Proposition: Concerning the Fibonacci sequence, for all $n \in \mathbb{N}$,

$$S_n : \frac{(\frac{1}{2}(1+\sqrt{5}))^n + (\frac{1}{2}(1-\sqrt{5}))^n}{\sqrt{5}}.$$

Proof. (Strong Induction).

Basis step. Suppose $n \in \mathbb{N}$ where $n \leq 2$. Observe that

$$F_1 = 1 = \frac{\frac{1}{2} - \frac{1}{2} + \frac{2}{2}\sqrt{5}}{\sqrt{5}} = \frac{\frac{1}{2} + \sqrt{5} - \frac{1}{2} + \sqrt{5}}{\sqrt{5}} = \frac{(\frac{1}{2}(1+\sqrt{5}))^1 - (\frac{1}{2}(1-\sqrt{5}))^1}{\sqrt{5}}.$$

$$F_2 = 1 = \frac{\frac{1}{4} + \frac{2}{4}\sqrt{5} + \frac{5}{4} - \frac{1}{4} + \frac{2}{4}\sqrt{5} - \frac{5}{4}}{\sqrt{5}} = \frac{\frac{1}{4}(1^2 + 2\sqrt{5} + \sqrt{5}^2) - \frac{1}{4}(1^2 - 2\sqrt{5} + \sqrt{5}^2)}{\sqrt{5}} = \frac{(\frac{1}{2}(1+\sqrt{5}))^2 - (\frac{1}{2}(1-\sqrt{5}))^2}{\sqrt{5}}.$$

Thus S_1 and S_2.

Inductive step. Suppose S_k for $k \in \mathbb{N}$ where $2 \leq k$.

We now show S_k implies S_{k+1}. Observe that

$$\begin{aligned}
\frac{(\frac{1}{2}(1+\sqrt{5}))^{k+1} - (\frac{1}{2}(1-\sqrt{5}))^{k+1}}{\sqrt{5}} &= \frac{(\frac{1}{2}(1+\sqrt{5}))^{k-1}(\frac{1}{2}(1+\sqrt{5}))^2 - (\frac{1}{2}(1-\sqrt{5}))^{k-1}(\frac{1}{2}(1-\sqrt{5}))^2}{\sqrt{5}} \qquad (1) \\
&= \frac{(\frac{1}{2}(1+\sqrt{5}))^{k-1}(\frac{1}{4}(1^2 + 2\sqrt{5} + \sqrt{5}^2)) - (\frac{1}{2}(1-\sqrt{5}))^{k-1}(\frac{1}{4}(1^2 - 2\sqrt{5} + \sqrt{5}^2))}{\sqrt{5}} \qquad (2) \\
&= \frac{(\frac{1}{2}(1+\sqrt{5}))^{k-1}(\frac{1}{4}(6 + 2\sqrt{5})) - (\frac{1}{2}(1-\sqrt{5}))^{k-1}(\frac{1}{4}(6 - 2\sqrt{5}))}{\sqrt{5}} \qquad (3) \\
&= \frac{(\frac{1}{2}(1+\sqrt{5}))^{k-1}(\frac{1}{2}(2 + 1 + \sqrt{5})) - (\frac{1}{2}(1-\sqrt{5}))^{k-1}(\frac{1}{2}(2 - 1 - \sqrt{5}))}{\sqrt{5}} \qquad (4) \\
&= \frac{(\frac{1}{2}(1+\sqrt{5}))^{k-1}(1 + \frac{1}{2}(1 + \sqrt{5})) - (\frac{1}{2}(1-\sqrt{5}))^{k-1}(1 + \frac{1}{2}(1 - \sqrt{5}))}{\sqrt{5}} \qquad (5) \\
&= \frac{(\frac{1}{2}(1+\sqrt{5}))^{k} + (\frac{1}{2}(1+\sqrt{5}))^{k-1} - (\frac{1}{2}(1-\sqrt{5}))^{k} - (\frac{1}{2}(1-\sqrt{5}))^{k-1}}{\sqrt{5}} \qquad (6) \\
&= \frac{(\frac{1}{2}(1+\sqrt{5}))^{k} - (\frac{1}{2}(1-\sqrt{5}))^{k}}{\sqrt{5}} + \frac{(\frac{1}{2}(1+\sqrt{5}))^{k-1} - (\frac{1}{2}(1-\sqrt{5}))^{k-1}}{\sqrt{5}} \qquad (7) \\
&= F_k + F_{k-1} = F_{k+1}. \qquad (8)
\end{aligned}$$

Thus S_{k+1}. It follows by mathematical induction that S_n for all $n \in \mathbb{N}$. $\blacksquare$