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Lemma 1: For any numbers a and k, $P(n)$: $k(x-a)^n = kx^n + g(x)$ where $g(x)$ is some polynomial of degree $n-1$.

Proof. (Induction).

Suppose $n = 1$. Then $k(x-a)^1 = kx - a = kx^1 + g(x)$ where $g(x) = -a$.

Now suppose $n \in \mathbb{N}$ and $P(n)$. Hence $k(x-a)^n = kx^n + g(x)$. Observe that $k(x-a)^{n+1} = k(x-a)^n(x-a) = (kx^n+g(x))(x-a) = kx^{n+1} + xg(x) - akx^n - ag(x)$. Note that, since $g(x)$ is degree $n-1$, $g(x)*x$ is degree n. Also $-akx^n$ and $-ag(x)$ are degree n and $n-1$, respectively. Then $k(x-a)^{n+1} = kx^{n+1} + h(x)$ where $h(x) = xg(x) - akx^n - ag(x)$ and h is degree n. Thus $P(n+1)$.

Therefore, it follows by induction that for any numbers a and k, $k(x-a)^n = kx^n + g(x)$ where $g(x)$ is degree $n-1$.

(a) Prove that $P(n)$: For any polynomial function f, and any number a, there is a polynomial function g and a number b such that $f(x) = (x-a)g(x) + b$.

Proof. (Induction).

Suppose $n = 1$. Then $f(x) = a_1x + a_0$ for some numbers a_1, a_0. Note that $a_1x + a_0 = a_1x - a_1a + a_1a + a_0 = (x-a)(a_1) + a_1a + a_0 = (x-a)g(x) + b$ where $g(x) = a_1$ and $b = (a * a_1 + a_0)$. Thus $P(1)$.

Now suppose $n \in \mathbb{N}$ and $P(n)$. Hence $g(x) = (x-a)h(x) + b$ for any polynomial g of degree n. Let f be a polynomial of degree $n+1$, such that $f(x) = kx^{n+1} + g(x)$. By Lemma 1, there exists a polynomial h of degree n for which $kx^{n+1} + g(x) = k(x-a)^{n+1} - h(x) + g(x)$. Then $-h(x) + g(x)$ is a polynomial of degree n, hence $-h(x) + g(x) = (x-a)l(x) + b$ for any polynomial l of degree n. Then $f(x) = k(x-a)^{n+1} + (x-a)l(x) + b = (x-a)(k(x-a)^n + l(x)) + b$. Since $k(x-a)^n + l(x)$ is a polynomial of degree n, we get $P(n+1)$.

Therefore, it follows by induction that for any polynomial function f, and any number a, there is a polynomial function g and a number b such that $f(x) = (x-a)g(x) + b$.