6.

(i) $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$

Proof.

$$\begin{aligned}
(n+1)^4 &= n^4 + 4n^3 + 6n^2 + 4n + 1 && (1)\\
\sum_{i=1}^{n}(i+1)^4 &= \sum_{i=1}^{n} i^4 + 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + n && \text{Summing (1) for numbers } 1, ..., n\\
4\sum_{i=1}^{n} i^3 &= \sum_{i=1}^{n}(i+1)^4 - \sum_{i=1}^{n} i^4 - 6\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i - n\\
&= \sum_{i=2}^{n+1} i^4 - \sum_{i=1}^{n} i^4 - 6\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} - n\\
&= -1^4 + (n+1)^4 - n(n+1)(2n+1) - 2n(n+1) - n\\
&= -1 + (n^4 + 4n^3 + 6n^2 + 4n + 1) - (n^2 + n)(2n+1) - 2n^2 - 2n - n\\
&= -1 + n^4 + 4n^3 + 6n^2 + 4n + 1 - 2n^3 - n^2 - 2n^2 - n - 2n^2 - 2n - n\\
&= n^4 + 2n^3 + n^2\\
&= n^2(n^2 + 2n + 1)\\
\sum_{i=1}^{n} i^3 &= \frac{1}{4}n^2(n+1)^2\\
&= (\frac{1}{2}n(n+1))^2\\
\therefore \sum_{i=1}^{n} i^3 &= (\sum_{i=1}^{n} i)^2.
\end{aligned}$$

(ii) $\sum_{i=1}^{n} i^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$

Proof.

$$\begin{aligned}
(n+1)^5 &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 && (1)\\
\sum_{i=1}^{n}(i+1)^5 &= \sum_{i=1}^{n} i^5 + 5\sum_{i=1}^{n} i^4 + 10\sum_{i=1}^{n} i^3 + 10\sum_{i=1}^{n} i^2 + 5\sum_{i=1}^{n} i + n && \text{Summing (1) for numbers } 1, ..., n\\
5\sum_{i=1}^{n} i^4 &= \sum_{i=1}^{n}(i+1)^5 - \sum_{i=1}^{n} i^5 - 10\sum_{i=1}^{n} i^3 - 10\sum_{i=1}^{n} i^2 - 5\sum_{i=1}^{n} i - n\\
&= \sum_{i=2}^{n+1} i^5 - \sum_{i=1}^{n} i^5 - 10\frac{n^2(n+1)^2}{4} - 10\frac{n(n+1)(2n+1)}{6} - 5\frac{n(n+1)}{2} - n\\
&= -1^5 + (n+1)^5 - 5\frac{n^2(n+1)^2}{2} - 5\frac{n(n+1)(2n+1)}{3} - 5\frac{n(n+1)}{2} - n\\
&= -1 + (n+1)^5 - 5\frac{n^2(n^2+2n+1)}{2} - 5\frac{(n^2+n)(2n+1)}{3} - 5\frac{n^2+n}{2} - n\\
&= -1 + (n+1)^5 - 5\frac{n^4+2n^3+n^2}{2} - 5\frac{2n^3+3n^2+n}{3} - 5\frac{n^2+n}{2} - n\\
&= -1 + (n+1)^5 - \frac{5}{2}n^4 + (-\frac{5*2}{2} - \frac{5*2}{3})n^3 + (-\frac{5}{2} - \frac{5*3}{3} - \frac{5}{2})n^2 + (-\frac{5}{3} - \frac{5}{2} - 1)n\\
&= -1 + (n+1)^5 - \frac{5}{2}n^4 - \frac{25}{3}n^3 - 10n^2 - \frac{31}{6}n\\
&= -1 + n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - \frac{5}{2}n^4 - \frac{25}{3}n^3 - 10n^2 - \frac{31}{6}n\\
&= 1 - 1 + n^5 + (5 - \frac{5}{2})n^4 + (10 - \frac{25}{3})n^3 + (10-10)n^2 + (5 - \frac{31}{6})n\\
5\sum_{i=1}^{n} i^4 &= n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 + \frac{-1}{6}n\\
\therefore \sum_{i=1}^{n} i^4 &= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}
\end{aligned}$$

(iii) $\sum_{i=1}^{n} \dfrac{1}{i(i+1)} = 1 - \dfrac{1}{n+1}$

Proof.

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{i(i+1)} &= \sum_{i=1}^{n} \frac{(i+1)-i}{i(i+1)}\\
&= \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i+1}\\
&= \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=2}^{n+1} \frac{1}{i}\\
&= \frac{1}{1} - \frac{1}{n+1} + \sum_{i=2}^{n} \frac{1}{i} - \sum_{i=2}^{n} \frac{1}{i}\\
\therefore \sum_{i=1}^{n} \frac{1}{i(i+1)} &= 1 - \frac{1}{n+1}.
\end{aligned}$$

(iv) $\sum_{i=1}^{n} \frac{2i+1}{i^2(i+1)^2} = 1 - \frac{1}{(n+1)^2}$

Proof.

$$\begin{aligned}
\sum_{i=1}^{n} \frac{2i+1}{i^2(i+1)^2} &= \sum_{i=1}^{n} \frac{(i^2+2i+1)-i^2}{i^2(i^2+2i+1)} \\
&= \sum_{i=1}^{n} \frac{(i^2+2i+1)-i^2}{i^2(i^2+2i+1)} \\
&= \sum_{i=1}^{n} \frac{1}{i^2} - \sum_{i=1}^{n} \frac{1}{i^2+2i+1} \\
&= \sum_{i=1}^{n} \frac{1}{i^2} - \sum_{i=1}^{n} \frac{1}{(i+1)^2} \\
&= \sum_{i=1}^{n} \frac{1}{i^2} - \sum_{i=2}^{n+1} \frac{1}{i^2} \\
&= (\frac{1}{1})^2 - (\frac{1}{n+1})^2 + \sum_{i=2}^{n} \frac{1}{i^2} - \sum_{i=2}^{n} \frac{1}{i^2} \\
\therefore \sum_{i=1}^{n} \frac{2n+1}{n^2(n+1)^2} &= 1 - \frac{1}{(n+1)^2}.
\end{aligned}$$