7. Let $p, n \in \mathbb{N}$. Then $P(p) : \sum_{k=1}^{n} k^p = \dfrac{n^{p+1}}{p+1} + An^p + Bn^{p-1} + ...$

Proof. (Strong Induction).

Suppose $p = 1$. Observe that $\sum_{k=1}^{n} k^1 = \dfrac{n(n+1)}{2} = \dfrac{n^2}{2} + \dfrac{n}{2} = \dfrac{n^{1+1}}{1+1} + \dfrac{1}{2}n^1 + 0n^0$. Thus $P(1)$.

Now suppose $P(l)$ for all $l \in \mathbb{N}, l \leq p$ for some $p \in \mathbb{N}$. Hence $\sum_{k=1}^{n} k^p = \dfrac{n^{p+1}}{p+1} + An^p + Bn^{p-1} + ...$ Observe that

$$
\begin{aligned}
(n+1)^{p+2} &= \binom{p+2}{0} n^{p+2} + \binom{p+2}{1} n^{p+1} + \binom{p+2}{2} n^p + \binom{p+2}{3} n^{p-1} + ... \\
\binom{p+2}{1} n^{p+1} &= (n+1)^{p+2} - \binom{p+2}{0} n^{p+2} - \binom{p+2}{2} n^p - \binom{p+2}{3} n^{p-1} - ... \\
(p+2)n^{p+1} &= (n+1)^{p+2} - n^{p+2} - \binom{p+2}{2} n^p - \binom{p+2}{3} p^{p-1} - ... \\
\sum_{k=1}^{n} (p+2)n^{p+1} &= \sum_{k=1}^{n} (n+1)^{p+2} - \sum_{k=1}^{n} n^{p+2} - \sum_{k=1}^{n} \binom{p+2}{2} n^p - \sum_{k=1}^{n} \binom{p+2}{3} p^{p-1} - ... \\
(p+2) \sum_{k=1}^{n} n^{p+1} &= \sum_{k=2}^{n+1} n^{p+2} - \sum_{k=1}^{n} n^{p+2} - \binom{p+2}{2} \sum_{k=1}^{n} n^p - \binom{p+2}{3} \sum_{k=1}^{n} p^{p-1} - ... \\
&= (n+1)^{p+2} - 1^{p+2} - \binom{p+2}{2} \sum_{k=1}^{n} n^p - \binom{p+2}{3} \sum_{k=1}^{n} p^{p-1} - ... \\
&= n^{p+2} + Xn^{p+1} - \binom{p+2}{2} \sum_{k=1}^{n} n^p - \binom{p+2}{3} \sum_{k=1}^{n} p^{p-1} - ... \\
&\text{(Where } X \text{ is a number such that } X = \frac{(n+1)^{p+2} - n^{p+2}}{n^{p+1}} \iff (n+1)^{p+2} = n^{p+2} + Xn^{p+1}) \\
\sum_{k=1}^{n} n^{p+1} &= \frac{n^{p+2}}{p+2} + \frac{X}{p+2} n^{p+1} - \frac{\binom{p+2}{2}}{p+2} \sum_{k=1}^{n} n^p - \binom{p+2}{3} \sum_{k=1}^{n} p^{p-1} - ... \\
&= \frac{n^{p+2}}{p+2} + \frac{X}{p+2} n^{p+1} + \left[\frac{n^{p+1}}{p+1} + An^p + Bn^{p-1} + ...\right] \\
&= \frac{n^{p+2}}{p+2} + \left(\frac{X}{p+2} + \frac{1}{p+1}\right) n^{p+1} + An^p + Bn^{p-1} + ... \\
&= \frac{n^{p+2}}{p+2} + Yn^{p+1} + An^p + Bn^{p-1} + ... \qquad \left(Y = \frac{X}{p+2} + \frac{1}{p+1}\right)
\end{aligned}
$$

Thus $P(p+1)$. Therefore, by induction, $\sum_{k=1}^{n} k^p = \dfrac{n^{p+1}}{p+1} + An^p + Bn^{p-1} + ...$ for all $p, n \in \mathbb{N}$.