Proposition: S_n : If $n = 2^k - 1$ for some $k \in \mathbb{N}$, then $2 \nmid \binom{n}{m}$ for all $m \in \mathbb{N}$ where $m \leq n$.

Proof. (Smallest counterexample).

Suppose $n = 1$. Observe that $1 = 2^1 - 1 = 2^k - 1$ for $k = 1$, and $1 = \binom{1}{1} = \binom{n}{m}$ for $m \leq n$, thus S_1.

Suppose for the sake of contradiction that there exists n for which $\neg S_n$.
Note that $2 \nmid \binom{2^k-1}{1}$ and $2 \nmid \binom{2^k-1}{2^k-1}$.
Thus, let $n = 2^k - 1$ where $k \in \mathbb{N}$ be the smallest n for which $2 | \binom{n}{m}$ and $2 \nmid \binom{n}{m-1}$ for some $m \in \mathbb{N}$, WLOG. Observe that

$$\begin{aligned}
\binom{n}{m} + \binom{n}{m-1} &= \binom{n+1}{m} && \text{(Def. of Pascal's triangle)} \quad (1) \\
&= \binom{(2^k-1)+1}{m} && \text{(Inductive hypothesis)} \quad (2) \\
&= \frac{(2^k)!}{m!(2^k-m)!} && (3) \\
&= \frac{2^k(2^k-1)!}{m!(2^k-m)!} && (4) \\
&= 2\left(\frac{2^{k-1}(2^k-1)!}{m!(2^k-m)!}\right). && (5)
\end{aligned}$$

Thus $2 | \binom{n+1}{m}$. But the sum of an even and an odd number is odd, a contradiction.

It follows by mathematical induction that S_n. ∎