Chapter 1 Problems

1.

(i) Proposition: If $ax = a$ for some number $a \neq 0$, then $x = 1$.

Proof. Let $ax = a$ and $a \neq 0$ for some number a. Then there exists a^{-1} such that $(a^{-1}a)x = a^{-1}a \implies (1)x = 1 \implies x = 1$.

(ii) Proposition: Let $x, y \in \mathbb{R}$. Then $x^2 - y^2 = (x-y)(x+y)$.

Proof. Observe that $(x-y)(x+y) = x(x-y) + y(x-y) = x * x - x * y + y * x - y * y = x^2 - y^2$.

(iii) Proposition: Let $x, y \in \mathbb{R}$. If $x^2 = y^2$, then $x = y$ or $x = -y$.

Proof. Let $x^2 = y^2$. Observe that $x^2 = (\pm|x|)^2 = (\pm 1)^2|x|^2 = |x|^2$, hence $|x|^2 = y^2 \implies \sqrt{|x|^2} = \sqrt{y^2} \implies |x| = y \implies x = y$ or $x = -y$.

(iv) Proposition: Let $x, y \in \mathbb{R}$. Then $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$.

Proof.

$$\begin{aligned}
(x-y)(x^2+xy+y^2) &= (x-y)(x^2+(xy+y^2)) \\
&= x^2(x-y) + (xy+y^2)(x-y) \\
&= x^3 - x^2y + xy(x-y) + y^2(x-y) \\
&= x^3 - x^2y + x^2y - xy^2 + xy^2 - y^3 \\
&= x^3 - y^3.
\end{aligned}$$

(v) Proposition: Let $x, y \in \mathbb{R}$. Then $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + ... + xy^{n-2} + y^{n-1})$.

Proof. Note that for some function f with $k \in \mathbb{N}$ terms in standard form, the function $g = (x-y)f = xf - yf$ has $2k$ terms in standard form.

$$\begin{aligned}
(x-y)(x^{n-1} + x^{n-2}y + ... + xy^{n-2} + y^{n-1}) &= x^{n-1}(x-y) + x^{n-2}y(x-y) + ... + xy^{n-2}(x-y) + y^{n-1}(x-y) \\
&= x^n - x^{n-1}y + x^{n-1}y - x^{n-2}y^2 + ... + x^2y^{n-2} - xy^{n-1} + xy^{n-1} - y^n \qquad (1) \\
&= x^n - y^n.
\end{aligned}$$

(1) Note that the expression has an even number of terms due to multiplication by a binomial, hence each term x^iy^j in the expression, where $i, j \in N$ and $i, j < n$, has a $-x^iy^j$ pair. Since $x^iy^j - x^iy^j = 0$, all such terms are eliminated.

(vi) Proposition: Let $x, y \in \mathbb{R}$. Then $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$.

Proof.

$$\begin{aligned}
(x+y)(x^2-xy+y^2) &= (x+y)x^2 - (x+y)xy + (x+y)y^2 \\
&= x^3 + x^2y - (x^2y + xy^2) + xy^2 + y^3 \\
&= x^3 + x^2y - x^2y - xy^2 + xy^2 + y^3 \\
&= x^3 + y^3.
\end{aligned}$$

2. Since $x = y$ implies $x - y = 0$, division of $(x+y)(x-y)$ and $y(x-y)$ by $(x-y)$ is undefined.