Lemma: Let H be a subgroup of a group G and suppose that $g_1, g_2 \in G$.

The following conditions are equivalent:

(1) $g_1 H = g_2 H$

(2) $H g_1^{-1} = H g_2^{-1}$

(3) $g_1 H \subseteq g_2 H$

(4) $g_2 \in g_1 H$

(5) $g_1^{-1} g_2 \in H$

Proof:

(1) $\Rightarrow$ (2)

Let $g_1 H = g_2 H$

Let $x \in H g_1^{-1}$
$\Rightarrow \quad x = h g_1^{-1}$ for some $h \in H$
$\Rightarrow \quad x^{-1} = g_1 h^{-1}$
$\Rightarrow \quad x^{-1} = g_1 h^{-1} \in g_1 H$
$\Rightarrow \quad x^{-1} = g_1 h^{-1} \in g_2 H$
$\Rightarrow \quad x^{-1} = g_1 h^{-1} = g_2 h'$ for some $h' \in H$
$\Rightarrow \quad x = h'^{-1} g_2^{-1}$
$\Rightarrow \quad x \in H g_2^{-1}$
$\Rightarrow \quad H g_1^{-1} \subseteq H g_2^{-1}$

Similarly $H g_2^{-1} \subseteq H g_1^{-1}$

Hence $\boxed{H g_2^{-1} = H g_1^{-1}}$

(2) = ⟩(3)

  ut $x \in g_1 H$

   ⟹ ... ; $x = g_1 h$ ⟹ $\cancel{g_1^{-1}}\, x^{-1} = h^{-1} g_1^{-1} \in H g_1^{-1}$

   ⟹ $x^{-1} = h^{-1} g_1^{-1} \in H g_1^{-1} = H g_2^{-1}$

   ⟹ $x^{-1} = h^{-1} g_1^{-1} = h' g_2^{-1}$

   ⟹ $x = g_2 h'^{-1} \in g_2 H.$

(3) = ⟩(4)

   g_2 given tht $g_1 H \subseteq g_2 H$

  ⟹ $g_1 \cdot e = g_1 = g_2 h$

  ⟹ $g_2 = g_1 h^{-1} \in g_1 H.$

(4) = ⟩(5)

   $g_2 \in g_1 H$

  ⟹ $g_2 = g_1 h$

  ⟹ $g_1^{-1} g_2 = h \in H.$

(5) = ⟩(1)

   given tht $g_1^{-1} g_2 \in H$

 $x \in g_1 H$ ⟹ $x = g_1 h$ ⟹ $x^{-1} = h^{-1} g_1^{-1}$

         ⟹ $x^{-1} g_2 = \underset{\in H}{h^{-1}} \underset{\in H}{g_1^{-1} g_2}$

  ⟹ $x^{-1} g_2 \in H$ ⟹ $x^{-1} g_2 = h'$ ⟹ $x^{-1} = h' g_2^{-1}$

               ⟹ $x = g_2 h' \in g_2 H.$

    ⟹ $\boxed{g_1 H \subseteq g_2 H}$

 Now ut $x \in g_2 H$ ⟹ $x = g_2 h$ ⟹ $\cancel{x^{-1} = h^{-1} g_2^{-1}}\; g_1^{-1} x = g_1^{-1} g_2 h$

 ⟹ $g_1^{-1} x \in H$ ⟹ $g_1^{-1} x = h'$ ⟹ $x = g_1 h' \in g_1 H$ ⟹ $\boxed{g_2 H \subseteq g_1 H}$