21.

(a) If $x_i = \lambda y_i$ for all $i \in \mathbb{N}, i \leq n$ where $n \in \mathbb{N}$, then $\sum_{i=1}^{n} x_i y_i = \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}$.

Proof.

$$\begin{aligned}
\sum_{i=1}^{n} x_i y_i &= \sum_{i=1}^{n} (\lambda y_i) y_i \\
&= \lambda \sum_{i=1}^{n} y_i^2 \\
&= \sqrt{\lambda^2}\sqrt{\sum_{i=1}^{n} y_i^2}\sqrt{\sum_{i=1}^{n} y_i^2} \\
&= \sqrt{\sum_{i=1}^{n} \lambda^2 y_i^2}\sqrt{\sum_{i=1}^{n} y_i^2} \\
&= \sqrt{\sum_{i=1}^{n} (\lambda y_i)^2}\sqrt{\sum_{i=1}^{n} y_i^2} \\
&= \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}
\end{aligned}$$

(a).2 If $y_i = 0$ for all $i \in \mathbb{N}, i \leq n$ where $n \in \mathbb{N}$, then $\sum_{i=1}^{n} x_i y_i = \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}$.

Proof.

$$\begin{aligned}
\sum_{i=1}^{n} x_i y_i &= \sum_{i=1}^{n} x_i (0) \\
&= 0 \\
&= \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} 0^2}.
\end{aligned}$$

(a).3 If at least one of $y_i \neq 0$ for all $i \in \mathbb{N}, i \leq n$ where $n \in \mathbb{N}$, and there exists no number λ for which all of $x_i = \lambda y_i$, then $0 < \sum_{i=1}^{n} (\lambda y_i - x_i)^2$.

Proof.

Note that $0 \leq (\lambda y_i - x_i)^2$ for all i, so we know that $0 \leq \sum_{i=1}^{n} (\lambda y_i - x_i)^2$. Since there exists some x_i for which $x_i \neq \lambda y_i$, then there exists some $\lambda y_i - x_i \neq 0$, meaning $0 < (\lambda y_i - x_i)^2$. Therefore $0 < \sum_{i=1}^{n} (\lambda y_i - x_i)^2$.

(b) $\sum_{i=1}^n x_i y_i \le \sqrt{\sum_{i=1}^n x_i^2}\sqrt{\sum_{i=1}^n y_i^2}$.

Proof.

Note that $0 \le (x-y)^2 = x^2 - 2xy + y^2 \implies 2xy \le x^2 + y^2$, and let $x = \dfrac{x_i}{\sqrt{\sum_{j=1}^n x_j^2}}$ and $y = \dfrac{y_i}{\sqrt{\sum_{j=1}^n y_j^2}}$ for all $x \le n$. Then

$$\begin{aligned}
2xy &\le x^2 + y^2 \\
2\sum_{i=1}^n xy &\le \sum_{i=1}^n x^2 + \sum_{i=1}^n y^2 \\
2\sum_{i=1}^n \frac{x_i y_i}{\sqrt{\sum_{j=1}^n x_j^2}\sqrt{\sum_{j=1}^n y_j^2}} &\le \sum_{i=1}^n \frac{x_i^2}{\sqrt{\sum_{j=1}^n x_j^2}^2} + \sum_{i=1}^n \frac{y_i^2}{\sqrt{\sum_{j=1}^n y_j^2}^2} \\
\frac{2}{\sqrt{\sum_{j=1}^n x_j^2}\sqrt{\sum_{j=1}^n y_j^2}} \sum_{i=1}^n x_i y_i &\le \frac{1}{\sqrt{\sum_{j=1}^n x_j^2}^2} \sum_{i=1}^n x_i^2 + \frac{1}{\sqrt{\sum_{j=1}^n y_j^2}^2} \sum_{i=1}^n y_i^2 \\
\frac{2}{\sqrt{\sum_{j=1}^n x_j^2}\sqrt{\sum_{j=1}^n y_j^2}} \sum_{i=1}^n x_i y_i &\le \frac{1}{\sqrt{\sum_{j=1}^n x_j^2}^2} \sqrt{\sum_{i=1}^n x_i^2}^2 + \frac{1}{\sqrt{\sum_{j=1}^n y_j^2}^2} \sqrt{\sum_{i=1}^n y_i^2}^2 \\
\frac{2}{\sqrt{\sum_{j=1}^n x_j^2}\sqrt{\sum_{j=1}^n y_j^2}} \sum_{i=1}^n x_i y_i &\le 1 + 1 = 2 \\
\sum_{i=1}^n x_i y_i &\le \sqrt{\sum_{j=1}^n x_j^2}\sqrt{\sum_{j=1}^n y_j^2}.
\end{aligned}$$