Proposition: For all $n \in \mathbb{N}$, it follows that

$$S_n : \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)...\left(1-\frac{1}{2^n}\right) \geq \frac{1}{4}+\frac{1}{2^{k+1}}.$$

Proof. (Induction).

Basis step. Suppose $n = 1$.

Observe that $1 - \frac{1}{2^1} = \frac{1}{2} = \frac{2}{4} = \frac{1}{4} + \frac{1}{2^{1+1}}$.

Thus S_1.

Inductive step. Suppose S_k for $n \in \mathbb{N}$.

We now show S_k implies S_{k+1}. Observe that

$$\begin{aligned}
\frac{1}{4}+\frac{1}{2^{k+1}} &= \frac{1}{2}*\frac{1}{2}+\frac{1}{2}*\frac{1}{2^k} && (1)\\
&= \left(\frac{1}{2}+\frac{1}{2^k}\right)\frac{1}{2} && (2)\\
&\leq \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)...\left(1-\frac{1}{2^k}\right)\left(\frac{1}{2}\right) && (3)\\
&\leq \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)...\left(1-\frac{1}{2^k}\right)\left(\frac{1}{2}\right)\left(2-\frac{1}{2^k}\right) && (4)\\
&\leq \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)...\left(1-\frac{1}{2^k}\right)\left(2*\frac{1}{2}-\frac{1}{2}*\frac{1}{2^k}\right) && (5)\\
&\leq \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)...\left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{2^{k+1}}\right). && (6)
\end{aligned}$$

Thus S_{k+1}.

It follows by mathematical induction that S_n for all $n \in \mathbb{N}$. ■