2-32. Prove that for all $k \in \mathbb{N}$, $S_k : \sum_{i=1}^{k}(-1)^{i-1}i^2 = (-1)^{k-1}\dfrac{k(k+1)}{2}$.

Proof.

Suppose $k = 1$. Observe that $\sum_{i=1}^{k}(-1)^{i-1}i^2 = (-1)^0 1^2 = (-1)^0\dfrac{2}{2} = (-1)^{k-1}\dfrac{k(k+1)}{2}$, thus S_1.
Now suppose S_k for $k \in \mathbb{N}$. We will show $S_k \implies S_{k+1}$. Observe that

$$\begin{aligned}
\sum_{i=1}^{k+1}(-1)^{i-1}i^2 &= \sum_{i=1}^{k}(-1)^{i-1}i^2 + (-1)^k(k+1)^2 && (1)\\
&= (-1)^{k-1}\frac{k(k+1)}{2} + (-1)^k(k+1)^2 \qquad \text{(Inductive hypothesis)} && (2)\\
&= (-1)^{k-1}(k+1)\Big(\frac{k}{2} - (k+1)\Big) && (3)\\
&= (-1)^{k-1}(k+1)\Big(\frac{k}{2} - \frac{2(k+1)}{2}\Big) && (4)\\
&= (-1)^{k-1}(k+1)\Big(\frac{k-2k-2}{2}\Big) && (5)\\
&= (-1)^{k-1}(k+1)\frac{-(k+2)}{2} && (6)\\
&= (-1)^{(k+1)-1}\frac{(k+1)((k+1)+1)}{2}. && (7)
\end{aligned}$$

Thus S_{k+1}. It follows by induction that S_k for all $k \in \mathbb{N}$.

2-33. For $n, k \in \mathbb{Z}$, let $f(n,k) = f(n-1,k-2) + f(n-1,k-1) + f(n-1,k)$ and $f(1,1) = 1$. Note $f(a,b) = 0$ for $a < 1$ or $b \notin [1, 2a-1]$ for $a, b \in \mathbb{Z}$. Prove that for $n \geq 1$, it follows that $S_n : \sum_{i=1}^{2n-1} f(n,i) = 3^{n-1}$.

Proof.

Suppose $n = 1$. Observe that $\sum_{i=1}^{2(1)-1} f(1,i) = f(1,1) = 1 = 3^{(1)-1}$, thus S_1.
Now suppose S_n for $n \in \mathbb{N}$. We will show $S_n \implies S_{n+1}$. Observe that

$$\begin{aligned}
\sum_{i=1}^{2(n+1)-1} f(n+1,i) &= \sum_{i=1}^{2n+1} f(n,i-2) + \sum_{i=1}^{2n+1} f(n,i-1) + \sum_{i=1}^{2n+1} f(n,i) && (8)\\
&= \Big[f(n,-1) + f(n,0) + \sum_{i=1}^{2n-1} f(n,i)\Big] + \Big[f(n,0) + f(n,2n) + \sum_{i=1}^{2n-1} f(n,i)\Big] + \Big[f(n,2n) + f(n,2n+1) + \sum_{i=1}^{2n-1} f(n,i)\Big] && (9)\\
&= 3\sum_{i=1}^{2n-1} f(n,i) && (10)\\
&= 3(3^{n-1}) && (11)\\
&= 3^{(n+1)-1}. && (12)
\end{aligned}$$

Thus S_{n+1}. It follows by induction that S_k for all $k \in \mathbb{N}$.

2-34. *Soln.* On the nth day, we received the first gift n times, the second gift $n-1$ times, and so forth, up until the nth gift which we received once. Thus the total number of gifts is $1 + 2 + 3 + ... + (n-1) + n = \dfrac{n(n+1)}{2}$.

2-35. *Soln.*

(a) $T(n) = \sum_{i=1}^{n}\sum_{j=i}^{2i} 1$.

(b)

$$
\begin{aligned}
T(n) = \sum_{i=1}^{n}\sum_{j=i}^{2i} 1 &= \sum_{i=1}^{n}(2i-(i-1)) && (13)\\
&= \sum_{i=1}^{n}(i+1) && (14)\\
&= \sum_{i=1}^{n+1} i - 1 && (15)\\
&= \frac{(n+1)(n+2)}{2} - 1 && (16)\\
&= \frac{(n+1)(n+2)}{2} - \frac{2}{2} && (17)\\
&= \frac{n^2+3n+2-2}{2} && (18)\\
&= \frac{n(n+3)}{2}. && (19)
\end{aligned}
$$

2-36. *Soln.*
(a) $T(n) = \sum_{i=1}^{n/2}\sum_{j=i}^{n-i}\sum_{k=1}^{j} 1$.
(b)

$$
\begin{aligned}
T(n) = \sum_{i=1}^{n/2}\sum_{j=i}^{n-i}\sum_{k=1}^{j} 1 &= \sum_{i=1}^{n/2}\sum_{j=i}^{n-i} j && (20)\\
&= \sum_{i=1}^{n/2}\Big(\sum_{j=1}^{n-i} j - \sum_{j=1}^{i-1} j\Big) && (21)\\
&= \sum_{i=1}^{n/2}\Big(\frac{(n-i)(n-i+1)}{2} - \frac{i(i-1)}{2}\Big) && (22)\\
&= \sum_{i=1}^{n/2}\frac{n^2-ni+n-ni+i^2-i-i^2+i}{2} && (23)\\
&= \sum_{i=1}^{n/2}\frac{n^2-2ni+n}{2} && (24)\\
&= \sum_{i=1}^{n/2}\frac{n^2}{2} - \sum_{i=1}^{n/2}\frac{2ni}{2} + \sum_{i=1}^{n/2} n && (25)\\
&= \frac{n}{2}\big(\frac{n^2}{2}\big) - \frac{2n}{2}\big(\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2}\big) + n\big(\frac{n}{2}\big) && (26)\\
&= \frac{n^3}{4} - \frac{n^2}{4}\big(\frac{n}{2}+1\big) + \frac{n^2}{2} && (27)\\
&= \frac{n^3}{4} - \frac{n^3}{8} - \frac{n^2}{4} + \frac{n^2}{2} && (28)\\
&= \frac{2n^3-n^3-2n^2+4n^2}{8} && (29)\\
&= \frac{n^3+2n^2}{8} && (30)\\
&= \frac{n^2(n+2)}{8}. && (31)
\end{aligned}
$$

2-37. Let x and y be the largest n-digit number in base b, which is $b^n - 1$. If each x is added to the total sum as so: $(...((x+x)+x)+x...)+x$, and one addition of an n-digit number to another number takes n steps, then $f(n,b)$ totals to $n(b^n-1)$ steps. Thus $f(n,b) = O(nb^n)$.

2-38. Let x and y be n-digit numbers. One multiplication or addition of an n-digit number to another number takes n steps. The first x is multiplied by 1 digit in n steps and produces an $n+1$-digit number. The second x is multiplied by 2 digits in n steps, producing an $n+2$-digit number. This process continues until the nth x is multiplied by n digits in n steps, producing a $2n$-digit number. So n terms each take n steps per product and summing the terms takes a total of $(n+1)+(n+2)+...+2n = n^2+\frac{1}{2}n(n+1)$ steps, assuming each term p is added to the total sum as so: $(...((p+p)+p)+p...)+p$. Then this method takes $n^2+n^2+\frac{1}{2}n(n+1) = 2n^2+\frac{1}{2}n^2+\frac{1}{2}n \leq 3n^2$ steps, thus its run time is $O(n^2)$.