24/11/20

Probabilistic Graphical Models

Assignment - I

Team:-

- Akshay Joshi [2581346]
- Ankit Agrawal [2581532]
- Amr Amer [2572216]

I. Probabilities :-

1- For any 2 events E_1 & E_2

Prove : $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Solution :- Given that both events E_1 & E_2 are disjoint. We can write $E_1 \cup E_2$ as

$$\Rightarrow E_1 \cup E_2 = (E_1 - E_2) \cup (E_1 \cap E_2) \cup (E_2 - E_1)$$

Applying the probability axiom.

$$\Rightarrow P(E_1 \cup E_2) = P(E_1 - E_2) + P(E_1 \cap E_2) + P(E_2 - E_1)$$

WKT,

$$P(E_1 - E_2) = P(E_1) - P(E_1 \cap E_2)$$
$$P(E_2 - E_1) = P(E_2) - P(E_1 \cap E_2)$$

Thus,

$$\Rightarrow P(E_1 \cup E_2) = P(E_1) - P(E_1 \cap E_2) + \cancel{P(E_1 \cap E_2)} + P(E_2) - \cancel{P(E_1 \cap E_2)}$$

$$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Scanned with CamScanner

2. Baye's Law :-

→ Given, Kolmogorov's definition for conditional prob.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Rearrange the above equation

$$P(A \cap B) = P(A|B) \cdot P(B) \quad \text{--- } ①$$

or

$$P(A \cap B) = P(A) \cdot P(B|A) \quad \text{--- } ②$$

① = ② [Both are equal]

$$\therefore P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$$

$$\boxed{P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}}$$

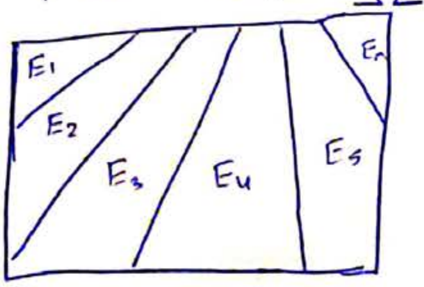
3. Law of Total Probability :-

→ Given, $E_1 \dots E_n$ are mutually disjoint events in probability space Ω. such that $\Omega = \bigcup_i^n E_i$

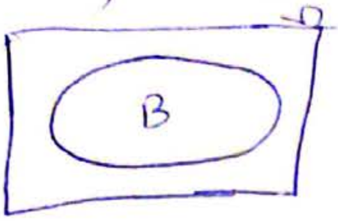
show that : $P(B) = \sum_{i=1}^{n} P(B \cap E_i) = \sum_{i=1}^{n} P(B|E_i) \cdot P(E_i)$

⇒ We can represent the events $E_1 \dots E_n$ as partitions in Ω space as :-

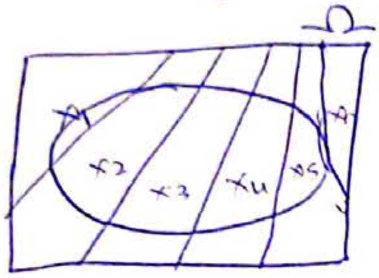
Ω



⇒ Any event B in Ω space can be visualized as



⇒ Pictorially we can represent total probability law as:



$X_1 = B \cap E_1$

$X_2 = B \cap E_2$

$\vdots \quad \vdots$

$X_n = B \cap E_n$

As $E_1 \ldots E_n$ are partitions in sample space Ω. we can write it as:

$$\Omega = \cup_i E_i$$

$$B = B \cap \Omega$$

$$= B \cap (\cup_i E_i)$$

$$= \cup_i (B \cap E_i) \qquad \text{[Distributive prop]}$$

Here, $B \cap E_i$ sets are disjoint. So we can apply 3rd probability axiom as:

$$P(B) = P(\cup_i (B \cap E_i))$$

$$= \sum_i P(B \cap E_i)$$

$$= \sum_i P(B \mid E_i) \cdot P(E_i)$$

Scanned with CamScanner

4. Linearity of Expectation

→ For any $x_1 \ldots x_n$ (discrete random variables)

$$\text{S.T} \quad E\left[\sum_{i=1}^{n} x_i\right] = \sum_{i=1}^{n} E[x_i]$$

Since it is given that the random variables are finite. for the sake of simplicity we can just use 2 variables to prove ~~that~~ the requirement, as anyway the solution for n finite variables can just be extended by Induction

- Let X & Y be any discrete random variables

so, $E(x+y) = E(x) + E(y)$ — ⊛ [assuming both are dependent]

WKT,

$$E[x] = \sum_x x \cdot P(X=x) \quad \text{— (1)}$$

$$E[y] = \sum_y y \cdot P(Y=y) \quad \text{— (2)}$$

continue with ⊛

$$E(x+y) = \sum_x \sum_y \left[(x+y) \cdot P(X=x, Y=y)\right]$$

$$= \sum_x \sum_y \left[x \cdot P(X=x, Y=y)\right] + \sum_x \sum_y \left[y \cdot P(X=x, Y=y)\right]$$

[Rearranging]

$$= \sum_x x \cdot \underbrace{\sum_y P(X=x, Y=y)}_{(1)/x} + \sum_y y \cdot \underbrace{\sum_x P(X=x, Y=y)}_{(2)/y}$$

$$= \sum_x x \cdot \underbrace{P(X=x)}_{(1)} + \sum_y y \cdot \underbrace{P(Y=y)}_{(2)}$$

$$= E[x] + E[y]$$

when random variable b/n $x_1 \ldots x_n$ →

$$= E[x_1] + E[x_2] + E[x_3] + E[x_n]$$

Scanned with CamScanner

5. Conditional Independence:-

Given:, X, Y, Z are disjoint subsets of random variables

- $X \perp Y \mid Z$ iff $P_{X,Y|Z}(x,y|z) = P_{X|Z}(x|z) \cdot P_{Y|Z}(y|z)$

show that: $X \perp Y \mid Z$ iff $P_{X,Y,Z}(x,y,z) = h(x,z) \cdot g(y,z)$

$\Rightarrow$ Lets start with the ~~factorization~~ of the L.H.S

$$P_{X,Y,Z}(x,y,z) = P_Z(z) \cdot P_{X,Y|Z}(x,y|z)$$
$$= P_{X|Z}(x|z) \cdot \underbrace{P_{Y|Z}(y|z) \cdot P_Z(z)}$$
Let, $g(y,z) =$
$$= \underbrace{P_{X|Z}(x|z)} \cdot g(y,z)$$
$$\boxed{= h(x,z) \cdot g(y,z)}$$

Now, proving the other way around.

Let, $h_1(z) = \sum_x h(x,z)$ — ①

$g_1(z) = \sum_y g(y,z)$ — ②

① $P_{X,Y|Z}(z,y|z) = \dfrac{P_{X,Y,Z}(x,y,z)}{\sum_{x,y} P_{X,Y,Z}(x,y,z)}$

= [can be written in terms of $h(x)$ & $g(x)$]

$= \dfrac{h(x,z) \cdot g(y,z)}{\sum_{x,y} h(x,z) \cdot g(y,z)} = \dfrac{h(x,z) \cdot g(y,z)}{h_1(z) \cdot g_1(z)}$ — ③

(ii) $P_{X|Z}(x|z) = \dfrac{P_{X,Z}(x,z)}{\sum_x P_{X,Z}(x,z)} = \dfrac{h(x,z)}{h_1(z)}$ (from ①) — ④

(iii) $P_{Y|Z}(y|z) = \dfrac{P_{Y,Z}(y,z)}{\sum_y P_{Y,Z}(y,z)} = \dfrac{g(y,z)}{g_1(z)}$ (from ②) — ⑤

Now, if we see carefully: product of ④ & ⑤ gives ③

$$\boxed{P_{X|Z}(x|z) \cdot P_{Y|Z}(y|z) = P_{X,Y|Z}(z,y|z)}$$ [From eqn ⑥]

Hence proved!

Scanned with CamScanner

2 Complexity Analysis:

(a) $p(x,y,z) \rightarrow p(x|y,z) \cdot p(y|z) \; p(z)$.

$\rightarrow 2^2 + 2^1 + 2^0$ (as binary states)

$\rightarrow \boxed{7}$.

Also can be written by $2^n - 1 \rightarrow 2^3 - 1 \rightarrow 7$

(b). $p(x,y,z) = p(x|y) \cdot p(y|z) \cdot p(z)$.

$\rightarrow 2^1 + 2^1 + 2^0$

$\rightarrow \boxed{5}$

(c). now, variables can take values $\{1, 2 \ldots N\}$.

so

(i) $p(x,y,z) = p(x|y,z) \cdot p(y|z) \cdot p(z)$

$\rightarrow \boxed{N^2 + N + 1}$

(ii) $p(x,y,z) = p(x|y) \cdot p(y|z) \cdot p(z)$

$\rightarrow N + N + 1$

$\rightarrow \boxed{2N+1}$

2(d) image $\rightarrow$ 1000×1000 pixels.

$\therefore$ total random variables = 1000^2.

can take values for $\{0 \ldots 255)$ i.e 256.

$\rightarrow$

now,

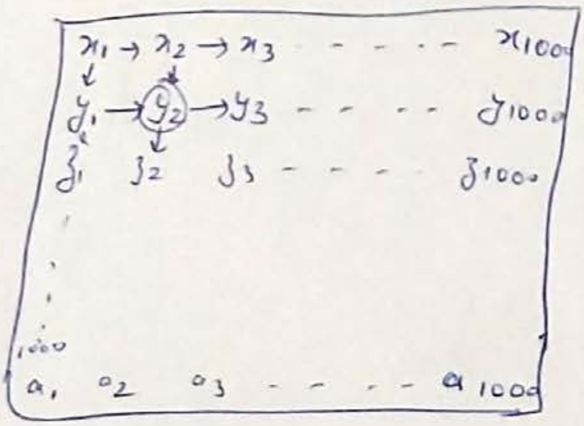
$$P\begin{pmatrix} x_1, x_2 \ldots x_{1m} \\ y_1, y_2 \ldots y_{1m} \end{pmatrix} = P\left(x_1 \middle| \begin{matrix} x_2, x_3 \ldots x_{1m} \\ y_1, y_2 \ldots y_{1m} \\ \vdots \\ 1000 \ldots 1000 \end{matrix}\right) \cdot P\left(x_2 \middle| \begin{matrix} x_3 \ldots x_{1m} \\ \vdots \\ \ldots 1000 \end{matrix}\right)$$

$$\Rightarrow \boxed{256^{1000^2-1} + 256^{1000^2-2} + \ldots + 256^{1000^2-1000^2}}$$

—

2e One way to represent the distribution more compactly is by performing reduction in inter-pixel redundancy.. i.e instead of making dependency for each pixel to each other, we just be dependent on its neighbouring pixels.

let us consider grid of 1000 × 1000 pixels as shown below:-



so for each pixel we look for its neighbour right and pixel below it.

(to remove redundancy).

so states for the first row:-

$$P(x_1 | x_2, y_1) \cdot P(x_2 | x_3, y_2) \ldots P(x_{1000-1} | x_{1000}, y_{1000-1})$$
$$\cdot P(x_{1000} | y_{1000}) \cdot P(y_{1000})$$

$$\Rightarrow 256^2 + 256^2 + \ldots 256^2 + 256 + 1$$

no of states $\Rightarrow$

$$\Rightarrow \boxed{999(256^2) + 256 + 1} \rightarrow \text{for first row.}$$ states

for 2nd row:-

$$P(y_1 | y_2, z_1) \cdot P(y_2 | y_3, z_2) \cdots P(y_{1000-1} | y_{1000}, z_{1000-1}) \cdot P(y_{1000} | z_{1000}) \cdot P(z_{1000})$$

$\Rightarrow$ no of states for 2nd row $\Rightarrow$ $999(256^2) + 256 + 1$.

This will be same for all row except the last row. $(1, 2, \ldots 999)$.

$\therefore$ for first 999 rows.

no of states $\Rightarrow$ $\sum_1^{999} \left(999(256^2) + 256 + 1\right)$

now for last row.

we have,

$$P(a_1 | a_2) \cdot P(a_2 | a_3) \cdots P(a_{1000-1} | a_{1000}) \cdot \cancel{P(a_{1000})}$$

as already considered.

$\Rightarrow$ $256 + 256 + 256 \ldots 256$

$\Rightarrow$ $999(256)$.

$\therefore$ total no. of states $\Rightarrow$

$$\boxed{\sum_1^{999} \left[999(256^2) + 256 + 1\right] + 999(256)}$$

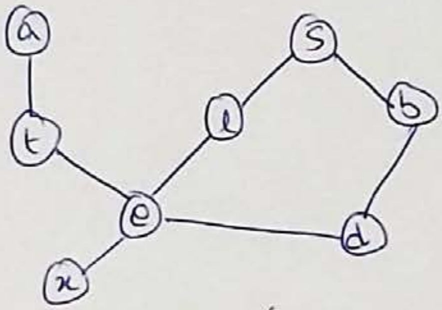
Clinic Chest Network:

3.1 $P(a) \cdot P(s) \cdot P(t|a) \cdot P(e|t,l) \cdot P(x|e) \cdot P(l|s) \cdot P(b|s) \cdot P(d|e,b)$

3.2 (a) $t \perp\!\!\!\perp s \mid d$ → $\boxed{\text{not}}$ conditional independence (NO)

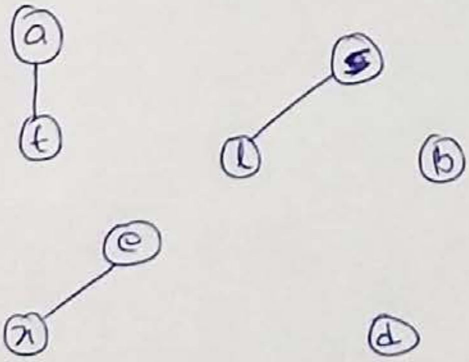
Using Graphical manipulations:

We get



hence not blocked.

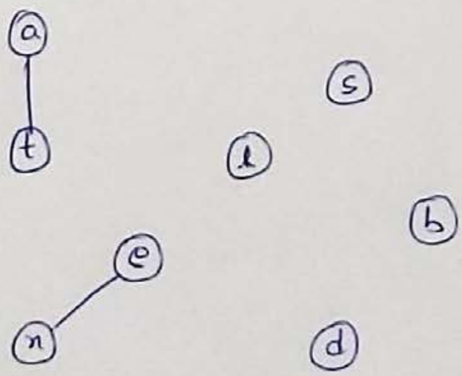
(b) $t \perp\!\!\!\perp s \mid b$ → $\boxed{\text{yes}}$ (it is conditional independence)



no path.

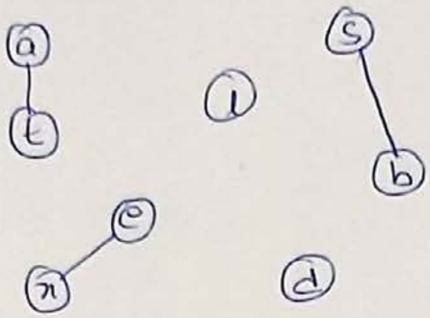
blocked ✓

(c) $l \perp\!\!\!\perp b \mid s$ → $\boxed{\text{yes}}$



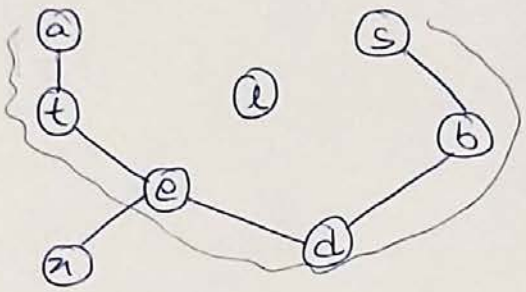
blocked.

(d) $a \perp\!\!\!\perp s \mid l \rightarrow \boxed{\text{yes}}$



blocked

(e) $a \perp\!\!\!\perp s \mid l, d \rightarrow \boxed{\text{no}}$



there is a path.

3.3 CPT

$$P(d) = P(d=1 \mid e=1, b=1) \cdot P(b=1 \mid s=1)\, P(s=1)\, P(e=1 \mid t=1, l=1) \cdot P(t=1 \mid a=1)\, P(a=1)\, P(l=1 \mid s=1)$$

$$= 0.9 * 0.6 * 0.5 * 1 * 0.05 * 0.01 * 0.1$$

$$\Rightarrow \underline{0.0000135}$$