If x, y are two independent variables and a variable z depends for its values on the values of x, y by a functional relation

$$z = f(x, y) \tag{3}$$

then we say z is a *function of* x, y. The ordered pair of numbers (x, y) is called a *point* and the aggregate of the pairs of numbers (x, y) is said to be the *domain* (or region) *of definition* of the function.

When the domain of definition is bounded by a closed curve C, it is said to be *closed* if f is defined for all points within and on the curve C; but *open or unclosed* when the function is defined for points within but not on the curve C.

1.2 The Neighbourhood of a Point

The set of values x_1, y_1 other than a, b that satisfy the conditions

$$|x_1 - a| < \delta, |y_1 - b| < \delta$$

where δ is an arbitrarily small positive number, is said to form a *neighbourhood* of the point $(a,$
Thus a neighbourhood is the square

$$(a - \delta, a + \delta; b - \delta, b + \delta)$$

where x takes any value from $a - \delta$ to $a + \delta$ except a, and y from $b - \delta$ to $b + \delta$ except b.

This is not the only way of specifying a neighbourhood of a point. There can be many other, thoug equivalent ways; for example the points inside the circle $x^2 + y^2 = \delta^2$ may be taken as a neighbourhoo of the point $(0, 0)$.

1.3 Limit Point

A point (ξ, η) is called a *limit point* or *a point of condensation* of a set of points S, if for ever neighbourhood of (ξ, η) contains an infinite number of points of S. The limit point itself may or may not b a point of the set. For example, the point $(0, 0)$ is a limit point of the set $\{(1/m, 1/n) : m, n \in \mathbf{N}\}$.

1.4 The Limit of a Function

A function f is said to tend to a limit l as a point (x, y) tends to the point (a, b) if for every arbitraril small positive number ε, there corresponds a positive number δ, such that

$$|f(x, y) - l| < \varepsilon,$$

for every point (x, y), [different from (a, b)] which satisfies

$$|x - a| < \delta, |y - b| < \delta$$

In other words, *a function f tends to a limit l, when (x, y) tends to (a, b) if for every positive numbe ε, there corresponds a neighbourhood N of (a, b) such that*

$$|f(x, y) - l| < \varepsilon,$$

for every point (x, y) other than (a, b) of the neighbourhood N.

Symbolically we then write

$$\lim_{(x, y) \to (a, b)} f(x, y) = l.$$

l is the *limit* (the *double limit* or the *simultaneous limit*) of f when x, y tend to a & b simultaneously.

Remark: The above definition implies that there must be no assumption of any relation between the independent variables as they tend to their respective limits.

For instance take $f(x, y)$ where

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

and find the limit when $(x, y) \to (0, 0)$.

If we put $y = m_1 x$ and let $x \to 0$, we get the limit to be equal to $\frac{m_1}{1 + m_1^2}$, while putting $y = m_2 x$ leads to a limit $\frac{m_2}{1 + m_2^2}$. Similarly letting $x \to 0$, while y remains constant or vice-versa leads to zero limit. Thus, we are led to erroneous results. Geometrically speaking when we approach the point (0, 0) along different paths, first along lines with slopes m_1 and m_2 and then along lines parallel to the coordinate axes, the function reaches different limits. The simultaneous limit postulates that by whatever path the point is approached, the function f attains the same limit. In general the determination whether a simultaneous limit exists or not is a difficult matter but very often a simple consideration enables us to show that the *limit does not exist.*

It may however be noted that

$$\lim_{(x,\, y)\to(a,\, b)} f(x, y) = l \quad \Rightarrow \quad \lim_{x\to a} f(x, b) = l = \lim_{y\to b} f(a, y)$$

Non-existence of limit. The above remark makes it amply clear that if $\lim_{(x,\, y)\to(a,\, b)} f(x, y) = l$ and if $y = \phi(x)$ is any function such that $\phi(x) \to b$, when $x \to a$, then $\lim_{x\to a} f(x, \phi(x))$ must exist and should be equal to l.

Thus, if we can find two functions $\phi_1(x)$ and $\phi_2(x)$ *such that the limits of* $f(x, \phi_1(x))$ *and* $f(x, \phi_2(x))$ *are different, then the simultaneous limit in question does not exist.*

Example 1(*a*). Let

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & \text{if } x^4 + y^2 \neq 0 \\ 0, & \text{if } x + y = 0 \end{cases}$$

If we approach the origin along any axis, $f(x, y) \equiv 0$.

If we approach (0, 0) along any line $y = mx$, then

$$f(x, y) = f(x, mx) = \frac{mx^3}{x^4 + m^2 x^2} = \frac{mx}{x^2 + m^2} \to 0, \text{ as } x \to 0$$

So any straight line approach gives,

$$\lim_{(x,\, y)\to(0,\, 0)} f(x, y) = 0$$

But putting $y = mx^2$,

$$\lim_{(x,\, y)\to(0,\, 0)} f(x, y) = \lim_{x\to 0} f(x, mx^2) = \frac{m}{1 + m^2}$$

which is different for the different m selected.

Hence, $\lim_{(x,\, y)\to(0,\, 0)} f(x, y)$ does not exist.

Thus, the function possesses no limit at the origin, but a straight line approach gives the limit zero.

Example 1 (b). Show that

$$\lim_{(x,\,y)\to(0,\,0)} \frac{2xy^2}{x^2+y^4} \text{ does not exist.}$$

- If we put $x = my^2$ and let $y \to 0$, we get

$$\lim_{y\to 0} \frac{2my^4}{(m^2+1)y^4} = \frac{2m}{1+m^2}$$

which is different for different values of m.

Hence, the limit does not exist.

Remark: It is pointed out earlier also that the determination of a simultaneous limit is a difficult matter but a simp consideration, as shown above, very often, enables us to show that the limit does not exist. We now show th sometimes it is possible to determine the simultaneous limit by changing to polars.

Example 2 (a). Show that

$$\lim_{(x,\,y)\to(0,\,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$$

- Put $x = r\cos\theta$, $y = r\sin\theta$

$$\left| xy\frac{x^2-y^2}{x^2+y^2} \right| = \left| r^2 \sin\theta \cos\theta \cos 2\theta \right|$$

$$= \left| \frac{r^2}{4} \sin 4\theta \right| \le \frac{r^2}{4} = \frac{x^2+y^2}{4} < \varepsilon,$$

if

$$\frac{x^2}{4} < \frac{\varepsilon}{2},\ \frac{y^2}{4} < \frac{\varepsilon}{2}$$

or if

$$|x| < \sqrt{2\varepsilon} = \delta,\ |y| < \sqrt{2\varepsilon} = \delta$$

Thus for $\varepsilon > 0, \exists\, \delta > 0$ such that

$$\left| xy\frac{x^2-y^2}{x^2+y^2} - 0 \right| < \varepsilon, \text{ when } |x| < \delta, |y| < \delta$$

$$\Rightarrow \quad \lim_{(x,\,y)\to(0,\,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$$

Example 2 (*b*). Show that

$$\lim_{(x,\, y)\to(0,\, 0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = 0$$

■ Since x, y are small

$$\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = \frac{(1+x^2y^2)^{1/2}-1}{x^2+y^2} \simeq \frac{\frac{1}{2}x^2y^2}{x^2+y^2}$$

Now changing to polars, we can show, as in the above example, that

$$\lim_{(x,\, y)\to(0,\, 0)} \frac{\frac{1}{2}x^2y^2}{x^2+y^2} = 0$$

Hence the required result.

Ex. 1. Show that

(*i*) $\lim_{(x,\, y)\to(0,\, 0)} \left(\frac{1}{|x|}+\frac{1}{|y|}\right) = \infty$, (*ii*) $\lim_{(x,\, y)\to(0,\, 0)} \frac{xy^2}{x^2+y^2} = 0$,

(*iii*) $\lim_{(x,\, y)\to(0,\, 0)} (x+y) = 0$, (*iv*) $\lim_{(x,\, y)\to(0,\, 0)} (1/xy)\sin(x^2y+xy^2) = 0$

Ex. 2. Show that the limit, when $(x, y) \to (0, 0)$ does not exist in each case

(*i*) $\lim \frac{2xy}{x^2+y^2}$, (*ii*) $\lim \frac{xy^3}{x^2+y^6}$,

(*iii*) $\lim \frac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$, (*iv*) $\lim \frac{x^3+y^3}{x-y}$

[*Hint*: (*iv*) Put $y = x - mx^3$].

Ex. 3. Show that the limit, when $(x, y) \to (0, 0)$ exist in each case.

(*i*) $\lim \frac{xy}{\sqrt{x^2+y^2}}$, (*ii*) $\lim \frac{x^3y^3}{x^2+y^2}$,

(*iii*) $\lim \frac{x^3-y^3}{x^2+y^2}$, (*iv*) $\lim \frac{x^4+y^4}{x^2+y^2}$.