

**Proposition:** Let  $A_1, A_2, \dots, A_n$  be sets in universe  $U$ , where  $n \geq 2$ . Then  $S_n : \overline{A_1 \cap A_2 \cap \dots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \dots \cup \overline{A_n}$ .

*Proof.* (Strong induction).

**Basis step.** Suppose  $n = 2$ . Observe that

$$\overline{A_1 \cap A_2} = \{x : x \in U \wedge x \notin (A_1 \cap A_2)\} \quad (\text{Def. of set complement}) \quad (1)$$

$$= \{x : x \in U \wedge \neg(x \in (A_1 \cap A_2))\} \quad (2)$$

$$= \{x : x \in U \wedge \neg(x \in A_1 \wedge x \in A_2)\} \quad (\text{Def. of set intersection}) \quad (3)$$

$$= \{x : x \in U \wedge (x \notin A_1 \vee x \notin A_2)\} \quad (\text{DeMorgan's law}) \quad (4)$$

$$= \{x : (x \in U \wedge x \notin A_1) \vee (x \in U \wedge x \notin A_2)\} \quad (\text{Distributive property}) \quad (5)$$

$$= \{x : x \in U \wedge x \notin A_1\} \cup \{x : x \in U \wedge x \notin A_2\} \quad (\text{Def. of set union}) \quad (6)$$

$$= \overline{A_1} \cup \overline{A_2}. \quad (7)$$

Thus  $S_2$ .

**Inductive step.** Suppose  $S_m$  for all  $m, n \in \mathbb{N}$  where  $2 \leq m \leq n$ .

We now show  $S_m$  implies  $S_{n+1}$ . Observe that

$$\overline{A_1 \cap A_2 \cap \dots \cap A_n \cap A_{n+1}} = (\overline{A_1 \cap A_2 \cap \dots \cap A_n}) \cup \overline{A_{n+1}} \quad (8)$$

$$= \overline{(A_1 \cap A_2 \cap \dots \cap A_n)} \cup \overline{A_{n+1}} \quad (\text{Inductive hypothesis}) \quad (9)$$

$$= \overline{A_1 \cap A_2 \cap \dots \cap A_n \cap A_{n+1}}. \quad (10)$$

Thus  $S_{n+1}$ .

It follows by mathematical induction that  $S_n$  for all sets  $A_1, A_2, \dots, A_n$  in universe  $U$ , where  $n \geq 2$ . ■