2-39.

(a) $xy = a^{log_a(x)}a^{log_a(y)} = a^{log_a(x)+log_a(y)} \iff log_a(xy) = log_a(x) + log_a(y)$.

(b) $x = a^{log_a(x)} \iff x^y = (a^{log_a(x)})^y = a^{ylog_a(x)} \iff log_a(x^y) = ylog_a(x)$.

(c) $x = a^{log_a(x)} = (b^{log_b(a)})^{log_a(x)} = b^{log_b(a)log_a(x)} \iff log_b(a)log_a(x) = log_b(x) \iff log_a(x) = \dfrac{log_b(x)}{log_b(a)}$.

(d) $y^{log_b(x)} = b^{log_b(y^{log_b(x)})} = b^{log_b(x)log_b(y)} = b^{log_b(x)^{log_b(y)}} = x^{log_b(y)}$.

2-40. For all $n, c \in \mathbb{N}, c > 1$, prove that $S_n : \lceil log_c(n+1) \rceil = \lfloor log_c(n) \rfloor + 1$.

Proof.

We consider two cases.

Case 1. Let $n = c^x$ for $x \in \mathbb{Z}, x \geq 0$ and $c \in \mathbb{N}, c > 1$. Observe that

$$
\begin{aligned}
2 \geq& 1 + \frac{1}{c^x} > 1 && (1)\\
log_c(2) \geq& log_c\Big(1 + \frac{1}{c^x}\Big) > log_c(1) = 0 && (2)\\
\lceil log_c(2) \rceil \geq& \left\lceil log_c\Big(1 + \frac{1}{c^x}\Big) \right\rceil > 0 && (3)\\
1 \geq& \left\lceil log_c\Big(1 + \frac{1}{c^x}\Big) \right\rceil > 0 && (4)\\
1 =& \left\lceil log_c(1 + \frac{1}{c^x}) \right\rceil && (5)\\
x + 1 =& x + \left\lceil log_c(1 + \frac{1}{c^x}) \right\rceil && (6)\\
=& \left\lceil x + log_c(1 + \frac{1}{c^x}) \right\rceil && (\forall b \in \mathbb{Z}, \lceil a + b \rceil = \lceil a \rceil + b) \quad (7)\\
\lfloor xlog_c(c) \rfloor + 1 =& \left\lceil xlog_c(c) + log_c(1 + \frac{1}{c^x}) \right\rceil && (a > 0 \implies log_a(a) = 1) \quad (8)\\
\lfloor log_c(c^x) \rfloor + 1 =& \left\lceil log_c(c^x) + log_c(1 + \frac{1}{c^x}) \right\rceil && (a \neq 0 \implies log(a^b) = blog(a)) \quad (9)\\
=& \left\lceil log_c\Big(c^x(1 + \frac{1}{c^x})\Big) \right\rceil && (\forall a, b \neq 0, log(a) + log(b) = log(ab)) \quad (10)\\
\lfloor log_c(c^x) \rfloor + 1 =& \lceil log_c(c^x + 1) \rceil . && (11)
\end{aligned}
$$

Thus S_n for $n = c^x$.

Case 2. Let $n, x, c \in \mathbb{N}, c > 1$ and $c^x > n \geq c^{x-1}$. Observe that

$$
\begin{aligned}
c^x >& n \geq c^{x-1} && (12)\\
x >& log_c(n) \geq x - 1 && (13)\\
x >& \lfloor log_c(n) \rfloor \geq x - 1 && (14)\\
x + 1 >& \lfloor log_c(n) \rfloor + 1 \geq x && (15)\\
x =& \lfloor log_c(n) \rfloor + 1. && (16)
\end{aligned}
$$

Note that $c^x > n \geq c^{x-1}$ implies $c^x \geq n + 1 > c^{x-1}$. Observe that

$$
\begin{aligned}
c^x \geq& n + 1 > c^{x-1} && (17)\\
x \geq& log_c(n+1) > x - 1 && (18)\\
x \geq& \lceil log_c(n+1) \rceil > x - 1 && (19)\\
x =& \lceil log_c(n+1) \rceil . && (20)
\end{aligned}
$$

Thus $x = \lfloor log_c(n) \rfloor + 1 = \lceil log_c(n+1) \rceil$ for $n \in \mathbb{N}$. It follows that S_n for all $n, c \in \mathbb{N}, c > 1$. ■

2-41. For all $n \in \mathbb{N}$, prove that n has $\lfloor log_2(n) \rfloor + 1$ digits in its binary representation.

Proof.

Suppose $n \in \mathbb{N}$ is an x-digit binary number where $x \in \mathbb{N}$. Note that the maximum x-digit binary number is $2^x - 1$, thus the minimum $(x+1)$-digit binary number is 2^x. Then consider two cases.

Case 1. Let $x = 1$. Observe that $n = 1 = 0 + 1 = \lfloor log_2(1) \rfloor + 1$, thus the statement is true.

Case 2. Let $x > 1$. Observe that

$$
\begin{aligned}
2^x > & n \geq 2^{x-1} & & (21) \\
x > & log_2(n) \geq x - 1 & (b \neq 0, log_2(2^b) = b) & \quad (22) \\
x > & \lfloor log_2(n) \rfloor \geq x - 1 & (\forall a \in \mathbb{N}, \lfloor a \rfloor = a) & \quad (23) \\
x + 1 > & \lfloor log_2(n) \rfloor + 1 \geq x & & (24) \\
x = & \lfloor log_2(n) \rfloor + 1. & & (25)
\end{aligned}
$$

Therefore for all $n \in \mathbb{N}$, n has $\lfloor log_2(n) \rfloor + 1$ digits in its binary representation. ■

2-42. *Soln.* Let $f(n) = nlog(\sqrt{n})$ and $g(n) = nlog(n)$. Then $f(n) = nlog(n^{\frac{1}{2}}) = \frac{1}{2}nlog(n) \geq c * g(n)$ for $c = \frac{1}{4}$, thus $f(n) = \Omega(g(n))$.