Proposition: For all $n \in \mathbb{N}$, it follows that

$$S_n : (1 + 2 + 3 + ... + n)^2 = 1^3 + 2^3 + 3^3 + ... + n^3.$$

Proof. (Induction).
Basis step. Let $n = 1$. Observe that $n^2 = 1^2 = 1^3 = n^3$, thus S_1.
Inductive step. Suppose S_k for $k \in \mathbb{N}$.

We now show S_k implies S_{k+1}. Observe that

$$\begin{aligned}
1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 &= (1 + 2 + 3 + ... + k)^2 + (k+1)^3 && (1)\\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{(Def. of triangular number)} \quad (2)\\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} && (3)\\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} && (4)\\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} && (5)\\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} && (6)\\
&= \frac{(k+1)^2(k+2)^2}{2^2} && (7)\\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2 && (8)\\
&= (1 + 2 + 3 + ... + k + (k+1))^2. && \text{(Def. of triangular number)} \quad (9)
\end{aligned}$$

Thus S_{k+1}.
It follows by mathematical induction that S_n for all $n \in \mathbb{N}$. ■