Kinematics and Kinetics

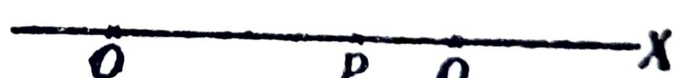
A. KINEMATICS : THE GEOMETRY OF MOTION

1·1. Displacement.

When a point changes its position relative to surrounding objects and occupies different positions at different times, it is said to be in **motion** relative to those objects. In this book we shall deal with motion of a *particle*, a particle being defined *as a geometrical point having a definite mass*. If a particle at any particular instant is at a point P and at any subsequent instant comes to a point Q, then the length PQ is the **displacement** of the particle during that interval. Thus the displacement of a moving particle is the *change of its position* and is measured by the distance moved by it during a particular interval. Displacement has both sense and magnitude.

1·2. Motion in a straight line : Velocity and Acceleration.

In order to determine the position of a moving particle or a point in a straight line, we must have a *fixed point* on the line. The position of the moving point is then determined with respect to that fixed point. Let O be the fixed point on the line and let P be the position of the moving point at any particular instant and Q its position at any subsequent instant.



Then PQ is the displacement of the moving point during that interval. If the time taken in moving from P to Q be denoted by t, then PQ/t measures the **average velocity** of the particle during the interval t. Thus the average velocity during a given interval is measured by the whole displacement divided by the whole time.

If a particle moves in a straight line and moves through equal distances in equal times, the particle is said to be moving with **uniform velocity**. If the velocity be uniform, then average velocity is the same as the uniform velocity. If, however, the velocity is variable, **it is defined as the rate of change of its displacement.**

Let a particle move along a straight line starting from a given point O on the line and let it come to the position P in time t where $OP=x$. Further suppose in a subsequent interval δt where δt is small, the particle moves through a distance $PQ(=\delta x)$ and comes to the position Q. Thus $\frac{\delta x}{\delta t}$ is the *average velocity* of the particle during the interval δt. As δt and consequently δx becomes smaller and smaller, the point Q approaches the point P and the quotient $\frac{\delta x}{\delta t}$ measures the rate of displacement of the particle. This quotient gives the velocity of the particle in the limit when $\delta t \to 0$. Thus if v be the velocity of the particle at time t, we have

$$v=\lim_{\delta t\to 0} \frac{\text{displacement in time } \delta t}{\delta t}$$

$$=\lim_{\delta t\to 0} \frac{\delta x}{\delta t}$$

$$=\frac{dx}{dt}$$

It is usual to denote differential co-efficients with regard to time by dots ; thus $\dot{x}$ means $\frac{dx}{dt}$ and $\ddot{x}$ means $\frac{d^2x}{dt^2}$.

Thus $$v=\frac{dx}{dt}=\dot{x}.$$

Velocity of a particle has got magnitude as well as direction. The magnitude is called the **speed**. If both speed and direction remain the same throughout a certain interval the velocity is uniform throughout that interval. If either of these changes, the velocity becomes variable.

Acceleration *of a moving particle is defined as the rate of change of velocity.*

If v be the velocity of the particle at time t when it is at the point P and $v+\delta v$ be its velocity at time $t+\delta t$ when it is at Q, then δv is the change of velocity in interval δt. Thus if f be the acceleration of the particle at time t, we have

$$f=\lim_{\delta t\to 0} \frac{\text{change of velocity in time } \delta t}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\delta v}{\delta t}$$

$$=\frac{dv}{dt}.$$

But $\qquad v=\dfrac{dx}{dt} \quad \therefore \quad f=\dfrac{d^2x}{dt^2}=\ddot{x}.$

Also $\qquad f=\lim\dfrac{\delta v}{\delta t} \qquad =\lim\dfrac{\delta v}{\delta x}\cdot\lim\dfrac{\delta x}{\delta t}$

$$=\frac{dv}{dx}\cdot\frac{dx}{dt}=v\frac{dv}{dx}.$$

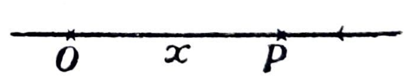
Thus any one of the three expressions $\dfrac{dv}{dt}$ or $\dfrac{d^2x}{dt^2}$ or $v\dfrac{dv}{dx}$ may be taken as the acceleration of the moving particle

Acceleration also has magnitude as well as direction. Negative acceleration is also known as **retardation.** Retardation implies decrease in the magnitude of velocity.

It should be very carefully noted that $\dfrac{dx}{dt}$ or $\dot{x}$ is the velocity of the particle *in the sense in which* x *increases*, similarly, $v\dfrac{dv}{dx}$ or $\ddot{x}$ is the acceleration of the particle *in the sense* x *increasing*.

Ex. 1. *Prove that, if a point moves with a velocity varying as any power not less than unity of its distance from a fixed point which it is approaching, it will take an infinite time to reach that point.*

Let O be the fixed point and P the position of the moving point at time t. If $OP=x$, the velocity is given to μx^n where μ is some constant. Since the point P is approaching O,



we must have $\qquad \dfrac{dx}{dt}=-\mu x^n.$

Therefore, $\qquad \dfrac{dx}{x^n}=-\mu dt.$

Integrating both sides, we get

$$-\frac{1}{(n-1)x^{n-1}}=-\mu t-C$$

where C is some constant, to be determined from the initial conditions of the problem.

Thus $\qquad \dfrac{1}{(n-1)x^{n-1}}=\mu t+C.$

If $n>1$, then when $x \to 0$, $t \to \infty$

i.e., the moving point will take an infinite time before it reaches the point O.

When $n=1$, the equation is $\dfrac{dx}{dt}=-\mu x$

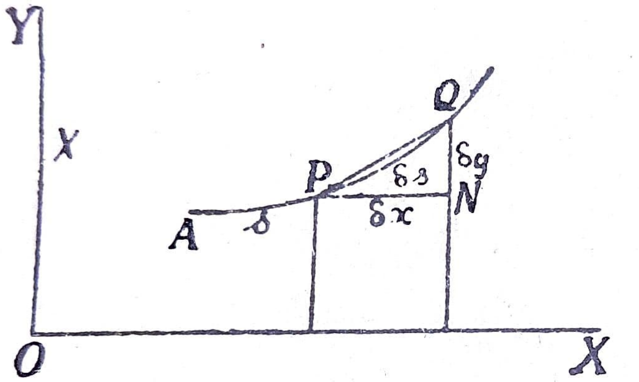
or
$$\frac{dx}{x}=-\mu t$$

Integrating we get $\qquad \log x=-\mu t-C$

From this equation also, we see that when $x \to 0$ (as $\log x \to -\infty$ $x \to 0$) ; $t \to \infty$. Hence the same result.

1·3. Motion in a plane ; Velocity and Acceleration.

When a particle moves in a plane, its path is in general a curve. To fix up the position of the particle on its path at any instant, we must have a fixed point on this curve. Let A be a fixed point on the curve and let P be the position of the particle at time t ; and let arc $AP=s$.



Let Q be the position of the particle at time $t+\delta t$, so that during the interval δt, the displacement of the particle is the chord PQ. Hence the average velocity of the particle during the interval δt

$$=\frac{\text{chord } PQ}{\delta t}.$$

As δt becomes smaller and smaller the point Q approaches the point P and the quotient $\dfrac{\text{chord } PQ}{\delta t}$ becomes the rate of displacement.

Hence the velocity of particle at time t

$$=\lim_{\delta t \to 0} \frac{\text{chord } PQ}{\delta t}$$

$$=\lim_{\delta t \to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta t}$$

$$=\lim_{\delta s \to 0} \frac{\text{chord } PQ}{\delta s} \times \lim_{\delta t \to 0} \frac{\delta s}{\delta t}$$

$$=1 \times \frac{ds}{dt}=\frac{ds}{dt}=\dot{s}$$

Further as the point Q approaches the point P, the chord PQ becomes the tangent to the curve at P. Hence *the direction of the velocity at P is along the tangent to the curve at P and the velocity is $\dot{s}$ in the sense in which s increases.*

Now the velocity of a particle moving in a plane can be resolved into two components with respect to two axes fixed in that plane. Let OX and OY be two rectangular axes fixed in the plane with O as origin. Let the co-ordinates of the points P and Q be (x, y) and $(x+\delta x, y+\delta y)$, so that the displacement PQ has two components $PN=\delta x$, parallel to OX and $QN=\delta y$, parallel to OY.

The component of velocity parallel to OX

$$=\text{rate of displacement parallel to } OX$$

$$=\lim_{\delta t\to 0}\frac{PN}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\delta x}{\delta t}=\frac{dx}{dt}=\dot{x}$$

The component of velocity parallel to OY

$$=\lim_{\delta t\to 0}\frac{QN}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\delta y}{\delta t}=\frac{dy}{dt}=\dot{y}.$$

Thus if x, y be the co-ordinates of a particle moving in a plane, referred to two rectangular axes fixed in that plane, the components of velocity of the particle are $\dot{x}$ and $\dot{y}$. Let these be represented by u and v so that $u=\dot{x}$, $v=\dot{y}$.

If V be the resultant velocity, then

$$V^2=\dot{x}^2+\dot{y}^2$$

and if ψ be the angle which the direction of motion makes with OX, then

$$\tan\psi=\frac{\dot{y}}{\dot{x}}=\frac{dy}{dx}$$

i.e., the direction of motion is the same as the direction of the tangent to the path.

We now find the components of acceleration of the particle parallel to OX and OY. Let (u, v) be the components of velocity of the particle at time t parallel to OX and OY and $(u+\delta u, v+\delta v)$ be those at time $t+\delta t$, so that changes in velocity during the interval δt parallel to OX, OY are δu, δv.

Hence the component of acceleration parallel to OX

$$=\lim_{\delta t\to 0}\frac{\text{change of velocity parallel to } OX \text{ in time } \delta t}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\delta u}{\delta t}=\frac{du}{dt}=\frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{d^2x}{dt^2}=\ddot{x},$$

and the component of acceleration parallel to OY

$$=\lim_{\delta t\to 0}\frac{\delta v}{\delta t}=\frac{dv}{dt}=\frac{d^2y}{dt^2}=\ddot{y}.$$

The resultant acceleration is $\sqrt{\ddot{x}^2+\ddot{y}^2}$ and its direction makes an angle $\tan^{-1}\dfrac{\ddot{y}}{\ddot{x}}$ with OX.

Further we can write

$$\ddot{x}=\frac{d^2x}{dt^2}=\frac{du}{dt}=\frac{du}{dx}\cdot\frac{dx}{dt}=u\frac{du}{dx}$$

and
$$\ddot{y}=\frac{d^2y}{dt^2}=\frac{dv}{dt}=\frac{dv}{dy}\cdot\frac{dy}{dt}=v\frac{dv}{dy}.$$

Thus components of acceleration can also be written as $u\dfrac{du}{dx}$, $v\dfrac{dv}{dy}$ where u, v are the components of velocity.

Ex. 2. *The law of motion in a straight line being $s=\frac{1}{2}vt$, prove that the acceleration is constant.*

$$s=\tfrac{1}{2}vt.$$

Differentiating with respect to t, we get

$$\dot{s}=\tfrac{1}{2}v+\tfrac{1}{2}\dot{v}t, \text{ but } \dot{s}=v,\ \dot{v}=\text{acceleration}=f, \text{ say}$$

then
$$v=\tfrac{1}{2}v+\tfrac{1}{2}ft$$

or
$$\tfrac{1}{2}v=\tfrac{1}{2}ft \quad\text{or}\quad v=ft$$

Differentiating again with respect to t, we get

$$\dot{v}=f+\dot{f}t$$

i.e.,
$$f=f+\dot{f}t$$

or
$$\dot{f}t=0 \quad\therefore\quad \dot{f}=0 \quad\therefore\quad f=\text{constant}.$$

Ex. 3. *If t be regarded as a function of the velocity v, prove that the rate of decrease of acceleration is given by $f^3\dfrac{d^2t}{dv^2}$, f being the acceleration.*

Rate of decrease of acceleration $=-\dfrac{df}{dt}$

Let $$t = F(v)$$

Differentiating with respect to t, we have

$$1 = F'(v)\frac{dv}{dt}$$

$\therefore$ $$f = \text{acceleration} = \frac{dv}{dt} = \frac{1}{F'(v)}.$$

Differentiating again

$$\frac{df}{dt} = -\frac{F''(v)}{\{F'(v)\}^2}\frac{dv}{dt} = -\frac{F''(v)}{\{F'(v)\}^3} = -f^3F''(v)$$

Again $$\frac{dt}{dv} = F'(v)$$

$\therefore$ $$\frac{d^2t}{dv^2} = F''(v)$$

Hence $$\frac{df}{dt} = -f^3\frac{d^2t}{dv^2}$$

i.e. $$-\frac{df}{dt} = f^3\frac{d^2t}{dv^2}.$$

4. Velocities of a moving point parallel to the axes of x and y are $u+ey$ and $v+ex$ respectively; show that the path is a conic section.

Here $$\frac{dx}{dt} = u+ey \text{ and } \frac{dy}{dt} = v+ex.$$

$\therefore$ dividing we get, $\dfrac{dy}{dx} = \dfrac{v+ex}{u+ey}$ or $(u+ey)\,dy = (v+ex)\,dx$.

Integrating, we get $(u+ey)^2 = (v+ex)^2 + C$ where C is some constant.

This represents a conic section.

EXAMPLES I (A)

1. The velocity of a particle moving in a straight line is given by the relation $v^2 = ax^2 + 2bx + c$. Prove that the acceleration varies as the distance from a fixed point in the line.

2. The co-ordinates of a moving point at time t are given by $x = a(2t + \sin 2t)$, $y = a(1 - \cos 2t)$; prove that its acceleration is constant.

3. If the time of a body's descent in a straight line towards a given point vary directly as the square of the distance fallen through, prove that the acceleration is inversely proportional to the cube of the distance fallen through.