Proposition: For all $k \in \mathbb{N}$, it follows that $S_n : \sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$.

Proof. (Induction).
Basis step. Suppose $n = 1$. Observe that $\sum_{i=1}^{1} i\binom{1}{i} = \binom{1}{1} = 1 = (1)2^{1-1}$. Thus S_1.
Inductive step. Suppose S_k for $k \in \mathbb{N}$.

We now show S_k implies S_{k+1}. Observe that

$$
\begin{aligned}
(k+1)2^k &= 2(k2^{k-1}) + 2^k && (1)\\
&= 2\sum_{i=1}^{k} i\binom{k}{i} + \sum_{i=0}^{k}\binom{k}{i} && (2)\\
&= \sum_{i=1}^{k} i\binom{k}{i} + \left[\binom{k}{1} + 2\binom{k}{2} + 3\binom{k}{3} + ... + k\binom{k}{k}\right] + \left[\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + ... + \binom{k}{k}\right] && (3)\\
&= \sum_{i=1}^{k} i\binom{k}{i} + \binom{k}{0} + \left[\binom{k}{1} + \binom{k}{1}\right] + \left[2\binom{k}{2} + \binom{k}{2}\right] + ... + \left[k\binom{k}{k} + \binom{k}{k}\right] && (4)\\
&= \sum_{i=1}^{k} i\binom{k}{i} + \binom{k}{0} + (1+1)\binom{k}{1} + (2+1)\binom{k}{2} + (3+1)\binom{k}{3} + ... + (k+1)\binom{k}{k} && (5)\\
&= \sum_{i=1}^{k} i\binom{k}{i} + \binom{k}{1-1} + 2\binom{k}{2-1} + 3\binom{k}{3-1} + 4\binom{k}{4-1} + ... + (k+1)\binom{k}{(k+1)-1} && (6)\\
&= \sum_{i=1}^{k} i\binom{k}{i} + \sum_{i=1}^{k+1} i\binom{k}{i-1} && (7)\\
&= \sum_{i=1}^{k+1} i\binom{k}{i} - (k+1)\binom{k}{k+1} + \sum_{i=1}^{k+1} i\binom{k}{i-1} && (8)\\
&= \sum_{i=1}^{k+1} i\binom{k}{i} + \sum_{i=1}^{k+1} i\binom{k}{i-1} && (9)\\
&= \sum_{i=1}^{k+1} i\binom{k+1}{i}. && (10)
\end{aligned}
$$

Thus S_{k+1}.
It follows by mathematical induction that S_n for all $n \in \mathbb{N}$. ∎