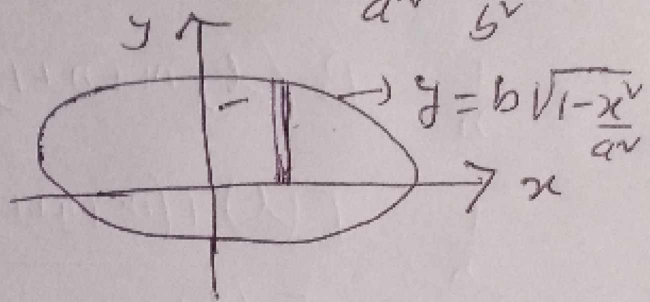
Evaluate

* Evaluate $\iint x^{m-1} y^{n-1}\, dx\, dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Here $\iint x^{m-1} y^{n-1}\, dx\, dy$

$$= \iint x^{m-1} y^{n-1}\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} y^{n-1}\, dy\, x^{m-1}\, dx$$

$$= \int_{x=0}^{a} \left.\frac{y^n}{n}\right|_0^{b\sqrt{1-\frac{x^2}{a^2}}} x^{m-1}\, dx$$

$$= \frac{b^n}{n} \int_0^a \left(1-\frac{x^2}{a^2}\right)^{\frac{n}{2}} x^{m-1}\, dx$$

put $\frac{x}{a} = \sqrt{t}$

x	0	a
t	0	1

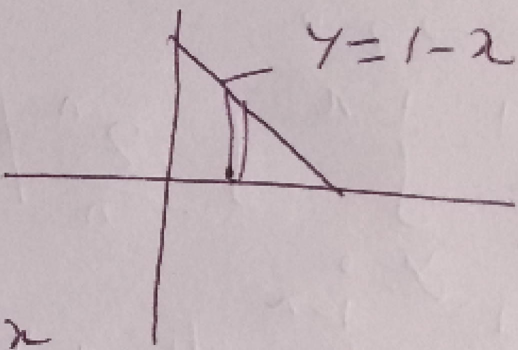
$$= \frac{b^n}{n} \int_0^1 (1-t)^{\frac{n}{2}} \left(a t^{\frac{1}{2}}\right)^{m-1} a \frac{1}{2\sqrt{t}}\, dt$$

$$= \frac{a^m b^n}{2n} \int_0^1 (1-t)^{\frac{n}{2}}\, t^{\frac{m}{2}-1}\, dt$$

$$= \frac{a^m b^n}{2n} B\left(\frac{m}{2}, \frac{n}{2}+1\right)$$

$$= \frac{a^m b^n}{2n} \frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{n}{2}+1\right)}$$

__Evaluate__ $\iint x^{l-1} y^{m-1}\,dx\,dy$ over $x \geq 0, y \geq 0$ and $x+y \leq 1$.



$$\int_{x=0}^{1}\int_{y=0}^{1-x} y^{m-1}\,dy\, x^{l-1}\,dx$$

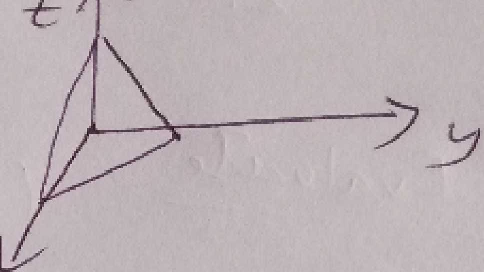
$$= \int_{x=0}^{1} \left.\frac{y^m}{m}\right|_0^{1-x} x^{l-1}\,dx$$

$$= \frac{1}{m}\int_{x=0}^{1} x^{l-1}(1-x)^m\,dx$$

$$= \frac{1}{m} B(l, m+1) = \frac{1}{m}\frac{\Gamma(l)\Gamma(m+1)}{\Gamma(l+m+1)}$$

This integral is known as Dirichlet integral over two dimensions.

Evaluate $\iiint x^{l-1} y^{m-1} z^{n-1}\,dx\,dy\,dz$ over ~~the~~ the region bounded by $x=0,\ y=0,\ z=0$ and $x+y+z \le 1$.



$$\therefore \iiint x^{l-1} y^{m-1} z^{n-1}\,dx\,dy\,dz$$

$$= \int_0^1 \int_{y=0}^{1-x} \cdot \int_{z=0}^{1-x-y} z^{n-1}\,dz\; y^{m-1}\,dy\; x^{l-1}\,dx$$

$$= \int_0^1 \int_{y=0}^{1-x} \int \frac{z^n}{n}\Big|_0^{1-x-y} y^{m-1}\,dy\; x^{l-1}\,dx$$

$$= \frac{1}{n}\int_0^1 \int_{y=0}^{1-x} (1-x-y)^n\, y^{m-1} x^{l-1}\,dx\,dy$$

put $y = (1-x)t$

$$= \frac{1}{n}\int_0^1 \int_{t=0}^{1} \left(1-x-(1-x)t\right)^n (1-x)^{m-1} t^{m-1} x^{l-1}\,dx\,(1-x)\,dt$$

$$= \frac{1}{n}\int_0^1 \int_{t=0}^{1} (1-x)^{n+m} x^{l-1} (1-t)^n t^{m-1}\,dx\,dt$$

$$= \frac{1}{n} \int_{x=0}^{1} x^{l-1} (1-x)^{m+n} dx \int_{t=0}^{1} t^{m-1} (1-t)^{n} dt$$

$$= \frac{1}{n} B(l, m+n+1) \quad B(m, n+1)$$

$$= \frac{1}{n} \frac{\Gamma(l)\Gamma(m+n+1)}{\Gamma(l+m+n+1)} \quad \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= \frac{1}{n} \frac{\Gamma(l)\Gamma(m)\, n\Gamma(n)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$$

This integral is called as Dirichlet integral in 3-dimension.

Prob Find the volume of the solid bounded by $x \geq 0$, $y \geq 0$, $z \geq 0$ and the surface

$$\left(\frac{x}{a}\right)^{1/2} + \left(\frac{y}{b}\right)^{1/2} + \left(\frac{z}{c}\right)^{1/2} = 1, \quad 0 < c < b < a < \infty.$$

$$V = \int_{x=0}^{a} \int_{y=0}^{b\left(1-\sqrt{\frac{x}{a}}\right)^2} c\left(1-\sqrt{\frac{x}{a}}-\sqrt{\frac{y}{b}}\right) dx\, dy.$$