Proposition: For all $n, m \in \mathbb{N}$, it follows that
$S_n : \sum_{i=0}^{n} i\binom{m+i}{m} = n\binom{m+n+1}{m+1} - \binom{m+n+1}{m+2}$.

Proof. (Induction).
Basis step. Suppose $n = 1$ and $m \in \mathbb{N}$. Observe that
$\sum_{i=0}^{1} i\binom{m+i}{m} = \binom{m+1}{m} = \frac{(m+1)!}{m!(m+1-m)!} = m+1 = m+2-1 = \frac{(m+2)!}{(m+1)!(m+2-(m+1))!} - 1 = \binom{m+1+1}{m+1} - \binom{m+1+1}{m+2}$.
Thus S_1.
Inductive step. Suppose S_n for $n, m \in \mathbb{N}$.
We now show S_n implies S_{n+1}. Observe that

$$
\begin{aligned}
\sum_{i=0}^{n+1} i\binom{m+1}{m} &= \left[\sum_{i=0}^{n} i\binom{m+1}{m}\right] + (n+1)\binom{m+n+1}{m} && (1)\\
&= n\binom{m+n+1}{m+1} - \binom{m+n+1}{m+2} + (n+1)\binom{m+n+1}{m} && (2)\\
&= n\binom{m+n+1}{m+1} + \binom{m+n+1}{m+1} - \binom{m+n+1}{m+1} - \binom{m+n+1}{m+2} + (n+1)\binom{m+n+1}{m} && (3)\\
&= (n+1)\binom{m+n+1}{m+1} + (n+1)\binom{m+n+1}{m} - \binom{m+n+1}{m+1} - \binom{m+n+1}{m+2} && (4)\\
&= (n+1)\left[\binom{m+n+1}{m+1} + \binom{m+n+1}{m}\right] - \left[\binom{m+n+1}{m+1} + \binom{m+n+1}{m+2}\right] && (5)\\
&= (n+1)\binom{m+n+2}{m+1} - \binom{m+n+2}{m+2} && \text{(Def. of Pascal's triangle)}\\
& && (6)\\
&= (n+1)\binom{m+(n+1)+1}{m+1} - \binom{m+(n+1)+1}{m+2}. && (7)
\end{aligned}
$$

Thus S_{n+1}.
It follows by mathematical induction that S_n for all $n, m \in \mathbb{N}$. ■