1. $[1] = \{1\}$ and $[2] = [3] = \{2, 3\}$ and $[4] = [5] = [6] = \{4, 5, 6\}$

2. $R = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, d), (d, a), (b, c), (c, b), (e, d), (d, e), (a, e), (e, a)\}$. Two equivalence classes: $[a] = [d] = [e] = \{a, d, e\}$ and $[b] = [c] = \{b, c\}$

3. $R = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, d), (d, a), (b, c), (c, b)\}$. Three equivalence classes: $[a] = [d] = \{a, d\}$ and $[b] = [c] = \{b, c\}$ and $[e] = \{e\}$

4. $[a] = [b] = [c] = [d] = [e] = \{a, d, e, c, b\}$, so R has one equivalence class.

5. $R_1 = \{(a, a), (b, b), (a, b), (b, a)\}$ and $R_2 = \{(a, a), (b, b)\}$

6.
$R_1 = \{(a, a), (b, b), (c, c)\}$
$R_2 = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$
$R_3 = \{(a, a), (b, b), (c, c), (a, c), (c, a)\}$
$R_4 = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$
$R_5 = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c), (c, a), (a, b), (b, a)\}$

7. Proposition The relation $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 2|3x - 5y\}$ on $\mathbb{Z}$ is an equivalence relation.
Proof.
First we show that the relation is reflexive. Let $x \in \mathbb{Z}$. Observe that $3x - 5x = -2x$. Since $2|2(-x)$ by def. of divisibility, it follows that xRx and thus R is reflexive.

Next we show that the relation is symmetric. Suppose xRy for some $y \in \mathbb{Z}$. We know $2|3x - 5y \iff 3x - 5y = 2a$ for some $a \in \mathbb{Z}$. Observe that $3x - 5y + (8y - 8x) = 2a + (8y - 8x) \iff 3y - 5x = 2a + 8y - 8x$. Since $2|2(a + 4y - 4x)$, it follows that yRx and thus R is symmetric.

Finally we show that the relation is transitive. Suppose xRy and yRz for some $z \in \mathbb{Z}$. We know $3x - 5y = 2a$, and $2|3y - 5z \iff 3y - 5z = 2b$ for some $b \in \mathbb{Z}$. Observe that $\big(3x - 5y\big) + \big(3y - 5z\big) = 2a + 2b \iff 3x - 5z = 2a + 2b + 2y$. Since $2|2(a + b + y)$, it follows that xRz and thus R is transitive.

Therefore the relation R on $\mathbb{Z}$ is an equivalence relation. ■

The equivalence classes of R are
$[0] = \{x \in \mathbb{Z} : 2|3x - 5(0)\} = \{x \in \mathbb{Z} : 2|3x\} = \{x \in \mathbb{Z} : 2|x\}$
$[1] = \{x \in \mathbb{Z} : 2|3x - 5(1)\} = \{x \in \mathbb{Z} : 2|3(x - 1) - 2\} = \{x \in \mathbb{Z} : 2|(x - 1)\} = \{x \in \mathbb{Z} : 2 \nmid x\}$

8. Proposition The relation $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 2|x^2 + y^2\}$ on $\mathbb{Z}$ is an equivalence relation.
Proof.
First we show that the relation is reflexive. Let $x \in \mathbb{Z}$. Observe that $x^2 + x^2 = 2x^2$. Since $2|2x^2$ by def. of divisibility, it follows that xRx and thus R is reflexive.

Next we show that the relation is symmetric. Suppose xRy for some $y \in \mathbb{Z}$. Observe that $2|x^2 + y^2 \iff 2|y^2 + x^2$. Thus

yRx, which means R is symmetric.

Finally we show that the relation is transitive. Suppose xRy and yRz for some $z \in \mathbb{Z}$. We know $2|x^2+y^2 \iff x^2+y^2 = 2a$, and $2|y^2+z^2 \iff y^2+z^2 = 2b$ for some $a, b \in \mathbb{Z}$. Observe that $x^2+y^2 - \Big(y^2+z^2\Big) + \Big(2z^2\Big) = 2a - \Big(2b\Big) + \Big(2z^2\Big) \iff x^2+z^2 = 2a-2b+2z^2$. Since $2|2(a-b+z^2)$, it follows that xRz and thus R is transitive.

Therefore the relation R on $\mathbb{Z}$ is an equivalence relation. ■

The equivalence classes of R are
$[0] = \{x \in \mathbb{Z} : 2|x^2+0^2\} = \{x \in \mathbb{Z} : 2|x^2\} = \{x \in \mathbb{Z} : 2|x\}$
$[1] = \{x \in \mathbb{Z} : 2|x^2+1^2\} = \{x \in \mathbb{Z} : 2 \nmid x^2 = \{x \in \mathbb{Z} : 2 \nmid x\}$

9. Proposition The relation $R = \{(x, y) \in \mathbb{Z} : 4|x+3y\}$ on $\mathbb{Z}$ is an equivalence relation.
Proof.
First we show that the relation is reflexive. Let $x \in \mathbb{Z}$. Observe that $x+3x = 4x$. Since $4|4x$ by def. of divisibility, it follows that xRx and thus R is reflexive.

Next we show that the relation is symmetric. Suppose xRy for some $y \in \mathbb{Z}$. We know $4|x+3y \iff 4a = x+3y$ for some $a \in \mathbb{Z}$. Observe that $x+3y+\Big(2x-2y\Big) = 4a+\Big(2x-2y\Big) \iff 3x+y = 4a+2x-2y$, thus $2|3x+y$. Since 3 is prime and $3 \nmid 2$, we get $2|x+y$. Note that $2|x+y$ only if x and y have the same parity, thus we consider two cases.
Case 1. Let $2|x \iff x = 2s$ and $2|y \iff y = 2t$. Then $3x+y = 4a+2\Big(2s\Big)-2\Big(2t\Big) = 4(a+s-t)$.
Case 2. Let $2 \nmid x \iff x = 2s+1$ and $2 \nmid y \iff y = 2t+1$. Then $3x+y = 4a+2\Big(2s+1\Big)-2\Big(2t+1\Big) = 4a+4s-4t$.
In both cases $4|4(a+s-t)$, thus yRx which means R is symmetric.

Finally we show that the relation is transitive. Suppose xRy and yRz for some $z \in \mathbb{Z}$. We know $4a = x+3y$, and $4|y+3z \iff 4b = y+3z$ for some $b \in \mathbb{Z}$. Observe that $x+3z = \Big(4a-3y\Big)+\Big(4b-y\Big) = 4a+4b-4y$. Since $4|4(a+b-y)$, it follows that xRz and thus R is transitive.

Therefore the relation R on $\mathbb{Z}$ is an equivalence relation. ■

The equivalence classes of R are
$[0] = \{x \in \mathbb{Z} : 4|x+3(0)\} = \{x \in \mathbb{Z} : x \equiv 0 \pmod 4\}$
$[1] = \{x \in \mathbb{Z} : 4|x+3(1)\} = \{x \in \mathbb{Z} : x \equiv 1 \pmod 4\}$
$[2] = \{x \in \mathbb{Z} : 4|x+3(2)\} = \{x \in \mathbb{Z} : 4|(x+2)+4\} = \{x \in \mathbb{Z} : x \equiv 2 \pmod 4\}$
$[3] = \{x \in \mathbb{Z} : 4|x+3(3)\} = \{x \in \mathbb{Z} : 4|(x+1)+2*4\} = \{x \in \mathbb{Z} : x \equiv 3 \pmod 4\}$

10. Proposition Suppose R and S are equivalence relations on A. Then $T = R \cap S = \{(x, y) \in A \times A : xRy \wedge xSy\}$ is also an equivalence relation on A.
Proof.
First we show that the relation is reflexive. Let $x \in A$. Since R and S are reflexive, xRx and xSx. Then xTx, and thus T is reflexive.

Next we show that the relation is symmetric. Suppose xTy, which means $xRy \wedge xSy$ for some $y \in A$. Since R and S are symmetric, $yRx \wedge ySx$. Then yTx, and thus T is symmetric.

Finally we show that the relation is transitive. Suppose xTy and yTz, which means $(xRy \wedge xSy)$ and $(yRz \wedge ySz)$ for some $z \in A$. Since R and S are transitive, $(xRy \wedge yRz) \implies xRz$, and $(xSy \wedge ySz) \implies xSz$. Then $(xRz \wedge xSz)$, so xTz, and thus T is transitive.

Therefore the relation T on R and S is an equivalence relation. ■

11. Proposition If R is an equivalence relation on an infinite set A, then R has infinitely many equivalence classes.
Disproof.
This proposition is false, we show this with a counterexample: If $R = \{(x, y) \in A \times A : 2|(x - y)\}$, then $[0] = [2] = [4]...[2n]$ and $[1] = [3] = [5]...[2n + 1]$ for $n \in \mathbb{Z}$. Thus R is a relation on an infinite set A, but has only two equivalence classes. ■

12. Proposition If R and S are two equivalence relations on set A, then $T = R \cup S = \{(x, y) \in A \times A : xRy \vee xSy\}$ is an equivalence relation on A.
Disproof.
This proposition is false, we show this with a counterexample: Let $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$ and $S = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$. Note that R and S are equivalence relations on A. Observe that $(aRb \vee aSb)$ and $(bRc \vee bSc)$, but not $(aRc \vee aSc)$. Thus aTb and bTc but not aTc, which means T is not transitive, and thus not an equivalence relation. ■

13. Note that each element in A is contained in some equivalence class. Since every equivalence class has m elements, and no two equivalence classes share a subset of elements, there are $\dfrac{|A|}{m}$ equivalence classes total. The amount of $(x, y) \in R \times R$ per equivalence class is m^2, since each of its elements relates to m elements in that class. Thus $|R| = \dfrac{|A|}{m} m^2 = |A|m$.

14. Proposition Suppose R is a symmetric and reflexive relation on a finite set A. The relation $S = \{(x, y) \in A \times A : n \in \mathbb{N}, (x_1, x_2, x_3, ...x_n \in A), (xRx_1 \wedge x_1Rx_2 \wedge ... \wedge x_{n-1}Rx_n \wedge x_nRy)\}$ is the smallest equivalence relation on A such that $S \subseteq R$.
Proof.
First we show that the relation is reflexive. Let $x \in A$. Since R is reflexive, xRx. Then xRx_1 and x_1Rx for $x_1 = x$, thus xSx, which means S is reflexive.

Next we show that the relation is symmetric. Suppose xSy for some $y \in A$. Since R is symmetric, $xRx_1 \wedge x_1Rx_2 \wedge ... \wedge x_{n-1}Rx_n \wedge x_nRy$ for $x_1, x_2, ..., x_n \in A$ and some $n \in \mathbb{N}$ implies that $yRx_n \wedge x_nRx_{n-1} \wedge ... \wedge x_2Rx_1 \wedge x_1Rx$. Thus ySx, which means S is symmetric.

Finally we show that the relation is transitive. Suppose xSy and ySz for some $z \in \mathbb{Z}$. Then $xRx_1 \wedge x_1Rx_2 \wedge ... \wedge x_nRy \wedge yRy_1 \wedge y_1Ry_2 \wedge ... \wedge y_mRz$ for $y_1, y_2, ...y_m \in A$ and some $m \in \mathbb{N}$. Then there exists $z_{i \in [1, n+m+1]} = x_1, x_2, ..., x_n, y, y_1, y_2, ...y_m \in A$ for which $xRz_1 \wedge z_1Rz_2 \wedge ... \wedge z_{n+m+1}Rz$, thus xSz which means S is transitive.

Thus the relation S on A is an equivalence relation.

Now we show that $R \subseteq S$. Suppose $(x, y) \in R$ for some $x, y \in A$, which implies xRy. Since R is symmetric, xRx_1 and x_1Ry for $x_1 = x$. Thus for any element in R, it is also in S, so $R \subseteq S$.

Next we show that S is the smallest equivalence relation on A such that $R \subseteq S$. Let $|x_{i\in[1,n]} \in A|$ be the smallest set such that $xRx_1 \wedge x_1Rx_2 \wedge ... \wedge x_nRy$ for any $x, y \in A$. Now suppose for the sake of contradiction that Q is an equivalence relation on A such that $R \subseteq S$ and $|Q| < |S|$. Then there exists some a such that $a \notin A$ for which $xRx_1 \wedge x_1Rx_2 \wedge ... \wedge x_iRa \wedge aRx_{i+1} \wedge ... \wedge x_mRy$ for an $m \in \mathbb{N}$ such that $m < n$. Then $Q \subseteq A \times A$ but there exists $(x_i, a) \in Q$ such that $(x_i, a) \notin A \times A$, a contradiction.

Therefore S is the smallest equivalence relation on a finite set A such that $S \subseteq R$, where R is a symmetric and reflexive relation on A. $\blacksquare$

15. Let $P = \{P_1, P_2, P_3, P_4\}$ be the set of equivalence classes of the equivalence relation R on A. For all $x \in A$, it follows that $x \in P_i$ for some (unique) $1 \leq i \leq 4$, thus $R = P_1^2 \cup P_2^2 \cup P_3^2 \cup P_4^2$. We know also that $A \times A = (P_1 \cup P_2 \cup P_3 \cup P_4)^2$ and consequently the equivalence class of $A \times A$ is $Q = \{P_1 \cup P_2 \cup P_3 \cup P_4\}$. Then any equivalence relation T on A of which R is a subset corresponds to a set of equivalence classes which contain from zero to three unions of some elements of P. Observe that we can form a set union between two elemets of P in $\binom{4}{2}$ ways. The remaining two elements in each case can either form a set union or not, so there are $\binom{4}{2}2 - 3$ sets of equivalence classes with set union(s) between two elements (Due to symmetry of $\{P_1 \cup P_2, P_3 \cup P_4\}$, $\{P_1 \cup P_3, P_2 \cup P_4\}$ and $\{P_1 \cup P_4, P_2 \cup P_3\}$ we counted them twice, hence subtract 3). We can also form a set union between three elements of P in $\binom{4}{3}$ ways. Finally, we count 2 more sets, P and Q themselves. The total amount of sets of equivalence classes, and thus total amount of equivalence relations T, is $2\binom{4}{2} - 3 + \binom{4}{3} + 2 = 15$.

16. Proposition The relation $R = \left\{\left(\frac{x}{y}, \frac{z}{w}\right) \in F \times F : xw = zy\right\}$ on F (page 213) is transitive.

Proof.
Suppose $\frac{a}{b}R\frac{c}{d}$ and $\frac{c}{d}R\frac{e}{f}$ for some $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$. We know $ad = cb$ and $cf = ed$. Substituting $c = \frac{ad}{b}$ into the latter, we get $f\frac{ad}{b} = ed \iff fa = eb$. Then $\frac{a}{b}R\frac{e}{f}$, and thus the relation R on F is transitive. $\blacksquare$