4.1 A Sufficient Condition for Differentiability

Theorem 2. *If (a, b) be a point of the domain of definition of a function f such that*

(*i*) f_x *is continuous at* (a, b),

(*ii*) f_y *exists at* (a, b),

then f is differentiable at (a, b).

The condition (*i*) implies that f_x exists in a certain neighbourhood $(a - \delta, a + \delta; b - \delta, b + \delta)$ of (a, b). Let $(a + h, b + k)$ be a point of this neighbourhood. Thus

Scanned with CamScanner

$$df = f(a+h, b+k) - f(a, b)$$

$$= f(a+h, b+k) - f(a, b+k) + f(a, b+k) - f(a, b) \quad \text{...(1)}$$

Since f_x exists in $(a-\delta, a+\delta; b-\delta, b+\delta)$, applying Lagrange's mean value theorem, we get

$$f(a+h, b+k) - f(a, b+k) = hf_x(a+\theta h, b+k) \quad \text{...(2)}$$

where $0 < \theta < 1$, and depends on h and k.

Again, since f_x is continuous at (a, b), therefore

$$\lim_{(h,k)\to(0,0)} f_x(a+\theta h, b+k) = f_x(a, b)$$

so that we can write

$$f_x(a+\theta h, b+k) = f_x(a, b) + \phi(h, k) \quad \text{...(3)}$$

where $\phi(h, k) \to 0$ as $(h, k) \to (0, 0)$.

Again, since by condition (*ii*), $f_y(a, b)$ exists, therefore

$$\lim_{k\to 0} \frac{f(a, b+k) - f(a, b)}{k} = f_y(a, b)$$

so that we can write

$$\frac{f(a, b+k) - f(a, b)}{k} = f_y(a, b) + \psi(k) \quad \text{...(4)}$$

where $\psi(k) \to 0$ as $k \to 0$.

$\therefore$ From (1), (2), (3) and (4), we get

$$df = hf_x(a, b) + kf_y(a, b) + h\phi(h, k) + k\psi(k)$$

$\Rightarrow$ f is differentiable at (a, b).

Note: In a similar way it can be shown that f is differentiable at (a, b), if f_x exists and f_y is continuous at (a, b). In fact, one of the partial derivatives is to be continuous and the other merely to exist at the point.

Remark: We have shown that the condition of existence of one partial derivative and the continuity of the other is sufficient to ensure that the function is differentiable but with the help of an example (Example I below) we now show that the condition of continuity is not necessary so that function may be differentiable even though none of the partial derivatives is continuous. However, if the function is not differentiable at a point, the partial derivatives cannot be continuous there at (Example II).

Example I. Consider the function

$$f(x, y) = \begin{cases} x^2 \sin 1/x + y^2 \sin 1/y, & \text{if } xy \neq 0 \\ x^2 \sin 1/x, & \text{if } x \neq 0 \text{ and } y = 0 \\ y^2 \sin 1/y, & \text{if } x = 0 \text{ and } y \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

Scanned with CamScanner

- The partial derivatives,

$$f_x(x, y) = \begin{cases} 2x \sin 1/x - \cos 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$f_y(x, y) = \begin{cases} 2y \sin 1/y - \cos 1/y, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}$$

are discontinuous at the origin, so that both the partial derivatives exist at the origin, but none is continuous there at.

Let us show that the function is differentiable at the origin. Here,

$$f(h, k) - f(0, 0) = h^2 \sin 1/h + k^2 \sin 1/k$$
$$= 0h + 0k + h(h \sin 1/h) + k(k \sin 1/k)$$

Now ($h \sin 1/h$) and ($k \sin 1/k$) both tend to zero when $(h, k) \to (0, 0)$ so that f is differentiable at the origin.

Example II. Prove that the function

$$f(x, y) = \sqrt{|xy|}$$

is not differentiable at the point (0, 0), but that f_x and f_y both exist at the origin and have the value 0. Hence deduce that these two partial derivatives are continuous except at the origin.

- Now at (0, 0),

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

If the function is differentiable at (0, 0) then by definition

$$f(h, k) - f(0, 0) = 0h + 0k + h\phi + k\psi$$

where ϕ and ψ are functions of h and k, and tend to zero as $(h, k) \to (0, 0)$.

Putting $h = \rho \cos\theta$, $k = \rho \sin\theta$ and dividing by ρ, we get

$$|\cos\theta \sin\theta|^{1/2} = \phi \cos\theta + \psi \sin\theta$$

Now for arbitrary θ, $\rho \to 0$ implies that $(h, k) \to (0, 0)$.

Taking the limit as $\rho \to 0$, we get

$$|\cos\theta \sin\theta|^{1/2} = 0,$$

which is impossible for all arbitrary θ.

Hence, the function is not differentiable at (0, 0) and consequently the partial derivatives f_x, f_y cannot be continuous at (0, 0), for otherwise the function would be differentiable there at,

Scanned with CamScanner

Let us now see that it is actually so

For $(x, y) \neq (0, 0)$.

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{|x+h||y|} - \sqrt{|x||y|}}{h}$$

$$= \lim_{h \to 0} \sqrt{|y|} \frac{|x+h| - |x|}{h\left[\sqrt{|x+h|} + \sqrt{|x|}\right]}$$

Now as $h \to 0$, we can take $x + h > 0$, *i.e.*, $|x + h| = x + h$, when $x > 0$ and $x + h < 0$ or $|x + h| = -(x + h)$, when $x < 0$.

$$\therefore \qquad f_x(x, y) = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{|y|}{|x|}}, & \text{when } x > 0 \\ -\dfrac{1}{2}\sqrt{\dfrac{|y|}{|x|}}, & \text{when } x < 0 \end{cases}$$

Similarly,

$$f_y(x, y) = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{|x|}{|y|}}, & \text{when } y > 0 \\ -\dfrac{1}{2}\sqrt{\dfrac{|x|}{|y|}}, & \text{when } y < 0 \end{cases}$$

which are, obviously, not continuous at the origin.

Example 14. Show that the function f, where

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } x^2 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

is differentiable at the origin.

- It may be easily shown that

$$f_x(0, 0) = 0 = f_y(0, 0)$$

Also when $x^2 + y^2 \neq 0$,

$$|f_x| = \frac{|x^4 y + 4x^2 y^3 - y^5|}{(x^2 + y^2)^2} \leq \frac{6(x^2 + y^2)^{5/2}}{(x^2 + y^2)^2} = 6(x^2 + y^2)^{1/2}$$

Scanned with CamScanner

Evidently

$$\lim_{(x,y)\to(0,0)} f_x(x, y) = 0 = f_x(0, 0)$$

Thus f_x is continuous at (0, 0) and $f_y(0, 0)$ exists.

$\Rightarrow$ f is differentiable at (0, 0).

4.2 Algebra of Differentiable Functions

If f and g are two functions differentiable at (a, b), then $f \pm g$, fg are differentiable at (a, b); f/g is differentiable at (a, b), if $g(a, b) \neq 0$, and

$$d(f \pm g) = df \pm dg$$

$$d(fg) = gdf + fdg$$

$$d(f/g) = (gdf + fdg)/g^2.$$

5. PARTIAL DERIVATIVES OF HIGHER ORDER

If a function f has partial derivatives of the first order at each point (x, y) of a certain region, then f_x, f_y are themselves functions of x, y and may also possess partial derivatives. These are called *second order partial derivatives of* f and are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = f_{x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = f_{y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

In a similar manner higher order partial derivatives are defined. For example $\dfrac{\partial^3 f}{\partial x \partial x \partial y} = f_{xxy}$ and so on.

The second order partial derivatives at a particular point (a, b) are often denoted by

$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{(a,b)}, \quad \frac{\partial^2 f(a, b)}{\partial x^2}, \quad f_{xx}(a, b) \text{ or } f_{x^2}(a, b)$$

$$\left[\frac{\partial^2 f}{\delta x \partial y}\right]_{(a,b)}, \quad \frac{\partial^2 f(a, b)}{\partial x \partial y} \text{ or } f_{xy}(a, b)$$

and so on.

Scanned with CamScanner

Thus

$$f_{xx}(a, b) = \lim_{h\to 0} \frac{f_x(a+h, b) - f_x(a, b)}{h}$$

$$f_{xy}(a, b) = \lim_{h\to 0} \frac{f_y(a+h, b) - f_y(a, b)}{h}$$

$$f_{yx}(a, b) = \lim_{k\to 0} \frac{f_x(a, b+k) - f_x(a, b)}{k}$$

$$f_{yy}(a, b) = \lim_{k\to 0} \frac{f_y(a, b+k) - f_y(a, b)}{k}$$

in case the limits exist.

5.1 Change in the Order of Partial Derivation

In most of the cases that occur in practice, a partial derivative has the same value in whatever order the different operations are performed. Thus, for example, it is usually found that

$$f_{xy} = f_{yx}, \quad f_{xyx} = f_{xxy}, \quad f_{xyxy} = f_{xxyy}$$

and one is often tempted to believe that it is always so. But it is not the case and there is no *a priori* reason why they should be equal. Let us now see why f_{xy} may be different from f_{yx} at some point (a, b) of the region.

Now

$$f_{xy}(a, b) = \lim_{h\to 0} \frac{f_y(a+h, b) - f_y(a, b)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\lim_{k\to 0} \frac{f(a+h, b+k) - f(a+h, b)}{k} - \lim_{k\to 0} \frac{f(a, b+k) - f(a, b)}{k}\right]$$

$$= \lim_{h\to 0}\lim_{k\to 0} \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk} = \lim_{h\to 0}\lim_{k\to 0} \frac{\phi(h, k)}{hk}$$

where $\phi(h, k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$.

Similarly,

$$f_{yx}(a, b) = \lim_{k\to 0}\lim_{h\to 0} \frac{\phi(h, k)}{hk}$$

Thus we see that $f_{xy}(a, b)$ and $f_{yx}(a, b)$ are the repeated limits of the same expression taken in different orders. There is therefore no *a priori* reason why they should always be equal.

Let us consider an example to show that f_{xy} may be different from f_{yx}.

Example 15. Let

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}, \ (x, y) \neq (0, 0), \ f(0, 0) = 0, \text{ then}$$

show that at the origin $f_{xy} \neq f_{yx}$.

CHAPTER 15

Scanned with CamScanner

■ Now

$$f_{xy}(0, 0) = \lim_{h\to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h}$$

$$f_y(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0$$

$$f_y(h, 0) = \lim_{k\to 0} \frac{f(h, k) - f(h, 0)}{k} = \lim_{k\to 0} \frac{hk(h^2 - k^2)}{k \cdot (h^2 + k^2)} = h$$

$\therefore$
$$f_{xy} = \lim_{h\to 0} \frac{h - 0}{h} = 1$$

Again

$$f_{yx}(0, 0) = \lim_{k\to 0} \frac{f_x(0, k) - f_x(0, 0)}{k}$$

But

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = 0$$

$$f_x(0, k) = \lim_{h\to 0} \frac{f(h, k) - f(0, k)}{h} = \lim_{h\to 0} \frac{hk(h^2 - k^2)}{h(h^2 + k^2)} = -k$$

$\therefore$
$$f_{yx}(0, 0) = \lim_{k\to 0} \frac{-k - 0}{k} = -1$$

$\therefore$
$$f_{xy}(0, 0) \neq f_{yx}(0, 0).$$

Example 16. Examine the equality of f_{xy} and f_{yx}, where

$$f(x, y) = x^3 y + e^{xy^2}$$

■ Now

$$f_y = x^3 + 2xye^{xy^2}$$

$\therefore$
$$f_{xy} = 3x^2 + 2ye^{xy^2} + 2xy^3 e^{xy^2}$$

Again

$$f_x = 3x^2 y + y^2 e^{xy^2}$$

$$f_{yx} = 3x^2 + 2ye^{xy^2} + 2xy^3 e^{xy^2}$$

$\Rightarrow$
$$f_{xy} = f_{yx}.$$

Scanned with CamScanner