2-21. *Soln.*
(a) $2n^2 + 1 \leq 3n^2 = c * n^2$ where $c = 3$, thus $2n^2 + 1 = O(n^2)$.

(b) $\sqrt{n} \gg log(n)$, thus $\sqrt{n} \neq O(log(n))$.

(c) $\sqrt{n} \gg log(n)$, thus $log(n) = O(\sqrt{n})$.

(d) $\sqrt{n} \gg log(n) \iff n^2(\sqrt{n} + 1) \gg n^2 log(n)$, thus $n^2(\sqrt{n} + 1) \neq O(n^2 log(n))$.

(e) $n^2 \gg \sqrt{n} \iff 3n^2 + \sqrt{n} \leq 4n^2 = c * n^2$ for $c = 4$, thus $3n^2 + \sqrt{n} = O(n^2)$.

(f) Applying L'Hopital's rule, $\lim_{n\to\infty} \dfrac{\sqrt{n}log(n)}{n} = \lim_{n\to\infty} \dfrac{1}{2\sqrt{n} * n} = 0$. Thus $n \gg \sqrt{n}log(n) \implies \sqrt{n}log(n) = O(n)$.

(g) $\lim_{n\to\infty} n^{-1/2} = \lim_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$ and $\lim_{n\to\infty} logn = \infty$, thus $log(n) \neq O(n^{-1/2})$.

2-22. *Soln.*
(a) $n^2 \gg n \iff f(n) \gg g(n)$, thus $f(n) = \Omega(g(n))$.

(b) $n \gg \sqrt{n} \iff n(n-1) \gg n\sqrt{n} \iff g(n) \gg f(n)$, thus $f(n) = O(g(n))$.

2-23. *Soln.*
(a) Yes. For example, bubblesort makes $n(n-1)/2$ comparisons in the worst case, but only $n-1$ comparisons on a sorted array. Thus it runs in $O(n^2)$ in the worst case but there exists a case for which it runs in $O(n)$.

(b) Yes, because $n \leq c * n^2$ for $c = 1$.

(c) Yes, it is possible that $n \geq c * n^2$ for small enough n and constants c.

(d) No, because $n^2 \gg n$, thus there exists no constant c for which $n \geq c * n^2$ for large enough n.

(e) Yes.
Let $f(n) = 100n^2$. Then $f(n) \leq c * n^2$ and $f(n) = 100n^2 \geq c * n^2$ for $c = 100$, thus $100n^2 = \Theta(n^2)$.
Let $f(n) = 20n^2 - nlogn$. Then $f(n) \leq c*n^2$ for $c = 20$, and $f(n) = 20n^2 - nlogn \geq c*n^2$ for $c = 19$, thus $20n^2 - nlogn = \Theta(n^2)$.

2-24. *Soln.*
(a) No. $3^n \gg 2^n$, thus $3^n \neq O(2^n)$.

(b) Yes. $log3^n = nlog3 \leq n2log2 = c * log2^n$ for $c = 2$, thus $log3^n = O(log2^n)$.

(c) Yes. $3^n \gg 2^n$, thus $3^n = \Omega(2^n)$.

(d) Yes. $log3^n = nlog3 \geq n2log2 = c * log2^n$ for $c = 2$, thus $log3^n = \Omega(log2^n)$.

2-25. *Soln.*
(a) $f(n) = \sum_{i=1}^{n} \frac{1}{i} \approx ln(n) = \Theta(g(n))$ for $g(n) = log(n)$.

(b) $f(n) = \sum_{i=1}^{n} \lceil \frac{1}{i} \rceil = n = \Theta(g(n))$ for $g(n) = n$.

(c) $f(n) = \sum_{i=1}^{n} log(i) \le c_1 * nlog(n)$ for $c_1 = 1$. Also

$$\begin{align}
\sum_{i=1}^{n} log(i) &\ge \sum_{i=1}^{n/2} log(i) \tag{1}\\
\iff f(n) &\ge \frac{n}{2} log \frac{n}{2} \tag{2}\\
&= \frac{n}{2}(log(n) - log2) \tag{3}\\
&\ge c_2 * nlog(n) \tag{4}
\end{align}$$

for $c_2 = \frac{1}{4}$. Thus $g(n) = nlogn$.

(d) $f(n) = log(n!) = log(\prod_{i=1}^{n} i) = \sum_{i=1}^{n} log(i)$, thus $g(n) = nlogn$ (Exercise 2-25. (c)).

2-26. *Soln.* $log_2(\sum_{i=0}^{n} 2^i) \ll n(log_2 n)^2 \ll n^2 log_2 n \ll \sum_{i=0}^{n} 2^i$.

2-27. *Soln.* $\sqrt{n}logn \ll 12n^{3/2} + 4n \ll n\sqrt{logn} \ll \sum_{i=1}^{n} \sqrt{i}$.

2-28. *Soln.*
(a)

$$\begin{align}
f(n) = \sum_{i=1}^{n}(3i^4 + 2i^3 - 19i + 20) = 3\sum_{i=1}^{n} i^4 + 2\sum_{i=1}^{n} i^3 - 19\sum_{i=1}^{n} i + 20n \tag{5}\\
\implies f(n) \le 3n^5 + 2n^4 - 19n^2 + 20n \le c_1 * n^5 \qquad \text{for } c_1 = 4. \tag{6}\\
\wedge f(n) \ge c_2 * n^5. \qquad \text{for } c_2 = 1 \tag{7}
\end{align}$$

Thus $g(n) = n^5$.

(b)

$$\begin{align}
f(n) = \sum_{i=1}^{n} \left(3(4^i) + 2(3^i) - i^{19} + 20\right) = 3\sum_{i=1}^{n} 4^i + 2\sum_{i=1}^{n} 3^i - \sum_{i=1}^{n} i^{19} + 20n \tag{8}\\
\implies f(n) \le 3 * 4^{n+1} + 2 * 3^{n+1} + 20n \le c_1 * 4^{n+1} \qquad \text{for } c_1 = 12 \tag{9}\\
\wedge f(n) \ge 3 * 4^{n+1} - n^{20} \ge c_2 * 4^{n+1}. \qquad \text{for } c_2 = \frac{1}{12} \tag{10}
\end{align}$$

Thus $g(n) = 4^{n+1}$.

(c) $f(n) = \sum_{i=1}^{n}(5^i + 3^{2i}) = \sum_{i=1}^{n} 5^i + \sum_{i=1}^{n} 9^i \le 5^{n+1} + 9^{n+1}$. Then $f(n) \le c_1 * 9^{n+1}$ for $c_1 = 9$. Also $f(n) \ge c_2 * 9^{n+1}$ for $c_2 = 1$. Thus $g(n) = 9^{n+1}$.

2-29. *Soln.* Note $f(n) = \sum_{i=1}^{n} 3^i = \dfrac{3^n - 1}{2}$.
(a) $\dfrac{3^n - 1}{2} \le c_1 * 3^{n-1}$ for $c_1 = 3$, and $\dfrac{2^n - 1}{2} \ge c_2 * 3^{n-1}$ for $c_2 = \frac{1}{2}$, thus $f(n) = \Theta(3^{n-1})$.

(b) $\dfrac{3^n - 1}{2} \le c_1 * 3^n$ for $c_1 = 1$, and $\dfrac{3^n - 1}{2} \ge c_2 * 3^n$ for $c_2 = \frac{1}{3}$, thus $f(n) = \Theta(3^n)$.

(c) $\dfrac{3^n - 1}{2} \le c_1 * 3^{n+1}$ for $c_1 = \frac{1}{3}$, and $\dfrac{3^n - 1}{2} \ge c_2 * 3^{n+1}$ for $c_3 = \frac{1}{9}$.

2-30. *Soln.*

(a) $1000 * 2^n + 4^n \le c_1 * 4^n$ for $c_1 = 4$, and $1000 * 2^n + 4^n \ge c_2 * 4^n$ for $c_2 = 1$, thus $f_1(n) = \Theta(4^n)$.

(b) $n \gg \sqrt{n} \gg nlogn \iff n + nlogn + \sqrt{n} \le c_1 * n$ for $c_1 = 2$, and $n + nlogn + \sqrt{n} \ge c_2 * n$ for $c_2 = 1$, thus $f_2(n) = \Theta(n)$.

(c) $(logn)^{10} \gg logn \iff 20logn + (logn)^{10} = f_3(n) \le c_1(logn)^{10}$ for $c_1 = 2$, and $20logn + (logn)^{10} \ge c_2(logn)^{10}$ for $c_2 = 1$, thus $f_3(n) = \Theta((logn)^{10})$.

(d) $n^{100} \gg 0.99^n \iff 0.99^n + n^{100} \le c_1 * n^{100}$ for $c_1 = 2$, and $0.99^n + n^{100} \ge c_2 * n^{100}$ for $c_2 = 1$, thus $f_4(n) = \Theta(n^{100})$.

2-31. *Soln.* (a) ω, (b) O, (c) o, Ω, (d) o, (e) ω, (f) Θ.