System of Linear Equations

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- ➢Gauss Elimination Method
- ➢LU Decomposition Method
- ➢Iterative Method
  - Jacobi Iteration method
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Jacobi Iteration method (method of successive displacement)

Suppose, the system of linear equation

$$a_{11}x_1+a_{12}x_2+a_{13}x_3+\dots\dots\dots +a_{1n}x_n =b_1$$
$$a_{21}x_1+a_{22}x_2+a_{23}x_3+\dots\dots\dots +a_{2n}x_n =b_2$$
$$\vdots$$
$$a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\dots\dots\dots +a_{nn}x_n =b_n$$

So, we have

$$x_1=1/a_{11}\,[b_1 -(a_{12}x_2+a_{13}x_3+\dots\dots\dots +a_{1n}x_n)]$$
$$x_2=1/a_{22}\,[b_2 -(a_{21}x_1+a_{23}x_3+\dots\dots\dots +a_{2n}x_n)]$$
$$\vdots$$
$$x_n=1/a_{nn}\,[b_n -(a_{n1}x_1+a_{n2}x_2+\dots\dots\dots +a_{nn-1}x_{n-1})]$$

At k^th^ iteration

$$x_1^{k+1} = 1/a_{11}\,[b_1 - (a_{12}x_2^k + a_{13}x_3^k + \ldots\ldots\ldots + a_{1n}x_n^k)]$$

Similarly,

$$x_2^{k+1} = 1/a_{22}\,[b_2 - (a_{21}x_1^k + a_{23}x_3^k + \ldots\ldots\ldots + a_{2n}x_n^k)]$$

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$$x_n^{k+1} = 1/a_{nn}\,[b_n - (a_{n1}x_1^k + a_{n2}x_2^k + \ldots\ldots\ldots + a_{nn-1}x_{n-1}^k)]$$

$k=0,1,2,\ldots.$

Jacobi Iteration method (method of successive desplacement) Cont..

One disadvantage in Jacobi method is even though we are knowing the updated value of x_1 but still using the old value in x_2, so approximation is slow.

Note: In the absence of any better estimates, initial approximation can be taken as zero.

Solve the following linear system of equations using Jacobi Iteration method

$20x_1+x_2-2x_3=17$

$3x_1+20x_2-x_3=-18$

$2x_1-3x_2+20x_3=25$

We have,

$$x_1 = \frac{1}{20}(17 - x_2 + 2x_3)$$

$$x_2 = \frac{1}{20}(-18 - 3x_1 + x_3)$$

$$x_3 = \frac{1}{20}(25 - 2x_1 + 3x_2)$$

Solve the following linear system of equations using Jacobi Iteration method

$20x_1+x_2-2x_3=17$

$3x_1+20x_2-x_3=-18$

$2x_1-3x_2+20x_3=25$

Take initial approximation x_1^0, x_2^0, x_3^0 as zero

$$x_1^1 = \frac{1}{20}(17 - x_2^0 + 2x_3^0)$$

$$x_2^1 = \frac{1}{20}(-18 - 3x_1^0 + x_3^0)$$

$$x_3^1 = \frac{1}{20}(25 - 2x_1^0 + 3x_2^0)$$

so

$$x_1^1 = \frac{1}{20}(17) = 0.85$$

$$x_2^1 = \frac{1}{20}(-18) = -0.9$$

$$x_3^1 = \frac{1}{20}(25) = 1.25$$

Second iteration

$$x_1^2 = \frac{1}{20}(17 - x_2^1 + 2x_3^1)$$

$$x_2^2 = \frac{1}{20}(-18 - 3x_1^1 + x_3^1)$$

$$x_3^2 = \frac{1}{20}(25 - 2x_1^1 + 3x_2^1)$$

so

$$x_1^2 = \frac{1}{20}(17 + 0.9 + 2 \times 1.25) = 1.02$$

$$x_2^2 = \frac{1}{20}(-18 - 3 \times 0.85 + 1.25) = -0.965$$

$$x_3^2 = \frac{1}{20}(25 - 2 \times 0.85 + 3 \times 1.25) = 1.3525$$

Third iteration

$$x_1^3 = ?$$
$$x_2^3 = ?$$
$$x_3^3 = ?$$

Fourth iteration

$$x_1^4 = ?$$
$$x_2^4 = ?$$
$$x_3^4 = ?$$

Fifth iteration

$$x_1^5 = ?$$
$$x_2^5 = ?$$
$$x_3^5 = ?$$

Sixth iteration

$$x_1^6 = ?$$
$$x_2^6 = ?$$
$$x_3^6 = ?$$

$x_1=1$, $x_2=-1$, $x_3=1$

Gauss Seidel iteration method

Suppose, we have system of linear equation

$$a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots\ldots.. +a_{1n}x_n =b_1$$

$$a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots\ldots.. +a_{2n}x_n =b_2$$

$$\vdots$$

$$a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\ldots\ldots.. +a_{nn}x_n =b_n$$

So we have

$$\Longrightarrow \quad x_1^{k+1} =1/a_{11}\,[b_1 -(a_{12}x_2^{k}+a_{13}x_3^{k}+\ldots\ldots.. +a_{1n}x_n^{k})]$$

$$x_2^{k+1}= 1/a_{22}\,[b_2 -(a_{21}x_1^{k+1}+ a_{23}x_3^{k}+\ldots\ldots.. +a_{1n}x_n^{k})]$$

$$\ldots\ldots..$$

$$x_n^{k+1}= 1/a_{nn}\,[b_n -(a_{n1}x_1^{k+1}+ a_{n2}x_2^{k+1}+\ldots\ldots.. +a_{nn-1}x_{n-1}^{k+1})]$$

Solve the following linear system of equations using Gauss Seidel method

$20x_1+x_2-2x_3=17$

$3x_1+20x_2-x_3=-18$

$2x_1-3x_2+20x_3=25$

Take initial approximation x_1^0, x_2^0, x_3^0 as zero

First iteration

$$x_1^1 = \frac{1}{20}(17 - x_2^0 + 2x_3^0)$$

$$x_2^1 = \frac{1}{20}(-18 - 3x_1^1 + x_3^0)$$

$$x_3^1 = \frac{1}{20}(25 - 2x_1^1 + 3x_2^1)$$

so

$$x_1^1 = \frac{1}{20}(17) = 0.85$$

$$x_2^1 = \frac{1}{20}(-18 - 3 \times 0.85 + 0) = -1.0275$$

$$x_3^1 = \frac{1}{20}(25 - 2 \times 0.85 + 3 \times (-1.0275)) = 1.0109$$

Second iteration

$$x_1^2 = \frac{1}{20}(17 - x_2^1 + 2x_3^1)$$

$$x_2^2 = \frac{1}{20}(-18 - 3x_1^2 + x_3^1)$$

$$x_3^2 = \frac{1}{20}(25 - 2x_1^2 + 3x_2^2)$$

so

$$x_1^2 = \frac{1}{20}(17 + 1.0275 + 2 \times 1.0109) = 1.0025$$

$$x_2^2 = \frac{1}{20}(-18 - 3 \times 1.0025 + 1.0109) = -0.9998$$

$$x_3^2 = \frac{1}{20}(25 - 2 \times 1.0025 + 3 \times (-0.9998)) = 0.9998$$

Third or fourth iteration

$$x_1^3 = 1$$
$$x_2^3 = -1$$
$$x_3^3 = 1$$

Practice Problems

1. Solve the following linear system of equations using Jacobi iteration method

$$5x+2y+z=12$$
$$x+4y+2z=15$$
$$x+2y+5z=20$$

2. Solve the following linear system of equations using Gauss Seidel method

$$10x+y+z=12$$
$$2x+10y+z=13$$
$$2x+2y+10z=14$$

Suggested books

1. Numerical Methods by **S.R.K Lyenger & R.K. Jain.**

2. Numerical Analysis by **Richard L. Burden.**

3. Introductory methods of Numerical analysis by **S.S. Sastry**.

Thank you