5. For $a, b, c, d \in \mathbb{R}$,

(i) Proposition: If $a < b$ and $c < d$, then $a + c < b + d$.

Proof. $a < b \Longrightarrow a+c < b+c$ and $c < d \Longrightarrow b+c < b+d$. Since $a+c < b+c < b+d$, it follows that $a+c < b+d$ if $a < b$ and $c < d$.

(ii) Proposition: If $a < b$ then $-b < -a$.

Proof. $a < b \Longrightarrow \Big(a - (a+b) < b - (a+b)\Big) \Longrightarrow \Big(a - a - b < b - a - b\Big) \Longrightarrow -b < -a$.

(iii) Proposition: If $a < b$ and $c > d$, then $a - c < b - d$.

Proof. $a < b \Longrightarrow \Big(a - c < b - c\Big)$ and $c > d \Longrightarrow \Big(-c < -d\Big) \Longrightarrow \Big(b - c < b - d\Big)$. Since $a - c < b - c < b - d$, it follows that $a - c < b - d$ for $a < b$ and $c > d$.

(iv) Proposition: If $a < b$ and $c > 0$, then $ac < bc$

Proof. $a < b \Longrightarrow a + a < b + b$. Hence $\sum_{i=1}^{c} a < \sum_{i=1}^{c} b$, meaning $ac < bc$.

(v) Proposition: If $a < b$ and $c < 0$, then $ac > bc$

Proof. $a < b \Longrightarrow \Big(\sum_{i=1}^{c} a < \sum_{i=1}^{c} b\Big) \Longrightarrow \Big(-\sum_{i=1}^{|c|} a > -\sum_{i=1}^{|c|} b\Big) \Longrightarrow -|c|a > -|c|b$. Since $c = -|a|$, it follows that $ac > bc$.

(vi) Proposition: If $a > 1$, then $a^2 > a$

Proof. $a > 1 > 0 \Longrightarrow a * a > 1 * a \Longrightarrow a^2 > a$.

(vii) Proposition: If $0 < a < 1$, then $a^2 < a$.

Proof. $0 < a < 1 \Longrightarrow a = \dfrac{\pm x}{\pm y}$ for $x, y \in \mathbb{N}$ where $x < y$. Observe that

$$\begin{aligned} x < y &\Longrightarrow x^2 < xy \\ &\Longrightarrow \frac{x^2}{y^2} < \frac{x}{y} \\ &\Longrightarrow \frac{\pm x}{\pm y} * \frac{\pm x}{\pm y} < \frac{\pm x}{\pm y} \\ &\Longrightarrow a^2 < a. \end{aligned}$$

(viii) Proposition: If $0 \le a < b$ and $0 \le c < d$, then $ac < bd$.

Proof. Observe that $\Big((0 \le c) \wedge (0 \le a < b)\Big) \Longrightarrow (0 \le ac \le bc)$, and $\Big((0 < b) \wedge (0 \le c < d)\Big) \Longrightarrow (0 \le bc < bd)$. Thus $(0 \le ac \le bc < bd) \Longrightarrow (ac < bd)$.

(ix) Proposition: If $0 \le a < b$, then $a^2 < b^2$.

Proof. Observe that $(0 \le a < b) \Longrightarrow (a * a < b * b) \Longrightarrow (a^2 < b^2)$.

(x) Proposition: If $a, b \ge 0$ and $a^2 < b^2$, then $a < b$.

Proof. Note that $a \ge 0 \Longrightarrow a = \sqrt{a^2}$, and similarly $b = \sqrt{b^2}$. Thus $a^2 < b^2 \Longrightarrow \sqrt{a^2} < \sqrt{b^2} \Longrightarrow a < b$.