1-8. *Soln.*

(a) $f(n) = \log 2^n = 2\log n \leq 2(\log n + 5) = g(n)$, thus $f(n) \leq c_1 * g(n)$ for $c_1 = 2$. Also $f(n) = 2\log n \geq \log n + 5$, thus $f(n) \geq c_2 * f(g)$ for $c_2 = 1$. Therefore $f(n) = \Theta(g(n))$.

(b)

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \frac{\sqrt{n}}{logn^2} && (1)\\
&= \frac{1}{2} \lim_{n\to\infty} \frac{\sqrt{n}}{logn} && (2)\\
&= \frac{1}{2} \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} && \text{(L'Hopital's rule)} \quad (3)\\
&= \frac{1}{2} \lim_{n\to\infty} \frac{n}{2\sqrt{n}} && (4)\\
&= \frac{1}{4} \lim_{n\to\infty} n * n^{-\frac{1}{2}} && (5)\\
&= \frac{1}{4} \lim_{n\to\infty} n^{\frac{1}{2}} = \infty. && (6)
\end{aligned}$$

Thus $f(n) \gg g(n)$, which implies $f(n) = \Omega(g(n))$.

(c) $\lim_{n\to\infty} \dfrac{f(n)}{g(n)} = \lim_{n\to\infty} \dfrac{logn^2}{logn} = \lim_{n\to\infty} logn = \infty$, thus $f(n) \gg g(n)$ which implies $f(n) = \Omega(g(n))$.

(d)

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{n}{(logn)^2} && (7)\\
&= \left(\lim_{n\to\infty} \frac{\sqrt{n}}{logn}\right)^2 && (8)\\
&= \left(\lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}}\right)^2 && \text{(L'Hopital's rule)} \quad (9)\\
&= \left(\lim_{n\to\infty} \frac{n}{2\sqrt{n}}\right)^2 && (10)\\
&= \left(\frac{1}{2} \lim_{n\to\infty} \frac{1}{\frac{1}{n}}\right)^2 && \text{(L'Hopital's rule)} \quad (11)\\
&= \left(\frac{1}{2} \lim_{n\to\infty} n\right)^2 = \infty. && (12)
\end{aligned}$$

Thus $f(n) \gg g(n)$, which implies $f(n) = \Omega(g(n))$.

(e)

$$\begin{aligned}
\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \frac{nlogn+n}{logn} && (13)\\
&= \lim_{n\to\infty}\frac{nlogn}{logn}+\lim_{n\to\infty}\frac{n}{logn} && (14)\\
&= \lim_{n\to\infty}\frac{nlogn}{logn}+\lim_{n\to\infty}\frac{1}{\frac{1}{n}} && \text{(L'Hopital's rule)} \quad (15)\\
&= \lim_{n\to\infty} n+\lim_{n\to\infty} n=\infty. && (16)
\end{aligned}$$

Thus $f(n) \gg g(n)$, which implies $f(n) = \Omega(g(n))$.

(f) $f(n) = 10 \geq log10 = c_1 * g(n)$ when $c_1 = 1$ and $f(n) = 10 \leq 10log10 = c_2 * g(n)$ when $c_2 = 10$, thus $f(n) = \Theta(g(n))$.

(g)

$$\begin{aligned}
\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \lim_{n\to\infty}\frac{2^n}{10n^2} && (17)\\
&= \frac{1}{10}\lim_{n\to\infty}\frac{2^n log2}{2n} && \text{(L'Hopital's rule)} \quad (18)\\
&= \frac{1}{10}\Big(\lim_{n\to\infty}\frac{2^n}{n} * \lim_{n\to\infty}\frac{log2}{2}\Big) && (19)\\
&= \Big(\lim_{n\to\infty} 2^n log2\Big)\frac{log2}{20} && \text{(L'Hopital's rule)} \quad (20)\\
&= \Big(\lim_{n\to\infty} 2^n\Big)\frac{log^2 2}{20} = \infty && (21)
\end{aligned}$$

Thus $f(n) \gg g(n)$, which implies $f(n) = \Omega(g(n))$.

(g)

$$\begin{aligned}
\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \lim_{n\to\infty}\frac{2^n}{3^n} && (22)\\
&= \Big(\lim_{n\to\infty}\frac{2}{3}\Big)^n && (23)\\
&= \Big(\frac{2}{3}\Big)^n = 0. && (24)
\end{aligned}$$

Thus $g(n) \gg f(n)$, which implies $f(n) = O(g(n))$.