B.Sc. Mathematics – 3rd Semester

MTM 302 – Differential Equations

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<u>Part II</u>

Equations of the First Order, but not of the First Degree

Let us consider an equation of the first order and n^{th} degree of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + ---- + P_{n-1} p + P_n = 0,$$

where $p = \dfrac{dy}{dx}$ and $P_1, P_2, P_3, ---, P_n$ are functions of x and y.

<u>Equation solvable for *p*:</u>

Let us suppose that the left hand side of the above equation can be expressed as a product of n linear factors in p, that is, the above equation can be put in the form

$$(p - f_1)(p - f_2)(p - f_3)\cdots\cdots(p - f_n) = 0,$$

where $f_1, f_2, f_3, \cdots\cdots, f_n$ are functions of x and y.

Thus, any relation in x and y which will make one or more of the factors zero, will be a solution of the equation. Hence each factor equated to zero will give a solution of the above equation.

Let the solution p be

$$F_1(x,y,c_1)=0,\ F_2(x,y,c_2)=0,\ldots\ldots..,\ F_n(x,y,c_n)=0.$$

Including all p variable solution in a single relation

$$F_1(x,y,c_1)\,F_2(x,y,c_2)\ldots\ldots.\,F_n(x,y,c_n)=0.$$

Since in the solution of first order ordinary differential equation only one arbitrary constant involved. Then the general solution of the above equation is

$F_1(x,y,c)\,F_2(x,y,c)\ldots\ldots F_n(x,y,c)=0$, where c is an arbitrary constant.

It is important to note that the degree of p in the given equation should be the same as that of c in its general solution.

Example: Solve $p^2-2xp-3x^2=0$.

Solution: We can write the equation as

$$(p-x)(p+3x)=0$$

The first factor when equated to zero, gives

$$p-x=0 \Rightarrow \frac{dy}{dx}=x$$

and its solution is $2y-x^2+c_1=0$

For 2nd factor, we have $p+3x=0$

$\Rightarrow \frac{dy}{dx}=-3x$ and its solution is $2y+3x^2+c_2=0$

Hence the general solution is

$$\left(2y-x^2+c\right)\left(2y+3x^2+c\right)=0.$$

Example: Solve $p^3-\left(x^2+xy+y^2\right)p^2+\left(x^3y+x^2y^2+xy^3\right)p-x^3y^3=0$.

Solution: On factorization, the given equation becomes

$$\left(p-x^2\right)\left(p-xy\right)\left(p-y^2\right)=0.$$

For the first factor $p-x^2=0\Rightarrow\frac{dy}{dx}=x^2\Rightarrow 3y-x^3+c_1=0$,

2nd factor $p-xy=0\Rightarrow\frac{dy}{dx}=-xy\Rightarrow e^{\frac{1}{2}x^2}+c_2y=0$

3rd factor $p-y^2=0\Rightarrow\frac{dy}{dx}=y^2\Rightarrow\frac{dy}{y^2}=dx$

$$\Rightarrow -\frac{1}{y} = x + c_3 \Rightarrow xy + c_3 y + 1 = 0$$

Hence the general solution is

$$\left(3y - x^3 + c\right)\left(e^{\frac{1}{2}x^2} + cy\right)\left(xy + cy + 1\right) = 0,\ c \text{ is an arbitrary constant.}$$

Examples: (i) $p^2 - p\left(e^x + e^{-x}\right) + 1 = 0$

(ii) $p\left(p - y\right) = x\left(x + y\right)$

(iii) $x + yp^2 = p\left(1 + xy\right)$

(iv) $p^3 + 2xp^2 - y^2 p^2 - 2xy^2 p = 0$

<u>Equation solvable for *y*:</u>

If the differential equation be solvable for y, then it may be put in the form

$$y = f\left(x, p\right) \tag{1}$$

Differentiating both sides of (1) w.r.t x we get $p = F\left(x, p, \dfrac{dp}{dx}\right)$.

This is an equation in two variables x and p and it can be solved to get a solution of the form

$$\phi\left(x, p, c\right) = 0 \tag{2}$$

Eliminating p from (1) and (2), we shall get the required solution, which will be a relation connecting x, y and an arbitrary constant c.

***Equation not containing x*:**

Consider the equation of the form $f(y, p) = 0$ If (it be solvable for p then)

it can be put as $p = \frac{dy}{dx} = F(y)$

and its solution will be $\int \frac{dy}{F(y)} = x + c$.

Again, if it is solvable for y, then we can have $y = G(p)$.

Then by previous method it can be solved.

Example: Solve: $y + px = x^4 p^2$

Solution: The equation can be written as $y = -px + x^4 p^2$.

Differentiating both sides w.r.t x we have

$$p = -p - x\frac{dp}{dx} + 4x^3 p^2 + 2x^4 p\frac{dp}{dx}$$

$$\Rightarrow 2p + x\frac{dp}{dx} = 2px^3\left(2p + x\frac{dp}{dx}\right)$$

$$\Rightarrow \left(2p + x\frac{dp}{dx}\right)\left(1 - 2px^3\right) = 0$$

$$\therefore \; 2p + x\frac{dp}{dx} = 0 \;\text{ if }\; 1 - 2px^3 \neq 0$$

$$\Rightarrow \frac{2}{x}dx + \frac{dp}{p} = 0$$

Integrating we get

$$\log\, x^2 + \log\, p = \log\, c$$

$\Rightarrow px^2 = c$, where c is an arbitrary constant.

$$\Rightarrow p = \frac{c}{x^2}$$

Putting this in the given equation, we get

$$y=-\frac{cx}{x^2}+x^4\frac{c^2}{x^4} \Rightarrow xy=c^2x-c$$

Now if $1-2px^3=0$ then $p=\dfrac{1}{2x^3}=\dfrac{1}{2}x^{-3}$

Then from the given we can have

$$y=-\frac{1}{2}x^{-3}.x+x^4\left(\frac{1}{4}x^{-6}\right) \Rightarrow y=-\frac{1}{2}x^{-2}+\frac{1}{4}x^{-2}=-\frac{1}{4}x^{-2} \Rightarrow 4x^2y+1=0$$

This relation satisfies the given equation. This cannot be deduced from the general solution and also it is not containing any arbitrary constant.

So this is a singular solution of the given equation.

Example: Solve: $y = p\tan p + \log\cos p$

Solution: Given equation, $y = p\tan p + \log\cos p$ (1)

Differentiating the given equation w.r.t x we get

$$p = \left(\tan p + p\sec^2 p - \tan p\right)\frac{dp}{dx} = p\sec^2 p\frac{dp}{dx}$$

$$\Rightarrow dx = \sec^2 p\,dp$$

Integrating, $\tan p = x + c$

$$\Rightarrow p = \tan^{-1}\left(x + c\right) \qquad (2)$$

Now eliminating p from (1) and (2) we get the general solution.

Equation solvable for *x*:

If the differential equation be solvable for x, then it can be written as

$$x = f(y, p) \tag{1}$$

Differentiating w.r.t y we get $\dfrac{1}{p} = F\left(y, p, \dfrac{dp}{dy}\right).$

Now this is an equation in two variables y and p and it can be solved to get a solution of the form

$$\phi(y, p, c) = 0 \tag{2}$$

Then eliminate p from (1) and (2) we get the required solution.

Equation not containing *y*:

Consider the equation of the form $f\left(x, p\right) = 0$.

If it can be put in the form $p = F\left(x\right)$.

Then its solution will be $y = \int F(x)dx + c$.

Now, if it can be written in form $x = G\left(p\right)$, then it can be solved by the previous method.

Example: Solve: $ayp^2+(2x-b)p-y=0$ where $a>0$ and b are known constants.

Solution: We have $2x=\frac{y}{p}-ayp+b$.

Differentiating w.r.t. y we get

$$\frac{2}{p}=\frac{1}{p}-\frac{y}{p^2}\frac{dp}{dy}-ap-ay\frac{dp}{dy}\Rightarrow\frac{1}{p}+ap+\left(\frac{y}{p^2}+ay\right)\frac{dp}{dy}=0$$

$$\Rightarrow p\left(a+\frac{1}{p^2}\right)+y\frac{dp}{dy}\left(a+\frac{1}{p^2}\right)=0\Rightarrow\left(a+\frac{1}{p^2}\right)\left(p+y\frac{dp}{dy}\right)=0.$$

Since $a>0$, so we have $p+y\dfrac{dp}{dy}=0$,

which gives $\dfrac{dp}{p}+\dfrac{dy}{y}=0 \Rightarrow py=c$

$$\Rightarrow p=\frac{c}{y};\ c \text{ is an arbitrary constant} \qquad (1)$$

Eliminating p from the given equation using (1) we have

$$ay\frac{c^2}{y^2}+(2x-b)\frac{c}{y}-y=0$$

$$\Rightarrow ac^2+(2x-b)c-y^2=0.$$

Example: Solve: $x+\frac{p}{\sqrt{1+p^2}}=a$, a being a given constant.

Solution: The equation given as: $x=-\frac{p}{\sqrt{1+p^2}}+a$

Differentiating w.r.t. y we get

$$\frac{1}{p}=-\left(\frac{1}{\sqrt{1+p^2}}-\frac{p^2}{\left(1+p^2\right)^{3/2}}\right)\frac{dp}{dy}\Rightarrow\frac{1}{p}=-\left(\frac{1}{\left(1+p^2\right)^{3/2}}\right)\frac{dp}{dy}$$

$$\Rightarrow dy=-\frac{p}{\left(1+p^2\right)^{3/2}}dp$$

Integrating we get $$y+c=\frac{1}{\sqrt{1+p^2}}$$

$$\Rightarrow (y+c)^2=\frac{1}{1+p^2} \qquad (1)$$

Also from the given equation we have

$$(x-a)^2=\frac{p}{1+p^2} \qquad (2)$$

Eliminating p from (1) and (2) we get

$$(x-a)^2+(y+c)^2=1.$$

Examples:

(1) $y = p\sin x + \cos x - 1$

(2) $y = 2px + \tan^{-1}\left(xp^2\right)$

(3) $x = 4p + 4p^3$

(4) $y = (1+p)x + ap^2$

Clairout's equation:

If differential equation of the form $y = px + f(p)$ is known as Clairout's equation.

To solve the equation we have to differentiate both sides of the equation with respect to x and obtain

$$p = p + \left(x + f'(p)\right)\frac{dp}{dx} \Rightarrow \left(x + f'(p)\right)\frac{dp}{dx} = 0$$

$$\Rightarrow \text{Either } \frac{dp}{dx} = 0 \quad \text{or} \quad x + f'(p) = 0$$

$$\Rightarrow \text{Either } p = \text{constant} \quad \text{or} \quad x + f'(p) = 0$$

Now, eliminating p from the given equation and $p = c$ we have

$$y = cx + f(c).$$

This is the general solution of the given equation.

Eliminating p from the given equation and $x + f'(p) = 0$ we get the singular solution of the equation which is a relation between x and y without any arbitrary constant.

The relation between the two solutions, if both exists:

The first solution $y = cx + f(c)$------ (1) represents a family of straight lines. If the family of straight lines has an envelope, then it can be found by differentiating both of the (1) with respect to c which gives $0 = x + f'(c)$ ----- (2) and then eliminating c between (1) and (2).

Now, elimination of c from (1) and (2) is precisely the same as that of p between $y = px + f(p)$ and $0 = x + f'(p)$.

Thus, the curve given by the latter is the envelope of the family of straight lines represented by the general solution, if these lines have an envelope.

Example: Obtain the complete primitive and the singular solution of the equation $y = px + \sqrt{1+p^2}$.

Solution: The equation is in the Clairout's form.

So differentiate with respect to x we get

$$p = p + \left(x + \frac{p}{\sqrt{1+p^2}} \right) \frac{dp}{dx} \Rightarrow \left(x + \frac{p}{\sqrt{1+p^2}} \right) \frac{dp}{dx} = 0.$$

Therefore either

$$\frac{dp}{dx} = 0 \Rightarrow p = c \qquad (1)$$

Or, $\quad x+\frac{p}{\sqrt{1+p}}=0 \Rightarrow x=-\frac{p}{\sqrt{1+p^2}}$ (2)

Eliminating p from the given equation and (1) we get

$$y=cx+\sqrt{1+c^2}.$$

This is the required complete primitive.

Now eliminating p from the given equation and (2) we get

$$x^2+y^2=\frac{p^2}{1+p^2}+\left(-\frac{p^2}{\sqrt{1+p^2}}+\sqrt{1+p^2}\right)^2=\frac{p^2}{1+p^2}+\frac{1}{1+p^2}=\frac{1+p^2}{1+p^2}=1$$

$\Rightarrow x^2+y^2=1$. This is the required singular solution.

Singular Solution

Sometime a solution of a differential equation can be found without involving any arbitrary constant and which is, in general, not a particular solution case of general solution. We have named such a solution as a singular solution.

Let us consider a differential equation $f(x, y, p) = 0$ whose general solution be $\varphi(x, y, c) = 0$, c being an arbitrary constant.

Now, the c-discriminant is obtained by eliminating c between

$$\varphi(x, y, c) = 0 \quad \text{and} \quad \frac{\partial \varphi}{\partial c} = 0.$$

The p-discriminant is obtained by eliminating p between the equations

$$f(x, y, p) = 0 \text{ and } \frac{\partial f}{\partial p} = 0.$$

Evidently, the c-discriminant is the locus of the points for each of which $\varphi(x, y, c) = 0$ has equal values of c and the p-discriminant is the locus of the points for each of which $f(x, y, p) = 0$ has equal values of p. The equation $f(x, y, p) = 0$ and its solution $\varphi(x, y, c) = 0$ are of same degree in p and c respectively and hence if there be a p-discriminant, then there must be a c-discriminant.

Envelope:

By assigning all possible values of c in $\varphi(x, y, c) = 0$ we obtain an infinite number of curves of same kind. The successive values of c differ by infinitesimal amounts. When these curves are drawn, the curves corresponding to two consecutive values of c (called consecutive curves) intersect and the limiting position of these points of intersection generates the envelope of the system of curves.

The envelope of the system is the locus of the points of intersection of the consecutive curves of the system obtained by giving different values to c

in $\varphi(x,y,c)=0$ and its obtained by eliminating c between $\varphi(x,y,c)=0$ and $\dfrac{\partial\varphi}{\partial c}=0$. Envelope is then a part of the c-discriminant.

Again, the envelope is touched at any point on it by each curves of the system. Therefore x, y, p for any point on the envelope are identical with x, y, p of some point on one of the curves of the system. At the points of intersection of the consecutive curves the p's are equal. Thus the locus of the points where p's have equal values will include the envelope. The p-discriminant relation of $f(x,y,p)=0$ contains the equation of the envelope of the system of curves given by $\varphi(x, y, c) = 0$ which is its solution.

Thus, both the *c*-discriminant and the *p*-discriminant relation contain the equation of envelope. This is the singular solution of the differential equation.

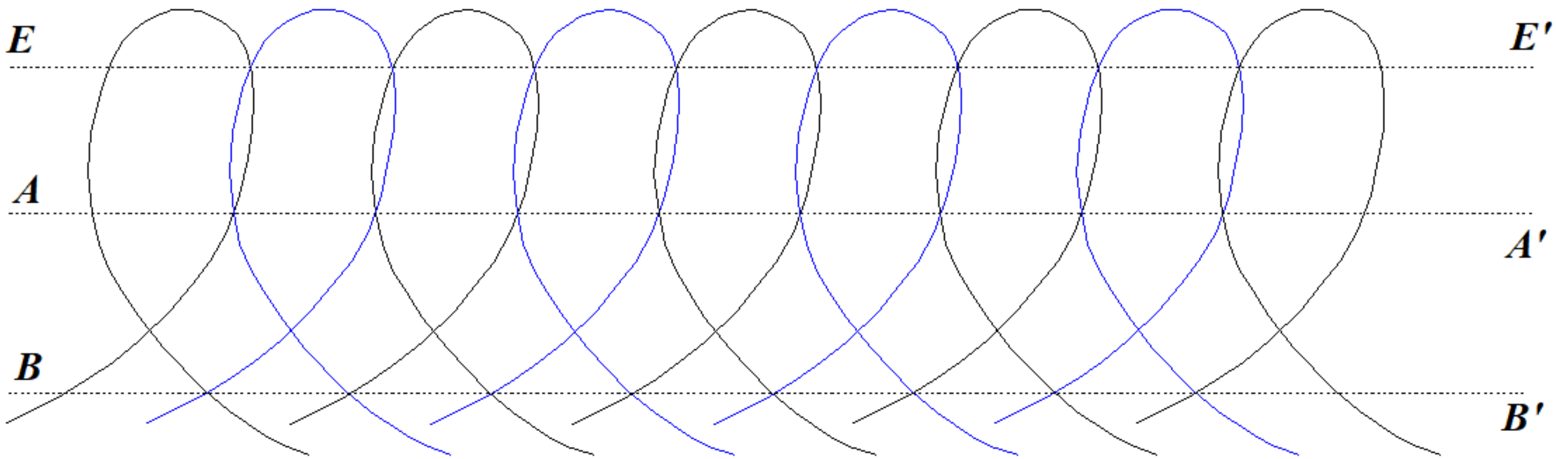
The *c*-discriminant relation may contain

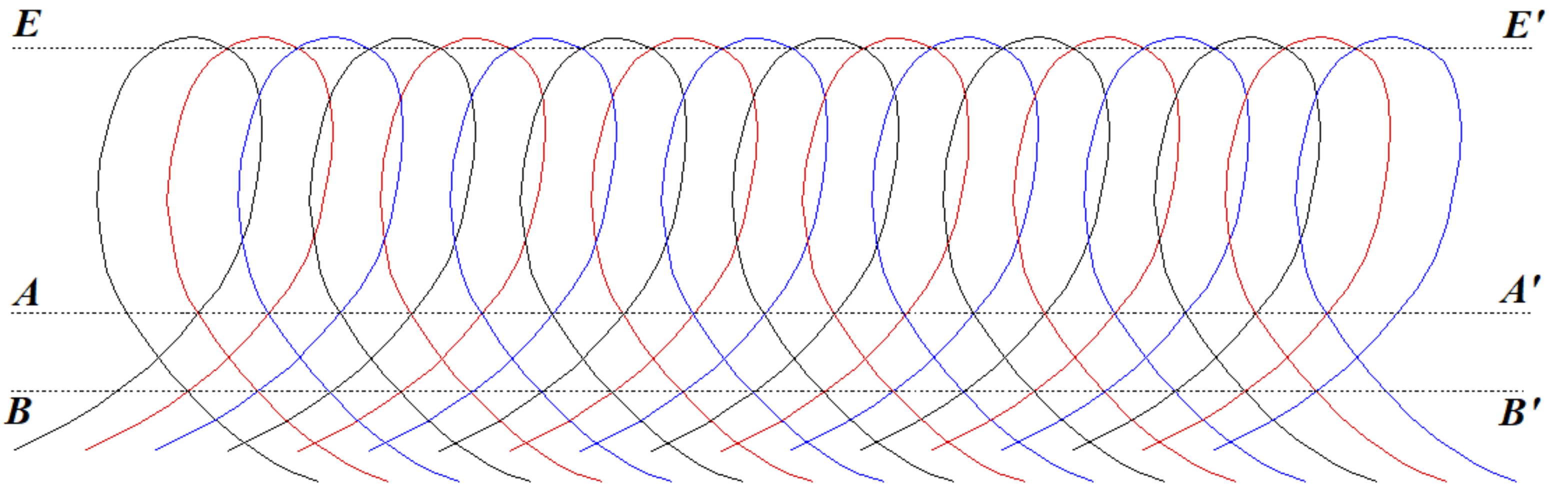
(1) The Envelope, (2) The Node-locus squared, (3) The Cusp-locus cubed

The *p*-discriminant relation may contain

(1) The Envelope, (2) The Tac-locus squared, (3) The Cusp-locus

$\underline{c\text{-discriminant}}$

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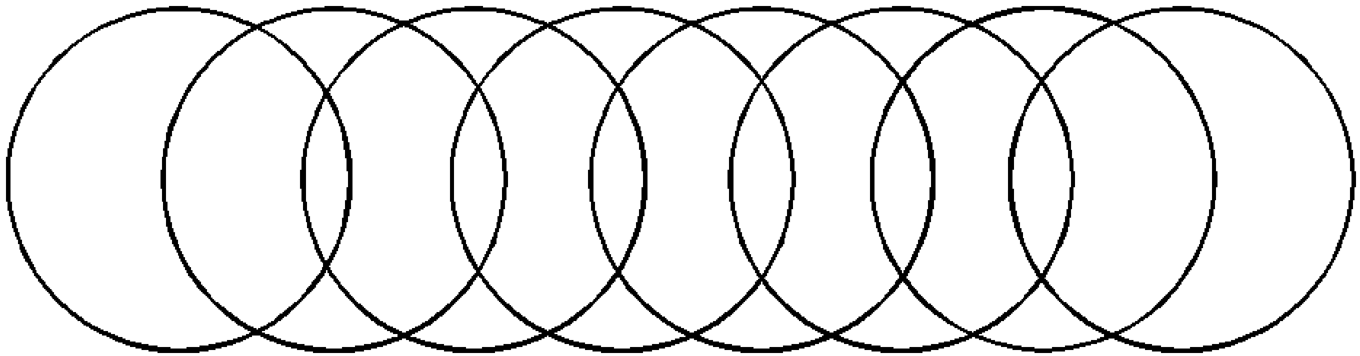
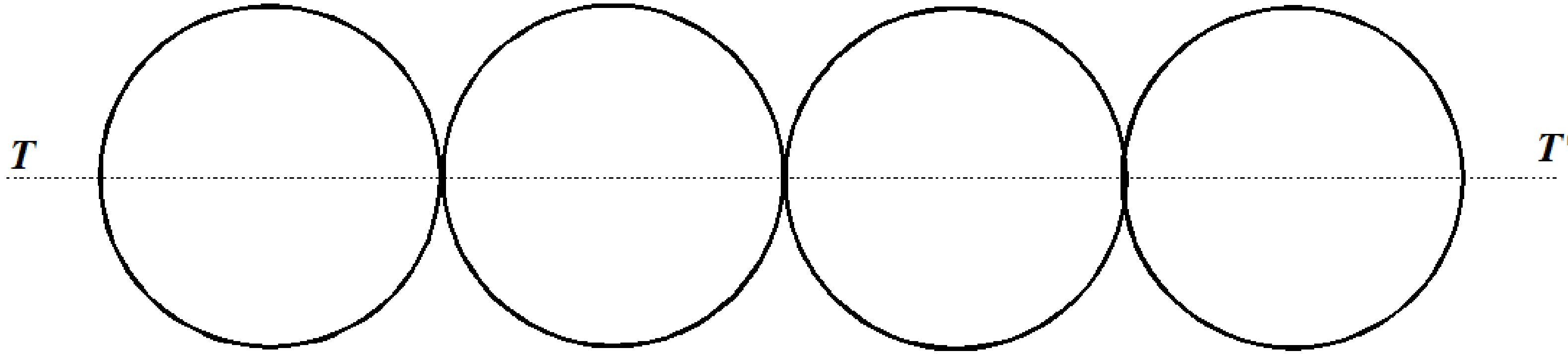
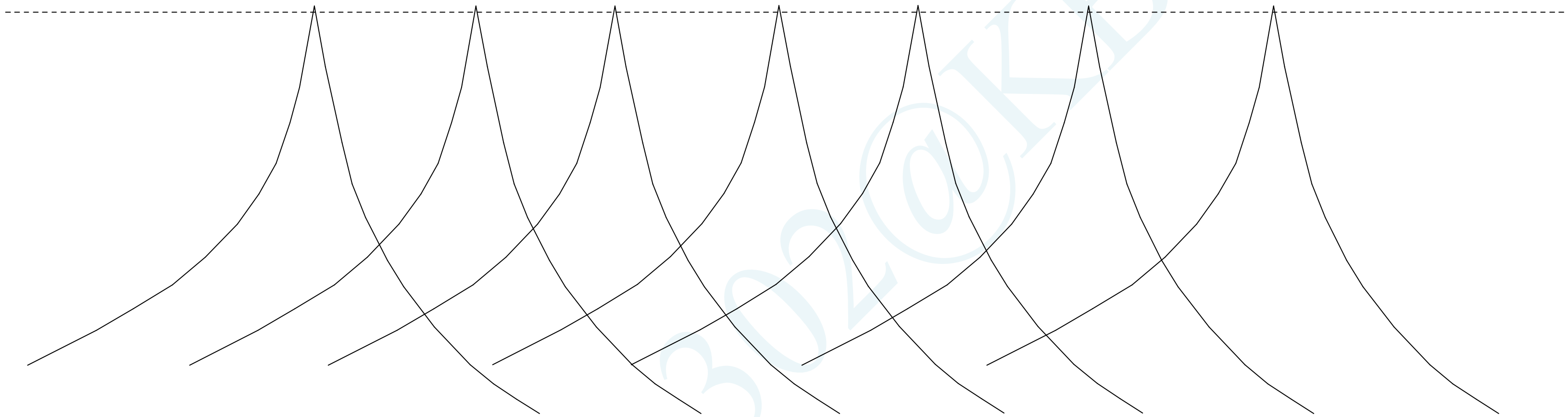
p-discriminant





Tac-locus



Cusp-locus

Thus using the symbols E, N, T, C for envelope, nodal locus (node), tac-locus and cusp locus we may summarize the above result as

c-discriminant relation $\rightarrow EN^2C^3 = 0$

p-discriminant relation $\rightarrow ET^2C = 0$

So the singular solution, i.e., the envelope is obtained as the common factor from c- and p-discriminant relations. It must also satisfy the differential equation.

Example: Examine the equation $y = 2px + p^2$ for singular solution.

Solution: We have

$$(p+x)^2 = y + x^2$$

$$p = \sqrt{y+x^2} - x$$

Putting $y = vx^2$, we get $x^2\dfrac{dv}{dx} + 2xv = x\left(\sqrt{1+v} - 1\right)$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{1+v} - 1 - 2v$$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{\sqrt{1+v} - 1 - 2v}$$

Putting $u^2 = 1 + v$, we have $2udu = dv$

So, $\dfrac{dx}{x} = \dfrac{2udu}{1+u-2u^2} \Rightarrow \dfrac{dx}{x} = -\dfrac{1}{2}\dfrac{(1-4u)du - du}{1+u-2u^2}$

Integrating $\log x = -\dfrac{1}{2}\log\left(1+u-2u^2\right) + \dfrac{1}{2}\int \dfrac{du}{1+u-2u^2} + \text{constant}$

$$\Rightarrow \log x = -\frac{1}{2}\log\left(1+u-2u^2\right) - \frac{1}{6}\log\frac{1-u}{1+2u} + \text{constant}$$

$$\Rightarrow x^6\left(1+u-2u^2\right)^3 \frac{1-u}{1+2u} = \text{constant}$$

$$\Rightarrow x^6(1-u)^4(1+2u)^2 = \text{constant} \Rightarrow x^3(1-u)^2(1+2u) = \text{constant} = c \text{ (say)}$$

$$\Rightarrow \left(1-3u^2+2u^3\right) = \frac{c}{x^3} \Rightarrow 1-3\left(1+\frac{y}{x^2}\right)+2\left(1+\frac{y}{x^2}\right)^{3/2} = \frac{c}{x}$$

Simplifying we get $\quad \left(2x^3+3xy+c\right)^2 = 4\left(x^2+y\right)^3$

The c-discriminant relation is given by $\left(x^2+y\right)^3 = 0$

and the p-discriminant relation is given by $x^2+y=0$

Hence, $x^2+y=0$ is the cusp-locus and the equation has no singular solution.

Example: Solve and examine for singular solution of the equation $xp^2-(x-a)^2=0$

Solution: We have from the given equation

$$p=\frac{x-a}{\sqrt{x}}=\sqrt{x}-ax^{-\frac{1}{2}}\Rightarrow dy=\left(\sqrt{x}-ax^{-\frac{1}{2}}\right)dx$$

Integrating $y=c+\left(\frac{2}{3}x^{3/2}-2ax^{1/2}\right)=c+\frac{2}{3}\sqrt{x}(x-3a)$

$$(y-c)^2=\frac{4}{9}x(x-3a)^2$$

So, the c-discriminant is given by $x(x-3a)^2=0$

and from the given equation the p-discriminant is $(x-a)^2=0$.

The given equation has no singular solution.

Example: Find the singular solution of the differential equation satisfied by the family of curves $c^2+2cy-x^2+1=0$.

Solution: The equation of the family of curves is

$$c^2+2cy-x^2+1=0 \qquad (1)$$

Differentiating both sides of (1) w.r.t. x and eliminating c, we get

$$\frac{x^2}{p^2}+2\frac{x}{p}y-x^2+1=0$$

$$\text{Or,} \quad x^2 + 2xyp + \left(1 - x^2\right) p^2 = 0 \qquad (2)$$

From (1) c-discriminant

$$x^2 + y^2 - 1 = 0.$$

From (2) p-discriminant

$$x^2\left(x^2 + y^2 - 1\right) = 0.$$

Hence, the singular solution is $x^2 + y^2 = 1$.

Examples: Solve and find the singular solutions of the following equations, if any,

(a) $3y = 2px - \dfrac{2p^2}{x}$

(b) $y^2\left(1+4p^2\right) - 2pxy - 1 = 0$

(c) $4xp^2 = (3x-1)^2$

(d) $x^3p^2 + x^2yp + a^3 = 0$

(e) $y^2\left(y - xp\right) = x^4p^2$

(f) $\left(1-p^2\right)^2 - e^{-2y} = p^2e^{-2x}$