4. Find all numbers x for which

(i) $4 - x < 3 - 2x$.

$$\begin{aligned} 4 - x < 3 - 2x &\implies 4 < 3 - 3x \\ &\implies 4 < 3(1 - x) \\ &\implies \frac{4}{3} - 1 < -x \\ &\implies 1 - \frac{4}{3} > x, x \in \mathbb{R}. \end{aligned}$$

(ii) $5 - x^2 < 8$

$(5 - x^2 < 8) \implies (-x^2 < 8 - 5 = 3) \implies (x^2 > -3)$. Notice that $x^2 = (\pm|x|)^2 = (\pm 1)^2|x|^2 = |x|^2 \geq 0$ for all $x \in \mathbb{R}$.

(iii) $5 - x^2 < -2$

$(5 - x^2 < -2) \implies (-x^2 < -2 - 5 = -7) \implies (x^2 > 7)$. Then $x > \sqrt{7}$ or $(-x > \sqrt{7}) \implies (x < -\sqrt{7})$ where $x \in \mathbb{R}$.

(iv) $(x - 1)(x - 3) > 0$

Note that $(ab > 0) \iff (a < 0 \wedge b < 0) \vee (a > 0 \wedge b > 0)$ for $a, b \in \mathbb{R}$. We consider two cases. Suppose $(x - 1) < 0 \wedge (x - 3) < 0$. Then $(x < 1) \wedge (x < 3), x \in \mathbb{R}$. Now suppose $(x - 1) > 0 \wedge (x - 3) > 0$. Then $(x > 1) \wedge (x > 3), x \in \mathbb{R}$.

(v) $x^2 - 2x + 2 > 0$

Let $x \in \mathbb{R}$. We consider two cases. Suppose $x > 0$, then

$$\begin{aligned} x^2 - 2x + 2 > 0 &\implies x^2 + 2 > 0 \\ &\implies x^2 > 2 \\ &\implies (x > 2) \vee (-x > 2) \\ &\implies (x > 2) \vee (x < -2). \end{aligned}$$

Since $x > 0$, $x < -2$ is not a solution. Thus $x > 2$.

Now suppose $x < 0$, then

$$\begin{aligned} x^2 - 2x + 2 > 0 &\implies x^2 - x - 2x + 2 > 0 \\ &\implies x(x - 1) - 2(x - 1) > 0 \\ &\implies (x - 2)(x - 1) > 0. \end{aligned}$$

Since the product of two numbers is positive when both numbers are positive or both numbers are negative, it follows that $(x - 2) > 0 \implies x > 2$ and $(x - 1) > 0 \implies x > 1$, or $(x - 2) < 0 \implies x < 2$ and $(x - 1) < 0 \implies x < 1$. Since $x < 0$, $x < 2$ is the only solution.

Combining solutions from both cases, we get $x > 2$ or $x < 2$ for $x \in \mathbb{R}$.

(vi) $x^2 + x + 1 > 2$

Let $x \in \mathbb{R}$. We consider two cases. Suppose $x > 0$, then

$$\begin{aligned}
x^2 + x + 1 > 2 &\implies x^2 + x > 1 \\
&\implies x^2 + x > 0 \\
&\implies x(x+1) > 0 \\
&\implies (x > 0 \wedge x + 1 > 0) \vee (x < 0 \wedge x + 1 < 0) \\
&\implies (x > 0 \wedge x > -1) \vee (x < 0 \wedge x < -1) \\
&\implies x > 0 \vee x < -1.
\end{aligned}$$

Since we supposed $x > 0$, $x < -1$ is not a solution. Now suppose $x < 0$. Then

$$\begin{aligned}
x^2 + x + 1 > 2 &\implies x^2 + x - 1 > 0 \\
&\implies x^2 > 0 \\
&\implies (x > 0) \vee (-x > 0) \\
&\impliedby (x > 0) \vee (x < 0).
\end{aligned}$$

Since we supposed $x < 0$, $x > 0$ is not a solution. Combining solutions from both cases, we get $x > 0$ or $x < 0$ for $x \in \mathbb{R}$.

(vii) $x^2 - x + 10 > 16$

$$\begin{aligned}
x^2 - x + 10 > 16 &\implies x^2 - x - 6 > 0 \\
&\implies x^2 - 3x + 2x - 6 > 0 \\
&\implies x(x-3) + 2(x-3) > 0 \\
&\implies (x+2)(x-3) > 0 \\
&\implies (x + 2 > 0 \wedge x - 3 > 0) \vee (x + 2 < 0 \wedge x - 3 < 0) \\
&\implies (x > -2 \wedge x > 3) \vee (x < -2 \wedge x < 3) \\
&\implies (x > 3) \vee (x < -2), x \in \mathbb{R}.
\end{aligned}$$

(viii) $x^2 + x + 1 > 0$

Let $x \in \mathbb{R}$. We consider two cases. Suppose $x > 0$, then

$$\begin{aligned}
x^2 + x + 1 > 0 &\implies x^2 + x + x + 1 > 0 \\
&\implies x(x+1) + 1(x+1) \\
&\implies (x+1)^2 > 0 \\
&\implies (x + 1 > 0 \vee -(x+1) > 0) \\
&\implies (x > -1 \vee -1 > x).
\end{aligned}$$

Since we supposed $x > 0$, $-1 > x$ is not a solution. Now suppose $x < 0$, then

$$\begin{aligned}
x^2 + x + 1 > 0 &\implies x^2 + 1 > 0 \\
&\implies x^2 > -1 \\
&\implies x^2 > 0 \\
&\implies (x > 0 \vee x < 0).
\end{aligned}$$

Since we supposed $x < 0$, $x > 0$ is not a solution. Combining solutions from both cases, we get $x > 0$ or $x < 0$ for $x \in \mathbb{R}$.

(ix) $(x - \pi)(x + 5)(x - 3) > 0$

$$\begin{aligned}(x - \pi)(x + 5)(x - 3) > 0 &\implies \big((x - \pi > 0) \land (x + 5 > 0) \land (x - 3 > 0)\big) \lor \big((x - \pi < 0) \land (x + 5 < 0) \land (x - 3 < 0)\big) \\ &\implies \big((x > \pi) \land (x > -5) \land (x > 3)\big) \lor \big((x < \pi) \land (x < -5) \land (x < 3)\big) \\ &\implies (x > \pi) \lor (x < -5), x \in \mathbb{R}.\end{aligned}$$

(x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$

Note that $\frac{1}{2} > \frac{1}{3} \implies \sqrt{2} = 2^{\frac{1}{2}} > 2^{\frac{1}{3}} = \sqrt[3]{2}$ (1). Observe that

$$\begin{aligned}(x - \sqrt[3]{2})(x - \sqrt{2}) > 0 &\implies \big((x - \sqrt[3]{2} > 0) \land (x - \sqrt{2} > 0)\big) \lor \big((x - \sqrt[3]{2} < 0) \land (x - \sqrt{2} < 0)\big) \\ &\implies \big((x > \sqrt[3]{2}) \land (x > \sqrt{2})\big) \lor \big((x < \sqrt[3]{2}) \land (x < \sqrt{2})\big) \\ &\implies \big((x > \sqrt[3]{2}) \land (x > \sqrt{2})\big) \lor \big((x < \sqrt[3]{2}) \land (x < \sqrt{2})\big) \\ &\implies (x > \sqrt{2}) \lor (x < \sqrt[3]{2}), x \in \mathbb{R}. \qquad (1)\end{aligned}$$

(xi) $2^x < 8$

Observe that $2^x < 8 = 2^3 \implies x < 3$.

(xii) $x + 3^x < 4$

Since $(1) + 3^{(1)} = 4$ and $x + 3^x > 4$ for all $x > 1, x \in \mathbb{R}$, we have that $x + 3^x < 4 \implies x < 1$.

(xiii) $\dfrac{1}{x} + \dfrac{1}{1 - x} > 0$

$\dfrac{1}{x} + \dfrac{1}{1 - x} = \dfrac{(1 - x) + x}{x(1 - x)} = \dfrac{1}{x(1 - x)}$. Note that $\dfrac{1}{x}$ and $\dfrac{1}{1 - x}$ are undefined for $x = 0$ and $1 - x = 0 \implies x = 1$, respectively. Then we consider three cases.

Suppose $x > 1$. Note that $x > 1 \implies (x = |x| > 1)$ and $0 > 1 - |x| = 1 - x$. Then $\dfrac{1}{x(1 - x)} = \dfrac{1}{|x|(1 - |x|)} < 0$.

Now suppose $0 < x < 1$. Note that $0 < x \implies (0 < |x| = x)$ and $(0 < x < 1) \implies (x = |x| < 1) \implies (0 < 1 - |x| = 1 - x)$. Then $\dfrac{1}{x(1 - x)} = \dfrac{1}{|x|(1 - |x|)} > 0$.

Now suppose $x < 0$. Note that $x < 0 \implies (x = -|x| < 0)$ and $1 - x = 1 + |x| > 0$. Then $\dfrac{1}{x(1 - x)} = \dfrac{1}{-|x|(1 + |x|)} < 0$.

Therefore $\dfrac{1}{x} + \dfrac{1}{1 - x} > 0$ for $0 < x < 1, x \in \mathbb{R}$.

(xiv) $\dfrac{x - 1}{x + 1} > 0$

Note that $\dfrac{x - 1}{x + 1}$ is undefined when $(x + 1 = 0) \implies (x = -1)$. We consider two cases. Suppose $x > -1$. Then

$$\begin{aligned}\frac{x - 1}{x + 1} > 0 &\implies x - 1 > 0 \\ &\implies x > 1.\end{aligned}$$

Now suppose $x < -1$. Then

$$\begin{aligned} \frac{x-1}{x+1} &= \frac{-|x|-1}{-|x|+1} \\ &= \frac{|x|+1}{|x|-1} > 0. \qquad (x < -1 \implies -x > 1 \implies |x| > 1 \implies |x|-1 > 0) \end{aligned}$$

Therefore $x > -1$ or $x < -1$ for $x \in \mathbb{R}$.