4

Applications to Integral Equations

4.1. Integral equations

Definition. *An integral equation is an equation in which an unknown function appears under integral sign.*

The most general type of linear integral equation is of the form

$$F(t) = G(t) + \lambda \int_a^b K(t,u)\, F(u)\, du, \qquad \ldots (1)$$

where the upper limit may be either variable or fixed. The function $G(t)$ and $K(t, u)$ are known functions, while $F(t)$ is to be determined. The function $K(t, u)$ is called the *kernel of the integral* equation.

If a and b are constants, the equation is known as *Fredholm integral equation.* If a is a constant while $b = t$, it is called *a Volterra integral equation.*

4.2. Some important results to be committed to memory

(i) Convolution (or Falting).

Definition. The convolution of $F(t)$ and $G(t)$ is denoted and defined as

$$F * G = \int_0^t F(x)\, G(t-x)\, dx \quad \text{or} \quad F * G = \int_0^t F(t-x)\, G(x)\, dx.$$

(ii) Convolution theorem or convolution property.

If $L\{F(t)\} = f(s)$ and $L\{G(t)\} = g(s)$, then $L^{-1}\{f(s)\, g(s)\} = \int_0^t F(x)\, G(t-x)dx = F * G$

or $$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(t-x)\, G(x) dx = F * G.$$

Moreover, $$L\{F * G\} = f(s)\, g(s) = L\{F(t)\} \times L\{G(t)\}.$$

i.e., $$L\left\{\int_0^t F(x)\, G(t-x)dx\right\} = L\left\{\int_0^t F(t-x)\, G(x)dx\right\} = f(s)\, g(s).$$

(iii) Volterra Integral equation of first and second kinds.

A linear integral equation of the form $G(t) = \lambda \int_a^t K(t,u)\, F(u)\, du$ is known as *Volterra integral equation of first kind.*

Again, a linear integral equation of the form $F(t) = G(t) + \lambda \int_a^t K(t,u)\, F(u) du$ is known as *Volterra integral equation of second kind.*

(iv) Integral equation of convolution type.

Definition. The integral equation $y(t) = f(t) + \int_0^t K(t-x)\, y(x) dx,$ in which the Kernel $K(t-x)$ is a function of the difference $t-x$ only, is known as *integral equation of the convolution type.*

Note. Using the definition of convolution, we may re-write the above integral equation as

$$y(t) = f(t) + K(t) * y(t).$$

4.3. Application of Laplace transform in solving Volterra integral equation with convolution type kernel. Working rule

(*i*) Consider the Volterra integral equation of the first kind

$$G(t) = \lambda \int_0^t K(t-u)\, F(u)\, du. \quad \text{or} \quad G(t) = \lambda\, K(t) * F(t). \quad \text{... (1)}$$

Let $\quad L\{G(t)\} = g(s), \quad L\{F(t)\} = f(s) \quad$ and $\quad L\{K(t)\} = k(s)$ ... (2)

Applying the Laplace transform of both sides of (1), we get

$$L\{G(t)\} = \lambda\, L\{K(t) * F(t)\}$$

or $\quad L\{G(t)\} = \lambda\, L\{K(t)\} \times L\{F(t)\}$, by the convolution theorem

or $\quad g(s) = \lambda\, k(s) f(s) \quad$ or $\quad f(s) = g(s)/\lambda\, k(s)$, by (2) ... (3)

Applying the inverse Laplace transform of both sides of (3), we get

$$L^{-1}\{f(s)\} = L^{-1}\left\{\frac{g(s)}{\lambda k(s)}\right\} \quad \text{or} \quad F(t) = L^{-1}\left\{\frac{g(s)}{\lambda k(s)}\right\}.$$

(*ii*) Consider the Volterra integral equation of the second kind

$$F(t) = G(t) + \lambda \int_0^t K(t-u)\, F(u)\, du.$$

or $$F(t) = G(t) + \lambda\, K(t) * F(t). \quad \text{... (1)}$$

Let $\quad L\{G(t)\} = g(s), \quad L\{F(t)\} = f(s) \quad$ and $\quad L\{K(t)\} = k(s)$. ... (2)

Applying the Laplace transform of both sides of (1), we get

$$L\{F(t)\} = L\{G(t)\} + \lambda\, L\{K(t) * F(t)\}$$

or $\quad L\{F(t)\} = L\{G(t)\} + \lambda\, L\{K(t)\} \times L\{F(t)\}$, using the convolution theorem

or $\quad f(s) = g(s) + \lambda\, k(s) f(s) \quad$ or $\quad [1 - \lambda\, k(s)] f(s) = g(s)$

or $\quad f(s) = g(s)/[1 - \lambda\, k(s)]$. ... (3)

Applying the inverse Laplace transform of both sides of (3), we get

$$F(t) = L^{-1}\{g(s)/[1 - \lambda\, k(s)]\}.$$

4.4. Abel's Integral equation.

Definition. An integral equation of the form $\displaystyle\int_0^t \frac{F(u)}{(t-u)^\alpha}\, du = G(t)$

is known as *Abel's integral equation*, where $F(t)$ is unknown function, $G(t)$ is a known function and α is a constant such that $0 < \alpha < 1$.

4.5. Integro-differential equation.

Definition. It is an equation in which various derivatives of unknown function $F(t)$ can also be present.

For example, $$F'(t) = F(t) + G(t) + \int_0^t K(t-u)\, F(u)\, du$$

is an *integro-differential equation*, where $F(t)$ is unknown function and $G(t)$ and $K(t-u)$ are known functions. The solution of such equation subject to given initial conditions can be easily obtained.

4.6. Solved examples based on integral equations

Ex.1. *Solve the integral equation* $F(t) = 1 + \int_0^t F(u) \sin(t-u)\, du$ *and verify your solution.*

[Osmania 2010; Kanpur 1995]

Sol. Given $$F(t) = 1 + \int_0^t F(u)\sin(t-u)\,du. \qquad \ldots (1)$$

Using the definition of convolution, (1) can be written as
$$F(t) = 1 + F(t) * \sin t. \qquad \ldots (2)$$

Let $L\{F(t)\} = f(s)$ Applying the Laplace transform of (2), we get
$$L\{F(t)\} = L\{1\} + L\{F(t) * \sin t\}$$

or $$L\{F(t)\} = (1/s) + L\{F(t)\} \times L\{\sin t\}, \text{ by the convolution theorem}$$

or $$f(s) = \frac{1}{s} + f(s) \times \frac{1}{s^2+1} \quad \text{or} \quad \left(1 - \frac{1}{s^2+1}\right) f(s) = \frac{1}{s}$$

or $$f(s) = (s^2+1)/s^3 = 1/s + 1/s^3. \qquad \ldots (3)$$

Applying the inverse Laplace transform of both sides of (3), we get
$$L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s^3}\right\} \quad \text{or} \quad F(t) = 1 + \frac{t^2}{2!} = 1 + \frac{t^2}{2}. \qquad \ldots (4)$$

Verification of solution (4): We wish to show that (4) satisfies the given integral equation (1).
From (4), $$F(u) = 1 + u^2/2$$

$\therefore$ R.H.S. of (1) $$= 1 + \int_0^t (1 + u^2/2)\sin(t-u)\,du$$

$$= 1 + \Big[(1+u^2/2)\cos(t-u)\Big]_0^t - \int_0^t u\cos(t-u)\,du, \text{ integrating by parts}$$

$$= 1 + 1 + \frac{t^2}{2} - \cos t - \left\{\Big[-u\sin(t-u)\Big]_0^t - \int_0^t 1\cdot\{-\sin(t-u)\}\,du\right\}$$

$$= 2 + \frac{t^2}{2} - \cos t - \int_0^t \sin(t-u)\,du.$$

$$= 2 + \frac{t^2}{2} - \cos t - \Big[\cos(t-u)\Big]_0^t = 2 + \frac{t^2}{2} - \cos t - (1 - \cos t)$$

$$= 1 + t^2/2 = F(t), \text{ using (4)}$$

$$= \text{L.H.S. of (4)}$$

Hence (4) is solution of given integral equation (1).

Ex.2. *Solve by the method of Laplace transform:* $3\sin 2x = y(x) + \int_0^x (x-t)y(t)\,dt.$

[Nagpur 2005; Lucknow 1994]

Sol. Note carefully that in the present problem we have x in place of t and t in place of u. So we shall make the corresponding changes in the usual solution and standard results.

Using the definition of convolution, the given equation can be written as
$$y(x) = 3\sin 2x - y(x) * x. \qquad \ldots (1)$$

Let $L\{y(x)\} = f(s)$. Applying the Laplace transformation of both sides of (1), we have
$$L\{y(x)\} = 3L\{\sin 2x\} - L\{y(x) * x\} \quad \text{or} \quad f(s) = 3 \times \{2/(s^2+2^2)\} - L\{y(x)\} \times L\{x\}$$

or $$f(s) = \frac{6}{s^2+4} - \frac{f(s)}{s^2} \quad \text{or} \quad f(s) = \frac{6s^2}{(s^2+1)(s^2+4)}$$

or $$f(s) = 2\{1/(s^2+1) - 1/(s^2+4)\} \qquad \text{... (2)}$$

Taking inverse Laplace transform of both sides of (2), we get

$$y(x) = 2\ \{\sin t - (\sin 2t)/2\} = 2 \sin t - \sin 2t.$$

Ex.3. *Solve the integral equation* $F(t) = t + 2\int_0^t F(u) \cos(t-u)\, du$ **[Kanpur 94]**

Sol. Given $$F(t) = t + 2\int_0^t F(u) \cos(t-u)\, du. \qquad \text{... (1)}$$

Using the definition of convolution, (1) can be re-written as

$$F(t) = t + 2\, F(t) * \cos t. \qquad \text{... (2)}$$

Let $L\{F(t)\} = f(s)$. Applying the Laplace transform of (1), we have

$$L\{F(t)\} = L\{t\} + 2\, L\{F(t) * \cos t\}$$

or $L\{F(t)\} = (1/s^2) + 2\, L\{F(t)\} \times L\{\cos t\}$, by the convolution theorem

or $$f(s) = \frac{1}{s^2} + 2\, f(s) \times \frac{s}{s^2+1} \qquad \text{or} \qquad f(s)\left[1 - \frac{2s}{s^2+1}\right] = \frac{1}{s^2}$$

or $$f(s) = \frac{s^2+1}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}. \qquad \text{... (3)}$$

Multiply both sides of (3) by s^2 and let $s \to 0$; then $B = \lim_{s\to 0} \dfrac{s^2+1}{(s-1)^2} = \dfrac{0+1}{(0-1)^2} = 1.$

Multiply both siders of (3) by $(s-1)^2$ and let $s \to 1$; then $D = \lim_{s\to 1} \dfrac{s^2+1}{s^2} = \dfrac{1+1}{1} = 2.$

Then, from (3), $$f(s) = \frac{s^2+1}{s^2(s-1)^2} = \frac{A}{s} + \frac{1}{s^2} + \frac{C}{s-1} + \frac{2}{(s-1)^2} \qquad \text{... (4)}$$

Multiplying both sides of (4) by $s^2(s-1)^2$, we have

$$s^2 + 1 = As(s-1)^2 + (s-1)^2 + C s^2 (s-1) + 2s^2.$$

or $$s^2 + 1 = As(s^2 - 2s + 1) + (s^2 - 2s + 1) + C(s^3 - s^2) + 2s^2. \qquad \text{... (5)}$$

Equating coefficients of s^3 and s on both sides of (5), we get

$A + C = 0$ and $A - 2 = 0$ so that $A = 2,$ $C = -2.$

Then, (3) and (4) $\Rightarrow$ $$f(s) = \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{2}{(s-1)^2}. \qquad \text{... (6)}$$

Taking inverse Laplace transform of both sides of (6), we get

or $F(t) = 2 + t - 2e^t L^{-1}\{1/s\} + 2e^t L^{-1}\{1/s^2\}$, using first shifting theorem

or $F(t) = 2 + t - 2e^t + 2e^t t = 2 + t - 2e^t(1-t)$

Ex.4. *Solve the integral equation:* $F(t) = t^2 + \int_0^t F(u) \sin(t-u)\, du$

Sol. Given $$F(t) = t^2 + \int_0^t F(u) \sin(t-u)\, du$$

or $$F(t) = t^2 + F(t) * \sin t, \text{ using definition of convolution} \qquad ...(1)$$

Let $L\{F(t)\} = f(s)$. Taking the Laplace transform of (1), we get

$$L\{F(t)\} = L\{t^2\} + L\{F(t) * \sin t\}$$

or $$f(s) = (2/s^3) + L\{F(t)\} \times L\{\sin t\}, \text{ by the convolution theorem}$$

or $$f(s) = \frac{2}{s^3} + f(s) \times \frac{1}{s^2+1} \qquad \text{or} \qquad f(s)\left[1 - \frac{1}{1+s^2}\right] = \frac{2}{s^3}$$

or $$\frac{s^2 f(s)}{1+s^2} = \frac{2}{s^3} \qquad \text{or} \qquad f(s) = \frac{2(1+s^2)}{s^5} = \frac{2}{s^5} + \frac{2}{s^3}.$$

Inverting, $$F(t) = 2 \times (t^2/2!) + 2 \times (t^4/4!) = t^2 + (t^4/12).$$

Ex.5. *Solve the following integral equations:*

(*a*) $F(t) = e^{-t} - 2\int_0^t F(u)\cos(t-u)\,du.$ **[S.V. Univ. (A.P.) 1997]**

(*b*) $e^{-x} = y(x) + 2\int_0^x \cos(x-t)\, y(t)\,dt.$

Sol. (*a*) Given $$F(t) = e^{-t} - 2\int_0^t F(u)\cos(t-u)\,du.$$

or $$F(t) = e^{t} - 2F(t) * \cos t, \text{ using definition of convolution.} \qquad ...(1)$$

Let $L\{F(t)\} = f(s)$. Taking the Laplace transform of both sides of (1), we get

$$L\{F(t)\} = L\{e^{-t}\} - 2L\{F(t) * \cos t\}$$

or $$f(s) = 1/(s+1) - 2L\{F(t)\} \times L\{\cos t\}, \text{ by the convolution theorem}$$

or $$f(s) = \frac{1}{s+1} - 2f(s) \times \frac{s}{s^2+1} \qquad \text{or} \qquad f(s)\left(1 + \frac{2s}{s^2+1}\right) = \frac{1}{s+1}$$

or $$\frac{(s+1)^2}{s^2+1} f(s) = \frac{1}{s+1} \qquad \text{or} \qquad f(s) = \frac{s^2+1}{(s+1)^3}$$

Inverting, $$F(t) = L^{-1}\left\{\frac{s^2+1}{(s+1)^3}\right\} = L^{-1}\left\{\frac{[(s+1)-1]^2+1}{(s+1)^3}\right\}$$

$$= e^{-t} L^{-1}\left\{\frac{(s-1)^2+1}{s^3}\right\}, \text{ by first shifting theorem}$$

$$= e^{-t} L^{-1}\left\{\frac{s^2-2s+2}{s^3}\right\} = e^{-t} L^{-1}\left\{\frac{1}{s} - \frac{2}{s^2} + \frac{2}{s^3}\right\}$$

$$= e^{-t}[1 - 2\times(t/1!) + 2\times(t^2/2!)] = e^{-t}(1 - 2t + t^2) = e^{-t}(1-t)^2.$$

(*b*) Here replace t by x and u by t in part (*a*) Also replace $F(t)$ by $y(x)$

Ex.6. *Solve the integral equation* $F(t) = a\sin t - 2\int_0^t F(u)\cos(t-u)\,du,$

[Ravishanker 2004]

Sol. Given $$F(t) = a \sin t - 2\int_0^t F(u) \cos(t-u)\,du$$

or $F(t) = a \sin t - 2\, F(t) * \cos t$, by definition of convolution ... (1)

Let $L\{F(t)\} = f(s)$. Taking the Laplace transform of both sides of (1), we get

$$L\{F(t)\} = a\, L\{\sin t\} - 2\, L\{F(t) * \cos t\}$$

or $f(s) = a/(1+s^2) - 2\, L\{F(t)\} \times L\{\cos t\}$, by the convolution theorem

or $$f(s) = \frac{a}{1+s^2} - 2 f(s) \times \frac{s}{s^2+1} \quad \text{or} \quad f(s)\left[1+\frac{2s}{s^2+1}\right] = \frac{a}{1+s^2}$$

or $$\frac{(s+1)^2}{s^2+1} f(s) = \frac{a}{1+s^2} \quad \text{or} \quad f(s) = \frac{a}{(s+1)^2}.$$

Inverting, $L^{-1}\{f(s)\} = a\, L^{-1}\{1/(s+1)^2 = a\, e^{-t} L^{-1}\{1/s^2\}$, using first shifting theorem

or $$F(t) = a\, e^{-t} \times (t/1!) = at\, e^{-t}.$$

Ex.7. *Solve the integral equation* $t = \int_0^t e^{t-u} F(u)\,du.$

Sol. Re-writing the given intergral equation, we get

or $t = e^t * F(t)$, by definition of convolution. ... (1)

Let $L\{F(t)\} = f(s)$. Taking Laplace transform of both sides of (1), we get

$L\{t\} = L\{e^t * F(t)\}$ or $L\{t\} = L\{e^t\} \times L\{F(t)\}$, using the convolution theorem

or $$\frac{1}{s^2} = \frac{1}{s-1} f(s) \quad \text{or} \quad f(s) = \frac{s-1}{s^2} = \frac{1}{s} - \frac{1}{s^2}.$$

Inverting, $L^{-1}\{f(s)\} = L^{-1}\{1/s\} - L^{-1}\{1/s^2\}$ or $F(t) = 1 - t.$

Ex.8. *Solve the integral equation* $\int_0^t F(u)\, F(t-u)\,du = 16 \sin 4t.$

[Kanpur 1995, S.V. Univ. (A.P.) 1997]

Sol. Re-writing the given integral equation, we have

$F(t) * F(t) = 16 \sin 4t$, by definition of convolution ... (1)

Let $L\{F(t)\} = f(s)$. Taking Laplace transform of both sides of (1), we get

$$L\{F(t) * F(t)\} = 16\, L\{\sin 4t\}$$

or $L\{F(t)\} \times L\{F(t)\} = 16\, L\{\sin 4t\}$, using the convolution theorem

or $$f(s) \times f(s) = 16\left(\frac{4}{s^2+4^2}\right) \quad \text{or} \quad f(s) = \pm \frac{8}{(s^2+4^2)^{1/2}}.$$

Inverting, $L^{-1}\{f(s)\} = \pm 8\, L^{-1}\left\{1/(s^2+4^2)^{1/2}\right\} = \pm 8\, J_0(4t)$, as $L\{J_0(at)\} = 1/(s^2+a^2)^{1/2}$

Ex.9. *Solve* $\sin t = \int_0^t J_0(t-u)\, F(u)\,du.$

Sol. Re-writing the given integral equation, we have

or $\sin t = J_0(t) * F(t)$, by definition of convolution. ... (1)

Let $L\{F(t)\} = f(s)$. Taking the Laplace transform of both sides of (1), we get

$$L\{\sin t\} = L\{J_0(t) * F(t)\}$$

or $1/(1+s^2) = L\{J_0(t)\} \times L\{F(t)\}$, using the convolution theorem