What is a matrix?

* A rectangular array of numbers is called a matrix.

Example - ?

Order of a matrix

If a matrix A has m rows and n columns then $m \times n$ is called the order of A.

Notation: In general a matrix is denoted as

$$\begin{bmatrix} * & * & \cdots & * \\ \vdots & & & \vdots \\ * & \cdots & & * \end{bmatrix} \text{ or } [a_{ij}]_{\substack{1 \le i \le m, \\ 1 \le j \le n}} \text{ or } (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

$$(a_{ij})_{m \times n}.$$

Equality of two matrices: Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices with same order (say $m \times n$). Then A and B are equal iff

$$a_{ij} = b_{ij} \quad \forall\, 1 \le i \le m,\ 1 \le j \le n.$$

Operation on Matrices:

Addition : Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices having same order (say, $m \times n$).

Then $A + B = (c_{ij})_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$.

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} e & 0 & i \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+e & 2 & 3+i \\ 4 & 4 & 2 \end{bmatrix}$$

Exercise : Let A, B, C be matrices with order of each matrix being $m \times n$. **Then**

(i) $A + B = B + A$

(ii) $A + (B + C) = (A + B) + C$.

Multiplication : Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{n \times p}$ be two matrices. Then the product of A and B (denoted by AB) is defined as the matrix

$$AB = (c_{ij})_{m \times p} \quad \text{with} \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Exercise: (1) Let $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{n \times p}$ and $C = (c_{ij})_{p \times r}$. Show $(AB)C = A(BC)$.

(2) Let A and B be matrices s.t. AB and BA are defined. Then it may happen that $AB \neq BA$ (produce example).

Transpose of a matrix:

Let $A = (a_{ij})_{m \times n}$ be a matrix. Then the transpose of A (denoted by A^t) is defined as

$$A^t = (b_{kl})_{n \times m}, \quad b_{kl} = a_{lk} \quad \begin{matrix} 1 \leq l \leq m \\ 1 \leq k \leq n \end{matrix}$$

Conjugate of a matrix:

Let $A = (a_{ij})_{m \times n}$ be a matrix. Then the conjugate of A (denoted by $\bar{A}$) is defined as

$$\bar{A} = (b_{ij})_{m \times n}, \quad b_{ij} = \overline{a_{ij}}.$$

Example :

$$A = \begin{bmatrix} i & 1 & 2 \\ 2 & 1+i & 3 \\ \cancel{e} & \cancel{\pi} & \cancel{0} \end{bmatrix} \qquad A = \begin{bmatrix} i & 1 & 2 \\ 2 & 1+i & 3 \end{bmatrix}$$

$$A^t = \begin{pmatrix} i & 2 \\ 1 & 1+i \\ 2 & 3 \end{pmatrix}.$$

$$\bar{A} = \begin{bmatrix} -i & 1 & 2 \\ 2 & 1-i & 3 \end{bmatrix}.$$

Some Special Matrices

(1) zero matrix

Let $A = [a_{ij}]_{m\times n}$ be a matrix. A is called a zero matrix if $a_{ij} = 0$ $\forall\, i \in \{1, \dots, m\}$
$j \in \{1, 2, \dots, n\}$.

Example

$[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are all zero matrices.

~~(2) Diagonal Matrix :~~

~~let $A = [a_{ij}]_m$.~~

(2) Square matrix:

An $m\times n$ matrix A is called a square matrix if $m = n$.

Example $\begin{bmatrix} 1 & 2 \\ e & \pi \end{bmatrix}$ ← a 2×2 square matrix.

$\begin{bmatrix} 1 & 2 & e \\ i & \pi & e \\ 0 & 1 & 7 \end{bmatrix}$ ← 3×3 square matrix.

(3) <u>Diagonal Matrix</u>:

let $A = [a_{ij}]_{m \times m}$ be a square matrix. A is a ~~square matrix~~ diagonal matrix if for $i \neq j$ $a_{ij} = 0$.
(Note: for $i = 1, \ldots, m$, a_{ii} are called diagonal elements.)

<u>Example</u>:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} e & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(4) <u>Identity Matrix</u>:

A diagonal matrix $A = [a_{ij}]_{m \times m}$ is called an identity matrix if for $i = 1, 2, \ldots, m$. $a_{ii} = 1$.

<u>Example</u>:

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots$$

(5) <u>Upper triangular Matrix</u>:

A square matrix $A = [a_{ij}]_{m \times m}$ is called an upper triangular matrix if for $j < i$, $a_{ij} = 0$.
(Entries below diagonal elements are zero).

Example :
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & e \\ 0 & 0 & \pi \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(6) Lower triangular matrix

$i < j \quad a_{ij} = 0.$

Find some examples of lower triangular matrix.

(7) Symmetric Matrix :

A square matrix $A = [a_{ij}]_{m \times m}$ is called a symmetric matrix if $A^t = A$.

Example
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & e & \pi \\ 1 & \pi & i \end{bmatrix}, \quad \begin{bmatrix} 1 & e \\ e & 2 \end{bmatrix}.$$

Properties : If A and B are symmetric matrices then A+B is also a symmetric matrix.

Exercise : Show that

$(A+B)^t = A^t + B^t$

$(A-B)^t = A^t - B^t$

$(\alpha A)^t = \alpha A^t$

$(A^t)^t = A.$

If A and B are symmetric matrices, then $(A+B)^t = A^t + B^t = A+B$, and so $A+B$ is also symmetric.

Exercise: $(AB)^t = B^t A^t$

Property: Let A and B be symmetric matrices. Then AB is symmetric iff $AB = BA$.

Pf: If AB is symmetric, then $(AB)^t = B^t A^t$
$= BA$.
(since $B^t = B$, $A^t = A$).

Now, if $AB = BA$ then $(AB)^t = B^t A^t$
$= BA = AB$.

8 Skew Symmetric Matrix:

A square matrix $A = [a_{ij}]_{m \times m}$ is called a skew symmetric matrix if $A^t = -A$.

Example: $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & e \\ -1 & -e & 0 \end{bmatrix}$.

Property: Diagonal entries of a skew symmetric matrix are all zero.

Pf: Note that $A^t = -A$ imply

$$a_{ij} = -a_{ji}$$

Hence for $j = i$

$$a_{ii} = -a_{ii}$$

But this is possible, only when $a_{ii} = 0$

Thm: Let A be a square matrix. Then A can be uniquely expressed as sum of a symmetric matrix and a skew symmetric matrix.

Meaning:

$$A = \overbrace{\underbrace{C}_{\text{Symmetric}} + \underbrace{D}_{\text{skew Symmetric}}}^{\text{Sum}}$$

Uniqueness:

$$A = \underbrace{E}_{\text{Symmetric}} + \underbrace{F}_{\text{skew Symmetric}}$$

$\Rightarrow$ $C = E$ and $D = F$.

Proof:

Step-1: $\longrightarrow$ $\frac{A+A^t}{2}$ is a symmetric matrix.

(How?) $\left(\frac{A+A^t}{2}\right)^t = \frac{1}{2}\left(A^t + (A^t)^t\right) = \frac{1}{2}(A+A^t)$

Hence $\frac{A+A^t}{2}$ is a symmetric matrix.

$\longrightarrow$ $\frac{A-A^t}{2}$ is a skew symmetric matrix.

$$\left(\frac{A-A^t}{2}\right)^t = \frac{1}{2}\left(A^t - (A^t)^t\right) = \frac{1}{2}(A^t - A)$$

$$= -\frac{1}{2}(A-A^t).$$

Choose $C = \frac{A+A^t}{2}$, $D = \frac{A-A^t}{2}$

Hence $A = \frac{A+A^t}{2} + \frac{A-A^t}{2}$

$= C + D.$

Step-2 (Uniqueness): let $A = E + F$, where E is symmetric and F is skew symmetric.

$A^t = E^t + F^t = E - F$

$\Rightarrow E = \frac{A+A^t}{2}$, $F = \frac{A-A^t}{2}$.

<u>Notation</u> : let A be an $m \times n$ matrix. Then conjugate transpose of A, ~~it's denoted by~~ $(\bar{A})^t$ is usually denoted by A^*.

<u>Example</u> :

$$A = \begin{bmatrix} 1-4i & i & 2 \\ 3 & 2+i & 0 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1+4i & -i & 2 \\ 3 & 2-i & 0 \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} 1+4i & 3 \\ -i & 2-i \\ 2 & 0 \end{bmatrix}$$

<u>Note that</u> $(\bar{A})^t = \overline{(A^t)}$.

<u>So</u>, $A^* = (\bar{A})^t = \overline{(A^t)}$

Hermitian Matrix and Skew Hermitian Matrix.

Let A be a square matrix. Then ~~conjugate~~

[A is called a Hermitian matrix if $A^* = A$.

[A is called skew-Hermitian if $A^* = -A$]

Example:

$$A = \begin{bmatrix} 2 & 4+i \\ 4-i & 3 \end{bmatrix} \qquad B = \begin{bmatrix} i & 1+i \\ -1+i & -i \end{bmatrix}$$

Properties: (1) The diagonal entries of a Hermitian matrix ~~must~~ are real numbers.

Let $z = a+ib$ be a diagonal elt.

$\Rightarrow$ $a+ib = a-ib$

$\Rightarrow$ $2ib = 0 \Rightarrow b = 0.$

$\Rightarrow$ $z = a$

(2) The diagonal entries of a skew Hermitian matrix are purely ~~imagnary~~ imaginary numbers.

$z = a+ib$

$\Rightarrow z = -\bar{z} \Rightarrow a+ib = -(a-ib) \Rightarrow 2a = 0 \Rightarrow a = 0.$

$\Rightarrow z = ib, \; b \in \mathbb{R}$

Thm: A is Hermitian (skew Hermitian) iff iA is skew Hermitian (Hermitian).

Proof:

Assume A is Hermitian.

Consider, $(iA)^* = -iA^* = -iA.$

$\Rightarrow$ iA is skew Hermitian

Conversely, Assume iA is skew Hermitian.

$\Rightarrow (iA)^* = -iA$

$\Rightarrow -iA^* = -iA$

$\Rightarrow A^* = A$

$\Rightarrow A$ is Hermitian.

Thm. A is Hermitian (skew Hermitian) iff $-iA$ is skew Hermitian (Hermitian)

Thm: Let A be a square matrix.

(1) If A is Hermitian, then $A = B + iC$ where B is real symmetric and C is real skew-symmetric.

(2) If A is skew Hermitian, then $A = B + iC$, where B is real skew symmetric and C is real symmetric.

symmetric and C.

Pf: (1) Let $A = B + iC$ and A is Hermitian.

Then $A^* = (B + iC)^* = B^* - iC^* = A$

$\Rightarrow$ $B + iC = B^* - iC^*$

Note that since B, and C are a real matrix,

so, $B^* = B^t$, $C^* = C^t$

So, $B + iC = B^* - iC^*$

$\Rightarrow$ $B = B^* = B^t$ and $C = -C^* = -C^t$

$\Rightarrow$ B is real symmetric and C is real skew symmetric.

Thm: Let A be a square matrix. Then $A = P + iQ$, where P and Q are Hermitian matrices.

Pf: We have:

$$A = \underbrace{P}_{\text{Hermitian}} + \underbrace{Q}_{\text{skew-Hermitian}}$$

$$\Rightarrow A = P + \frac{i}{i} Q$$

$$= P + \frac{1}{i}(iQ)$$

$$= P + -i(iQ)$$

$$= P + i(-iQ)$$

Thm: Let A be an $n \times n$ square matrix. Then

(1) $A + A^*$ is Hermitian and $A - A^*$ is skew Hermitian.

(2) There is one and only way to write A as sum of a Hermitian matrix and a skew Hermitian matrix.

Pf: (1) $(A + A^*)^* = A^* + A^{**} = A^* + A$

> **Ex:** Show that
> (1) $(A+B)^* = A^* + B^*$
> (2) $(A^*)^* = A$

$(A - A^*)^* = -(A - A^*)$

(2.) $A = \underbrace{\frac{A + A^*}{2}}_{E} + \underbrace{\frac{A - A^*}{2}}_{D}$

If $A = E + F \Rightarrow A^* = E - F$

$\Rightarrow E = \frac{A + A^*}{2} = G$

$\Rightarrow E = G, \quad F = D$

Result: Let A be a square matrix. Then there are real matrices B and C such that

$$A = B + iC$$

Pf: Let

$$A = \begin{bmatrix} a_{11}+ib_{11} & - & - & a_{1n}+ib_{1n} \\ | & & & \\ | & & & \\ | & & & \\ a_{n1}+ib_{n1} & & - & a_{nn}+ib_{nn} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} a_{11} & a_{12} & - & a_{1n} \\ | & & & \\ | & & & \\ | & & & a_{nn} \\ a_{n1} & - & - & \end{bmatrix} + i \begin{bmatrix} b_{11} & -- & b_{1n} \\ | & & \\ | & & \\ b_{n1} & -- & b_{nn} \end{bmatrix}$$

Note: B and C are unique such matrices such that $\underline{A = B + iC}$