# Solution Manual

***For Graph theory with Applications - Bondy and Murthy***

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# PREFACE

This book is a compilation of solutions to the problems in the classic "Graph Theory with Applications" by Bondy and Murthy. The author had written these during his first course on graph theory at Indian Institute of Technology, Madras, India.
Unfortunately the solutions to special sections on applications haven't been covered.
Any suggestions/ feedback / corrections are very appreciated and may be sent to shivanshu.ydv@gmail.com. Contributors will be acknowledged appropriately.

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# CHAPTER 1

# GRAPHS AND SUBGRAPHS

## Section 1.1

**1.1.1** Left as exercise to Chat-GPT

**1.1.2** Observe that the graph has three vertices of degree 4 and two of degree 3 connected to the former 3. With the idea in mind consider the following figure:

**1.1.3** Since any pair of vertices can share a maximum of one edge, the number of edges $\leq$ the number of ways of counting pairs from a set of size v = $\binom{n}{2}$

**1.2.1** In the given isomorphism, simply exchange the multiple edges to get a different mapping but by definition it is still an isomorphism.

**1.2.2** (a) Since there exists a bijection between V(H) and V(G), and E(H) and E(G), and the sets are finite, their cardinalities should be equal = number of elements. Follows from this.

**1.2.3** Name the vertices as follows In any isomorphism, b is mapped to 5, since they are the only vertices with multiple edges in their respective graphs. b is adjacent to c having a loop, whereas no such vertex adjacent to 5 exists, leading to a contradiction. Hence the graphs are non isomorphic.

**1.2.4** We enumerate by the number of edges in the graph starting from 0 to 6 = $\binom{4}{2}$

**1.2.5**

1.2.5 Suppose G and H are simple. Suppose there exists a bijection $\theta : V(G) \to V(H)$ such that $uv \in E(G) \Leftrightarrow \theta(u)\theta(v) \in E(H)$. Since graphs are simple they can be uniquely described by the vertices on which the edge is incident.
The bijection $\psi : E(G) \to E(H)$ defined by $\psi(uv) = \theta(u)\theta(v)$ thus makes sense. Clearly, we have bijections between the vetex and the edge sets satisfying the iosomorphism criteria.

If the graphs are isomorphic there exist bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_g(e) = uv \Leftrightarrow \psi_h(\phi(e)) = \theta(u)\theta(v) \in E(H)$. Thus $uv \in E(G) \Leftrightarrow \theta(u)\theta(v) \in E(H)$

1.2.6 figure
1.2.8 (a)Consider the two sets X and Y, all edges are representable as (x,y) x∈ X and y∈Y. The total number of such pairs = number of edges = $|X||Y| = mn$
(b) Suppose v = $|V(G)|$ and $v_1, v_2 = v - v_1$ are the number of vertices in the two partitions, the edges will be maximum when every vertex in one partition is connected to the every vertex in the other. From (a) number of edges = $v_1v_2 \leq (v_1 + v_2)^2/4 = v^2/4$ by AM-GM inequality.

1.3.1 (a) Since an edge can appear in a maximum of two rows or in a single row as a loop, the column sum is always 2.
(b) The column sums of A gives the number of vertices connected to that vertex, i.e, the degree of that vertex.

1.4.1 Consider a simple graph G with n vertices. Consider $K_n$ and a bijection $\theta$ between the vertex sets. Now if we remove the edges in $K_n$, such that 1.4.2

1.5.1

$$\delta \leq deg(v_i) \leq \Delta, 1 \leq i \leq |V(G)|$$
$$\Rightarrow \delta v \leq \Sigma_1 |V(G)| deg(v_i) \leq \Delta v$$
$$\Rightarrow \delta \leq 2\epsilon/v \leq \Delta$$

1.5.2

1.5.4 Suppose all the friends are repsented as vertices and friendships as edges of a simple graph G. The number of frienships per person is the degree corresponding to that vertex. If there are two or more people with zero friends the claim is correct. Suppose there are 0 or 1 people with zero friends. In the first case there are n-1 possible degrees (1 to n-1) for n people and n-2 possible degrees (1 to n-2) for n-1 people.In either case by PHP, there exist two people with same degree i.e., same number of friends. Hence Proved.

1.5.8 For two partitions X,Y of V(G) define same(X,Y) = number of edges between vertices in X+number of edges between vertices in Y.
If same(X,Y) = 0, then graph is bipartite and our claim is satisfied.
For all pairs of partitions consider the partition X*,Y* such that same(X*,Y*) is minimum.
Now for every vertex in X :
neighbours in X ≤ neighbours in Y, otherwise the partition X* {x} , Y*∪ {x} exists with same(X**,Y**)<same(X*,Y*), leading to a contradiction. Thus if the subgraph obtained by removing all the edges inside X and inside Y is H, same(X,Y) = 0 in H and $deg_H(v) \geq 1/2 deg_G(v)$ for all vertices v ∈V(G). Hence proved.

1.7.4 Since a k-regular graph is simple it cannot have loops or multiple edges, thus the second and third vertices of the cycle cannot overlap with the terminus.
Thus the only possbile four-cycle is a graph with four vertices and each having degree 2. Such a graph is a subgraph of the given graph by hypothesis.

Vertex a should be connected to exacly k -2 vertices other than b,c,d so that its degree is k. None of these vertices can be connected to b and so there exist additional k-2vertices so that its degree is k. Thus we have atleast k-2+k-2+4 = 2k vertices

1.5.2 $A^2[ii] = \Sigma_{k=1}^{\nu} A[ik] * A[ki]$.Since G is simple, $A[ki] = 1$ whenever there is an edge between i and k and 0 otherwise.
Thus the sum is adding 1 as many times as i is connected to another vertex.Since G is simple this is the number of edges that $v_i$ is connected to = its degree.
$MM'[ii] = \Sigma_{k=1}^{\epsilon} A[ik] * A[ki]$..Since G is simple, $A[ki] = 1$ whenever an edge is incident on i and 0 otherwise.Again this is the degree of $v_i$.

1.5.6 (b) Suppose d is graphic, then there exists a simple graph G with the degree sequence d. Consider $S_k = \{v_1, v_2, ...v_k\}, v_i \in V(G)$
For each vertex in $S_k$ let $t_i$ be the number of vertices connected in $S_k$ $u_i$ be the number of vertices connected in $G/\{S_k\}$.Since G is simple $d_i = t_i + u_i$ $\Sigma_{i=1}^{k} d_i = \Sigma_{i=1}^{k} t_i + \Sigma_{i=1}^{k} u_i$. The first term represents the degree sum inside the simple sg $S_k \leq 2\binom{k}{2} = k(k-1)$. The second term represents the degree sum of vertices in $G/\{S_k\}$.
For every vertex in $G/\{S_k\}$, number of vertices connected in $S_k$ is $\leq min(k, d_i)$ since a maximum of k vertices can be connected if degree is greater than k and degree otherwise.
Hence $\Sigma_{i=1}^{k} \leq k(k-1) + \Sigma_{i=k+1}^{n} min(k, d_i)$

1.2.12 The caveat here is, since the iso is onto G, E(G) has to be the same, i.e, for bijection $\theta(u)\theta(v)$ should also be in E(G) and if uv$\in$ E(G) then for some u'v' $\theta(u')\theta(v')$ should also be in E(G) i.e., the edge and vertex sets are still the same but the mapping relation between edges and vertices changes.
Basically realiasing the edges and vertices but the existence of edges is unique by original vertices

1.6.14 Claim: If there exists a path of distance d from a vertex to u then it is adjacent to u.
Trivial for d = 1. Suppose holds for d = k. If there exists a path with distance d = k+1, then by removing a terminal we get a path of length k, by hypothesis this means that the two are adjacent. Since graph is connected all vertices are connected to each other by paths of finite distance and hence the graph is complete leading to a contrn.

1.6.12 If there does not exist a vertex which is not adjacent to both u and v, then the maximum distance between two vertices is 3, leadin to a contradiction. Hence in $G^C$ there exists a vertex connected to both and the diameter is 2<3.

1.6.9 Claim: Any component of G-v should contain a neighbour of v. If not, then there exist vertices not connected to v in G or is a neighbour of v.
Since every neighbour of v in G-v has odd degree, number of neighbours in a component should be $\geq$2. Thus maximum number of components is 1/2d(v).

1.6.4 Suppose G is connected, If there exists a partition such that no edge has ends in different partitions, then any path must be entirely in one of the partitions, which means that G is disconnected leading to contradiction.
If for every parition of G, there exists an edge with ends in different partitions, Consider an arbitrary vertex s and the partition.
define $v_{i+1}$ =vertex connected to $v_i$ in the partition $\{v_1...i\}\&\{v_i+1...n\}$, with $v_1 = s$. Clearly there exists a path from every vertex

# CHAPTER 2

# TREES

## Section 2.1

**2.1.1** Suppose there exists a cycle in the graph. Since the graph is loopless every cycle should contain at least one vertex apart from the terminus.
Consider terminus $v_1$ and $v_2$ as a vertex adjacent to $v_1$. If G/{e} has no path between $v_1$ and $v_2$ then there cannot exist a cycle containing $v_2$ as an internal vertex. But since $v_1 e v_2$ already describes a unique path, this is not possible. Hence there are no cycles in the graph.

**2.1.2** Suppose one of the ends $v_1$ has degree more than 2. Then there exists a vertex which is a neighbour $v_0$ of this vertex and doesnot appear in the path (by property of tree). Clearly $v_0 ... v_2$ is also a valid path and has length greater than $v_1 ... v_2$ which contradicts the hypothesis. Thus both ends have degree one

**2.1.3** If there doesnot exist vertex with degree 1, $\delta \geq 2$ since G is connected and thus contains a cycle, which is false. Thus, there exists at least one vertex with degree 1. Consider the longest path from this vertex P = $v_1 v_2 ... v_n$ (P is finite).Since there are no cycles, $v_j \neq v_i$ for any $j > i$. If there exists a vertex u not in this path and adjacent to $v_n$, P+$uv_n$ is a longer path, and since $v_n$ should not be adjacent to any $v_i, i < n$ the degree of $v_n$ is 1.Thus there exist at least two vertices with degree 1

**2.1.4** There exists only one tree with 2 vertices and is a path by defintion. Suppose the claim holds for trees with k vertices. Consider one of the vertices which are ends of the longest path in G.
G/{v} is also a tree since only vertex is connected to v and as proved earlier only one connected and acyclic partition is generated.
Since G/{v} is a path, the longest path is G/{v} itself. If v is not connexted to one of the ends, there exist two vertices with degree one in G other than v, since removal of v does not affect their adjacency.
Thus we arrive at a contradiction.

**2.1.5** Suppose for some G, $|V(G)| = 1$.
(a) For v = 2, the claim holds trivially.Suppose (a) $\Rightarrow$ (b) for all graphs with v vertices. Now consider a connected graph G with v+1 vertices. By degree sum, there exists at least one vertex with degree 1. Let this vertex be u and its corresponding edge be e. G{u,e} has v vertices and v-1 edges, and is connected because v is of degree one and its removal does not affect the existence of paths connecting two vertices in G unless u is one of the terminal vertices. Thus G{u,e} has no cycles. If G is cyclic then u should be a part of a cycle meaning that it should be incident to at least two distinct edges, which leads to a contradiction. (b) Suppose G is acyclic and has v-1 vertices. Again using induction v= 2 holds trivially. Suppose claim holds for v vertices then there exists a degree one vertex say u with edge e. G/{u,e} should also be acyclic. By hypothesis G/{u,e} is connected. Consider the path from u's neighbour to a vertex w$\neq$ u,v be $P_w$ in G/{u,e}. Since v adjacent to u,there exists a path $Q$ from u to v, Clearly $P_w \cup Q$ is a valid path from u to w for all w $\in$ V(G). Thus G is connected and is a tree by defintion.
(c) Follows from (a) and (b)

**2.1.6** Let p vertices have degree 1.

$$\sum_{v \in V} d(v) \geq \Delta + p + 2(v - p - 1)$$

$$\sum_{v \in V} d(v) = 2v - 2 \geq \Delta + p + 2(v - p - 1)$$

$$\Rightarrow p \geq \Delta \geq k$$

**2.1.7** (a) If there exists a cycle in any of the components, the graph is cyclic which is not possible. Hence every component of a forest is connected and acyclic i.e., a tree.
(b) If G is a forest then

$$\epsilon = \sum_{c \in G} \epsilon_c = \sum_{c \in G} v_c - 1 = v - \omega$$

.
(b) For every component c in G, $\epsilon_c \geq v_c - 1$ (refer ...). Thus $\sum_{c \in G} \epsilon_c = \sum_{c \in G} v_c - 1 \geq v - \omega$.Equality holds when $\epsilon_c = v_c - 1$, for all c.
**Claim** All connected graphs with $\epsilon = v - 1$ are trees.
Proof: By induction on v. For v=2, trivial. Suppose claim holds true for v = k ($\geq$ 2). Consider a graph on k+1 vertices with k edges. From handshaking lemma one can conclude that there exists a vertex with degree 1.Let this vertex be v and then corresponding edge be e. G - v is connected, has k vertices and k-1 edges and is a tree. Since deg(v) = 1, it cannot be a part of a cycle and thus G does not have any cycles. Hence all connected graphs with v vertices and v-1 edges are trees.
Thus each component is a tree and G is a forest.

**2.1.8**

**2.1.9** We first prove the result for a tree, by induction on k.
Base case: k= 1. In this case there exist exactly two vertices of degree 1 and from 2.1.4, G is a path. Taking $P_1$ = G, E(G) = E($P_1$).
Suppose the result holds true for k = m. Suppose G has 2(m+1) vertices with odd degree. If there exists a vertex with odd degree adjacent to a pendant vertex, let the edge corresonding be e.G - e has 2m vertices of odd degree and m edge disjoint paths $P_i, i \leq m$.Let $P_{m+1} = e$, thus we have E(G) = $\bigcup P_i, i \leq m+1$. On the other hand, if only even degree vertices are adjacent to a pendant vertex.

**2.1.10**

**2.1.11**

**2.1.12**

# Section 2.2

**2.2.1** If G is a forest, every component in G is a tree. If there exists an edge uv in some component which is not a cut edge, there exists a path P from u to v, different from uv and hence P + uv is a cycle, contradicting our assumption hence every edge is a cut edge.
Suppose every edge in G is a cut edge. Consider a component G1 in G which has a cycle. There exist two adjacent vertices u and v. Since every component is connected, there exists a path from every vertex in G1 to either u or v, not containing uv. Since G1-uv contains a path from u to v, there exists a path between any two vertices. Thus G1 is connected, a contradiction.

**2.2.2** (a) If e is a cut edge of G, G-e is disconnected and there does not exist a spanning tree not containing G.
Now suppose all spanning trees contain e, and e is contained in a cycle. G-e is connected, since G is connected and there exists a spanning tree T in G-e. Clearly T is a spanning tree in G also, leading to a contradiction.Hence an edge contained in all spanning tree of G must be a cut edge.
(b)If e = vv is a loop, vev is a cycle and hence e$\notin$ any spanning tree.
Suppose there exists an edge not in any spanning tree. If e = uv cannot be a cut edge from (a).If e is not a cut edge consider a maximal set E of edges such that G/E is connected and e is the only path between u and v in G/E, since e itself such a set is well defined. e is clearly a cut edge of G/E and since G/E is connected there exists a spanning tree of G/V (and hence G) containing e.Thus e cannot be incident to two vertices $\Rightarrow$ e is a loop.

**2.2.3** Since no edge in G is a loop, it is contained in at least one spanning tree from 2.2.2(b). Since $\exists$ only one spanning tree T, all edges belong to T. Thus E(G) = E(T) and V(T) = V(G) by definition. Thus G = T.

**2.2.4** (a) Number of edges in $F \cup H$ is $v - \omega$, which for a fixed number of vertices is maximum

for $\omega = 1$. This means that $F \cup H$is connected and acylic in H, and hence its spanning tree.
(b) $\epsilon(F)$ =sum of edges in each component $= \Sigma c \in G v_c - 1 = v(G) - \omega(G)$

**2.2.5** We first prove the result for connected graphs, i.e., $\omega = 1$. The proof is by induction on $\epsilon - v$, which at hind sight might not be very intuitive. But given a moment of thought, e = v-1 corresponds to 0 cycles.

**Claim**

Any connected graph has at least $\epsilon - v + 1$ distinct cycles.

***Proof***

By induction on $\epsilon - v + 1$.Base case : 0. If $\epsilon = v - 1$, G is a tree since it is connected (refer ... ).Suppose all connected graphs with $\epsilon = v + k$ have k+1 distinct cycles. Consider a connected graph with $\epsilon = v + k + 1$. Since this grpah is not a tree, there exists an edge(link) which is not a cut edge. Let this edge be e.G - e has $\epsilon = v + k$, it has k+1 distinct cycles. G-e is connected and there is a path P between the ends of e. All cycles in G-e are cycles in G and P+e is a cycle in G different from all these. Hence there are at least k+2 distinct cycles.
Thus every connected graph contains at least $\epsilon - v + 1$ distinct cycles. ■

Consider a disconnected graph with components having edges $\epsilon_i$ and vertices $v_i$. Number of cycles $\geq \Sigma \epsilon_i - v_i + 1 \geq \epsilon - v + \omega$.

**2.2.6** (a) WLOG assume G is connected. Suppose there exists a cut edge e in G. G - e has exactly two components with one vertex in each component. Thus there exists an odd number (1) of vertices with odd degree in any of them, which is not possible. Hence G does not have a cut edge if all degrees are even. (Note that no cut edge is a loop so e = uv always)
(b) WLOG let G be connected. Suppose G has a cut edge e. G-e should have two components, both bipartite. (refer – ).If e = uv then deg(u) = deg(v) = k-1 and they belong to different components. Consider the component containing u. We have two disjoint partitions X and Y of this component such that every edge has an end in both. Thus number of edges = degree sum of X = degree sum of Y. WLOG let u $\in$ X then degree sum of X = nk + k-1 = mk-1 for m,n $\in N$, while degree sum of Y = pk for p $\in N$. This leads to a contradiction and hence G has no cut egde.

**2.2.7** (a) Any path from one tip to the other should pass through one of the vertices in the middle, the remaining vertices should either be connected to exactly one of the tips. Number of ways = 6*32 = 192.
(b) Each of the hanging vertices is a cut edge and should be present in a spanning tree. We should now find a subset of the pentagonal graph which is connected. Need verification – 18.

**2.2.8** If $[S, \overline{S}]$ is a bond and any of $G[S]$ is disconnected then there exists an edge e between at least two components of $G[S]$ such that G-$[S, \overline{S}]$+e is disconnected $\Rightarrow$ $[S, \overline{S}]$ is not

minimal.
If both $G[S]$ and $G[\overline{S}]$ are connected and $[S,\overline{S}]$ is not a bond, there exists an edge e$\in$ $[S,\overline{S}]$ such that G-$[S,\overline{S}]$+e is disconnected, which is false since $G[S]$ and $G[\overline{S}]$ are connected(adding an edge makes the whole graph connected)

**2.2.9** Let the components in be $G_i$ and the edge cut be E. Let the bond $b_{ij}$ be the set of edges between $G_i$ and $G_j$ in G. Let B = $\{b_{ij}, i, j \leq \omega\}$. By definition $\cup b_{ij} = E$ and they are disjoint. Since G/ $b_{ij}$ has two components, both connected, $b_{ij}$ is a bond. Hence proved.

**2.2.10** (a) Since G/$B_1$ is disconnected, $B_1 \cup B_2$ is an edge cut. If $B_1 \Delta B_2$ is not an edge cut, G/ $(B_1 \Delta B_2)$ is connected but G/ $(B_1 \cup B_2)$ , and hence $B_1 \cap B_2$ is an edge cut. If $B_1 = B_2$ and $B_1 \Delta B_2 = \phi$ , otherwise there exists an edge cut $B_1 \cap B_2$ having less cardinality than at least one of the two, violating the fact that they are bonds. Hence $B_1 \Delta B_2$ is an edge cut and as proved earlier is a union of disjoint bonds
(b) If $C_1 \cap C_2 = \phi$, done. Else between any two nearest common vertices, theier exists a path $P_1 \subset C_1$ and $P_2 \subset C_2$, $P_1 \cup P_2$ is a cycle. Let the set of all such cycles be S. Since $C_1$ and $C_2$ are cycles, every edge in $C_1 \Delta C_2$is in some cycle of S. By construction no two edges can be common in any two cycles and henc e$C_1 \Delta C_2$ is a union of disjoint cycles.

***Note that the statemnets below DONOT hold when $B_1 = B_2$ or $C_1 = C_2$ (c) If $e \in B_1 \cap B_2, B_1 \Delta B_2 \subset (B_1 \cup B_2)/\{e\}$ which from (a) contains a bond.Otherwise WLOG ,if $e \in B_1, e \notin B_2, B_2 \in (B_1 \cup B_2)/\{e\}$, which is a bond.
(d) Similar to above.

**2.2.11**

**2.2.12**

# Section 2.3

2.3.1 (a) Let v be one of the ends of a cut edge e. For any set E of edges in a graph G, $\omega(G - E) \geq \omega(G)$. Note that G-v = (G-e) - v.
Thus

$$\omega(G - v) = \omega((G - e) - v) \geq \omega(G - e) > \omega(G)$$

(b) Join two squares at a single vertex.

2.3.2 By induction on number of vertices.
For v = 2, G =$K_2$ and this case is trivial.
Suppose v>3. Consider a spanning tree T of G, every pendant vertex of T is not a cut vertex of G (refer proof of corollary 2.7). Thus any spanning tree of G must be a path.
Let P =$v_1 v_2 ... v_n$ be the spanning path, and $v_1, v_n$ are the only non cut vertices. If there exists an edge between two non adjacent vertices $v_i$ and $v_j$, $(j > i)$,$G - v_{i+1}$ is connected, since all of $v_{i+2}...v_n$ have a path to $v_j$. Since all vertices $v_1...v_{i-1}$ have a path to $v_i$ and $v_i$ is adjacent

to $v_j$, all vertices have a path to $v_j$ and hence any two vertices have a path between them. This leads to a contradiction since there exists another non cut vertex. Since V(P) = V(G) (spanning) and E(P) = E(G) (proved), P = G and G is a path.

# Section 2.4

**2.4.1**

**2.4.2**

**2.4.3** Left as an exercise to the reader ;)

**2.4.4**

**2.4.5**

# CHAPTER 3

# CONNECTIVITY

## Section 3.1

**3.1.1** (a) By definition $\kappa'(G) \geq k$, and at least k edges should be removed for disconnecting the graph and G is connected
Suppose $\omega(G - E') > 2$ then there exist at least three components in G-E'. Since $k > 0$there exists an edge in e E' having ends in different components. G-E'+e is still disconnected since $\omega(G - E' + e) > 1$. This implies that there exists an edge cut of size k-1 i.e., E'-e, which is a contradiction.
(b) Consdier 2 vertices mutually connected to 3 vertices

**3.1.2** $2\epsilon \geq \delta v \geq kv$

**3.1.3** (a) If $\delta = v - 1$, G is complete and $\kappa = v - 1$.
If $\delta = v - 2$, all vertices of degree v-1 should be removed from the graph otherwise the final graph will be connected since every other vertex would be connected to this vertex. There is at least one vertex with degree v-2 and another vertex, which is not connected to this one.
If any vertex other than the two remain they will be connected to both of them, making the final graph connected, hence the vetex cut should contain at least v-2 vertices $\Rightarrow \kappa = v - 2 = \delta$
(b) Take two triangles joint a point, $\kappa = 1$

**Claim**

If deg(u) + deg(v) $\geq$ v - 1 for a v-vertex graph, then its is connected.

***Proof***

**3.1.4** (a) If $v \geq 2$, G is connected since deg(u) + deg(v) $\geq$ v. (from above claim).Let E be a minimal edge cut. G/E should have exactly two components since if had more, there would be always be an edge between a pair of components in G and G/(E-e) would also be an edge cut, violating minimality.

Since E is minimal, no edge should have both its ends in the same component. Consider the smaller of the two components. If number of vertices is $\alpha 1 \leq \alpha \leq v/2$, the maximum degree of any vertex is $\alpha - 1$, since G is simple. Thus at least $\delta - \alpha + 1$ edges should be removed from every vertex.

$$\Rightarrow \kappa' \geq \alpha(\delta - \alpha + 1)$$

, note that under the given constraints, minimum value is $\delta$ and hence $\kappa' \geq \delta$.Also $\kappa' \leq \delta \Rightarrow \kappa' = \delta$
(b)

**3.1.5** Let the minimal vertex cut be V. G/V has v - $\kappa$ vertices with $\delta \geq (v+k-2)/2 - kappa$. The maximum number of vertices in the smallest component of any graph is v/2. Since G and hence G/V is simple $(v - \kappa)/2 - 1\delta \geq (v + k)/2 - \kappa - 1 \Rightarrow \kappa \geq k$.Hence G is k-connected.

**3.1.6** Consider the following cases :
(1) If $\kappa = 3, \kappa' = 3$
(2) If $\kappa = 2$, consider the minimal vertex cut = $\{v_1, v_2\}$ G $/\{v_1, v_2\}$ has three components, and G - $v_1$ and G-$v_2$ are connected,Any set containing edges from $v_1$ and $v_2$ to a particular component is an edge cut, $\kappa' = 2$,. If G $/\{v_1, v_2\}$ has two components,
(a) If $v_1, v_2$have an edge between them, E = edges from $v_1, v_2$ to any one component (card = 2)
(b) If they do not have an edge between them there is a component for $v_1$ and (may not be same) for $v_2$, having only one edge incident, E = edges from $v_1, v_2$ to the component having only one such edge.
In either case it is evident that E has two edges and is an edge cut $\Rightarrow \kappa' = 2$
(3) If $\kappa = 1$, let $v_1$ be a cut vertex. Wether G-$v_1$ has two or three components, there is a component with only one edge incident and this edge is a cut egde, hence $\kappa' = 1$.
Hence proved

**3.1.7**

# Section 3.2

**3.2.1** (If part) The removal of any edge does not make any two vertices disconnected since there exists another edge disjoint path connecting the two,hence the graph is 2-connected.
(Only if part ) Proof by induction on distance between two vertices:
Claim : All vertices u,v with d(u,v) =k,$k \in N$are connected by two edge disjoint paths in a 2-connected graph
Base case: If d(u,v) = 1, there shud exist a path from u to v in G-uv else G would be 1-connected.
Suppose hypothesis true for all u,v with d(u,v) = k.
Consider for some u,v d(u,v) = k+1. Since G is 2-connected there exists a path P from u to v. Let the vertex preceeding v in P be w. d(u,w) = k.Let th P-wv = Q and by

hypothesis we have an edge disjoint path R between u and w.There should exist a path S between u and v not containing wv since G is 2-connected. If x is the last vertex in $S \cap (Q \cup R)$, let R' be the section of R till x and S' be the section of S from x to v( note that this might be w also ). R' and S' are edge disjoint with Q and wv by construction and hence we have two edge disjoint paths from u to v, the result holds true for all values of d(u,v).

**3.2.2** Consider a pentagon with $v_1$ and $v_3$ adjacent.

**3.2.3** Consider a block B of G, by defintion at least two vertices must be removed to make G disconnected or trivial,hence $|V(B)| \geq 2$ Case 1: $|V(B)| = 2$. Since G has no even cycles there shouldnt be more than one edge between the two vertices of B. Thus B is isomorphic to $K_2$.
Case 2: $|V(B)| > 2$. Consider two vertices u,v in B. By mengers theorem they are part of a cycle which should be of odd length.
Call this cycle as C.
Claim: If G has an odd cycle there does not exist and path between any two vertices which is vertex disjoint with the cycle.
Proof : Let the cycle be $v_1 - v_2 - ... - v_{2m+1} - v_1$ and suppose there exists a path P between $v_i$ and $v_j$ disjoint with C.
Let the length of P be x. There exist two disjoint paths Q1 and Q2 of C with ends as $v_i$ and $v_j$ ,$Q_1 \cup Q_2 = C$. Since the parity of the lengths of Q1 and Q2 are different, one of $Q_1 \cup P$ and $Q_2 \cup P$ is of even length. WLOG, let Q1 be the path since $Q_1 \cup P = \phi$, it is a valid cycle containing $v_i$ and $v_j$. But this contradicts the fact that G has no even cycles.
If there exists a vertex v outside C in B, then there exists an edge from to some vertex u in C since B is connected. Also since u is not a cut-vertex, there should exist a path from u to som vertex w in C. WLOG consider the path having internal vertices not beloning to C. Let this path be R.Clearly R+uv is a path disjoint from C between two of its vertices, which is not possible, by claim above
Also there cannot exist an edge in B different from the edges of C, by the above claim.
Hence B = C, meaning that B is an odd cycle

**3.2.4** By induction on number of edges. Assume holds for number of edges < k. Consider a connected graph G, (not 2 connected) with k edges. G has a cut vertex x. Let the components of G - x be $C_1, C_2...C_m$. Let the set of edges incident on x, contained in some block of $C_i$ be $E_i$. Note that $E(G) = \cup(E(C_i) \cup E_i)$. Consider the graph $G_i = C_i + E_i + x$, If any of $G_i$ is not a block, there exists at least two blocks with exactly one cut vertex, with at least one of them different from x.If $G_i$ is a block then x is the only cut vertex. Thus for every $G_i$, there exists a block with only one cut vertex in **G** (careful!). Since x is a cut vertex, at least two $G_i$ and hence two blocks with exactly one cut vertex exist.

**3.2.5** Let G be connected WLOG, $\omega = 1$.Proof by induction on number of edges.
Base case $\epsilon = 1$ and G is $K_2$, number of blocks $= \omega + \Sigma(b(v) - 1)$
Suppose the above holds for all connected graphs with <k edges. Consider a connected

graph G with k edges. Repaeting the construction from 3.1.4, since the set of blocks is edge disjoint and every $G_i$ is edge disjoint and $\cup G_i = G$, the number of blocks = number of blocks in $G_i$. Since any block in G is contained entirely in a $G_i$,

$$\Sigma_{i \leq \omega(G-x)}(1 + \Sigma_{v \in G_i}(b(v) - 1))$$

$$= \omega + \Sigma_{v \in V(G), v \neq x}(b(v) - 1) + \Sigma b(x) - \omega = 1 + \Sigma_{v \in V(G)}(b(v) - 1)$$

(Note that since x is in every $G_i$)

**3.2.6**

**3.2.7**

**3.2.8** Algorithm :

# CHAPTER 4

# EULER TOURS AND HAMILTONIAN CYCLES

## Section 4.1

**4.1.1** The only graph with all even degrees is the first, hence eulerian.

**4.1.2** Fish!

**4.1.3** We prove by induction that for any number m of blocks, G is eulerain.
Base case :m =1: G is a block and hence every block of G is eulerian.
Suppose holds true for m = k(>2) then, Consider a graph with k+1 blocks.As proven earlier there exists a block with exactly one cut vertex. Consider G-B+x. Since every vertex other than x has even degree in B (none of the other vertices belong to any other block), x also has even degree in B and hence in G-B+x. Thus G-B+x is eulerian. B also is eulerian. Consider the eulerian paths P1 and P2 in G-B+x and B with terminal vertex x. By definition P1+P2 is eulerian and hence G with k+1 blocks is also eulerian.

**4.1.4** From previous theorem, every block in an Eulerian graph is Eulerian. Also since no two blocks can have two common edges (their union would be a block violtaing maximality), all blocks of a graph are edge disjoint.
Every block $B_i$ then has a cycle $C_i$ such that all the edges occur exactly once. Thus E(G) = $\Cup E(C_i)$

**4.1.5** Proof by induction. For k = 1. Let the two vertices with odd degree be u and v. G+uv is Eulerian and since G is connected, there exists a closed Eulerian trail C.C-uv is a trail. E(G) = C-uv = $Q_1$.
Suppose holds for k = m. Consider a graph with 2(m+1) vertices of odd degree. Again there exist two vertices u and v in G with odd degree.G+uv has 2m vertices of odd degree and E(G) = $\Cup E(Q_i)$ for m edge disjoint trails. If uv $\in Q_j, Q_j - uv$ is either a trail or two edge disjoint trails (by definition of a trail). For the former, if the trail is Q, consider the first vertices x,y in Q, then we have edge disjoint trails xy and Q - xy. In either case we have m+1 edge disjoint trails and hence the statement holds for all k$\in N$.

**4.1.6** If G - v is a forest, we prove that every trail with origin v can be extended to an Euler tour of G. Induction on number of edges. For $\epsilon = 2$, if G - v is a forest it is K2, and the claim is true. Assume hypothesis to be true for all graphs with <k edges.Let G be an eulerian graph with k edges.
If for some v,G - v is a forest, we have two cases
Note that all components of G are trees and at least two pendant vertices exist.Since G is eulerian all degrees are even and hence at least two edges exist that are incident on v and vertices of this component. Case1 : All components have exactly two edges incident on v from its vertices.
SHOULD TRY AGAIN – something much simpler exists

# Section 4.2

**4.2.1** (a) If G is not 2-connected, it has a cut vertex, there exists a set S = cutvertex v of G st $\omega(G-S) > |S|$,which means G is not Hamiltonian.
Alternately, if G is Hamiltonian then for any two vertices there exist two disjoint paths.
(b) If Suppose G is Hamiltonian, then there exists a cycle C = $v_1 v_2 ... v_n v_1$, since G is bipartite, every odd indexed vertices and even indexed vertices should belong to different partitions. Clearly n is not odd since G cannot have odd cycles, leading to the contradiction that $|X| = |Y|$, as the number of even numbers and odd numbers are same from 1 to 2k, $k \in N$.
Alternately if G is Hamiltonian then it has a connected 2-regular subgraph which is also bipartite. This implies $|X| = |Y|$.

**4.2.2** Since G +uv is bipartite with odd vertices, there cannot be a cycle containing all the vertices exactly once thus such a path is not possible.

**4.2.3** If G has a Hamiltonian path ending at u and v, if uv $\in$E(G) then G is Hamiltonian, hence $\omega(G-S) \leq |S| \leq |S|+1$
Otherwise G+uv is Hamilotonian,and

$$\omega(G-S+uv) \leq \omega(G-S) \leq \omega(G-S+uv)+1 \leq |S|+1$$

**4.2.4** (A modified version of the theorem given earlier). If c(G) is complete, G has Hamiltonian cycle and hence a Hamiltonian path.
Suppose c(G) is not complete, there exists a pair of non adjacent vertices u and v such that deg(u) + deg(v) < n, choose u and v to be of maximal degree and WLOG, deg(u) $\leq$ deg(v).

**4.2.5** (a) WLOG let $v_i$ be the vertex with degree $d_i$, clearly $deg_{G^c} v_i = d'_{v-i+1}$ if $d_m \geq d'_m = n-1-d_{v-m+1}$ for all $m \leq v/2$, thus $d_m + d_{v-m+2} \geq v-1$.
Now suppose there exists $m < (v+1)/2 \Rightarrow m \leq (v-1)/2 \leq v/2$,with$d_m < m$and$d_{v-m+1} < v-m \Rightarrow d_m + d_{v-m+1} < v$, leading to a contradiction, hence from 4.2.4, the graph has a Hamiltonian path.
(b) Since G is self complementary, the degree sequences of G and $G^c$ are the same, from (a), G has a Hamiltonian path.

**4.2.6** 4.2.4

**4.2.7** P(v) : For a simple graph with v vertices and $\epsilon > \binom{v-1}{2} + 1$, c(G) is complete. (v≥3)
For v = 3, the graph is K3 and hence c(G) is complete.
Suppose true for v = k, consider a graph with k + 1 vertices and $\epsilon > \binom{k}{2} + 1$. If $\delta = k$, G is complete and hence c(G) is complete.
If $\delta = k$, consider a vertex v with min degree, $\epsilon(G - v) > \binom{k}{2} + 1 - k + 1 > \binom{k-1}{2} + 1$.
Since deg of any vertex in c(G-v) is ≤ deg in c(G), if an edge is added in c(G-v) from G-v, it should also be added in c(G) from G.
Thus c(G-v) is complete ⇒ c(G) - v is complete. Note that $deg_G(v) \geq 2$, since G is simple. If there does not exist an edge between v and any other vertex u in c(G), we would have $deg(u) + deg(v) \geq k + 1$ which is not possible. Hence every vertex is adjacent to all other vertices in c(G), making it complete. By 4.4 G is Hamiltonian.

**4.2.8** 4.2.6

**4.2.9**

**Claim**

If G is simple with $v \geq 3$ and $\epsilon > \binom{v-1}{2} + 1$, cl(G) is complete

***Proof***

Proof by induction. If v = 3, $\epsilon > 2$ thus G and hence its closure are complete. Suppose true for v =k.Consider a simple G with v = k+1 and $\epsilon > \binom{k}{2} + 1$. If $\delta(G) = k$,G and hence its closure are complete. Otherwise consider a vertex with degree $\delta$, G - v has $\epsilon > \binom{k}{2} - k + 1 + 1 = \binom{k-1}{2} + 1$. Thus closure of G -v is complete. Since for any vertex u$deg_G(u) \geq deg_{G-v}(u)$, any two adjacent vertices in cl(G -v) should be adjacent in cl(G)[a]. Also $deg_{cl(G)}(u) + deg_{cl(G)}(v) \geq k + 1$,and hence every vertex is adjacent to v in cl(G). Thus cl(G) is complete. ■

[a]For a more precise argument we can argue that apply the sequence of steps from G-v to cl(G-v) is applicable to getting cl(G) from G

Now the rest of the proof directly follows from the corollaries.

**4.2.10** Proof by induction on number

**4.2.11**

**4.2.12**

**4.2.13**

**4.2.14** (a) Suppose there is a Hamiltonian cycle that contains $e_1$ but not $e_2$. Formally .. the explanation is based on the fact that exactly two edges in G shud belong to the HC.

(b)There shud exist pairs of edges between the two components otherwise no Hamiltonian cycle can come back to the same vertex.
If If e $\notin$ HC then exactly two such edges exist.
Case 1: If bottom two edges are included, the second component can be replaced by edge, from (a) such a Hamiltonian cycle wouldnt contain this section of the graph.
If $e_1$ and $e_3$ are in the cycle the same holds for the other component.
Now suppose $e_1$and$e_2$ are included, again replace the second component by an edge try again for diagonal

**4.2.15**

# CHAPTER 5

# MATCHINGS

## Section 5.1

**5.1.1** (a) Proof by induction:
Base case k = 2. Take the alterante edges, forms perfect matching.
Assume holds for k = m-1, m > 2, Let S = { $(x_1, x_2...x_{m-1}, 0), x_i \in \{0, 1\}\}$and T = { $(x_1, x_2...x_{m-1}, 1), x_i \in \{0, 1\}\}$ , by definition S and T are disjoint m-1 cubes. By induction hypothesis S and T contain perfect matchings $M_s$ and $M_t$. The matching $M_s \cup M_t$ is a perfect matching of G, since every vertex belongs either to S or to T and no edges in $M_s$ and $M_t$ are incident to the same vertex. Thus all k cubes have a perfect matching.

(b) For $K_{2n}$, we can partition the matchings into 2n-1 disjoint sets such that $v_1$ is adjacent to $v_i, 1 < i \leq 2n$. For any $v_i$ chosen, none of the edges incident on $v_1$ and $v_i$ belong to the matching and hence total number of ways to construct a matching containing this edge is the number of perfect matchings in the graph G-$\{v_1, v_i\}$. Thus number of perfect matchings for $K_{2n} = t_n = (2n-1) * t_{n-1}$.
Note that $t_1 = 1(K_2)$. Thus by induction one can prove that

$$t_n = (2n-1)(2n-3)...(3)(1) = \frac{(2n)!}{2^n(n!)}$$

For $K_{n,n}$, let the bipartitions be X and Y. We partition the set of matchings into disjoint sets such that $v_1 - u_i$ is contained in the matching, $v_1 \in X$ and $u_i \in Y$. The number of ways of choosing $u_i$ is n and it then remains to construct a matching for the graph G-$\{v_1, u_i\}$, which is isomorphic to $K_{n-1,n-1}$. Hence $t_n = (n) * t_{n-1}$, with $t_1 = 1$.

$$\Rightarrow t_n = n!$$

**5.1.2** Since a perfect matching can exist in graphs with even vertices we prove by induction on even number of vertices :
Base case: v= 2. M = E(G) is the only perfect matching.
Suppose all trees with even number of vertices < 2m have only one perfect matching. Consider a tree T with 2m vertices. Suppose it has a perfect matching M. If v is

a pendant vertex with neighbour u, vu $\in$ M. None of the other edges incident on u belong to the a perfect matching M of T. Thus M should be a union of uv and perfect matchings in components of G - $\{u, v\}$. Since each of these perfect matchings is unique, M is unique.

**5.1.3** (Refer to figure 5.10 in the textbook for k = 3) If k is even, a k+1 complete graph, is k regular but has no perfect matching since number of vertices is odd.
If k is odd, consider the following construction.
Let $G_i, i \in [k]$ be k+1 complete graphs. Let $G_i'$be the graph obtained by removing an edge in $G_i$ and connecting the two vertices to $v_i, i \in [k]$
Let there be k-2 vertices $u_i, i \in [k-2]$. Join every $v_i$ to $u_j$ and let this be the final graph G". Note that each of the $G_i'$ have odd number of vertices and there should exist at least one vertex that is adjacent to some vertex $\notin G_i'$. Since $v_i$ is the only such vertex and it is incident to some $u_j$. Therefore every $v_i$ is incident to a distinct $u_j$. But this leads to a contradiction since $|V| > |U|$

**5.1.4** If there exists a perfect matching then the second player can always choose the second vertex of the edge corresponding to the vertex chosen by the first player at any step. By induction one can prove that for any number of vertices player 2 will always win and hence 1 does not have a winning strategy. Thus if G player 1 has a winning strategy then G has no perfect matching. (contrapostion)

If G does not have a perfect matching, consider the following strategy for player 1: If there exists an isolated vertex choose it. Othewise identify a maximum matching M*, since it is not perfect, there exists a vertex v not saturated by M*. Choose this vertex on the first step. Any vertex adjacent to v must be saturated by M*. Player 2 thus chooses a saturated vertex. In all subsequent steps choose the vertex adjacent to vertex chosen by player 2 (in M*). Since there is no M* augmenting path, player 2 always chooses a vertex saturated by M* and the correspoding vertex adjacent in M* can be chosen by 1. Hence player 1 can never lose.

**5.1.5** (a) (i) For $K_{n,n}$ Let $H_i$ be a spanning subgraph of G such that $v_1 - v_i \in H_i$ and $v_k - v_{(\text{k+i-1})\text{mod n}} \in H_i, \forall 1 < k \leq n$. Note that this is a 1 factor and

$$\bigcup_{1 \leq i \leq n} H_i = G$$

For $K_{2n}$

(ii) (will add figure) Note that there should be exactly three factors 1,2,3. Two of these factors should have a pair of non adjacent edges say 1,2 (WLOG) in the outer pentagon. Note that any vertex should have exactly one edge incident from each factor since 3 regular. Now there are three junctions of 1 and 2 in the outer pentagon, meaning three edges incident from the pentagon to the star from 3. This means every edge in the star is ajdacent to some edge in 3, meaning no other edge can belong to 3, leading to a contradiction.
**A neater argument** from here Consider the union of two factors, H. H is a 2-regular

spanning subgraph of G, and every two regular graph is a union of cycles. The peterson graph has a girth of 5. Since H as 10 edges it is therefore a disjoint union of two 5-cycles or a 10-cycle.Since the Peterson graph is not Hamiltonian, H should be a union of two 5 cycles, but H has a 1 factor i.e, perfect matching leading to a contradiction.

(b) (i) If there is a 2 factor, there exist 3 vertices which should be adjacent to the same vertex leading to a constradiction.
(ii) Graph is Hamiltonian and has a 2 factor. (iii) Since graph is not Hamiltonian, its 2 factor (say H) should be a union of two disjoint cycles. Since the girth is 4, H should be a union of two 4 cycles. Since G is bipartite each cycle should contain 2 vertices from each partition, leading to a contradiction.

(c) If $\delta \geq (v/2) + 1$, then G has a hamiltonian cycle C = $v_1 v_2 ... v_{2m} v_1$. Let G' = G/ S, where S = $\{v_1 - v_2, v_3 - v_4, ...v_{2m+1} - v_{2m}\}$.
Note that $\delta_{G'} \geq (v/2)$ hence G' is also Hamiltonian and there exists another Hamiltonian cycle C' in G' and hence G. Hence $C' \cup S$ is a three regular spanning subgraph of G, i.e, a 3 factor

**5.1.6** We try to prove something simpler.

## Claim

$K_{2n+1}$ has n edge disjoint hamiltonian cycles

***Proof***

Consider the following construction. Choose $v_1$ arbitrarily. Let $C_m = u_1 - u_2..u_k$ (where $u_k$ is the first repeating vertex) be a spanning subgraph such that $u_1 = v_1$ and $u_{i+1} = v_{(j+2m)mod(2n+1)}$ if $u_i = v_j$.

If a vertex $u_i = v_j = u_k, k > i$ then (i + 2p)mod (2n+1) = i for some p $\Rightarrow$ 2p = (2n+1)k, $\Rightarrow p \geq 2n + 1$, this means $v_1$ is the first vertex to repeat and every $C_m$ is of the form $u_1 - u_2..u_1$ .
Since at least 2n+1 distinct vertices have appeared before the last one, it is a Hamiltonian cycle. Each of the $C_m, m < n$ are edge disjoint otherwise for two $m_1, m_2 < n$ $(i + 2m_1)mod(2n + 1) = (i + 2m_2)mod(2n + 1) \Rightarrow (2n + 1)|(2m_1 - 2m_2)$, which is false.
Hence we have n edge disjoint Hamiltonian cycles. ■

From above claim, we have n 2-factors.

## Theorem 0.1: Hall

A bipartite graph (X,Y) contains a matching that saturates every vertex in X iff for all $S \subseteq X, |N_G(S)| \geq |S|$

***Proof.***

The if part if fairly straight forward since there exists a unique vertex in Y for every vertex in S.

Only if : Suppose there exists a maximum matching M* not containing a vertex u. Consider

S = set of all vertices in X connected to u by an M* alternating path and T = set of all vertices in Y connected to u by an M* alternating path

Note that every vertex in S or T should be saturated by M*, else we would have an M* augmenting path.
Every vertex in S has at least one vertex in T. Now, every vertex in T should be adjacent to some vertex in S/{u} by an edge $\in$ M*,and vice versa. Thus $|S/\{u\}| = |T| = |S| - 1$.
If there is a vertex n in N(G) not in T, it wouldnt be incident an edge in M*, and there is an s in S such that n is adjacent to s but this means that there is an M* augmenting path.
Thus $|T| = |N_G(S)| < |S|$, leading to a contradiction. ■

**5.2.1** Let G be the graph where each tile is a vertex and adjacency of tiles implies existence of an edge and vice versa. Clearly G is bipartite (think of chess board) with the corner squares in the same component. Removal of two such vertices implies the components no longer have same number of vertices.
The process of covering with 1*2 is equivalent to getting a perfect matching in G, this is possible only if both components have same number of vertices.

**5.2.2** (a) If G(X,Y) has a perfect matching then there exists a matching that saturates every vertex in any of the components, let some $S \cup X = S_1$ and $S \cup Y = S_2$, $|N(S)| = N(S_1) + N(S_2) \geq |S_1| + |S_2| = |S|$, since $N(S_1) \cup N(S_2) = S_1 \cup S_2 = \phi$
For the converse it can be shown that there exist matchings that saturate all the vertices of X and of Y, clearly this possible only when $|X| = |Y|$. Consider a matching M that saturates all the vertices of X, each vertex is ajdacent to a unique vertex in Y and since $|X| = |Y|$, all vertices in Y are also saturated. Hence M is perfect.

(b) Consider a Triangle with an edge from one of the vertices to a vertex outside the triangle.

**5.2.3** (a)Since a k-regular bipartite graph has a perfect matching, it is one factorable.
(b) Every vertex has even degree and hence G is eulerian.

**5.2.4** (Only if )If there exists a representative set then obviously $| \cup A_i| > |J|$ for all subsets of S.

(If) Construct a bipartite graph with vertices in X representing elements from S= $\{1, 2, ...m\}$ and vertices in Y representing elements from $\cup A_i$, such that an edge xy exists only between i and the elements of $A_i$. By hall's theorem, if $| \cup A_i| > |J|$ for all subsets of S, there exists a matching saturating every vertex in X and thus a representative set corresponding to the ends of these edges in Y exists.

**5.2.5** Consider a bipartite graph G(X,Y) with vertices in X $=\{x_i\}$ where $x_i$represents row i and Y $= \{y_j\}$ where $y_j$ represents column j . E(G) $= \{x_i - y_j\}$ iff $M_{ij} = 1$in the matrix M. The minimum number of lines containing all the 1's is the cardinality of minimum vertex cover = cardinality of the maximum matching = maximum number of 1's, no two of which are in the same line.

**5.2.6** (a)

**5.2.7** Only if part is trivial.
If $|N(S)| \geq |S|$ for all $S \subset X$ in G(X,Y), suppose there exists a maximum mathcing not saturating all vertices, then by Konig's theorem there exists a covering with cardinality less than that of X, and hence there exists a vertex in X not part of the minimum covering $\Rightarrow$ not adjacent to any vertex in Y, leading to a contradiction. Hence proved.

**5.2.8** Note: Since the author had done a course on group theory and the methods used were algebraic, he cannot come up with a graph theoretic proof of 5.2.9 :)

**5.3.1** Again the only if part is trivial.
We first prove the result for bipartite graphs having $|X| = |Y|$. If there does not exist a matching not saturating all the vertices of X, then G has no perfect matching.
Hence there is a set S of V(G), st $o(G - S) > |S|$. Consider such a set S, $G - S$ has at least $|S| + 1$ components with odd vertices.

**5.3.2**

**5.3.3**

**5.3.4**

**5.3.5** (a) (Not sure about the direction of the question) For odd number of vertices, complete graphs are maximal.
For even number of vertices, consider an isolated vertex and n-1 complete graph.

(b) Suppose G is simple and v is even.

# CHAPTER 6

# EDGE COLORING

## Section 6.1

**6.1.1** WLOG assume max(m,n) = n = $\Delta$. Consider the set S of colors $\{1, 2, 3..n\}$. If X and Y are the two bipartitions of $K_{m,n}$, assign $x_i y_j$ the color (i+j-1)modn. Clearly, for any $x_i$ or $y_j$,colors of edges incident on it are all different. Since $\chi' \geq n, \chi' = n = \Delta$

**6.1.2** Suppose it is 3-chromatic, by pigeon hole principle we can say that at least five (15/3) edges are independent i.e, are in a matching. Since the size of a maximum matching is 5, we can say that there are exactly 3 maximum matchings which constitute G. This means G is one factorable, which is proved to be false in (ref req)

**6.1.3** Let G be a bipartite graph, consider the recursive procedure, with $H_0 = G$:
If there exists a vertex with degree less than $\Delta$, Let $H_i'(X', Y')$ isomorphic to $H_i(X, Y)$, and the graph obtained by joining any $x_i \in X$ to corresponding $x_i' \in X'$ whenever deg($x_i < \Delta$) and $y_i \in Y$ to corresponding $y_i' \in Y'$ whenever deg($y_i < \Delta$)

Since $\Delta$ is finite, and the degrees in $H_{i+1}$ are $\geq$ those in $H_i$, there exists a $\Delta$ regular bipartite graph obtained by this procedure and G is its subgraph.

(b)Let G(X,Y) be a $\Delta$ regular bipartite graph. Claim : All $\Delta$ regular bipartite graph are $\Delta$ chromatic.
Proof by induction on $\Delta = 1$, G is 1 colorable.
Assume true for $\Delta = k$. Consider a k+1 regular bipartite graph. By Hall's perfect matching, there exists a perfect mathcing M and $\Delta(G - M) = k$. By induction hypothesis, G-M is k edge colorable and assign a k+1 th color to all edges of M we have a proper k+1 edge coloring in G.
Since any bipartite graph with is a subgraph of a $\Delta$ regular bp graph, it is $\Delta$ colorable $\Rightarrow \chi' \leq \Delta$. Since $\chi' \geq \Delta, \chi' = \Delta$.

**6.1.4**

**6.1.5** By the statement of 1.5.8, consider a bipartite subgraph H of G such that $deg_H(v) \geq$

$deg_G(v)/2 \Rightarrow deg_G(v) - deg_H(v) \leq deg_G(v)/2$ Since $\Delta = 3$, $deg_G(v) - deg_H(v) \leq 1$. Consider an edge in G but not in H, the corresponding vertices have no other edge in G which is not in H. Since H is 3 edge colorable, this edge can be assigned a 4th color. This is a proper 4 edge coloring of G $\Rightarrow \chi' \leq 4$.

**6.1.6**

**6.2.1**

**6.2.2** Let the degree of the graph be d.$\epsilon = dv/2$. Since $\Delta = d$, if $\chi' = \Delta$, there exists a mathcing of size $\lceil v/2 \rceil$. Since v is odd, $2 * \lceil v/2 \rceil > v$ by PigeonHole principle, leading to a contradiction. Thus $\chi' = \Delta + 1$ from Gupta Vizing theorem.

**6.2.3** (a) Similar to 6.2.2, use PHP and divide $\epsilon$ by $\Delta$ to arrive at a contradiction.

(b) Both follow trivially from simple algebra

**6.2.4** (a) Let $H_0 = K_{\Delta+1}$. Consider the following procedure on $H_i$: If there exist two vertices u,v in G with multiple edges but only one edge in $H_i$, there exists a pair (x,y) in $H_i$ such that for every edge other than this edge, u is adjacent to x and v is adjacent to y. Let $H_{i+1}$ be the graph obtained by swapping the edges between all such pairs and u,v so that the number of edges between u,v in $H_{i+1}$ is same as that in G.
Since $\Delta$ is finite the procedure ends in a $\Delta$ regular graph $H_k$ such that G is a subgraph.

(b) Since a 2k regular graph is 2 factorable, every factor can be colored in a maximum of 3 colors, assigned a different set of colors to these k = $\Delta/2$ factors, $\chi' \leq 3\Delta/2$.

**6.2.5** Let the colors be 1,2,3. Note that each of these color sets forms a perfect matching. Taking the union of any two of these gives a 2-regular subgraph of G, which is a union of disjoint cycles. Since G is uniquely colorable, there cannot be more than one such cycle, otherwise two colors in one of the cycles can be interchanged giving rise to another proper coloring in G. Thus G has a cycle spanning all vertices $\Rightarrow$ G is Hamiltonian.

**6.2.6** (a) Note that the operation is equivalent to take another copy of G and joining corresponding vertices. Consider a minimum coloring of G and G'(its copy) Since there are a maximum of $\Delta(G)$ colors at a vertex in G (and G'), assigning the edge joining this vertex to its corresponding vertex a different color results in a $\Delta(G) + 1 = \Delta(G \times K_2)$ coloring of the graph, since each of G and G' are $\Delta(G) + 1$colorable.

(b) (H shudnt be disconnected) Consider a minimum proper coloring of

**6.2.7** I dont know

# CHAPTER 7

## INDEPENDENT SETS AND CLIQUES

**7.1.1** (a) If G is bipartite, every subgraph of G is bipartite. For any bipartite graph, each bipartition is an independent set. If X is the larger one, $1/2v(H) \leq |X| \leq \alpha(H)$.Thus for every subgraph, $\alpha(H) \geq 1/2v(H)$
If for every subgraph H, $\alpha(H) \geq 1/2v(H)$, we will prove that G does not contain any odd cycles. Suppose G contains an odd cycle C of size 2n+1. Since every vertex should be at least one vertex apart in the cycle, we cannot have an independent set of size n+1 or larger. Thus $\alpha(C) \leq n$ but according to our assumption, $\alpha(C) \geq n+1$. Thus G does not contain an odd cycle and is bipartite.

(b) For a biparite graph, any subgraph is bipartite and hence $\alpha(H) = \beta'(H)$.
Now suppose $\alpha(H) = \beta'(H)$ for every subgraph H in G. Suppose G contains an odd cycle C of size 2n+1, then $\alpha(C) \leq n$ (proved in (a)). Consider a minimum edge cover in C, by hypothesis, it has $\beta'(H) = \alpha(H) \leq n$ egdes and can cover $\leq 2n < 2n+1$ vertices leading to a contradiction. Hence G is bipartite.

**7.1.2** (a) Since $\alpha + \beta = v$, follows from 7.1.2
(b)Some wierd theorem

# CHAPTER 8

# VERTEX COLORINGS

## section 8.1

**8.1.1** Consider a maximum clique in G, with cardinality x. Note that the maximum degree sum of this set can be $\Delta x \leq v(x-1)$. The RHS comes from the fact that when counting the degree sum, we should include the edges from a vertex in the clique to another vertex. Any vertex inside the clique contributes exactly x-1 egdes whereas a vertex outside the clique should not contribute more than $x-1$ egdes otherwise it would be adjacent to all of the vertices in the clique, violating its maximality. Thus we have,

$$2\epsilon/vx \leq \Delta x \leq v(x-1)$$

$$\Rightarrow 2\epsilon/v^2 \leq 1 - 1/x$$

$$\Rightarrow v^2/(v^2 - 2\epsilon) \leq x \leq \chi$$

**8.1.2** (Adopted from ) Suppose $\chi > 5$. Consider the color classes by 6 of the colors and call them 1,2,3,4,5,6. Note that the subgraphs induced by vertices in classes 1,2 and 3 should contain an odd cycle else these can be assigned fewer than 3 colors. A similar argument follows for the subgraph induced by vertices in classes 4,5 and 6. Since no vertex is common between any color class, there exist two odd cycles that do not have a common vertex, leading to a contradiction.

**8.1.3** Since there exist at least $\chi$ vertices of degree $\chi - 1$, $\min(d_i + 1, i) = i \forall i \leq \chi$. Consider the maximum i such that $\min(d_i + 1, i) = i$ be $i'$, if $i' < n$, $\min(d_i + 1, i) = d_i + 1, \forall i' > i$ Since $d_i + 1$ is a decreasing sequence, the values hereafter decrease and are less than $i'$. Hence max(min$(d_i + 1, i)$) = i' $\geq \chi$.

**8.1.4** (a) Statement does not hold for a complete graph

(b) Let $\chi_1, \chi_2$ be the chromatic numbers of $G$ and $G^c$ respectively.
Suppose $\chi_1 + \chi_2 > n + 1$. Note that the first $\chi_2$ elements in the degree sequence of $G^c$ should be $\geq \chi_2 - 1$ meaning that the last $\chi_2$ elements in the degree sequence of G should be $\leq n - \chi_2$. If $\chi_2 + \chi_1 > n + 1$, the $\chi_1 - 1^{th}$ element is among the last $\chi_2$ elements and its degree should be $\leq n - \chi_2$. But the first $\chi_1$ elements have degree

$\geq \chi_1 - 1$, $\chi_1 - 1 \leq n - chi_2 \Rightarrow chi_1 + chi_2 \leq n + 1$, contrary to our assumption. There is an alternate proof using induction (and is simpler according to me)

**8.1.5** Let the $\chi$ critical subgraph be J, $\chi \leq 1 + \delta(J) \leq 1 + \max(\delta(H))$

**8.1.6** (Adopted from hints) Let C' be a coloring of G with the property that all its color classes have size $\geq 2$ with the minimum number of colors. Suppose C' has more than k colors. Consider a k-coloring of G, with color classes $V_i, 1 \leq i \leq k$. WLOG let $|V_1| = 1$. Let this unique vertex be $v_1$.

**8.1.7** 7

**8.1.8**

**8.1.9** (a) Suppose N(u) $\subseteq$ N(v). Since v$\notin$N(v), v$\notin$N(u). Let G be k-critical $\Rightarrow$ G - u is k-1 colorable. Assign u the same color as v in G-u, this is a valid k-1 coloring in G, which is false since G is k-chromatic.

(b) Note that for a k-critical graph, $\delta \geq k - 1$. Suppose G has k+1 vertices and is k-critical. If G were complete, $\chi = k + 1$. Thus there exist non adjacent u and v $\in V(G)$ such that they hasve at least k-1 neighbours. Since there are exactly k-1 vertices remaining, N(u) = N(v)$\Rightarrow$ N(u) $\subset$N(v), leading to a contradiction as proved in (a).

**8.1.10** (a) Any two vertices u $\in G_1$ and v $\in G_2$, cannot have the same color in $G_1.G_2$. Hence the coloring of $G_1.G_2$ is equivalent to coloring them independently with different colors. Thus $\chi(G_1.G_2) = \chi(G_1) + \chi(G_2)$
(b) Let G = $G_1.G_2$.
If $G_1, G_2$ are critical, let H be a subgraph of G, and $H_1 = H \cup G_1$ and $H_2 = H \cup G_2$

$$\chi(H) \leq \chi(H_1.H_2) = \chi(H_1) + \chi(H_2) < \chi(G_1) + \chi(G_2) = \chi(G)$$

Now if G is critical, consider any subgraph H of $G_1$, $H.G_2$ is a subgraph of $G_2$ and $\chi(H) + \chi(G_2) < \chi(G_1) + \chi(G_2) \Rightarrow \chi(H) < \chi(G_1)$. Thus $G_1$ is critical. Similarly prove for $G_2$.

**8.1.11** 11

**8.1.12** 12

**8.1.13**

## Section 8.2

**8.2.1** If Brook's theorem holds, then if G is k-critical and not complete. G cannot be an odd cycle since k$\geq$4. Thus $k \leq \Delta$. Thus there exists a vertex with degree at least $\chi$. Since $\delta \geq k - 1$ for a k critical graph, $2\epsilon \geq (v - 1)(k - 1) + k = v(k - 1) + 1$.

Now suppose whenever G is k-critical ($k \geq 4$) and not complete then $2\epsilon \geq v(k-1)+1$ holds true. We are required to prove that whenever G is not an odd cycle or a complete graph, $k \leq \Delta$.
Consider a connected graph that is not an odd cycle or complete graph. If G is 1 or 2 chromatic, $k \leq \Delta$. If G is 3 chromatic $\Delta \geq 3 = k$ since G contains an odd cycle and there should exist an edge other than the edges in the cycle ( G is connected ).
For $k \geq 4$. Consider a k-critical subgraph H of G, $2\epsilon_H \geq v(k-1)+1$. If $k > \Delta$, $2\epsilon_G \geq 2\epsilon_H \geq v\Delta + 1$ which is false.

**8.2.2** WLOG, let G be connected.
Since any edge can be adjacent to a maximum of 4 edges, the degree of a vertex in the line graph L(G) $\leq 4$. If L(G) is complete, it can either be $K_3$ or $K_4$ which means it has $\leq$4 egdes and hence $\chi' \leq 4$ Note that L(G) cannot be 5 complete. Since $\Delta = 3$,$\exists$ an 3-cycle in L(G) which means it cannot be a bigger odd cycle.
Since G is connected,for any two edges there is always a path from one vertex of an edge to a vertex of the other edge. Considering the edges in the path as adjacent vertices, there exists a path in L(G) between the vertices representing these edges. Thus L(G) is simple and connected with $\Delta = 4$. $\chi'(G) = \chi(L(G)) \leq 4$, from brook's theorem.