Proposition: For all $n, m \in \mathbb{N}$, it follows that S_n : if $2|n$ and $2|m$, then $2|\binom{n}{m}$.

Proof. (Strong Induction).
Note that $\binom{n}{m} = 0 \Longrightarrow 2|\binom{n}{m}$ for $m < 0$ or $m > n$, thus we consider $0 < m < n$.
Basis step. Suppose $n = 2$ and $m = 1$. Observe that $\binom{2}{1} = 2$, thus S_2. Now suppose $n = 4$ and $m = 1$ or $m = 3$. Observe that $\binom{4}{3} = 4$ and $\binom{4}{1} = 4$, thus S_4.
Inductive step. Suppose S_{n-2} for $n, m \in \mathbb{N}$ where $2|n$ and $6 \leq n$ and $2 \nmid m$. Note that $\binom{n}{m} = n \Longrightarrow 2|\binom{n}{m}$ for $m = 1$. Thus we consider $1 < m < n - 2$. We now show S_{n-2} implies S_n. Observe that

$$\begin{aligned}
\binom{n}{m} &= \binom{n-1}{m} + \binom{n-1}{m-1} && (1)\\
&= \binom{n-2}{m} + \binom{n-2}{m-1} + \binom{n-2}{m-1} + \binom{n-2}{m-2} \qquad \text{(Def. of Pascal's triangle)} && (2)\\
&= \binom{n-2}{m} + 2\binom{n-2}{m-1} + \binom{n-2}{m-2} \qquad . && (3)
\end{aligned}$$

Since $2|2\binom{n-2}{m-1}$, it follows that the second term is even. Note that $2|n-2$ implies $2|n$, and $2 \nmid m$ implies $2 \nmid m - 2$, thus the first and third terms are even. Since the sum of three even terms is even, it follows that $\binom{n}{m}$ is even.
It follows by mathematical induction that S_n for all $n, m \in \mathbb{N}$ where $m < n$. ■