



# 卡尔曼滤波基础

- 卡尔曼滤波的应用框架
- 信号与观测模型
- 算法推导
- 算法总结



# 1. 卡尔曼滤波的应用框架

1960年, R.E.Kalman 发表了一篇著名的文章



R. E. KALMAN

Research Institute for Advanced Study,<sup>2</sup>  
Baltimore, Md.

R. E. KALMAN

RESEARCH INSTITUTE FOR ADVANCED STUDY,<sup>2</sup>  
BALTIMORE, MD.

## A New Approach to Linear Filtering and Prediction Problems<sup>1</sup>

The classical filtering and prediction problem is re-examined using the state-space representation of random processes and the "state transition" method of analysis of dynamic systems. New results are:

- (1) The formulation and methods of solution of the problem apply without modification to stationary and nonstationary statistics and to growing-memory and finite-memory filters.
- (2) A nonlinear difference (or differential) equation is derived for the covariance matrix of the optimal estimation error. From the solution of this equation the coefficients of the difference (or differential) equation of the optimal linear filter are obtained without further calculation.
- (3) The filtering problem is shown to be the dual of the time-varying regulator problem. The new method developed here is applied to both well-known problems, clarifying and extending earlier results.

The discussion is largely self-contained and proceeds from first principles; basic concepts of the theory of random processes are reviewed in the appendix.

### Introduction

Important class of theoretical and practical communication and control is of a statistical nature. are: (i) Prediction of random signals; (ii) separation of signals from random noise; (iii) detection of signals (pulses, sinusoids) in the presence of random noise. Wiener [1]<sup>2</sup> showed that problems (i) and (ii) can be solved by the so-called Wiener-Hopf integral equation; he also gave a method (spectral factorization) for the solution of this equation. For problem (iii) the so-called Kalman filter [3] is used.

Present methods for solving the Wiener problem are subject to a number of limitations which seriously curtail their practical usefulness:

- (1) The optimal filter is specified by its impulse response. It is not a simple task to synthesize the filter from such data.
- (2) Numerical determination of the optimal impulse response is often quite involved and poorly suited to machine computation. The situation gets rapidly worse with increasing complexity of the problem.
- (3) Important generalizations (e.g., growing-memory filters, nonstationary prediction) require new derivations, frequently of considerable difficulty because of nonstationarity.
- (4) The mathematics of the derivations are not transparent. Fundamental assumptions and their consequences tend to be obscured.

This paper introduces a new look at this whole assortment of problems, side-stepping the difficulties just mentioned. The following are the highlights of the paper:

- (5) *Optimal Estimation and Orthogonal Projections*. The Wiener problem is approached from the point of view of conditional distributions and expectations. In this way, basic facts of the Wiener theory are quickly obtained; the scope of the results and the fundamental assumptions appear clearly. It is seen that all statistical calculations and results are based on first and second order averages; no other statistical data are needed. Thus the problem is greatly simplified.

It is eliminated. This method is well known in theory (see pp. 75-78 and 144-155 of Doob [15]) and of Loève [16]) but has not yet been used extensively in the literature. Following, in particular, Kalman [3], arbitrary random signals are represented as order average statistical properties) as the output of a linear system excited by independent or uncorrelated noise ("white noise"). This is a standard trick in the applications of the Wiener theory [2-7]. The new idea differs from the conventional one only in the way these dynamic systems are described. We shall use the concepts of state and state transition. In other words, linear systems will be specified by systems of first-order difference (or differential) equations. This point of view natural

## A New Approach to Linear Filtering and Prediction Problems<sup>1</sup>

## New Results in Linear Filtering and Prediction Theory

<sup>1</sup> Individual contributions of the author and of the Society. Kalman's research at ASME, Washington, February 23, 1960. Paper No. 58-120-11.



# 1. 卡尔曼滤波的应用框架

## Kalman简介



1930出生于匈牙利布达佩斯

1953年MIT获电机工程学士学位，翌年获硕士学位

1957年在美国哥伦比亚大学获博士学位

1964-1971任职斯坦福大学

1971-1992任职佛罗里达数学系统理论中心

2008年获Charles Stark Draper 奖

2009获美国国家科学奖章

2016年7月2日去世



# 1. 卡尔曼滤波的应用框架

卡尔曼突出贡献：

提出了线性系统滤波的新方法，超越了维纳等研究的线性滤波理论，解决了非平稳、多输入

多输出线性系统滤波问题，使工程实现成为可能。成就了过去60年间的许多基本技术，如航空航天、导航、雷达、通信、机器视觉等。





# 1. 卡尔曼滤波的应用框架

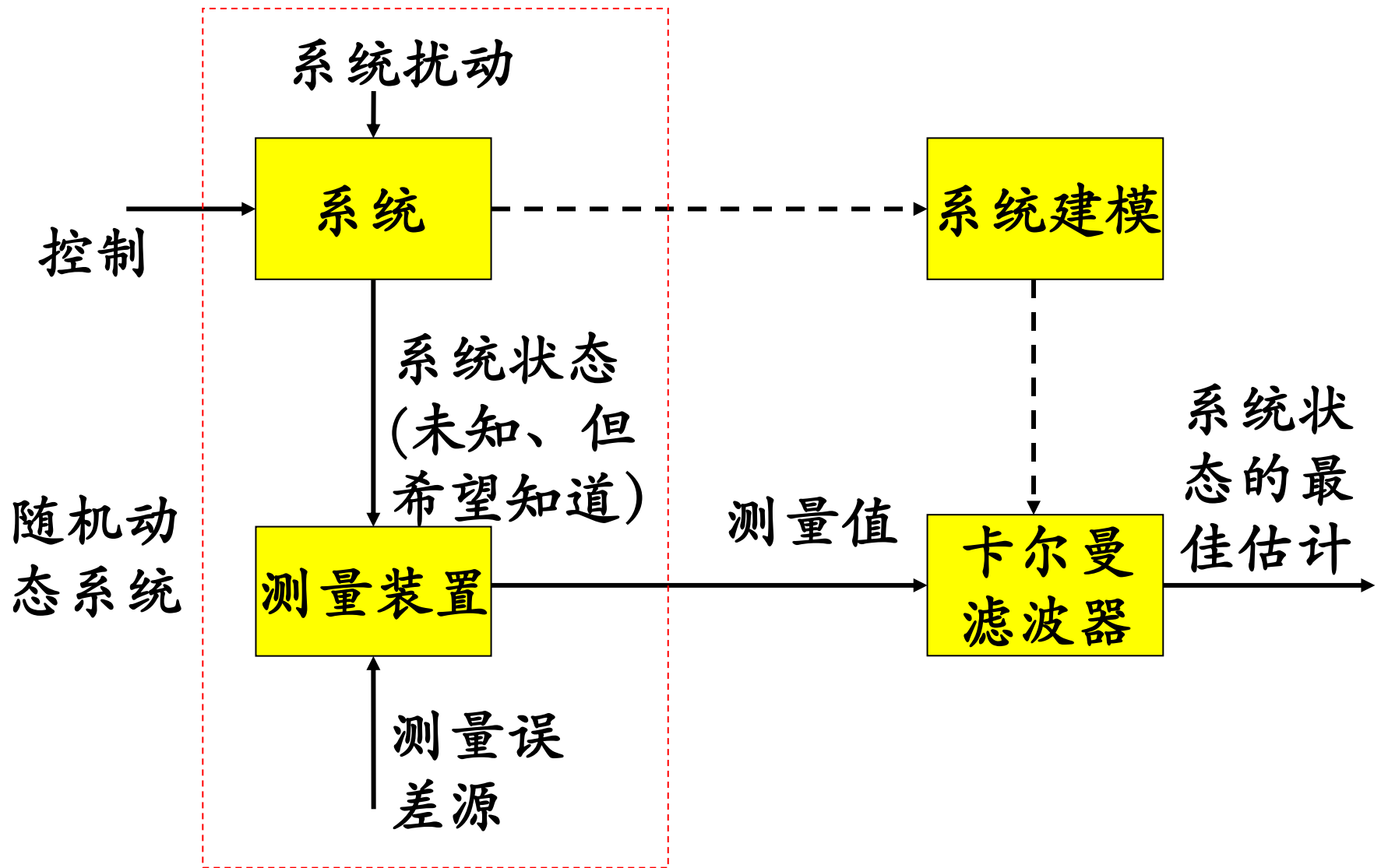
卡尔曼滤波是一组递推的数据处理算法，这组算法提供了离散线性系统状态的线性最小均方估计的有效计算方法。

其有效性体现在：

- 能提供对系统过去、现在和未来状态的估计，  
甚至当系统精细的特性未知的情况下也能如此
- 能估计非平稳过程
- 能估计矢量过程（适用于多输入多输出系统）

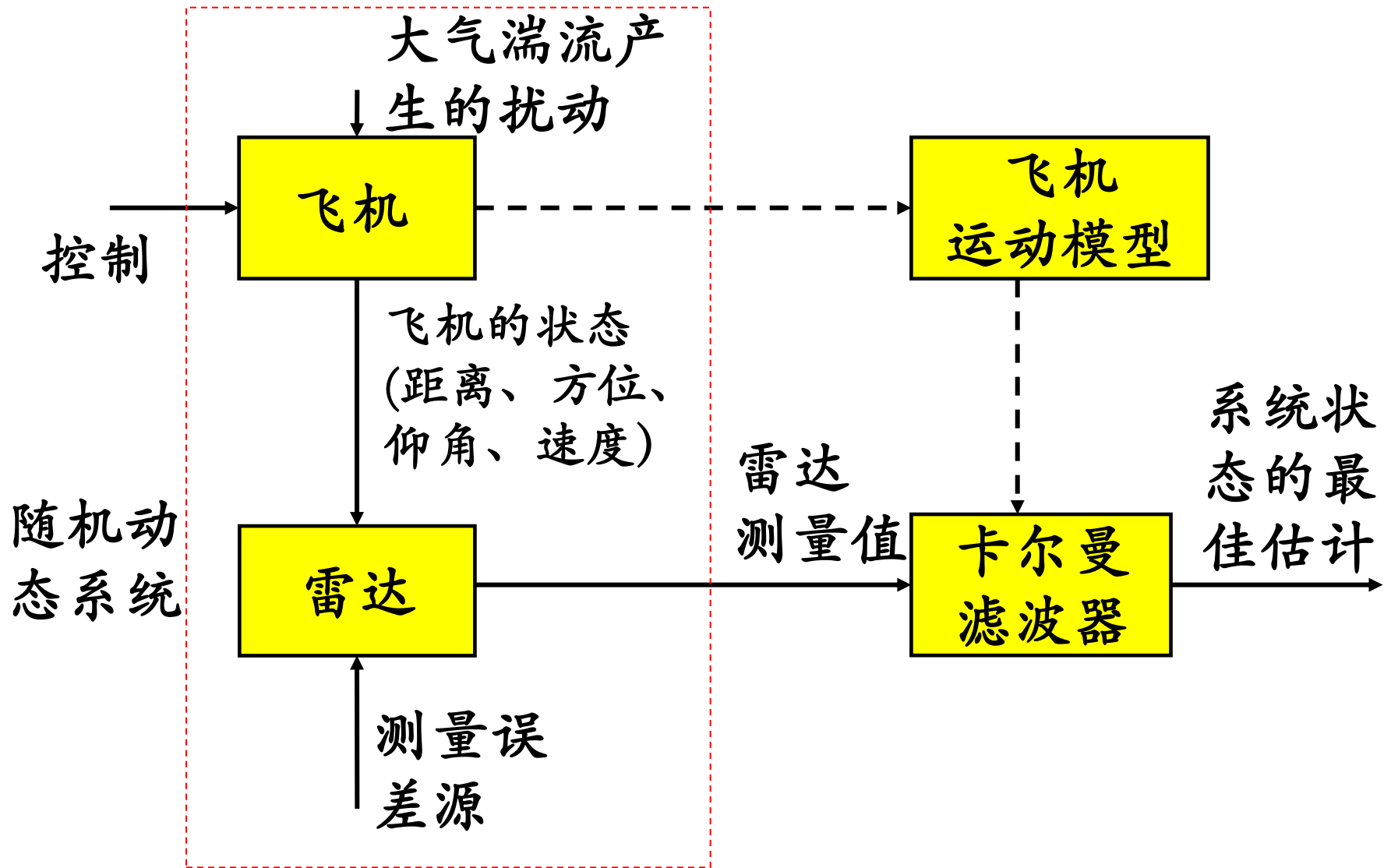


# 1. 卡尔曼滤波的应用框架





# 1. 卡尔曼滤波的应用框架





# 1. 卡尔曼滤波的应用框架

## 典型的应用领域

- 目标跟踪
- 导航
- 控制
- 弹道导弹弹道估计
- 火炮控制；
- 通信信道均衡
- 气象预报
- ...





## 2. 信号模型与观测模型

信号模型:  $\mathbf{x}[k+1] = \mathbf{\Phi}[k+1, k]\mathbf{x}[k] + \mathbf{\Gamma}[k]\mathbf{n}[k]$

$\mathbf{x}$  :  $M \times 1$  的状态矢量

$\mathbf{\Phi}$  :  $M \times M$  维的矩阵, 称为状态转移矩阵

$\mathbf{n}$  :  $p \times 1$  维的矢量, 称为系统扰动噪声

$\mathbf{\Gamma}$  :  $M \times p$  维的矩阵, 称为系统扰动矩阵

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观测模型:  $\mathbf{z}[k] = \mathbf{H}[k]\mathbf{x}[k] + \mathbf{w}[k]$

$\mathbf{z}$  :  $N \times 1$  的观测矢量

$\mathbf{H}$  :  $N \times M$  维的矩阵, 称为测量矩阵

$\mathbf{w}$  :  $N \times 1$  的观测噪声矢量



## 2. 信号模型与观测模型

$\mathbf{n}[k], \mathbf{w}[k]$  假定为不相关的白噪声，统计特性为：

$$E(\mathbf{n}[k]) = \mathbf{0}, E(\mathbf{n}[k]\mathbf{n}^T[l]) = \mathbf{Q}[k]\delta_{kl}$$

$$E(\mathbf{w}[k]) = \mathbf{0}, E(\mathbf{w}[k]\mathbf{w}^T[l]) = \mathbf{R}[k]\delta_{kl}$$

$$E(\mathbf{n}[k]\mathbf{w}^T[l]) = \mathbf{0}$$

系统的起始条件：

$$E(\mathbf{x}[k_0]) = \boldsymbol{\mu}_x[k_0]$$

$$E\{(\mathbf{x}[k_0] - \boldsymbol{\mu}_x[k_0])(\mathbf{x}[k_0] - \boldsymbol{\mu}_x[k_0])^T\} = \mathbf{P}_x[k_0]$$

$$E\{(\mathbf{x}[k_0] - \boldsymbol{\mu}_x[k_0])\mathbf{n}^T[k]\} = \mathbf{0}$$

$$E\{(\mathbf{x}[k_0] - \boldsymbol{\mu}_x[k_0])\mathbf{w}^T[k]\} = \mathbf{0}$$



### 3. 算法推导

设观测为  $\{\mathbf{z}[k_0], \mathbf{z}[k_0 + 1], \dots, \mathbf{z}[k]\}$  , 这组观测用  $\mathbf{z}^k$  表示。

$$\mathbf{z}^k = \begin{bmatrix} (\mathbf{z}^{k-1})^T & \mathbf{z}^T[k] \end{bmatrix}^T$$

$\mathbf{z}^{k-1}$  表示  $k-1$  以前的观测数据集  $\{\mathbf{z}[k_0], \mathbf{z}[k_0 + 1], \dots, \mathbf{z}[k - 1]\}$

$\mathbf{x}[j]$  的线性最小均方估计为

$$\hat{\mathbf{x}}[j / k] = \hat{E}(\mathbf{x}[j] | \mathbf{z}^k)$$

$j=k$  表示滤波,  $j=k+1$  表示一步预测



### 3. 算法推导

考虑滤波问题，由正交投影性质

$$\begin{aligned}\hat{\mathbf{x}}[k/k] &= \hat{E}(\mathbf{x}[k] | \mathbf{z}^k) \\ &= \hat{E}(\mathbf{x}[k] | \mathbf{z}^{k-1}) + \hat{E}(\tilde{\mathbf{x}}[k/k-1] | \mathbf{v}[k]) \\ &= \hat{\mathbf{x}}[k/k-1] + \underbrace{E(\tilde{\mathbf{x}}[k/k-1] \mathbf{v}^T[k]) \{E(\mathbf{v}[k] \mathbf{v}^T[k])\}^{-1}}_{\mathbf{K}[k]} \mathbf{v}[k]\end{aligned}$$

状态的一步预测误差： $\tilde{\mathbf{x}}[k/k-1] = \mathbf{x}[k] - \hat{E}[\mathbf{x}[k] | \mathbf{z}^{k-1}]$

$$= \mathbf{x}[k] - \hat{\mathbf{x}}[k/k-1]$$

观测的预测误差：

$$\mathbf{v}[k] = \mathbf{z}[k] - \hat{E}[\mathbf{z}[k] | \mathbf{z}^{k-1}] = \mathbf{z}[k] - \hat{\mathbf{z}}[k/k-1]$$

$$\hat{\mathbf{x}}[k/k] = \hat{\mathbf{x}}[k/k-1] + \mathbf{K}[k](\mathbf{z}[k] - \hat{\mathbf{z}}[k/k-1])$$



### 3. 算法推导

$$\begin{aligned}\hat{\mathbf{z}}[k / k - 1] &= \hat{E}[\mathbf{z}[k] | \mathbf{z}^{k-1}] \\ &= \hat{E}(\mathbf{H}[k]\mathbf{x}[k] + \mathbf{w}[k] | \mathbf{z}^{k-1}) \\ &= \mathbf{H}[k]\hat{E}(\mathbf{x}[k] | \mathbf{z}^{k-1}) + \hat{E}(\mathbf{w}[k] | \mathbf{z}^{k-1}) \\ &= \mathbf{H}[k]\hat{\mathbf{x}}[k / k - 1]\end{aligned}$$

本质是根据模型进行预测： $\mathbf{z}[k] = \mathbf{H}[k]\mathbf{x}[k] + \mathbf{w}[k]$

$$\begin{aligned}\mathbf{v}[k] &= \mathbf{z}[k] - \hat{\mathbf{z}}[k / k - 1] \\ &= \mathbf{H}[k]\mathbf{x}[k] + \mathbf{w}[k] - \mathbf{H}[k]\hat{\mathbf{x}}[k / k - 1] \\ &= \mathbf{H}[k]\tilde{\mathbf{x}}[k / k - 1] + \mathbf{w}[k]\end{aligned}$$

$$\mathbf{v}[k] = \mathbf{H}[k]\tilde{\mathbf{x}}[k / k - 1] + \mathbf{w}[k]$$



### 3. 算法推导

**K[k]的推导:**  $\mathbf{K}[k] = E(\tilde{\mathbf{x}}[k / k - 1] \mathbf{v}^T[k]) \{E(\mathbf{v}[k] \mathbf{v}^T[k])\}^{-1}$

$$\begin{aligned} E(\tilde{\mathbf{x}}[k / k - 1] \mathbf{v}^T[k]) &= E\{\tilde{\mathbf{x}}[k / k - 1](\mathbf{H}[k]\tilde{\mathbf{x}}[k / k - 1] + \mathbf{w}[k])^T\} \\ &= \mathbf{P}_{\tilde{\mathbf{x}}}[k / k - 1] \mathbf{H}^T[k] \end{aligned}$$

$$\begin{aligned} &E(\mathbf{v}[k] \mathbf{v}^T[k]) \\ &= E\{(\mathbf{H}[k]\tilde{\mathbf{x}}[k / k - 1] + \mathbf{w}[k])(\mathbf{H}[k]\tilde{\mathbf{x}}[k / k - 1] + \mathbf{w}[k])^T\} \\ &= \mathbf{H}[k] \mathbf{P}_{\tilde{\mathbf{x}}}[k / k - 1] \mathbf{H}^T[k] + \mathbf{R}[k] \end{aligned}$$

$$\mathbf{K}[k] = \mathbf{P}_{\tilde{\mathbf{x}}}[k / k - 1] \mathbf{H}^T[k] (\mathbf{H}[k] \mathbf{P}_{\tilde{\mathbf{x}}}[k / k - 1] \mathbf{H}^T[k] + \mathbf{R}[k])^{-1}$$



### 3. 算法推导

预测均方误差差阵  $\mathbf{P}_{\tilde{\mathbf{x}}}[k / k - 1]$  的推导:

$$\begin{aligned}\hat{\mathbf{x}}[k / k - 1] &= \hat{E}(\mathbf{x}[k] | \mathbf{z}^{k-1}) \\ &= \hat{E}(\Phi[k, k - 1]\mathbf{x}[k - 1] + \Gamma[k - 1]\mathbf{n}[k - 1] | \mathbf{z}^{k-1}) \\ &= \Phi[k, k - 1]\hat{E}(\mathbf{x}[k - 1] | \mathbf{z}^{k-1}) \\ &= \Phi[k, k - 1]\hat{\mathbf{x}}[k - 1 / k - 1]\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{x}}[k / k - 1] &= \mathbf{x}[k] - \hat{\mathbf{x}}[k / k - 1] \\ &= \Phi[k, k - 1]\mathbf{x}[k - 1] + \Gamma[k - 1]\mathbf{n}[k - 1] \\ &\quad - \Phi[k, k - 1]\hat{\mathbf{x}}[k - 1 / k - 1] \\ &= \Phi[k, k - 1]\tilde{\mathbf{x}}[k - 1 / k - 1] + \Gamma[k - 1]\mathbf{n}[k - 1]\end{aligned}$$

$$\begin{aligned}\mathbf{P}_{\tilde{\mathbf{x}}}[k / k - 1] &= \Phi[k, k - 1]\mathbf{P}_{\tilde{\mathbf{x}}}[k - 1 / k - 1]\Phi^T[k, k - 1] \\ &\quad + \Gamma[k - 1]\mathbf{Q}[k - 1]\Gamma^T[k - 1]\end{aligned}$$



### 3. 算法推导

滤波均方误差阵  $\mathbf{P}_{\tilde{\mathbf{x}}}[k/k]$  的推导:

$$\begin{aligned}\tilde{\mathbf{x}}[k/k] &= \mathbf{x}[k] - \hat{\mathbf{x}}[k/k] \\ &= \mathbf{x}[k] - \left\{ \hat{\mathbf{x}}[k/k-1] + \underbrace{\mathbf{K}[k](\mathbf{z}[k] - \hat{\mathbf{z}}[k/k-1])}_{\mathbf{H}[k]\tilde{\mathbf{x}}[k/k-1] + \mathbf{w}[k]} \right\} \\ &= \tilde{\mathbf{x}}[k/k-1] - \mathbf{K}[k](\mathbf{H}[k]\tilde{\mathbf{x}}[k/k-1] + \mathbf{w}[k]) \\ &= (\mathbf{I} - \mathbf{K}[k]\mathbf{H}[k])\tilde{\mathbf{x}}[k/k-1] - \mathbf{K}[k]\mathbf{w}[k]\end{aligned}$$





### 3. 算法推导

$$\tilde{\mathbf{x}}[k / k] = (\mathbf{I} - \mathbf{K}[k]\mathbf{H}[k])\tilde{\mathbf{x}}[k / k - 1] - \mathbf{K}[k]\mathbf{w}[k]$$

$$\begin{aligned}\mathbf{P}_{\tilde{\mathbf{x}}}[k / k] &= (\mathbf{I} - \mathbf{K}[k]\mathbf{H}[k])\mathbf{P}_{\tilde{\mathbf{x}}}[k / k - 1](\mathbf{I} - \mathbf{K}[k]\mathbf{H}[k])^T \\ &\quad + \mathbf{K}[k]\mathbf{R}[k]\mathbf{K}^T[k]\end{aligned}$$

$$\begin{aligned}&= (\mathbf{I} - \mathbf{K}[k]\mathbf{H}[k])\mathbf{P}_{\tilde{\mathbf{x}}}[k / k - 1] - (\mathbf{I} - \mathbf{K}[k]\mathbf{H}[k])\mathbf{P}_{\tilde{\mathbf{x}}}[k / k - 1]\mathbf{H}^T[k]\mathbf{K}^T[k] \\ &\quad + \mathbf{K}[k]\mathbf{R}[k]\mathbf{K}^T[k] \\ &= (\mathbf{I} - \mathbf{K}[k]\mathbf{H}[k])\mathbf{P}_{\tilde{\mathbf{x}}}[k / k - 1] - \mathbf{P}_{\tilde{\mathbf{x}}}[k / k - 1]\mathbf{H}^T[k]\mathbf{K}^T[k] \\ &\quad + \mathbf{K}[k]\mathbf{H}[k]\mathbf{P}_{\tilde{\mathbf{x}}}[k / k - 1]\mathbf{H}^T[k]\mathbf{K}^T[k] + \mathbf{K}[k]\mathbf{R}[k]\mathbf{K}^T[k]\end{aligned}$$



### 3. 算法推导

$$\mathbf{P}_{\tilde{x}}[k / k] = (\mathbf{I} - \mathbf{K}[k]\mathbf{H}[k])\mathbf{P}_{\tilde{x}}[k / k - 1] - \mathbf{P}_{\tilde{x}}[k / k - 1]\mathbf{H}^T[k]\mathbf{K}^T[k] \\ + \underbrace{\mathbf{K}[k]\mathbf{H}[k]\mathbf{P}_{\tilde{x}}[k / k - 1]\mathbf{H}^T[k]\mathbf{K}^T[k] + \mathbf{K}[k]\mathbf{R}[k]\mathbf{K}^T[k]}$$

$$\mathbf{K}[k]\left\{\mathbf{H}[k]\mathbf{P}_{\tilde{x}}[k / k - 1]\mathbf{H}^T[k] + \mathbf{R}[k]\right\}\mathbf{K}^T[k]$$

$$\rightarrow \mathbf{P}_{\tilde{x}}[k / k - 1]\mathbf{H}^T[k](\mathbf{H}[k]\mathbf{P}_{\tilde{x}}[k / k - 1]\mathbf{H}^T[k] + \mathbf{R}[k])^{-1}$$

$$\mathbf{P}_{\tilde{x}}[k / k] = (\mathbf{I} - \mathbf{K}[k]\mathbf{H}[k])\mathbf{P}_{\tilde{x}}[k / k - 1]$$



## 4. 算法总结

预测： $\hat{\mathbf{x}}[k / k - 1] = \mathbf{\Phi}[k, k - 1]\hat{\mathbf{x}}[k - 1 / k - 1]$

预测均方误差阵：

$$\mathbf{P}_{\tilde{x}}[k / k - 1] =$$

$$\mathbf{\Phi}[k, k - 1]\mathbf{P}_{\tilde{x}}[k - 1 / k - 1]\mathbf{\Phi}^T[k, k - 1] + \mathbf{\Gamma}[k - 1]\mathbf{Q}[k - 1]\mathbf{\Gamma}^T[k - 1]$$

增益： $\mathbf{K}[k] = \mathbf{P}_{\tilde{x}}[k / k - 1]\mathbf{H}^T[k](\mathbf{H}[k]\mathbf{P}_{\tilde{x}}[k / k - 1]\mathbf{H}^T[k] + \mathbf{R}[k])^{-1}$

滤波： $\hat{\mathbf{x}}[k / k] = \hat{\mathbf{x}}[k / k - 1] + \mathbf{K}[k](\mathbf{z}[k] - \mathbf{H}[k]\hat{\mathbf{x}}[k / k - 1])$

滤波误差方差阵： $\mathbf{P}_{\tilde{x}}[k / k] = (\mathbf{I} - \mathbf{K}[k]\mathbf{H}[k])\mathbf{P}_{\tilde{x}}[k / k - 1]$

起始条件： $\hat{\mathbf{x}}[k_0 / k_0] = \boldsymbol{\mu}_x[k_0] \quad \mathbf{P}_{\tilde{x}}[k_0 / k_0] = \mathbf{P}_x[k_0]$



## 几点说明:

(1) 关于预测, 预测是根据模型中确定性的部分来预测的, 而白噪声是不可预测的。

信号模型:  $\mathbf{x}[k] = \mathbf{\Phi}[k, k-1]\mathbf{x}[k-1] + \mathbf{\Gamma}[k-1]\mathbf{n}[k-1]$

一步预测:  $\hat{\mathbf{x}}[k / k-1] = \mathbf{\Phi}[k, k-1]\hat{\mathbf{x}}[k-1 / k-1]$

观测模型:  $\mathbf{z}[k] = \mathbf{H}[k]\mathbf{x}[k] + \mathbf{w}[k]$

观测预测:  $\hat{\mathbf{z}}[k / k-1] = \mathbf{H}[k]\hat{\mathbf{x}}[k / k-1]$



## 几点说明:

(2) 观测的预测误差称为新息 (Innovation)

$$\mathbf{v}[k] = \mathbf{z}[k] - \hat{\mathbf{z}}[k / k - 1] = \mathbf{H}[k] \tilde{\mathbf{x}}[k / k - 1] + \mathbf{w}[k]$$

新息的方差阵:

$$E(\mathbf{v}[k] \mathbf{v}^T[k]) = \underbrace{\mathbf{H}[k] \mathbf{P}_{\tilde{\mathbf{x}}}[k / k - 1] \mathbf{H}^T[k] + \mathbf{R}[k]}_{\mathbf{P}_v[k]}$$

(3) 滤波可以表示为预测加修正项

$$\hat{\mathbf{x}}[k / k] = \hat{\mathbf{x}}[k / k - 1] + \mathbf{K}[k] \mathbf{v}[k]$$

卡尔曼增益