B.Sc. Mathematics – 2nd Semester

MTB 202 – Statics and Dynamics

by

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Part – I

Forces in Two-Dimension

<u>Equilibrium of forces acting at a point</u>

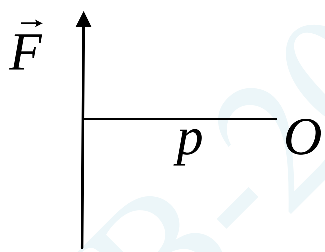
Analytical conditions: If a member of forces act at point then, they are in equilibrium if and only if algebraic sums of their resolved parts along two perpendicular directions through the point are zero.

$$R = \sqrt{R_x^2 + R_y^2},\ R_x = 0,\ R_y = 0.$$

Geometrical conditions: If any number of forces acting at a point can be represented in magnitude and direction by the sides of a polygon, taken in order, then the forces will be in equilibrium.

<u>Moment of a force about a point</u>

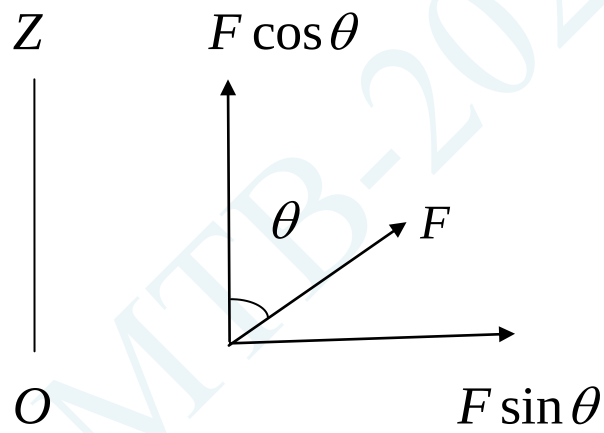
The moment of a force about a point is defined as the product of the force and the perpendicular distance of the point from the line of action of the force.



If $\vec{F}$ be the force and O be the point, then $F.p$ is the moment of the force about O, where p is the perpendicular distance of O from $\vec{F}$.$[|\vec{F}|=F]$

Moment of force about a line

Let $\vec{F}$ be any force and OZ be any line (in space). Let θ be the angle between them. Let us resolve $\vec{F}$ into $F\cos\theta$ parallel to OZ and $F\sin\theta$ perpendicular to OZ. $[|\vec{F}|=F]$



Let p be the shortest distance between the line of action of $\vec{F}$ and OZ. Then $F\sin\theta.p$ is called the moment of F about OZ.

"The moment of a force about a line is the product of the resolve part of the force perpendicular to the line and the shortest distance between the line of action of the force and the line".

It is evident that if the line of action of the force intersects the line about which the moment is considered, then the shortest distance between them is zero. Hence the moment of the force about the line is equal to zero $\left[F\sin\theta.0=0\right]$.

On the other hand, if the force is parallel to the line then $\theta = 0$ and hence the moment of the force about the line is equal to zero $\left[F \sin 0 . p = 0\right]$.

Theorem: If two forces act at a point, the algebraic sum of their moments about any line in space is equal to the moment of their resultant about the same line.

Note: This theorem holds for any number of concurrent forces.

Couples

Two equal unlike parallel forces whose lines of action are not the same, form a Couple.

A Couple is called a torque by some writers; by others, the word torque is used to denote the moment of a Couple.

Examples

(i) The forces by the hands to the handle of a car while driving,

(ii) The forces by the fingers to the key during locking,

etc.

The arm of a couple is the perpendicular distance between the lines of action of the two forces which form the couple.

The moment of a couple is the product of one of the forces forming the couple and the arm of the couple.

If the force and the arm of a couple are P and p respectively, then the couple is denoted by (P, p) and its moment is $P.p$.

Couples acting in the same plane and having equal moments are equivalent. Consequently, couples of equal but opposite moments in the same plane are balance each other.

Resultant of Couples (Coplaner)

Let us consider the couples (P, p), (Q, q), (R, r), $(S, s), \ldots$, where $P, Q, R, S, \ldots$ are the forces of the couples and $p, q, r, s, \ldots$ are their arms. [As the couple of equal moments in a plane are equivalent]. Now, replace the couple (Q, q) by a couple whose components have the same lines of action as those of (P, p). If X be the magnitude of the force of this couple, then we must have

$$X.p = Q.q$$

So that, $X = \dfrac{Qq}{p}$.

Similarly, replace the couples (R,r), (S,s),… by couples whose forces act in the same line as that of (P,p). We are then left with a single couple the magnitude of whose force is

$P+Q.\frac{q}{p}+R.\frac{r}{p}+S.\frac{s}{p}+...$ and the moment of the couple is

$$\left(P+Q.\frac{q}{p}+R.\frac{r}{p}+S.\frac{s}{p}+...\right)p=P.p+Q.q+R.r+S.s+...$$

<u>Theorem:</u> The resultant of any number of couples acting in the same plane on a rigid body is a couple whose moment is equal to the algebraic sum of the moments of the couples.

To give a couple in a plane it is not necessary to give the positions or magnitude of the forces of the couple but only the moment of the couple should be given.

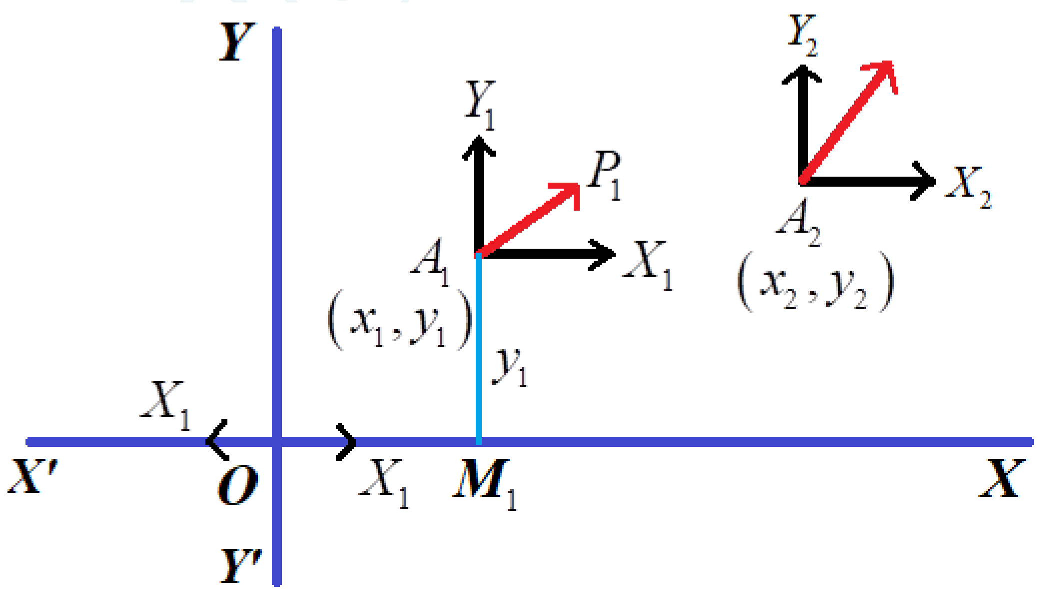
Again, couples acting in parallel planes and have equal moments are equivalent. So, it is not even necessary to state the plane of the couples it is enough to give the direction of the normal to its plane.

Thus, a couple is completely determined by the direction of the normal to its plane of action and the moment with its sense.

Reduction of Coplanar Forces

Theorem: A system of forces acting in one plane at different points of a rigid body can be reduce to a single force through a given point and couple.

Proof: Let the forces $P_1, P_2, P_3, \ldots$ acts at points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ of the body, the coordinates of the points being given



with reference to rectangular axes OX and OY through a given point O.

Consider first the force P_1 acting at $A_1(x_1, y_1)$. Let it be resolved into two forces X_1 and Y_1 parallel to the coordinate axes. At O introduce two equal and opposite forces X_1, one along OX and other along $O\boldsymbol{X'}$.

This will have no effect on the body. Now the forces X_1 at A_1 and X_1 at O along OX' from a couple of moment $-X_1y_1$ and we are left with the force X_1 along OX.

Thus the force X_1 at A_1 is equivalent to a force X_1 at O along OX and a couple of moment $-X_1y_1$.

Similarly by introducing at O equal forces Y_1 along OY and OY', it is easy to see that the force Y_1 at A_1 is equivalent to a force Y_1 at O along OY and a couple of moment Y_1x_1.

It follows, therefore, that the force P_1 at A_1 is equivalent to force X_1 and Y_1 at O along OX and OY respectively and a couple of moment $Y_1x_1 - X_1y_1$.

Proceeding in the same way with the remaining forces we see that the given system of forces is equivalent to

$$R_x = X_1 + X_2 + X_3 + --- = \Sigma X_i \quad \text{along } OX$$

and $$R_y = Y_1 + Y_2 + Y_3 + --- = \Sigma Y_i \quad \text{along } OY,$$

and a couple of moment

$$G = \left(Y_1 x_1 - X_1 y_1\right) + \left(Y_2 x_2 - X_2 y_2\right) + --- = \Sigma\left(Y_i x_i - X_i y_i\right) \qquad (1)$$

The forces R_x and R_y can be compounded into a single force through O of magnitude R given by

$$R^2 = R_x{}^2 + R_y{}^2 \qquad (2)$$

acting at an angle $\theta = \tan^{-1}\left({R_y}/{R_x} \right)$ with the X-axis.

Thus the system of forces can be reduced to a single force R through O and a couple of moment G.

It is evident that G depends upon the position O of the given point while R does not.

<u>Theorem</u>: A system of forces acting in one plane at different points of a rigid body can be reduced to a single force, or a couple.

<u>Proof</u>: We have already seen that a system of coplanar forces acting in a body in general can be reduced to a single force R and a couple of moment G, given by equation (2) and (1).

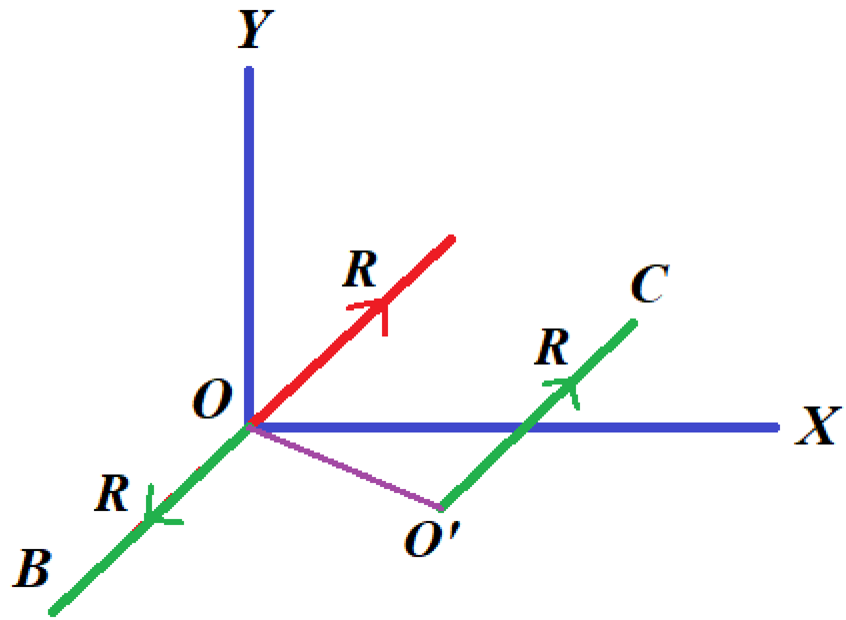
If $R = 0$. Then the forces reduce to a couple and the theorem is proved. But if $R \neq 0$, we shall show that the force R and the couple of moment G can be reduced to a single force R acting in the same direction but in a different line.

Replace the couple G by two equal and opposite forces of magnitude R, one along OB in the direction opposite to R and the other along $O'C$, where OO' is the perpendicular to OB i.e., the arm of the couple.

Then $OO' = \dfrac{G}{R}$

i.e. $OO' = \dfrac{\Sigma\left(Y_1x_1 - X_1y_1\right)}{\sqrt{R_x^{\ 2} + R_y^{\ 2}}}$

The forces at O balance each other and we are left with the force R acting at O' along $O'C$.



Equation of the resultant

Now we obtain the equation of the final resultant of a system of coplanar forces.

Let $P_1, P_2, P_3, ...$be the forces acting at points $(x_1, y_1), (x_2, y_2), (x_3, y_3),$ …… of a body.

Let G be the moment of all the forces about the origin.

So $G = \Sigma(Y_1 x_1 - X_1 y_1)$ [similar as previous theorem]

If the coordinates of a point Q be given by (ξ, η) the moment of force P_1 about Q is

$$Y_1(x_1-\xi)-X_1(y_1-\eta)$$

i.e., $$(Y_1x_1-X_1y_1)-Y_1\xi+X_1\eta$$

Writing the moment of the other forces similarly, we see that the total moment G' of all the forces about the point Q is given by

$$G'=G-\xi\Sigma Y_1+\eta\Sigma X_1=G-\xi R_y+\eta R_x.$$

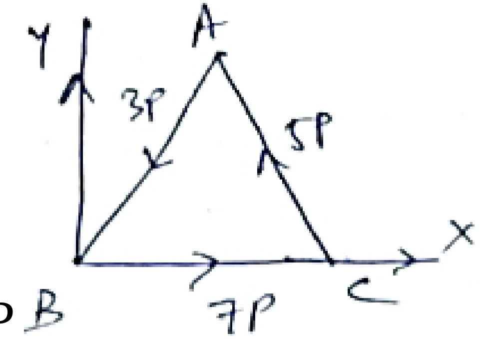
If the point Q lies on the resultant, then we have $G'=0$.

Hence, $G-\xi R_y+\eta R_x=0$.

Replacing ξ,η by x, y we can see that the equation line of action is same as found before.

<u>Example</u>: Forces equal to 3*P*, 7*P* and 5*P* act along the sides *AB*, *BC* and *CA* of an equilateral triangle *ABC*, find the magnitude and direction of the resultant.

<u>Solution</u>: Let *O* be the centroid of the equilateral. Let us consider the rectangular axes at *B*, *X*-axis along *BC* and *Y*-axis perpendicular to *BC*.



Then $R_x = 7P\cos 0° + 5P\cos 120° + 3P\cos 240°$

$$= 7P - \frac{5}{2}P - \frac{3}{2}P = 3P$$

$$R_y = 0 + 5P\sin 120° + 3P\sin 240° = \frac{5\sqrt{3}}{2}P - \frac{3\sqrt{3}}{2}P = \sqrt{3}P$$

Therefore the magnitude of the resultant is

$$R = \sqrt{(3P)^2 + \left(\sqrt{3}P\right)^2} = \sqrt{12}P$$

and it makes an angle $\tan^{-1}\left(\dfrac{\sqrt{3}P}{3P}\right) = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = 30^\circ$ with BC.

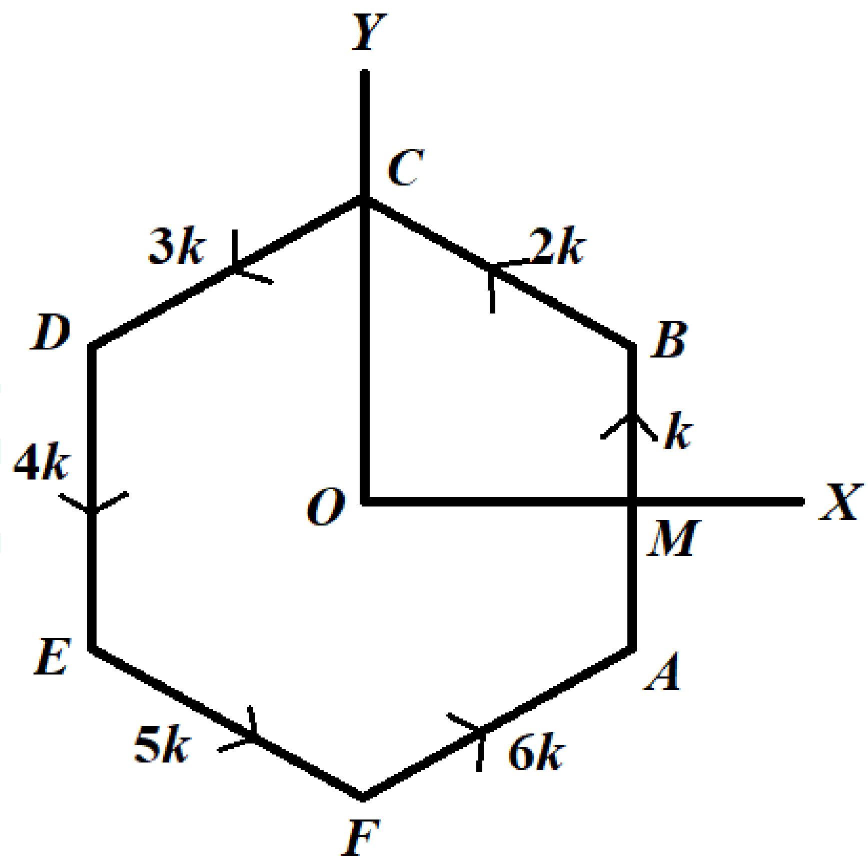
Example: If six forces of relative magnitude 1, 2, 3, 4, 5 and 6 act along the sides of a regular hexagon, taken in order, show that the single equivalent force is F relative magnitude 6, and that it acts along a line parallel to the force 5 at a distance from the centre of the hexagon $\frac{7}{2}$ times the distance of the sides from the centre.

Solution: Let $ABCDEF$ be a regular hexagon of side $2a$. Suppose that forces of magnitude k, $2k$, $3k$, $4k$, $5k$ and $6k$ act along the sides AB, BC, CD, DE, EF and FA respectively.

Let O be the centre of the hexagon. Choose through O a set of rectangular axes OMX and OCY.

It is easy to see that the six forces are inclined to OMX at the angles $\frac{1}{2}\pi$, $\frac{5}{6}\pi$, $\frac{7}{6}\pi$, $\frac{9}{6}\pi$, $\frac{11}{6}\pi$ and $\frac{13}{6}\pi$.

Hence the sums of resolved parts along the axes are



$$R_x = k\left(\cos\frac{\pi}{2} + 2\cos\frac{5}{6}\pi + 3\cos\frac{7}{6}\pi + 4\cos\frac{9}{6}\pi + 5\cos\frac{11}{6}\pi + 6\cos\frac{13}{6}\pi\right) = 3\sqrt{3}k$$

$$R_y = k\left(\sin\frac{\pi}{2} + 2\sin\frac{5}{6}\pi + 3\sin\frac{7}{6}\pi + 4\sin\frac{9}{6}\pi + 5\sin\frac{11}{6}\pi + 6\sin\frac{13}{6}\pi\right) = -3k$$

Therefore, the magnitude of the resultant

$$= \sqrt{\left(3\sqrt{3}k\right)^2 + (-3k)^2} = 6k$$

and it makes an angle $\tan^{-1}\left(\frac{-3k}{3\sqrt{3}k}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \frac{11}{6}\pi$ with OMX.

It is therefore, parallel to the forces $5k$ acting along EF. Further, since all the sides of the hexagon are at a distance OM from O, the algebraic sum of the moments of forces about O

$$=(k+2k+3k+4k+5k+6k)OM=21k.OM$$

Hence the distance of the resultant from the centre

$$=21k.OM/6k=\frac{7}{2}OM\ .$$

Example: Three forces P, Q, R act along the sides of the triangle formed by the lines: $x+y=1,\ y-x=1,\ y=2$, taken in order. Find the equation of the line of action of their resultant.

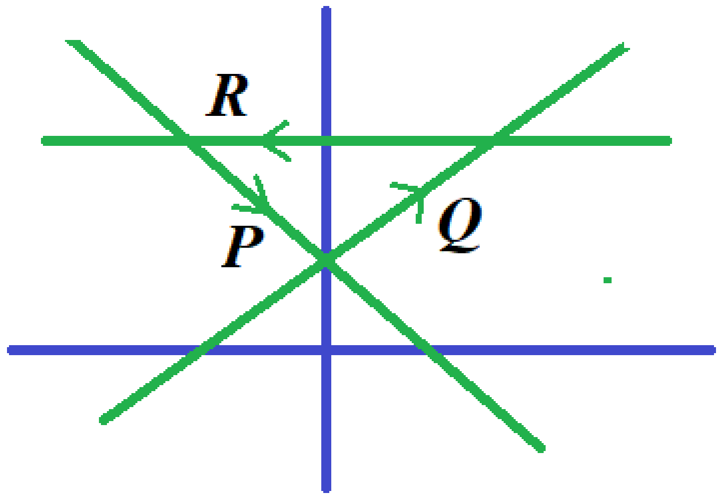
Solution: Let the equation of line of action of the resultant force is

$$xR_y - yR_x = G$$

We have $R_x = P\cos(-45°) + Q\cos 45° - R\cos 0° = \left(P + Q - \sqrt{2}R\right)\dfrac{1}{\sqrt{2}}$

$$R_y = P\sin(-45°) + Q\sin 45° - R\sin 0° = (-P + Q)\frac{1}{\sqrt{2}}$$

$$G = -\frac{P}{\sqrt{2}} - \frac{Q}{\sqrt{2}} + 2R$$
$$= \left(-P - Q + 2\sqrt{2}R\right) \frac{1}{\sqrt{2}}$$



Therefore the required equation of the resultant is

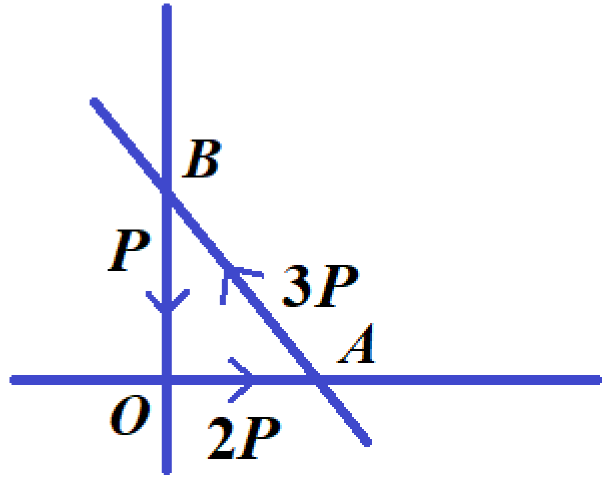
$$x(-P+Q)/\sqrt{2} - y\left(P+Q-\sqrt{2}R\right)/\sqrt{2} = \left(-P-Q+2\sqrt{2}R\right)/\sqrt{2}$$

$$\Rightarrow x(-P+Q) - y\left(P+Q-\sqrt{2}R\right) = \left(-P-Q+2\sqrt{2}R\right)$$

$$\Rightarrow (x+y-1)P + (y-x-1)Q - (y-2)\sqrt{2}R = 0$$

Example: Forces P, $2P$, $3P$ act along the sides of a triangle formed by the lines $x=0,\ y=0$ and $3x+4y=5$.

Solution:



$$R_x = 2P - 3P\left(\frac{4}{5}\right) = -\frac{2}{5}P$$

$$R_y = -P + 3P\frac{3}{5} = \frac{4}{5}P$$

$$G = 3P.1 = 3P$$

$$R = \frac{2}{5}\sqrt{5}P, \quad xR_y - yR_x = G \Rightarrow 4x + 2y = 15$$

Example: Forces proportional to 1, 2, 3 and 4 act along the sides *AB*, *BC*, *AD* and *DC* respectively of a square *ABCD* the length of whose sides is 2 ft. Find the magnitude and line of action of their resultant.

Example: Forces of relative magnitude 5, 1, 1, -3 act along the side *AB*, *BC*, *CD*, *DA* respectively of a square *ABCD*. Show that the system is equivalent to a single force and illustrate in a diagram its magnitude, direction and line of action.

Example: *ABCDEF* is a regular hexagon and *O* is its centre. Forces of magnitude 1, 2, 3, 4, 5, 6 acts in the lines *AB*, *CB*, *CD*, *ED*, *EF*, and *AF* in the sense indicated by the order of the letters. Reduce the system to a force at O and a couple, and find the point in AB through which the single resultant passes.

General conditions of equilibrium:

(1) A system of forces in a plane is in equilibrium if the algebraic sums of the resolved parts in any two perpendicular directions vanish and if the algebraic sum of the moments about any point also vanishes.

$$\text{i.e., } R_x = 0,\ R_y = 0,\quad G = 0.$$

(2) A system of forces in a plane is in equilibrium if the algebraic sums of the moments of all the forces with respect to each of three non-collinear points is zero.

Without loss of generality, let the three points be $0, (\xi, \eta), (\xi', \eta')$.

$$G = 0$$

$$G' = G - \xi R_y + \eta R_x = 0$$

$$G'' = G - \xi' R_y + \eta' R_x = 0$$

Since (ξ, η) and (ξ', η') are non-collinear, $R_x = R_y = 0$.

Example: Two systems of forces *P*, *Q*, *R* and P', Q', R' act along the sides *BC*, *CA*, *AB* of a triangle *ABC*, prove that their resultants will be parallel if $(QR' - Q'R)\sin A + (RP' - R'P)\sin B + (PQ' - P'Q)\sin C = 0$.

Solution: For the first system the sum resolved parts of the forces along the axes over

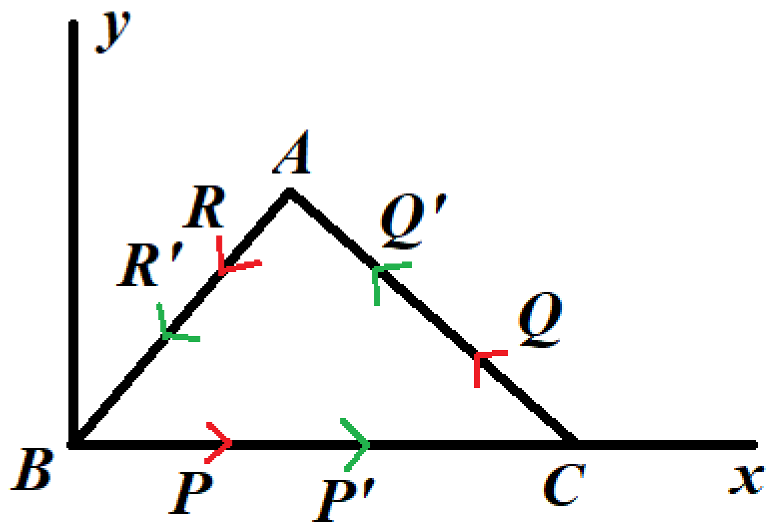
$$X = P - Q\cos C - R\cos B \text{ and } Y = Q\sin C - R\sin B,$$

where *x*-axis is taken along *BC* and *y*-axis is taken perpendicular to *BC* with *B* as origin.

Similarly, for the end system

$X' = P' - Q'\cos C - R'\,\mathrm{Cos}\,B$ and

$Y' = Q'\sin C - R'\sin B$



The resultants are to be parallel if

$$\frac{Y}{X} = \frac{Y'}{X'} \Rightarrow YX' = XY'$$

i.e., $\left(Q\sin C - R\sin B\right)\left(P' - Q'\cos C - R'\cos B\right)$

$$= \left(P - Q\cos C - R\cos B\right)\left(Q'\sin C - R'\sin B\right)$$

$$\Rightarrow P'Q\sin C - QQ'\cos C\sin C - QR'\cos B\sin C - P'R\sin B$$
$$-Q'R\cos C\sin B + RR'\cos B\sin B$$

$$= PQ'\sin C - PR'\sin B - QQ'\cos C\sin C + QR'\cos C\sin B$$
$$-Q'R\cos B\sin C + RR'\cos B\sin B$$

$$\Rightarrow \left(PQ' - P'Q\right)\sin C + \left(RP' - R'P\right)\sin B + QR'\left(\sin B\cos C + \cos B\sin C\right)$$
$$-Q'R\left(\sin B\cos C + \cos B\sin C\right) = 0$$

$$\Rightarrow \left(PQ' - P'Q\right)\sin C + \left(RP' - R'P\right)\sin B + \left(QR' - Q'R\right)\sin\left(B + C\right) = 0$$

$$\Rightarrow \left(PQ' - P'Q\right)\sin C + \left(RP' - R'P\right)\sin B + \left(QR' - Q'R\right)\sin A = 0$$

Hence the results.

Example: The moments of a system of coplanar forces about three collinear points *A*, *B*, *C* in the plane are G_1, G_2, G_3. Prove that $G_1BC + G_2CA + G_3AB = 0$.

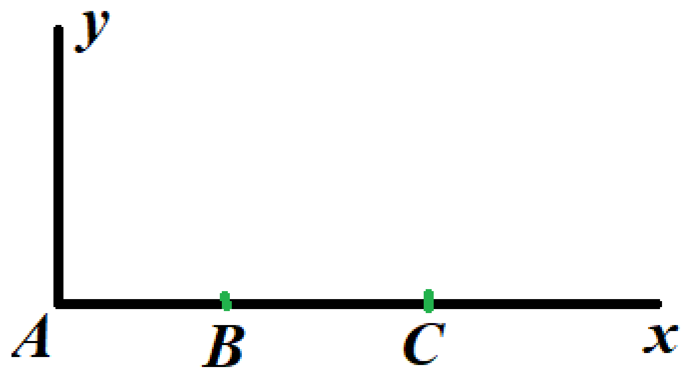
Solution: Without loss of generality, we can consider the point *A* be the origin and the line *ABC* be the *x*-axis. Also let the system of forces be reduced to a force with components R_x and R_y along the axes and a couple of moment *G*. Let the coordinates of *B* and *C* be $(b,0)$ and $(c,0)$.

Hence, we have

$G_1 = G$ for point A

$G_2 = G - bR_y$ for point B

$G_3 = G - cR_y$ for point C



Now

$G_1BC + G_2CA + G_3AB$

$= G(c-b) + (G - bR_y)(-c) + (G - cR_y)b$ [as $BC = c-b,\ CA = -c,\ AB = b$]

$= Gc - Gb - Gc + bcR_y + Gb - bcR_y = 0.$ Hence the result is proved.

Example: A system of forces in the plane of a triangle *ABC* is equivalent to a single force at *A*, acting long the internal bisector of the angle *BAC*, and a couple of moment G_1. If the moments of forces about *B* and *C* are G_2 and G_3 respectively, then show that $(b+c)G_1 = bG_2 + cG_3$, where *b*=*AC* and *c*=*AB*

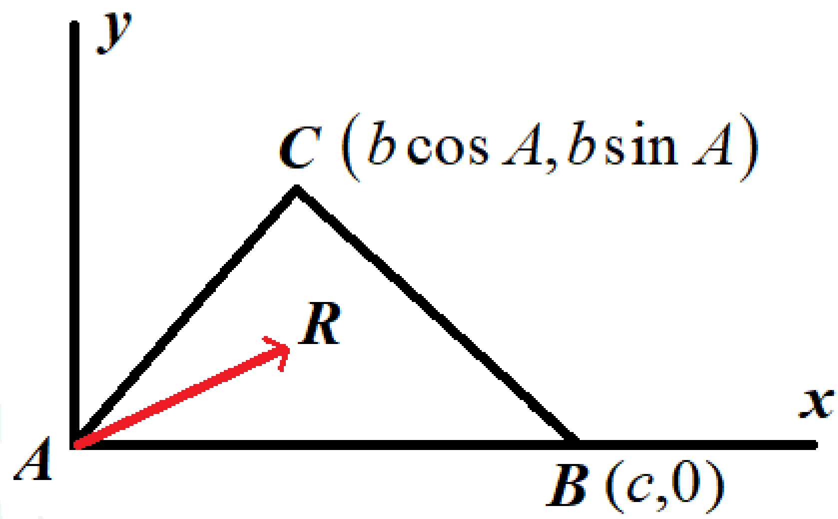
Solution: Without the loss of generality, we can take *A* as the origin, *AB* as the *x* axis and *y* axis is perpendicular to *AB* as shown in the figure.

It is given that the single resultant, say, *R* is along the internal bisector of angle *A*.

$$R_x = R\cos\frac{A}{2} \quad \text{and} \quad R_y = R\sin\frac{A}{2}$$

Also, we have $AB=c$, $CA=b$.

The coordinate of B is $(c,0)$ and that of C is $(b\cos A, b\sin A)$



Now according to the given condition we have

$$G_2 = G_1 - cR\sin\frac{A}{2}$$

$$G_3 = G_1 - (b\cos A)R\sin\frac{A}{2} + (b\sin A)R\cos\frac{A}{2}$$

Now $bG_2 + cG_3 = (b+c)G_1 - bcR\sin\frac{A}{2} + Rbc\left[\sin A\cos\frac{A}{2} - \cos A\sin\frac{A}{2}\right]$

i.e., $bG_2 + cG_3 = (b+c)G_1 - bcR\sin\frac{A}{2} + bcR\sin\left(A - \frac{A}{2}\right)$

$\Rightarrow bG_2 + cG_3 = (b+c)G_1$ (Proved)