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(i) Proposition: $|xy| = |x| * |y|$

Proof.

We consider two cases. Let $x \geq 0$ and $y \geq 0$, or $x \leq 0$ and $y \leq 0$. Since $xy \geq 0$, it follows that $|xy| = xy = |x| * |y|$. Now let $x > 0$ and $y < 0$, WLOG. Since $x = |x|$, it follows that $|xy| = x * |y| = |x| * |y|$. Therefore $|xy| = |x| * |y|$.

(ii) Proposition: $\frac{1}{|x|} = \left|\frac{1}{x}\right|, x \neq 0$

Proof.

We consider two cases. Let $x > 0$. Since $|x| = x$, it follows that $\frac{1}{|x|} = \frac{1}{x} = \left|\frac{1}{x}\right|$. Now let $x < 0$. Since $|x| = -x$, it follows that $\frac{1}{|x|} = -\frac{1}{x} = \left|\frac{1}{x}\right|$. Therefore $\frac{1}{|x|} = \left|\frac{1}{x}\right|$.

(iii) Proposition: $\frac{|x|}{|y|} = \left|\frac{x}{y}\right|, y \neq 0.$

Proof.

Observe that $\left|\frac{x}{y}\right| = |x| * \left|\frac{1}{y}\right| = |x| * \frac{1}{|y|} = \frac{|x|}{|y|}$.

(iv) Proposition: $|x - y| \leq |x| + |y|$

Proof. $|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|$ by Theorem 1.

(v) Proposition: $|x| - |y| \leq |x - y|$

Proof.

By Theorem 1, $|(x - y) + y| \leq |x - y| + |y|$. Then

$$\begin{aligned} |(x-y)+y| \leq |x-y| + |y| &\implies |x| \leq |x-y| + |y| \\ &\implies |x| - |y| \leq |x-y|. \end{aligned}$$

(vi) Proposition: $|(|x| - |y|)| \leq |x - y|$

Proof.

By Proposition (v), $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |y - x|$. Observe that

$$\begin{aligned} |y| - |x| &\leq |y - x| \\ -(|x| - |y|) &\leq |-(x - y)| \\ -(|x| - |y|) &\leq |x - y|. \end{aligned}$$

Thus $\pm(|x| - |y|) = |(|x| - |y|)| \leq |x - y|$.

(vii) Proposition: $|x + y + z| \leq |x| + |y| + |z|$

Proof.

By Theorem 1, $|x + (y + z)| \leq |x| + |y + z|$, WLOG. Since $|y + z| \leq |y| + |z|$, it follows that $|x| + |y + z| \leq |x| + |y| + |z|$. Therefore $|x + y + z| \leq |x| + |y| + |z|$.