{n-1}, i.e.

    $T_n(x) = 2^{n-1} \prod_{k=0}^{n} (x - x_k)$.
  - Induction step: $T_{n+1} = 2xT_n - T_{n-1}$

    $\implies T_{n+1} = 2x2^{n-1} \prod_{k=0}^{n} (x - x_k) - T_{n-1}$

    $= 2^n x \prod_{k=0}^{n} (x - x_k) - T_{n-1}$

    As $T_n = 2xT_{n-1} - Tn - 2 \implies degree(T_{n-1}) < degree(T_n)$

    Therefore, $degree(T_{n+1}) = degree(x \prod_{k=0}^{n} (x - x_k)) = n + 1$ and the leading coefficient is 2^n. Hence, proved.

- Let x_k be any zero of T_{n+1}
  $T_{n+1}(x_k) = \cos((n + 1) \arccos(x_k))$
  Let $\theta = \arccos(x_k) \implies x_k = \cos(\theta)$
  As $\cos(\theta) \in [-1, 1]$, $\implies x_k \in [-1, 1]$
  Therefore all zeros of T_{n+1} lie in $[-1, 1]$.

  $T_{n+1} = 0 \implies \cos((n + 1) \arccos(x_k)) = 0$

  $\implies (n + 1) \arccos(x_k) = \pi(k + 1) - \pi/2 = \pi(k + 1) - \pi/2 = \frac{(2k + 1)\pi}{2}$, $k \in Z$

  $\implies \arccos(x_k) = \frac{2k + 1}{2n + 2}\pi$

  $\implies x_k = \cos(\frac{2k + 1}{2n + 2}\pi)$, $k \in \{0, 1, 2 \cdots n\}$ (Principal Solutions)

- As T_{n+1} has exactly $n+1$ distinct roots, it must intersect with the $x - axis$, $n+1$ times, and as the function is continous, it must change its sign each time it intersects with the horizontal axis. Thereforce, T_{n+1} changes its sign $n + 1$ times, and as it is an equioscillating function, we can say that it alternates between ± 1 exactly $n + 1$ times.

- We know that T_{n+1} has degree $n + 1$ and leading coefficient 2^n, thereforce we can write it in the form of its leading coefficient multiplied by a monic polynomial, i.e.

  $T_{n+1} = 2^n \prod_{k=0}^{n} (x - x_k)$

$\implies \prod_{k=0}^{n}(x - x_k) = \frac{T_{n+1}}{2^n}$
$\implies |\prod_{k=0}^{n}(x - x_k)| = \frac{|T_{n+1}|}{2^n}$
As $|T_{n+1}| \leq 1$,
$\implies |\prod_{k=0}^{n}(x - x_k)| \leq \frac{1}{2^n}, \forall x \in [-1, 1]$

- Let $y_{k}{}_{k=0}^{n} \in [-1, 1]$ be any set of nodes and $x_{k}{}_{k=0}^{n+1} \in [-1, 1]$ be the nodes where $|T_{n+1}(x_k)| = 1$.
  Let $P_{n+1} = \prod_{k=0}^{n}(x - y_k)$ and $|P_{n+1}| \leq 2^{-n}$.
  Now, let $T_{n+1}(x_j) = 1\ ,\ 0 \leq j \leq n$

  $\implies T_{n+1}(x_{j+1}) = 1$

  $\implies P_{n+1}(x_j) \leq 2^{-n}T_{n+1}(x_j)$
  And, $P_{n+1}(x_{j+1}) \geq 2^{-n}T_{n+1}(x_{j+1})$
  Assumption: $P_{n+1} \neq 2^{-n}T_{n+1}$ i.e., the two polynomials are distinct, and will be equal only in finite number of points.

  Now, as P_{n+1} is continous, we can say that for some $\zeta \in [x_j, x_{j+1}]$ considering the above relations, $P_{n+1}(\zeta) = 2^{-n}T_{n+1}(\zeta)$. Considering all $n+2$ extremum points of T_{n+1}, we can say that, $P_{n+1}(x) = 2^{-n}T_{n+1}(x)$ at at least $n+1$ points.

  Therefore,

  $F(x) = P_{n+1}(x) - 2^{-n}T_{n+1}(x)$

  will have at least $n+1$ roots. Thus $F(x)$ will be changing its sign at least $n+1$ times in $[-1.1]$. This follows directly from our assumption that P_{n+1} and $2^{-n}T_{n+1}$ will be equal only in a finite number of points. But $F(x) \in P_n$, i.e. it has degree n. Therefore we have a contradition, and our assumption is false.

  Thus, $P_{n+1} = 2^{-n}T_{n+1}$
  $\implies F(x) = 0$

  We have now proved that $2^{-n}T_{n+1}$ is the only polynomial P_{n+1} such that, $|P_{n+1}| \leq 2^{-n}$. Thus, the minimum of $\max_{x\in[-1.1]} \prod_{k=0}^{n}(x - z_k)$ will be given by Chebyshev nodes, i.e. when $(z_k)_{k=0}^{n}$ are Chebyshev nodes. And, that minimum value will be $\max_{x\in[-1.1]} \prod_{k=0}^{n}(x - z_k) = 2^{-n}$