BPT-401 Date - 09/03/2021

1. Let $\vec{r}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z) and r be its length.

calculate

(a) $\bar{\nabla}(r^2)$ (b) $\bar{\nabla}\left(\frac{1}{r}\right)$ (c) $\bar{\nabla}(r^n)$

2. Sketch the vector function

$$\vec{v} = \frac{\hat{r}}{r^2}$$

$$\bar{\nabla}\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial x}$$

Compute its divergence. Explain your result.

Solution

$\cdot (x, y, z)$ $\qquad\qquad$ $\cdot (x', y', z')$

$$\vec{r} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$$

$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

(a) $$\vec{\nabla}(r^2) = \hat{x}\frac{\partial}{\partial x}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]$$

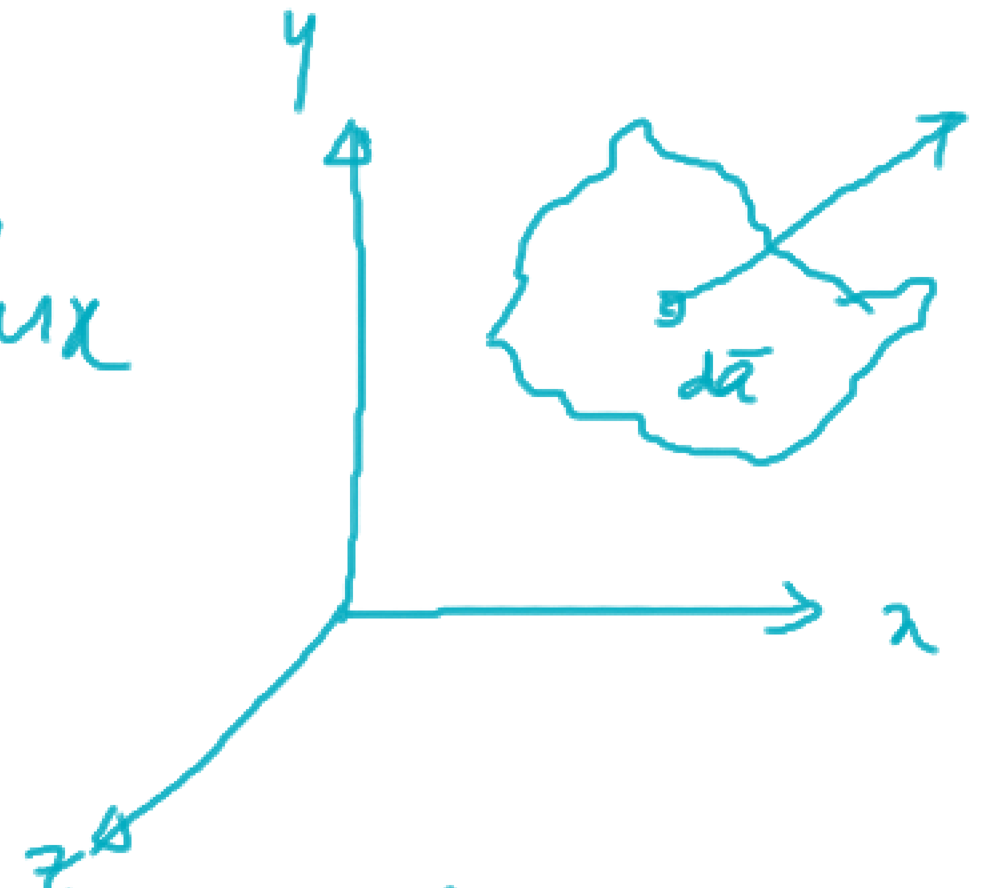
$$+ \hat{y}\frac{\partial}{\partial y}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right] + \hat{z}\frac{\partial}{\partial z}\left[r^2\right]$$

$$= \hat{x}\,2\cdot(x-x') + \hat{y}\,2(y-y') + \hat{z}\cdot 2\cdot(z-z')$$

$$= 2\left[(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}\right] = 2\vec{r}$$

Surface integrals:

$$\int_S \bar{v} \cdot d\bar{a} = \text{flux}$$



For closed surface, $d\bar{a}$ always points outward normal.

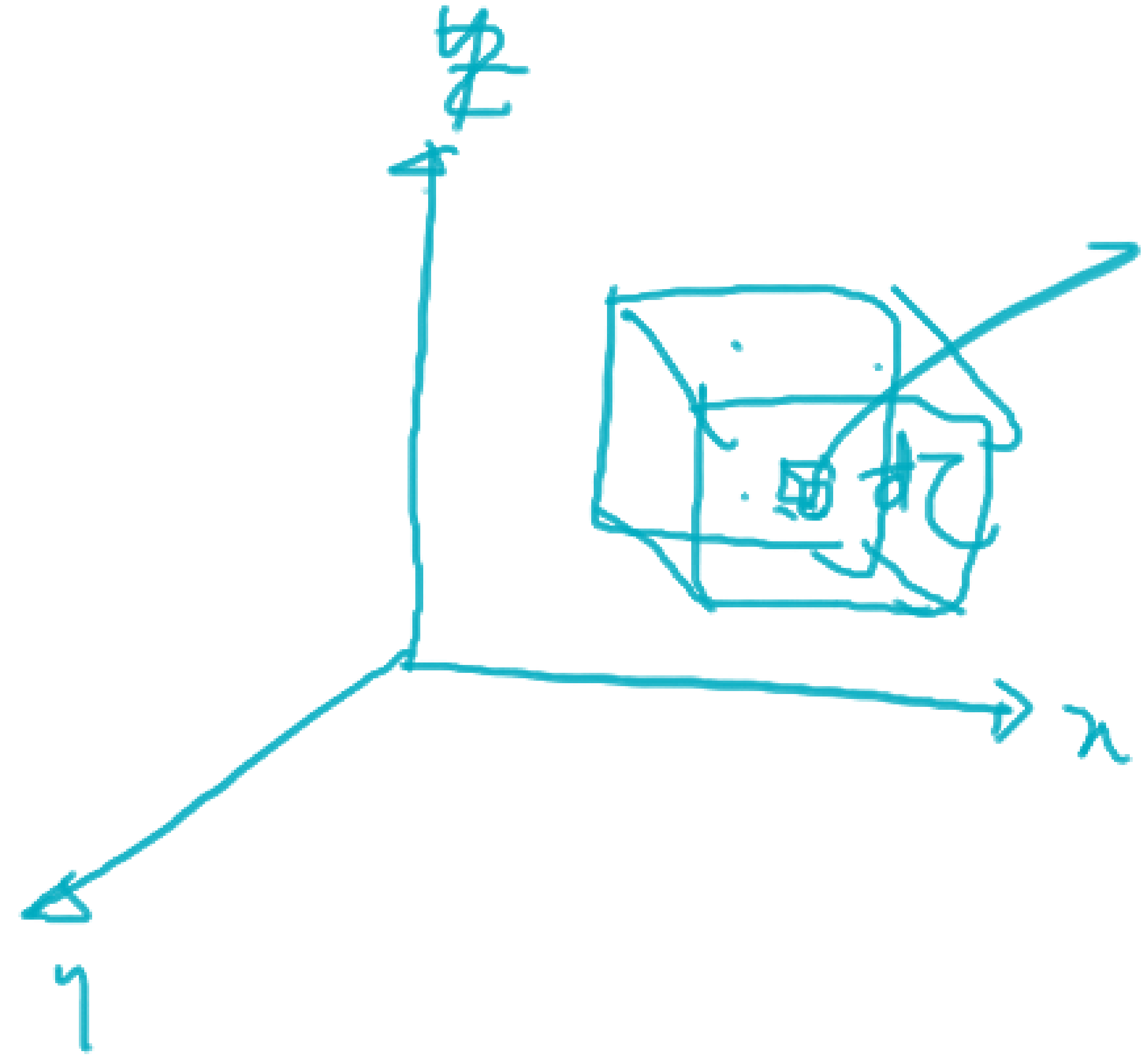
$\bar{v} \to$ represents flow fluid

Volume integral :

$T(\vec{r})$

$$\int_V T \, d\tau$$

$$d\tau = dx\,dy\,dz$$

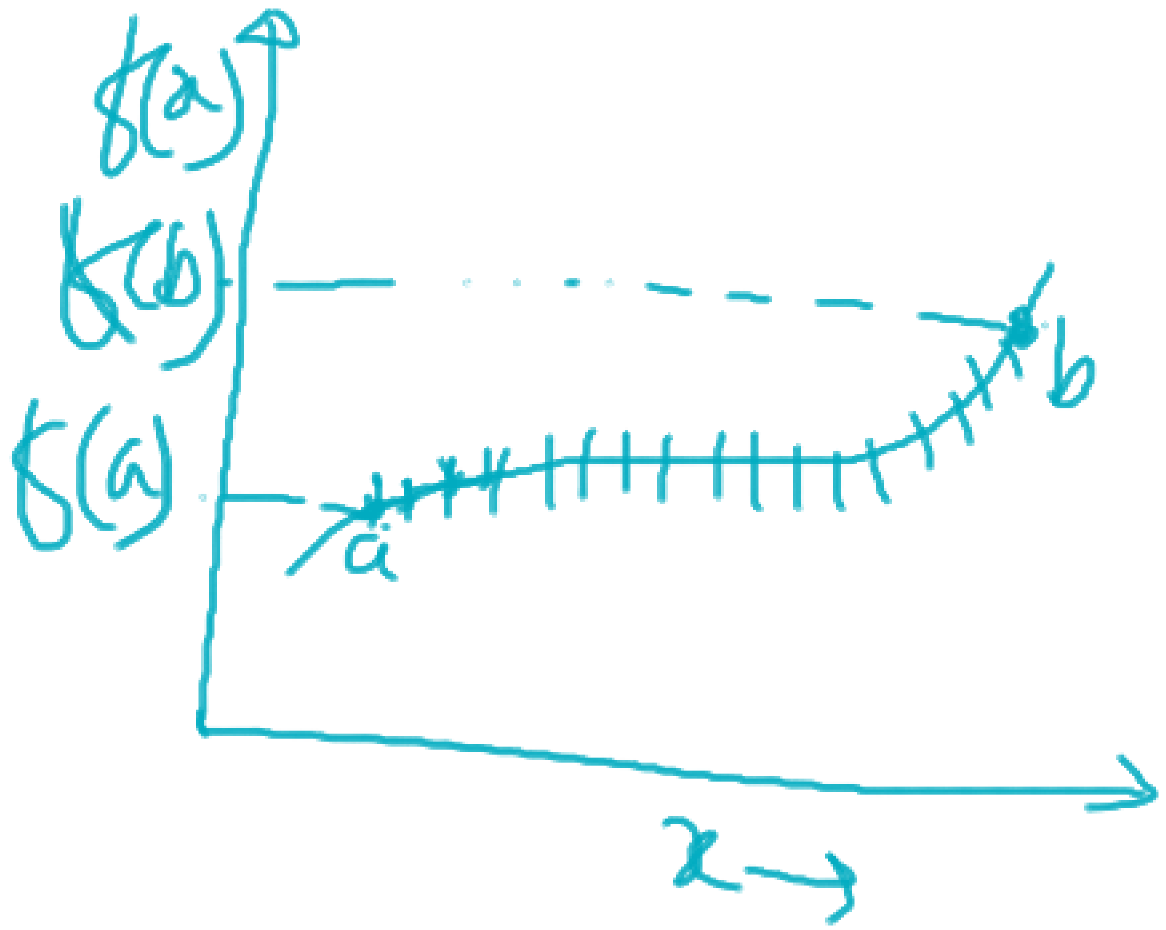


$$\int_V \bar{v} \, d\tau$$

The fundamental Theorem of calculus:

$f(x)$

$$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$$



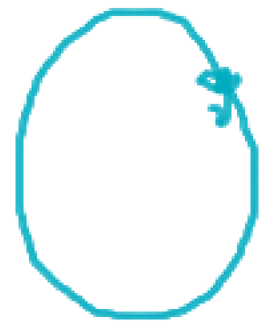
(x, y, z)

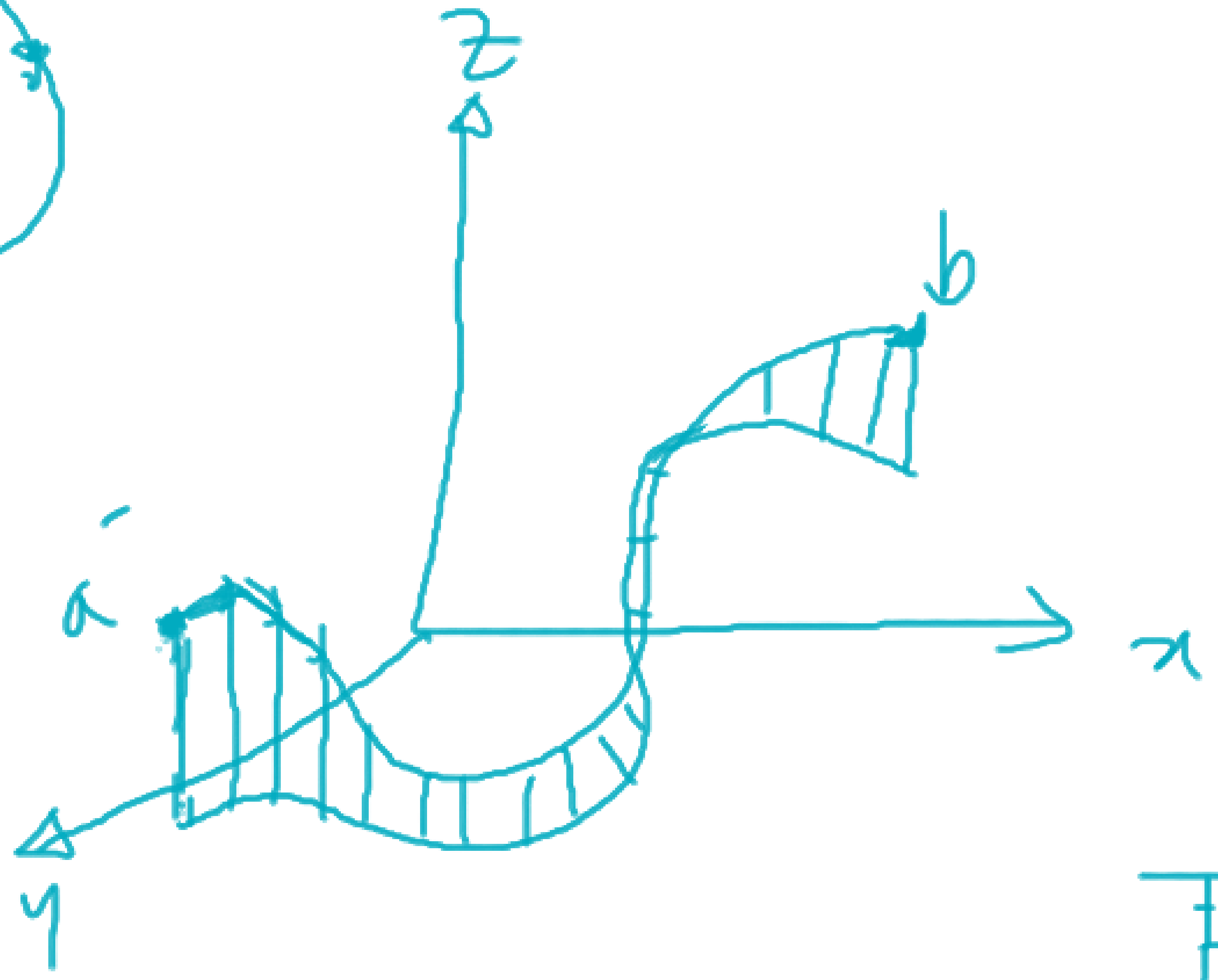
$\bar{\nabla}$

- gradient $(\bar{\nabla} T)$
- divergence $(\bar{\nabla} \cdot \bar{A})$
- curl $(\bar{\nabla} \times \bar{A})$

Gradient:

$T(x, y, z)$





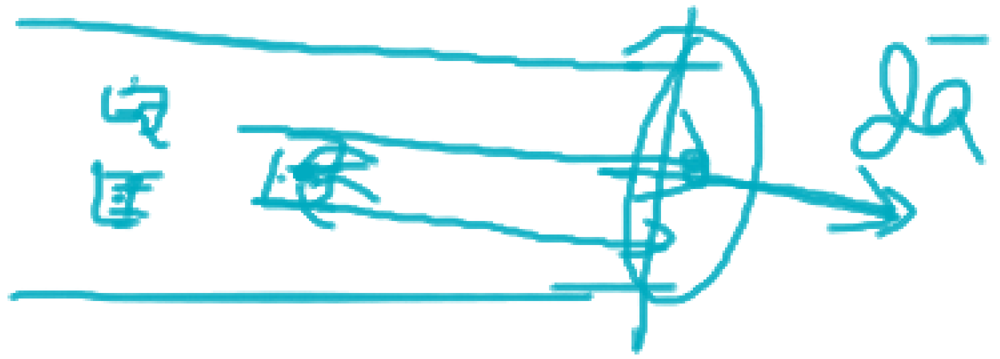
$$dT_1 = (\bar{\nabla} T) \cdot d\bar{l}_1$$

$$\int_a^b (\bar{\nabla} T) \cdot d\bar{l} = T(b) - T(a)$$

$$\oint (\bar{\nabla} T) \cdot d\bar{l} = 0$$

Gauss's theorem or divergence theorem :

$$\boxed{\int_V (\bar{\nabla} \cdot \bar{v})\, d\tau = \oint_S \bar{v} \cdot d\bar{a}} \Rightarrow \boxed{\text{flux}}$$

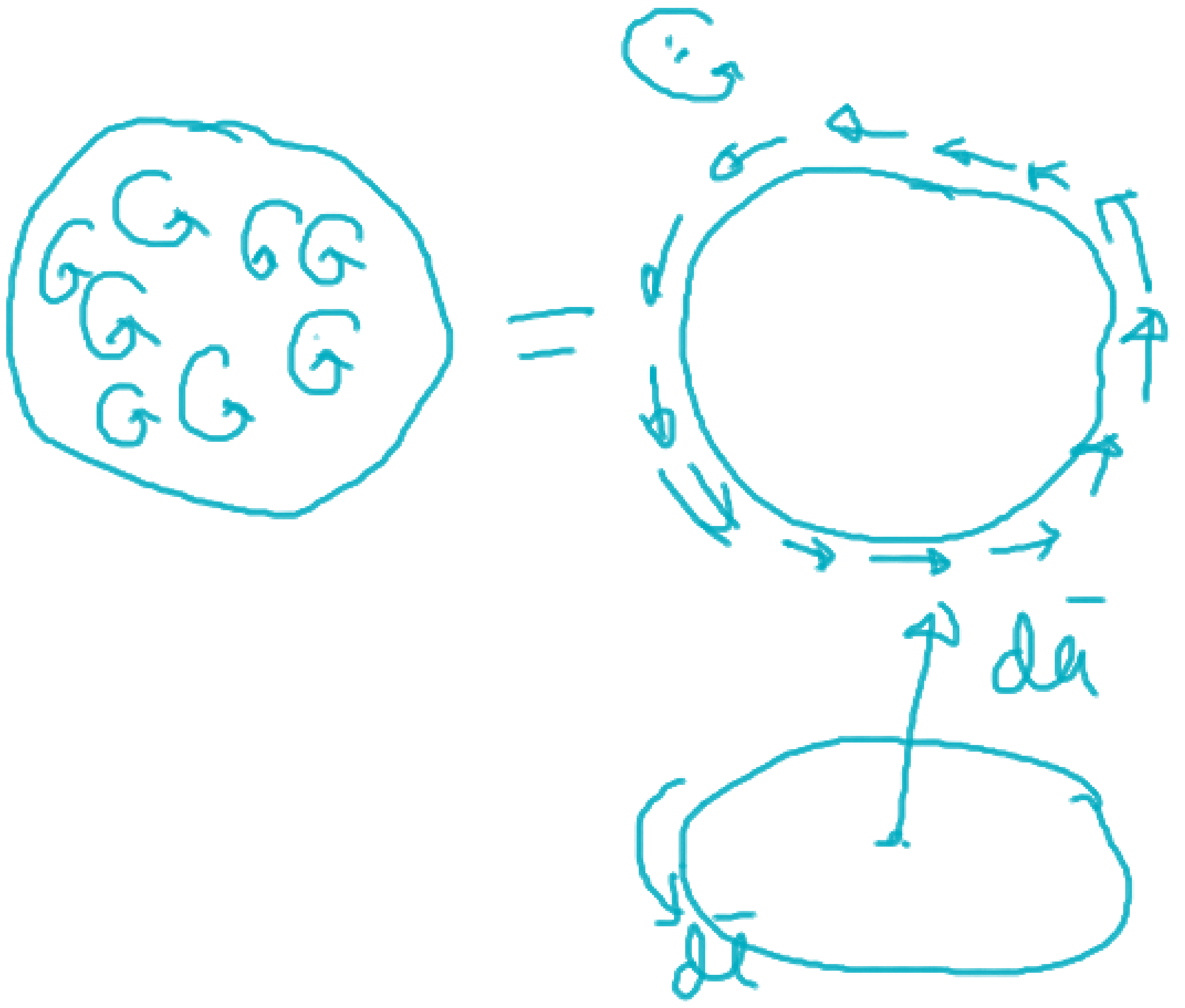


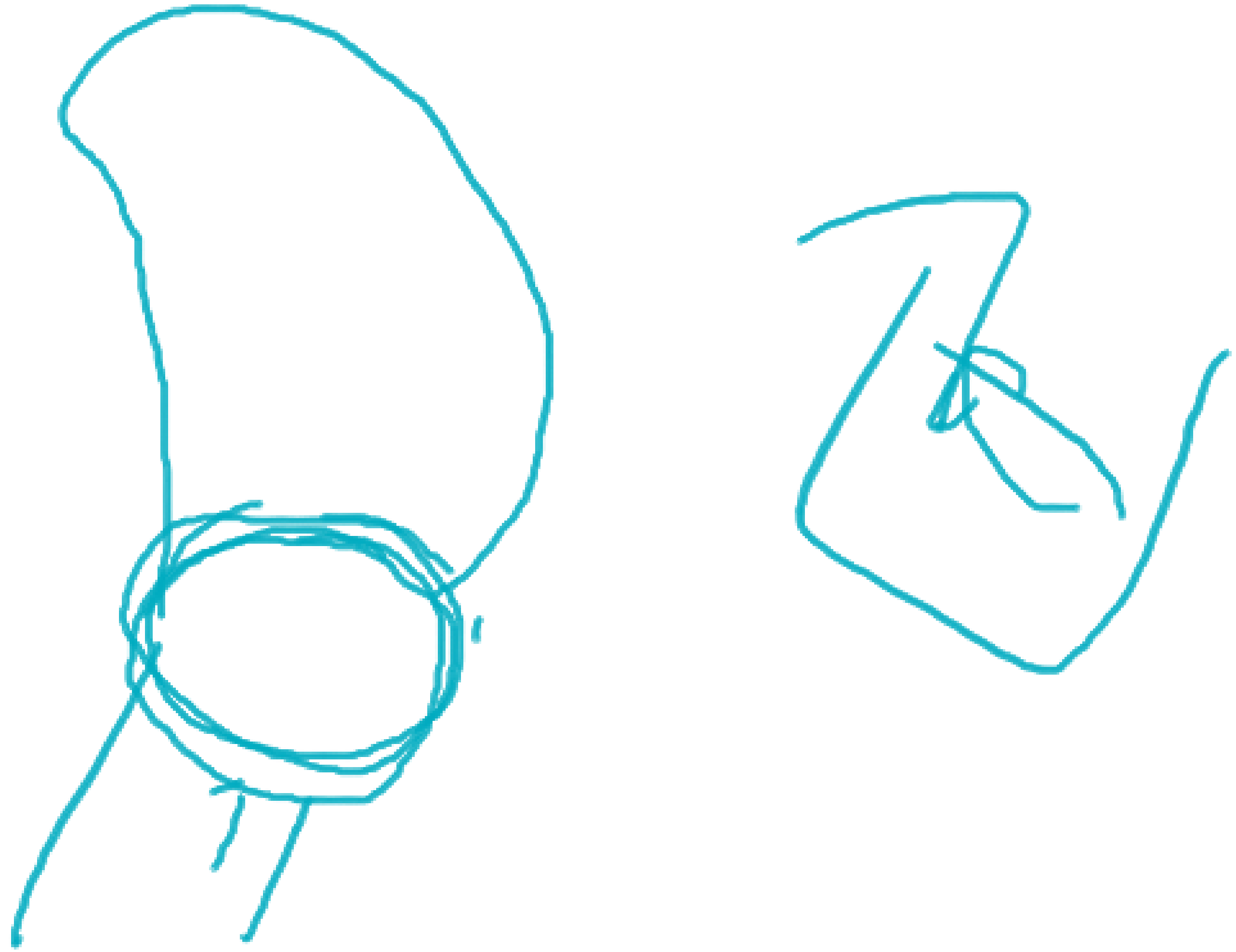
$(\bar{\nabla} \cdot \bar{v}) \Rightarrow$

$\int$ (faucets within volume) $= \oint$ (flow out through surface)

stokes theorem

$$\int_S (\bar{\nabla} \times \bar{v}) \cdot d\bar{a} = \oint_P \bar{v} \cdot d\bar{l}$$





$\int (\bar{\nabla} \times \bar{v})\, d\bar{a}$ depends only on the boundary line

$\oint (\bar{\nabla} \times \bar{v})\, d\bar{a} = 0$