Numerical Computing

Root Finding Methods

1. *Bisection* Method

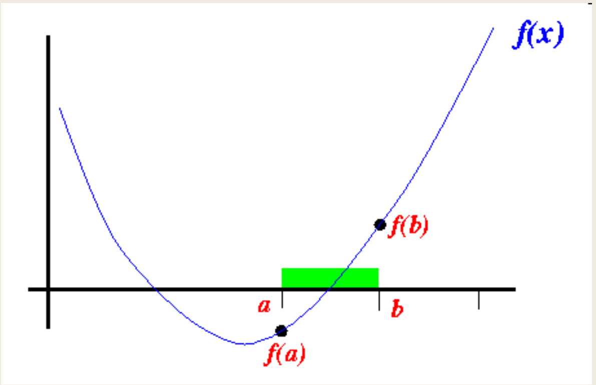
2. *Regula Falsi* method

3. *Secant* Method

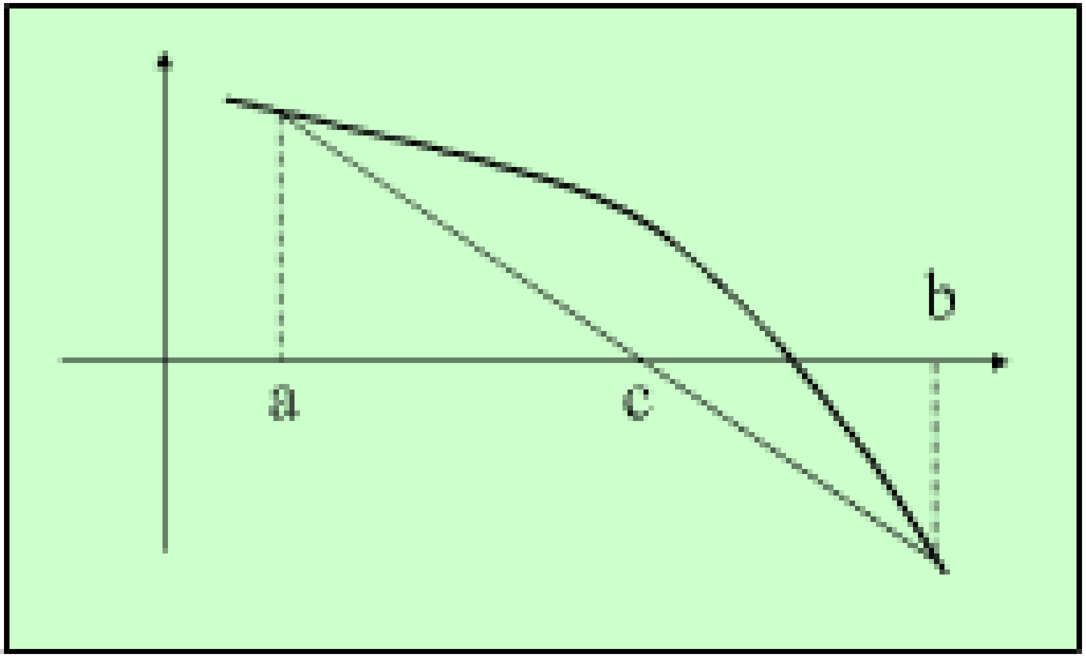
4. *Newton Raphson* method

Bisection Method

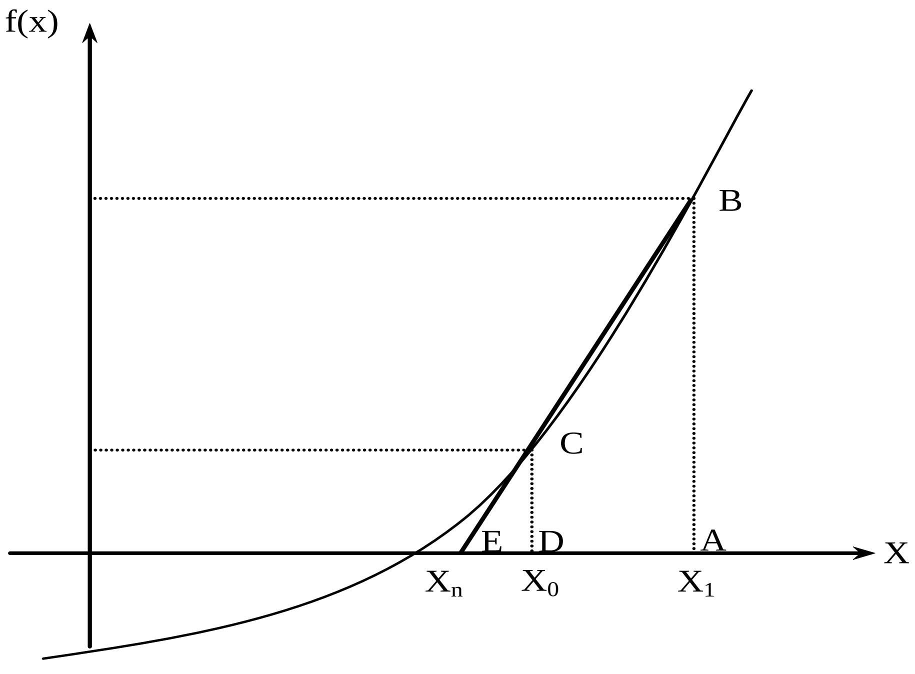
- sign of *f(m)* ≠ sign of *f(a)* , we *proceed* with the search in the *new* interval [*a*..*b*]:



The False-Position Method (Regula-Falsi) Cont..



Secant Method



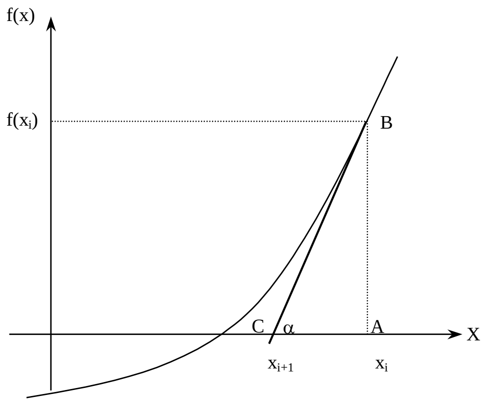
Newton Raphson's Method

Let x0 be an approximate root of the equation f(x)=0.

If x1=x0+h be the exact root, then f(x1)=0

Expanding f(x1)=f(x0+h) by taylor's series

f(x0+h)=0



$$f(x_0)+hf'(x_0)+\frac{h^2 f''(x_0)}{2!}+........=0$$

Since h is small, neglecting h^2 and higher power of h, we get

$$f(x_0)+hf'(x_0)=0$$

or $$h=-\frac{f(x_0)}{f'(x_0)}$$

A closer approximation to the root is given by

x1=x0+h

A closer approximation to root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

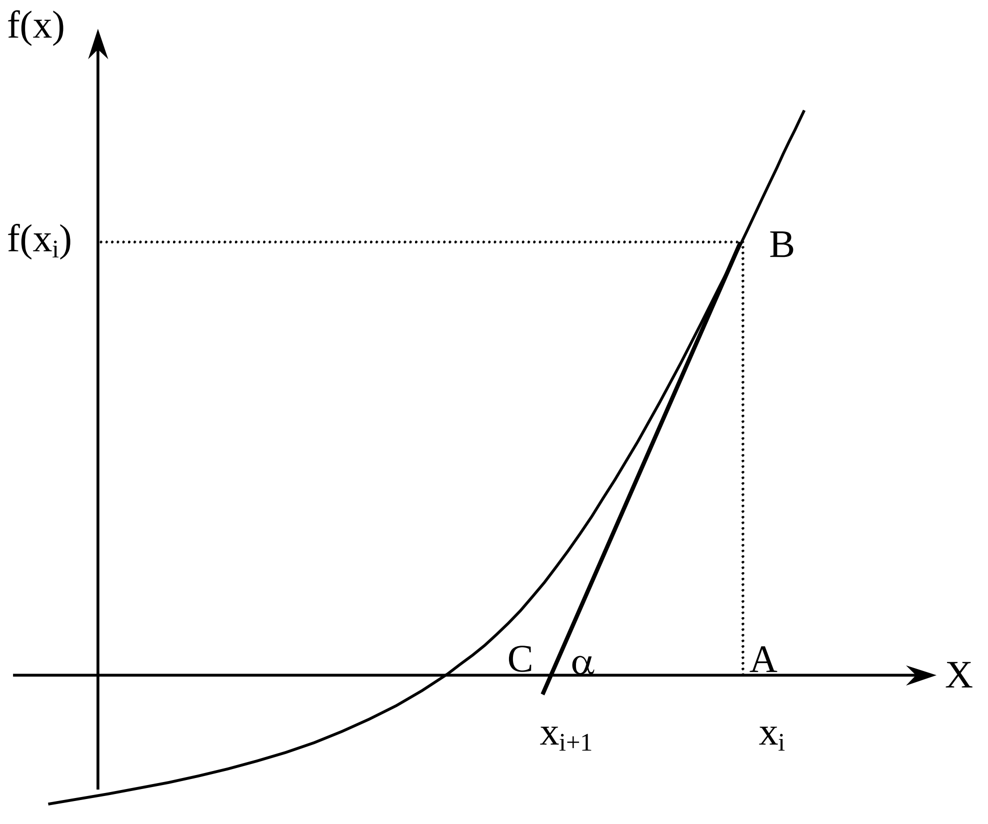
Similarly, starting with x1, a still better approximation x2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general, nth approximation can be given by

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n=0,1,2\ldots.}$$

Derivation



$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Derivation of the Newton-Raphson method.

A real root of the equation x^3-5x+1=0 lies in the interval (0, 1). Perform four iterations of the Newton Raphson method.

f(x) = x^3-5x+1=0

The smalest root lies in the interval (0,1). Take the initial approximation as x0=0.5.

We have, f(x)=x^3-5x+1 and f'(x)=3x^2-5

Using Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^3 - 5x_n + 1)}{3x_n^2 - 5} = \frac{2x_n^3 - 1}{3x_n^2 - 5}, n = 0,1,2....$$

Starting with x0=0.5, we obtain, x1=0.176471

$x_2=0.201568$
$x_3=0.201640$
$x_4=0.201640$

The exact value correct to six decimal places is 0.201640

Practice problems

1. Find the real root of the equation 3x-cos x-1=0 using Newton Raphson method, correct to 3 decimal places.

2. Obtain the cube root of 12 correct to five decimal places by Newton Raphson method

Convergence of Root finding methods

Order of an iterative method-

An iterative method is of order p if

$$|\varepsilon_{n+1}| \leq c\,|\varepsilon_n|^p$$

Where, ‘c’ is called the asymptotic error constant.

Note: Error at any iteration can be defined as

$$\varepsilon_n = X_n - \varepsilon$$

Where, ε is the exact root, ε_n is th error between X_n and ε

Rate of convergence of Newton Raphson method

Let $\varepsilon_{n+1} = x_{n+1} - \varepsilon$ and $\varepsilon_n = x_n - \varepsilon$,

$$\Rightarrow x_{n+1} = \varepsilon_{n+1} + \varepsilon \text{ and } x_n = \varepsilon_n + \varepsilon$$

Therefore by Newton Raphson method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\varepsilon_{n+1} + \varepsilon = \varepsilon_n + \varepsilon - \frac{f(\varepsilon_n + \varepsilon)}{f\,'(\varepsilon_n + \varepsilon)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\varepsilon_n + \varepsilon)}{f\,'(\varepsilon_n + \varepsilon)}$$

$$By..taylor...series...$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{\left[f(\varepsilon) + \varepsilon_n f'(\varepsilon) + \frac{\varepsilon_n^2}{2} f''(\varepsilon) + ... \right]}{\left[f'(\varepsilon) + \varepsilon_n f''(\varepsilon) + \frac{\varepsilon_n^2}{2} f'''(\varepsilon) + ... \right]}$$

$$as...\varepsilon...is...the...exact....root...so...f(\varepsilon) = 0$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{\left[0 + \varepsilon_n f'(\varepsilon) + \frac{\varepsilon_n^2}{2} f''(\varepsilon) + ... \right]}{\left[f'(\varepsilon) + \varepsilon_n f''(\varepsilon) + \frac{\varepsilon_n^2}{2} f'''(\varepsilon) + ... \right]}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{1}{f'(\varepsilon)} \frac{\left[\varepsilon_n f'(\varepsilon) + \frac{\varepsilon_n^2}{2} f''(\varepsilon) + \ldots\ldots\right]\left[1 + \varepsilon_n \frac{f''(\varepsilon)}{f'(\varepsilon)} + \frac{\varepsilon_n^2}{2} \frac{f'''(\varepsilon)}{f'(\varepsilon)} + \ldots\ldots\right]^{-1}}{}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{1}{f'(\varepsilon)} \frac{\left[\varepsilon_n f'(\varepsilon) + \frac{\varepsilon_n^2}{2} f''(\varepsilon) + \ldots\ldots\right]\left[1 - \varepsilon_n \frac{f''(\varepsilon)}{f'(\varepsilon)} + \ldots\ldots\right]}{}$$

$$\varepsilon_{n+1} = \varepsilon_n - \left[\varepsilon_n \frac{f'(\varepsilon)}{f'(\varepsilon)} + \frac{\varepsilon_n^2}{2} \frac{f''(\varepsilon)}{f'(\varepsilon)} + \ldots\ldots\right]\left[1 - \varepsilon_n \frac{f''(\varepsilon)}{f'(\varepsilon)} + \ldots\ldots\right]$$

$$\varepsilon_{n+1} = \varepsilon_n - \left[\varepsilon_n + \frac{\varepsilon_n^2}{2} \frac{f''(\varepsilon)}{f'(\varepsilon)} + \ldots\ldots\right]\left[1 - \varepsilon_n \frac{f''(\varepsilon)}{f'(\varepsilon)} + \ldots\ldots\right]$$

$$\varepsilon_{n+1} = \varepsilon_n - \left[\varepsilon_n - \frac{\varepsilon_n^2}{} \frac{f''(\varepsilon)}{f'(\varepsilon)} + \frac{\varepsilon_n^2}{2} \frac{f''(\varepsilon)}{f'(\varepsilon)} - \frac{\varepsilon_n^3}{2} \left(\frac{f''(\varepsilon)}{f'(\varepsilon)}\right)^2\right]$$

$$\varepsilon_{n+1} = \varepsilon_n - \left[\varepsilon_n - \frac{\varepsilon_n^2}{2} \frac{f''(\varepsilon)}{f'(\varepsilon)} + o(\varepsilon_n^3) \right]$$

$$\varepsilon_{n+1} = \varepsilon_n - \varepsilon_n + \frac{\varepsilon_n^2}{2} \frac{f''(\varepsilon)}{f'(\varepsilon)} + o(\varepsilon_n^3)$$

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2} \frac{f''(\varepsilon)}{f'(\varepsilon)} + o(\varepsilon_n^3)$$

$as \ldots \varepsilon_n \ldots is \ldots small \ldots \varepsilon_n^3 \ldots will \ldots be \ldots much \ldots smaller \ldots\ldots$

$. neglecting \ldots higher \ldots order \ldots term \ldots$

$$\varepsilon_{n+1} = c\varepsilon_n^2$$

$$\Rightarrow p = 2$$

Find the rate of Convergence of Secant Method.

Suggested books

1. Numerical Methods by **S.R.K Lyenger & R.K. Jain.**

2. Numerical Analysis by **Richard L. Burden.**

3. Introductory methods of Numerical analysis by **S.S. Sastry**.

Thank you