12.

(a).1 If a is rational and b is irrational, then $a + b$ is irrational.

Proof. (Contradiction).

Suppose a is rational and b is irrational but $a+b$ is rational. Then $a+b = \frac{n}{m}$ for some integers n and m. Thus $b = \frac{n}{m} - a = \frac{n+ma}{m}$. Since n, m, and a are rational, it follows that $\frac{n+ma}{m}$ is rational. But b is irrational, a contradiction.

(a).2 If a is irrational and b is irrational, then $a + b$ must be irrational.

Disproof. (Counterexample).

Let $a = \sqrt{2}$ and $b = 0 - \sqrt{2}$. Note that, since $\sqrt{2}$ is irrational and 0 is rational, it follows that a and b are irrational. Then $a + b = \sqrt{2} - \sqrt{2} = 0$. But 0 is rational, thus there exist irrational a and b for which $a + b$ is rational.

(b) If a is rational and b is irrational, then ab is irrational.

Proof. (Contradiction).

Suppose a is rational and b is irrational but ab is rational. Hence $a = \frac{x}{y}$ and $ab = \frac{n}{m}$ for some integers x, y, n, m. Then $ab = \frac{x}{y}b = \frac{n}{m}$, so $b = \frac{ny}{mx}$. Since n, y, m, x are all rational, it follows that $\frac{ny}{mx}$ is rational. But b is irrational, a contradiction.

(c) Is there a number a such that a^2 is irrational but a^4 is rational?

First we show that $\sqrt[4]{2}$ is irrational. Suppose for the sake of contradiction that $\sqrt[4]{2}$ is rational, hence $\sqrt[4]{2} = \frac{n}{m}$ for some integers n, m. Then $\sqrt{2} = (\frac{n}{m})^2$. Since $(\frac{n}{m})^2$ is rational, but we know $\sqrt{2}$ is irrational, we have a contradiction.

Hence $\sqrt[4]{2}$ is irrational. Then $\sqrt[4]{2}^2 = \sqrt{2}$ is irrational, but $\sqrt[4]{2}^4 = 2$ is rational.

(d) Are there two irrational numbers whose sum and product are both irrational?

$\sqrt{2} + \sqrt{3}$ is irrational by Proposition 14.b, and $\sqrt{2}\sqrt{3} = \sqrt{6}$ is irrational by Proposition 13.a. Thus $\sqrt{2}$ and $\sqrt{3}$ are two such numbers.