Constructive Approximation Theory
Assignment 9

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1 Aliasing of Trigonometric functions

$C_k = \cos(2k\pi x)$ For $n \geq 1$ and $0 \leq m \leq n$,

Let $k = Nn \pm m$ where $N = 1, 2, \cdots$

For the $n+1$ grid, let $x = \frac{j}{n}$ where $j = 0, 1, \cdots n$

$C_k = \cos(2(Nn \pm m)\pi\frac{j}{n}) = \cancel{\cos(2Nj\pi)}\cos(2m\pi\frac{j}{n}) \mp \cancel{\sin(2Nj\pi)\sin(2m\pi\frac{j}{n})} = C_m$

Thus,

$$C_m = C_{n\pm m} = C_{2n\pm m} = C_{3n\pm m}\cdots$$

Similarly, for $S_k = \sin(2k\pi x)$

$S_k = \sin(2(Nn \pm m)\pi\frac{j}{n}) = \cancel{\sin(2Nj\pi)\cos(2m\pi\frac{j}{n})} \pm \cancel{\cos(2Nj\pi)}\sin(2m\pi\frac{j}{n}) = \pm S_m$
Thus

$$S_m = \pm S_{n\pm m} = \pm S_{2n\pm m} = \pm S_{3n\pm m}\cdots$$

1.1 Aliasing of Chebyshev polynomials

$T_k = \cos(k \arccos(x))$ For For $n \geq 1$ and $0 \leq m \leq n$

Let $k = 2Nn \pm m$ where $N = 1, 2, \cdots$

For the $n+1$ grid, let $x = \cos(\frac{j\pi}{n})$ where $j = 0, 1, \cdots n$

$T_k = \cos(k\frac{j\pi}{n}) = \cos((2Nn \pm m)\frac{j\pi}{n}) = \cancel{\cos(2Nj\pi)}\cos(m\pi\frac{j}{n}) \mp \cancel{\sin(2Nj\pi)\sin(m\pi\frac{j}{n})} = T_m$

Thus,

$$T_m = T_{2n \pm m} = T_{4n \pm m} = T_{6n \pm m}$$

For $k = 2n \pm m \implies m = \pm k \mod (2n)$

Therefore, $T_k = T_m$ for $m = \pm k \mod (2n)$

We will use the following modular arithmetic identity for the next proof:

$(a + b) \mod (n) = (a \mod (n) + b \mod (n)) \mod (n)$ ♠

Now, for
$m = |(k + n - 1) \mod (2n) - (n - 1)|$
$\implies \pm m = (k + n - 1) \mod (2n) - (n - 1)$
$\implies \pm m = (k + n - 1) \mod (2n) - (n - 1) \mod (2n)$
(Because $n - 1 < 2n \implies (n - 1) \mod (2n) = n - 1$)

$\implies \pm m \mod (2n) = ((k + n - 1) \mod (2n) - (n - 1) \mod (2n)) \mod (2n)$
$\implies \pm m = ((k + n - 1) \mod (2n) - (n - 1) \mod (2n)) \mod (2n)$
(Because $m < 2n \implies m \mod (2n) = m$)

$\implies \pm m = (k + \cancel{n - 1 - n + 1}) \mod (2n)$
(Using the identity ♠)

$\implies \pm m = k \mod (2n)$
$\implies m = \pm k \mod (2n)$

And for this relation, we have already proved that $T_k = T_m$

2 Aliasing of Chebyshev coefficients

Let

$$f = \sum_{k=0}^{\infty} a_k T_k(x)$$

And, Let

$$c_0 = a_0 + a_{2n} + a_{4n} + a_{6n} + \cdots$$

$$c_n = a_n + a_{3n} + a_{5n} + a_{7n} + \cdots$$

For $0 < k < n$

$$c_k = a_k + (a_{2n+k} + a_{4n+k} + a_{6n+k} + \cdots) + (a_{2n-k} + a_{4n-k} + a_{6n-k} + \cdots)$$

Let x_j be $n+1$ Chebyshev nodes. Multiplying $T_k(x_j)$ with each c_k and considering that

$$T_m = T_{2n\pm m} = T_{4n\pm m} = T_{6n\pm m}$$

we get

$$c_0T_0(x_j) = (a_0 + a_{2n} + a_{4n} + a_{6n} + \cdots)T_0(x_j) = a_0T_0(x_j) + a_{2n}T_{2n}(x_j) + a_{4n}T_{4n}(x_j)\cdots$$
$$c_nT_n(x_j) = (a_n + a_{3n} + a_{5n} + a_{7n} + \cdots)T_n(x_j) == a_nT_n(x_j) + a_{3n}T_{3n}(x_j) + a_{5n}T_{5n}(x_j)\cdots$$

$$c_kT_k(x_j) = (a_k + (a_{2n+k} + a_{4n+k} + a_{6n+k} + \cdots) + (a_{2n-k} + a_{4n-k} + a_{6n-k} + \cdots))\,T_k(x_j)$$
$$\implies c_kT_k(x_j) = a_kT_k(x_j) + (a_{2n+k}T_{2n+k}(x_j) + a_{4n+k}T_{4n+k}(x_j) + \cdots) + (a_{2n-k}T_{2n-k}(x_j) + a_{4n-k}T_{4n-k}(x_j) + \cdots)$$

Now considering the sum,

$$p_n(x_j) = \sum_{k=0}^{n} c_kT_k(x_j)$$

is a polynomial of degree n, and expanding the coefficients and rearranging the terms,

$$p_n(x_j) = \sum_{k=0}^{\infty} a_kT_k(x_j) = f(x_j)$$

Thus, at the Chebsyshev nodes, x_j

$f(x_j) = p_n(x_j)$

Thus, p_n is an n-degree interpolant of f, matching it at $n+1$ points, for the coefficients c_k. Aliasing of Chebyshev coefficients is hence proven.

3 Chebyshev approximation bounds

$$|a_k| \leq \frac{2M}{\rho^k}$$

Now,

$$|f - f_n| = \left|\sum_{k=n+1}^{\infty} a_kT_k\right| \leq \left|\sum_{k=n+1}^{\infty} a_k\right| \leq \sum_{k=n+1}^{\infty} |a_k| \leq \sum_{k=n+1}^{\infty} \frac{2M}{\rho^k} = \frac{2M\rho^{-(n+1)}}{1-\dfrac{1}{\rho}} = \frac{2M\rho^{-n}}{\rho - 1}$$

$$\implies ||f - f_n|| \leq \frac{2M\rho^{-n}}{\rho - 1}$$

Also,

$$|f-p_n| = \left|\sum_{k=0}^{n}(a_k-c_k)T_k + \sum_{k=n+1}^{\infty} a_kT_k\right| \leq \left|\sum_{k=0}^{n}(a_k-c_k)T_k\right| + \left|\sum_{k=n+1}^{\infty} a_kT_k\right|$$

$$\implies |f-p_n| \leq \left|\sum_{k=0}^{n}(a_k-c_k)T_k\right| + \frac{2M\rho^{-n}}{\rho-1} \leq \left|\sum_{k=0}^{n}(a_k-c_k)\right| + \frac{2M\rho^{-n}}{\rho-1}$$

$$\implies |f-p_n| \leq \left|\sum_{k=0}^{n} a_k - \sum_{k=0}^{n} c_k\right| + \frac{2M\rho^{-n}}{\rho-1} = \left|\sum_{k=0}^{n} a_k - \sum_{k=0}^{\infty} a_k\right| + \frac{2M\rho^{-n}}{\rho-1}$$

$$\implies |f-p_n| \leq \left|-\sum_{k=n+1}^{\infty} a_k\right| + \frac{2M\rho^{-n}}{\rho-1} \leq \frac{4M\rho^{-n}}{\rho-1}$$

$$\implies ||f-p_n|| \leq \frac{4M\rho^{-n}}{\rho-1}$$