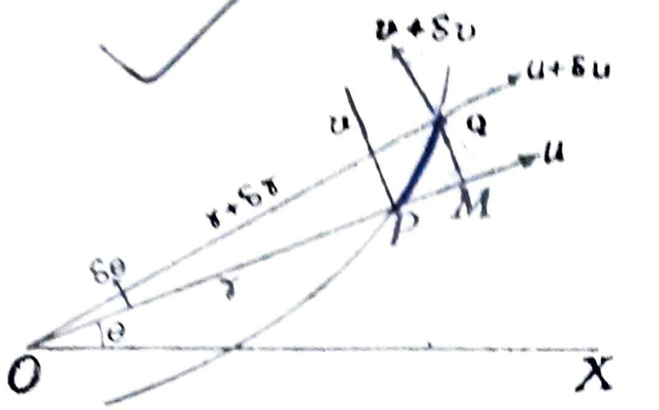
1·4. Radial and Transverse Components of Velocity and Acceleration.

A particle is moving in a plane curve, to find components of velocity and acceleration at time t along and perpendicular to the radius vector drawn from a fixed point in the plane.



Take the fixed point O as the pole and line OX as the initial line. Let P be the position of the particle at time t, its co-ordinates be (r, θ) and Q be the position at time $t+\delta t$, its co-ordinates $(r+\delta r, \theta+\delta\theta)$, so that the chord PQ is the displacement in time δt. Draw QM perpendicular from Q to OP so that PM and QM are the components of the displacement PQ along and perpendicular to OP. Let u, v be the components of velocity along and perpendicular to OP.

Then

radial velocity

$$u = \lim_{\delta t \to 0} \frac{\text{displacement along } OP \text{ in time } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{PM}{\delta t} = \lim_{\delta t \to 0} \frac{OM - OP}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{OQ \cos \delta\theta - OP}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r) \cos \delta\theta - r}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\, 1 - r}{\delta t},$$ small quantities of above the first order being neglected.

$$= \lim_{\delta t \to 0} \frac{\delta r}{\delta t} = \frac{dr}{dt} = \dot{r}$$ radial velocity

transverse.

$$v = \lim_{\delta t \to 0} \frac{\text{displacement perp. to } OP \text{ in time } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{QM}{\delta t} = \lim_{\delta t \to 0} \frac{OQ \sin \delta\theta}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r) \sin \delta\theta}{\delta t} = \lim_{\delta t \to 0} \frac{(r+\delta r) \sin \delta\theta}{\delta\theta} \cdot \frac{\delta\theta}{\delta t}$$

$$= \lim_{\delta t \to 0} (r+\delta r)\frac{\delta\theta}{\delta t} \text{ because } \lim_{\delta\theta\to 0}\frac{\sin\delta\theta}{\delta\theta}=1$$

$$= \lim_{\delta t\to 0}\frac{r\delta\theta}{\delta t}, \text{ neglecting the other term}$$

$$= r\frac{d\theta}{dt}=r\dot{\theta}.$$

transverse velocity

Thus the components of velocity along and perpendicular to the radius vector are $\dot{r}$ and $r\dot{\theta}$, in the senses in which r and θ increase. These are called the radial and transverse or cross-radial components of velocity.

Now let the components of velocity along and perpendicular to OQ be $u+\delta u$, $v+\delta v$; (u, v) being those along and perpendicular to OP.

Thus the change of velocity along OP in time δt

$$= (u+\delta u)\cos\delta\theta-(v+\delta v)\sin\delta\theta-u$$

$$=(u+\delta u)\,.\,1-(v+\delta v)\,.\,\delta\theta-u, \quad \text{neglecting higher powers of } \delta\theta$$

$$=\delta u-v\delta\theta, \quad \text{neglecting the other term.}$$

Similarly the change of velocity perpendicular to OP in time δt

$$=(u+\delta u)\sin\delta\theta+(v+\delta v)\cos\delta\theta-v$$

$$=(u+\delta u)\,\delta\theta+(v+\delta v)\,.\,1-v, \text{ as before}$$

$$=u\delta\theta+\delta v, \quad \text{neglecting the other term.}$$

Therefore, radial acceleration

$$= \lim_{\delta t\to 0}\frac{\text{change of velocity along } OP \text{ in time } \delta t}{\delta t}$$

$$= \lim_{\delta t\to 0}\frac{\delta u - v\delta\theta}{\delta t}$$

$$=\frac{du}{dt}-v\frac{d\theta}{dt}$$

$$=\frac{d}{dt}\left(\frac{dr}{dt}\right)-r\frac{d\theta}{dt}\cdot\frac{d\theta}{dt}$$

get

$$=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=\boxed{\ddot{r}-r\dot{\theta}^2.}$$

Transverse acceleration

$$=\lim_{\delta t\to 0}\frac{\text{change of velocity perp. to } OP \text{ in time } \delta t}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{u\delta\theta+\delta v}{\delta t}$$

$$=u\frac{d\theta}{dt}+\frac{dv}{dt}$$

$$=\frac{dr}{dt}\cdot\frac{d\theta}{dt}+\frac{d}{dt}\left(r\frac{d\theta}{dt}\right)$$

$$=\frac{dr}{dt}\cdot\frac{d\theta}{dt}+\frac{dr}{dt}\cdot\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}$$

$$=r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\cdot\frac{d\theta}{dt}$$

$$=r\ddot{\theta}+2\dot{r}\dot{\theta}$$

also $$=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

$$\boxed{=\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}).}$$

Thus the components of acceleration are $\ddot{r}-r\dot{\theta}^2$ along OP in the sense r increasing and $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$ perp. to OP in the sense θ increasing.

Cor. If the particle describes a circle of radius a, then $r=a$ =constant, so that $u=\dot{r}=0$, $v=r\dot{\theta}=a\dot{\theta}$;

Radial acc. $=\ddot{r}-r\dot{\theta}^2=0-a\dot{\theta}^2=-a\dot{\theta}^2$, *i.e.*, $a\dot{\theta}^2$ towards the centre

and Transverse acc. $=r\ddot{\theta}+2\dot{r}\dot{\theta}=a\ddot{\theta}$, *i.e.*, tangentially.

Alternative method. We have

$$x=r\cos\theta\ .\ y=r\sin\theta$$

$$\therefore\quad \left.\begin{aligned}\dot{x}&=\dot{r}\cos\theta-r\sin\theta\,\dot{\theta}\\ \dot{y}&=\dot{r}\sin\theta+r\cos\theta\,\dot{\theta}\end{aligned}\right\}\quad\ldots(1)$$

and $$\left.\begin{aligned}\ddot{x}&=\ddot{r}\cos\theta-2\dot{r}\dot{\theta}\sin\theta-r\dot{\theta}^2\cos\theta-r\ddot{\theta}\sin\theta\\ \ddot{y}&=\ddot{r}\sin\theta+2\dot{r}\dot{\theta}\cos\theta-r\dot{\theta}^2\sin\theta+r\ddot{\theta}\cos\theta\end{aligned}\right\}\quad\ldots(2)$$

Relation between R & T and T & N

Radial velocity $=\dot{x}\cos\theta+\dot{y}\sin\theta=\dot{r}$,

Transverse velocity $=\dot{y}\cos\theta-\dot{x}\sin\theta=r\dot{\theta}$, from (1)

Radial acceleration $=\ddot{x}\cos\theta+\ddot{y}\sin\theta=\ddot{r}-r\dot{\theta}^2$, from (1)

Transverse acceleration $=\ddot{y}\cos\theta-\ddot{x}\sin\theta=2\dot{r}\dot{\theta}+r\ddot{\theta}$, from (2)

$$=\frac{1}{r}\cdot\frac{d}{dt}(r^2\dot{\theta}).$$ from (

δt

gmp

Ex. 4. *If the radial and transverse velocities of a particle are always proportional to each other, show that the path is an equiangular spiral.*

Here $\quad \dfrac{dr}{dt}=kr\dfrac{d\theta}{dt}$ where k is some constant

or $\quad \dfrac{dr}{r}=k\,d\theta.$

Integrating we get, $\log r=k\theta+C$ where C is some constant

or $\quad r=ae^{k\theta}$ where a is also a constant.

This is an equiangular spiral.

Ex. 5 *The velocities of a particle along and perpendicular to the radius from a fixed origin are λr and $\mu\theta$; find the path and show that the acceleration, along and perpendicular to the radius vector, are*

$$\lambda^2 r-\frac{\mu^2\theta^2}{r} \text{ and } \mu\theta\left(\lambda+\frac{\mu}{r}\right).$$

Here $\quad \dfrac{dr}{dt}=\lambda r$ and $r\dfrac{d\theta}{dt}=\mu\theta.$

Dividing we get, $\dfrac{r\,d\theta}{dr}=\dfrac{\mu\theta}{\lambda r}$ or $\dfrac{\mu}{\lambda}\dfrac{dr}{r^2}=\dfrac{d\theta}{\theta}.$

Integrating we get

$$-\frac{\mu}{\lambda}\cdot\frac{1}{r}=\log\theta+C, \text{ where } C \text{ is a constant.}$$

This gives the path.

Radial acceleration $=\ddot{r}-r\dot{\theta}^2.$

$$=\lambda\dot{r}-r.\frac{\mu^2\theta^2}{r^2}, \quad \because \dot{r}=\lambda r \quad \therefore \ddot{r}=\lambda\dot{r}=\lambda^2 r$$

$$=\lambda^2 r-\frac{\mu^2\theta^2}{r}, \qquad \text{and } \dot{\theta}=\frac{\mu\theta}{r}.$$

Transverse acceleration

$$=2\dot{r}\dot{\theta}+r\ddot{\theta}$$

$$=2\lambda r.\frac{\mu\theta}{r}+r\ddot{\theta}$$

By differentiating $r\dot{\theta}=\mu\theta$

we get $\quad r\ddot{\theta}+\dot{r}\dot{\theta}=\mu\dot{\theta}$

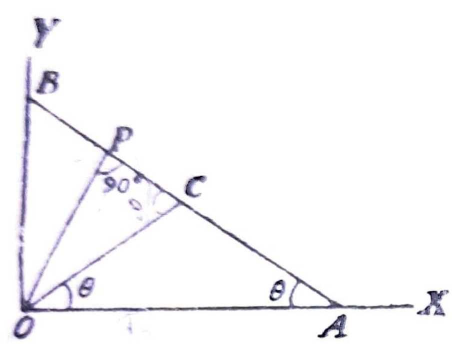
i.e.,
$$r\ddot{\theta}=\mu.\frac{\mu\theta}{r}-\lambda r.\frac{\mu\theta}{r}=\frac{\mu\theta}{r}(\mu-\lambda r).$$

Hence transverse acc. $=2\lambda\mu\theta+\dfrac{\mu\theta}{r}(\mu-\lambda r)$

$$=\mu\theta\left(2\lambda+\frac{\mu}{r}-\lambda\right)=\mu\theta\left(\lambda+\frac{\mu}{r}\right).$$

Ex. 6. *A straight line of constant length moves with its ends on two fixed rectangular axes OX, OY and P is the foot of the perpendicular from O on the straight line. Show that the velocity of P perpendicular to OP is $OP.\dot{\theta}$ and along OP is $2\,CP.\dot{\theta}$ where C is the middle point of the line and θ is the angle COX.*

Since AOB is a right angled triangle, C is the middle point of the hypotenuse, $OC=CA=CB=a$ if $AB=2a$.



$$\angle COA=\theta=\angle CAO$$

$$\angle POX=90-\theta$$

and $PO=OC\cos(90-2\theta)=a\sin 2\theta$

$\therefore$ Polar co-ordinates of P are

$$(a\sin 2\theta,\ 90-\theta)$$

Also $CP=a\cos 2\theta$

$\therefore$ Velocity of P along OP

$$=\frac{d}{dt}(a\sin 2\theta)=2a\cos 2\theta\dot{\theta}$$

$$=2.CP\dot{\theta}.$$

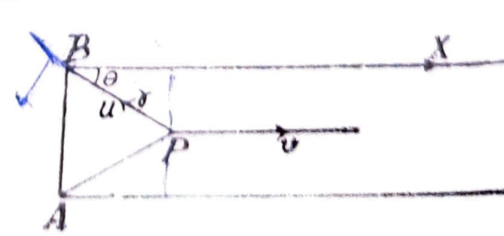
Velocity of P perp. to $OP=a\sin 2\theta\dfrac{d}{dt}(90-\theta)$

$$=-OP.\dot{\theta},$$

i.e., velocity along $PA=OP.\dot{\theta}$.

Ex. 7. *A boat which is rowed with constant velocity U starts from a point A on the bank of a river which flows with a constant velocity V; and it points always towards a point B on the other bank exactly opposite to A; find the equation of the path of the boat. If $V=U$, show that the path is a parabola whose focus is B.*

Let P be the position of the boat at time t. It has two velocities, U towards B and V downstream. Take the opposite bank as initial line and B as pole and P the point (r, θ), its components of velocity are $\dot{r}$ along BP produced and $r\dot{\theta}$ perpendicular to BP.



Hence resolving along and perp. to BP, we get

$$\dot{r}=V\cos\theta-U$$

and $$r\dot{\theta} = -V \sin\theta$$

Dividing we get $$\frac{r\,d\theta}{dr} = -\frac{V \sin\theta}{V\cos\theta - U}$$

or $$\frac{dr}{r} = \left(-\cot\theta + \frac{U}{V}\operatorname{cosec}\theta\right) d\theta.$$

Integrating we get $\log r = \log \operatorname{cosec}\theta + \frac{U}{V} \log \tan \theta/2 + C_1$

where C_1 is constant.

This gives the path.

If $V=U$, it becomes $\log r = \log \operatorname{cosec}\theta + \log \tan \theta/2 + C_1$

or $$\log r = \log\left(\frac{1}{2\cos^2\theta/2}\right) + C_1$$

or $$2r\cos^2\theta/2 = A \text{ where } A \text{ is also a constant}$$

i.e., $\frac{A}{r} = 2\cos^2\theta/2 = 1+\cos\theta$ which is a parabola, with B as focus.

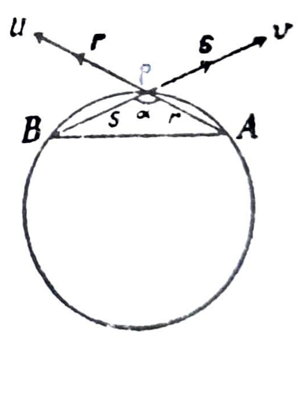
Ex. 8. *A and B are two fixed points on the circumference of a circle and the distances from A and B of any other point P on the circumference are r and s respectively. If u and v are the components of P's velocity, as it moves round the circumference, along AP and BP respectively prove that α being the angle APB*

$$u \sin^2\alpha = \dot{r} - \dot{s}\cos\alpha,$$

$v^2 = u^2 \sin^2\theta + 2gh$

$$v \sin^2\alpha = \dot{s} - \dot{r}\cos\alpha.$$

and deduce that $\quad ur + vs = 0$



Resolving along AP,

$$\dot{r} = u - v\cos(180-\alpha) = u + v\cos\alpha$$

and resolving along BP

$$\dot{s} = v - u\cos(180-\alpha) = v + u\cos\alpha$$

$\therefore$ $$\dot{r} - \dot{s}\cos\alpha = u\sin^2\alpha$$

and $$\dot{s} - \dot{r}\cos\alpha = v\sin^2\alpha.$$

Also $$(ur+vs)\sin^2\alpha = r\dot{r} - r\dot{s}\cos\alpha + s\dot{s} - s\dot{r}\cos\alpha$$

$$= \tfrac{1}{2}\frac{d}{dt}(r^2+s^2) - \cos\alpha\,\frac{d}{dt}(rs)$$

$$= \tfrac{1}{2}\frac{d}{dt}(r^2+s^2-2rs\cos\alpha)$$

$$= \tfrac{1}{2}\frac{d}{dt}AB = 0, \text{ hence}$$