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IPCV Assignment 5

1) Keys Interpolation:

→ Given,

- 1-D Keys synthesis function :

$$\varphi_{int}(x) = \begin{cases} \frac{3}{2}|x|^3 - \frac{5}{2}x^2 + 1, & \text{if } |x| < 1, \\ -\frac{1}{2}|x|^3 + \frac{5}{2}x^2 - 4|x| + 2, & \text{if } 1 \le |x| < 2, \\ 0 & \text{else} \end{cases}$$

- w/ $(a = -\frac{1}{2})$

(a) To show;

$$\sum_{K \in Z} \varphi_{int}(x - K) = 1 \qquad \forall x \in R$$

⇒ using the hint & try to make a case distinction b/n $x \in Z$ & $x \notin Z$

- In interpolation without loss of generality, we assume that the regular sampling step is unity. We ask the function φ_{int} to satisfy the interpolation property which must vanish for all integer arguments except at origin $(x \in Z)$. Here it must assume a unit value. — (*)

(i) when $x \in Z$

Using the Keys synthesis function for different values of x

- if $|x| < 1$ condition

$$\varphi_{int}(x) = \frac{3}{2}|x|^3 - \frac{5}{2}x^2 + 1$$

for the above condition the possible values of x are : 0

i.e $\varphi_{int}(0) = \frac{3}{2}|0|^3 - \frac{5}{2}|0|^2 + 1$

$= 0 + 0 + 1$

$\varphi_{int}(0) = \underline{1}$

- if $1 \leq |x| < 2$

$\varphi_{int}(x) = \frac{-1}{2}|x|^3 + \frac{5}{2}x^2 - 4|x| + 2$

The possible values of x are : $+1$ & -1
(using ✱ - interpolation property)

- $\varphi_{int}(+1) = \frac{-1}{2}|1|^3 + \frac{5}{2}|1|^2 - 4|1| + 2$

$= -0.5 + 2.5 - 4 + 2$

$= \underline{0}$

same

- $\varphi_{int}(-1) = \frac{-1}{2}|-1|^3 + \frac{5}{2}|1|^2 - 4|-1| + 2$

$= \underline{0}$

- if else

$\varphi_{int}(x) = 0$

The possible values of x for this condition are all the integers above/equal to $+2$ & less than -3

- $\varphi_{int}(+2) = 0$
- $\varphi_{int}(+3) = 0$
- $\varphi_{int}(-2) = 0$
- $\varphi_{int}(-3) = 0$

(P.T.O)

$$\therefore \sum_{K\in Z} \varphi_{int}(x-K) = 1 + 0 + 0 + 0 + 0 + 0 + 0 = \underline{\underline{1}} \quad (\forall x \in R)$$

(ii) when $x \notin Z$

Since x doesn't belong to Z, the evaluation positions of the keys function cannot be computed as of the previous case.

- Let x be the center of the function, now to find the distance of x to the next integer [across either directions], let us assume a value p

i.e $K - x = p$ (assumption)

Repeating the same steps as in case i

- if $|x| < 1$

$$\varphi_{int}(x) = \frac{3}{2}|x|^3 - \frac{5}{2}x^2 + 1$$

- $\varphi_{int}(3) = \frac{3}{2}|3-K-x|^3 - \frac{5}{2}(3-K-x)^2 + 1$

$$= \frac{3}{2}|3-p|^3 - \frac{5}{2}(3-p)^2 + 1$$

$$= \frac{3}{2}|p-3|^3 + \frac{5}{2}(p+3)^2 + 1$$

$$= \frac{3}{2}\left(p^3 - 3p^2(3) + 3p(3)^2 - (3)^2\right) + \frac{5}{2}(p^2 + 9 + 6p) + 1$$

(P.T.O)

- $\varphi int(1-k-x) = \frac{3}{2}|1-k-x|^3 - \frac{5}{2}(1-k-x)^2 + 1$

$$= \frac{3}{2}|1-p|^3 - \frac{5}{2}(1-p)^2 + 1$$

$$= \frac{3}{2}(+1 + 3p^2 - 3p + 1) - \frac{5}{2}(1 - 2p + p^2) + 1$$

$$= \frac{3}{2} + \frac{9p^2}{2} - \frac{9p}{2} + \frac{3}{2} - \frac{5}{2} + \frac{10p}{2} - \frac{5p^2}{2} + 1$$

$$= \left(\frac{3}{2} - \frac{5}{2} + \frac{9p^2}{2} - \frac{5p^2}{2} - \frac{9p}{2} + \frac{10p}{2} + 1\right)$$

$$= \cancel{-1} + \frac{\cancel{4}^{2}p^2}{\cancel{2}} + \frac{p}{2} + \cancel{1}$$

$$= \underline{\underline{2p^2 + p/2}}$$

- $\varphi int(0-(k-x)) = \frac{3}{2}|0-k-x|^3 - \frac{5}{2}(0-k-x)^2 + 1$

$$= \frac{3}{2}|-p|^3 - \frac{5}{2}(-p)^2 + 1$$

$$= \underline{\underline{\frac{3p^3}{2} + \frac{5p^2}{2} + 1}}$$

- if $\underline{1 \le |x| < 2}$

$\Rightarrow \varphi int(x) = -1/2|x|^3 + 5/2\, x^2 - 4|x| + 2$

- $\varphi int(2-(k-x)) = \frac{-1}{2}|2-p|^3 + \frac{5}{2}(2-p)^2 - 4(2-p) + 2$

$$= \frac{-1}{2}(8 + 6p^2 - 12p - p^3) + \frac{5}{2}(4 - 4p + p^2) -$$

$$4(2-p) + 2$$

$$= -\frac{\cancel{8}^{4}}{\cancel{2}} - \frac{\cancel{6}^{3}p^2}{\cancel{2}} + \frac{\cancel{12}^{6}p}{2} + \frac{p^3}{2} + \frac{\cancel{20}^{10}}{\cancel{2}} - \frac{\cancel{20}^{10}p}{\cancel{2}} + \frac{5p^2}{\cancel{2}} -$$

$$8 \quad -4p+2$$

$$= (\cancel{-4-8+2+10})^{\to 0} + p^3/2 - 3p^2 + \frac{5}{2}p^2 + (\cancel{6p - 10p + 4p})^{\to 0}$$

$$= 0 + \frac{p^3}{2} - 0.5p^2$$

* we have already covered $(1-p)$ in condition $|x|<1$, so now

- $\varphi_{int}(-1-p) = -\frac{1}{2}|-1-p|^3 + \frac{5}{2}(-1-p)^2 - 4(-1-p)+2$

$$= -\frac{1}{2}(1+3p^2+3p+p^3) + \frac{5}{2}(1+2p+p^2) + 4 + 4p + 2$$

$$= -\frac{1}{2} - \frac{3p^2}{2} - \frac{3p}{2} - \frac{p^3}{2} + \frac{5}{2} + \frac{\cancel{10}^{5}p}{\cancel{2}} + \frac{5p^2}{2} + 4 + 4p + 2$$

$$= \left(-0.5 + 2.5 - 4 + 2 - \frac{3p^2}{2} + \frac{5p^2}{2} - \frac{p^3}{2} - \frac{3p}{2} + 5p\right)$$

$$= 0 + p^2 - p^3/2 + 3.5p$$

- if others / else
  $\varphi_{int} = 0$

- $\varphi_{int}(3-p) = \underline{0}$

* we have already covered $(2-p)$ in the previous step, so now:

- $\varphi_{int}(-2-p) = \underline{0}$

$\therefore$ Finally, $\sum_{k\in Z} \varphi_{int}(x-k) = 2p^2 + p/2 + \frac{3p^3}{2} + \frac{5p^2}{2} + 1 + \frac{p^3}{2}$
$- 0.5p^2 + 0 + p^3 - p^3/2 + 3.5p$
$+ 0 + 0$

$$= (2p^2 + \cancel{2.5p^2 - 0.5p^2 - 2p^2}^{\to 0} + \cancel{\frac{3p^3}{2} - \frac{p^3}{2} + \frac{p^3}{2} + p^3}^{\to 0} + 1$$

$$\sum_{x \in \mathbb{Z}} \psi_{int}^{(x-k)} = \underline{\underline{1}} \quad \text{(proved)}$$

(b) show that, partition of unity property guarantees an exact reproduction of a constant signal.

⇒ Reference :- The finite element method : a practical course (page number : 56)

The shape function is given as :

$$f(x) = \sum_{j}^{k} p_j(x) . \beta_j \quad , \quad \text{if } k \leq nd \quad \text{—— (*)}$$

The above equation can be written using the basis terms because $p_j(x)$ are monomials

i.e $$f(x) = \sum_{j}^{nd} p_j(x) . \beta_j$$

$$= p^T(x) . \alpha$$

here,

$$\alpha = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ 0 \end{bmatrix}$$

* we can obtain the vector of the function using the n nodes in the support domain of x, i.e :

$$V = \begin{bmatrix} p_1(x_1) & \cdots & p_{nd}(x_1) \\ p_2(x_2) & \cdots & p_{nd}(x_2) \\ \vdots & \cdots & \\ p_1(x_{nd}) & \cdots & p_{nd}(x_{nd}) \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ 0 \end{Bmatrix}$$

$$= \underline{\underline{P\alpha}} \quad \text{—— (1)}$$

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using ① in the reconstruction eqn

i.e $u^h(x) = P^T(x)\, P^{-1} V$

$= P^T(x) . \cancel{P^{-1}} \cancel{P} . \alpha$

$= P^T(x) . \alpha$

$= \sum_{j}^{K} P_j(x) . \alpha_j$ — ②

$= \underline{f(x)}$

eqn ① & ② are same

∴ This shows that unity property guarantees an exact reproduction of a constant signal or a shape function

2] Quadratic B-Spline Interpolation

→ Given, 1-D Interpolation data

- $x_1 = 1 \Rightarrow f_1 = 30$
- $x_2 = 2 \Rightarrow f_2 = 20$
- $x_3 = 3 \Rightarrow f_3 = 90$

Function: $\beta_2(x) = \begin{cases} 3/4 - x^2 & , \text{for } |x| < 1/2 \\ 1/2\,(3/2 - |x|)^2 & , \text{for } 1/2 \leq |x| < 3/2 \\ 0 & \text{else} \end{cases}$

(d) These are the following reasons for Quadratic B-splines to be unpopular in practice:

- Even degree B-spline curves are not suitable for curve & surface interpolation problem, this is because it is harder to determine the control vertices of the curve.
- In practice, the transformation functions of

even & odd degree B-splines before differently.

- The quadratic curve in B-splins are planar whereas cubic curve in cubic splines are non-planar
- Even though the computational complexity of both cubic & B-splines is similar, cubic splines produce smoother & better results!
- Cubic splines have a continous 2nd derivative while the quadratic splines have only a continous 1st derivative.

ⓐ Setup linear system & verify the system with $(c_1, c_2, c_3)^T = (40, 0, 120)^T$.

$\Rightarrow$ - if $|x| < 1/2$

In this condition the possible values of x are 0

so, $\beta_2(0) = 3/4 - (0)^2$
$= 3/4$

- if $1/2 \leq |x| < 3/2$

The possible values for x are 1

$$\beta_2(1) = 1/2\left(\frac{3}{2} - |x|\right)^2$$

$$= 1/2\left(3/2 - |1|\right)^2$$

$$= 1/2\left(1/2\right)^2$$

$$= 1/8$$

- if (other / else)

$\beta_2(2) = \underline{0}$

$\therefore$ The linear system is

$$\begin{pmatrix} 3/4 & 1/8 & 0 \\ 1/8 & 3/4 & 1/8 \\ 0 & 1/8 & 3/4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 30 \\ 20 \\ 90 \end{pmatrix}$$

[Used online calculator to perform matrix vector multiplication]

- Verify $(40, 0, 120)^T$ solves the system.

$$= \begin{pmatrix} 3/4 & 1/8 & 0 \\ 1/8 & 3/4 & 1/8 \\ 0 & 1/8 & 3/4 \end{pmatrix} \begin{pmatrix} 40 \\ 0 \\ 120 \end{pmatrix}$$

$$= \begin{pmatrix} 0.75 \times 40 + 0.125 \times 0 + 0 \times 120 \\ 0.125 \times 40 + 0.75 \times 0 + 0.125 \times 120 \\ 0 \times 40 + 0.125 \times 0 + 0.75 \times 120 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 30 \\ 20 \\ 90 \end{pmatrix}}$$

(b) Analytic expression of the interpolant.

Given, we have $x_k = k-1$ instead of $x_k = k$ i.e left shift by 1.

$$I(x) = \sum_{k=1}^{3} c_k \beta_2 (x - (k-1))$$

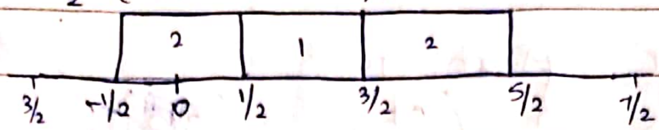
$$= 40 \beta_2 (x-1) + \underline{0} \, \beta_2 (x-2) + 120 \beta_2 (x-3)$$

(P.T.O)

• when $x-1$:

↳ $30 - 40(x-1)^2$

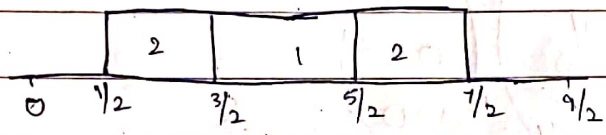
↳ $\frac{40}{2}\left(3/2 - |x-1|\right)^2$



• when $x-2$:

↳ 0

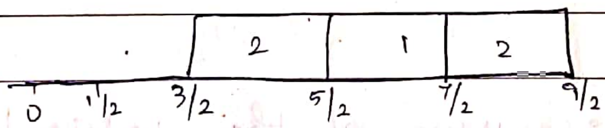
↳ 0



• when $x-3$:

↳ $\frac{40}{4} - 120(x-3)^2$

↳ $\frac{120}{2}\left(\frac{3}{2} - |x-3|\right)^2$



$$\therefore B_2(x) = \begin{cases} 20(3/2 - |x-1|)^2 + 0 & , x \in [-1/2, 1/2] \\ 30 - 40(x-1)^2 + 0 & , x \in [1/2, 3/2] \\ 20(3/2 - |x-1|)^2 + 60(3/2 - |x-3|)^2 & , x \in [3/2, 5/2] \\ 40/4 - 120(x-3)^2 & , x \in [5/2, 7/2] \\ 60(3/2 - |x-3|)^2 & , x \in [7/2, 9/2] \\ 0 & , \text{else} \end{cases}$$

∴ evaluating the function at the points:
$0, 3/2, 2$ & 4

- $I(0) = 20\left(\frac{3}{2} - |x-1|\right)^2$

$= 20\left(\frac{3}{2} - |0-1|\right)^2$

$= 20(0.25)$

$= \underline{\underline{5}}$

$\underline{\underline{I(0)=5}}$

IIly,

- $I(3/2) = \underline{\underline{20}}$
- $I(2) = \underline{\underline{20}}$
- $I(4) = \underline{\underline{15}}$

(c) Evaluate the function at $0, 3/2, 2$ & 4 & compare with results of (b)

given,

$$f(x) = 40x^2 - 130x + 120$$

- when $x=0$

$f(0) = 40(0)^2 - 130(0) + 120 = \underline{\underline{120}}$

- when $x = 3/2$

$f(3/2) = 40(3/2)^2 - 130(3/2) + 120 = \underline{\underline{15}}$

- when $x=2$

$f(2) = 40(2)^2 - 130(2) + 120 = \underline{\underline{20}}$

- when $x=4$

$f(4) = 40(4)^2 - 130(4) + 120 = \underline{\underline{240}}$