9·2. Method of virtual work. The method of virtual work is as follows. In any given problem, we imagine the body to be displaced a little and then find the work done during the displacement. The condition of equilibrium is obtained by equating to zero the toal sum of the works done.

Since the body is not actually displaced, the work done is called *virtual work*. The virtual work that is calculated is the amount of work that would have been done if the displacements had actually been made.

We shall now formally enunciate the principle of virtual work and establish the same. S nce the proof is simpler for the case of a particle acted upon by a number of forces, we consider this first. In § 9·5, the case of rigid body will be dealt with.

9·3. Principle of virtual work for a system of coplanar forces acting on a particle. *The necessary and sufficient condition that a particle acted upon by a number of coplanar forces, be in equilibrium is that the sum of the virtual works done by the forces in a small displacement, consistent with the geometrical conditions of the system, is zero.*

Kanpur, B.A. 1984, *B.Sc.* 1985; *Gorakhpur*, 1983; *Bundelkhand* 1984]

Let any number of forces F_1, F_2, F_3, ... act on the particle whose actual position is given by O (Fig. 83).

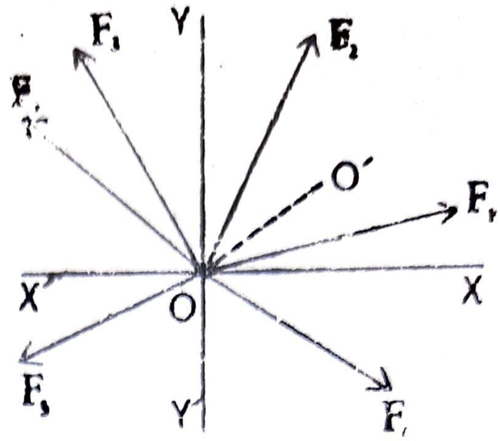

Fig. 83

Through O draw two rectangular axes OX and OY. Let the components of the force F_1 along OX and OY be X_1 and Y_1, of the force F_2, X_2 and Y_2, and so on.

We shall first show the necessity of the condition. In other words, we shall show that if a system of coplanar forces acting on a particle be in equilibrium and the particle undergoes a small displacement consistent with the geometrical conditions of the system, the algebraic sum of the virtual works done by the different forces is zero.

Suppose that OO' represents an arbitrary small displacement and that the coordinates of O' are $(\alpha, \beta,)$. Since, by § 7·2, the work done by a force is equal to the algebraic sum of the works done by its components, the virtual work of the force F is

$$X_1\alpha+Y_1\beta.$$

Similarly the virtual works of the other forces F_2, F_3, etc. are

$$X_2\alpha+Y_2\beta,\ X_3\alpha+Y_3\beta, \text{ etc.}$$

Hence the algebraic sum of the virtual works done by the different forces is

$$(X_1\alpha+Y_1\beta)+(X_2\alpha+Y_2\beta)+(X_3\alpha+Y_3\beta)+\ldots,$$

i.e.

$$\alpha\Sigma X_1+\beta\Sigma Y_1.$$

Now the particle is in equilibrium under the action of the forces F_1, F_2, F_3, ... so that $\Sigma X_1=0$ and $\Sigma Y_1=0$ (see § 2·6).

Therefore $$\alpha\Sigma X_1+\beta\Sigma Y_1=0.$$

Thus the condition that the algebraic sum of the virtual works done by the different forces is zero is necessary.

To establish the sufficiency of the condition we have to show that if the algebraic sum of the virtual works done by a system of coplanar forces acting on a particle be zero for all arbitrary small displacements, the particle is in equilibrium. Since for different displacements we will get different values for α and β, it is given here that for all small values of α and β,

$$\alpha\Sigma X_1+\beta\Sigma Y_1=0. \qquad \ldots \quad (1)$$

Now α and β are independent of each other. Suppose that α' and β are the projections of a new small displacement on the axes of x and y. Then we shall have

$$\alpha'\Sigma X_1+\beta\Sigma Y_1=0. \quad \ldots \quad (2)$$

Subtracting (2) from (1), we get $(\alpha-\alpha')\Sigma X_1=0$.

But $\alpha-\alpha'$ is not zero, therefore $\Sigma X_1=0$.

In the same way by taking a displacement which has for its component displacements α, β' along the coordinate axes, we can show that $\Sigma Y_1=0$. Hence $\Sigma X_1=0$ and $\Sigma Y_1=0$ and the particle is in equilibrium.

The equation (1) is known as *the equation of virtual work*.

9·4. Unwanted forces. Supppose that a force F acts on a particle at the point O in a certain direction and that the displacement OO' is at right angles to the direction of F. Since the projection of OO' in the direction of the force F is zero, the virtual work of the force is zero, and this force will not enter into the equation of virtual work. The principle involved in this statement is important in solving problems on equilibrium. If we want to get rid of a certian unknown force, we imagine a displacement at right angles to the line of action of that force provided this displacement is not inconsistent with the geometrical condition of the system.

As an illustration let us consider the following problem.

EXAMPLE. *A heavy particle O is placed in a given position inside a rough circular tube whose centre is C* (Fig. 84). *To find the horizontal force P necessary to keep the particle from falling.*

Suppose that the force just drags the particle up the tube. The reaction R will pass through C and the frictional force μR will act in the direction OB at right angles to OC. The resultant of the frictional force and the reaction will be a force acting, in the case of limiting equilibrium, at an angle λ to OC.

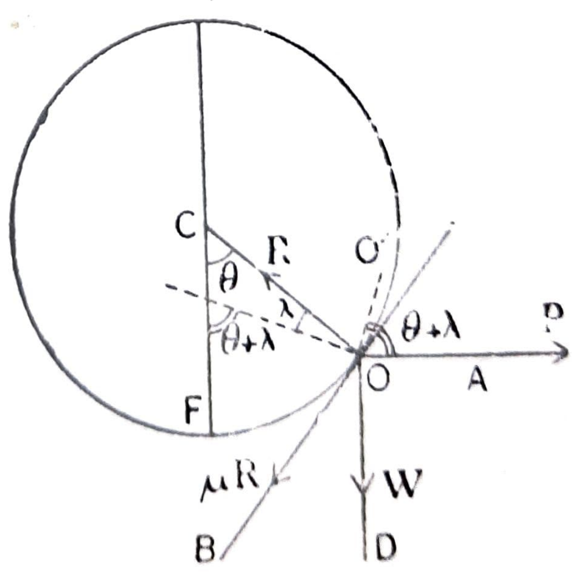

Fig. 84

To apply the principle of virtual work in finding the magnitude of the required force P, we consider a displacement δl at right-angles to the direction of this resultant, so that it may not appear in the equation of virtual work. If δl is geometrically represented by OO', the projections of δl along OA and OD are

$$\delta l \cos(\theta+\lambda) \text{ and } -\lambda l \sin(\theta+\lambda).$$

Hence the equation of virtual work is

$$P\,\delta l \cos(\theta+\lambda) - W\,\delta l \sin(\theta+\lambda) = 0,$$

i. e.

$$\{P\cos(\theta+\lambda) - W\sin(\theta+\lambda)\}\,\delta l = 0.$$

Since $\delta l \neq 0$, we have $P = W\tan(\theta+\lambda)$.

In particular, if the circular tube be smooth. we give a displacement along the tangent at O, since that would eliminate the reaction R. Putting $\lambda = 0$ above, we find that the force required to balance in this case is $P = W\tan\theta$.