

Example: Recall a function  $f: X \rightarrow Y$  is one-one  
 iff  $\forall x, y \in X \quad f(x) = f(y) \Rightarrow x = y$ ,  
 'f' is said to be onto iff  $\forall y \in Y, \exists x \in X$  s.t.  
 $f(x) = y$ .

—:

let  $X$  be a non-empty set. let  $S(X)$  denote  
 the collection of all bijections defined on  $X$ , i.e.

$$S(X) := \{ f: X \rightarrow X; f \text{ is 1-1 and onto} \}.$$

Claim:  $S(X)$  is a group w.r.t. the function  
 composition.

(1)  $S(X)$  is non-empty, because Identity  
 map,  $\text{Id} \in S(X)$ .

(2) let  $f, g \in S(X)$ . We show that

$$f \circ g \in S(X).$$

(i)  $f \circ g$  is one-one.

$$f \circ g(x) = f \circ g(y)$$

$$\Rightarrow f(g(x)) = f(g(y))$$

$$\Rightarrow g(x) = g(y)$$

$$\Rightarrow x = y.$$

(ii)  $f \circ g$  is onto.  
 let  $y \in X$

Choose  $x = g^{-1} \circ f^{-1}(y)$

Then  $f \circ g(x) = f \circ g(g^{-1} \circ f^{-1}(y)) = y.$

(5) Hence, function composition,  $\circ$ , is indeed a binary operation on  $X$ .

(3) Clearly,  $\forall f, g, h \in S(X)$

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

(4) Identity map play the role of identity element.

(5) Since Each  $f \in S(X)$  is a bijection.

Hence  $f^{-1}$  plays the inverse of  $f \in S(X)$ .

Hence  $(S(X), \circ)$  is a group.

#  $(S(X), \circ)$  is called a permutation group.



Now assume  $X = \{1\}$ .

$$S(X) = ?$$

$$\text{Let } X = \{1, 2, 3\}.$$

$$S(X) = ?$$

$$S(X) = \{f_1, f_2, f_3, f_4, f_5, f_6\}.$$

$$\text{--- } f_1(1)=1 \quad f_1(2)=2 \quad f_1(3)=3 \quad \text{ ~~$f_1(4)=4$~~   ~~$f_1(5)=5$~~   ~~$f_1(6)=6$~~  }$$

$$\text{--- } f_2(1)=2, f_2(2)=1, f_2(3)=3, \text{  ~~$f_2(4)=4$~~ ,  ~~$f_2(5)=5$~~ ,  ~~$f_2(6)=6$~~  }$$

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$$f_6 = ? \text{ ---}$$

Let  $X = \{1, 2, \dots, n\}$ . Then no. of elements in  $S(X)$ ,

$$|S(X)| = n!$$

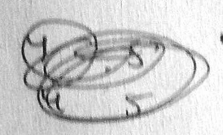
Another, representation of  $S(X)$ . Let  $X = \{1, 2, \dots, n\}$ .

Let  $f \in S(X)$ . Then  $f$  can be re-written as

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ f(1) & f(2) & f(3) & \dots & f(n) \end{pmatrix}.$$

Note that in the representation of 'f', no number in  $\{1, 2, \dots, n\}$  is repeated in upper and lower row.

Example:  $S(X) = 9$   $X = \{1, 2, 3\}$ .

$$f_1 \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$


$$f_2 \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$f_3 \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad f(1)=3, f(2)=2, f(3)=1$$

$$f_4 \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad f(1)=2, f(2)=3, f(3)=1$$

$$f_5 \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$f_6 \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Thm: No. of elements in  $S(X)$  for  $X = \{1, 2, \dots, n\}$  is  $n!$ .

pf: ~~to~~ For a bijection  $f$ ,

$f(1)$  has  $n$  choices.

$f(2)$  has  $n-1$  choices

$f(3)$  has  $n-2$  choices

$f(n)$  has 1 choice

No. of possible  $f$ .

$n \times (n-1) \times \dots \times 1$

$= n!$