6. Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$.

(a) Proposition: Let $0 \leq x < y$. Then $x^n < y^n$.

Proof.

$(0 \leq x < y) \implies (0 \leq x * x < y * y) \implies \left(\prod_{i=1}^{n} x < \prod_{i=1}^{n} y \right) \implies (x^n < y^n).$

(b) Proposition: Let $x < y$ and n is odd. Then $x^n < y^n$.

Proof.

First we show that $(x < y \implies x|x| < y|y|)$. Observe that

$$\begin{aligned}
(0 < x < y) &\implies 0 < (|x|)|x| < (|y|)|y| \\
(x < 0 < y) &\implies (-|x|)|x| < 0 < (|y|)|y| \\
(x < y < 0) &\implies \Big(-|x| < -|y| \wedge |x| > |y| \Big) \implies (-|x|)|x| < (-|y|)|y| < 0.
\end{aligned}$$

Now we show $x^n < y^n$. Note that since n is odd, $n = 2m + 1$ for some $m \in \mathbb{Z}$.

$$\begin{aligned}
\Big((x < y) \wedge (x|x| < y|y|) \Big) &\implies x(|x|)^{2m} < y(|y|)^{2m} \\
x(x)^{2m} &< y(y)^{2m} \\
x^{2m+1} &< y^{2m+1} \\
\therefore x^n &< y^n.
\end{aligned}$$

(c) Proposition: If $x^n = y^n$ and n is odd, then $x = y$.

Proof. (Contrapositive).

Suppose $x \neq y$. Since $x * x \neq y * y$, it follows that $x^n \neq y^n$ for any n. Thus ($x^n \neq y^n \vee n$ is not odd) is a true statement, meaning $\neg(x = y \wedge n$ is odd).

(d) Proposition: If $x^n = y^n$ and n is even, then $x = y$ or $x = -y$.

Proof. (Contrapositive).

Suppose $\neg(x = y \vee x = -y)$, and let n be an even number $2m$. Then $x \neq y \wedge x \neq -y$, so $\Big(x * x \neq (\pm y) * (\pm y) = y * y \Big) \implies \Big(x^n \neq y^n \Big)$. Thus the statement ($x^n \neq y^n \vee n$ is not even) is true, meaning $\neg(x^n = y^n \wedge n$ is even).