1-7. *Soln.*

(a) Let $2^{n+1} \le c * 2^n$ for some constant c. Observe that $2^{n+1} \le 2 * 2^n = c * 2^n$ for $c = 2$, thus $2^{n+1} = O(2^n)$.

(b) For all $n \in \mathbb{N}$, it follows that $2n > n$, which implies $2^{2n} \gg 2^n$. Thus there exists no constant c for which $2^{2n} \le c * 2^n$. Therefore $2^{2n} \neq O(2^n)$.

1-8. *Soln.*

(a) $f(n) = \log 2^n = 2\log n \le 2(\log n + 5) = g(n)$, thus $f(n) \le c_1 * g(n)$ for $c_1 = 2$. Also $f(n) = 2\log n \ge \log n + 5$, thus $f(n) \ge c_2 * g(n)$ for $c_2 = 1$. Therefore $f(n) = \Theta(g(n))$.

(b) $\sqrt{n} \gg logn \iff \sqrt{n} \gg 2logn = logn^2 \iff f(n) \gg g(n)$, thus $f(n) = \Omega(g(n))$.

(c) $log^2 n \gg logn \iff f(n) \gg g(n)$, thus $f(n) = \Omega(g(n))$.

(d) $\sqrt{n} \gg logn \iff n \gg (logn)^2 \iff f(n) \gg g(n)$, thus $f(n) = \Omega(g(n))$.

(e) $n \gg logn \iff n(logn + 1) \gg logn \iff f(n) \gg g(n)$, thus $f(n) = \Omega(g(n))$.

(f) $f(n) = 10 \ge log10 = c_1 * g(n)$ when $c_1 = 1$, and $f(n) = 10 \le 10log10 = c_2 * g(n)$ when $c_2 = 10$, thus $f(n) = \Theta(g(n))$.

(g) $2^n \gg n^2 \iff 2^n \gg 10n^2 \iff f(n) \gg g(n)$, thus $f(n) = \Omega(g(n))$.

(g) $3^n \gg 2^n \iff g(n) \gg f(n)$, thus $f(n) = O(g(n))$.

1-9. *Soln.*

(a)

$$
\begin{aligned}
n^2 \gg n &\iff n^2 - n \gg n && (1)\\
&\iff (n^2 - n)\frac{1}{2} \gg 6n && (2)\\
&\iff f(n) \gg g(n) && (3)
\end{aligned}
$$

Thus $g(n) = O(f(n))$.

(b)

$$
\begin{aligned}
n^2 \gg n \gg \sqrt{n} &\iff n + \sqrt{n} \ll n^2 && (4)\\
&\iff n + 2\sqrt{n} \ll n^2 && (5)\\
&\iff f(n) \ll g(n) && (6)
\end{aligned}
$$

Thus $f(n) = O(g(n))$.

(c) $\sqrt{n} \gg logn \Longleftrightarrow n\sqrt{n}\frac{1}{2} \gg nlogn \Longleftrightarrow g(n) \gg f(n)$, thus $f(n) = O(g(n))$.

(d) $n \gg \sqrt{n} \Longleftrightarrow logn + n \gg \sqrt{n} \Longleftrightarrow f(n) \gg g(n)$, thus $g(n) = O(f(n))$.

(e) $log^2 n \gg logn \Longleftrightarrow 2log^2 n \gg logn + 1 \Longleftrightarrow f(n) \gg g(n)$, thus $g(n) = O(g(n))$.

(f)

$$\begin{aligned} n \gg logn &\Longleftrightarrow n - 1 \gg logn + 1 && (7)\\ &\Longleftrightarrow n(n-1) \gg n(logn + 1) && (8)\\ &\Longleftrightarrow (n^2 - n)\frac{1}{2} \gg 4nlogn + n && (9)\\ &\Longleftrightarrow g(n) \gg f(n) && (10) \end{aligned}$$

Thus $f(n) = O(g(n))$.

1-10. *Proof.*
$n^3 - 3n^2 - n + 1 \geq c_1 * n^3$ for $c_1 = \frac{1}{2}$ and $n \geq 10$. Also $n^3 - 3n^2 - n_1 \leq c_2 * n^3$ where $c_2 = 1$. Thus $n^3 - 3n^2 - n + 1 = \Theta(n^3)$.

1-11. *Proof.*
$2^n \gg n^2$, thus $n^2 = O(2^n)$.

1-12. *Soln.*
(a) $f(n) = n^2 + n + 1 \leq 2n^3 = c * g(n)$ where $c = 1$ for $n > 1$.

(b) $f(n) = n\sqrt{n} + n^2 \leq 2n^2 = g(n)$ where $c = 2$ for $n > 1$.

(c) $f(n) = n^2 - n + 1 \leq n^2 = c * g(n)$ where $c = 2$ for $n > 1$.

1-13. *Proof.*
Let $c \in \mathbb{R}$ for which $f_1(n) \leq c * g_1(n)$ and $f_2(n) \leq c * g_2(n)$. Then $f_1(n) + f_2(n) \leq c * g_1(n) + c * g_2(n) = c(g_1(n) + g_2(n)$. Therefore $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$.

1-14. *Proof.*
Let $c \in \mathbb{R}$ for which $f_1(n) \geq c * g_1(n)$ and $f_2(n) \geq c * g_2(n)$. Then $f_1(n) + f_2(n) \geq c * g_1(n) + c * g_2(n) = f_1(n) + f_2(n) \geq c(g_1(n) + g_2(n))$. Therefore $f_1(n) + f_2(n) = \Omega(g_1(n) + g_2(n))$.

1-15. *Proof.*
Let $c \in \mathbb{R}$ for which $f_1(n) \leq c * g_1(n)$ and $f_2(n) \leq c * g_2(n)$. Then $f_1(n) * f_2(n) \leq (c * g_1(n))(c * g_2(n)) = c^2(g_1(n) * g_2(n))$. Therefore $f_1(n) + f_2(n) = O(g_1(n) * g_2(n))$.

1-16. *Proof.*

Let $c = max(\{a_0, a_1, a_2, ..., a_k\} \in \mathbb{R})$. Note that $a_i n^i \leq c * n^i$ for all $i \geq 0$. Then

$$\begin{aligned} a_k n^k + a_{k-1}n^{k-1} + ... + a_1 n + a_0 &\leq cn^k + cn^{k-1} + ... + cn + c && (11)\\ &= c(n^k + n^{k-1} + ... + n + 1) && (12)\\ &\leq 2cn^k && (13)\\ &= b * n^k && (14)\end{aligned}$$

where $b = 2c$. Therefore $a_k n^k + a_{k-1}n^{k-1} + ... + a_1 n + a_0 = O(n^k)$.

1-17. *Proof.*

Observe that $(n+a)^b \leq (2n)^b = 2^b n^b = c_1 * n^b$ for $c_1 = 2^b$, and $(n+a)^b \geq (\frac{1}{2}n)^b = 2^{-b}n^b = c_2 * n^b$ for $c_2 = 2^{-b}$, thus $(n+a)^b = \Theta(n^b)$.

1-18. *Soln.*

$log(log(n)) \ll ln(n) \equiv log(n) \ll log^2(n) \ll \sqrt{n} \ll n \ll nlog(n) \ll n^{1+w} \ll n^2 \equiv n^2 + log(n) \ll n^3 \ll 7n^5 + n - n^3 \ll 2^n \equiv 2^{n-1} \ll e^n \ll n!$.

1-19. *Soln.*

$6 \ll \frac{1}{3^n} log(log(n)) \ll log(n) \equiv ln(n) \ll log^2(n) \ll n^{\frac{1}{3}} + log(n)\sqrt{n} \ll \frac{n}{log(n)} \ll n \ll nlog(n) \ll n^2 + log(n) \equiv n^2 \ll n^3 \ll 7n^5 - n^3 + n\frac{3^n}{2^n} \ll 2^n \ll n!$.

1-20. *Soln.*

(a) Let $f(n) = n$ and $g(n) = n!$. Since $n! \gg n \iff g(n) \gg f(n)$, it follows that $f(n) = o(g(n))$ but $f(n) \neq \Theta(g(n))$.

(d) Let $g(n) = n$ and $f(n) = n^2$. Since $n^2 \gg n \iff f(n) \gg g(n)$, it follows that $f(n) = \Omega(g(n))$ but $f(n) \neq O(g(n))$.