15. $x, y \neq 0$

Proposition: $0 < x^2 + xy + y^2$

Proof.

Consider two cases. When x and y are both negative or both positive, $xy > 0$, so clearly $0 < x^2 + xy + y^2$. Now x and y have opposite signs, so that $xy < 0$. Observe that

$$\begin{aligned} xy < 0 \leq (x+y)^2 &\implies xy < x^2 + 2xy + y^2 \\ &\implies -xy < x^2 + y^2 \\ &\implies 0 < x^2 + xy + y^2. \end{aligned}$$

Proposition: $0 < x^4 + x^3y + x^2y^2 + xy^3 + y^4$

Proof.

Consider three cases. Let $x = y$. Observe that

$$\begin{aligned} \pm x &> 0 \\ x^4 &> 0 \\ x^4 + x^3x + x^2x^2 + x^3x + x^4 &> 0 \\ x^4 + x^3y + x^2y^2 + y^3x + y^4 &> 0. \end{aligned}$$

Now let $x > y$, then

$$\begin{aligned} x^5 &> y^5 && \text{By Exercise 6.b} \\ x^5 - y^5 &> 0 \\ (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4) &> 0 && \text{By Exercise 1.v} \\ x^4 + x^3y + x^2y^2 + xy^3 + y^4 &> 0. && x > y \implies (x-y) > 0 \end{aligned}$$

Lastly, let $x < y$. Then

$$\begin{aligned} x^5 &< y^5 \\ x^5 - y^5 &< 0 \\ (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4) &< 0 \\ x^4 + x^3y + x^2y^2 + xy^3 + y^4 &> 0. && x < y \implies (x-y) < 0 \end{aligned}$$

Therefore $0 < x^4 + x^3y + x^2y^2 + xy^3 + y^4$.