18.

(a) Proposition: Suppose $b^2 - 4c \geq 0$. Then $x = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$ satisfy $x^2 + bx + c = 0$.

Proof.

Let $x = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$. By substitution,

$$\begin{aligned} x^2 + bx + c &= \Big(\frac{-b \pm \sqrt{b^2 - 4c}}{2}\Big)^2 + b\Big(\frac{-b \pm \sqrt{b^2 - 4c}}{2}\Big) + c \\ &= \frac{1}{4}(-b \pm \sqrt{b^2 - 4c})^2 + \frac{2}{4}b(-b \pm \sqrt{b^2 - 4c}) + c \\ &= \frac{1}{4}(b^2 \mp 2b\sqrt{b^2 - 4c} + b^2 - 4c) + \frac{1}{4}(-2b^2 \pm 2b\sqrt{b^2 - 4c}) + c \\ &= \frac{1}{4}(b^2 + b^2 - 2b^2 \mp 2b\sqrt{b^2 - 4c} \pm 2b\sqrt{b^2 - 4c} - 4c) + c \\ &= \frac{1}{4}(-4c) + c \\ &= -c + c \\ &= 0. \end{aligned}$$

(b) Proposition If $b^2 - 4c < 0$, then there are no numbers x satisfying $x^2 + bx + c = 0$. In fact, if $b^2 - 4c < 0$, then $x^2 + bx + c > 0$ for all x.

Proof.

Suppose for the sake of contradiction that $b^2 - 4c < 0$ and there exists a number x satisfying $x^2 + bx + c = 0$. Note that

$$\begin{aligned} 0 &= x^2 + bx + c \\ &= (x^2 + \frac{b}{2}x + \frac{b}{2}x + \frac{b^2}{4} - \frac{b^2}{4}) + c \\ &= (x^2 + \frac{b}{2}x + \frac{b}{2}x + \frac{b^2}{4}) - \frac{b^2}{4} + c \\ &= (x + \frac{b}{2})^2 - \frac{b^2}{4} + c \\ \frac{b^2}{4} - c &= (x + \frac{b}{2})^2. \end{aligned}$$

Since $b^2 - 4c < 0$, we know $\frac{b^2}{4} - c < 0$. But $(x + \frac{b}{2})^2 \geq 0$. Thus $\frac{b^2}{4} - c \neq (x + \frac{b}{2})^2$, a contradiction. In fact, since $-\frac{b^2}{4} + c > 0$ and $(x + \frac{b}{2})^2 \geq 0$, it follows that $x^2 + bx + c = (x + \frac{b}{2})^2 - \frac{b^2}{4} + c > 0$.

(c) Proposition If $x \neq 0$ or $y \neq 0$, then $x^2 + xy + y^2 > 0$.

Proof.

By Proposition 18.b, $y^2 - 4(y^2) = -3y^2 < 0$ implies $x^2 + xy + y^2 > 0$.

(d) Proposition When $a < 2$ or $a > -2$, and x and y are not both zero, $x^2 + axy + y^2 > 0$.

Proof.

WLOG, let $y \neq 0$. By Proposition 18.b, $x^2 + axy + y^2 > 0$ if $(ay)^2 - 4y^2 < 0$. Note that $(ax)^2 - 4x^2 < 0$ if we instead let $x \neq 0$. Observe that $(ay)^2 - 4y^2 = y^2(a^2 - 4) < 0$. Since $y^2 > 0$, $a^2 - 4 < 0$, so $a^2 < 4$. Thus $a < 2$ or $-a < 2 \Longrightarrow a > -2$.

(e).1 Proposition The minimal value of $x^2 + bx + c$ is $-\frac{b^2}{4} + c$.

Proof.

$$\begin{aligned}
x^2+bx+c &= (x^2+\frac{b}{2}x+\frac{b}{2}x+\frac{b^2}{4}-\frac{b^2}{4})+c \\
&= (x^2+\frac{b}{2}x+\frac{b}{2}x+\frac{b^2}{4})-\frac{b^2}{4}+c \\
&= (x+\frac{b}{2})^2-\frac{b^2}{4}+c.
\end{aligned}$$

The minimal value of $(x+\frac{b}{2})^2 \geq 0$ is 0 when $x=-\frac{b}{2}$. Thus the minimal value of x^2+bx+c is $(0)-\frac{b^2}{4}+c=-\frac{b^2}{4}+c$.

(e).2 Proposition The minimal value of ax^2+bx+c is $-\frac{b^2}{4}+c$.

Proof.

$$\begin{aligned}
ax^2+bx+c &= a(x^2+\frac{b}{a}x)+c \\
&= a(x^2+\frac{b}{2a}x+\frac{b}{2a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2})+c \\
&= a(x^2+\frac{b}{2a}x+\frac{b}{2a}x+\frac{b^2}{4a^2})-\frac{b^2}{4a}+c \\
&= a(x+\frac{b}{2a})^2-\frac{b^2}{4a}+c.
\end{aligned}$$

The minimal value of $(x+\frac{b}{2a})^2 \geq 0$ is 0 when $x=-\frac{b}{2a}$. Thus the minimal value of ax^2+bx+c is $a(0)-\frac{b^2}{4a}+c=-\frac{b^2}{4a}+c$.