Constructive Approximation Theory
Assignment 4

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1 Young's inequality

To prove:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Given:

$$a, b \geq 0, \ , \frac{1}{p} + \frac{1}{q} = 1$$

Proof:

Let us consider the function $f(x) = ln(x)$ between points x_1 and x_2. As $ln(x)$ is a strictly concave function, all the function values between x_1 and x_2 will be greater than or equal to the straight line joining x_1 and x_2, i.e. the $ln(x)$ curve will be above the straight line.

We can write any point between x_1 and x_2 as,

$x_t = tx_1 + (1-t)x_2, \ t \in [0, 1]$

Value of the straight line (secant line) at x_t is $tln(x_1) + (1-t)ln(x_2)$

Therefore, we can write,

$tln(x_1) + (1-t)ln(x_2) \leq ln(tx_1 + (1-t)x_2)$
$\implies e^{tln(x_1)}e^{(1-t)ln(x_2)} \leq tx_1 + (1-t)x_2$
$\implies x_1^t x_2^{(1-t)} \leq tx_1 + (1-t)x_2$

As $\frac{1}{p} + \frac{1}{q} = 1, \implies p, q \geq 1, \implies \frac{1}{p}, \frac{1}{q} \in [0, 1]$

So, let $t = \frac{1}{p}. \implies 1 - t = \frac{1}{q}$

And, let $x_1 = a^p, x_2 = b^q$

$$\implies (a^p)^{\frac{1}{p}}(b^q)^{\frac{1}{q}} \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$$\implies ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

2 Holder's inequality

Let,

$$a = \frac{|f(x)|}{||f||_p}$$

$$b = \frac{|g(x)|}{||g||_q},$$

and, $\frac{1}{p} + \frac{1}{q} = 1$

Then, by Young's Inequality

$$\frac{|f(x)g(x)|}{||f||_p||g||_q} \leq \frac{|f(x)|^p}{p||f||_p^p} + \frac{|g(x)|^q}{q||g||_q^q}$$

Integrating both sides,

$$\frac{||fg||_1}{||f||_p||g||_q} \leq \frac{||f||_p^p}{p||f||_p^p} + \frac{||g||_q^q}{q||g||_q^q} = \frac{1}{p} + \frac{1}{q} = 1$$

$$\implies \frac{||fg||_1}{||f||_p||g||_q} \leq 1$$

$$\implies ||fg||_1 \leq ||f||_p||g||_q$$

3 To prove:

$$L^p([-1,1]) \supset L^q([-1,1]).\ for\ p < q$$

Proof:

Considering $\frac{1}{q/p} + \frac{1}{q/(q-p)} = 1$, and applying Holder's inequality for $|f|^p$ and 1,

$$\int_{-1}^{1} |f|^p.1dx \leq \left(\int_{-1}^{1} |f|^{p\frac{q}{p}}dx\right)^{\frac{p}{q}} \left(\int_{-1}^{1} dx\right)^{\frac{q-p}{q}}$$

$$\implies \int_{-1}^{1} |f|^p dx \leq \left(\int_{-1}^{1} |f|^q dx\right)^{\frac{p}{q}} 2^{1-\frac{p}{q}}$$

As $p < q \implies 1 - \frac{p}{q} \in [0,1] \implies 2^{1-\frac{p}{q}} \in [1,2]$

Therefore, we can write,

$$\int_{-1}^{1} |f|^p dx \leq \left(\int_{-1}^{1} |f|^q dx\right)^{\frac{p}{q}}$$

$$\implies \left(\int_{-1}^{1} |f|^p dx\right)^{\frac{1}{p}} \leq \left(\int_{-1}^{1} |f|^q dx\right)^{\frac{1}{q}}$$

$$\implies ||f||_p \leq ||f||_q$$

$$\implies L^p([-1,1]) \supset L^q([-1,1])$$

3.1 To find:

Function $f \in L^p([-1,1]) \backslash L^q([-1,1]), p < q$

Let us consider the function

$$f(x) = \begin{cases} |x|^{\frac{-1}{p+1}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \tag{1}$$

Now,

$$||f||_p = \left(\int_{-1}^{1} |x|^{\frac{-p}{p+1}} dx\right)^{\frac{1}{p}} = \left(2\int_{0}^{1} x^{\frac{-p}{p+1}} dx\right)^{\frac{1}{p}}$$

$$= \left(\left[2(p+1)x^{\frac{1}{p+1}}\right]_0^1\right)^{\frac{1}{p}} = (2(p+1))^{\frac{1}{p}} < \infty$$

$$\implies f \in L^p([-1,1])$$

Now,

$$||f||_q = \left(2\int_{0}^{1} x^{\frac{-q}{p+1}} dx\right)^{\frac{1}{q}}$$

If $q = p+1$, $\displaystyle\int_{0}^{1} x^{\frac{-q}{p+1}} = ln(1) - ln(0) \to \infty$

$$\implies ||f||_q \to \infty$$

If $q > p+1$

$$||f||_q = 2\left(\frac{-q+p+1}{p+1}\left[x^{\frac{-q+p+1}{p+1}}\right]_0^1\right)^{\frac{1}{q}}$$

As $q > p+1 \implies \dfrac{q-p-1}{p+1} > 0$

Let $\dfrac{q-p-1}{p+1} = c$

Therefore, $||f||_q = 2\left(\frac{q-p-1}{p+1}\left[\frac{1}{x^c}\right]_1^0\right)^{\frac{1}{q}}$

As $c > 0$, $\frac{1}{x^c} \to \infty$ as $x \to 0 \implies ||f||_q \to \infty$

Therefore,
$f \in L^p([-1,1]) \backslash L^q([-1,1])$

4 Uniform convergence implies convergence in p norm

If the function $f_n - f$ converges uniformly in $[-1.1]$, we can write

$|f_n - f| < \epsilon$ for any $\epsilon > 0$
$\implies |f_n - f| \to 0$
$\implies \max_{x\in[-1,1]} |f_n - f| \to 0$
$\implies |f_n - f|_\infty \to 0$

As $p < \infty$ we can write, $L^p([-1,1]) \supset L^\infty([-1,1])$

$\implies |f_n - f|_p \leq |f_n - f|_\infty$
$\implies |f_n - f|_p \to 0$

5 Runge Approximation Using Bernstein

As the interval is $[-1,1]$ we modified the Bernstein polynomial as follows

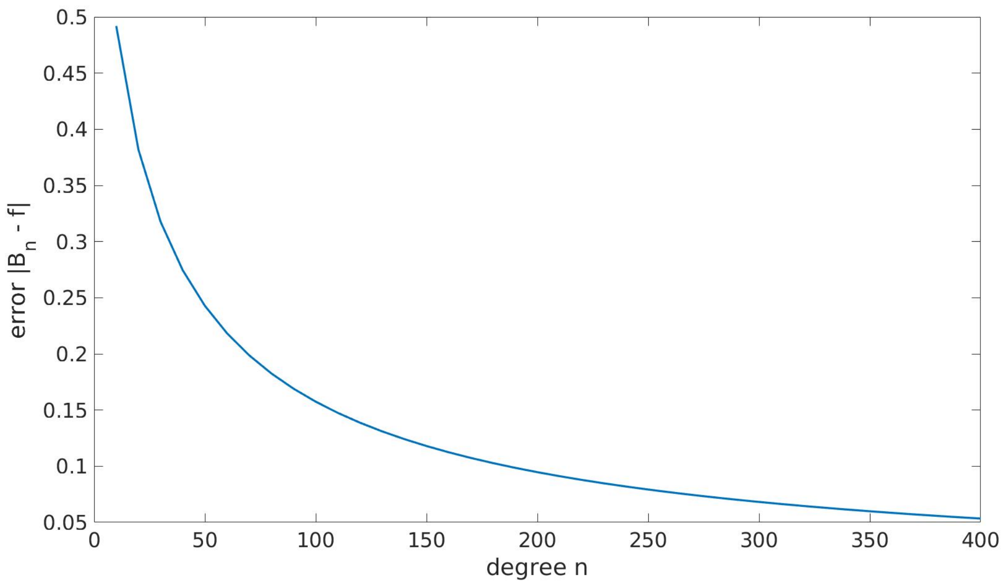
$$B(f;x) = \sum_0^n \binom{n}{r}(x/2+0.5)^r(1-(x/2+0.5))^{(n-r)} f(2r/n-1)$$

The plots are given below.

Table 1: Error values for the Bernstein approximation

n	e_n
10.0	0.4919
20.0	0.3819
30.0	0.3176
40.0	0.2744
50.0	0.2427
60.0	0.2183
70.0	0.1987
80.0	0.1825
90.0	0.169
100.0	0.1574
110.0	0.1474
120.0	0.1387
130.0	0.1309
140.0	0.1241
150.0	0.1179
160.0	0.1123
170.0	0.1073
180.0	0.1027
190.0	0.0985
200.0	0.0946
210.0	0.0911
220.0	0.0878
230.0	0.0847
240.0	0.0819
250.0	0.0792
260.0	0.0767
270.0	0.0744
280.0	0.0722
290.0	0.0701
300.0	0.0681
310.0	0.0663
320.0	0.0646
330.0	0.0629
340.0	0.0613
350.0	0.0598
360.0	0.0584
370.0	0.0571
380.0	0.0558
390.0	0.0545
400.0	0.0534

Figure 1: Error behavior for the Bernstein interpolant



From the plot, we can conclude that the error behavior exhibits exponential decay (decreases more slowly as we increase n).

Figure 2: Degree 10

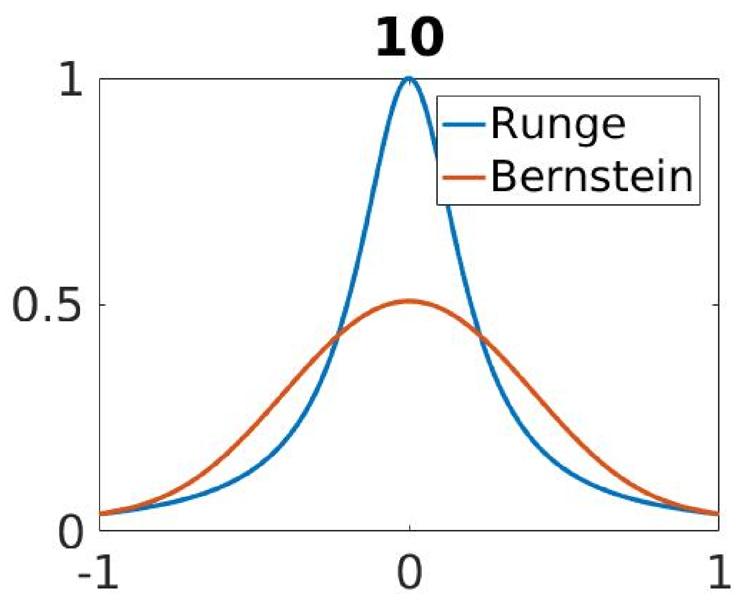


Figure 3: Degree 20

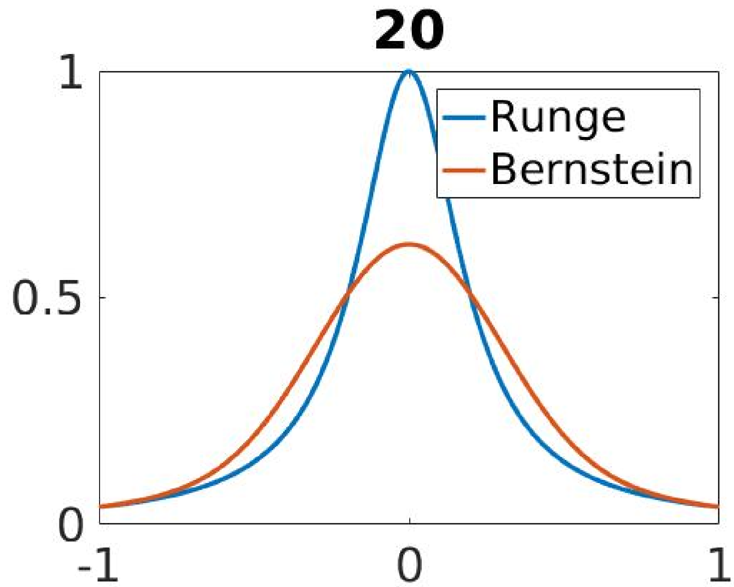


Figure 4: Degree 50

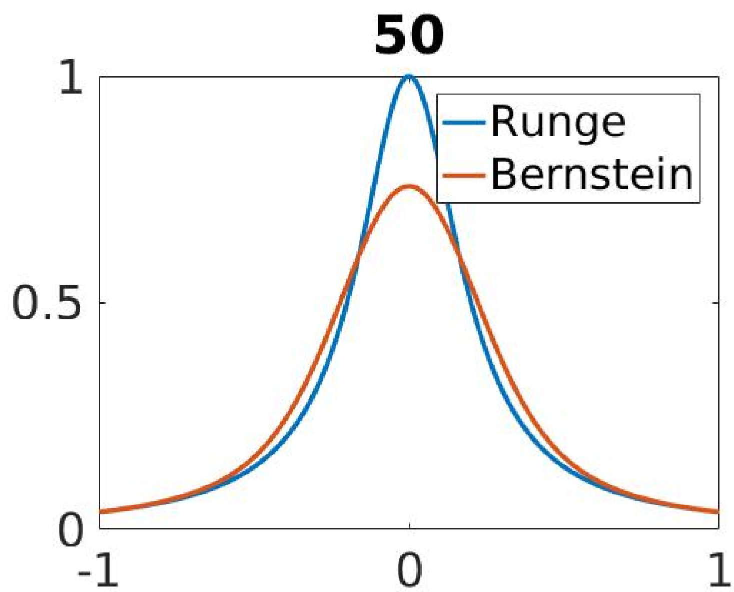


Figure 5: Degree 100

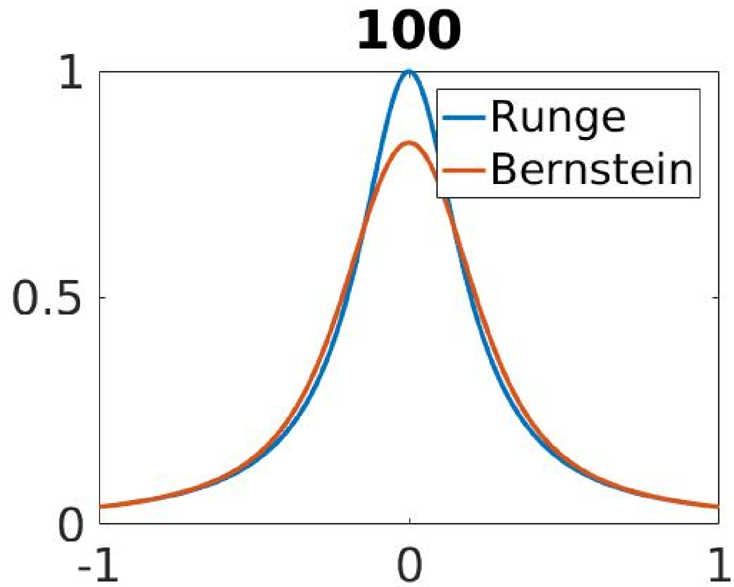


Figure 6: Degree 200