Lecture: Motion Inside a Smooth Vertical Circle

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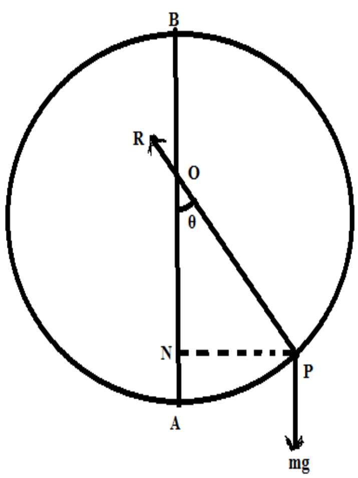
Motion on the inside of a smooth vertical circle

To discuss the motion of a particle projected from the lowest point with velocity u and moves along the inside of a smooth vertical circle.

Let A and B be the lowest and highest points of the circle, P be the position of the particle at time t where $\angle AOP = \theta$, O being the center of the circle. Let v be the velocity at P and PN be the perpendicular to AB. By the equation of energy



$$v^2 = u^2 - 2g \cdot AN$$

$$v^2 = u^2 - 2gr(1 - \cos\theta) \text{where } r = \text{radius of the circle.}$$

Equation of motion along the normal is

$$\frac{mv^2}{r} = R - mg\cos\theta.$$

$$\therefore R = m\left(\frac{v^2}{r} + g\cos\theta\right) = m\left[\frac{u^2}{r} - 2g(1-\cos\theta) + g\cos\theta\right]$$

$$= m\left[\frac{u^2}{r} - 2g + 3g\cos\theta\right].$$

v vanishes when

$$u^2 = 2gr(1-\cos\theta) \text{ or } \cos\theta = 1 - \frac{u^2}{2gr}. \tag{1}$$

R vanishes when

$$\frac{u^2}{r} = 2g - 3g\cos\theta \text{ or } \cos\theta = \frac{1}{3}\left(2 - \frac{u^2}{gr}\right). \tag{2}$$

If v vanishes before R vanishes, i.e., if $\cos\theta$ from (1) is greater than $\cos\theta$ from (2) then

$$1 - \frac{u^2}{2gr} > \frac{1}{3}\left(2 - \frac{u^2}{gr}\right)$$

$$\Rightarrow 3(2gr - u^2) > 2(2gr - u^2) \text{ or } u^2 < 2gr.$$

If $u^2 = 2gr$, v and R vanish simultaneously and the particle will rise up to the horizontal diameter of the circle.

Therefore when $u^2 < 2gr$, v will vanish in some point in the first quadrant, R will not vanish, therefore the particle will stop there and will come down, pass through the lowest point and will go up in the fourth quadrant and rise up to the same height and will then come down and so on. Thus the particle will oscillate about the lowest point. When $u^2 = 2gr$, the particle will go up to the end of the horizontal diameter and will oscillate through a quadrant on each side of the vertical.

If R vanishes before v vanishes we will have

$$1 - \frac{u^2}{2gr} < \frac{1}{3}\left(2 - \frac{u^2}{2gr}\right)$$

$$\therefore u^2 > 2gr.$$

The particle will rise above the horizontal diameter.

The particle will go right round the circle if both v and R do not vanish at the highest point when $\theta = \pi$, i.e., if $u^2 > 4gr$ and if $u^2 > 5gr$, the particle will describe the complete circle. Thus if $u^2 > 2gr$ but $< 5gr$, the particle will go above the horizontal diameter but will not go up to the highest point i.e., the particle will leave the circle at some point in the second quadrant and will afterwards move in a parabola. Therefore, we obtain the following results :

1. when $u^2 < 2gr$, the particle will oscillate on each side of the lowest point,
2. when $u^2 > 2gr$ but $< 5gr$, the particle leaves the circle and describe a parabola.
3. when $u^2 > 5gr$, the particle makes complete revolutions.

The reaction at A is

$$R_0 = m\left[\frac{u^2}{r} - 2g + 3g\right] = m\left(\frac{u^2}{r} + g\right).$$

Hence when $u^2 > 5gr$, $R_0 > 6mg$, i.e., the pressure at the lowest point must be greater than $6mg$, if the particle is to go right round the circle.

Cor. In the case of a bead moving in a circular wire or a particle moving inside a circular tube, the condition for complete revolution is that $v > 0$ when $\theta = \pi$, i.e., $u^2 > 4gr$.

Example. A particle is free to move on a smooth vertical circular wire of radius r. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time

$$\sqrt{\frac{r}{g}}\log(\sqrt{5} + \sqrt{6}).$$

Solution. Here $u^2 = 4gr \Rightarrow v^2 = u^2 - 2gr(1 - \cos\theta) = 2gr(1 + \cos\theta) = 4gr\cos^2\frac{\theta}{2}$.

$$R = \frac{mv^2}{r} + mg\cos\theta$$

$$\text{where } R = 0, v^2 = -gr\cos\theta$$

If at $\theta = \theta_1$, $R = 0$ then $-gr\cos\theta_1 = 2gr(1+\cos\theta_1)$

$$\cos\theta_1 = -\frac{2}{3}$$

Also,

$$v = 2\sqrt{gr}\cos\frac{\theta}{2}$$

Hence

$$r\dot{\theta} = 2\sqrt{gr}\cos\frac{\theta}{2}$$

or

$$\sqrt{\frac{g}{r}}t = \int_0^{\theta_1}\frac{d\theta}{2\cos\frac{\theta}{2}}$$

$$= \log\left(\tan\frac{\theta}{2} + \sec\frac{\theta}{2}\right)\Big|_0^{\theta_1}$$

$$= \log\left(\tan\frac{\theta_1}{2} + \sec\frac{\theta_1}{2}\right)$$

Now,

$$\because \cos\theta_1 = -\frac{2}{3} \Rightarrow 2\cos^2\frac{\theta_1}{2} - 1 = -\frac{2}{3}$$

$$\therefore \cos^2\frac{\theta_1}{2} = \frac{1}{6} \Rightarrow \sec\frac{\theta_1}{2} = \sqrt{6}, \tan\frac{\theta_1}{2} = \sqrt{5}.$$

Hence $t = \sqrt{\frac{r}{g}}\log(\sqrt{5}+\sqrt{6})$.

Example. A heavy bead slides on a smooth fixed circular wire of radius a. If it be projected from the lowest point, with a velocity just sufficient to carry it to the highest point, prove that the radius through the bead in time t will turn through an angle $2\tan^{-1}\left(\sinh t\sqrt{\frac{g}{a}}\right)$.

Solution. The velocity v at a point whose angular distance from the lowest point of the circle is θ is given by

$$v^2 = u^2 - 2ag(1-\cos\theta)$$

We are given that $v = 0$ when $\theta = \pi$, so

$$0 = u^2 - 2ag(1 - \cos \pi) \text{ or } u^2 = 4ag$$

$$\therefore v^2 = 4ag - 2ag(1 - \cos\theta) = 2ag(1 + \cos\theta) = 4ag\cos^2\frac{\theta}{2}$$

$$\Rightarrow v = 2\sqrt{ag}\cos\frac{\theta}{2}$$

For circle $s = a\theta \therefore \frac{ds}{dt} = a\frac{d\theta}{dt}$. From these two equations we have

$$a\frac{d\theta}{dt} = 2\sqrt{ag}\cos\frac{\theta}{2}$$

Integrating we get

$$2\sqrt{\frac{g}{a}}\int_0^t dt = \int_0^\theta \sec\frac{\theta}{2}d\theta$$

$$\sqrt{\frac{g}{a}}t = 2\log\left(\sec\frac{\theta}{2} + \tan\frac{\theta}{2}\right)$$

$$\sec\frac{\theta}{2} + \tan\frac{\theta}{2} = e^{\phi} \text{ where } \phi = \sqrt{\frac{g}{a}}t$$

$$\sec\frac{\theta}{2} - \tan\frac{\theta}{2} = e^{-\phi}$$

$$\sinh\phi = \frac{e^{\phi} - e^{-\phi}}{2} = \tan\frac{\theta}{2}$$

$$\Rightarrow \theta = 2\tan^{-1}\left(\sinh t\sqrt{\frac{g}{a}}\right).$$