Proposition: For all $n \in \mathbb{Z}$ where $n \geq 0$, it follows that
$S_n : (x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + ... + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$.

Proof. (Induction).
(1) Let $n = 0$. Observe that $(x+y)^n = (x+y)^0 = 1 = 1 + 0 * 1 = \binom{0}{0}x^0 + \binom{0}{1}y^0$. Thus S_1.
(2) Let $n = k$ where $k \geq 0$. Suppose S_k.
Note that we have $k+1$ terms on RHS of S_k.
We now show S_k implies S_{k+1}. Observe that

$$
\begin{aligned}
(x+y)^{k+1} &= (x+y)(x+y)^k && (1)\\
&= (x+y)\left(\binom{k}{0}x^k + \binom{k}{1}x^{k-1}y + \binom{k}{2}x^{k-2}y^2 + ... + \binom{k}{k-1}xy^{k-1} + \binom{k}{k}y^k\right) && (2)\\
&= (x+y)\binom{k}{0}x^k + (x+y)\binom{k}{1}x^{k-1}y + (x+y)\binom{k}{2}x^{k-2}y^2 + ... + (x+y)\binom{k}{k-1}xy^{k-1} + (x+y)\binom{k}{k}y^k && (3)\\
&= \binom{k}{0}x^{k+1} + \binom{k}{0}x^k y + \binom{k}{1}x^k y + \binom{k}{1}x^{k-1}y^2 + ... + \binom{k}{k-1}x^2y^{k-1} + \binom{k}{k-1}xy^k + \binom{k}{k}xy^k + \binom{k}{k}y^{k+1} && (4)\\
&\text{(Note that we now have } 2(k+1) \text{ terms on RHS, so we group terms as such,)} && (5)\\
&= \binom{k}{0}x^{k+1} + \left(\binom{k}{0}x^k y + \binom{k}{1}x^k y\right) + ... + \left(\binom{k}{k-1}xy^k + \binom{k}{k}xy^k\right) + \binom{k}{k}y^{k+1} && (6)\\
&= \binom{k}{0}x^{k+1} + x^k y\left(\binom{k}{0} + \binom{k}{1}\right) + ... + xy^k\left(\binom{k}{k-1} + \binom{k}{k}\right) + \binom{k}{k}y^{k+1} && (7)\\
&\text{(Applying } \binom{k+1}{i+1} = \binom{k}{i} + \binom{k}{i+1} \text{ from def. of Pascal's triangle,)} && (8)\\
&= \binom{k}{0}x^{k+1} + \binom{k+1}{1}x^k y + ... + \binom{k+1}{k}xy^k + \binom{k}{k}y^{k+1} && (9)\\
&\text{(Applying } \binom{k}{0} = \binom{k}{k} = 1 = \binom{k+1}{0} = \binom{k+1}{k+1} \text{ rule,)} && (10)\\
&= \binom{k+1}{0}x^{k+1} + \binom{k+1}{1}x^k y + ... + \binom{k+1}{k}xy^k + \binom{k+1}{k+1}y^{k+1}. && (11)
\end{aligned}
$$

Thus S_{k+1}.
It follows by mathematical induction that S_n for all $n \in \mathbb{Z}$ where $n \geq 0$. ∎