Let $Ax = b$ be a system of linear equation, where A is an $m \times n$ matrix, x is a $n \times 1$ matrix and b is a $m \times 1$ matrix. Let A has a right inverse B, ~~then~~ so that $AB = I_m$, then $x = Bb$ is a solution of the system $Ax = b$.

For example:

$$Ax = ABb = I_m b = b$$

In particular, if A is invertible square matrix then it has only inverse A^{-1}, and $x = A^{-1}b$ is the only solution of the system.

Defn: An elementary matrix is a matrix obtained from the identity matrix I_n by executing only one elementary row operation.

Example:

(1) $\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Interestingly, If E is an elementary matrix obtained by executing a certain elementary row operation on the identity matrix I_m, then for any $m \times n$ matrix, A the product EA is exactly the matrix that is obtained when the same elementary row operation in E is executed on A.

Illustration:

Let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ be a 3×1 column matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to -2R_1 + R_2} E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Eb = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix}$$

Example:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, Each elementary matrix is invertible.

Theorem: For a square matrix A of order 'n', the following statements are equivalent:

1. A is invertible.
2. A has rank n.
3. A is row-equivalent to identity matrix.
4. A is product of elementary matrix.

Theorem: The following statements are equivalent for a square matrix A of order 'n'.

1. A is invertible.
2. $Ax = 0$ has only trivial solution
3. $Ax = b$ has a solution x for every b.

Method of finding A^{-1}

Let $Ax = b$ be a system of linear equations.

Here, A is a square matrix.

Further, assume that A is invertible.

Since: by previous thm A is now equivalent to the identity matrix.

So, there are k elementary matrices, say $E_k, E_{k-1}, \ldots, E_2, E_1$ such that

$$E_k E_{k-1} \cdots E_2 E_1 A = I.$$

We also know that $x = A^{-1}b$ is the unique solution of the system $Ax = b$.

So,

$$Ax = b$$

$$Ax = Ib$$

$$\underbrace{E_k E_{k-1} \cdots E_2 E_1 A}_{I} x = \underbrace{E_k \cdots E_1 I}_{A^{-1}} b$$

$$x = A^{-1}b$$

This process is called Gauss-Jordan method to compute inverse of a matrix.

Example: Compute A^{-1} by Gauss-Jordan method.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -3 & 2 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 4 & -1 \\ 0 & -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \xrightarrow{R_2 \to (-1)R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 4 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right].$$

So, $A^{-1} = \begin{bmatrix} 6 & 4 & -1 \\ -1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}$

Example: Write the system linear equations

$$x + 2y + 2z = 10$$
$$2x - 2y + 3z = 1$$
$$4x - 3y + 5z = 4$$

in matrix form $Ax = b$ and solve it by finding A^{-1}.

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 3 \\ 4 & -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 4 \end{bmatrix}.$$

A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & -2 & 3 & 0 & 1 & 0 \\ 4 & -3 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_4 \to R_4 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -6 & -1 & -2 & 1 & 0 \\ 0 & -11 & -3 & -4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + \frac{-11}{6} R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -6 & -1 & -2 & 1 & 0 \\ 0 & 0 & -7/6 & -2/6 & -11/6 & 1 \end{array}\right] \xrightarrow{R_2 \to -1/6 R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1/6 & 2/6 & -1/6 & 0 \\ 0 & 0 & -7/6 & -2/6 & -1/6 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 5/3 & 2/6 & 2/6 & 0 \\ 0 & 1 & 1/6 & 2/6 & -1/6 & 0 \\ 0 & 0 & -7/6 & -2/6 & -1/6 & 1 \end{array}\right]$$

~~$R_2 \to R_2 + 7$~~

$$R_3 \longrightarrow \frac{-6}{7} R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 5/3 & 2/6 & 2/6 & 0 \\ 0 & 1 & 1/6 & 2/6 & -1/6 & 0 \\ 0 & 0 & 1 & 2/7 & 11/7 & -6/7 \end{array}\right]$$

$R_2 \longrightarrow R_2 - 1/6 R_3$

$R_1 \longrightarrow R_1 - 5/3 R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right] \quad x = A^{-1} b$$

Example: compute A^{-1}

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 4 \\ 2 & 4 & 6 \end{bmatrix}$$