Proposition: For all $n, m, p \in \mathbb{Z}$ where $n, m, p \geq 0$, it follows that $S_n : \sum_{i=0}^{p} \binom{m}{i}\binom{n}{p-i} = \binom{m+n}{p}$.

Proof. (Induction).
Basis step. Suppose $n = 0$ and $m, p \in \mathbb{Z}$ where $m, p \geq 0$.
Observe that $\sum_{i=0}^{p} \binom{m}{i}\binom{0}{p-i} = \binom{m}{p}\binom{0}{p-p} = \binom{m}{p}(1) = \binom{m+0}{p}$. Thus S_0.
Inductive step. Suppose S_n for any $n, m, p \in \mathbb{Z}$ where $n, m, p \geq 0$.
We now show S_n implies S_{n+1}. Observe that

$$\begin{aligned}
\binom{m+n+1}{p} &= \binom{m+n}{p} + \binom{m+n}{p-1} && \text{(Def. of Pascal's triangle)} \quad (1)\\
&= \sum_{i=0}^{p} \binom{m}{i}\binom{n}{p-i} + \sum_{i=0}^{p-1} \binom{m}{i}\binom{n}{p-i-1} && \text{(Inductive hypothesis)} \quad (2)\\
&= \sum_{i=0}^{p} \left[\binom{m}{i}\binom{n}{p-i} + \binom{n}{i}\binom{n}{p-i-1}\right] - \binom{m}{p}\binom{n}{p-p-1} && (3)\\
&= \sum_{i=0}^{p} \binom{m}{i}\left(\binom{n}{p-i} + \binom{n}{p-i-1}\right) - \binom{m}{p}(0) && (4)\\
&= \sum_{i=0}^{p} \binom{m}{i}\binom{n+1}{p-i}. && (5)
\end{aligned}$$

Thus S_{n+1}.
It follows by mathematical induction that S_n for all $n, m, p \in \mathbb{Z}$ where $n, m, p \geq 0$. ■