System of Linear Equations

Systems of Linear Equations

Suppose, we have system of linear equation

$$a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots\ldots\ldots +a_{1n}x_n =b_1$$
$$a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots\ldots\ldots +a_{2n}x_n =b_2$$
$$\vdots$$
$$a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\ldots\ldots\ldots +a_{nn}x_n =b_n$$

Let $A=\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & & & & \\ . & & & & \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{bmatrix}$ $X=\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$ $b=\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}$

$$Ax=b$$

A Matrix is an array of numbers:

$$A = \begin{bmatrix} 4 & 10 & 5 \\ 5 & 12 & 20 \end{bmatrix}$$

(This one has 2 Rows and 3 Columns)

Types of Matrix (Cont…)

The **Main Diagonal** starts at the top left and goes down to the right:

$$\begin{bmatrix} \color{red}{7} & 6 & 4 \\ 4 & \color{red}{2} & -2 \\ 3 & 0 & \color{red}{9} \end{bmatrix}$$

Another example:

$$\begin{bmatrix} \color{red}{6} & 4 & 24 \\ 1 & \color{red}{-9} & 8 \end{bmatrix}$$

Types of Matrix (Cont…)

A **Transpose** is where we swap entries across the main diagonal (rows become columns) like this:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 4 & -9 \\ 24 & 8 \end{bmatrix}$$

The main diagonal stays the same.

Types of Matrix (Cont…)

Square

A **square** matrix has the same number of rows as columns.

$$\begin{bmatrix} 2 & 0 \\ 1 & 8 \end{bmatrix}$$

A square matrix (2 rows, 2 columns)

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \\ 3 & 0 & 7 \end{bmatrix}$$

Also a square matrix (3 rows, 3 columns)

Types of Matrix (Cont…)

Identity Matrix

An **Identity Matrix** has **1**s on the main diagonal and **0**s everywhere else:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3×3 Identity Matrix

- It is square (same number of rows as columns)
- It can be large or small (2×2, 100×100, ... whatever)
- Its symbol is the capital letter **I**

It is the matrix equivalent of the number "1", when we multiply with it the original is unchanged:

$$A \times I = A$$

$$I \times A = A$$

Types of Matrix (Cont…)

Diagonal Matrix

A diagonal matrix has zero anywhere not on the main diagonal:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A diagonal matrix

Scalar Matrix

A scalar matrix has all main diagonal entries the same, with zero everywhere else:

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

A scalar matrix

Types of Matrix (Cont…)

Triangular Matrix

Lower triangular is when all entries above the main diagonal are zero:

$$\begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & 0 \\ 7 & 6 & -3 \end{bmatrix}$$

A lower triangular matrix

Upper triangular is when all entries below the main diagonal are zero:

$$\begin{bmatrix} 2 & -2 & 7 \\ 0 & 4 & 11 \\ 0 & 0 & 5 \end{bmatrix}$$

An upper triangular matrix

Types of Matrix (Cont…)

Zero Matrix (Null Matrix)

Zeros just everywhere:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Zero matrix

Symmetric

In a Symmetric matrix matching entries either side of the main diagonal are **equal**, like this:

$$\begin{bmatrix} 3 & 2 & 11 & 5 \\ 2 & 9 & -1 & 6 \\ 11 & -1 & 0 & 7 \\ 5 & 6 & 7 & 9 \end{bmatrix}$$

Symmetric matrix

It must be square, and is equal to its own transpose

$$A = A^T$$

Types of Matrix (Cont…)

Skew-Symmetric Matrix

Square Matrix A is said to be skew-symmetric if $a_{ij}=-a_{ji}$ for all i and j.
In other words, we can say that matrix A is said to be skew-symmetric if transpose of matrix A is equal to negative of Matrix A i.e ($\mathbf{A^T=-A}$).
Note that all the main diagonal elements in skew-symmetric matrix are zero.

Lets take an example of **matrix** $A = \begin{bmatrix} 0 & -5 & 4 \\ 5 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix}$.

It is **skew-symmetric matrix** because $a_{ij} = -a_{ji}$ for all i and j.
Example, a_{12} = -5 and a_{21}=5 which means $a_{12} = -a_{21}$. Similarly, this condition holds true for all other values of i and j.

We can also verify that **Transpose** of **Matrix A** is equal to negative of **matrix A i.e** $A^T = -A$.

$$A^T = \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & -5 & 4 \\ 5 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix}.$$

We can clearly see that $A^T = -A$ which makes **A skew-symmetric matrix**.

Types of Matrix (Cont…)

Hermitian

A Hermitian matrix is symmetric except for the imaginary parts that swap sign across the main diagonal:

$$\begin{bmatrix} 3 & 2+3i & -2i & 5-i \\ 2-3i & 9 & 12 & 1+4i \\ 2i & 12 & 1 & 7 \\ 5+i & 1-4i & 7 & 12 \end{bmatrix}$$

Hermitian matrix

See how **+i** changes to **−i** and vice versa?

Changing the sign of the second part is called the conjugate, and so the correct definition is:

A Hermitian matrix is equal to its own **conjugate transpose**:

$$A = \overline{A^T}$$

This also means the main diagonal entries must be purely real (to be their own conjugate).

It is named after French mathematician Charles Hermite.

Types of Matrix (Cont…)

Orthogonal Matrix: A square matrix with real numbers or values is termed as an orthogonal matrix if its transpose is equal to the inverse matrix of it. In other words, the product of a square orthogonal matrix and its transpose will always give an identity matrix.

Suppose A is the square matrix with real values, of order n × n.
Also, let A^T is the transpose matrix of A. Then according to the definition:
If, $A^T = A^{-1}$ condition is satisfied,
then **$A \times A^T = I$**

Example:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Systems of Linear Equations

Suppose, we have system of linear equation

$$a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots\ldots\ldots +a_{1n}x_n =b_1$$
$$a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots\ldots\ldots +a_{2n}x_n =b_2$$
$$\vdots$$
$$a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\ldots\ldots\ldots +a_{nn}x_n =b_n$$

Let $A=\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & & & & \\ . & & & & \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{bmatrix}$ $X=\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$ $b=\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}$

Ax=b

For the given system of linear equations, we have

$[A \mid b]$: Augmented matrix

Case 1: if $|A| \neq 0$ $\Longrightarrow A^{-1}$ exist.

$\Longrightarrow x = A^{-1} b$

Case 2: if $b=0$: Homogeneous system

& if $|A| \neq 0$, $\Longrightarrow x = A^{-1} b = 0$, we will get a trivial solution i.e. 0

Consistency and inconsistency of System of Linear equation

Consistency: If solution exists for $Ax=b$

Inconsistency: If solution does not exists for $Ax=b$

For Augmented matrix, $[A \mid b]$, $n=$ no. of unknowns

1. If $\text{rank}(A) = \text{rank}([A \mid b]) = n$

$\Longrightarrow$ System is consistent and has unique solution.

2. If $\text{rank}(A) = \text{rank}([A \mid b]) = n1 < n$

$\Longrightarrow$ system is consistent and has infinite number of solutions.

3. If $\text{rank}(A) \neq \text{rank}([A \mid b])$

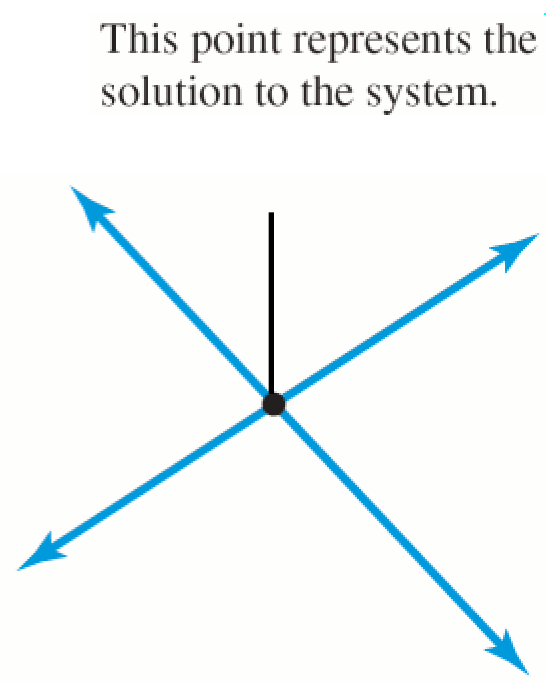
$\Longrightarrow$ System is inconsistent and has no solution.

A **system of equations** is a group of two or more equations. To solve a system of equations means to find values for the variables that satisfy all of the equations in the system.

Systems of equations can involve any number of equations and variables; however, we will limit ourselves to situations containing two variables in this section.

Systems of Linear Equations

Graphing a system of two linear equations in two unknowns gives one of three possible situations:

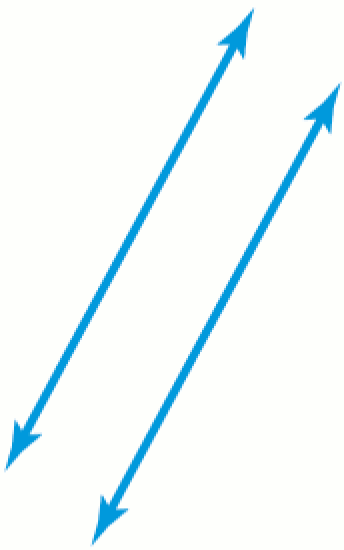


Case 1
Intersecting Lines
One Solution

Case 1: Lines intersecting in a single point. The ordered pair that represents this point is the *unique solution* for the system.

$3x+y=17$
$4x-y=18$

Systems of Linear Equations



Case 2
Parallel Lines
No Solutions

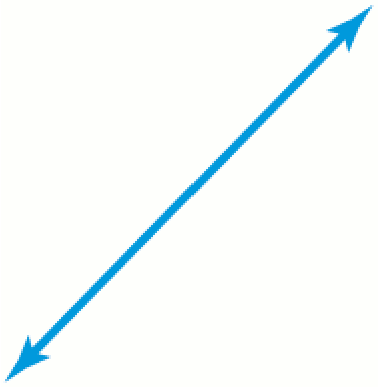
Case 2: Lines that are distinct parallel lines and therefore don't intersect at all. Because the lines have no common points, this means that the system has *no solutions*.

$3x+2y=5$
$6x+4y=8$

Systems of Linear Equations

Case 3: Two lines that are the same line. The lines have an infinite number of points in common, so the system will have *an infinite number of solutions*.

$$2x+4y=8$$
$$x+2y=4$$



Case 3
Same Line
Infinite Number
of Solutions

Systems of Linear Equations

In numerical methods, there are two types of methods:

•*Direct Method*

•*Iterative Method*

Systems of Linear Equations

Example 1: For System of linear equation Ax=b,

If A is Diagonal Matrix,

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & . & 0 \\ 0 & 0 & 0 & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_n \end{bmatrix}$$

$$x_n = \frac{b_n}{a_{nn}}$$

Systems of Linear Equations

Example 2: For System of linear equation Ax=b,

If A is Lower Triangular Matrix,

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ . & . & . & . \\ l_{n1} & l_{n2} & . & l_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_n \end{bmatrix}$$

$$x_1 = \frac{b_1}{l_{11}}, x_2 = \frac{1}{l_{22}}(b_2 - l_{12}x_1)$$

$$x_n = \frac{1}{l_{22}}(b_n - l_{1n}x_1 - l_{2n}x_2.....)$$

This is called forward substitution method.

Systems of Linear Equations

Example 3: For System of linear equation Ax=b,

If A is Upper Triangular Matrix,

$$\begin{bmatrix} U_{11} & U_{12} & .... & U_{1n} \\ 0 & U_{22} & .... & U_{2n} \\ . & . & . & . \\ 0 & 0 & . & U_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_n \end{bmatrix}$$

$$x_n = \frac{b_n}{U_{nn}}$$

This is called Backward substitution method.

Gauss Elimination Method

A set of *n* equations and *n* unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \qquad \ldots(1)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \qquad \ldots(2)$$

$$\begin{matrix} . & & . \\ . & & . \\ . & & . \end{matrix}$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

(n-1) steps of forward elimination

Forward Elimination

Step 1

For Equation 2, divide Equation 1 by a_{11} and multiply by a_{21}.

$$\left[\frac{a_{21}}{a_{11}}\right](a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1)$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Forward Elimination

Subtract the result from Equation 2.

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$- \quad a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

or $$a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2$$

Forward Elimination

Repeat this procedure for the remaining equations to reduce the set of equations as

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n &= b'_3 \\
\vdots \qquad \vdots \qquad\qquad \vdots \\
a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n &= b'_n
\end{aligned}
$$

End of Step 1

Forward Elimination

Step 2

Now eliminate x2 from the last (n-2) equations.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a''_{n3}x_3 + \ldots + a''_{nn}x_n = b''_n$$

End of Step 2

Forward Elimination

Repeat this process till we finally obtain upper triangular form:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

End of Step (n-1)

Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

Back Substitution Starting Eqns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Where, $a_{nn}^{(n-1)}$ indicates that the element a_{nn} has changed (n-1) times

Back Substitution

The required solution is obtained from the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Then, x_{n-1}, x_{n-2}…….. x_1 *can be calculated.*

Example-1 Use Gauss Elimination method to solve

$2x_1+x_2+x_3=10$ ………..(1)

$3x_1+2x_2+3x_3=18$ ………..(2)

$x_1+4x_2+9x_3=16$ ………..(3)

Solution: At first eliminate x_1 from eq. (2) and (3)

Multiply eq.(1) by (3/2) and subtract from eq(2)

$$3x_1+2x_2+3x_3=18$$

$$-\quad [3x_1+(3/2)x_2+(3/2)x_3=10(3/2)]$$

$$(2-3/2)x_2+(3-3/2)x_3=18-15$$

$$(1/2)x_2+(3/2)x_3=3$$

$$x_2+3x_3=6 \qquad \text{………(4)}$$

Multiply eq.(1) by (1/2) and subtract from eq(3)

$$x_1+4x_2+9x_3=16$$

$$-\quad [x_1+(1/2)x_2+(1/2)x_3=10(1/2)]$$

$$(4-1/2)x_2+(9-1/2)x_3=16-5$$

$$(7/2)x_2+(17/2)x_3=11$$

$$7x_2+17x_3=22 \qquad \text{………(5)}$$

Solution (Cont……)

Now eliminate x_2 from eq. (5)

Multiply eq.(4) by (7) and subtract from eq(5)

$$7x_2+17x_3 = 22$$

$$- \quad [7x_2+21x_3 = 42]$$

$$(17-21)x_3 = 22-42$$

$$(-4)x_3 = -20$$

$$\boxed{x_3 = 5}$$

Now from eq.(4)

$$x_2+3x_3 = 6$$

$$x_2+3\times5 = 6$$

$$\boxed{x_2 = 6-15 = -9}$$

Now from eq.(1)

$$2x_1+x_2+x_3 = 10$$

$$2x_1-9+5 = 10$$

$$\boxed{x_1 = 7}$$

Pivot

If, in the system of linear equation pivot (a_{11}) is zero or very small compared to other coefficients of the equation, then we can find the largest available coefficient in the column below the pivot equation & then interchange the two rows.

In this way we can find the pivot equation with a non zero pivot.

Note: if we search both column and rows for largest element, the procedure is called "Complete pivoting"

Example-1 Use Gauss Elimination method with partial pivoting to solve

$x_1+2x_2+4x_3=9$ ………..(1)

$2x_1+5x_2-3x_3=9$ ………..(2)

$5x_1+12x_2-2x_3=25$ ………..(3)

$$\begin{bmatrix} 1 & 2 & 4 & | & 9 \\ 2 & 5 & -3 & | & 9 \\ 5 & 12 & -2 & | & 25 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 12 & -2 & | & 25 \\ 2 & 5 & -3 & | & 9 \\ 1 & 2 & 4 & | & 9 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 12/5 & -2/5 & | & 5 \\ 2 & 5 & -3 & | & 9 \\ 1 & 2 & 4 & | & 9 \end{bmatrix} R_1 \rightarrow R_1/5$$

$$\begin{bmatrix} 1 & 12/5 & -2/5 & | & 5 \\ 0 & 1/5 & -11/5 & | & -1 \\ 0 & -2/5 & 22/5 & | & 4 \end{bmatrix} R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & 12/5 & -2/5 & | & 5 \\ 0 & -2/5 & 22/5 & | & 4 \\ 0 & 1/5 & -11/5 & | & -1 \end{bmatrix} R_2 \to R_3$$

$$\begin{bmatrix} 1 & 12/5 & -2/5 & | & 5 \\ 0 & 1 & -11 & | & -10 \\ 0 & 1/5 & -11/5 & | & -1 \end{bmatrix} R_2 \to R_2/(-2/5)$$

$$\begin{bmatrix} 1 & 12/5 & -2/5 & | & 5 \\ 0 & 1 & -11 & | & -10 \\ 0 & 0 & 0 & | & 1 \end{bmatrix} R_3 \to R_3 - R_2(1/5)$$

Since $\text{rank(A)} \neq \text{rank(A | b)}$

$2 \neq 3$

So no solution exists

Practice Problems

1. Solve the following linear system of equation using Gauss elimination method:

$$y+z=2$$
$$2x+3z=5$$
$$x+y+z=3$$

2. Use Gauss Elimination method with partial pivoting to solve

$$2x+3y+z=9$$
$$x+2y+3z=6$$
$$3x+y+2z=8$$

Suggested books

1. Numerical Methods by **S.R.K Lyenger & R.K. Jain.**

2. Numerical Analysis by **Richard L. Burden.**

3. Introductory methods of Numerical analysis by **S.S. Sastry**.

Thank you