B.Sc. Mathematics – 3rd Semester

MTM 302 – Differential Equations

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A relation/equation connecting independent variable(s), dependent variable(s) and one or more of their differential coefficients or differentials is called Differential Equation.

In a differential equation if there is only one independent variable, then it is called Ordinary Differential Equation and those involving more than one independent variables are called Partial Differential Equations.

A total different equation contains two or more dependent variables together with their differentials or differential coefficients with respect to a single independent variable.

When in an ordinary or partial differential equations the dependent variable and its derivatives occur to the first degree only and not as higher powers or products, the equation is called **linear**, otherwise it is **non-linear**.

Order and Degree of a Differential Equation:

The order of a differential equation is the order of the highest ordered differential coefficient involving in it, while degree of an equation is the greatest exponent of the highest ordered derivative when the equation has been made rational and integral as far as the derivatives are concerned.

Examples: Find the order and degree of following equations

(i) $\frac{dy}{dx}+y=x^3$, (ii) $\left(\frac{dy}{dx}\right)^2-2\sqrt{y}=\sin x$, (iii) $\frac{d^2y}{dx^2}=7$,

(iv) $5\frac{d^2y}{dx^2}+\sqrt{\frac{dy}{dx}}=3yx$, (v) $\frac{d^3y}{dx^3}+\left(\frac{d^2y}{dx^2}\right)^2-2\frac{dy}{dx}=e^x$.

Solution:

(i) order 1 and degree 1, (ii) order 1 and degree 2, (iii) order 2 and degree 1,

(iv) order 2 and degree 2, (v) order 3 and degree 1.

Formulation of Differential Equation:

Consider a relation $y = cx$, where c is an arbitrary constant.

Differentiating both sides w.r.t. x, we get

$\frac{dy}{dx} = c$. So we have $\frac{dy}{dx} = \frac{y}{x}$, which is of first order and first degree.

Again consider $y = A\cos(x+B)$, where A and B are an arbitrary constants

$\frac{dy}{dx} = -A\sin(x+B)$, $\frac{d^2y}{dx^2} = -A\cos(x+B)$

So, $\frac{d^2y}{dx^2} + y = 0$, which is of 2nd order and 1st degree.

$$f_1\left(x, y, \frac{dy}{dx}\right) = 0 \Leftrightarrow g_1(x, y, c) = 0$$

$$f_2\left(x, y, \frac{d^2y}{dx^2}\right) = 0 \Leftrightarrow g_2(x, y, A, B) = 0$$

This shows that a solution of the n^{th} order ODE should involve n arbitrary constants.

<u>Examples:</u> Determine the differential equations, whose primitives are

(i) $y = ax + b$, (ii) $y = kx + k - k^2$

(iii) $y^3 = 2kx + k^3$, (iv) $x\cos\theta + y\sin\theta = a$

(a, b, k are arbitrary constants)

General, Particular and Singular Solutions:

The solution which contains a number of arbitrary constant equal to the order of the differential equation is called the general solution or the complete primitive or the complete integral. Solutions obtained from the general solution by giving particular values to the arbitrary constants are called particular solutions.

In some special cases, we get solutions of a differential equation which can't be derived from its general solution by giving particular values to the arbitrary constants. These solutions are known as singular solutions.

Part I

Equations of First Order and First Degree

An ordinary differential equation of first order and first degree

$\frac{dy}{dx} = f(x, y)$ can always be written as $Mdx + Ndy = 0$, where M and N are functions of x and y.

Assuming that the equation has a solution, we shall discuss methods by which the general solution of these equations can be found in terms of known functions. We classify these equations according to the methods by which they are solved. These classifications are:

(i) Equations solvable by separation of variables,

(ii) Homogeneous equations

(iii) Exact equations

(iv) Linear equations

Equations solvable by separation of variables:

If the equation $Mdx + Ndy = 0$ can be put in the form $f_1(x)dx + f_2(y)dy = 0$, that is , in the separated variables form, then the equation can be solved easily by integrating each term separately. The general solution of the above equation is $\int f_1(x)dx + \int f_2(y)dy = c$, where c is an arbitrary constant. By giving a particular value to c, we shall get a particular solution.

Example: Solve $x^2\dfrac{dy}{dx}+y=1$

Solution: we have from the equation,

$$x^2\frac{dy}{dx}=1-y \quad \text{or} \quad \frac{dx}{x^2}=\frac{dy}{1-y}$$

The variables have been separated.

Now integrating both sides, we get the general solution as

$$-\frac{1}{x}=-\log(1-y)+\log c \qquad \text{(where } c \text{ is an arbitrary constant)}$$

$$\text{or, } \log\frac{1-y}{c}=\frac{1}{x} \Rightarrow 1-y=ce^{\frac{1}{x}} \Rightarrow y=1-ce^{\frac{1}{x}}$$

Examples: Solve the following equations

(a) $x\sqrt{y}dx+(1+y)\sqrt{1+x}dy=0$

(b) $(x+y)^2\dfrac{dy}{dx}=a^2$

(c) $ydx+\left(1+x^2\right)\tan^{-1}xdy=0$

(d) $\operatorname{cosec}x\log ydy+x^2y^2dx=0$

(e) $\sqrt{1-x^2}dy+\sqrt{1-y^2}dx=0$

(f) $x\log xdy+\sqrt{1-y^2}dx=0$

(g) $(e^x+1)ydy=(y^2+1)e^xdx$, given that $y=0$ at $x=0$.

Solutions:

(a) We have $\dfrac{x}{\sqrt{1+x}}dx+\dfrac{1+y}{\sqrt{y}}dy=0$

$$\Rightarrow\left(\sqrt{1+x}-\frac{1}{\sqrt{1+x}}\right)dx+\left(\frac{1}{\sqrt{y}}+\sqrt{y}\right)dy=0$$

Integrating, we get the general solution as

$$\frac{2}{3}(1+x)^{3/2}-2\sqrt{1+x}+2\sqrt{y}+\frac{2}{3}y^{3/2}=c,$$

where c is an arbitrary constant.

(b) Put $x+y=v$, $\dfrac{dy}{dx}=\dfrac{dv}{dx}-1$

So, we have $v^2\left(\dfrac{dv}{dx}-1\right)=a^2 \Rightarrow \dfrac{dv}{dx}=1+\dfrac{a^2}{v^2}=\dfrac{v^2+a^2}{v^2}$

$$\Rightarrow \frac{v^2 dv}{v^2+a^2}=dx \Rightarrow \left(1-\frac{a^2}{v^2+a^2}\right)dv=dx$$

Integrating we get

$v-\dfrac{a^2}{a}\tan^{-1}\dfrac{v}{a}=x+c$, where c is an arbitrary constant.

Thus, $x+y-a\tan^{-1}\left(\dfrac{x+y}{a}\right)=x+c \Rightarrow y=a\tan^{-1}\left(\dfrac{x+y}{a}\right)+c$

(c) $\dfrac{dx}{\left(1+x^2\right)\tan^{-1}x}+\dfrac{dy}{y}=0$

Putting $\tan^{-1}x=z$, $dz=\dfrac{dx}{1+x^2}$, we have $\dfrac{dz}{z}+\dfrac{dy}{y}=0$.

Integrating, we have

$\log z+\log y=\log c$, where c is an arbitrary constant.

$\Rightarrow yz=c$

$\Rightarrow y\tan^{-1}x=c$

(d) $\dfrac{\log y}{y^2}dy+\dfrac{x^2}{\operatorname{cosec} x}dx=0$

Put $\log y=z$, i.e. $y=e^z$ and $dz=\dfrac{dy}{y}$,

So, we have $ze^{-z}dz+x^2\sin dx=0$

Integrating, we have

$\int ze^{-z}dz+\int x^2\sin dx=c$, where c is an arbitrary constant

$$\Rightarrow -ze^{-z}+\int e^{-z}dz+\left(-x^2\cos x\right)+2\int x\cos xdx=0$$

$$\Rightarrow -ze^{-z}-e^{-z}-x^2\cos x+2x\sin x-2\int \sin xdx=c$$

$$\Rightarrow ze^{-z}+e^{-z}+x^2\cos x-2x\sin x-2\cos x=c$$

$$\Rightarrow \frac{\log y}{y}+\frac{1}{y}+x^2\cos x-2x\sin x-2\cos x=c$$

$$\Rightarrow \frac{1}{y}(1+\log y)+\left(x^2-2\right)\cos x-2x\sin x=c$$

Homogenous Equations:

If the equation $Mdx + Ndy = 0$ $\left[\text{or, } \frac{dy}{dx} = f(x, y)\right]$ can be put in the form $\frac{dy}{dx} = f_1\left(\frac{y}{x}\right)$, or, if in the equation $Mdx + Ndy = 0$, M and N be homogenous functions of x and y of same degree, then the equation is called homogenous. In this case, the substitution of $y = vx$, where v is a function x, enables us to separate the variables. By this substitution, we change the variable y, such that $\frac{dy}{dx}=v+x\frac{dv}{dx}$. Then after integration, v is replaced by $\frac{y}{x}$.

Examples: Solve $x^2ydx-\left(x^3+y^3\right)dy=0$

Solve: Here M and N are homogenous functions of x and y of degree 3.

The given equation may be written as $\frac{dy}{dx}=\frac{x^2y}{x^3+y^3}$.

Let $y=vx$, Then $\frac{dy}{dx}=v+x\frac{dv}{dx}$

Then the equation becomes

$$v+x\frac{dv}{dx}=\frac{x^3v}{x^3+x^3v^3}=\frac{v}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v}{1+v^3}-v=-\frac{v^4}{1+v^3}$$

$$\Rightarrow \frac{1+v^3}{v^4}dv=-\frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{v^4}+\frac{1}{v}\right)dv=-\frac{dx}{x}$$

Integrating, we get $\Rightarrow -\frac{1}{3v^3}+\log v=-\log x+\log c$, c is being a constant.

$$\Rightarrow \log\frac{vx}{c}=\frac{1}{3v^3}$$

$$\Rightarrow \frac{vx}{c}=e^{\frac{1}{3v^3}}$$

Putting $y=vx$, we get $y=ce^{\frac{x^3}{3y^3}}$

Examples: (a) $y^2dx+\left(xy+x^2\right)dy=0$

(b) $x\dfrac{dy}{dx}+\dfrac{y^2}{x}=y$

(c) $xy\dfrac{dy}{dx}-y^2=\left(x+y\right)^2e^{-\frac{y}{x}}$

Non–homogenous equation reducible to homogenous form:

Consider a non-homogenous equation of the form

$$(a_1x+b_1y+c_1)dx=(a_2x+b_2y+c_2)dy,$$

i.e., $\dfrac{dy}{dx}=\dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, in which $\dfrac{a_1}{a_2}\neq\dfrac{b_1}{b_2}$ (1)

This equation can be made homogenous by the substitution $x=x'+h$ and $y=y'+k$, where h and k are constants and so chosen that

$$\left.\begin{aligned} a_1h+b_1k+c_1&=0\\ a_2h+b_2k+c_2&=0 \end{aligned}\right\} \qquad (2)$$

Then equation (1) is reduced to the homogenous equation (of degree 1)

$$\frac{dy'}{dx'}=\frac{a_1x'+b_1y'}{a_2x'+b_2y'}$$

which can be solved using substitution $y'=vx'$ as before.

Now, if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}$, then the substitution $a_1x+b_1y=v \Rightarrow a_1+b_1\dfrac{dy}{dx}=\dfrac{dv}{dx}$

transforms the equation to a form, which can be easily solved.

Example: Solve $\left(x+2y-3\right)dx=\left(2x+y-3\right)dy$.

Solution: We have $\dfrac{dy}{dx}=\dfrac{x+2y-3}{2x+y-3}$

Let us put $x=x'+h, y=y'+k$, so that becomes $\dfrac{dy'}{dx'}$ and the equation

becomes $\dfrac{dy'}{dx'}=\dfrac{x'+2y'+\left(h+2k-3\right)}{2x'+y'+\left(2h+k-3\right)}$

Let us choose h and k such that

$$h+2k-3=0 \text{ and } 2h+k-3=0$$

which gives $h=k=1$ and the equation becomes

$$\frac{dy'}{dx'}=\frac{x'+2y'}{2x'+y'}$$, which is homogenous.

We now put $y'=vx'$, so that $\frac{dy'}{dx'}=v+x'\frac{dv}{dx'}$

The equation becomes $v+x'\frac{dv}{dx'}=\frac{1+2v}{2+v}$

$$\Rightarrow x'\frac{dv}{dx'}=\frac{1-v^2}{2+v}\Rightarrow\frac{2+v}{1-v^2}dv=\frac{dx'}{x'}$$

$$\Rightarrow\left[\frac{2}{1-v^2}-\frac{1}{2}\frac{(-2v)}{1-v^2}\right]dv=\frac{dx'}{x'}$$

$$\Rightarrow\left[\frac{1}{1+v}-\frac{1}{1-v}-\frac{1}{2}\frac{(-2v)}{1-v^2}\right]dv=\frac{dx'}{x'}$$

Integrating we get $\log\left(\frac{1+v}{1-v}\right)-\frac{1}{2}\log\left(1-v^2\right)=\log\left(cx'\right)$

Putting $v=\frac{y'}{x'}$, we get

$$\frac{1+\frac{y'}{x'}}{1-\frac{y'}{x'}}\cdot\frac{1}{\sqrt{\left(1-\frac{y'^2}{x'^2}\right)}}=cx'$$

$$\Rightarrow \frac{x'+y'}{x'-y'} \cdot \frac{x'}{\sqrt{\left(x'^2-y'^2\right)}} = cx'$$

Now Putting $x' = x-1,\ y' = y-1$

$$\frac{x+y-2}{x-y} \cdot \frac{1}{\sqrt{(x-1)^2-(y-1)^2}} = c, \text{ where } c \text{ is an arbitrary constant.}$$

Example: Solve $\dfrac{dy}{dx} = \dfrac{x+y+1}{3x+3y+1}$

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{(x+y)+1}{3(x+y)+1} \tag{1}$$

Let $x+y=v \Rightarrow 1+\frac{dy}{dx}=\frac{dv}{dx}$

Then (1) becomes

$$\frac{dv}{dx}-1=\frac{v+1}{3v+1} \Rightarrow \frac{dv}{dx}=\frac{v+1+3v+1}{3v+1}=\frac{4v+2}{3v+1}$$

$$\Rightarrow \frac{3v+1}{2v+1}dv=2dx \Rightarrow \frac{\frac{3}{2}(2v+1)-\frac{1}{2}}{2v+1}dv=2dx$$

$$\Rightarrow \left(\frac{3}{2}-\frac{1/2}{2v+1}\right)dv=2dx.$$

Integrating both sides, we get

$$\frac{3}{2}v-\frac{1}{4}\log\left(2v+1\right)=2x+c$$

$$\Rightarrow \frac{3}{2}\left(x+y\right)-\frac{1}{4}\log\left(2x+2y+1\right)=2x+c$$

$\Rightarrow \log\left(2x+2y+1\right)+2x-6y=c_1$, where c_1 is an arbitrary constant.

Examples: Solve the following equations:

(i) $\left(3y-7x+7\right)dx+\left(7y-x+3\right)dy=0$.

(ii) $\left(7x+4y-4\right)dx+\left(4x+5y-5\right)dy=0$

(iii) $\left(6x+9y-7\right)dx=\left(2x+3y-6\right)dy$

Exact Equations:

If the differential equation $Mdx + Ndy = 0$ can be expressed in the form $du = 0$, where u is a function of x and y, without multiplying by any other function, then the differential equation $Mdx + Ndy = 0$ is said to be an exact differential equation and its general solution can be written as $u(x, y) = c$, where c is an arbitrary constant.

Some Exact Equations:

(1) $xdy + ydx = d(xy) = 0$

(2) $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right) = 0$

(3) $\dfrac{xdy - ydx}{y^2} = d\left(\dfrac{x}{y}\right) = 0$

(4) $\dfrac{xdy - ydx}{xy} = d\left(\log\dfrac{y}{x}\right) = 0$

(5) $\dfrac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right) = 0$

(6) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right) = 0$

(7) $\dfrac{2xydy - y^2dx}{x^2} = d\left(\dfrac{y^2}{x}\right) = 0$

(8) $\dfrac{y^2dx + 2xydy}{x^2y^4} = d\left(-\dfrac{1}{xy^2}\right) = 0$

(9) $\dfrac{xdy + ydx}{\sqrt{1 - x^2y^2}} = d\left[\sin^{-1}(xy)\right] = 0$

Theorem: The necessary and sufficient condition for the ordinary differential equation $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. [we assume that functions M and N have continuous partial derivatives]

Solution of Exact Equation; $Mdx + Ndy = 0$**:**

Integrate the terms of Mdx considering y as constant, then integrate those terms of Ndy which do not contain x and then equate the sum of these integrals to a constant.

Example: Examine whether the equation $\left(a^2-2xy-y^2\right)dx-(x+y)^2 dy=0$ is exact or not. If it be exact, then find its primitive.

Solution: Here, $M=a^2-2xy-y^2$ and $N=-\left(x+y\right)^2$

So, $\dfrac{\partial M}{\partial y}=-2x-2y,\ \dfrac{\partial N}{\partial x}=-2\left(x+y\right)$

Hence, the equation is exact. Thus, the primitive of the equation is $\int\left(a^2-2xy-y^2\right)dx+\int\left(-y^2\right)dy=c$, where y is considered as constant in 1^{st} integral.

$$\Rightarrow a^2x - x^2y - xy^2 - \frac{1}{3}y^3 = c$$, where c is an arbitrary constant.

Example: Solve $ye^{xy}dx + \left(xe^{xy} - \sin y\right)dy = 0$

Solution: Here, $M = ye^{xy}, N = xe^{xy} - \sin y$

$$\frac{\partial M}{\partial y} = e^{xy} + xye^{xy}, \frac{\partial N}{\partial x} = e^{xy} + xye^{xy}.$$

Thus this equation is exact.

Hence, the solution is $\int ye^{xy}dx + \int(-\sin y)dy = c$

$\Rightarrow e^{xy} + \cos y = c$, where c is an arbitrary constant.

Example: Solve $\dfrac{dy}{dx}+\dfrac{2x+3y+5}{3x+8y-2}=0$

Solution:

$$3xdy+8ydy-2dy+2xdx+3ydx+5dx=0$$

$$\Rightarrow d\left(x^2\right)+4d\left(y^2\right)+3d\left(xy\right)+5dx-2dy=0$$

Integrating we get

$x^2+4y^2+3xy+5x-2y=c$ (c is an arbitrary constant).

Example: Solve $\left(1-x^2\right)\dfrac{dy}{dx}-2xy=x-x^3$.

Solution: $\left(1-x^2\right)dy-2xydx=\left(x-x^3\right)dx$

$$\Rightarrow d\left\{\left(1-x^2\right)y\right\} = d\left(\frac{x^2}{2}-\frac{x^4}{4}\right)$$

$$\Rightarrow \left(1-x^2\right)y = \left(\frac{x^2}{2}-\frac{x^4}{4}\right)+c$$ (c is an arbitrary constant).

Example: Solve $ydx + xdy = xy(dy - dx)$.

Solution: $$ydx + xdy = xy(dy - dx)$$

$$\Rightarrow \frac{d(xy)}{xy} = d(y-x)$$

$$\Rightarrow \log(xy) = (y-x) + \log c \Rightarrow xy = ce^{y-x}$$

(c is an arbitrary constant).

Examples: Solve the following

(a) $\sin x \dfrac{dy}{dx} + x^3 = y\cos x$

(b) $x^2 \dfrac{dy}{dx} + xy + 2\sqrt{1 - x^2 y^2} = 0$

(c) $\left(x^2 + y^2\right)dx + \left(2xy + \cos y\right)dy = 0.$

Integrating factor:

Some equations which are not exact, but can be made exact by multiplying it by some function of x and y. This function is called integrating factor.

Theorem: The number of integrating factor of an equation $Mdx + Ndy = 0$, which has a solution, is infinite.

Proof: Let $\mu(x, y)$ be an integrating factor of the equation $Mdx + Ndy = 0$ so that $\mu(x, y)(Mdx + Ndy) = du$.

Hence $u(x,y)=c$ is solution of equation. Now, if $f(u)$ be any function of u then $\mu(x,y)f(u)(Mdx+Ndy)=f(u)du$.

Now $f(u)du$ can be easily integrable and we have

$$\mu(x,y)f(u)(Mdx+Ndy)=dv$$

So, $\mu(x,y)f(u)$ is also integrating factor of the equation $Mdx+Ndy=0$, where f is arbitrary function u.

Hence the number of integrating factor is infinite.

Rules of Finding Integrating Factor:

Let us consider an equation $Mdx + Ndy = 0$ with $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

(I) If $Mx + Ny \neq 0$ and the given equation is homogeneous then $\frac{1}{Mx + Ny}$ is an integrating factor of the equation.

Example: Solve $\left(x^2 y - 2xy^2\right)dx + \left(3x^2 y - x^3\right)dy = 0$

Solution: $\left(x^2 y - 2xy^2\right)dx + \left(3x^2 y - x^3\right)dy = 0$

Here $\frac{\partial M}{\partial y} = x^2 - 4xy$ and $\frac{\partial N}{\partial x} = 6xy - 3x^2 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

So the equation is not exact, but it is a homogeneous equation (M and N are homogeneous function of x, y).

Hence $$\frac{1}{Mx+Ny}=\frac{1}{\left(x^2y-2xy^2\right)x+\left(3x^2y-x^3\right)y}$$

$$=\frac{1}{x^2y^2}$$

will be an integrating factor. Multiplying both side by $\frac{1}{x^2y^2}$, we get

$$\frac{1}{y}dx-\frac{2}{x}dx+\frac{3}{y}dy-\frac{x}{y^2}dy=0$$

$$\Rightarrow \frac{ydx - xdy}{y^2} - 2d(\log x) + 3d(\log y) = 0$$

$$\Rightarrow d\left(\frac{x}{y}\right) + d(3\log y - 2\log x) = 0$$

Therefore, the primitive is

$\frac{x}{y} + 3\log y - 2\log x = c$, where c is arbitrary constant.

(II) If $Mx - Ny \neq 0$ and the equation can be written as $\{f(xy)\}ydx + \{g(xy)\}xdy = 0$, then $\dfrac{1}{Mx - Ny}$ is an integrating factor of the equation.

Example: Solve $(xy\sin xy + \cos xy)ydx + (xy\sin xy - \cos xy)xdy = 0$

Solution: Here $M = (xy\sin xy + \cos xy)y = f(xy)y$

and $N = (xy\sin xy - \cos xy)x = g(xy)x$.

Also, $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ and $Mx - Ny = 2xy\cos xy$

So, $\dfrac{1}{2xy\cos xy}$ is an integrating factor.

Multiplying both sides of the given equation with integrating factor, we get

$$\tan xy\left(ydx+xdy\right)+\frac{dx}{x}-\frac{dy}{y}=0$$

$$\Rightarrow \tan xyd\left(xy\right)+d\left(\log x\right)-d\left(\log y\right)=0\ .$$

Integrating we have $\log\left(\sec xy\right)+\log x-\log y=\log c$

$$\Rightarrow x\sec xy=cy,$$ where c is an arbitrary constant.

(III) If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ be an function of x only, say, $f(x)$ then $e^{\int f(x)dx}$ is an integrating factor of the equation.

Proof: Let μ be an integrating factor. So $(\mu M)dx+(\mu N)dy=0$ is an exact equation. Hence $\frac{\partial}{\partial y}(\mu M)=\frac{\partial}{\partial x}(\mu N)$

$$\Rightarrow M\frac{\partial \mu}{\partial y}-N\frac{\partial \mu}{\partial x}+\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=0$$

Suppose μ is function of x only then $\frac{\partial \mu}{\partial y}=0$ so we have

$$N\frac{d\mu}{dx}=\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\Rightarrow\frac{d\mu}{\mu}=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx$$

Since μ is function of x only, $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ also the same.

Hence integrating we get $\log\mu=\int\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx$

$$\Rightarrow\mu=e^{\int f(x)dx},\quad \text{where } f(x)=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right).$$

Example: Find the solution of the equation

$$\left(x^2+y^2+2x\right)dx+2ydy=0.$$

Solution: Here $\dfrac{\partial M}{\partial y}=2y$ and $\dfrac{\partial N}{\partial x}=0$, so the equation is not exact.

Now, $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)=\dfrac{2y}{2y}=1$, which can be taken as function of x only.

Hence $e^{\int dx}=e^x$ is an integrating factor. So, after multiplying e^x on both sides of the equation, it becomes

$$\left(x^2+2x\right)e^x dx+\left(y^2e^x dx+2ye^x dy\right)=0$$

$$\Rightarrow d\left(x^2e^x\right)+d\left(y^2e^x\right)=0$$

Integrating we get

$$e^x\left(x^2+y^2\right)=c, \text{ where } c \text{ is an arbitrary constant.}$$

(IV) If $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$, be an function of y only say $g\left(y\right)$ then $e^{\int g(y)dy}$ is an integrating factor of the equation.

Example: Solve $\left(3x^2y^4+2xy\right)dx+\left(2x^3y^3-x^2\right)dy=0$.

Solution: Here $\frac{\partial M}{\partial y}=12x^2y^3+2x$ and $\frac{\partial N}{\partial x}=6x^2y^3-2x$ so it is not an exact equation as $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ but we have

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{xy\left(3xy^3+2\right)}\left(6x^2y^3-2x-12x^2y^3-2x\right)$$

$$=\frac{-6x^2y^3-4x}{xy\left(3xy^3+2\right)}=\frac{-2x\left(3xy^3+2\right)}{xy\left(3xy^3+2\right)}=-\frac{2}{y}$$

which is function of y only.

Hence $e^{\int -\frac{2}{y}dy} = e^{-2\log y} = \dfrac{1}{y^2}$ is an integrating factor of the D.E.

So, multiplying both sides of the given equation by $\dfrac{1}{y^2}$, we get

$$3x^2y^2dx + 2\frac{x}{y}dx + 2x^3ydy - \frac{x^2}{y^2}dy = 0$$

$$\Rightarrow \left(3x^2y^2dx + 2x^3ydy\right) + \left(2\frac{x}{y}dx - \frac{x^2}{y^2}dy\right) = 0$$

$$\Rightarrow d\left(x^3y^2\right) + \frac{2xydx - x^2dy}{y^2} = 0$$

$$\Rightarrow d\left(x^3y^2\right)+d\left(\frac{x^2}{y}\right)=0$$

Integrating we get

$$x^3y^2+\frac{x^2}{y}=c, \; c \text{ being an arbitrary constant.}$$

<u>(V)</u> If the equation be of the form $x^a y^b(mydx+nxdy)=0$, where a, b, m, n are constants, then $x^{km-a-1}y^{kn-b-1}$ is an integrating factor of the equation.

Example: Solve $\left(2x^2y-3y^4\right)dx+\left(3x^3+2xy^3\right)dy=0$.

Solution: The given equation can be written as

$$x^2\left(2ydx+3xdy\right)+y^3\left(-3ydx+2xdy\right)=0$$

For first part: $a_1=2,\ b_1=0,\ m_1=2,\ n_1=3$

So, IF is $x^{2k_1-2-1}y^{3k_1-1}\Rightarrow x^{2k_1-3}y^{3k_1-1}$

For second part: $a_2=0,\ b_2=3,\ m_2=-3,\ n_2=2$

So, IF is $x^{-3k_2-1}y^{2k_2-3-1}\Rightarrow x^{-3k_2-1}y^{2k_2-4}$

Thus we have $2k_1-3=-3k_2-1$ and $3k_1-1=2k_2-4$

$\Rightarrow k_1=-5/13$ and $k_2=12/13$

and the common IF is $x^{-49/13}y^{-28/13}$.

Hence the given equation becomes

$$\left(2x^{-23/13}y^{-15/13}-3x^{-49/13}y^{24/13}\right)dx+\left(3x^{-10/13}y^{-28/13}+2x^{-36/13}y^{11/13}\right)dy=0$$

which is an exact equation (to be verified) and its general solution is

$$5x^{-36/13}y^{24/13}-12x^{-10/13}y^{-15/13}=c,$$ where c is an arbitrary constant.

<u>Examples:</u> Solve the following equations

(a) $(1+xy)\,ydx+(1-xy)\,xdy=0$

(b) $(x^4+y^4)dx-x^4y^3dy=0$

(c) $\left(xy^2 - e^{1/x^3}\right)dx - x^2 y\,dy = 0$

(d) $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}\left(x + xy^2\right)dy = 0$

(e) $\left(y^3 - 2x^2 y\right)dx - \left(2xy^2 - x^3\right)dy = 0$

<u>Examples:</u> (a) Prove that e^{x^2} is an integrating factor of the equation $\left(x^2 + xy^4\right)dx + 2y^3 dy = 0$ and hence solve it.

(b) If $x^\alpha y^\beta$ be an integrating factor of the equation $(2y\,dx + 3x\,dy) + 2xy(3y\,dx + 4x\,dy) = 0$, then find α, β and hence solve it.

Linear equations:

An equation of the form $\frac{dy}{dx}+Py=Q$, where P and Q are functions of x only (or constant) is called a linear equation of first order in y.

The dependent variable and its derivative in such equation occur in the first degree only and not as higher powers or product.

If both P and Q be constant then the variables can be easily separated and the equation may be solved.

Let R be an integrating factor of the above equation, so that the left hand side of the equation $R\frac{dy}{dx}+RPy=RQ$ become an exact differential of some product.

Since the first term is $R\frac{dy}{dx}$ so it can only be derived by differentiating Ry.

Now $\frac{d}{dx}(Ry)=R\frac{dy}{dx}+y\frac{dR}{dx}$.

Thus, we have $R\frac{dy}{dx}+y\frac{dR}{dx}=R\frac{dy}{dx}+RPy$

$$\Rightarrow RP = \frac{dR}{dx} \Rightarrow \frac{dR}{R} = Pdx \;\; \Rightarrow \log R = \int Pdx \Rightarrow R = e^{\int Pdx}.$$

Hence the equation reduces to

$$e^{\int Pdx}\frac{dy}{dx} + Pye^{\int Pdx} = Qe^{\int Pdx}$$

$$\Rightarrow \frac{d}{dx}\left(ye^{\int Pdx}\right) = Qe^{\int Pdx} \Rightarrow d\left(ye^{\int Pdx}\right) = Qe^{\int Pdx}dx$$

Integrating we get primitive of the equation as

$\Rightarrow ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$, where c is an arbitrary constant.

Sometimes an equation may be linear in x, where y is the independent variable. The form of such an equation is $\frac{dx}{dy}+P_1x=Q_1$ where P_1 and Q_1 are function of y only (or constant). The general solution of that equation will be $xe^{\int P_1 dy}=\int Q_1 e^{\int P_1 dy}dy+c_1$, c_1 is an arbitrary constant.

Example: Solve $\frac{dy}{dx}+\frac{4x}{x^2+1}y=\frac{1}{\left(x^2+1\right)^3}$

Solution: The equation is linear in y and here

$$P = \frac{4x}{x^2+1},\ Q = \frac{1}{\left(x^2+1\right)^3}$$

So, the integrating factor is

$$e^{\int P dx} = e^{\int \frac{4x}{x^2+1}dx} = e^{2\log(x^2+1)} = \left(x^2+1\right)^2.$$

Hence, the general solution of the equation is given by

$$y\left(x^2+1\right)^2 = \int \frac{\left(x^2+1\right)^2}{\left(x^2+1\right)^3} dx + c = \int \frac{1}{x^2+1} dx + c,$$

i.e., $y\left(x^2+1\right)^2 = \tan^{-1} x + c$, where c is an arbitrary constant.

Example: Solve $\left(1+y^2\right)+\left(x-e^{-\tan^{-1}y}\right)\dfrac{dy}{dx}=0$

Solution: The equation can be written as

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{-\tan^{-1}y}}{1+y^2}$$

It is linear equation in x and $P=\dfrac{1}{1+y^2}$, $Q=\dfrac{e^{-\tan^{-1}y}}{1+y^2}$

Hence the integrating factor is $e^{\int\frac{1}{1+y^2}}=e^{\tan^{-1}y}$.

Thus the general solution is given by $xe^{\tan^{-1}y}=\int\dfrac{1}{1+y^2}dy+c$

$\Rightarrow xe^{\tan^{-1} y} = \tan^{-1} y + c$, where c is an arbitrary constant.

Example: Solve $\dfrac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$

Solution: From the given equation, we have $\sec^2 y \dfrac{dy}{dx} + 2x\tan y = x^3$.

Putting $\tan y = v$, we get $\dfrac{dv}{dx} = \sec^2 y \dfrac{dy}{dx}$

and the equation becomes $\dfrac{dv}{dx} + 2xv = x^3$

This is a linear equation in v and its integrating factor is $e^{\int 2x dx} = e^{x^2}$

Thus, its solution is $ve^{x^2} = \int x^3 e^{x^2} dx + c$

$$\Rightarrow ve^{x^2} = \frac{1}{2}\int ze^z dz + c, \quad \text{where } x^2 = z$$

$$\Rightarrow ve^{x^2} = \frac{1}{2}\left(ze^z - e^z\right) + c$$

$$\Rightarrow e^{x^2} \tan y = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + c$$

$\Rightarrow \tan y = \frac{1}{2}\left(x^2 - 1\right) + ce^{-x^2}$, where c is an arbitrary constant.

Equations Reducible to Linear Form:

Let us consider the equation $\frac{dy}{dx}+Py=Qy^n$, which is known as Bernoulli's equation in which P and Q are function of x alone or constant.

So, we can write $y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$.

Now, let $v=y^{1-n}$.

Then $\frac{dv}{dx}=(1-n)y^{-n}\frac{dy}{dx}$ and the above equation reduces to

$$\frac{1}{(1-n)}\frac{dv}{dx}+Pv=Q\text{, i.e., }\frac{dv}{dx}+(1-n)Pv=(1-n)Q$$

which is linear equation in v and its integrating factor will be $e^{(1-n)\int P dx}$

Therefore, the solution is given by

$$ve^{(1-n)\int P dx} = (1-n)\int Qe^{(1-n)\int P dx} dx + c$$

To get the solution of the given equation we have to put $v = y^{1-n}$.

<u>Example:</u> Solve $\dfrac{dy}{dx} + y\cos x = y^n \sin 2x$

<u>Solution:</u> We can rewrite the equation as

$$y^{-n}\frac{dy}{dx} + y^{1-n}\cos x = \sin 2x .$$

Put $v = y^{1-n}$, then $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ and the equation becomes

$$\frac{1}{(1-n)}\frac{dv}{dx} + v\cos x = \sin 2x, \text{ i.e., } \frac{dv}{dx} + (1-n)v\cos x = (1-n)\sin 2x.$$

This is linear equation in *v* and its integrating factor will be

$$e^{(1-n)\int \cos x dx} = e^{(1-n)\sin x}.$$

So, the solution of the given equation is

$$\begin{aligned} ve^{(1-n)\sin x} &= 2(1-n)\int e^{(1-n)\sin x}\sin x\cos x dx + c \\ &= 2(1-n)\int e^{(1-n)z} z dz + c, \qquad \text{where } z = \sin x \end{aligned}$$

$$\Rightarrow ve^{(1-n)z} = 2(1-n)\left[\frac{z}{1-n}e^{(1-n)z} - \frac{1}{(1-n)^2}e^{(1-n)z}\right] + c$$

Hence the required solution of the given equation is

$$\Rightarrow y^{1-n}e^{(1-n)\sin x} = 2e^{(1-n)\sin x}\sin x - \frac{2e^{(1-n)\sin x}}{1-n} + c$$

$$\Rightarrow y^{1-n} = 2\sin x - \frac{2}{1-n} + ce^{(n-1)\sin x},$$ c being arbitrary constant.

Example: Solve $y + 2\dfrac{dy}{dx} = y^3(x-1)$.

Solution: we have $y^{-3}\dfrac{dy}{dx} + \dfrac{1}{2}y^{-2} = \dfrac{1}{2}(x-1)$

Putting $v = y^{-2}$ we have $-\dfrac{1}{2}\dfrac{dv}{dx} + \dfrac{1}{2}v = \dfrac{1}{2}(x-1) \Rightarrow \dfrac{dv}{dx} - v = (1-x)$

I.F. e^{-x} and the solution will be

$$ve^{-x} = \int (1-x)e^{-x}dx + c$$

$$= -e^{-x} + xe^{-x} + e^{-x} + c = xe^{-x} + c$$

So, $y^{-2}e^{-x} = xe^{-x} + c \Rightarrow \dfrac{1}{y^2} = x + ce^x$

$\Rightarrow y^2(x+ce^x) = 1$, c being arbitrary constant.

Examples: Solve the followings

(a) $\left(x^2y^3+xy\right)dy=dx$

(b) $x^2y-x^3\dfrac{dy}{dx}=y^4\cos x$

(c) $\cos^2 x\dfrac{dy}{dx}+y=\tan x$

(d) $\left(x+y+1\right)dy=dx$

(e) $\dfrac{dy}{dx}+y\dfrac{df}{dx}=f\left(x\right)\dfrac{df}{dx}$