16.

a) IF $n, m \in \mathbb{N}$, AND $\frac{m^2}{n^2} < 2$, THEN $(m+2n)^2/(m+n)^2 > 2$.

PROOF.

$$\begin{aligned} m^2/n^2 < 2 \Rightarrow \quad & m^2 < 2n^2 & (1^*) \\ \Rightarrow \quad & m^2 - 2n^2 < 0 \\ \Rightarrow \quad & 2m^2 + 4mn + 2n^2 < m^2 + 4mn + 4n^2 \\ \Rightarrow \quad & 2(m^2 + 2mn + n^2) < m(m+2n) + 2n(m+2n) \\ \Rightarrow \quad & 2(m+n)^2 < (m+2n)^2 \\ \Rightarrow \quad & 2 < (m+2n)^2/(m+n)^2. \quad \square \end{aligned}$$

a).2 IF $n, m \in \mathbb{N}$, AND $m^2/n^2 < 2$, THEN

PROOF.

NOTE THAT $n, m \in \mathbb{N} \Rightarrow (m+n)^2 > n^2$ AND $0 < m^2 < 2n^2$. THEN:

$$\begin{aligned} (2n^2 - m^2)(m+n)^2 > (2n^2 - m^2)n^2 \\ = (4n^2 - 2n^2 + m^2 - 2m^2 + 4nm - 4nm)n^2 \quad (2^*) \\ = (m^2 + 4nm + 4n^2 - 2(m^2 + 2nm + n^2))n^2 \\ = ((m+2n)^2 - 2(m+n)^2)n^2. \end{aligned}$$

(DIVIDING BOTH SIDES BY $(m+n)^2 n^2$): (3^*)

$$\begin{aligned} \Rightarrow (2n^2 - m^2)/n^2 > (m+2n)^2/(m+n)^2 - 2(m+n)^2/(m+n)^2 \\ 2 - m^2/n^2 > (m+2n)^2/(m+n)^2 - 2. \quad \square \end{aligned}$$

b) IF $n, m \in \mathbb{N}$, AND $m^2/n^2 > 2$, THEN $(m+2n)^2/(m+n)^2 < 2$.

a).1 $m^2/n^2 > 2 \Rightarrow m^2 > 2n^2$. REVERSING THE INEQUALITY STARTING LINE (1^*), WE GET $2 > (m+2n)^2/(m+n)^2$. $\square$

a).2 NOTE THAT $n, m \in \mathbb{N}$ AND $m^2/n^2 > 2$ IMPLY $(m+n)^2 > n^2$ AND $0 < 2n^2 < m^2$. THEN:

$$\begin{aligned} & (m^2 - 2n^2)(m+n)^2 > (m^2 - 2n^2)n^2 \\ \Rightarrow \quad & (2n^2 - m^2)(m+n)^2 < (2n^2 - m^2)n^2 & \text{(MULTIPLYING BY } -1) \\ & \qquad = ((m+2n)^2 - 2(m+n)^2)n^2. & (\text{BY } (2^*)) \\ \Rightarrow \quad & 2n^2/n^2 - m^2/n^2 < (m+2n)^2/(m+n)^2 - 2(m+n)^2/(m+n)^2. & (\text{BY } (3^*)) \\ \Rightarrow \quad & 2 - m^2/n^2 < (m+2n)^2/(m+n)^2 - 2. \quad \square \end{aligned}$$

c) IF $n, m \in \mathbb{N}$ AND $m/n < \sqrt{2}$, THEN THERE EXIST $n', m' \in \mathbb{N}$ SUCH THAT $m/n < m'/n' < \sqrt{2}$.

PROOF.

NOTE THAT $m/n < \sqrt{2} \Rightarrow m^2/n^2 < 2 \Rightarrow m^2 < 2n^2$, AND $\frac{(m+2n)^2}{(m+n)^2} > 2$, BY PROPOSITION 16.a.1.

THEN $[(m+2n) + 2(m+n)]^2 / [(m+2n) + (m+n)]^2$
$= (4n + 3m)^2/(2m+3n)^2 < 2$,
BY PROPOSITION 16.b.

THUS $m^2/n^2 < 2 < (m+2n)^2/(m+n)^2$, AND $(4n+3m)^2/(2m+3n)^2 < 2 < (m+2n)^2/(m+n)^2$. NEXT, NOTE THAT

$$\begin{aligned} & m^2 < 2n^2 \Rightarrow 2m^2 < 4n^2 \\ \Rightarrow \quad & 2m^2 + 3mn < 4n^2 + 3mn \\ \Rightarrow \quad & [m(2m+3n)]^2 < [n(4n+3m)]^2 \\ \Rightarrow \quad & m^2/n^2 < (4n+3m)^2/(2m+3n)^2. \\ \Rightarrow \end{aligned}$$

COMBINING THE 3 INEQUALITIES, WE GET $m^2/n^2 < (4n+3m)^2/(2m+3n)^2 < 2$.