19.

(a).1 Proposition: If $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$, then then equality holds in the Schwarz inequality, $x_1y_1+x_2y_2 \leq \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$.

Proof.

$$\begin{aligned} x_1y_1 + x_2y_2 &= (\lambda y_1)y_1 + (\lambda y_2)y_2 \\ &= \lambda(y_1^2+y_2^2) \\ &= \sqrt{\lambda^2}\sqrt{y_1^2+y_2^2}\sqrt{y_1^2+y_2^2} \\ &= \sqrt{\lambda^2 y_1^2+\lambda^2 y_2^2}\sqrt{y_1^2+y_2^2} \\ &= \sqrt{(\lambda y_1)^2+(\lambda y_2)^2}\sqrt{y_1^2+y_2^2} \\ &= \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}. \end{aligned}$$

(a).2 Proposition: If $y_1 = y_2 = 0$, then $x_1y_1 + x_2y_2 = \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$.

Proof.

$$\begin{aligned} x_1y_1 + x_2y_2 &= x_1(0) + x_2(0) \\ &= 0 \\ &= \sqrt{x_1^2+x_2^2}\sqrt{(0)^2+(0)^2} \\ &= \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}. \end{aligned}$$

(a).3 Proposition: If $y_1 \neq 0$ or $y_2 \neq 0$, and there exists no number λ for which $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$, then $0 < (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2$.

Proof.

Note that $0 \leq (\lambda y_1 - x_1)^2$ and $0 \leq (\lambda y_2 - x_2)^2$. Since $x_1 \neq \lambda y_1$ and $x_2 \neq \lambda y_2$ for any number λ, it would follow that $y_1 = y_2 = 0$ and $x_1 = x_2 = 0$ if $(\lambda y_1 - x_1)^2 = 0$ and $(\lambda y_2 - x_2)^2 = 0$. But we assumed $y_1 \neq 0$ or $y_2 \neq 0$, thus $(\lambda y_1 - x_1)^2 > 0$ or $(\lambda y_2 - x_2)^2 > 0$, meaning $0 < (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2$.

(b) Proposition: $x_1y_1 + x_2y_2 \leq \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$.

Proof.

Note that $\left(0 \leq (x-y)^2 = x^2 - 2xy + y^2\right) \implies \left(2xy \leq x^2 + y^2\right)$ for any numbers x and y. Let $x = \dfrac{x_1}{\sqrt{x_1^2+x_2^2}}$ and $y = \dfrac{y_1}{\sqrt{y_1^2+y_2^2}}$. Observe that

$$\begin{aligned} 2\frac{x_1}{\sqrt{x_1^2+x_2^2}}\frac{y_1}{\sqrt{y_1^2+y_2^2}} &\leq \left(\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right)^2 + \left(\frac{y_1}{\sqrt{y_1^2+y_2^2}}\right)^2 \\ \frac{2x_1y_1}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} &\leq \frac{x_1^2}{x_1^2+x_2^2} + \frac{y_1^2}{y_1^2+y_2^2} \end{aligned}$$

Similarly, $\frac{2x_2y_2}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} \leq \frac{x_2^2}{x_1^2+x_2^2} + \frac{y_2^2}{y_1^2+y_2^2}$. Adding the inequalities,

$$\begin{aligned}
\frac{2x_1y_1}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} + \frac{2x_2y_2}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} &\leq \frac{x_1^2}{x_1^2+x_2^2} + \frac{y_1^2}{y_1^2+y_2^2} + \frac{x_2^2}{x_1^2+x_2^2} + \frac{y_2^2}{y_1^2+y_2^2} \\
\frac{2x_1y_1+2x_2y_2}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} &\leq \frac{x_1^2+x_2^2}{x_1^2+x_2^2} + \frac{y_1^2+y_2^2}{y_1^2+y_2^2} \\
\frac{2(x_1y_1+x_2y_2)}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} &\leq 2 \\
x_1y_1+x_2y_2 &\leq \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}.
\end{aligned}$$

(c) Proposition: $x_1y_1+x_2y_2 \leq \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$.

Proof.

First we show that $(x_1^2+x_2^2)(y_1^2+y_2^2) = (x_1y_1+x_2y_2)^2 + (x_1y_2+x_2y_1)^2$:

$$\begin{aligned}
(x_1^2+x_2^2)(y_1^2+y_2^2) &= x_1^2y_1^2 + x_1^2y_2^2 + x_2^2y_1^2 + x_2^2y_2^2 \\
&= x_1^2y_1^2 + x_1^2y_2^2 + x_2^2y_1^2 + x_2^2y_2^2 + 2x_1y_1x_2y_2 - 2x_1y_1x_2y_2 \\
&= (x_1y_1)^2 + 2(x_1y_1)(x_2y_2) + (x_2y_2)^2 + (x_1y_2)^2 - 2(x_1y_2)(x_2y_1) + (x_2y_1)^2 \\
&= (x_1y_1+x_2y_2)^2 + (x_1y_2+x_2y_1)^2.
\end{aligned}$$

Note that $(x_1y_1+x_2y_2)^2 \geq 0$ implies that $(x_1y_1+x_2y_2)^2 + (x_1y_2+x_2y_1)^2 \geq (x_1y_2+x_2y_1)^2$. Thus

$$\begin{aligned}
(x_1y_1+x_2y_2)^2 + (x_1y_2+x_2y_1)^2 = (x_1^2+x_2^2)(y_1^2+y_2^2) &\geq (x_1y_2+x_2y_1)^2 \\
\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2} &\geq x_1y_2+x_2y_1.
\end{aligned}$$

(d) Deduce from each of the above proofs that $0 = (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2$ implies $y_1 = y_2 = 0$ or there exists a number $\lambda \geq 0$ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

In (a), this statement is true by Propositions (a).1 and (a).2.

In (b), it must hold that $\frac{2x_2y_2}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} = \frac{x_2^2}{x_1^2+x_2^2} + \frac{y_2^2}{y_1^2+y_2^2}$ and $\frac{2x_1y_1}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} = \frac{x_1^2}{x_1^2+x_2^2} + \frac{y_1^2}{y_1^2+y_2^2}$. Thus, $0 \leq (x-y)^2$, the inequality that the two were derived from, becomes $0 = (x-y)^2$, which means $y = \frac{y_i}{\sqrt{y_1^2+y_2^2}} = \frac{y_i\lambda}{\sqrt{\lambda^2}\sqrt{(y_1)^2+(y_2)^2}} = \frac{y_i\lambda}{\sqrt{(\lambda y_1)^2+(\lambda y_2)^2}} = \frac{x_i}{\sqrt{x_1^2+x_2^2}} = x$. Thus $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

In (c), it must hold that $(x_1y_1+x_2y_2)^2 = 0$. Then $x_1y_1 = -x_2y_2$, which means that $y_1 = y_2 = 0$ is a solution.