

Proposition: Concerning the Fibonacci sequence, for all $n \in \mathbb{N}$,

$$S_n : \frac{(\frac{1}{2}(1 + \sqrt{5}))^n + (\frac{1}{2}(1 - \sqrt{5}))^n}{\sqrt{5}}.$$

Proof. (Strong Induction).

Basis step. Suppose $n \in \mathbb{N}$ where $n \leq 2$. Observe that

$$F_1 = 1 = \frac{\frac{1}{2} - \frac{1}{2} + \frac{2}{2}\sqrt{5}}{\sqrt{5}} = \frac{\frac{1}{2} + \frac{1}{2}\sqrt{5} - \frac{1}{2} + \frac{1}{2}\sqrt{5}}{\sqrt{5}} = \frac{(\frac{1}{2}(1 + \sqrt{5}))^1 - (\frac{1}{2}(1 - \sqrt{5}))^1}{\sqrt{5}}.$$

$$F_2 = 1 = \frac{\frac{1}{4} + \frac{2}{4}\sqrt{5} + \frac{5}{4} - \frac{1}{4} + \frac{2}{4}\sqrt{5} - \frac{5}{4}}{\sqrt{5}} = \frac{\frac{1}{4}(1^2 + 2\sqrt{5} + \sqrt{5}^2) - \frac{1}{4}(1^2 - 2\sqrt{5} + \sqrt{5}^2)}{\sqrt{5}} = \frac{(\frac{1}{2}(1 + \sqrt{5}))^2 - (\frac{1}{2}(1 - \sqrt{5}))^2}{\sqrt{5}}.$$

Thus S_1 and S_2 .

Inductive step. Suppose S_k for $k \in \mathbb{N}$ where $2 \leq k$.

We now show S_k implies S_{k+1} . Observe that

$$\frac{(\frac{1}{2}(1 + \sqrt{5}))^{k+1} - (\frac{1}{2}(1 - \sqrt{5}))^{k+1}}{\sqrt{5}} = \frac{(\frac{1}{2}(1 + \sqrt{5}))^{k-1}(\frac{1}{2}(1 + \sqrt{5}))^2 - (\frac{1}{2}(1 - \sqrt{5}))^{k-1}(\frac{1}{2}(1 - \sqrt{5}))^2}{\sqrt{5}} \quad (1)$$

$$= \frac{(\frac{1}{2}(1 + \sqrt{5}))^{k-1}(\frac{1}{4}(1^2 + 2\sqrt{5} + \sqrt{5}^2)) - (\frac{1}{2}(1 - \sqrt{5}))^{k-1}(\frac{1}{4}(1^2 - 2\sqrt{5} + \sqrt{5}^2))}{\sqrt{5}} \quad (2)$$

$$= \frac{(\frac{1}{2}(1 + \sqrt{5}))^{k-1}(\frac{1}{4}(6 + 2\sqrt{5})) - (\frac{1}{2}(1 - \sqrt{5}))^{k-1}(\frac{1}{4}(6 - 2\sqrt{5}))}{\sqrt{5}} \quad (3)$$

$$= \frac{(\frac{1}{2}(1 + \sqrt{5}))^{k-1}(\frac{1}{2}(2 + 1 + \sqrt{5})) - (\frac{1}{2}(1 - \sqrt{5}))^{k-1}(\frac{1}{2}(2 - 1 - \sqrt{5}))}{\sqrt{5}} \quad (4)$$

$$= \frac{(\frac{1}{2}(1 + \sqrt{5}))^{k-1}(1 + \frac{1}{2}(1 + \sqrt{5})) - (\frac{1}{2}(1 - \sqrt{5}))^{k-1}(1 + \frac{1}{2}(1 - \sqrt{5}))}{\sqrt{5}} \quad (5)$$

$$= \frac{(\frac{1}{2}(1 + \sqrt{5}))^k + (\frac{1}{2}(1 + \sqrt{5}))^{k-1} - (\frac{1}{2}(1 - \sqrt{5}))^k - (\frac{1}{2}(1 - \sqrt{5}))^{k-1}}{\sqrt{5}} \quad (6)$$

$$= \frac{(\frac{1}{2}(1 + \sqrt{5}))^k - (\frac{1}{2}(1 - \sqrt{5}))^k}{\sqrt{5}} + \frac{(\frac{1}{2}(1 + \sqrt{5}))^{k-1} - (\frac{1}{2}(1 - \sqrt{5}))^{k-1}}{\sqrt{5}} \quad (7)$$

$$= F_k + F_{k-1} = F_{k+1}. \quad (8)$$

Thus S_{k+1} . It follows by mathematical induction that S_n for all $n \in \mathbb{N}$. ■