1. Reflexive, symmetric, transitive.

2. Not reflexive ($\neg(aRa)$), not symmetric ($aRb \wedge \neg(bRa)$), transitive.

3. Not reflexive ($\neg(aRa)$), not symmetric ($aRb \wedge \neg(bRa)$), not transitive ($cRb \wedge bRc \wedge \neg(cRc)$).

4. Reflexive, symmetric, transitive.

5. Not reflexive ($\neg(1R1)$), symmetric, transitive.

6. Reflexive, symmetric, transitive. The relation R on $\mathbb{Z}$ is $=$.

7.

	$R = \varnothing$	$R = A \times A - \{\emptyset\}$
Reflexive	F	T
Symmetric	T	T
Transitive	T	T

	$R = \{(b,a)\} \vee R = \{(a,b)\}$	$R = \{(a,a)\} \vee R = \{(b,b)\}$
Reflexive	F	F
Symmetric	F	T
Transitive	T	T

	$R = \{(a,b),(b,a)\}$	$R = \{(a,a),(b,b)\}$
Reflexive	F	T
Symmetric	T	T
Transitive	F	T

	$R = \{(a,b),(b,a),(a,a)\}$	$R = \{(a,b),(b,a),(b,b)\}$	$R = \{(a,a),(b,b),(a,b)\}$	$R = \{(a,a),(b,b),(b,a)\}$
Reflexive	F	F	T	T
Symmetric	T	T	F	F
Transitive	F	F	F	F

	$R = \{(b,a),(a,a)$	$R = \{(a,b),(a,a)\}$	$R = \{(a,b),(b,b)\}$	$R = \{(b,a),(b,b)\}$
Reflexive	F	F	F	F
Symmetric	F	F	F	F
Transitive	T	T	T	T

8. $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : |x-y| < 1\}$. Reflexive, symmetric, transitive. The relation R on $\mathbb{Z}$ is $=$.

9. $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod 2\}$. Reflexive, symmetric, transitive. The relation R on $\mathbb{Z}$ is $2|(x-y)$.

10. Not reflexive ($\forall x \in A, (x,x) \notin \varnothing$), symmetric, transitive.

11. Reflexive, symmetric, transitive.

12. Proposition The relation $|$ (divides) on the set $\mathbb{Z}$ is reflexive and transitive.

Proof.
First we show that the relation is reflexive. Let $x \in \mathbb{Z}$, and observe that $x = x(1)$, so $x|x$ by def. of divisibility. Thus $|$ is reflexive.

Next we show that the relation is transitive. Suppose $x|y$ and $y|z$ for $y, z \in \mathbb{Z}$. Then $y = xa$ for any $a \in \mathbb{Z}$. It follows that $xa|z$, so $z = xab$ for some $b \in \mathbb{Z}$. Thus $x|z$, which means $|$ is transitive.

Therefore $|$ on the set $\mathbb{Z}$ is reflexive and transitive. ■

13. Proposition The relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$ on $\mathbb{R}$ is reflexive, symmetric and transitive.

Proof.

First we show that the relation is reflexive. Let $x \in \mathbb{R}$. Observe that $x - x = 0$, so $x - x \in \mathbb{Z}$ and consequently xRx. Thus R is reflexive.

Next we show that the relation is symmetric. Suppose xRy for some $y \in \mathbb{R}$, which means $x - y = a$ for any $a \in \mathbb{Z}$. Multiplying both sides by -1 gives $y - x = -a$. Then $y - x \in \mathbb{Z}$, which implies yRx. Thus R is symmetric.

Finally we show that the relation is transitive. Suppose xRy and yRz for any $z \in \mathbb{R}$. Then $x - y = a$ and $y - z = c$ for some $c \in \mathbb{Z}$. Adding the two equations, we get $(x - y) + (y - z) = a + b$ so $x - z = a + b$. Then $x - z \in \mathbb{Z}$, which implies xRz. Thus R is transitive.

Therefore $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$ on $\mathbb{R}$ is reflexive, symmetric and transitive. ■

14. Proposition Suppose R is a symmetric and transitive relation on a set A. If there exists an element $a \in A$ for which aRx for all $x \in A$, then R is reflexive.

Proof.

Suppose there exists $a \in A$ for which aRx for all $x \in A$. Since R is symmetric, xRa and aRx. Since R is transitive, xRx. Thus R is reflexive. Therefore R is reflexive on set A if R is symmetric and transitive, and there exists $a \in A$ for which aRx for all $x \in A$. ■

15. Conjecture If a relation is symmetric and transitive, then it is also reflexive.

Disroof.

This conjecture is false. We show this with a counterexample. Let $A = \{a, b, c\}$ and $R = \{(a, b), (b, a), (a, a), (b, b)\}$ and $x, y \in A$. Observe that xRy implies yRx, thus R is symmetric. Also xRy and yRx implies xRx, thus R is transitive. But it is not the case that cRc, thus R is not reflexive. ■

16. Proposition Let R be a relation on $\mathbb{Z}$ such that xRy if and only if $x^2 \equiv y^2 \pmod 4$. Then R is reflexive, symmetric and transitive.

Proof.

First we show that the relation is reflexive. Let $x \in \mathbb{Z}$. Observe that $4|0$ implies $4|(x^2 - x^2)$ by def. of divisiblity, so $x^2 \equiv x^2 \pmod 4$ by def. of congruence modulo 4. Thus R is reflexive.

Next we show that the relation is symmetric. Suppose xRy for any $y \in \mathbb{Z}$. It follows that $x^2 \equiv y^2 \pmod 4$. Then $x^2 = 4a + c$ and $y^2 = 4b + c$ for some $a, b, c \in \mathbb{Z}$ where $0 \leq c < 4$ by def. of division algorithm. Solving for c, we get $x^2 - 4a = y^2 - 4b$, so $x^2 - y^2 = 4a - 4b$. Multiplying both sides by -1, we get $y^2 - x^2 = 4b - 4a$, thus $4|(y^2 - x^2)$. Thus $y^2 \equiv x^2 \pmod 4$, which implies yRx and so R is symmetric.

Finally we show that R is transitive. Suppose xRy and yRz for any $z \in \mathbb{Z}$. Then $x^2 \equiv y^2 \pmod 4$ and $y^2 \equiv z^2 \pmod 4$, which implies $4|(x^2 - y^2)$ and $4|(y^2 - z^2)$ and consequently $x^2 - y^2 = 4s$ and $y^2 - z^2 = 4t$ for some $s, t \in \mathbb{Z}$. Adding the two equations, we get $(x^2 - y^2) + (y^2 - z^2) = 4s + 4t$, so $x^2 - z^2 = 4(s + t)$. Thus $4|(x^2 - z^2)$, which implies $x^2 \equiv z^2 \pmod 4$. Since xRz, it follows that R is transitive.

Therefore the relation R on $\mathbb{Z}$ such that xRy if and only if $x^2 \equiv y^2 \pmod 4$ is reflexive, symmetric and transitive. ■

17. Proposition Let R be a relation on $\mathbb{Z}$ such that xRy if and only if $|x - y| \leq 1$. Then R is reflexive and symmetric, but not transitive.

Proof.

First we show that the relation is reflexive. Let $x \in \mathbb{Z}$. Since $|x - x| = 0 \leq 1$, it follows that xRx. Thus R is reflexive.

Next we show that the relation is symmetric. Suppose xRy for any $y \in \mathbb{Z}$. Observe that $|x - y| \leq 1$ implies that $-1 \leq x - y \leq 1$. Multiplying all sides by -1 we get $1 \geq y - x \geq -1$, which implies that $|y - x| \leq 1$. It follows that yRx, thus R is symmetric.

Finally, we show that the relation is not transitive with a counterexample. Observe that $(-2)R(-1)$ and $(-1)R0$, but it is not the case that $(-2)R0$. Thus there exists some $a, b, c \in \mathbb{Z}$ for which aRb and bRc but not aRc, which means R is not transitive.

Therefore the relation R on $\mathbb{Z}$ such that xRy if and only if $|x - y| \leq 1$ is reflexive and symmetric but not transitive. ■

18.

	$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 2\|a \wedge 2\|b\}$		
Reflexive	F	T	F
Symmetric	T	T	T
Transitive	T	F	T