Interpolation

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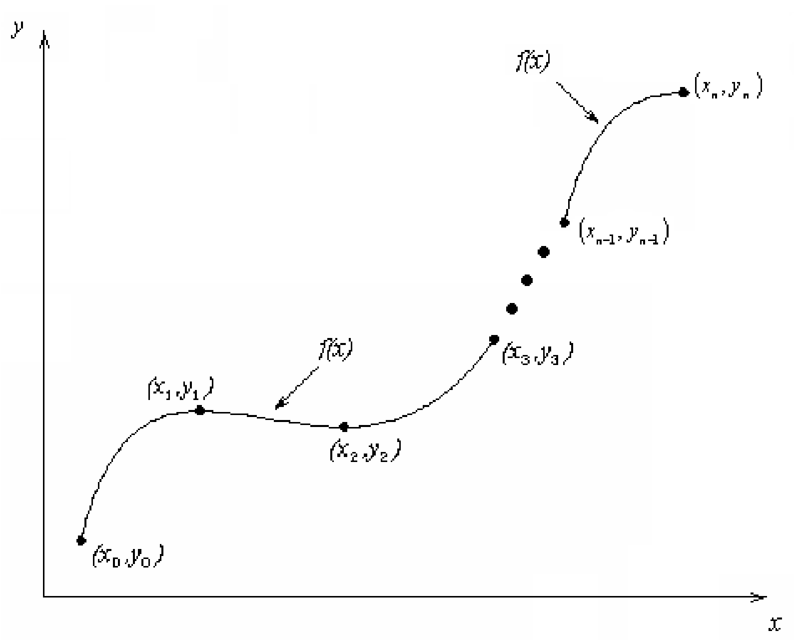
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of ‘y’ at a value of ‘x’ that is not given.



Interpolations

Polynomials are the most common choice of interpolation because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Lagrange's Interpolation

If y=f(x) takes the values y0, y1, y2……yn , corresponding to $x_0, x_1, x_2 \ldots\ldots x_n$ then

$$f(x)=\frac{(x-x_1)(x-x_2).....(x-x_n)}{(x_0-x_1)(x_0-x_2).....(x_0-x_n)}\times y_0+\frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2).....(x_1-x_n)}\times y_1+.........$$

$$\frac{(x-x_0)(x-x_1).....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1).....(x_n-x_{n-1})}\times y_n$$

Proof

Let y=f(x) be a function which takes the data points $(x_0, y_0), (x_1, y_1) ..... (x_n, y_n)$ since there are *n+1* pairs of values of x. we can represent f(x) by a polynomial in x of degree n. Let this polynomial be of the form

$$
\begin{aligned}
y = f(x) &= a_0(x - x_1)(x - x_2).....(x - x_n) + \\
&a_1(x - x_0)(x - x_2).....(x - x_n) + \\
&a_2(x - x_0)(x - x_1)(x - x_3) + ...(x - x_n) + ... + \qquad \ldots\ldots\ldots(1) \\
&a_n(x - x_0)(x - x_1)(x - x_2) + ...(x - x_{n-1})
\end{aligned}
$$

Lagrange's Interpolation (Cont..)

Putting $x=x_0$ and $y=y_0$ in equation (1)

$y_0=a_0(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)$

$$a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2).....(x_0 - x_n)}$$

Similarly putting $x=x_1$ and $y=y_1$ in equation (1), we have

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2).....(x_1 - x_n)}$$

Proceeding the same way, we find a_2, a_3, …..a_n.

Lagrange's Interpolation (Cont..)

Substituting the values of a_0, a_1…..a_n in equation (1), we get

$$f(x) = \frac{(x-x_1)(x-x_2).....(x-x_n)}{(x_0-x_1)(x_0-x_2).....(x_0-x_n)} \times y_0 + \frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2).....(x_1-x_n)} \times y_1 + \text{..........}$$

$$\text{............} \frac{(x-x_0)(x-x_1).....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1).....(x_n-x_{n-1})} \times y_n$$

Lagrange's formula can be applied whether the values x_i are equally spaced or not. It is easy to remember but cumbersome to apply.

Lagrange's Interpolation:Summary

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Example

Given the values

x	f(x)
5	150
7	392
11	1452
13	2366
17	5202

Evaluate f(9), using Lagrange's formula.

Solution

$$f(x) = \sum_{i=0}^{3} L_i(x) f(x_i)$$

x0=5, x1=7, x2=13, x3=17

f(x0)=150, f(x1)=392, f(x2)=1452, f(x3)=5202

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392 +$$

$$\frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$$= 810$$

Practice Problems

1. The following table is given

X	0	1	2	5
f(x)	2	3	12	147

What is the form of f(x)?

2. Apply Lagrange's formula to find f(5) given that f(1)=2, f(2)=4, f(3)=8, f(4)=16, f(7)=128.

Suggested books

1. Numerical Methods by **S.R.K Lyenger & R.K. Jain.**

2. Numerical Analysis by **Richard L. Burden.**

3. Introductory methods of Numerical analysis by **S.S. Sastry**.

Thank you