3. Induction

1-10. For $n \in \mathbb{Z}, n \geq 0$, prove that $S_n : \sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$.

Proof. Suppose $n = 0$. Observe that $\sum_{i=1}^{n} i = 0 = \dfrac{(0)(n+1)}{2}$, thus S_0.
Now suppose S_k for some $k \in \mathbb{Z}, k \geq 0$. We now show $S_k \implies S_{k+1}$. Observe that

$$
\begin{aligned}
\sum_{i=1}^{k+1} i &= (k+1) + \sum_{i=1}^{k} i && (1)\\
&= (k+1) + \frac{k(k+1)}{2} && \text{(Inductive hypothesis)} \quad (2)\\
&= \frac{2(k+1) + k(k+1)}{2} && (3)\\
&= \frac{(k+1)(k+2)}{2} && (4)\\
&= \frac{(k+1)((k+1)+1)}{2}. && (5)
\end{aligned}
$$

Thus S_{k+1}. It follows by induction that S_n for all $n \in \mathbb{Z}, n \geq 0$.■

1-11. For $n \in \mathbb{Z}, n \geq 0$, prove that $S_n : \sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Proof. Suppose $n = 0$. Observe that $\sum_{i=1}^{0} i^2 = 0 = \dfrac{(0)(n+1)(2n+1)}{6}$, thus S_0.
Now suppose S_k for some $k \in \mathbb{Z}, k \geq 0$. We now show $S_k \implies S_{k+1}$. Observe that

$$
\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= (k+1)^2 + \sum_{i=1}^{k} i^2 && (6)\\
&= (k+1)^2 + \frac{k(k+1)(2k+1)}{6} && \text{(Inductive hypothesis)} \quad (7)\\
&= \frac{6(k+1)^2 + k(k+1)(2k+1)}{6} && (8)\\
&= \frac{(k+1)\big(6(k+1) + k(2k+1)\big)}{6} && (9)\\
&= \frac{(k+1)(2k^2 + 4k + 3k + 6)}{6} && (10)\\
&= \frac{(k+1)(k+2)(2k+3)}{6} && (11)\\
&= \frac{(k+1)\big((k+1)+1\big)\big(2(k+1)+1\big)}{6}. && (12)
\end{aligned}
$$

Thus S_{k+1}. It follows by induction that S_n for all $n \in \mathbb{Z}, n \geq 0$.■

1-12. For $n \in \mathbb{Z}, n \geq 0$, prove that $S_n : \sum_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$.

Proof. Suppose $n = 0$. Observe that $\sum_{i=1}^{0} i^3 = 0 = \dfrac{(0)^2(n+1)^2}{4}$, thus S_0. Now suppose S_k for some $k \in \mathbb{Z}, k \geq 0$. We now show $S_k \implies S_{k+1}$. Observe that

$$
\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= (k+1)^3 + \sum_{i=1}^{k} && (13)\\
&= (k+1)^3 + \frac{k^2(k+1)^2}{4} && \text{(Inductive hypothesis)} \quad (14)\\
&= \frac{4(k+1)^3 + k^2(k+1)^2}{4} && (15)\\
&= \frac{(k+1)^2\big(4(k+1)+k^2\big)}{4} && (16)\\
&= \frac{(k+1)^2(k^2+4k+4)}{4} && (17)\\
&= \frac{(k+1)^2(k+2)^2}{4} && (18)\\
&= \frac{\Big((k+1)\big((k+1)+1\big)\Big)^2}{4}. && (19)
\end{aligned}
$$

Thus S_{k+1}. It follows by induction that S_n for all $n \in \mathbb{Z}, n \geq 0$. ■

1-13. For $n \in \mathbb{N}$, prove that $S_n : \sum_{i=1}^{n} i(i+1)(i+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

Proof. Suppose $n = 1$. Observe that $\sum_{i=1}^{1} i(i+1)(i+2) = (1)(1+1)(1+2) = 6 = \frac{24}{4} = \dfrac{(1)(1+1)(1+2)(1+3)}{4}$, thus S_1. Now suppose S_k for some $k \in \mathbb{N}$. We now show $S_k \implies S_{k+1}$. Observe that

$$
\begin{aligned}
\sum_{i=1}^{k+1} i(i+1)(i+2) &= (k+1)(k+2)(k+3) + \sum_{i=1}^{k} i(i+1)(i+2) && (20)\\
&= (k+1)(k+2)(k+3) + \frac{k(k+1)(k+2)(k+3)}{4} && \text{(Inductive hypothesis)} \quad (21)\\
&= \frac{4(k+1)(k+2)(k+3) + k(k+1)(k+2)(k+3)}{4} && (22)\\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{4} && (23)\\
&= \frac{(k+1)\big((k+1)+1\big)\big((k+1)+2\big)\big((k+1)+3\big)}{4}. && (24)
\end{aligned}
$$

Thus S_{k+1}. It follows by induction that S_n for all $n \in \mathbb{N}$.■

1-14. For $n \in \mathbb{N}$ and $a \in \mathbb{R}, a \neq 1$, prove that $S_n : \sum_{i=0}^{n} a^i = \dfrac{a^{n+1}-1}{a-1}$.

Proof. Suppose $n = 1$ and $a \in \mathbb{R}$.
Observe that $\sum_{i=0}^{1} a^i = a^0 + a^1 = a + 1 = \dfrac{(a+1)(a-1)}{a-1} = \dfrac{a^{(1)+1}-1}{a-1}$, thus S_1.
Now suppose S_k for $k \in \mathbb{N}$ and $a \in \mathbb{R}$. We now show $S_k \implies S_{k+1}$. Observe that

$$\begin{aligned}
\sum_{i=0}^{k+1} a^i &= a^{k+1} + \sum_{i=0}^{k} a^i && (25)\\
&= a^{k+1} + \frac{a^{k+1}-1}{a-1} && \text{(Inductive hypothesis)} \quad (26)\\
&= \frac{a^{k+1}(a-1) + a^{k+1} - 1}{a-1} && (27)\\
&= \frac{a^{(k+1)+1} - a^{k+1} + a^{k+1} - 1}{a-1} && (28)\\
&= \frac{a^{(k+1)+1}-1}{a-1}. && (29)
\end{aligned}$$

Thus S_{k+1}. It follows by induction that S_n for all $n \in \mathbb{N}$ and $a \in \mathbb{R}$. ■

1-15. For $n \in \mathbb{N}$, prove that $S_n : \sum_{i=1}^{n} \dfrac{1}{i(i+1)} = \dfrac{n}{n+1}$.

Proof. Suppose $n = 1$. Observe that $\sum_{i=1}^{1} \dfrac{1}{i(i+1)} = \dfrac{1}{1+1}$, thus S_1.
Now suppose S_k for some $k \in \mathbb{N}$. We now show $S_k \implies S_{k+1}$. Observe that

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \frac{1}{(k+1)(k+2)} + \sum_{i=1}^{k} \frac{1}{i(i+1)} && (30)\\
&= \frac{1}{(k+1)(k+2)} + \frac{k}{k+1} && \text{(Inductive hypothesis)} \quad (31)\\
&= \frac{1 + k(k+2)}{(k+1)(k+2)} && (32)\\
&= \frac{k^2 + 2k + 1}{(k+1)(k+2)} && (33)\\
&= \frac{(k+1)^2}{(k+1)(k+2)} && (34)\\
&= \frac{k+1}{\big((k+1)+1\big)}. && (35)
\end{aligned}$$

Thus S_{k+1}. It follows by induction that S_n for all $n \in \mathbb{N}$. ■

1-16. For $n \in \mathbb{Z}, n \geq 0$, prove that $3|n^3 + 2n$.

Proof. Suppose $n = 0$. Observe that $3|3(0) \iff 3|0 \iff 3|(0)^3 + 2(0)$, thus S_0.
Now suppose S_k for some $k \in \mathbb{Z}, k \geq 0$. Note $k^3 + 2k = 3x$ for some $x \in \mathbb{Z}, x \geq 0$ by def. of divisibility.
We now show $S_k \implies S_{k+1}$. Observe that

$$\begin{aligned}
(k+1)^3 + 2(k+1) &= (k^3 + 3k^2 + 3k + 1) + (2k + 2) && (36)\\
&= (k^3 + 2k) + 3k^2 + 3k + 3 && (37)\\
&= 3x + 3k^2 + 2k + 3 \quad \text{(Inductive hypothesis)} && (38)\\
&= 3(x + k^2 + k + 1). && (39)
\end{aligned}$$

Since $3|3(x + k^2 + k + 1)$, we have that S_{k+1}. It follows by induction that S_n for all $n \in \mathbb{Z}, n \geq 0$. ■

1-17. For $n \in \mathbb{N}$, prove that S_n : a tree with n vertices has exactly $n - 1$ edges.

Proof. Suppose $n = 1$. Observe that 1 vertex has 0 edges, thus S_1.
Now suppose S_k for some $k \in \mathbb{N}$. Connecting a leaf node to a tree with k vertices and $k - 1$ edges produces a tree with $k + 1$ vertices and k edges. Thus $S_k \implies S_{k+1}$. It follows by induction that S_n for all $n \in \mathbb{N}$.■

1-18. For $n \in \mathbb{N}$, prove that $S_n : \sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$.

Proof. Suppose $n = 1$. Observe that $\sum_{i=1}^{1} i^3 = 1^3 = 1^2 = \left(\sum_{i=1}^{1} i\right)^2$, thus S_1.
Now suppose S_k for some $k \in \mathbb{N}$. We now show $S_k \implies S_{k+1}$. Observe that

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= (k+1)^3 + \sum_{i=1}^{k} i^3 && (40)\\
&= (k+1)^3 + \left(\sum_{i=1}^{k} i\right)^2 \quad \text{(Inductive hypothesis)} && (41)\\
&= (k+1)^3 + \frac{\big(k(k+1)\big)^2}{2^2} \quad \text{(Def. of triangular number)} && (42)\\
&= \frac{4(k+1)^3 + k^2(k+1)^2}{4} && (43)\\
&= \frac{(k+1)^2\big(4(k+1) + k^2\big)}{4} && (44)\\
&= \frac{(k+1)^2(k+2)^2}{4} && (45)\\
&= \frac{\Big((k+1)\big((k+1)+1\big)\Big)^2}{2^2} && (46)\\
&= \left(\sum_{i=1}^{k+1} i\right)^2. \quad \text{(Def. of triangular number)} && (47)
\end{aligned}$$

Thus S_{k+1}. It follows by mathematical induction that S_n for all $n \in \mathbb{N}$. ■