

Chapter 9 Problems

Quantum Computing: An Applied Approach

1. This problem guides you through how expectation values of single qubit operators can be computed on a quantum computer.
 - (a) Let $|\psi\rangle$ be the state of a qubit. From Born's rule, show that the probability of measuring the zero state is $p(0) = |\langle 0|\psi\rangle|^2$.
 - (b) Similar to above, what is the probability of measuring the one state, $p(1)$?
 - (c) Define the projectors $\Pi_0 = |0\rangle\langle 0|$ and $\Pi_1 = |1\rangle\langle 1|$. Show that $p(0) = \langle \psi|\Pi_0|\psi\rangle$ and $p(1) = \langle \psi|\Pi_1|\psi\rangle$.
 - (d) Prove that $Z = \Pi_0 - \Pi_1$.
 - (e) Prove that $\langle \psi|Z|\psi\rangle = p(0) - p(1)$. Thus, by measuring in the Z -basis (many times), we can estimate the expectation of Pauli Z by subtracting the frequency of the outcomes.
 - (f) Suppose now we wanted to measure $\langle \psi|X|\psi\rangle$. Show that this can be done with the above steps by adding a Hadamard gate before measuring in the computational basis. *Hint: $HZH = X$.*
 - (g) The same idea above can be applied to measuring $\langle \psi|Y|\psi\rangle$. What gate(s) should be added before measuring in the computational basis to achieve this?
2. Problem 1 demonstrates how we can measure single qubit expectation values. Consider the qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $\alpha = 3/5$ and $\beta = 4/5$.
 - (a) Compute analytically $\langle X\rangle$, $\langle Y\rangle$, and $\langle Z\rangle$ where angled brackets denote expectation value in ψ .
 - (b) Design a quantum circuit to prepare the state $|\psi\rangle$.
 - (c) Design a quantum circuit to estimate $\langle X\rangle$. How does the accuracy improve as you increase the number of repetitions of the circuit?
 - (d) Do the above for Y and Z .
3. Problem on measuring generalized Pauli operators.

4. Prove that generalized Pauli operators form a basis for all unitaries. Combined with problems 1-3, this shows that we can estimate expectation values of any unitary matrix (although we may not be able to do so efficiently).
5. Express the following single-qubit unitaries as a sum of Pauli operators.
 - (a) H .
 - (b) S .
 - (c) T .
 - (d) $R_Y(\theta)$ for $\theta = \pi/4$.
6. Suppose we wish to use quantum phase estimation (QPE) to estimate an eigenvalue of a unitary U , and that controlled- U can be implemented in a circuit with u gates. If we use n qubits of precision, how many total gates are required in the circuit? Make a plot of the number of gates for, say, $1 \leq n \leq 10$.
7. *Iterative quantum phase estimation* (IQPE) is a modification of the phase estimation algorithm to use fewer resources. In phase estimation, we use a *single* circuit with n qubits of precision to estimate an eigenvalue using n bits. In *iterative* quantum phase estimation, we use n circuits with *one* qubit of precision to estimate an eigenvalue using n bits.

For the following three parts, let U be a unitary on m qubits, and let

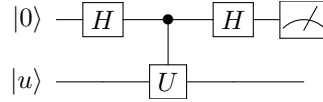
$$U|u\rangle = e^{2\pi i\phi}|u\rangle. \quad (1)$$

Further, suppose that ϕ has the binary decimal expansion

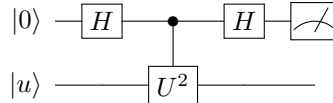
$$\phi = 0.\phi_1\phi_2\cdots\phi_n, \quad (2)$$

i.e. $\phi = \sum_{j=1}^n \phi_j 2^{-j}$

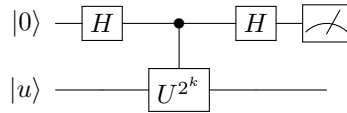
- (a) Show that the following circuit estimates ϕ_1 .



- (b) Show that the following circuit estimates ϕ_2 .



- (c) Show that the following circuit estimates ϕ_k .



- (d) Argue that n circuits with the above structure are sufficient to estimate $\phi = \phi_1\phi_2 \cdots \phi_n$.
 - (e) What is the largest number of gates required in a single circuit? Assume that controlled- U can be implemented in a circuit with u gates.
 - (f) What is the total number of gates required across *all* n circuits? How does this answer compare with Question 7? Explain.
8. Design a *biased* quantum random bit generator algorithm which produces $|0\rangle$ with probability 25 percent and $|1\rangle$ with probability 75 percent.
 9. Describe how a classical computer generates “random” numbers. How can we test for the randomness of a bitstring created in this manner?
 10. Suggest at least two more methods of generating a random number if you did not have access to a quantum computer