3. Def. of binomial coefficient: For $0 \le k \le n$, $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ is the binomial coefficient. For $k < 0$ and $k > n$, $\binom{n}{k} = 0$ is the binomial coefficient.

(a) $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Proof.

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} && \text{Def. of binomial coefficient.}\\
&= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}\\
&= \frac{n!}{(k-1)!(n-k+1)(n-k)!} + \frac{n!}{k(k-1)!(n-k)!}\\
&= \frac{k*n! + (n-k+1)*n!}{k(n-k+1)(k-1)!(n-k)!}\\
&= \frac{n!(k+n-k+1)}{k!(n-k+1)!}\\
\therefore \binom{n}{k-1} + \binom{n}{k} &= \frac{(n+1)!}{k!(n-k+1)!}.
\end{aligned}$$

(b) For all $n \in \mathbb{Z}, 0 \le n$ and $k \in \mathbb{Z}, 0 \le k \le n$, it follows that $P(n) : \binom{n}{k} \in \mathbb{N}$.

Proof. (Strong Induction).

Let $n = 0$, so $k = 0$. Observe that $\binom{1}{0} = \dfrac{1!}{0!(1-0)!} = 1 = \dfrac{0!}{0!(0-0)!} + 0 = \binom{0}{0} + \binom{0}{-1}$. Thus $P(0)$.

Now let n be some number such that $0 \le n, n \in \mathbb{Z}$, and suppose $P(n)$ for any $k \in \mathbb{Z}$ such that $0 \le k \le n$. Hence $\binom{n}{k} \in \mathbb{N}$. Then by Proposition 3.a, $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$. Since $\binom{n}{k} \in \mathbb{N}$ for all k by our assumption, and the sum of two natural numbers (or a natural number with zero, when $k = 0 \implies \binom{n}{k-1} = 0$) is a natural number, it follows that $\binom{n+1}{k} \in \mathbb{N}$. Therefore, for all $n \in \mathbb{Z}, 0 \le n$ and $k \in \mathbb{Z}, 0 \le k \le n$, it follows by induction that $P(n) : \binom{n}{k} \in \mathbb{N}$.

(c) Lemma: For all $n \in \mathbb{Z}, 0 \le n$ and $k \in \mathbb{Z}, 0 \le k \le n$, it follows that $\binom{n}{k}$ is the number of sets of k integers each chosen from the set $S = \{1, ..., n\}$.

Proof.

Suppose we are removing k elements from the set $S = \{1, ..., n\}$. For the 1st turn, we have $|S| = n$ elements to choose from. For the 2nd, we have $n-1$ choices, etc., until the kth turn, when we have $n-k+1$ elements to choose from. Thus there are $n(n-1)...(n-k+1) = \dfrac{n(n-1)...(n-k+1)(n-k)!}{(n-k)!} = \dfrac{n!}{(n-k)!}$ ways to choose k elements from S. Note that for each possible (unique) set of k chosen elements, we have removed them from S multiple times in different order. For the 1st element in one such set, we can remove it on one of k turns. For the 2nd, one of $k-1$ turns, etc., until the kth element, when we have only 1 turn to choose from. Thus we divide the number of ways to choose k elements from S by $k!$, the number of orders of each set of k elements, to count the number of (unique) such sets. Therefore the answer is $\dfrac{n!}{(n-k)!k!} = \binom{n}{k}$.

(c) Proposition: For all $n \in \mathbb{Z}, 0 \le n$ and $k \in \mathbb{Z}, 0 \le k \le n$, it follows that $\binom{n}{k} \in \mathbb{N}$.

Proof.

We consider two cases. Let $n = 0$, so $k = 0$. Then $\binom{n}{k} = \binom{0}{0} = 1$, thus $\binom{n}{k} \in \mathbb{N}$. Now let $n \in \mathbb{N}$, so $0 \le k \le n, k \in \mathbb{Z}$. By Lemma 3.d, $\binom{n}{k}$ is the number of sets of k integers each chosen from the set $S = \{1, ..., n\}$. Since there is at least one subset of k integers in S (Note when $k = 0$, the subset is $\emptyset$), it follows that $\binom{n}{k} \ge 1$. Therefore for all $n \in \mathbb{Z}, 0 \le n$ and $k \in \mathbb{Z}, 0 \le k \le n$, it follows that $\binom{n}{k} \in \mathbb{N}$.

(d) If $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, then $P(n) : (a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$.

Proof.

Let $a, b \in \mathbb{R}$. We will use induction on n. Suppose $n = 1$, then $(a+b)^1 = a + b = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1 = \sum_{i=0}^{1} \binom{1}{i} a^{1-i} b^i$. Thus $P(1)$. Now suppose $P(n)$ for some $n \in \mathbb{N}$. Hence $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$. Observe that

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b) \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \\
&= a \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i + b \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \\
&= \sum_{i=0}^{n} \binom{n}{i} a^{n-i+1} b^i + \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^{i+1}
\end{aligned}$$

For the 2nd term, let $i' = i - 1$. When $i' = 0$, then $i = 1$, and when $i' = n$, then $i = n + 1$. Thus

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} a^{n-i+1} b^i + \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^{i+1} &= \sum_{i=0}^{n} \binom{n}{i} a^{n-i+1} b^i + \sum_{i=1}^{(n+1)} \binom{n}{(i-1)} a^{n-(i-1)} b^{(i-1)+1} \\
&= \sum_{i=0}^{n} \binom{n}{i} a^{n-i+1} b^i + \Big[\sum_{i=0}^{n+1} \binom{n}{i-1} a^{n-i+1} b^i\Big] - \binom{n}{0-1} a^{n-0+1} b^0 \\
&= \sum_{i=0}^{n} \binom{n}{i} a^{n-i+1} b^i + \sum_{i=0}^{n+1} \binom{n}{i-1} a^{n-i+1} b^i && \binom{n}{-1} = 0 \\
&= \Big[\sum_{i=0}^{n+1} \binom{n}{i} a^{n-i+1} b^i\Big] - \binom{n}{n+1} a^{n-(n+1)+1} b^{n+1} + \sum_{i=0}^{n+1} \binom{n}{i-1} a^{n-i+1} b^i \\
&= \sum_{i=0}^{n+1} \binom{n}{i} a^{n-i+1} b^i + \sum_{i=0}^{n+1} \binom{n}{i-1} a^{n-i+1} b^i && \binom{n}{n+1} = 0 \\
&= \sum_{i=0}^{n+1} \Big[\binom{n}{i} a^{n-i+1} b^i + \binom{n}{i-1} a^{n-i+1} b^i\Big] \\
&= \sum_{i=0}^{n+1} a^{n-i+1} b^i \Big[\binom{n}{i} + \binom{n}{i-1}\Big] \\
&= \sum_{i=0}^{n+1} \binom{n+1}{i} a^{(n+1)-i} b^i && \text{By Proposition 3.a}
\end{aligned}$$

Thus $P(n+1)$. Therefore, by induction, if $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, then $P(n) : (a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$.

(e)

(i) $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

Proof.

By Lemma 3.d, we know that $\binom{n}{k}$ is the number of k-element subsets of the set $S = \{1, ...n\}$. Thus the sum of the number of i-element subsets for all lengths i $(0 \le i \le n)$ is the total number of subsets of S.

Another way to count the total number of subsets of S is to note that for each of its n elements, we can make 2 new subsets: one with the element included, and one without it. Thus there are 2^n subsets.

Therefore, $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

(ii) $\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$

Proof.

$$\begin{aligned}
\sum_{i=0}^{n}(-1)^i\binom{n}{i} &= \sum_{i=0}^{n}(-1)^i\Big[\binom{n-1}{i-1}+\binom{n-1}{i}\Big] && \text{by Proposition 3.a}\\
&= \sum_{i=0}^{n}(-1)^i\binom{n-1}{i-1}+\sum_{i=0}^{n}(-1)^i\binom{n-1}{i}
\end{aligned}$$

For the first term, let $i' = i+1$. When $i' = 0$, then $i = -1$, and when $i' = n$, then $i = n-1$. Hence

$$\begin{aligned}
\sum_{i=0}^{n}(-1)^i\binom{n-1}{i-1}+\sum_{i=0}^{n}(-1)^i\binom{n-1}{i} &= \sum_{i=-1}^{n-1}(-1)^{i+1}\binom{n-1}{(i+1)-1}+\sum_{i=0}^{n}(-1)^i\binom{n-1}{i}\\
&= \sum_{i=0}^{n}(-1)^{i+1}\binom{n-1}{i}+(-1)^{(-1)+1}\binom{n-1}{-1}-(-1)^{n+1}\binom{n-1}{n}+\sum_{i=0}^{n}(-1)^i\binom{n-1}{i}\\
&= \sum_{i=0}^{n}(-1)^{i+1}\binom{n-1}{i}+\sum_{i=0}^{n}(-1)^i\binom{n-1}{i} && \binom{n-1}{-1}\\
&= \sum_{i=0}^{n}\binom{n-1}{i}((-1)^{i+1}+(-1)^i)\\
&= \sum_{i=0}^{n}\binom{n-1}{i}(-1)^i((-1)+1) = 0
\end{aligned}$$

Therefore $\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0$.

(iii) $\sum_{odd\,i}\binom{n}{i} = 2^{n-1}$

Proof.

Note that if $x \in \mathbb{Z}$ is even, $\dfrac{1-(-1)^x}{2} = \dfrac{1-(1)}{2} = 0$, and if x is odd, $\dfrac{1-(-1)^x}{2} = \dfrac{1-(-1)}{2} = 1$. Then

$$\begin{aligned}
\sum_{odd\,i}\binom{n}{i} &= \sum_{i=0}^{n}\frac{1-(-1)^i}{2}\binom{n}{i}\\
&= \sum_{i=0}^{n}\Big[\frac{1}{2}\binom{n}{i}-\frac{1}{2}(-1)^i\binom{n}{i}\Big]\\
&= \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}-\frac{1}{2}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\\
&= \frac{1}{2}2^n-\frac{1}{2}(0) && \text{By Propositions 3.e.i and 3.e.ii}\\
\therefore \sum_{odd\,i}\binom{n}{i} &= 2^{n-1}.
\end{aligned}$$

(iv) $\sum_{even\,i}\binom{n}{i} = 2^{n-1}$.

Proof.

$$\begin{aligned}
\sum_{even\,i}\binom{n}{i} &= \sum_{i=0}^{n}\binom{n}{i}-\sum_{odd\,i}^{n}\binom{n}{i}\\
&= 2^n-2^{n-1} && \text{By Propositions 3.e.i and 3.e.iii}\\
&= 2^{n-1}(2-1)\\
\therefore \sum_{even\,i}\binom{n}{i} &= 2^{n-1}.
\end{aligned}$$