**2·42. Analytical method.** Let $P_1$, $P_2$, $P_3$, ..., be any number of forces acting through the point $O$. Choose a set of rectangular coordinate axes $X'OX$ and $Y'OY$ through $O$ (Fig. 5). Let the forces be inclined at angles $\theta_1$, $\theta_2$, $\theta_3$, ..., to the positive direction $OX$ of the axis of $x$.

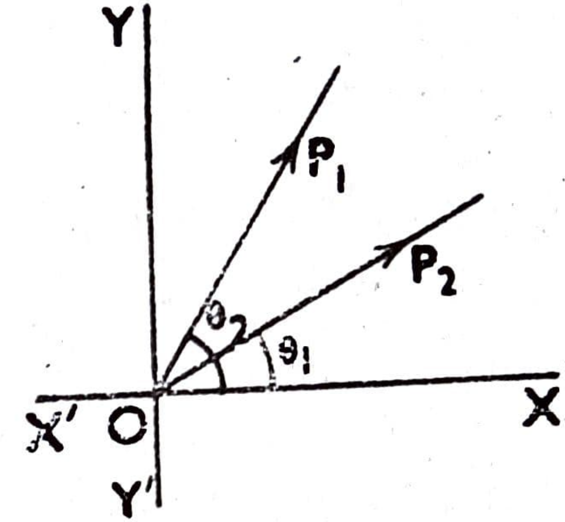


Fig. 5

Resolving the forces along $OX$ and $OY$, we get, if we denote the resolved parts of the resultant along $OX$ and $OY$ by $R_x$ and $R_y$ respectively,

$$R_x = P_1 \cos\theta_1 + P_2 \cos\theta_2 + P_3 \cos\theta_3 + \ldots \quad \ldots (1)$$

and $$R_y = P_1 \sin\theta_1 + P_2 \sin\theta_2 + P_3 \sin\theta_3 + \ldots \quad \ldots (2)$$

Hence the resultant $R$ of the given forces is

$$R = \sqrt{\{(R_x)^2 + (R_y)^2\}} \quad \ldots (3)$$

and is inclined to the axis of $x$ at an angle $\theta$, given by

$$\theta = \tan^{-1} (R_y/R_x). \qquad (4)$$

The equation (3) may be expressed in terms of the forces $P_1, P_2, \ldots$. Substituting the values of $R_x$ and $R_y$ from (1) and (2), we get

$$\begin{aligned}
R^2 &= (R_x)^2 + (R_y)^2 \\
&= (P_1 \cos \theta_1 + P_2 \cos \theta_2 + \ldots)^2 + (P_1 \sin \theta_1 + P_2 \sin \theta_3 + \ldots)^2 \\
&= [P_1^2 \cos^2 \theta_1 + P_2^2 \cos^2 \theta_2 + \ldots + 2P_1P_2 \cos \theta_1 \cos\theta_2 + \ldots] \\
&\quad + [P_1^2 \sin^2 \theta_1 + P_2^2 \sin^2 \theta_2 + \ldots + 2P_1P_2 \sin \theta_1 \sin \theta_2 + \ldots] \\
&= P_1^2(\cos^2 \theta_1 + \sin^2 \theta_1) + P_2^2 (\cos^2 \theta_2 + \sin^2 \theta_2) + \ldots \\
&\qquad + 2P_1P_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + \ldots \\
&= P_1^2 + P_2^2 + \ldots + 2P_1P_2 \cos (\theta_2 - \theta_1) + \ldots \\
&= \Sigma P_1^2 + 2\Sigma P_1P_2 \cos \phi_{12},
\end{aligned}$$

where $\phi_{12}$ $(= \theta_2 - \theta_1)$ denotes the angle between the forces $P_1$ and $P_2$.

This theorem is a generalisation of the theorem of the parallelogram of forces. For, if only two forces, $P_1$ and $P_2$, represented by $OA$ and $OB$ (see the Fig. in § 2·41), are given then $R$, as given by the above equation, is represented by $OC$, the diagonal of the parallelogram $OACB$.

**Ex.** *ABCDEF is a regular hexagon. Show that the resultant of forces represented by* AB, 2AC, 3AD, 4AE and 5AF *is represented by* $\sqrt{(351)}\, AB$, *and find its direction.*

In the regular hexagon, the lines of action of the given forces AB, 2AC, 3AD, 4AE, 5AF, are as marked by the arrows. It is clear from the figure (Fig. 6) that for the hexagon $ABCDEF$,

$$AC = AE = 2AB \sin 60° = \sqrt{3} AB,$$

and $$AD = AB \sec 60° = 2AB.$$

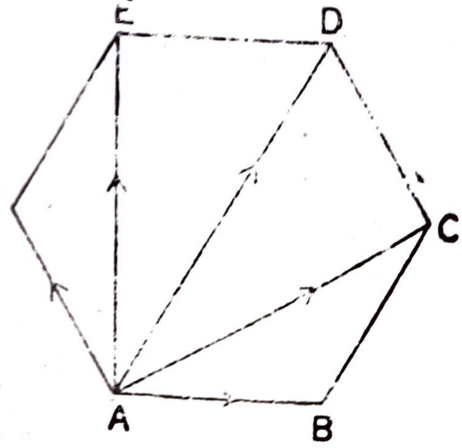

Fig. 6

Thus if a side of the hexagon be $l$, the magnitudes of the forces are

$$l,\ 2\sqrt{3}l,\ 6l,\ 4\sqrt{3}l \text{ and } 5l.$$

Now let us choose the perpendicular lines $AB$ and $AE$ as co-ordinate axes through $A$. Resolving the forces along these two lines, we have

$$\begin{aligned}
R_x &= l + 2\sqrt{3}l \cos 30° + 6l \cos 60° + 5l \cos 120° \\
&= l + 3l + 3l - (5/2)l = \tfrac{1}{2} \cdot 9l,
\end{aligned}$$

and $R_y = 2\sqrt{3}l \sin 30° + 6l \sin 60° + 4\sqrt{3}l + 5l \sin 120°$

$= \sqrt{3}l + 3\sqrt{3}l + 4\sqrt{3}l + (5\sqrt{3}/2)l = \frac{1}{2} . 21\sqrt{3}l.$

Hence the resultant $R$ is given by

$$R = \tfrac{1}{2}\sqrt{\{81 + 441.3\}}l = \tfrac{3}{2}\sqrt{(9+147)}l = \tfrac{3}{2}\sqrt{(156)}l$$
$$= 3\sqrt{(39)}l = \sqrt{(351)}l,$$

and its inclination to the line $AB$ is

$$\theta = \tan^{-1}(R_y/R_x) = \tan^{-1}(7/\sqrt{3}).$$

## Examples 2(b)

1. $ABCD$ is a quadrilateral and forces acting at a point are represented in magnitude and direction by BA, BC, CD, and DA. Find their resultant.

2. $ABCDE$ is a polygon. Forces acting on a particle are represented in magnitude and direction by AB, AE, BC, DC, ED and AC. Find their resultant.

3. Three forces $P, Q, R$ in a plane act on a particle, the angles between $R$ and $Q$, $P$, and $R$ and $P$ and $Q$ being $\alpha$, $\beta$ and $\gamma$ respectively. Show that their resultant is equal to

$$\sqrt{\{P^2 + Q^2 + R^2 + 2QR \cos \alpha + 2RP \cos \beta + 2PQ \cos \gamma\}}.$$

4. If forces of magnitudes $P$, $Q$ and $R$ act at a point parallel to and in the directions of the sides $BC$, $CA$ and $AB$ of a triangle $ABC$ respectively, prove that the magnitude of their resultant is

$$\sqrt{(P^2 + Q^2 + R^2 - 2QR \cos A - 2RP \cos B + 2PQ \cos C)}.$$

5. Three forces acting at a point are parallel to the sides of a triangle $ABC$, taken in order, and proportional to the cosines of the opposite angles. Show that their resultant is proportional to

$$\sqrt{(1 - 8 \cos A \cos B \cos C)}.$$

6. $ABCDEF$ is a regular hexagon. Find the resultant of the forces represented by AB, AC, AD, AE and AF.

7. Forces 2, $\sqrt{3}$, 5, $\sqrt{3}$, 2 lbs. respectively act at one of the angular points of a regular hexagon, towards the five others in order. Find the magnitude and direction of the resultant.

# Coplanar Forces

**4·1. Reduction of Coplanar Forces. Theorem.** *A system offorces acting in one plane at different points of a rigid body can bereduced to a single force through a given point, and a couple.*

[*Meerut*, 1982]

Let the forces $P_1, P_2, P_3, \ldots$ act at points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3), \ldots$, of the body, the coordinates of the points being given with reference to rectangular axes $OX$ and $OY$ through a given point $O$ (Fig. 27).

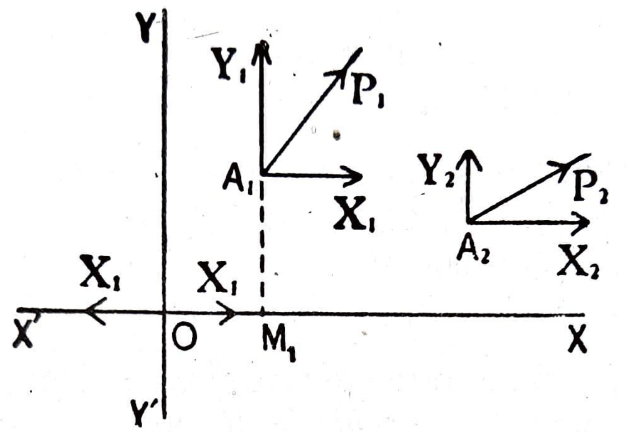


Fig. 27

Consider first the force $P_1$ acting at $A_1$ $(x_1, y_1)$. Let it be resolved into two forces $X_1$ and $Y_1$ parallel to the coordinate axes. At $O$ introduce two equal and opposite forces $X_1$, one along $OX$ and another along $OX'$. This will have no effect on the body. Now the forces $X_1$ at $A_1$ and $X_1$ at $O$ along $OX'$ form a couple of moment* $-X_1 \cdot A_1M_1$, i.e. $-X_1y_1$, and we are left with the force $X_1$ along $OX$. Thus the force $X_1$ at $A_1$ is equivalent to a force $X_1$ at $O$ along $OX$ and a couple of moment $-X_1y_1$.

Similarly by introducing at $O$ equal forces $Y_1$ along $OY$ and $OY'$, it is easy to see that the force $Y_1$ at $A_1$ is equivalent to a force $Y_1$ at $O$ along $OY$ and a couple of moment $Y_1x_1$.

It follows, therefore, that the force $P_1$ at $A_1$ is equivalent to forces $X_1$ and $Y_1$ at $O$ along the axes $OX$ and $OY$ respectively and a couple of moment $Y_1x_1 - X_1y_1$

*The negative sign is prefixed since the tendency of the couple is to rotate the body clockwise.

Proceeding in the same way with the remaining forces we see that the given system of forces is equivalent to forces

$$R_x = X_1 + X_2 + X_3 + \ldots = \Sigma X_1, \text{ along } OX,$$

and $$R_y = Y_1 + Y_2 + Y_3 + \ldots = \Sigma Y_1, \text{ along } OY,$$

and a couple of moment

$$G = (Y_1 x_1 - X_1 y_1) + (Y_2 x_2 - X_2 y_2) + \ldots = \Sigma (Y_1 x_1 - X_1 y_1). \quad \ldots (1)$$

The forces $R_x$ and $R_y$ can be compounded into a single force through $O$ of magnitude $R$ given by

$$R^2 = (R_x)^2 + (R_y)^2, \quad \ldots (2)$$

acting at an angle $\theta = \tan^{-1} (R_y / R_x)$ with the axis of $X$.

Thus the system of forces can be reduced to a single force $R$ through $O$ and a couple of moment $G$.

It is evident that $G$ depends upon the position $O$ of the given point while $R$ does not.

**Exercise.** Reduce the system of coplanar forces acting on a rigid body to a single force, acting through an arbitrary point, and a couple. Find the necessary conditions for the equilibrium of the rigid body. [ *Gorakhpur*, 1985]

**4·11. Further reduction of coplanar forces. Theorem.** *A system of forces acting in one plane at different points of a rigid body can be reduced to a single force, or a couple.*

We have just seen that a system of forces in general can be reduced to a single force $R$ and a couple of moment $G$, given by equations (2) and (1) of § 4·1.

If $R=0$, the forces reduce to a couple. But if $R \neq 0$, we will show that the force $R$ and the couple $G$ can be reduced to a single force $R$ acting in the same direction but in a different line.

Replace the couple $G$ by two equal and opposite forces of magnitude $R$, one along $OB$ in the direction opposite to $R$ and the other along $O'C$, where $OO'$ is perpendicular to $OB$, ($O'$ lies to the right or left of $OB$ depending upon the sign of $G$), as shown in Fig. 28.

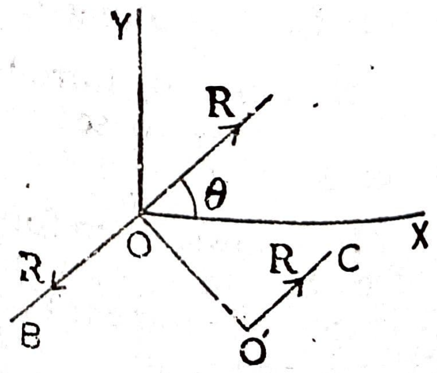

Fig. 28

We then have $OO' . R = G$, i.e.

$$OO' = \frac{\Sigma (Y_1 x_1 - X_1 y_1)}{\sqrt{\{(R_x)^2 + (R_y)^2\}}}.$$

The forces at $O$ balance each other and we are left with the force $R$ acting at $O'$ along $O'C$.

The equation of the line $O'C$ may be obtained as follows.

Since $\tan\theta = R_y/R_x$, we have $\sin\theta = R_y/R$ and $\cos\theta = R_x/R$. The line $O'C$ is therefore

$$x\cos\{-(\tfrac{1}{2}\pi-\theta)\}+y\sin\{-(\tfrac{1}{2}\pi-\theta)\}=OO',$$

i.e. $$x\sin\theta - y\cos\theta = OO',$$

i.e. $$xR_y - yR_x = G.$$

**Exercise.** Find the equation of the resultant of any number of coplanar forces acting on a rigid body. [*Gorakhpur*, 1983]

**4·12. Equation of the resultant.** We may also obtain the equation of the final resultant as follows.

Let $G$ be the moment of all the forces about the origin, i.e. in the notation of § 4·1,

$$G=\Sigma(Y_1x_1-X_1y_1).$$

If the coordinates of a point $Q$ be given by $(\xi, \eta)$, the moment of the force $P_1$ about $Q$ is

$$Y_1(x_1-\xi)-X_1(y_1-\eta),$$

i.e. $$(Y_1x_1-X_1y_1)-Y_1\xi+X_1\,\eta.$$

Writing the moments of the other forces similarly, we see that the total moment $G'$ of all the forces about the point $Q$ is given by

$$G'=G-\xi\Sigma Y_1+\eta\Sigma X_1=G-\xi R_y+\eta R_x.$$

If now the point $Q$ lies on the resultant, we have $G'=0$,

i.e. $$G-\xi R_y+\eta R_x=0.$$

Replacing $\xi$, $\eta$ by the current coordinates $x$, $y$ in this result we obtain the same line of action of the resultant, as in § 4·11.

**Ex. 1.** *The algebraic sums of the moments of a system of coplanar forces about points whose coordinates are* (1, 0), (0, 2) *and* (2, 3), *referred to rectangular axes, are* $G_1$, $G_2$ *and* $G_3$ *respectively. Find the tangent of the angle which the direction of the resultant force makes with the axis of* $x$. [B.H.U., 1977]

Let $R$ represent the magnitude of the resultant force and let its equation be

$$y-mx-c=0,$$

where $c$ is positive. Then

$$G_1=\frac{-m-c}{\sqrt{(1+m^2)}}R,\ G_2=\frac{2-c}{\sqrt{(1+m^2)}}R \text{ and } G_3=\frac{3-2m-c}{\sqrt{(1+m^2)}}R.$$