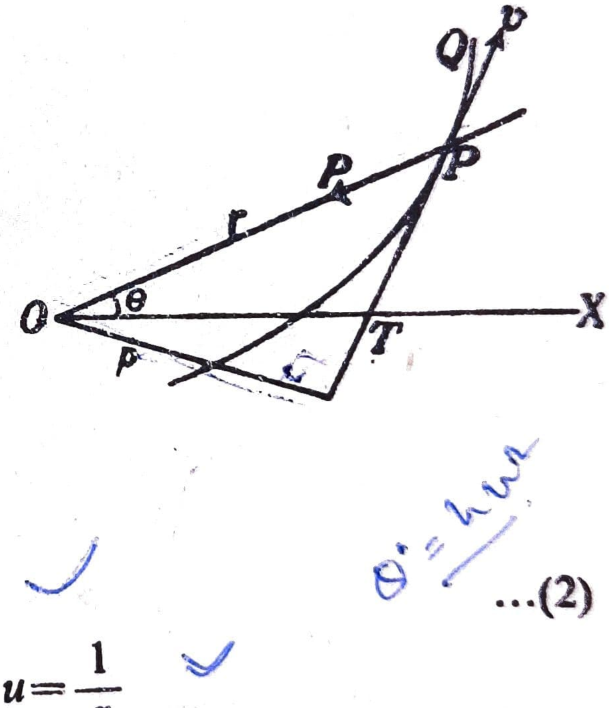
Central Forces

8·1. *A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane ; to discuss the motion.*

Take O as origin and a fixed line OX as initial line and let the polar co-ordinates of P, the position of the particle at time t, be (r, θ).

Let P be the acceleration of the particle towards O, *i.e.*, force per unit mass acting on the particle.



The radial equation of motion is

$$\ddot{r} - r\dot{\theta}^2 = -P. \qquad \ldots(1)$$

Since there is no acceleration perp. to OP, the cross-radial equation is

Transverse

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

i.e., $$r^2\dot{\theta} = \text{const.} = h, \text{ say} \qquad \ldots(2)$$

$\dot\theta = hu^2$

$$\dot{\theta} = \frac{h}{r^2} = hu^2 \text{ if } u = \frac{1}{r}$$

$$\dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right) = \dot{\theta}\frac{d}{d\theta}\left(\frac{1}{u}\right) = -\dot{\theta}\cdot\frac{1}{u^2}\frac{du}{d\theta}$$

$$= -hu^2\frac{1}{u^2}\frac{du}{d\theta} = -h\frac{du}{d\theta}$$

$\dot r = -h\frac{du}{d\theta}$

$\frac{d\theta}{dt} \cdot dt$

and
$$\ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = \dot{\theta}\frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)$$
$$= -h\dot{\theta}\frac{d^2u}{d\theta^2} = -h^2u^2\frac{d^2u}{d\theta^2}.$$

Hence (1) becomes
$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4 = -P$$

i.e.,
$$\boxed{P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right).} \quad \ldots(3)$$

If p be the perpendicular from O upon the tangent at P,

then
$$\boxed{\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.}$$

$p = h^2u^2($

Differentiating both sides with respect to θ, we get
$$-\frac{2}{p^3}\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}$$

i.e.,
$$-\frac{1}{p^3}\frac{dp}{dr}\frac{dr}{d\theta} = \left(u + \frac{d^2u}{d\theta^2}\right)\frac{du}{d\theta}$$

i.e.,
$$-\frac{1}{p^3}\frac{dp}{dr} \times \left(-\frac{1}{u^2}\frac{du}{d\theta}\right) = \left(u + \frac{d^2u}{d\theta^2}\right)\frac{du}{d\theta}$$

or
$$\frac{1}{p^3}\frac{dp}{dr} = u^2\left(u + \frac{d^2u}{d\theta^2}\right).$$

Hence (3) gives
$$\boxed{P = \frac{h^2}{p^3}\frac{dp}{dr}.} \quad \ldots(4)$$

If Q be the position of the particle at time $t + \delta t$ where $\angle POQ = \delta\theta$

then
$$\text{area } POQ = \tfrac{1}{2}.OP.OQ \sin POQ$$
$$= \tfrac{1}{2}r.(r + \delta r)\sin\delta\theta$$

$\therefore$
$$\text{Areal velocity} = \lim_{\delta t \to 0}\frac{\text{area } POQ}{\delta t}$$
$$= \lim_{\delta t \to 0}\frac{\tfrac{1}{2}r(r + \delta r)\sin\delta\theta}{\delta t}$$
$$= \tfrac{1}{2}r^2\dot{\theta} \text{ in the limit}$$
$$\boxed{= \tfrac{1}{2}h.}$$

If v be the velocity at P, it has components $\dot{x}$, $\dot{y}$ parallel to OX and perpendicularly and componets $\dot{r}$, $r\dot{\theta}$ along OP and perpendicularly.

Taking moment of the velocity about O, we get

$$vp = r^2\dot{\theta} = x\dot{y} - \dot{x}y$$

This relation is an identity and holds good for any curve.

When the curve is a central orbit, this relation becomes

$$vp = r^2\dot{\theta} = x\dot{y} - \dot{x}y = \text{const.} = h.$$

Thus $$\boxed{v^2 = \frac{h^2}{p^2} = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right].}$$

Application. (*a*) *Given the orbit in polar co-ordinates, to find the force.*

(1) *If the central orbit is an ellipse, the focus being the centre of force, find the law of force.* 92

The equation to the ellipse with focus as pole is

$$\frac{l}{r} = 1 + e\cos\theta \quad \therefore \quad u = \frac{1}{l} + \frac{e}{l}\cos\theta$$

$$\therefore \quad \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta.$$

Hence $$P = h^2u^2\left[\frac{d^2u}{d\theta^2} + u\right] = \frac{h^2}{l}u^2 = \frac{\mu}{r^2} \text{ say}$$

where $$\mu = \frac{h^2}{l} \text{ i.e., } h = \sqrt{\mu l}.$$

Thus the central force varies inversely as the square of the distance from the focus

Also $$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$$

$$= h^2\left[\left(\frac{1}{l} + \frac{e}{l}\cos\theta\right)^2 + \left(\frac{e}{l}\sin\theta\right)^2\right]$$

$$= \frac{\mu}{l}[1 + 2e\cos\theta + e^2]$$

$$= \mu\left[2\,\frac{1 + e\cos\theta}{l} - \frac{1 - e^2}{l}\right]$$

$$= \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

for l = semi-latus rectum $= \dfrac{b^2}{a} = a(1 - e^2)$

$2a$ being the major axis of the ellipse.

If T be the periodic time *i.e.*, the time the particle takes to describe the whole arc of the ellipse, we have

$\frac{1}{2}h.T$ = area of the ellipse $=\pi ab$, since $\frac{1}{2}h$ is the areal velocity *i.e.*, $\frac{h}{2}$ is the area described in unit time.

Also
$$h^2=\mu l=\mu\,\frac{b^2}{a}$$

$$\therefore \qquad T=\frac{2\pi ab}{h}=\frac{2\pi ab}{\sqrt{\mu\,\frac{b^2}{a}}}=\frac{2\pi}{\sqrt{\mu}}\,a^{\frac{3}{2}}$$

(2) *If the central orbit is $r^n=a^n\cos n\theta$ under a force towards the pole, to find the law of force.*

Here
$$u^n a^n \cos n\theta=1$$

Taking logarithmic differential, we get
$$\frac{du}{d\theta}=u\tan n\theta$$

$$\therefore \qquad \frac{d^2u}{d\theta^2}=\frac{du}{d\theta}\tan n\theta+nu\sec^2 n\theta=u\,(\tan^2 n\theta+n\sec^2 n\theta)$$

$$\therefore \qquad \frac{d^2u}{d\theta^2}+u=u(n+1)\sec^2 n\theta=(n+1)a^{2n}u^{2n+1}$$

Hence
$$P=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=(n+1)h^2a^{2n}u^{2n+3}$$

Also taking logarithmic differential of $r^n=a^n\cos n\theta$, we get
$$\frac{n}{r}\cdot\frac{dr}{d\theta}=-n\tan n\theta$$

or $\cot\phi=-\tan n\theta$ where ϕ is the angle between the tangent and radius vector

$$\therefore \qquad \phi=\frac{\pi}{2}+n\theta$$

Now $p=r\sin\phi=r\cos n\theta=\dfrac{r^{n+1}}{a^n}$.

Hence
$$\frac{dp}{dr}=(n+1)\frac{r^n}{a^n}$$

$$\therefore \quad P=\frac{h^2}{p^3}\,\frac{dp}{dr}=\frac{h^2a^{3n}}{r^{3n+3}}\,(n+1)\frac{r^n}{a^n}=\frac{h^2a^{2n}(n+1)}{r^{2n+3}}.$$

(*b*) *Given the central force as a function of r, to find the orbit.*

(1) ***If the central force varies inversely as the square of the distance from a fixed point, to find the orbit.*** 92

Here $$P=\frac{\mu}{r^2}$$

$$\therefore \quad \frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}=\frac{\mu u^2}{h^2u^2}=\frac{\mu}{h^2}$$

or $$\frac{d^2u}{d\theta^2}=-\left(u-\frac{\mu}{h^2}\right)$$

solution of which is $u-\dfrac{\mu}{h^2}=A\cos\theta+B\sin\theta$

or $$u=\frac{\mu}{h^2}+A_1\cos(\theta-\alpha)$$

or $$\frac{h^2/\mu}{r}=1+C\cos(\theta-\alpha),\text{ putting } \frac{A_1h^2}{\mu}=C,$$

i.e., $$\frac{l}{r}=1+C\cos(\theta-\alpha)$$

where $$l=\frac{h^2}{\mu}$$

or $$h^2=\mu l$$

Thus the orbit is a conic, focus being the pole.

Q (2) ***If the central force varies as the distance from a fixed point, to find the orbit.***

Here $$P=\mu r$$

$$\therefore \quad \frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}=\frac{\mu}{h^2u^3} = \frac{\mu r}{h^2u^2} = \frac{\mu}{h^2u^3}$$

Multiplying by $2\dfrac{du}{d\theta}$, we get,

$$2\frac{d^2u}{d\theta^2}\frac{du}{d\theta}+2u\frac{du}{d\theta}=\frac{2\mu}{h^2u^3}\frac{du}{d\theta}$$

Integrating, we get

$$\left(\frac{du}{d\theta}\right)^2+u^2=-\frac{\mu}{h^2u^2}+C \text{ where } C \text{ is a const.}$$

or $$\int\frac{u\,du}{\sqrt{\left(Cu^2-\frac{\mu}{h^2}-u^4\right)}}=\theta+\text{const.}$$

Putting $u^2=z$, we get

$$\int \frac{dz}{\sqrt{\left(Cz-\frac{\mu}{h^2}-z^2\right)}}=2\theta+\alpha \text{ where } \alpha \text{ is a constant}$$

or
$$2\theta+\alpha=\int \frac{dz}{\sqrt{\{(a-z)(z-b)\}}}$$

where
$$a+b=C,\ ab=\frac{\mu}{h^2}$$

Putting $z=a\sin^2\lambda+b\cos^2\lambda$, we get

$$2\theta+\alpha=\int \frac{2(a-b)\sin\lambda\cos\lambda\,d\lambda}{\sqrt{\{(a-b)\cos^2\lambda.(a-b)\sin^2\lambda}}=2\lambda$$

Choosing $\alpha=0$, we get $\theta=\lambda$

Hence $u^2=z=a\sin^2\theta+b\cos^2\theta$

or
$$\frac{1}{r^2}=a\sin^2\theta+b\cos^2\theta$$

or
$$ay^2+bx^2=1$$

Thus the orbit is a conic, centre being the pole.

(3) *If the central force varies inversely as the cube of the distance from a fixed point, to find the orbit.*

Here
$$P=\frac{\mu}{r^3}$$

$$\therefore \quad \frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}\ u$$

or
$$\frac{d^2u}{d\theta^2}=-\left(1-\frac{\mu}{h^2}\right)u$$

If $\frac{\mu}{h^2}>1$, then $\frac{d^2u}{d\theta^2}=\left(\frac{\mu}{h^2}-1\right)u=n^2u$ say

where
$$n^2=\frac{\mu}{h^2}-1$$

Its solution is $u=A\cosh n\theta+B\sinh n\theta$

If $\frac{\mu}{h^2}=1$, then $\frac{d^2u}{d\theta^2}=0$ so that $u=A\theta+B$.

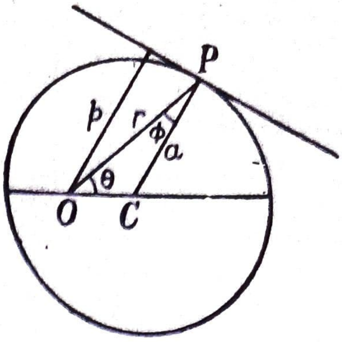
If $\frac{\mu}{h^2}<1$, then $\frac{d^2u}{d\theta^2}=-n^2u$ so that $u=A\cos n\theta+B\sin n\theta$.

8·2. (p, r) equations of some curves.

(1) *Circle...pole at any point.*

C is centre of the circle, O the pole.

$OC = c$.

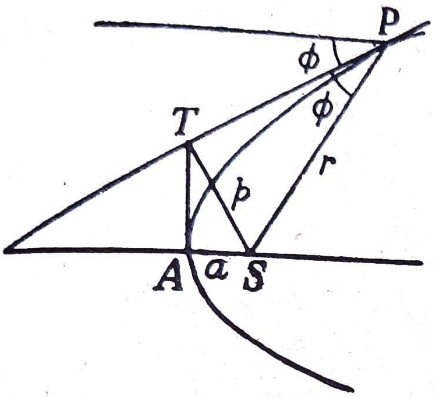


$$p = r \sin (90 - \phi) = r \cos \phi$$

$$= r \cdot \frac{a^2 + r^2 - c^2}{2ar}$$

i.e., $\boxed{r^2 + a^2 - c^2 = 2ap.}$

If pole be on the circumference, $c = a$, then $\boxed{r^2 = 2ap.}$



(2) *Parabola...pole at focus.*

If ST is perpendicular from focus S to the tangent at P, then T will lie on AT, the tangent at the vertex A.

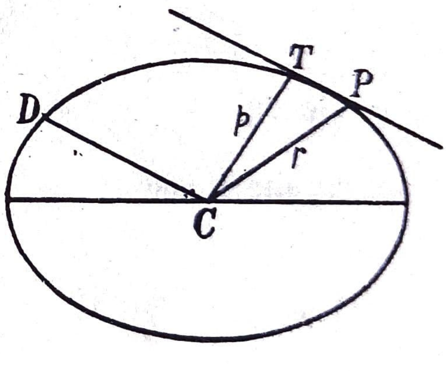
Then $\angle SPT = \phi = \angle ATS$

$$\therefore \quad \frac{p}{r} = \sin \phi = \frac{a}{p}$$

or $\boxed{p^2 = ar.}$

(3) *Ellipse* or *Hyperbola...pole at centre.*

If CD is semi-conjugate to CP,



then $\quad CP^2 + CD^2 = a^2 + b^2$

and $\quad CD.CT = ab$

i.e., $\quad CD^2 = a^2 + b^2 - r^2$

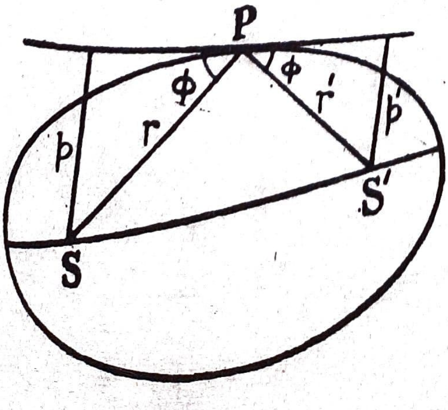
and $\quad CD = \dfrac{ab}{p}$

$$\therefore \quad \boxed{\frac{a^2b^2}{p^2} = a^2 + b^2 - r^2.}$$

This is for ellipse

For hyperbola, change the sign of b^2.

(4) *Ellipse or Hyperbola...pole at focus.*



$$\sin \phi = \frac{p}{r} = \frac{p'}{r'}$$

or $\quad \dfrac{p'}{p} = \dfrac{r'}{r}$

$$\therefore \quad \frac{pp'}{p^2} = \frac{r'}{r} = \frac{2a - r}{r}$$

because $\boxed{r + r' = 2a}$

Aiso $$pp'=b^2$$

$\therefore$ $$\boxed{\frac{b^2}{p^2}=\frac{2a}{r}-1}$$

This is for ellipse.

For hyperbola, the equation is

$$\frac{b^2}{p^2}=\frac{2a}{r}+1 \text{ for the nearer branch}$$

and $$\frac{b^2}{p^2}=1-\frac{2a}{r} \text{ for the farther branch.}$$

(5) *Equiangular spiral.*

Here the angle ϕ, the angle between the radius vector and the tangent, is constant, equal to α say

$\therefore$ $$\boxed{p=r\sin\alpha.}$$

(6) $r^n=a^n\cos n\theta.$

Taking log. differential, we get

$$\frac{1}{r}\frac{dr}{d\theta}=-\tan n\theta$$

i.e., $$\cot\phi=-\tan n\theta$$

$\therefore$ $$\phi=90^\circ+\theta$$

Hence $$p=r\sin\phi=r\cos n\theta=\frac{r^{n+1}}{a^n}$$

Applications

(1) *Orbit is a circle under a force to a point on the circumference.*

Here the equation to the circle is $r^2=2ap$

$\therefore$ $$P=\frac{h^2}{p^3}\frac{dp}{dr}=\frac{h^2}{p^3}\cdot\frac{r}{a}=\frac{8a^2h^2}{r^5}=\frac{\mu}{r^5}$$

where $$\mu=8a^2h^2$$

Also $$v^2=\frac{h^2}{p^2}=\frac{4a^2h^2}{r^4}=\frac{\mu}{2r^4}$$

(2) *Orbit is an ellipse under a force to the centre.*

Ellipse is $$\frac{a^2b^2}{p^2}=a^2+b^2-r^2$$

$$P=\frac{h^2}{p^3}\frac{dp}{dr}=\frac{h^2r}{a^2b^2}=\mu r$$

where $$\mu = \frac{h^2}{a^2b^2}$$

Also $$v^2 = \frac{h^2}{p^2} = \frac{b^2(a^2+b^2-r^2)}{a^2b^2}$$

$$= \mu(a^2+b^2-r^2) = \mu.CD^2$$

where CD is the semi-conjugate diameter to CP.

(3) ***Orbit*** $r^n = a^n \cos n\theta$ *under a force to the pole.*

Here $$p = \frac{r^{n+1}}{a^n}$$

$\therefore$ $$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{(n+1)h^2a^{2n}}{r^{2n+3}}$$

(4) ***Orbit is an ellipse under a force to the focus.***

Here ellipse is $$\frac{b^2}{p^2} = \frac{2a}{r} - 1$$

$\therefore$ $$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{ah^2}{b^2r^2} = \frac{\mu}{r^2}$$

where $$h^2 = \mu\frac{b^2}{a} = \mu l, \; l \text{ being semi-latus rectum}$$

Also $$v^2 = \frac{h^2}{p^2} = \frac{\mu b^2}{ap^2} = \frac{\mu}{a}\left(\frac{2a}{r} - 1\right) = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

(5) ***Orbit is a parabola under a force to the focus.***

Here parabola is $p^2 = ar$

$\therefore$ $$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{ah^2}{2p^4} = \frac{ah^2}{2a^2r^2} = \frac{h^2}{2ar^2} = \frac{\mu}{r^2}$$

where $$h^2 = 2a\mu$$

Also $$v^2 = \frac{h^2}{p^2} = \frac{2a\mu}{ar} = \frac{2\mu}{r}.$$

EXAMPLES VIII (A)

Prove the following cases of central orbits :

(*i*) Equiangular Spiral $r = ae^{\theta \cot \alpha}$ or $p = r \sin \alpha$

$$P = \frac{\mu}{r^3}$$

(*ii*) Bernoulli's Lemniscate $r^2 = a^2 \cos 2\theta$, $P = \frac{\mu}{r^7}$

(*iii*) $\frac{a}{r} = e^{n\theta}, n\theta, \cosh n\theta$ or $\sinh n\theta$, $P = \frac{\mu}{r^3}$