

16.

a) IF  $n, m \in \mathbb{N}$ , AND  $\frac{m^2}{n^2} < 2$ , THEN  $(m+2n)^2 / (m+n)^2 > 2$ .

PROOF.

$$\begin{aligned}
 m^2 / n^2 < 2 &\Rightarrow m^2 < 2n^2 & (1^*) \\
 &\Rightarrow m^2 - 2n^2 < 0 \\
 &\Rightarrow 2m^2 + 4mn + 2n^2 < m^2 + 4mn + 4n^2 \\
 &\Rightarrow 2(m^2 + 2mn + n^2) < m(m+2n) + 2n(m+2n) \\
 &\Rightarrow 2(m+n)^2 < (m+2n)^2 \\
 &\Rightarrow 2 < (m+2n)^2 / (m+n)^2. \quad \square
 \end{aligned}$$

a).2 IF  $n, m \in \mathbb{N}$ , AND  $m^2 / n^2 < 2$ , THEN

PROOF.

NOTE THAT  $n, m \in \mathbb{N} \Rightarrow (m+n)^2 > n^2$  AND  $0 < m^2 < 2n^2$ . THEN:

$$\begin{aligned}
 (2n^2 - m^2)(m+n)^2 &> (2n^2 - m^2)n^2 \\
 &= (4n^2 - 2n^2 + m^2 - 2m^2 + 4nm - 4nm)n^2 \quad (2^*) \\
 &= (m^2 + 4nm + 4n^2 - 2(m^2 + 2nm + n^2))n^2 \\
 &= ((m+2n)^2 - 2(m+n)^2)n^2.
 \end{aligned}$$

(DIVIDING BOTH SIDES BY  $(m+n)^2 n^2$ ):

$$\begin{aligned}
 &\Rightarrow (2n^2 - m^2) / n^2 > (m+2n)^2 / (m+n)^2 - 2(m+n)^2 / (m+n)^2 \\
 &2 - m^2 / n^2 > (m+2n)^2 / (m+n)^2 - 2. \quad \square
 \end{aligned}$$

b) IF  $n, m \in \mathbb{N}$ , AND  $m^2 / n^2 > 2$ , THEN  $(m+2n)^2 / (m+n)^2 < 2$ .a).1  $m^2 / n^2 > 2 \Rightarrow m^2 > 2n^2$ . REVERSING THE INEQUALITY STARTING LINE (1\*), WE GET  $2 > (m+2n)^2 / (m+n)^2$ .  $\square$ a).2 NOTE THAT  $n, m \in \mathbb{N}$  AND  $m^2 / n^2 > 2$  IMPLY  $(m+n)^2 > n^2$  AND  $0 < 2n^2 < m^2$ . THEN:

$$\begin{aligned}
 &\Rightarrow (m^2 - 2n^2)(m+n)^2 > (m^2 - 2n^2)n^2 \\
 &\Rightarrow (2n^2 - m^2)(m+n)^2 < (2n^2 - m^2)n^2 \quad (\text{MULTIPLYING BY } -1) \\
 &\quad = ((m+2n)^2 - 2(m+n)^2)n^2 \quad (\text{BY } (2^*)) \\
 &\Rightarrow 2n^2 / n^2 - m^2 / n^2 < (m+2n)^2 / (m+n)^2 - 2(m+n)^2 / (m+n)^2 \quad (\text{BY } (3^*)) \\
 &\Rightarrow 2 - m^2 / n^2 < (m+2n)^2 / (m+n)^2 - 2. \quad \square
 \end{aligned}$$

c) IF  $n, m \in \mathbb{N}$  AND  $m/n < \sqrt{2}$ , THEN THERE EXIST  $n', m' \in \mathbb{N}$  SUCH THAT  $m/n < m'/n' < \sqrt{2}$ .

PROOF.

NOTE THAT  $m/n < \sqrt{2} \Rightarrow m^2 / n^2 < 2 \Rightarrow m^2 < 2n^2$ , AND  $\frac{(m+2n)^2}{(m+n)^2} > 2$ , BY PROPOSITION 16.a.1.

$$\begin{aligned}
 &\text{THEN } [(m+2n) + 2(m+n)]^2 / [(m+2n) + (m+n)]^2 \\
 &\quad = (4n+3m)^2 / (2m+3n)^2 < 2, \\
 &\text{BY PROPOSITION 16.b.}
 \end{aligned}$$

THUS  $m^2 / n^2 < 2 < (m+2n)^2 / (m+n)^2$ , AND  $(4n+3m)^2 / (2m+3n)^2 < 2 < (m+2n)^2 / (m+n)^2$ . NEXT, NOTE THAT

$$\begin{aligned}
 m^2 < 2n^2 &\Rightarrow 2m^2 < 4n^2 \\
 &\Rightarrow 2m^2 + 3mn < 4n^2 + 3mn \\
 &\Rightarrow [m(2m+3n)]^2 < [n(4n+3m)]^2 \\
 &\Rightarrow m^2 / n^2 < (4n+3m)^2 / (2m+3n)^2. \\
 &\Rightarrow
 \end{aligned}$$

COMBINING THE 3 INEQUALITIES, WE GET  $m^2 / n^2 < (4n+3m)^2 / (2m+3n)^2 < 2$ .  
 $\therefore$  IF  $m/n < \sqrt{2}$ , THEN THERE EXIST  $m' = 4n+3m$  AND  $n' = 2m+3n$  SUCH THAT  $m/n < m'/n' < \sqrt{2}$ .  $\square$