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(a) Let $x = 3$ or $x = 5$ or $x = 6$. Then $\sqrt{x}$ is irrational.

Proof. (Contradiction).

Suppose for the sake of contradiction that $\sqrt{x}$ is rational. Hence $\sqrt{x} = \dfrac{n}{m}$ for some integers n and m such that they share no common factor. Then $x = \dfrac{n^2}{m^2} \implies m^2x = n^2$. Since $x|n^2$, and the prime factorization of x (either $x = 3$ or $x = 5$ or $x = 6 = 2*3$) contains no square of a prime, it follows that $x|n$. Hence $n = xk$ for some integer k. Then $m^2x = (xk)^2 = x^2k^2 \implies m^2 = xk^2$. Thus $x|m$. Then n and m share a common factor of x, a contradiction.

Note that this proof would not work for if $x = 4$. Since the prime factorization of 4 is 2^2, then $x|n^2$ does not imply that $x|n$. For a counterexample, suppose $n = 2$. Then $4|2^2 = n^2$, but $4 \nmid n$.

(b) Let $x = 2$ or $x = 3$. Then $\sqrt[3]{x}$ is irrational.

Proof. (Contradiction).

Suppose for the sake of contradiction that $\sqrt[3]{x}$ is rational. Hence $\sqrt[3]{x} = \dfrac{n}{m}$ for some integers n and m such that they share no common factor. Then $x = \dfrac{n^3}{m^3} \implies m^3x = n^3$. Since $x|n^3$, and x is prime (as $x = 2$ or $x = 3$), it follows that $x|n$. Hence $n = xk$ for some integer k. Then $m^3x = (xk)^3 = x^3k^3 \implies m^3 = x^2k^3$. Since $x^2|m^3$, and the prime factorization of x^2 (either $x^2 = 2^2$ or $x^2 = 3^2$) contains no cube of a prime, it follows that $x|m$. Then n and m share a common factor of x, a contradiction.