Proposition: For all $n \in \mathbb{N}$, it follows that
$S_n : \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 ... + \binom{n}{n}^2 = \binom{2n}{n}$.

Proof. (Induction).
Basis step. Suppose $n = 1$. Observe that $\binom{1}{0}^2 + \binom{1}{1}^2 = 1 + 1 = 2 = \binom{2(1)}{1}$, thus S_1.
Inductive step. Suppose S_n for $n \in \mathbb{N}$.

Note that we will use the equation $\sum_{i=0}^{p} \binom{m}{i}\binom{n}{p-i} = \binom{m+n}{p}$ (Ch. 10 Exercise 38).

We will now show S_n implies S_{n+1}. Observe that

$$\binom{2(n+1)}{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i}\binom{n+1}{n-i+1} \qquad \text{(Ch. 10 Exercise 38)} \tag{1}$$

$$= \binom{n+1}{0}\binom{n+1}{n+1} + \binom{n+1}{1}\binom{n+1}{n} + ... + \binom{n+1}{n+1}\binom{n+1}{n-(n+1)+1} \tag{2}$$

$$= \binom{n+1}{0}^2 + \binom{n+1}{1}^2 + \binom{n+1}{3}^2 + ... + \binom{n+1}{n+1}^2 \qquad \left(\text{Apply } \binom{a}{b} = \binom{a}{a-b} \text{ property}\right). \tag{3}$$

Thus S_{n+1}.
It follows by mathematical induction that S_n for all $n \in \mathbb{N}$. ∎