2. JACOBIANS

For further development of the subject, acquaintance with the notion of Jacobians is necessary. We shall now define a Jacobian and also prove some of its important properties.

If $u_1, u_2, \ldots, u_n$ be n differentiable functions of n variables $x_1, x_2, \ldots, x_n$, then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the *Jacobian* or the *Functional Determinant* of the functions $u_1, u_2, \ldots, u_n$ with respect to $x_1, x_2, \ldots, x_n$ and is denoted by

$$\frac{\partial (u_1, u_2, \ldots, u_n)}{\partial (x_1, x_2, \ldots, x_n)} \text{ or } J\left(\frac{u_1, u_2, \ldots, u_n}{x_1, x_2, \ldots, x_n}\right)$$

2.1 Some Properties

Jacobians have the remarkable property of behaving like the derivatives of functions of one variable. A few of the important relations are given here and the proofs depend upon the algebra of determinants.

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For $n = 1$, the determinant is simply $\dfrac{\partial y_1}{\partial x_1}$ or $\dfrac{dy_1}{dx_1}$, the derivative of y_1 with respect to x_1; the first of the notations for a Jacobian is suggested by a certain analogy between the properties of the Jacobian and the derivative.

Theorem 1. *If $u_1, u_2, \ldots, u_n$ are functions of $y_1, y_2, \ldots, y_n$ and $y_1, y_2, \ldots, y_n$ are themselves functions of $x_1, x_2, \ldots, x_n$, then*

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(y_1, y_2, \ldots, y_n)} \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} \qquad \ldots(1)$$

For $n = 1$, the theorem reduces to the usual notation

$$\frac{du_1}{dx_1} = \frac{du_1}{dy_1}\frac{dy_1}{dx_1}$$

The proof of the theorem depends on the "row by column" rule of multiplication of determinants combined with the rule for the derivative of a function of a function.

Thus for determinants on the right hand side of (1), rth row of the first is $\dfrac{\partial u_r}{\partial y_1}, \dfrac{\partial u_r}{\partial y_2}, \ldots, \dfrac{\partial u_r}{\partial y_n}$, sth column of the second is $\dfrac{\partial y_1}{\partial x_s}, \dfrac{\partial y_2}{\partial x_s}, \ldots, \dfrac{\partial y_n}{\partial x_s}$, so that the element in the rth row and the sth column of the product is

$$\frac{\partial u_r}{\partial y_1}\frac{\partial y_1}{\partial x_s} + \frac{\partial u_r}{\partial y_2}\frac{\partial y_2}{\partial x_s} + \ldots + \frac{\partial u_r}{\partial y_n}\frac{\partial y_n}{\partial x_s}$$

and this is equal to $\dfrac{\partial u_r}{\partial x_s}$, which is the element in the rth row and the sth column of the Jacobian on the left hand side. Hence the theorem.

Corollary. If $x_r = u_r$, $r = 1, 2, \ldots, n$ and assuming the existence of inverse functions $x_1, x_2, \ldots, x_n$ (that is, assuming that the equations which define $y_1, y_2, \ldots, y_n$ as functions of $x_1, x_2, \ldots, x_n$ determine $x_1, x_2, \ldots, x_n$ as functions of $y_1, y_2, \ldots, y_n$) we find

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} \cdot \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(x_1, x_2, \ldots, x_n)} = 1 \qquad \ldots(2)$$

since $\dfrac{\partial x_i}{\partial x_j} = 0$, for $i \neq j = 1$, for $i = j$

Theorem 2. *If $y_1, y_2, \ldots, y_n$ are determined as functions of $x_1, x_2, \ldots, x_n$ by the equations*

$$\phi_r(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = 0, \; r = 1, 2, \ldots, n$$

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then

$$\frac{\partial(\phi_1, \phi_2, \ldots, \phi_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\partial(\phi_1, \phi_2, \ldots, \phi_n)}{\partial(y_1, y_2, \ldots, y_n)} \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} \quad \ldots(3)$$

[Theorem 1 is a particular form of this theorem.]

Differentiating the equations $\phi_r = 0$ with respect to x_s, we get

$$\frac{\partial \phi_r}{\partial x_s} + \frac{\partial \phi_r}{\partial y_1}\frac{\partial y_1}{\partial x_s} + \frac{\partial \phi_r}{\partial y_2}\frac{\partial y_2}{\partial x_s} + \ldots + \frac{\partial \phi_r}{\partial y_n}\frac{\partial y_n}{\partial x_s} = 0$$

or

$$\frac{\partial \phi_r}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_s} + \frac{\partial \phi_r}{\partial y_2}\frac{\partial y_2}{\partial x_s} + \ldots + \frac{\partial \phi_r}{\partial y_n}\frac{\partial y_n}{\partial x_s} = -\frac{\partial \phi_r}{\partial x_s}$$

so that the element in the rth row and the sth column of the determinant which is the product of the two determinants on the right of (3) is $-\dfrac{\partial \phi_r}{\partial x_s}$, from which the result follows.

Theorem 3. *(i) If $y_{m+1}, y_{m+2}, \ldots, y_n$ are constant with respect to $x_1, x_2, \ldots, x_m$, or (ii) if $y_1, y_2, \ldots, y_m$ are constant with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then*

$$\frac{\partial(y_1, y_2, \ldots, y_m, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_m, \ldots, x_n)} = \frac{\partial(y_1, y_2, \ldots, y_m)}{\partial(x_1, x_2, \ldots, x_m)} \cdot \frac{\partial(y_{m+1}, y_{m+2}, \ldots, y_n)}{\partial(x_{m+1}, x_{m+2}, \ldots, x_n)} \quad \ldots(4)$$

(*i*) $\dfrac{\partial y_r}{\partial x_s} = 0$, when $r = m+1, m+2, \ldots, n$; $s = 1, 2, \ldots, m$.

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} & \frac{\partial y_1}{\partial x_{m+1}} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_m} & \frac{\partial y_2}{\partial x_{m+1}} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \vdots\ \vdots & & \vdots & \vdots\ \vdots & \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_m} & \frac{\partial y_m}{\partial x_{m+1}} & \cdots & \frac{\partial y_m}{\partial x_n} \\ \frac{\partial y_{m+1}}{\partial x_1} & \frac{\partial y_{m+1}}{\partial x_2} & \cdots & \frac{\partial y_{m+1}}{\partial x_m} & \frac{\partial y_{m+1}}{\partial x_{m+1}} & \cdots & \frac{\partial y_{m+1}}{\partial x_n} \\ \vdots & \vdots & \vdots\ \vdots & & \vdots & \vdots\ \vdots & \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_m} & \frac{\partial y_n}{\partial x_{m+1}} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

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$$
= \begin{vmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} & \frac{\partial y_1}{\partial x_{m+1}} & \cdots & \frac{\partial y_m}{\partial x_n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_m} & \frac{\partial y_m}{\partial x_{m+1}} & \cdots & \frac{\partial y_m}{\partial x_n} \\
0 & 0 & \cdots & 0 & \frac{\partial y_{m+1}}{\partial x_{m+1}} & \cdots & \frac{\partial y_{m+1}}{\partial x_n} \\
0 & 0 & \cdots & 0 & \frac{\partial y_{m+2}}{\partial x_{m+1}} & \cdots & \frac{\partial y_{m+2}}{\partial x_n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \frac{\partial y_n}{\partial x_{m+1}} & \cdots & \frac{\partial y_n}{\partial x_n}
\end{vmatrix}
$$

$$
= \frac{\partial(y_1, y_2, \ldots, y_m)}{\partial(x_1, x_2, \ldots, x_n)} \cdot \frac{\partial(y_{m+1}, y_{m+2}, \ldots, y_n)}{\partial(x_{m+1}, x_{m+2}, \ldots, x_n)}
$$

(ii) may also be proved similarly.

Corollary. *In particular,*

$$
\frac{\partial(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n)}{\partial(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)} = \frac{\partial(y_1, \ldots, y_m)}{\partial(x_1, \ldots, x_m)} \qquad \ldots(5
$$

Theorem 4. *If u, v are functions of ξ, η, ζ, and the variables ξ, η, ζ, are themselves functions c the independent variables x and y, then*

$$
\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(\xi, \eta)} \cdot \frac{\partial(\xi, \eta)}{\partial(x, y)} + \frac{\partial(u, v)}{\partial(\eta, \xi)} \cdot \frac{\partial(\eta, \xi)}{\partial(x, y)} + \frac{\partial(u, v)}{\partial(\zeta, \xi)} \cdot \frac{\partial(\zeta, \xi)}{\partial(x, y)} \qquad \ldots(6
$$

We have

$$
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} \qquad \ldots(
$$

$$
\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y} \qquad \ldots(
$$

and if we substitute these values in the Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$, we get

$$
\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial \xi}\frac{\partial(\xi, v)}{\partial(x, y)} + \frac{\partial u}{\partial \eta}\frac{\partial(\eta, v)}{\partial(x, y)} + \frac{\partial u}{\partial \zeta}\frac{\partial(\zeta, v)}{\partial(x, y)} \qquad \ldots
$$

which is a linear expression of the Jacobians of $(\xi, v), (\eta, v)$ and (ζ, v) with respect to x and y.

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Now in each Jacobian on the right of equation (9), substitute the expressions for $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ which are similar to (7) and (8). Each of these Jacobians will be given as a linear expression of the Jacobians of (ξ, η), (η, ζ) and (ζ, ξ) since those of (ξ, ξ), (η, η) and (ζ, ζ) have two identical parallel lines and so vanish. Thus we see that the terms which involve the Jacobian of (ξ, η) are

$$\frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta}\frac{\partial(\xi, \eta)}{\partial(x, y)} + \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi}\frac{\partial(\eta, \xi)}{\partial(x, y)}$$

which is equal to $\frac{\partial(u, v)}{\partial(\xi, \eta)}\frac{\partial(\xi, \eta)}{\partial(x, y)}$, the first terms on the right of equation (6).

Similarly we obtain the remaining two terms and the formula is established.

Example 3. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$$

■
$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

Adding $(\cos\phi)\, R_1$ to $(\sin\phi)\, R_2$,

$$= \frac{r^2\sin\theta}{\sin\phi}\begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta & \cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

Example 4. If $y_1 + y_2 + \dots + y_n = x_1$, $y_2 + y_3 + \dots + y_n = x_1x_2$, …, $y_r + y_{r+1} + \dots + y_n = x_1x_2 \dots x_r$, …, $y_n = x_1x_2 \dots x_n$, then show that

$$\frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)} = x_1^{n-1}x_2^{n-2} \dots x_{n-2}^2 x_{n-1}.$$

■ Solving for $y_1, y_2, \dots, y_n$, we get

$$y_1 = x_1 - x_1x_2 = x_1(1 - x_2)$$
$$y_2 = x_1x_2 - x_1x_2x_3 = x_1x_2(1 - x_3)$$
$$\vdots$$
$$y_{n-1} = x_1x_2 \dots x_{n-1}(1 - x_n)$$
$$y_n = x_1x_2 \dots x_n$$

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$$\therefore \quad \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} =$$

$$\begin{vmatrix} 1-x_2 & -x_1 & 0 & \ldots & 0 \\ x_2(1-x_3) & x_1(1-x_3) & -x_1x_2 & \ldots & 0 \\ x_2x_3(1-x_4) & x_1x_3(1-x_4) & x_1x_2(1-x_4) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_2x_3\ldots x_{n-1}(1-x_n) & x_1x_3\ldots x_{n-1}(1-x_n) & x_1x_2x_4\ldots x_{n-1}(1-x_n) & \ldots & x_1x_2\ldots x_{n-1} \\ x_3x_3\ldots x_n & x_1x_2x_4\ldots x_n & x_1x_3\ldots x_n & \ldots & x_1x_2\ldots x_{n-1} \end{vmatrix}$$

Adding R_n to R_{n-1}, then R_{n-1} to R_{n-2}, ..., then R_2 to R_1 and expanding by last column

$$= (x_1x_2\ldots x_{n-1})\ (x_1x_2\ldots x_{n-2})\ldots(x_1x_2)(x_1)$$

$$= x_1^{n-1}x_2^{n-2}\ldots x_{n-2}^2x_{n-1}$$

Example 5. The roots of the equation in λ

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$$

are u, v, w. Prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

■ Here u, v, w are roots of the equation

$$\lambda^3 - (x+y+z)\lambda^2 + (x^2+y^2+z^2)\lambda - \frac{1}{3}(x^3+y^3+z^3) = 0$$

Let $$x+y+z = \xi,\ x^2+y^2+z^2 = \eta,\ \frac{1}{3}(x^3+y^3+z^3) = \zeta \qquad \ldots(1)$$

and $$u+v+w = \xi,\ vw+wu+uv = \eta,\ uvw = \zeta \qquad \ldots(2)$$

Hence from (1),

$$\frac{\partial(\xi, \eta, \zeta)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ x^2 & y^2 & z^2 \end{vmatrix} \qquad \ldots(3)$$

$$= 2(y-z)(z-x)(x-y)$$

and from (2),

$$\frac{\partial(\xi, \eta, \zeta)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix} \qquad \ldots(4)$$

$$= -(v-w)(w-u)(u-v)$$

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Hence from (3) and (4) and using theorem 1, we get

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{\partial(u, v, w)}{\partial(\xi, \eta, \zeta)} \cdot \frac{\partial(\xi, \eta, \zeta)}{\partial(x, y, z)} = -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

Example 6. If $y_r = \dfrac{u_r}{u}$, $r = 1, 2, \ldots, n$, and if u and u_r are functions of the n independent variables x_1, $x_2, \ldots, x_n$, prove that

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{1}{u^{n+1}}\begin{vmatrix} u & \dfrac{\partial u}{\partial x_1} & \dfrac{\partial u}{\partial x_2} & \cdots & \dfrac{\partial u}{\partial x_n} \\ u_1 & \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\ u_2 & \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_n & \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

■ Now

$$\frac{\partial y_r}{\partial x_s} = \frac{1}{u}\frac{\partial u_r}{\partial x_s} - \frac{u_r}{u^2}\frac{\partial u}{\partial x_s}$$

$$\therefore \quad \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \dfrac{1}{u}\dfrac{\partial u_1}{\partial x_1} - \dfrac{u_1}{u^2}\dfrac{\partial u}{\partial x_1} & \cdots & \dfrac{1}{u}\dfrac{\partial u_1}{\partial x_n} - \dfrac{u_1}{u^2}\dfrac{\partial u}{\partial x_n} \\ \dfrac{1}{u}\dfrac{\partial u_2}{\partial x_1} - \dfrac{u_2}{u^2}\dfrac{\partial u}{\partial x_1} & \cdots & \dfrac{1}{u}\dfrac{\partial u_2}{\partial x_n} - \dfrac{u_2}{u^2}\dfrac{\partial u}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \dfrac{1}{u}\dfrac{\partial u_n}{\partial x_1} - \dfrac{u_n}{u^2}\dfrac{\partial u}{\partial x_1} & \cdots & \dfrac{1}{u}\dfrac{\partial u_n}{\partial x_n} - \dfrac{u_n}{u^2}\dfrac{\partial u}{\partial x_n} \end{vmatrix}$$

Taking out $\dfrac{1}{u}$ from each column and bordering the determinant, we get

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{1}{u^n}\begin{vmatrix} 1 & 0 & \cdots & 0 \\ u_1 & \dfrac{\partial u_1}{\partial x_1} - \dfrac{u_1}{u}\dfrac{\partial u}{\partial x_1} & \cdots & \dfrac{\partial u_1}{\partial x_n} - \dfrac{u_1}{u}\dfrac{\partial u}{\partial x_n} \\ u_2 & \dfrac{\partial u_2}{\partial x_1} - \dfrac{u_2}{u}\dfrac{\partial u}{\partial x_1} & \cdots & \dfrac{\partial u_2}{\partial x_n} - \dfrac{u_2}{u}\dfrac{\partial u}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ u_n & \dfrac{\partial u_n}{\partial x_1} - \dfrac{u_n}{u}\dfrac{\partial u}{\partial x_1} & \cdots & \dfrac{\partial u_n}{\partial x_n} - \dfrac{u_n}{u}\dfrac{\partial u}{\partial x_n} \end{vmatrix}$$

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Operating

$$C_2 + \frac{1}{u}\frac{\partial u}{\partial x_1}C_1,\ C_3 + \frac{1}{u}\frac{\partial u}{\partial x_2}C_1,\ \ldots,\ C_{n+1} + \frac{1}{u}\frac{\partial u}{\partial x_n}C_1$$

$$= \frac{1}{u^n}\begin{vmatrix} 1 & \frac{1}{u}\frac{\partial u}{\partial x_1} & \cdots & \frac{1}{u}\frac{\partial u}{\partial x_n} \\ u_1 & \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ u_2 & \frac{\partial u_2}{\partial x_1} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ u_n & \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{1}{u^{n-1}}\begin{vmatrix} u & \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n} \\ u_1 & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ u_2 & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_n & \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Example 7. If $u = \dfrac{x^2 + y^2 + z^2}{x}$, $v = \dfrac{x^2 + y^2 + z^2}{y}$, and $w = \dfrac{x^2 + y^2 + z^2}{z}$ find $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$.

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 - \frac{y^2 + z^2}{x^2} & \frac{2y}{x} & \frac{2z}{x} \\ \frac{2x}{y} & 1 - \frac{x^2 + z^2}{y^2} & \frac{2z}{y} \\ \frac{2x}{z} & \frac{2y}{z} & 1 - \frac{x^2 + y^2}{z^2} \end{vmatrix}$$

Applying $C_1 \to C_1 + \frac{y}{x}C_2 + \frac{z}{x}C_3$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{x^2 + y^2 + z^2}{x^2} & \frac{2y}{x} & \frac{2z}{x} \\ \frac{x^2 + y^2 + z^2}{xy} & 1 - \frac{x^2 + z^2}{y^2} & \frac{2z}{y} \\ \frac{x^2 + y^2 + z^2}{xz} & \frac{2z}{z} & 1 - \frac{x^2 + y^2}{z^2} \end{vmatrix}$$

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$$= \frac{(x^2+y^2+z^2)}{x^2 \cdot xy \cdot xz} \begin{vmatrix} 1 & 2xy & 2xz \\ 1 & xy - \frac{x}{y}(x^2+z^2) & 2xz \\ 1 & 2yz & xz - \frac{x}{z}(x^2+y^2) \end{vmatrix}$$

$$= \frac{(x^2+y^2+z^2)}{x^4 yz} \begin{vmatrix} 1 & 2xy & 2xz \\ 0 & -\frac{x(x^2+y^2+z^2)}{y} & 0 \\ 0 & 0 & -\frac{x}{z}(x^2+y^2+z^2) \end{vmatrix}$$

$$= \frac{(x^2+y^2+z^2)^3}{x^2y^2z^2}$$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{x^2y^2z^2}{(x^2+y^2+z^2)^3}$$

Ex. 1. If $u = \cos x$, $v = \sin x \cos y$, $w = \sin x \sin y \cos z$, then show that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^2 \sin^3 x \sin^2 y \sin z.$$

Ex. 2. If $u = a \cosh x \cos y$, $v = a \sinh x \sin y$, then show that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{2}a^2(\cosh 2x - \cos 2y).$$

Ex. 3. If $x + y + z = u$, $y + z = uv$, $z = uvw$, then show that

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v.$$

Ex. 4. If α, β, γ are the roots of the equation in t, such that

$$\frac{u}{a+t} + \frac{v}{b+t} + \frac{w}{c+t} = 1,$$

then prove that

$$\frac{\partial(u, v, w)}{\partial(\alpha, \beta, \gamma)} = -\frac{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}{(b-c)(c-a)(a-b)}.$$

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