Proposition: For all $r, n \in \mathbb{N}$ where $r \leq n$, it follows that $S_n : \sum_{i=0}^{n} \binom{i}{r} = \binom{n+1}{r+1}$.

Proof. (Induction).
Basis step. Suppose $n = 1$, then $r = 1$. Observe that $\sum_{i=0}^{1} \binom{i}{r} = \binom{0}{1} + \binom{1}{1} = 0 + 1 = \binom{1+1}{1+1}$. Thus S_1.
Inductive step. Suppose S_k for $r, k \in \mathbb{N}$ and $r \leq k$.

We now show S_k implies S_{k+1}. Observe that

$$\sum_{i=0}^{k+1} \binom{i}{r} = \left[\sum_{i=0}^{k} \binom{i}{r}\right] + \binom{k+1}{r} \tag{1}$$

$$= \binom{k+1}{r+1} + \binom{k+1}{r} \tag{2}$$

$$= \binom{(k+1)+1}{r+1} \qquad \text{(Def. of Pascal's triangle).} \tag{3}$$

Thus S_{k+1}.

It follows by mathematical induction that S_n for all $r, n \in \mathbb{N}$ where $r \leq n$. ■