Proposition: For all $n \in \mathbb{N}$, it follows that $S_n : 2|F_n \iff 3|n$.

Proof. (Smallest counterexample).

Suppose $n \in \mathbb{N}$ where $n \leq 2$.
Case 1. Let $n = 1$. Observe that $3 \nmid 1$ and $2 \nmid 1 \iff 2 \nmid F_1$, thus S_1.
Case 2. Let $n = 2$. Observe that $3 \nmid 2$ and $2 \nmid 1 \iff 2 \nmid F_2$, thus S_2.

Suppose for the sake of contradiction that there exists n for which $\neg S_n$.
Let $k > 2$ be the smallest $k \in \mathbb{N}$ for which $2|F_k$ but $3 \nmid k$, so $F_k = 2x$ for some $x \in \mathbb{N}$.
Then we consider two cases for S_{k-1}.
Case 1. Let $2|F_{k-1}$ and $3|k-1$.
Then $F_{k-1} = 2y$ and $k - 1 = 3z$ for some $y, z \in \mathbb{N}$. Observe that

$$F_k = F_{k-1} + F_{k-2} \iff 2x = 2y + F_{k-2} \tag{1}$$
$$\iff F_{k-2} = 2(x - y). \tag{2}$$

Thus $2|F_{k-2}$ which implies $k - 2 = 3w$ for some $x \in \mathbb{N}$. Observe that

$$k - 2 = (k - 1) - 1 \iff 3w = 3z - 1 \tag{3}$$
$$\iff 3(z - w) = 1. \tag{4}$$

But $3 \nmid 1$, a contradiction.
Case 2. Let $2 \nmid F_{k-1}$ and $3 \nmid k - 1$.
Note that $k = 3a + r$ for $a, r \in \mathbb{N}$ where $1 \leq r \leq 2$ by def. of division algorithm.
Then $F_{k-1} = 2y + 1$ for some $y \in \mathbb{N}$ and $k - 1 = 3a - (r + 1)$ where $1 \leq r + 1 \leq 2$. Observe that

$$F_k = F_{k-1} + F_{k-2} \iff 2x = (2y + 1) + F_{k-2} \tag{5}$$
$$\iff F_{k-2} = 2x - 2y - 1 = 2(2x - y) - 1. \tag{6}$$

Thus $2 \nmid F_{k-2}$, which implies $3 \nmid k - 2$ so $k - 2 = 3a - (r + 2)$ where $1 \leq r + 2 \leq 2$, a contradiction.

It follows that S_n for all $n \in \mathbb{N}$. ■