24. Let $a_1 + ... + a_k$ denote $a_1 + (a_2 + ... + a_{k-2} + (a_{k-1} + a_k)...)$.

(a) Proposition: $(a_1 + ... + a_{k-1}) + a_k = a_1 + ... + a_k$ for all $k \geq 3$.

Proof. (Induction).

For the basis case, let $k = 3$. Clearly, $(x + y) = x + y$ for any numbers x and y. Thus $(a_1 + a_2) + a_3 = a_1 + a_2 + a_3 = a_1 + (a_2 + a_3) = a_1 + ... + a_3$.

Now let $(a_1 + ... + a_{k-1}) + a_k = a_1 + ... + a_k$ for some $k \geq 3$. Then

$$\begin{aligned}(a_1 + ... + a_k) + a_{k+1} &= a_1 + a_2 + ... + a_k + a_{k+1} \\ &= a_1 + a_2 + ... + (a_k + a_{k+1}) \\ &= a_1 + (a_2 + ... + (a_k + a_{k+1})) \\ &= a_1 + ... + a_{k+1}.\end{aligned}$$

Thus $(a_1 + ... + a_k) + a_{k+1} = a_1 + ... + a_{k+1}$. By induction, $(a_1 + ... + a_{k-1}) + a_k = a_1 + ... + a_k$ for all $k \geq 3$.

(b) Proposition: If $n - 2 > k > 1$, then $(a_1 + ... + a_k) + (a_{k+1} + ... + a_n) = a_1 + ... + a_n$ for all n and k.

Proof.

Let $n \geq 4$. We will do induction on k.

For the basis case, let $k = 2$. Then $(a_1 + a_2) + (a_3 + ... + a_n) = a_1 + a_2 + a_3 + ... + a_n = a_1 + ... + a_n$ by Proposition 24.a.

Now let $(a_1 + ... + a_k) + (a_{k+1} + ... + a_n) = a_1 + ... + a_n$ for some n and k such that $n - 3 > k > 2$. Then

$$\begin{aligned}(a_1 + ... + a_{k+1}) + (a_{k+2} + ... + a_n) &= (a_1 + ... + a_k) + a_{k+1} + (a_{k+2} + ... + a_n) && \text{by Proposition 24.a} \\ &= (a_1 + ... + a_k) + (a_{k+1} + ... + a_n) \\ &= a_1 + ... + a_n.\end{aligned}$$

Thus $(a_1+...+a_{k+1})+(a_{k+2}+...+a_n) = a_1+...+a_n$. Therefore, if $n-2 > k > 1$, then $(a_1+...+a_k)+(a_{k+1}+...+a_n) = a_1+...+a_n$ for all n and k, by induction.

(c) Proposition: Let $s(a_1, ..., a_k)$ be some sum formed by $a_1 + ... + a_k$, where $k > 1$. Then $s(a_1, ...a_k) = a_1 + ... + a_k$.

Proof. (Induction).

For the basis case, let $k = 2$. Then $a_1 + a_2 = a_2 + a_1 = s(a_1, a_2)$.

Now let $s(a_1, ...a_k) = a_1 + ... + a_k$ for some $k > 1$. Observe that

$$\begin{aligned}s(a_1, ..., a_{k+1}) &= s(a_1, ..., a_k, a_{k+1}) \\ &= s(a_1, ..., a_k) + a_{k+1} \\ &= (a_1 + ... + a_k) + a_{k+1} \\ &= a_1 + ... + a_{k+1}. && \text{by Proposition 24.a}\end{aligned}$$

Thus $s(a_1, ..., a_{k+1}) = a_1 + ... + a_{k+1}$. Therefore, if $k > 1$, then $s(a_1, ...a_k) = a_1 + ... + a_k$, by induction.