Solutions to Exercises of Section IV.1.

1. We find $v(\{1,2\}) = -v(\{3\}) = 4.4$, $v(\{1,3\}) = -v(\{2\}) = 4$, $v(\{2.3\}) = -v(\{1\}) = 1.5$, and $v(\emptyset) = v(N) = 0$.

(I,II) \ III	1	2
1,1	−1	−3
1,2	4	5
2,1	−3	−2
2,2	6	2

(I,III) \ II	1	2
1,1	−1	−3
1,2	4	5
2,1	2	6
2,2	−2	−3

(II,III) \ I	1	2
1,1	2	1
1,2	−1	−12
2,1	−1	4
2,2	−10	1

2. $v(\emptyset) = 0$, $v(1) = 6/10$, $v(2) = 2$, $v(3) = 1$, $v(12) = 5$, $v(13) = 4$, $v(23) = 3$ and $v(123) = 16$.

I \ (II,III)	1, 1	1, 2	2, 1	2, 2
1	1	3	−1	3
2	−1	1	7	3

II \ (I,III)	1, 1	1, 2	2, 1	2, 2
1	2	0	2	0
2	6	2	5	2

III \ (I,II)	1, 1	1, 2	2, 1	2, 2
1	1	−3	4	4
2	1	1	3	1

(I,II) \ III	1	2
1,1	3	3
1,2	5	5
2,1	1	1
2,2	12	5

(I,III) \ II	1	2
1,1	2	−4
1,2	4	4
2,1	3	11
2,2	4	4

(II,III) \ I	1	2
1,1	3	6
1,2	1	3
2,1	3	9
2,2	3	3

3. (a) $v(\emptyset) = 0$, $v(1) = 2$, $v(2) = 2$ and $v(12) = 9$.

(b) Player 2's threat strategy and MM (safety level) strategy are the same, column 2. Player 1's threat strategy is row 1, while his MM strategy is row 2. His threat is not believable, whereas 2's threat is very believable. In addition, (row 2, col 1) is a PSE.

(c) $\sigma = 9$ and $\Delta = \begin{pmatrix} -2 & 3 \\ -2 & 1 \end{pmatrix}$, so $\delta = -2$. The TU-value is $\phi = ((9-2)/2, (7-2)/2) = (7/2, 11/2)$. Player 1 gets 5, but has to make a side payment of $3/2$ to Player 2.

(d) The strategy spaces are taken to be $X_1 = \{\{1\},\{1,2\}\}$ and $X_2 = \{\{2\},\{1,2\}\}$. The bimatrix therefore is 2 by 2:

$$\begin{array}{c c} & \begin{array}{cc} \{2\} & \{1,2\} \end{array} \\ \begin{array}{c} \{1\} \\ \{1,2\} \end{array} & \begin{pmatrix} (2,2) & (2,2) \\ (2,2) & (9/2,9/2) \end{pmatrix} \end{array}$$

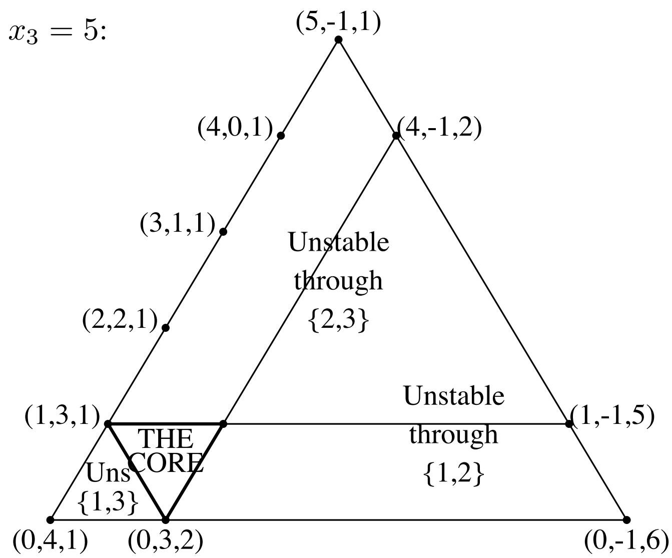
4. $N = \{1,2,3\}$. $v(\emptyset) = 0$. $v(\{1\}) = 1$, because Player 2 can always choose j within 1 of i. Similarly, $v(\{2\}) = 1$ because Player 1 can choose $i = 5$ say, and then Player 3 can choose k within 1 of j. $v(\{3\}) = 4$ is achieved by choosing $i = 5$ and $j = 10$, say. Similarly, $v(\{1,2\}) = 10$, $v(\{1,3\}) = 10$, $v(\{2,3\}) = 14$, and $v(N) = 18$.

Solutions to Exercises of Section IV.2.

1. A constant-sum game has $v(S) + v(N - S) = v(N)$ for all coalitions S. Therefore, a two-person constant-sum game has $v(\{1\}) + v(\{2\}) = v(\{1, 2\})$ and so is inessential.

2. We have $v(\{1\}) = \mathrm{Val}\begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} = 8/5$, $v(\{2\}) = \mathrm{Val}\begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix} = 8/5$, and $v(\{1, 2\}) = 6$. The set of imputations is $\{(x_1, x_2) : x_1 + x_2 = 6, x_1 \geq 8/5, x_2 \geq 8/5\}$, the line segment from $(8/5, 22/5)$ to $(22/5, 8/5)$. The core is the same set. In fact, for all 2-person games, the core is always the whole set of imputations.

3. On the plane $x_1 + x_2 + x_3 = 5$:



4. (a) The set of imputations is $\{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 3, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$. The core is

$$\{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 3,\ x_1 + x_2 \geq a,\ x_1 + x_3 \geq a,\ x_2 + x_3 \geq a,\ \ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\}.$$

For values of $a \leq 2$, the point $(1, 1, 1)$ is in the core. For values of $a > 2$, the core is empty, since summing the corresponding sides of the inequalities $x_1 + x_2 \geq a$, $x_1 + x_3 \geq a$, and $x_2 + x_3 \geq a$, gives $6 = 2(x_1 + x_2 + x_3) \geq 3a > 6$, a contradiction.

(b) The core consists of points (x_1, x_2, x_3, x_4) such that

$$\begin{array}{lllll}
x_1 \geq 0 & x_1 + x_2 \geq a & x_2 + x_3 \geq a & x_1 + x_2 + x_3 \geq b & x_1 + x_2 + x_3 + x_4 = 4. \\
x_2 \geq 0 & x_1 + x_3 \geq a & x_2 + x_4 \geq a & x_1 + x_2 + x_4 \geq b & \\
x_3 \geq 0 & x_1 + x_4 \geq a & x_3 + x_4 \geq a & x_1 + x_3 + x_4 \geq b & \\
x_4 \geq 0 & & & x_2 + x_3 + x_4 \geq b &
\end{array}$$

If $a \leq 2$ and $b \leq 3$, then $(1, 1, 1, 1)$ is in the core.
If $a > 2$, summing the inequalities involving a gives $12 = 3(x_1 + x_2 + x_3 + x_4) \geq 6a > 12$,

so the core is empty.
If $b > 3$, summing the inequalities involving b gives $12 = 3(x_1 + x_2 + x_3 + x_4) \geq 4b > 12$, so the core is empty.
Thus the core is non-empty if and only if $a \leq 2$ and $b \leq 3$. If v is superadditive, then a cannot be greater than two, so the condition reduces to $b \leq 3$.

(c) The core is nonempty if and only if $f(k)/k \leq f(n)/n$ for all $k = 1, \ldots, n$. To see this, note that if $f(k)/k \leq f(n)/n$ for all $k = 1, \ldots, n$, then $\boldsymbol{x} = (f(n)/n, \ldots, f(n)/n)$ is in the core since for any coalition S of size $|S| = k$, we have $\sum_{i \in S} x_i = kf(n)/n \leq f(k) = v(S)$. On the other hand, suppose that $f(k)/k > f(n)/n$ for some k. Then for any imputation $\boldsymbol{x}$, the coalition S consisting of those players with the k smallest x_i satisfies $(1/k)\sum_{i \in S} x_i \leq (1/n)\sum_{i \in N} x_i = f(n)/n < f(k)/k$. This means that $\boldsymbol{x}$ is unstable through S, and so cannot be in the core.

5. Suppose $\boldsymbol{x} = (x_1, \ldots, x_n)$ is in the core. For $\boldsymbol{x}$ to be stable, we must have $\sum_{j \neq i} x_j \geq v(N - \{i\})$ for all i. This is equivalent to $x_i \leq \delta_i$ for all i. However, summing over i gives $v(N) = \sum_1^n x_i \leq \sum_1^n \delta_i < v(N)$, a contradiction. Thus, the core must be empty.

6. Suppose Player 1 is a dummy. If $\boldsymbol{x} = (x_1, \ldots, x_n)$ is in the core, it is an imputation, so $x_1 \geq v(\{1\}) = 0$. Suppose $x_1 > 0$. Then, $v(N - \{x_1\}) = v(N) = \sum_1^n x_i > \sum_2^n x_i$. So $\boldsymbol{x}$ is unstable through $N - \{x_1\}$ and cannot be in the core.

7.(a) If Players 1 and 2 are in P, and Players 3 and 4 are in Q, then $v(i) = 0$ for all i, $v(ij) = 1$ for all ij except $ij = 12$ and $ij = 34$, $v(ijk) = 1$ and $v(1234) = 2$. The core, where all coalitions are satisfied, is

$$\begin{aligned} C = \{(x_1, x_2, x_3, x_4) :\, & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_1 + x_2 + x_3 + x_4 = 2, \\ & x_1 + x_3 \geq 1, x_1 + x_4 \geq 1, x_2 + x_3 \geq 1, x_2 + x_4 \geq 1\} \end{aligned}$$

Since $x_1 + x_3 \geq 1$, $x_2 + x_4 \geq 1$ and $x_1 + x_2 + x_3 + x_4 = 2$, we must have $x_1 + x_3 = 1$, $x_2 + x_4 = 1$. Similarly, we have $x_1 + x_4 = 1$, $x_2 + x_3 = 1$. The core is therefore $C = \{(x_1, x_1, 1 - x_1, 1 - x_1) : 0 \leq x_1 \leq 1$. This is the line segment joining the points $(0, 0, 1, 1)$ and $(1, 1, 0, 0)$.

(b) If Players 1 and 2 are in P, and Players 3, 4 and 5 are in Q, then the core is

$$\begin{aligned} C = \{(x_1, x_2, x_3, x_4, x_5) :\, & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, \\ & x_1 + x_3 \geq 1, x_1 + x_4 \geq 1, x_1 + x_5 \geq 1, x_2 + x_3 \geq 1, x_2 + x_4 \geq 1, \\ & x_2 + x_5 \geq 1, x_1 + x_2 + x_3 + x_4 \geq 2, x_1 + x_2 + x_3 + x_5 \geq 2, \\ & x_1 + x_2 + x_4 + x_5 \geq 2, x_1 + x_2 + x_3 + x_4 + x_5 = 2.\} \end{aligned}$$

These inequalities imply that $x_3 = x_4 = x_5 = 0$, and therefore that $x_1 = x_2 = 1$. The core consists of the single point, $C = \{(1, 1, 0, 0, 0)\}$.

(c) If $|P| < |Q|$, the core is the imputation, $\boldsymbol{x}_P$, with $x_i = 1$ for $i \in P$ and $x_i = 0$ for $i \in Q$. If $|P| > |Q|$, the core is the imputation, $\boldsymbol{x}_Q$, with $x_i = 1$ for $i \in Q$ and $x_i = 0$ for $i \in P$. If $|P| = |Q|$, the core is the line segment joining $\boldsymbol{x}_P$ and $\boldsymbol{x}_Q$.

8. In the core, we have $x_i + x_k \geq 1$ for $i = 1, 2$ and $k = 3, 4, 5$. Also we have $x_i + x_j + x_k \geq 2$ for $i = 1, 2$ and $j, k = 3, 4, 5$. Finally, we have $x_1 + x_2 + x_3 + x_4 + x_5 = 3$.

Since, $x_1 + x_3 \geq 1$ and $x_2 + x_4 + x_5 \geq 2$, and $x_1 + x_2 + x_3 + x_4 + x_5 = 3$, we must have equality: $x_1 + x_3 = 1$ and $x_2 + x_4 + x_5 = 2$. Similarly, $x_1 + x_3 + x_4 = 2$, which with $x_1 + x_3 = 1$ implies $x_4 = 1$, etc. The core consists of the single point, $(0, 0, 1, 1, 1)$.

Solutions to Exercises of Section IV.3.

1. No player can get anything acting alone, so $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$. Players 2 and 3 can do nothing together, $v(\{2,3\}) = 0$, but $v(\{1,2\}) = 30$ and $v(\{1,3\}) = 40$. The object is also worth 40 to the grand coalition, $v(\{1,2,3\}) = 40$. (Player 3 will take the object and replace it by \$40, and the players must now decide how to split this money.) To find the Shapley value using Equation (4), we first find $c_\emptyset = c_{\{1\}} = c_{\{2\}} = c_{\{3\}} = c_{\{2,3\}} = 0$, and $c_{\{1,2\}} = 30$, $c_{\{1,3\}} = 40$ and $c_{\{1,2,3\}} = v(\{1,2,3\})-$ the sum of the above, so $c_{\{1,2,3\}} = 40 - 40 - 30 = -30$. Thus we have $v = 30w_{\{1,2\}} + 40w_{\{1,3\}} - 30w_{\{1,2,3\}}$, and consequently, $\phi(v) = 30\phi(w_{\{1,2\}}) + 40\phi(w_{\{1,3\}}) - 30\phi(w_{\{1,2,3\}})$.

From this we may compute

$$\begin{aligned}
\phi_1(v) &= 30/2 + 40/2 - 30/3 = 25\\
\phi_2(v) &= 30/2 + 0 - 30/3 = 5\\
\phi_3(v) &= 0 + 40/2 - 30/3 = 10.
\end{aligned}$$

So 3 gets the painting for \$30, of which \$25 goes to 1 and \$5 to 2. The core is

$$\begin{aligned}
C = \{(x_1,x_2,x_3) : x_1 \ge 30, x_2 \ge 0, x_3 \ge 0, x_1 + x_2 \ge 30,&\\
x_1 + x_3 \ge 40, x_2 + x_3 \ge 0, x_1 + x_2 + x_3 = 40\}&\\
= \{(x_1,x_2,x_3) : x_2 = 0, 30 \le x_1 \le 40, x_3 = 40 - x_1\}.&
\end{aligned}$$

The core gives player 2 nothing, while the Shapley value gives him 5. Without 2 present, 1 and 3 would probably agree on a price of 20. With 2 present, 1 is in a better bargaining position as he can play 3 off against 2.

2. We compute the Shapley value using Theorem 2. $\phi_1(v) = (1/3)\cdot 1 + (1/6)\cdot 2 + (1/6)\cdot 3 + (1/3)\cdot 3 = 13/6$. $\phi_2(v) = (1/3)\cdot 0 + (1/6)\cdot 1 + (1/6)\cdot 7 + (1/3)\cdot 7 = 22/6 = 11/3$. $\phi_3(v) = (1/3)\cdot(-4) + (1/6)\cdot(-2) + (1/6)\cdot 3 + (1/3)\cdot 4 = 1/6$.

3. $\sum_{j\in N}\phi_j(v) = v(N)$ from Axiom 1. We must show $\phi_i(v) \ge v(\{i\})$ for all $i \in N$. Since v is superadditive, $v(\{i\}) + v(S - \{i\}) \le v(S)$ for all S containing i. But since $\phi_i(v)$ is an average of numbers, $v(S) - v(S - \{i\})$, each of which is at least $v(\{i\})$, $\phi_i(v)$ itself must be at least $v(\{i\})$.

4.(a) By the symmetry axiom, $\phi_2(v) = \phi_3(v) = \cdots = \phi_n(v)$. If $1 \in S$, then $v(S) - v(S - \{i\}) = \begin{cases} 1 & \text{if } i \in S\\ 0 & \text{if } i \notin S\end{cases}$. Therefore $\phi_2(v)$ is just the probability that 1 comes before 2 in the random ordering of the players. This probability is 1/2 by symmetry. So $\phi_2(v) = \phi_3(v) = \cdots = \phi_n(v) = 1/2$, and $\phi_1(v) = n - \phi_2(v) - \phi_3(v) - \cdots - \phi_n(v) = (n+1)/2$.

One may also use Equation (4) to compute the Shapley value. First show $c_{\{1\}} = c_{\{1,j\}} = 1$ for all $j = 2,\ldots,n$, and all other $c_S = 0$. Hence, $v = w_{\{1\}} + \sum_{j=2}^n w_{\{1,j\}}$, from which follows $\phi_1(v) = 1 + (n-1)/2 = (n+1)/2$ and $\phi_j(v) = 1/2$ for $j = 2,\ldots,n$.

(b) By the symmetry axiom, $\phi_1(v) = \phi_2(v)$ and $\phi_3(v) = \cdots = \phi_n(v)$. This time $\phi_3(v)$ is the probability that 1 or 2 is chosen before 3 in the random ordering. This is 1 minus the probability that 3 is chosen before 1 and 2, namely, $1 - (1/3) = 2/3$. So $\phi_3(v) = \cdots = \phi_n(v) = 2/3$ and $\phi_1(v) = \phi_2(v) = (1/2)(n - (n-2)(2/3)) = (n+4)/6$.

(c) By the symmetry axiom, $\phi_1 = \phi_2$ and $\phi_3 = \cdots = \phi_n$. If $3 \in S$, then $v(S) - v(S - \{3\})$ is 1 if 1 and 2 are in S and 0 otherwise. This implies that ϕ_3 is just the probability that 3 enters after both 1 and 2. Since each of 1, 2 and 3 have the same probability of entering after the other two, $\phi_3 = 1/3$. Then since $2\phi_1 + (n-2)\phi_3 = n$, we have $\phi_1 = \phi_2 = (n+1)/3$, and $\phi_3 = \cdots = \phi_n = 1/3$.

5 The answer is no. Here is a counterexample with $n = 4$ players. The minimal winning coalitions are $\{1,2\}$ and $\{3,4\}$. To be a weighted voting game, with weights w_i and quota q, we must have $w_1 + w_2 > q$ and $w_3 + w_4 > q$, so that $w_1 + w_2 + w_3 + w_4 > 2q$. On the other hand, $\{1,3\}$ and $\{2,4\}$ are losing coalitions so that $w_1 + w_3 \le q$ and $w_2 + w_4 \le q$. This gives $q_1 + q_2 + q_3 + q_4 \le 2q$, a contradiction.

6. The winning coalitions are $\{2,3\}$, $\{2,4\}$, $\{3,4\}$ and all supersets. It appears that 1 is a dummy, so $\phi_1(v) = 0$. Also, $\phi_2(v) = \phi_3(v) = \phi_4(v)$ from symmetry. Since the sum of the $\phi_i(v)$ must be 1, we have $\phi(v) = (0, 1/3, 1/3, 1/3)$.

7. Player 2 can be in only two winning coalitions that would be losing without him, namely, $S = \{1,2\}$ and $S = \{2,3,\ldots,n\}$. Hence,

$$\phi_2(v) = \frac{(2-1)!\,(n-2)!}{n!} + \frac{(n-2)!\,(n-(n-1))!}{n!} = \frac{2(n-2)!}{n!} = \frac{2}{n(n-1)}.$$

By symmetry,

$$\phi_3(v) = \cdots = \phi_n(v) = \frac{2}{n(n-1)}$$

and

$$\phi_1(v) = 1 - \phi_2(v) - \cdots - \phi_n(v) = 1 - (n-1)\frac{2}{n(n-1)} = \frac{n-2}{n}.$$

8. Let 1,2,3,4 denote the stockholders and let c denote the chairman of the board. The coalitions winning with 1 but losing without 1 are $\{1,4,c\}$ and $\{1,2,3\}$. So

$$\phi_1(v) = \frac{2!\,2!}{5!} + \frac{2!\,2!}{5!} = \frac{1}{30} + \frac{1}{30} = \frac{2}{30}.$$

The corresponding coalitions for 2 are $\{2,3,c\}$, $\{2,4\}$, $\{1,2,3\}$, $\{1,2,3,c\}$, $\{2,4,c\}$, and $\{1,2,4\}$. So $\phi_2(v) = 4(1/30) + 2(1/20) = 7/30$. Similarly, $\phi_3(v) = 7/30$, $\phi_4(v) = 12/30$, and $\phi_c(v) = 2/30$. The Shapley value is $(2/30, 7/30, 7/30, 12/30, 2/30)$, or, in terms of percentage power $(6.7\%, 23.3\%, 23.3\%, 40\%, 6.7\%)$.

9. (a) Let 1 denote the large party and 2, 3, 4 denote the smaller parties. Then the only winning coalitions that become losing without 2 are $S = \{1,2\}$ and $S = \{2,3,4\}$. Hence,

$$\phi_2(v) = \frac{1!\,2!}{4!} + \frac{2!\,1!}{4!} = \frac{1}{6}.$$

By symmetry, $\phi_3(v) = \phi_4(v) = 1/6$ also, and hence $\phi_1(v) = 1/2$. The large party has half the power.

(b) Let 1, 2 denote the large parties and 3, 4, 5 denote the smaller ones. The only coalitions winning with 3 and losing without 3 are $\{1,3,4\}$, $\{1,3,5\}$, $\{2,3,4\}$, and $\{2,3,5\}$. Hence,

$$\phi_3(v) = 4\frac{2!\,2!}{5!} = \frac{2}{15}.$$

By symmetry, $\phi_4(v) = \phi_5(v) = 2/15$, and hence $\phi_1(v) = \phi_2(v) = 3/10$. The two large coalitions are less powerful than their size indicates.

10. There are three coalitions that are winning with 6 but losing without 6: $\{1,2,6\}$, $\{1,3,5,6\}$, $\{2,3,5,6\}$. Hence,

$$\phi_6(v) = \frac{2!\,3!}{6!} + 2\frac{3!\,2!}{6!} = \frac{3}{60}.$$

By symmetry, $\phi_5(v) = 3/60$. There are 7 coalitions winning with 4 but losing without 4: $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, $\{1,3,4,5\}$, $\{1,3,4,6\}$, $\{2,3,4,5\}$, $\{2,3,4,6\}$. Hence, $\phi_4(v) = 7/60$. There are 11 coalitions winning with 3 but losing without 3: $\{1,2,3\}$, $\{1,3,4\}$, $\{2,3,4\}$, $\{1,3,4,5\}$, $\{1,3,4,6\}$, $\{1,3,5,6\}$, $\{2,3,4,5\}$, $\{2,3,4,6\}$, $\{2,3,5,6\}$, $\{1,3,4,5,6\}$, $\{2,3,4,5,6\}$. Hence,

$$\phi_3(v) = 3\frac{2!\,3!}{6!} + 6\frac{3!\,2!}{6!} + 2\frac{4!\,1!}{6!} = \frac{13}{60}.$$

Since $\phi_1(v) = \phi_2(v)$ by symmetry, we find that $\phi_1(v) = \phi_2(v) = 17/60$. The Shapley value is $(17/60, 17/60, 13/60, 7/60, 3/60, 3/60)$. In terms of percentage power, this is $(28.3\%, 28.3\%, 21.7\%, 11.7\%, 5\%, 5\%)$, which is much closer to the original intention found in Table 1.

11. Let $\phi_A(v)$ denote the Shapley value for A, of one of the big five, and let $\phi_a(v)$ denote the Shapley value of a, one of the smaller members. We must have $5\phi_A(v) + 10\phi_a(v) = 1$. Let us find $\phi_a(v)$. The only losing coalitions that become winning, when a is added to it, are the coalitions consisting of the big five and three of the other smaller nations. Thus in the random ordering, a must come in ninth and find all members of the big five there already. The number of such coalitions is the number of ways of choosing the three smaller nation members out of the remaining nine, namely $\binom{9}{3} = 9!/3!\,6!$. Thus

$$\phi_a(v) = \frac{8!\,6!}{15!} \cdot \frac{9!}{3!\,6!} = \frac{4}{15 \cdot 13 \cdot 11} = .001865\cdots.$$

Thus the 10 smaller nations have only 1.865% of the power, and each of the big five nations has 19.627% of the power.

12. The trip to A and return costs 14, so the value of A acting alone is $v(A) = 20 - 14 = 6$. Similarly, $v(B) = 20 - 16 = 4$, and $v(C) = 20 - 12 = 8$. If A and B combine forces, the travel cost is 17, the cost of the trip to A then to B and return. So, $v(AB) = 40 - 17 = 23$. Similarly, $v(AC) = 40 - 17 = 23$ and $v(BC) = 40 - 18 = 22$. If all three cities combine, the least travel cost is obtained using the route from H to A to B to C and return (or the reverse), for a total cost of 19. So $v(ABC) = 60 - 19 = 41$. From these, we may compute the Shapley value as

$$\begin{aligned}
\phi_A(v) &= \frac{1}{3}\cdot 6 + \frac{1}{6}\cdot 19 + \frac{1}{6}\cdot 15 + \frac{1}{3}\cdot 19 = 14 \\
\phi_B(v) &= \frac{1}{3}\cdot 4 + \frac{1}{6}\cdot 17 + \frac{1}{6}\cdot 14 + \frac{1}{3}\cdot 18 = 12.5 \\
\phi_C(v) &= \frac{1}{3}\cdot 8 + \frac{1}{6}\cdot 17 + \frac{1}{6}\cdot 18 + \frac{1}{3}\cdot 18 = 14.5
\end{aligned}$$

Thus, we require A to pay $20-14 = 6$, B to pay $20-12.5 = 7.5$, and C to pay $20-14.5 = 5.5$ for a total of 19.

13. Consider a permutation of the n players, $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, and let π' denote the reverse permutation, $\pi' = (\pi_n, \ldots, \pi_2, \pi_1)$. Consider player i and let z_i denote the sum of the contributions player i makes to the coalitions when he enters for these two permutations. Below, it is shown that $z_i = x_i$ if $i \in B$ and $z_i = y_i$ if $i \in C$. If so, then the Shapley value for player i in this game is $x_i/2$ if $i \in B$ and $y_i/2$ if $i \in C$.

Let S_0 denote the coalition player i finds upon entering, when the players enter in the order given by π, and let $S = S_0 \cup \{i\}$. Then if the players enter in the reverse order π', player i will find coalition $\bar{S}$ (the complement of S) there when he enters and he increases it to $\bar{S}_0$. The amount player i contributes to the grand coalition is $v(S) - v(S_0)$ when entering in the order given by π and $v(\bar{S}_0) - v(\bar{S})$ when entering in the order given by π. The sum is therefore

$$z_i = v(S) - v(S_0) + v(\bar{S}_0) - v(\bar{S}) = [v(S) - v(\bar{S})] - [v(\bar{S}_0) - v(S_0)]$$

Letting T denote the common value $T = \sum_B x_j = \sum_C y_j = v(N)$, we find

$$\begin{aligned}
v(S) - v(\bar{S}) &= \min\{\sum_{j\in S\cap B} x_j, \sum_{k\in S\cap C} y_k\} - \min\{\sum_{j\in \bar{S}\cap B} x_j, \sum_{k\in \bar{S}\cap C} y_k\} \\
&= \min\{\sum_{j\in S\cap B} x_j, \sum_{k\in S\cap C} y_k\} - \min\{T - \sum_{j\in S\cap B} x_j, T - \sum_{k\in S\cap C} y_k\} \\
&= \min\{\sum_{j\in S\cap B} x_j, \sum_{k\in S\cap C} y_k\} - T + \max\{\sum_{j\in S\cap B} x_j, \sum_{k\in S\cap C} y_k\} \\
&= \sum_{j\in S\cap B} x_j + \sum_{k\in S\cap C} y_k - T.
\end{aligned}$$

Similarly,

$$v(\bar{S}_0) - v(S_0) = \sum_{j \in \bar{S}_0 \cap B} x_j + \sum_{k \in \bar{S}_0 \cap C} y_k - T.$$

Since S plus $\bar{S}_0$ is the set of all players but including player i twice, the sum of the two previous displays is equal to $T + x_i + T - 2T = x_i$ if $i \in B$, and $T + T + y_i - 2T = y_i$ if $i \in C$.

14. (a) We compute the Shapley value, ϕ, by the method of Theorem 1. We have $c_\emptyset = 0$. For singleton coalitions, we have $c_{\{1\}} = 1$ and $c_{\{j\}} = 0$ for $j \neq 1$. For coalitions of two players, we have $c_{\{1,2\}} = 2 - 1 = 1$ and $c_{\{i,j\}} = 0$ for all other $i < j$. Continuing in the same way we find

$$c_{\{12\cdots k\}} = 1 \quad \text{for } k = 1, \ldots, n, \quad \text{and} \quad c_S = 0 \quad \text{for all other coalitions, } S.$$

We can check this by checking that $v(T) = \sum_S c_S w_S(T) = \sum_{k=1}^n w_{\{1,2,\ldots,k\}}(T)$. From this, we may conclude that $\phi_i(v) = \sum_{S \text{ containing } i} c_S/|S|$, or

$$\begin{aligned}
\phi_1(v) &= 1 + (1/2) + (1/3) + \cdots + (1/n) \\
\phi_2(v) &= \phantom{1 + {}} (1/2) + (1/3) + \cdots + (1/n) \\
\phi_3(v) &= \phantom{1 + (1/2) + {}} (1/3) + \cdots + (1/n) \\
&\cdots \\
\phi_n(v) &= \phantom{1 + (1/2) + (1/3) + \cdots + {}} (1/n)
\end{aligned}$$

(b) Similarly, $c_{\{12\cdots k\}} = a_k - a_{k-1}$ for $k = 1, \ldots, n$, and $c_S = 0$ for all other coalitions, S, where $a_0 = 0$. Then, $\phi_i(v) = \sum_i^n (a_j - a_{j-1})/j$ for $i = 1, \ldots, n$.

15.(a) Let S be an arbitrary coalition and let $m = \max\{i : i \in S\}$. Then,

$$v_k(S) = \begin{cases} -(c_k - c_{k-1}) & \text{if } k \leq m \\ 0 & \text{if } k > m \end{cases}$$

So $\sum_{k=1}^n v_k(S) = \sum_{k=1}^m -(c_k - c_{k-1}) = -c_m = v(S)$.

(b) Since the Shapley value is additive, $\phi_i(v) = \sum_{k=1}^n \phi_i(v_k)$. To compute $\phi_i(v_k)$, note that $\phi_i(v_k) = 0$ if $i < k$ and $\phi_i(v_k) = -(c_k - c_{k-1})P(i,k)$ for $i \geq k$, where $P(i,k)$ represents the probability that in a random ordering of the players into the grand coalition, player i is the first member of S_k to appear. $P(i,k)$ is just the probability that i is first in a random ordering of memberw of S_k, and so $P(i,k) = 1/(n-k+1)$, since there are $n - k + 1$ players in S_k. Therefore,

$$\phi_i(v) = \sum_{k=1}^{i} \frac{-(c_k - c_{k-1})}{n - k + 1}.$$

Thus, player 1 pays c_1/n, player 2 pays $c_1/n + (c_2 - c_1)/(n-1)$, etc. Since all n players use the first part of the airfield, each player pays c_1/n for this. Since players 2 through n use the second part of the airfield, they each pay $(c_2 - c_1)/(n-1)$, and so on.

16. The characteristic function is $v(S) = \begin{cases} 0 & \text{if } 0 \notin S \text{ or if } S = \{0\} \\ a_{k(S)} & \text{otherwise} \end{cases}$, where $k(S) = \min\{i : i \in S - \{0\}\}$. For $i \neq 0$,

$$\phi_i(v) = \sum_{S \in \mathcal{S}_i} \frac{|S|-1)!(m+1-|S|)!}{(m+1)!}(a_i - a_{k(S-\{i\})})$$

where $\mathcal{S}_i = \{S \subset N : 0 \in S, i \in S, k(S) = i\}$. This is because $v(S) - v(S - \{i\}) = 0$ unless $S \in \mathcal{S}$. Then,

$$\phi_i(v) = a_i \left[\sum_{S \in \mathcal{S}_i} \frac{|S|-1)!(m+1-|S|)!}{(m+1)!}\right] + \sum_{j=i+1}^{m} a_j \left[\sum_{S \in \mathcal{S}_{i,j}} \frac{|S|-1)!(m+1-|S|)!}{(m+1)!}\right]$$

where $\mathcal{S}_{i,j} = \{S \subset N : 0 \in S, i \in S, k(S - \{i\}) = j\}$.

The coefficient of a_i is just the probability that in a random ordering of all $m+1$ players, i enters after 0 but before $1, \ldots, i-1$. This is the same as the probability that in a random ordering of $0, 1, \ldots, i$, 0 enters first and i second, namely $1/((i+1)i)$.

The coefficient of a_j is just the probability that in a random ordering of all $m+1$ players, i enters after 0 and j but before $1, \ldots, j-1$. This is the same as the probability that in a random ordering of $0, 1, \ldots, j$, i enters third after 0 and j in some order, namely $2/((j+1)j(j-1))$. This gives

$$\phi_i(v) = \frac{a_i}{(i+1)i} - \sum_{j=i+1}^{m} \frac{2a_j}{(j+1)j(j-1)}.$$

Similarly,

$$\begin{aligned} \phi_0(v) &= \sum_{j=1}^{m} a_j P(0 \text{ enters after } j \text{ but before } 1, \ldots, j-1) \\ &= \sum_{j=1}^{m} \frac{a_j}{(j+1)j}. \end{aligned}$$

17.(a) If $v(N - \{i\}) = 1$ and if $\boldsymbol{x}$ is in the core, then $1 = v(N - \{i\}) \leq \sum_1^n x_j - x_i = 1 - x_i$, so $x_i = 0$. Thus, any player without veto power gets zero at every core point. If there are no veto players, then there can be no core points since we must have $\sum_1^n x_i = 1$.

(b) If i is a veto player, then $\boldsymbol{x} = \boldsymbol{e}_i$, the ith unit vector, is a core point, since if S is a winning coalition, then $i \in S$ and $\sum_{i \in S} x_i = 1 \geq v(S) = 1$, and if S is losing, then certainly $\sum_{i \in S} \geq v(T) = 0$.

(c) The core is the set of all vectors, $\boldsymbol{x}$, such that $\sum_1^n x_i = 1$, $x_i \geq 0$ for all $i \in N$, and $x_i = 0$ if i is not a veto player.

Solutions to Exercises of Section IV.4

1. The core is the set of imputations, $\boldsymbol{x}$, such that the excesses, $e(\boldsymbol{x}, S)$, are negative or zero for all coalitions, S. The nucleolus is an imputation that minimizes the largest of the excesses. If the core is not empty, there is an imputation, $\boldsymbol{x}$, with $e(\boldsymbol{x}, S) \le 0$ for all S. Therefore the nucleolus also satisfies $e(\boldsymbol{x}, S) \le 0$ for all S and so is in the core.

2. A constant-sum game satisfies $v(S) + v(S^c) = v(N)$ for all coalitions, S. The Shapley value for Player 1 in a three person game is

$$
\begin{aligned}
\phi_1 &= \frac{1}{3}v(\{1\}) + \frac{1}{6}[v(\{1,2\}) - v(\{2\})] + \frac{1}{6}[v(\{1,3\}) - v(\{3\})] + \frac{1}{3}[v(N) - v(\{2,3\})] \\
&= \frac{1}{3}v(\{1\}) + \frac{1}{6}[v(\{N\}) - v(\{3\}) - v(\{2\})] + \frac{1}{6}[v(\{N\}) - v(\{2\}) - v(\{3\})] + \frac{1}{3}v(\{1\}) \\
&= \frac{1}{3}[v(N) + 2v(\{1\}) - v(\{2\}) - v(\{3\})]
\end{aligned}
$$

and similarly for the other two players. The excess, $e(\boldsymbol{x}, \{1\}) = v(\{1\}) - x_1$, is the negative of the excess, $e(\boldsymbol{x}, \{2,3\}) = v(\{2,3\}) - x_2 - x_3 = v(N) - v(\{1\}) - x_1 - x_2 - x_3 + x_1 = -v(\{1\}) + x_1$, since $x_1 + x_2 + x_3 = v(N)$. Since $x_i \ge v(\{i\})$ for any imputation, the maximum excess is $\max\{x_1 - v(\{1\}), x_2 - v(\{2\}), x_3 - v(\{3\})\}$. This can be made a minimum by making all three terms equal: $x_1 - v(\{1\}) = x_2 - v(\{2\}) = x_3 - v(\{3\})$ which, together with $x_1 + x_2 + x_3 = v(N)$, determines the x_i to be the same as for the Shapley value.

3. (a) The core is the set of vectors (x_1, x_2, x_3) of non-negative numbers satisfying $x_1 + x_2 + x_3 = 1200$, $x_1 + x_2 \ge 1200$, $x_1 + x_3 \ge 1200$, and $x_2 + x_3 \ge 0$. If non-negative numbers satisfy $x_1 + x_2 + x_3 = 1200$ and $x_1 + x_2 \ge 1200$, we must have $x_3 = 0$. Similarly, we must have $x_2 = 0$. Therefore $x_1 = 1200$. The core consists of the single point $(1200, 0, 0)$. Since the nucleolus is in the core and the core consists of one point, that point must be the nucleolus.

(b) If the players enter the grand coalition in a random order, Player B can win only if Player A enters first and B second. This happens with probability $1/6$. The amount won is 1200. So $\phi_B = (1/6)1200 = 200$. Similarly, $\phi_C = 200$, and then $\phi_A = 1200 - 200 - 200 = 800$.

(c) The Shapley value seems more reasonable to me. There is a danger that B and C will combine to demand say 1000, (500 each), so some payment to one or the other or both seems reasonable. It does not seem reasonable that A can play B and C against each other to be able to pay practically nothing.

4. The core consists of points $(x, 0, 40 - x)$ for $30 \le x \le 40$. We might try $(30, 0, 10)$ as a guess at the nucleolus. In the table below, we see the maximum excess is zero. The excess for either of the coalitions $\{2\}$ and $\{2,3\}$ cannot be made smaller without making

the other larger, so $x_2 = 0$. The excess for $\{1,3\}$ can be made smaller by increasing x_1. This increases the excess for $\{3\}$. These are equal for $x_1 = 35$. This gives $(35, 0, 5)$ as the nucleolus.

Coalition	Excess	$(30, 0, 10)$	$(35, 0, 5)$
$\{1\}$	$-x_1$	-30	-35
$\{2\}$	$-x_2$	0	0
$\{3\}$	$-x_3$	-10	-5
$\{1,2\}$	$30 - x_1 - x_2$	0	-5
$\{1,3\}$	$40 - x_1 - x_3$	0	0
$\{2,3\}$	$-x_2 - x_3$	-10	-5

The Shapley value is $(25, 5, 10)$. Player 2 receives 5 for just being there (to help Player 1). Since the Shapley value is not in the core and the core is not empty, we know that the nucleolus cannot be equal to the Shapley value. The nucleolus is always in the core when the core is not empty.

5. The Shapley value was found to be $(13/6, 22/6, 1/6)$ so we might try $(2, 3, 1)$ as an initial guess at the nucleolus. The largest excess occurs at either of the coalitions $\{1\}$ and $\{2,3\}$. One cannot be made larger without making the other smaller. So $x_1 = 2$ in the nucleolus. The next largest excess occurs at $\{2\}$ and $\{1,2\}$. These can be made smaller by making x_2 larger. This increases the excess of $\{1,3\}$. These are equal at $x_2 = 3.5$ and $x_3 = .5$. The nucleolus is $(2, 3.5, .5)$.

Coalition	Excess	$(2, 3, 1)$	$(2, 3.5, .5)$
$\{1\}$	$1 - x_1$	-1	-1
$\{2\}$	$-x_2$	-3	-3.5
$\{3\}$	$-4 - x_3$	-5	-4.5
$\{1,2\}$	$2 - x_1 - x_2$	-3	-3.5
$\{1,3\}$	$-1 - x_1 - x_3$	-4	-3.5
$\{2,3\}$	$2 - x_2 - x_3$	-1	-1

6. Since the characteristic function is symmetric in players 2 through n, we may assume the nucleolus is of the form $(x_1, x, x, \ldots, x)$ for some x_1 and x. To be an imputation we must have $x_1 + (n-1)x = v(N) = n$, so $x_1 = n - (n-1)x$. The excess for S not containing 1 is $e(\boldsymbol{x}, S) = -|S|x$. The excess for S containing 1 is $|S| - x_1 - (|S|-1)x = -(n-|S|) + (n-|S|)x$. The smallest maximum excess is certainly less than 0 (since it is for $x = 0$) so we can see that $0 < x < 1$. The largest excess for S not containing 1 is $-x$ (when $|S| = 1$). The largest excess for S containing 1 is $-(1-x)$ (when $|S| = n-1$). The largest of these two is smallest when x is chosen to make then equal. This gives $x = 1/2$. Hence $x_1 = (n+1)/2$ and the nucleolus is the same as the Shapley value, $((n+1)/2, 1/2, ..., 1/2)$.

7. Player 1 is a dummy, so he gets zero. Players 2, 3 and 4 are symmetric, so they get the same amount, say x. Since the sum must be $v(N) = 1$, we have $3x = 1$ or $x = 1/3$. This gives $(0, 1/3, 1/3, 1/3)$ as the nucleolus.

8.(a) No player can profit without the others so $v(A) = v(B) = v(C) = 0$. Players A and B can build a road for 18 and receive 19 in return so $v(AB) = 19 - 18 = 1$. Similarly, $v(AC) = 0$, $v(BC) = 6$, and $v(ABC) = 8$, the latter requiring a road of cost 19.

(b) The Shapley values are: $\phi_A = (1/3)0 + (1/6)1 + (1/6)0 + (1/3)2 = 5/6$, $\phi_B = (1/3)0+(1/6)1+(1/6)6+(1/3)8 = 23/6$, and $\phi_C = (1/3)0+(1/6)0+(1/6)6+(1/3)7 = 20/6$. To build the road Player A pays $10-5/6 = 9+(1/6)$, Player B pays $9-(23/6) = 5+(1/6)$, and Player C pays $8 - (20/6) = 4 + (2/3)$, for a total of 19.

(c) Based on the Shapley value we might try a first guess of $(1, 4, 3)$. We must always have $x_1 + x_2 + x_3 = 8$. The maximum excess occurs for A and BC. One cannot be made smaller without making the other larger, so $x_1 = 1$. The next largest excess occurs for C, so we must make x_3 larger. But as C is made smaller, B and AC get larger. We choose $x_3 = 3.5$ because then all three will be equal. Thus $(1, 3.5, 3.5)$ is the nucleolus. Player A pays 9, Player B pays 5.5 and Player C pays 4.5 for a total of 19.

Coalition	Excess	$(1,4,3)$	$(1,3.5,3.5)$
$\{A\}$	$-x_1$	-1	-1
$\{B\}$	$-x_2$	-4	-3.5
$\{C\}$	$-x_3$	-3	-3.5
$\{A,B\}$	$1-x_1-x_2$	-4	-3.5
$\{A,C\}$	$-x_1-x_3$	-4	-4.5
$\{B,C\}$	$6-x_2-x_3$	-1	-1

9. (a) $\phi_1 = 3/2$, $\phi_2 = \phi_3 = \phi_4 = 1/2$.

(b) $\nu_1 = 3/2$, $\nu_2 = \nu_3 = \nu_4 = 1/2$.

(c) If 1 enters the coalition and finds k peasants there, he wins $f(k)$. He is equally likely to enter at any of the positions 1 through $m+1$, so his expected payoff is $\phi_1 = (f(1)+\cdots+f(m))/(m+1)$. The other players are symmetric and so receive equal amounts, $\phi_2 = \cdots = \phi_m = (f(m) - \phi_1)/m$.

(d) Players 2 through $m+1$ are symmetric, so the nucleolus must be of the form $\nu = (f(m) - my, y, y, \ldots, y)$. The largest excess, $e(\nu, S)$ for S not containing 1 occurs at $|S| = 1$ with value $-y$, decreasing in y. The largest excess for S not containing 1 is $\max_{0\le k<m}[f(k) - f(m) + (m-k)y]$, increasing in y. The maximum excess is minimized when these are equal:

$$\max_{0\le k<m}[-(f(m) - f(k)) + (m-k-1)y] = 0$$

These are lines with positive slope starting at a negative value. Therefore this equation is satisfied at the first root, $y = \min_{0\le k<m}[(f(m) - f(k))/(m-k+1)]$. Thus, $\nu_2 = \cdots = \nu_{m+1} = \min\{(f(m) - f(k))/(m+1-k) : 0 \le k < m\}$, and $\nu_1 = f(m) - m * \nu_2$.

10.(a) Let us write the value in terms of profit. This normalizes the game so that $v(A) = v(B) = v(C) = v(D) = 0$. In addition, we have $v(AB) = v(CD) = 0$, $v(AC) = 4$,

$v(AD) = 8$, $v(BC) = 3$ and $v(BD) = 5$. Finally, $v(ABC) = 4$, $v(ABD) = 8$, $v(ACD) = 8$, $v(BCD) = 5$ and $v(ABCD) = 11$.

(b) $c_a = c_b = c_C = c_D = c_{AB} = c_{CD} = 0$, $c_{AC} = 4$, $c_{AD} = 8$, $c_{BC} = 3$, $c_{BD} = 5$, $c_{ABC} = -3$, $c_{ABD} = -5$, $c_{ACD} = -4$, $c_{BCD} = -3$, $c_{ABCD} = 6$. Therefore, the Shapley value is $\phi_A = 3.5$, $\phi_B = 1.833$, $\phi_C = 1.667$, $\phi_D = 4.0$.

(c) The nucleolus is $(3.5, 1.5, 1.5, 3.5)$. Under the nucleolus, A receives 13.5 for his house, B receives 21.5 for his house, C gets B's house for 21.5 and D gets A's house for 13.5. Under the Shapley value, A receives 13.5 for his house, B receives 21.833 for his house, C pays 21.333 and gets B's house and D pays 14 and gets A's house.

Coalition	Excess	$(3.5, 1.5, 1.5, 4.5)$
$\{A\}$	$-x_1$	-3.5
$\{B\}$	$-x_2$	-1.5
$\{C\}$	$-x_3$	-1.5
$\{D\}$	$-x_4$	-4.5
$\{A,B\}$	$-x_1 - x_2$	-5
$\{A,C\}$	$4 - x_1 - x_3$	-1
$\{A,D\}$	$8 - x_1 - x_4$	0
$\{B,C\}$	$3 - x_2 - x_3$	0
$\{B,D\}$	$5 - x_2 - x_4$	-1
$\{C,D\}$	$-x_3 - x_4$	-6
$\{A,B,C\}$	$x_4 - 7$	-2.5
$\{A,B,D\}$	$x_3 - 3$	-1.5
$\{A,C,D\}$	$x_2 - 3$	-1.5
$\{B,C,D\}$	$x_1 - 6$	-2.5

11. The nucleolus will certainly have (*) $x_1 \ge x_2 \ge \cdots \ge x_n$. Therefore, the maximum excess for a coalition S such that $v(S) = k$ occurs when $S = \{1, \ldots, k\}$, except for $k = 0$, when it occurs at $S = \{n\}$. The problem therefore reduces to minimizing

$$\max\{1 - x_1, 2 - x_1 - x_2, \ldots, (n-1) - x_1 - \cdots - x_{n-1}, -x_n\}$$

subject to (*) and $\sum_{i=1}^{n} x_i = n$. The last two are minimized when they are equal, giving $x_n = .5$. Then the previous one is minimized when $x_{n-1} = .5$, and so on down to $x_3 = .5$. Then the first two are minimized when they are equal, giving $x_2 = 1$, and therefore $x_1 = n/2$.

12. Since the game is symmetric in players 2 through n, we have $\nu_2 = \nu_3 = \cdots = \nu_n$. If $1 \in S$, the biggest excess, $e(\boldsymbol{\nu}, S)$, occurs when $S = \{1, 2\}$, in which case, $e(\boldsymbol{\nu}, S) = 1 - \nu_1 - \nu_2$. If $1 \notin S$, the biggest excess (and only positive excess) occurs at $S = \{2, \ldots, n\}$, in which case $e(\boldsymbol{\nu}, S) = 1 - \nu_2 - \cdots - \nu_n = 1 - (n-1)\nu_2$. The largest excess is minimized if these are equal: $1 - \nu_1 - \nu_2 = 1 - (n-1)\nu_2$. Since $\nu_1 + (n-1)\nu_2 = 1$, we may solve to find $\nu_2 = 1/(2n-1)$ and $\nu_1 = n/(2n-1)$.