Constructive Approximation Theory
Assignment 6

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1 Theorem:

$$|f(x) - T_n(x)| \leq \lambda \left(\frac{K}{n}\right)^{p+1} \text{ for all } x \in \Re$$

For:

$$|f^{(p)}(x) - f^{(p)}(y)| \leq \lambda |x - y| \text{ for all } x, y \in \Re$$

$$K = \frac{\pi^4 ln(2)}{4}$$

- **To prove (Drichlet Kernel):**

$$D_m(x) = \frac{\sin((m+1/2)x)}{\sin(x/2)} = 1 + 2\sum_{k=1}^{m} \cos(kx)$$

Proof by induction:

- Basis: For m = 1,

$$\frac{\sin(\frac{3x}{2})}{\sin(\frac{x}{2})} = \frac{\sin(x)\cos(\frac{x}{2})}{\sin(\frac{x}{2})} + \frac{\cos(x)\cancel{\sin(\frac{x}{2})}}{\cancel{\sin(\frac{x}{2})}}$$
$$= \frac{2\cos(\frac{x}{2})\cos(\frac{x}{2})\cancel{\sin(\frac{x}{2})}}{\cancel{\sin(\frac{x}{2})}} + \cos(x)$$
$$2\cos^2(\tfrac{x}{2}) + \cos(x)$$
$$= 1 + \cos(x) + \cos(x) = 1 + 2\cos(x)$$

So the identity holds for m = 1.

- Induction Hypothesis: Let, for m = n

$$\frac{\sin((n+1/2)x)}{\sin(x/2)} = 1 + 2\sum_{k=1}^{n} \cos(kx)$$

– Induction Step: For m = n + 1

$$\frac{\sin((n+1+1/2)x)}{\sin(x/2)} = \frac{\sin((n+\frac{1}{2}x)+x)}{\sin(x/2)}$$

$$= \frac{\sin((n+\frac{1}{2}x))}{\sin(x/2)}\cos(x) + \frac{\cos((n+\frac{1}{2}x))}{\cancel{\sin(\frac{x}{2})}} 2\cancel{\sin(\frac{x}{2})}\cos(\frac{x}{2})$$

Using Induction Hypothesis

$$= \left(1 + 2\sum_{k=1}^{n}\cos(kx)\right)\cos(x) + \cos((n+1)x) + \cos(nx)$$

$$= \cos(x) + 2\sum_{k=1}^{n}\cos(kx)\cos(x) + \cos((n+1)x) + \cos(nx)$$

$$= \cos(x) + \sum_{k=1}^{n}(\cos((k+1)x) + \cos((k-1)x)) + \cos((n+1)x) + \cos(nx)$$

$$= \cos(x) +$$
$$(1 + \cos(x) + 2\cos(2x) + 2\cos(3x)\cdots 2\cos((n-1)x) + \cos(nx) + \cos((n+1)x))$$
$$+ \cos((n+1)x) + \cos(nx)$$

$$= 1 + 2\sum_{k=1}^{n+1}\cos(kx)$$

Hence, proved.

- **To Prove:**

$$f_m(x) = \frac{\sin^2(mx/2)}{\sin^2(x/2)} = m + 2\sum_{k=1}^{m-1}(m-k)\cos(kx)$$

Proof by induction:

– Basis: For m = 1, result is trivial, our identity holds.

For m = 2, $\frac{\sin^2(x)}{\sin^2(x/2)} = \frac{4\cancel{sin^2(x/2)}\cos^2(x/2)}{\cancel{sin^2(x/2)}}$

$= 2(\cos(2x+1))$

$= 2 + 2cos(2x)$

– Indution Hypothesis: Let, for m = n,

$$\frac{\sin^2(nx/2)}{\sin^2(x/2)} = n + 2\sum_{k=1}^{n-1}(n-k)\cos(kx)$$

− Induction Step: For m = n + 1.

$$\frac{\sin^2((n+1)x/2)}{\sin^2(x/2)} = \frac{(\sin(nx/2)\cos(x/2)+\cos(nx/2)\sin(x/2))^2}{\sin^2(x/2)}$$
$$= \frac{\sin^2(nx/2)}{\sin^2(x/2)}\cos^2(x/2) + \frac{\cancel{\sin^2(x/2)}}{\cancel{\sin^2(x/2)}}\cos^2(nx/2) + 2\frac{\sin(nx/2)\cos(x/2)\cos(nx/2)\cancel{\sin(x/2)}}{\sin^{\cancel{2}}(x/2)}$$

$$= \left(\frac{\sin^2(nx/2)}{\sin^2(x/2)}\right)\cos^2(x/2) + \cos^2(nx/2) + 2\frac{\sin(mx)\cos(x/2)}{\sin(x/2)}$$

Using Induction Hypothesis

$$= \left(n + 2\sum_{k=1}^{n-1}(n-k)\cos(kx)\right)\cos^2(x/2)+\cos^2(nx/2)+\left(\frac{1}{2}\frac{\sin((n+\frac{1}{2})x)}{\sin(x/2)}\right)+\left(\frac{1}{2}\frac{\sin((n-\frac{1}{2})x)}{\sin(x/2)}\right)$$

Using Drichlet Kernel formula

$$= \left(n\cos^2(x/2) + 2\sum_{k=1}^{n-1}(n-k)\cos(kx)\cos^2(x/2)\right) + \cos^2(nx/2) +$$
$$\left(\frac{1}{2} + \sum_{k=1}^{n}\cos(kx)\right) + \left(\frac{1}{2} + \sum_{k=1}^{n-1}\cos(kx)\right)$$

$$= n\left(\frac{\cos(x)+1}{2}\right) + \sum_{k=1}^{n-1}(n-k)\cos(kx)\,(\cos(x)+1) + \cos^2(x/2) + 1 + 2\sum_{k=1}^{n-1}\cos(kx) +$$
$$\cos(nx)$$

$$= \frac{n\cos(x)}{2} + \frac{n}{2} + \sum_{k=1}^{n-1}(n-k)\cos(kx)\,(\cos(x)+1) + \frac{\cos(nx)}{2} + \frac{1}{2} + 1 + 2\sum_{k=1}^{n-1}\cos(kx) + \cos(nx)$$
$$= \frac{n\cos(x)}{2} + \frac{n}{2} + n\sum_{k=1}^{n-1}\cos(kx)\cos(x) - \sum_{k=1}^{n-1}k\cos(kx)\cos(x) + n\sum_{k=1}^{n-1}\cos(kx) - \sum_{k=1}^{n-1}k\cos(kx) +$$
$$\frac{\cos(nx)}{2} + \frac{1}{2} + 1 + 2\sum_{k=1}^{n-1}\cos(kx) + \cos(nx)$$

$$= \frac{n\cos(x)}{2} + \frac{n}{2} + n\sum_{k=1}^{n-1}\frac{(\cos((k+1)x) + \cos((k-1)x)}{2} - \sum_{k=1}^{n-1} k\frac{(\cos((k+1)x) + \cos((k-1)x)}{2} +$$

$$n\sum_{k=1}^{n-1}\cos(kx) - \sum_{k=1}^{n-1} k\cos(kx) + \frac{\cos(nx)}{2} + \frac{1}{2} + 1 + 2\sum_{k=1}^{n-1}\cos(kx) + \cos(nx)$$

$$= \frac{n}{2} + \frac{n\cos(x)}{2} +$$

$$n\left(\frac{1}{2} + \frac{\cos(x)}{2} + \cos(2x) + \cos(3x) + \cdots \cos((n-2)x) + \frac{\cos((n-1)x)}{2} + \frac{\cos(nx)}{2}\right)$$

$$-\left(\cancel{\frac{1}{2}} + \cos(x) + 2\cos(x) + \cdots (n-2)\cos((n-2)x) + \frac{n-2}{2}\cos((n-1)x) + \frac{n-1}{2}\cos(nx)\right)$$

$$+ n\sum_{k=1}^{n-1}\cos(kx) - \sum_{k=1}^{n-1} k\cos(kx) + \frac{\cos(nx)}{2} + \cancel{\frac{1}{2}} + 1 + 2\sum_{k=1}^{n-1}\cos(kx) + \cos(nx)$$

$$= n + n\sum_{k=1}^{n-2}\cos(kx) + n\frac{\cos((n-1)x)}{2} + n\frac{\cos(nx)}{2} - \sum_{k=1}^{n-2}\cos(kx) - (n-2)\cos((n-2)x)$$

$$- \frac{n-2}{2}\cos((n-1)x) - \frac{n-1}{2}\cos(nx) + n\sum_{k=1}^{n-1}\cos(kx) - \sum_{k=1}^{n-1} k\cos(kx)$$

$$+ \frac{\cos(nx)}{2} + 1 + 2\sum_{k=1}^{n-1}\cos(kx) + \cos(nx)$$

$$= n + 1 + 2n\sum_{k=1}^{n-2}\cos(kx) + n\cos((n-1)x) + n\frac{\cos((n-1)x)}{2} + n\frac{\cos(nx)}{2}$$

$$- \frac{n-2}{2}\cos((n-1)x) - \frac{n-1}{2}\cos(nx) + \frac{\cos(nx)}{2} + \cos(nx) - 2\sum_{k=1}^{n-2}\cos(kx)$$

$$- (n-1)\cos((n-1)x) + 2\sum_{k=1}^{n-1}\cos(kx)$$

$$= n + 1 + 2n\sum_{k=1}^{n-2}\cos(kx) + (n+1)\cos((n-1)x)$$

$$- 2\sum_{k=1}^{n-2}\cos(kx) - (n-1)\cos((n-1)x) + 2\sum_{k=1}^{n}\cos(kx)$$

Adding the remaining required terms to the sums, and subtracting them from the rest of the terms

$$= n + 1 + 2n\sum_{k=1}^{n}\cos(kx) - 2\sum_{k=1}^{n} k\cos(kx) + 2\sum_{k=1}^{n}\cos(kx)$$

$$+ \cancel{(n+1)\cos((n-1)x)} - \cancel{(n-1)\cos((n-1)x)} - \cancel{2n\cos((n-1)x)}$$

$$- \cancel{2n\cos(nx)} + \cancel{2(n-1)\cos((n-1)x)} + \cancel{2n\cos(nx)}$$

$$= n + 1 + 2\sum_{k=1}^{n}(n+1-k)\cos(kx)$$

Hence, proved.

- **To Prove:**

$$g_m(x) = \frac{\sin^4(mx/2)}{\sin^4(x/2)} = \sum_{k=0}^{2m-2} a_k \cos(kx)$$

Proof:

$$\frac{\sin^4(mx/2)}{\sin^4(x/2)} = \frac{\sin^2(mx/2)}{\sin^2(x/2)}\frac{\sin^2(mx/2)}{\sin^2(x/2)}$$

$$= \left(m + 2\sum_{k=1}^{m-1}(m-k)\cos(kx)\right)\left(m + 2\sum_{k=1}^{m-1}(m-k)\cos(kx)\right)$$

$$= \left(m + \textstyle\sum_{k=1}^{m-1} b_k \cos(kx)\right)\left(m + \displaystyle\sum_{k=1}^{m-1} b_k \cos(kx)\right)$$

$$= m^2 + 2m\sum_{k=1}^{m-1} b_k \cos(kx) + \left(\sum_{k=1}^{m-1} b_k \cos(kx)\right)\left(\sum_{k=1}^{m-1} b_k \cos(kx)\right)$$

$$= m^2 + 2m\sum_{k=1}^{m-1} b_k \cos(kx) + \sum_{k=1}^{m-1}\sum_{j=1}^{m-1} c_{kj}\cos(kx)\cos(jx)$$

$$= m^2 + 2m\sum_{k=1}^{m-1} b_k \cos(kx) + \sum_{k=1}^{m-1}\sum_{j=1}^{m-1} c_{kj}\left(\frac{\cos((k+j)x) + \cos((k-j)x)}{2}\right)$$

$$= m^2 + 2m\sum_{k=1}^{m-1} b_k \cos(kx) + \sum_{k=1}^{2m-2} d_{kj}\cos(kx)$$

$$= \sum_{k=0}^{2m-2} a_k \cos(kx)$$

-

$$h_m = \int_{-\pi/2}^{\pi/2} g_m(2x)dx$$

$$\tilde{f}(x) = \frac{1}{h_m} \int_{-\pi/2}^{\pi/2} f(x+2u) g_m(2u) du$$

Taking $v = x + 2u, \implies du = \frac{dv}{2}$

$$\tilde{f}(x) = \frac{1}{2h_m} \int_{x-\pi}^{x+\pi} f(v) g_m(v-x) dv$$

- Let $G(a) = \int_{a-\pi}^{a+\pi} f(v) g_m(v-x) dv$
  $\implies \frac{d(G(a))}{da} = f(a+\pi) g_m(a+\pi-x) - f(a-\pi) g_m(a-\pi-x)$

  As $f(x)$ and $g_m(x)$ are 2π periodic,

  $f(a+\pi) g_m(a+\pi-x) = f(a-\pi) g_m(a-\pi-x)$
  $\implies \frac{d(G(a))}{da} = 0 \implies \frac{1}{2h_m} \int_{a-\pi}^{a+\pi} f(v) g_m(v-x) dv = \frac{1}{2h_m} \int_{-\pi}^{\pi} f(v) g_m(v-x) dv$
  Replacing a with x, we can write,

  $\tilde{f}(x) = \frac{1}{2h_m} \int_{a-\pi}^{x+\pi} f(v) g_m(v-x) dv = \frac{1}{2h_m} \int_{-\pi}^{\pi} f(v) g_m(v-x) dv$

- $\tilde{f}(x) = \frac{1}{2h_m} \int_{-\pi}^{\pi} f(v) g_m(v-x) dv$
  $= \sum_{k=0}^{2m-2} \frac{a_k}{2h_m} \int_{-\pi}^{\pi} f(v) \cos(k(v-x)) dv$
  $= \sum_{k=0}^{2m-2} \frac{a_k}{2h_m} \int_{-\pi}^{\pi} f(v) \left(\cos(kv)\cos(kx) + \sin(kv)\sin(kx)\right) dv$
  $= \sum_{k=0}^{2m-2} \left(\left(\frac{a_k}{2h_m} \int_{-\pi}^{\pi} \cos(kv) f(v) \right) \cos(kx) + \left(\frac{a_k}{2h_m} \int_{-\pi}^{\pi} \sin(kv) f(v) \right) \sin(kx) \right)$
  $= \sum_{k=0}^{2m-2} \left(c_k \cos(kv) + d_k \sin(kv) \right)$

  Where,
  $c_k = \frac{a_k}{2h_m} \int_{-\pi}^{\pi} \cos(kv) f(v)$, and

$d_k = \frac{a_k}{2h_m}\int_{-\pi}^{\pi} \sin(kv) f(v)$

As c_k and d_k are constants with respect to x, we can conclude that $\tilde{f}(x)$ is a trigonometric polynomial of order $2m-2$.

- $\left|\tilde{f}(x) - f(x)\right| = \left| \left(\frac{1}{h_m}\int_{-\pi/2}^{\pi/2} f(x+2u) g_m(2u) du \right) - f(x) \right|$

$= \left| \frac{1}{h_m}\int_{-\pi/2}^{\pi/2} f(x+2u) g_m(2u) du - f(x) \frac{\int_{-\pi/2}^{\pi/2} g_m(2u) du}{h_m} \right|$

$= \frac{1}{h_m}\int_{-\pi/2}^{\pi/2} |(f(x+2u) - f(x))| g_m(2u) du,$

Modulus sign goes inside the integral because $g_m(2u) \geq 0$

$\implies \left|\tilde{f}(x) - f(x)\right| \leq \left| \frac{1}{h_m}\int_{-\pi/2}^{\pi/2} 2|u|\lambda g_m(2u) du = 2\lambda \frac{\cancel{2}\int_0^{\pi/2} u g_m(2u) du}{\cancel{2}\int_0^{\pi/2} g_m(2u) du}\right.$

- Let $Si(x) = \int_0^x \frac{\sin(s)}{s} ds$

We use Laplace transform to evaluate the improper integral $Si(\infty)$

Thus, $Si(\infty) = \int_0^\infty \frac{\sin(s)}{s} ds$

$= \int_0^\infty \mathcal{L}\left\{\sin(t)\right\}(s) ds$

$= \int_0^\infty \frac{1}{s^2+1} ds = \arctan(s)\Big|_0^\infty = \frac{\pi}{2}$

- Let $Si_k(x) = \int_0^x \frac{\sin(ks)}{s} ds$

Let $z = ks \implies ds = dz/s$

$\implies Si_k(x) = \int_0^x \frac{\sin(z)}{\frac{z}{\cancel{k}}} \frac{dz}{\cancel{k}} = \int_0^\infty \frac{\sin(z)}{z} dz$

Thus $Si_k(x) = Si(x)$

- Let $F(a) = \int_0^\infty \frac{\sin^4(as)}{s^4} ds$

$F(0) = 0$

$F'(a) = \int_0^\infty \frac{4\sin^3(ax)\cos(ax)}{x^3} dx$

$$= \int_0^\infty \frac{(1-\cos(2ax))\sin(2ax)}{x^3} dx$$
$$= \frac{1}{2}\int_0^\infty \frac{2\sin(2ax)-\sin(4ax)}{x^3} dx$$

$$F'(0) = 0$$

$$F''(a) = 2\int_0^\infty \frac{\cos(2ax)-\cos(4ax)}{x^2} dx$$

$$F''(0) = 0$$

$$F'''(a) = 2\int_0^\infty \frac{-2\sin(2ax)+4\sin(4ax)}{x} dx$$
$$= 4\left(2\int_0^\infty \frac{\sin(4ax)}{x} dx - \int_0^\infty \frac{\sin(2ax)}{x} dx\right)$$

Using earlier proof that $Si_k(x) = Si(x)$ and $Si(\infty) = \pi/2$

$$\implies F'''(a) = 4\int_0^\infty \frac{\sin(x)}{x} dx = 2\pi$$

$$\implies F''(a) = 2\pi a$$

$$\implies F'(a) = \pi a^2$$

$$\implies F(a) = \frac{\pi a^3}{3}$$

$$\implies F(1) = \frac{\pi}{3}$$

$$\implies \int_0^\infty \frac{\sin^4(x)}{x^4} dx = \frac{\pi}{3}$$

- $$I = \int_0^\infty \frac{\sin^4(x)}{x^3} dx$$

  We will evaluate this integral analytically, using Laplace Transforms, using the following formulas.

$$\int_0^\infty F(u)g(u)du = \int_0^\infty f(u)G(u)du$$

Where,

$$F(u) = \mathcal{L}\left[f(t)\right]$$

And,

$$G(u) = \mathcal{L}\left[g(t)\right]$$

Let $f(t) = \sin^4(t) = \dfrac{e^{it} - e^{-it}}{(2i)^4}$

Now, $F(u) = \mathcal{L}\left[f(t)\right] = \displaystyle\int_0^\infty f(t)e^{-ut}dt$

$= \dfrac{1}{16}\displaystyle\int_0^\infty \left(e^{i4t} - 4e^{i2t} + 6 - 4e^{-i2t} + e^{-i4t}\right)e^{-ut}dt$

$= \dfrac{1}{16}\left(\dfrac{1}{u-4i} - 4\dfrac{1}{u-2i} + \dfrac{6}{u} - 4\dfrac{1}{u+2i} + \dfrac{1}{u+4i}\right)$

$= \dfrac{1}{8}\left(\dfrac{u}{u^2+16} - 4\dfrac{u}{u^2+4} + \dfrac{3}{u}\right)$

Let $G(u) = \dfrac{1}{u^3} = \mathcal{L}\left[g(t)\right]$

Using the Laplace Transform formula,

$$\mathcal{L}\left[\frac{t^{n-1}}{(n-1)!}\right] = \frac{1}{s^n}$$

We get,

$$g(u) = \frac{u^2}{2}$$

Thus, $F(u)g(u) = \dfrac{1}{16}\dfrac{u^3}{u^2+16} - \dfrac{1}{4}\dfrac{u^3}{u^2+4} + \dfrac{3}{16}u$

$= \dfrac{u}{16} - \dfrac{u}{u^2+16} - \dfrac{u}{4} + \dfrac{u}{u^2+4} + \dfrac{3}{16}u$

$= \dfrac{u}{u^2+4} - \dfrac{u}{u^2+16}$

Now,

$\displaystyle\int_0^\infty F(u)g(u)du = \int_0^\infty f(u)G(u)du$

$\displaystyle\implies \int_0^\infty \frac{\sin^4(u)}{u^3}du = \int_0^\infty F(u)g(u)du = \lim_{N\to\infty}\left(\int_0^N \frac{u}{u^2+4} - \int_0^N \frac{u}{u^2+16}\right)$

$$= \frac{1}{2}\lim_{N\to\infty}\big(\ln(N^2+4)-\ln(4)-\ln(N^2+16)+\ln(16)\big)$$
$$= \frac{1}{2}(\ln(16)-\ln(4))$$
$$= \ln(2)$$

- $\displaystyle\int_{m\pi/2}^{\infty}\frac{\sin^4(x)}{x^4}dx = \sum_{k=m}^{\infty}\int_{k\pi/2}^{(k+1)\pi/2}\frac{\sin^4(x)}{x^4}dx < \sum_{k=m}^{\infty}\int_{k\pi/2}^{(k+1)\pi/2}\frac{\sin^4(x)}{(k\pi/2)^4}dx = \sum_{k=m}^{\infty}\frac{16}{k^4\pi^4}\frac{3\pi}{16}$

  $$\left(\int_{k\pi/2}^{(k+1)\pi/2}\sin^4(x)dx = \frac{3\pi}{16},\ \text{for } k\in Z\right)$$
  $$\implies \int_{m\pi/2}^{\infty}\frac{\sin^4(x)}{x^4}dx < \frac{3}{\pi^3}\sum_{k=m}^{\infty}\frac{1}{k^4}$$

  As $\frac{1}{k^4}$ is a strictly decreasing function in $(0,\infty]$, we can write,
  $$\frac{1}{m^4} < \int_{m-1}^{m}\frac{1}{k^4}dk$$
  $$\implies \sum_{k=m}^{\infty}\frac{1}{k^4} < \int_{m-1}^{\infty}\frac{1}{k^4}dk = \frac{1}{3(m-1)^3}$$
  Thus,
  $$\int_{m\pi/2}^{\infty}\frac{\sin^4(x)}{x^4}dx < \frac{1}{\pi^3(m-1)^3}$$
  $$\left(\frac{m}{m-1}\right)^3 \leq 8 \text{ for } m\geq 2$$

  Therefore, for $m\geq 2$,
  $$\int_{m\pi/2}^{\infty}\frac{\sin^4(x)}{x^4}dx < \frac{8}{\pi^3m^3}$$

- $\displaystyle\int_0^{\pi/2} g_m(2u)du = \int_0^{\pi/2}\frac{\sin^4(mu)}{\sin^4(u)}du$

  $$= \frac{1}{m}\int_0^{m\pi/2}\frac{\sin^4(t)}{\sin^4(t/m)}dt \text{ (Substituting } t=mu\text{)}$$
  Now,
  $$\frac{1}{m}\int_0^{m\pi/2}\frac{\sin^4(t)}{\sin^4(t/m)}dt \leq \frac{1}{m}\int_0^{m\pi/2}\frac{\sin^4(t)}{(t/m)^4}dt \text{ (Because } (t/m)^4\geq\sin^4(t/m))$$
  $$\implies \int_0^{\pi/2}\frac{\sin^4(mu)}{\sin^4(u)}du \leq m^3\int_0^{m\pi/2}\frac{\sin^4(t)}{t^4}dt$$

Now, $\displaystyle\int_{m\pi/2}^{\infty} \frac{\sin^4(x)}{x^4}dx < \frac{8}{3\pi^3 m^3}$ (Given)

$$\implies \int_0^{\infty} \frac{\sin^4(x)}{x^4}dx - \int_0^{m\pi/2} \frac{\sin^4(x)}{x^4}dx < \frac{8}{3\pi^3 m^3}$$

$$\implies \pi/3 - \int_0^{m\pi/2} \frac{\sin^4(x)}{x^4}dx < \frac{8}{3\pi^3 m^3}$$

$$\implies \int_0^{m\pi/2} \frac{\sin^4(x)}{x^4}dx > \left(\frac{\pi}{3} - \frac{8}{3\pi^3 m^3}\right)$$

$$\implies m^3 \int_0^{m\pi/2} \frac{\sin^4(x)}{x^4}dx > m^3 \left(\frac{\pi}{3} - \frac{8}{3\pi^3 m^3}\right)$$

- $\displaystyle\int_0^{\pi/2} u g_m(2u)du = \int_0^{\pi/2} \frac{u\sin^4(mu)}{\sin^4(u)}du < \frac{\pi^4}{16}\int_0^{\pi/2} \frac{\cancel{u}\sin^4(mu)}{u^{\cancel{4}3}}du$ (Because $\sin(u) \geq \dfrac{u}{\pi/2} \implies \sin^4(u) \geq \dfrac{u^4}{(\pi/2)^4}$ in $[0, \pi/2]$. $\sin(u)$ is a concave curve, and $\dfrac{u}{\pi/2}$ will be under it always in $[(0, \pi/2]$)

  Now, $\displaystyle\frac{\pi^4}{16}\int_0^{\pi/2} \frac{\sin^4(mu)}{u^3}du = \frac{\pi^4 m^2}{16}\int_0^{m\pi/2} \frac{\sin^4(t)}{t^3}dt$ (By substituting $t = mu$)

  $\displaystyle\int_0^{m\pi/2} \frac{\sin^4(t)}{t^3}dt < \int_0^{\infty} \frac{\sin^4(t)}{t^3}dt$

  $\displaystyle\implies \frac{\pi^4 m^2}{16}\int_0^{m\pi/2} \frac{\sin^4(t)}{t^3}dt < \frac{\pi^4 m^2 \ln(2)}{16}$

2 Fourier series approximation using MATLAB:

$$g(x) = x(1-x)$$

$$f(x) = g(x - [x]), \quad \text{where} \quad [x] = floor(x)$$

We approximate $f(x)$ using Fourier series, $F(x)$, truncated upto n terms, for different values of n. The co-efficients were computed using the formulas:

$$F(x) = a_0 + \sum_{k=1}^{k} (a_k \cos(2kx\pi) + b_n \sin(2kx\pi))$$

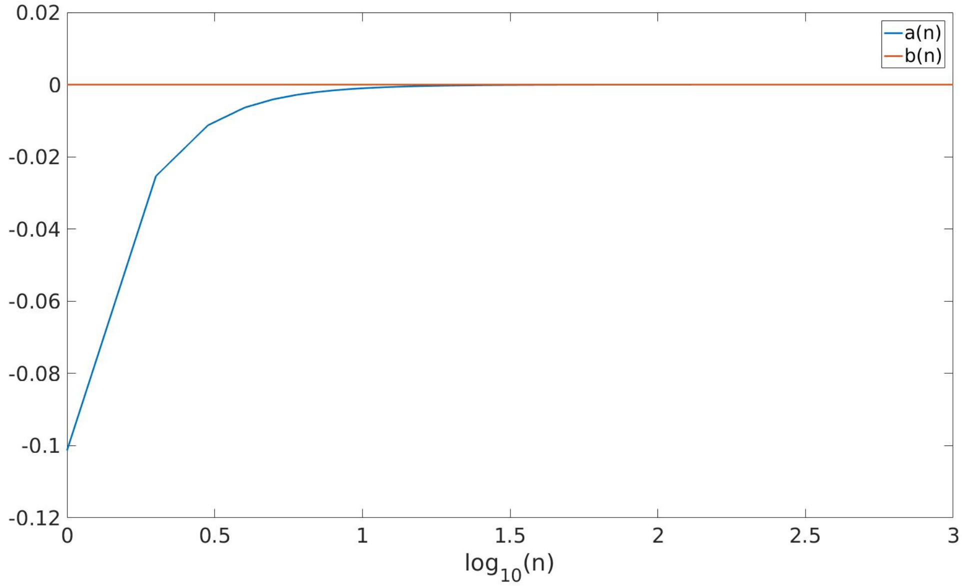
$$a_0 = \int_{-0.5}^{0.5} f(x)dx$$

$$a_n = 2\int_{-0.5}^{0.5} f(x)\cos(2kx\pi)dx$$

$$b_n = 2\int_{-0.5}^{0.5} f(x)\sin(2kx\pi)dx$$

We used MATLAB's *trapz* (trapezoidal numerical integration) function to compute the integral, with step size of 0.001

Figure 1: Decay of n^{th} Fourier co-efficients



Observations: The second co-efficient (b_n) remains zero, while the co-efficient (a_n) decays exponentially.