Motion in Rotating Frame

Rotation of a Vector in two dimensions: Let a vector $\vec{A}$ be defined in a two-dimensional plane OXY. The Vector depends upon its magnitude A and direction of unit vector $\hat{e}$ both. Let at any time t, its line of action, OX' makes ϕ angle with OX.

$$\vec{A} = A\hat{e} \quad \text{——— (1)}$$

$$\frac{d\vec{A}}{d\phi} = \hat{e}\frac{dA}{d\phi} + A\frac{d\hat{e}}{d\phi} \quad \text{——— (2)}$$

Taking dot product of eq (2) with $\hat{e}$, then

$$\hat{e}.\frac{d\vec{A}}{d\phi} = (\hat{e}.\hat{e})\frac{dA}{d\phi} + \left(\hat{e}.\frac{d\hat{e}}{d\phi}\right)A \Rightarrow \hat{e}.\frac{d\vec{A}}{d\phi} = \frac{dA}{d\phi} + A\left(\hat{e}.\frac{d\hat{e}}{d\phi}\right) \quad \text{— (3)}$$

$$2\frac{dA}{d\phi} = \frac{1}{A}\frac{dA^2}{d\phi} = \frac{1}{A}\frac{d}{d\phi}(\vec{A}.\vec{A}) = \frac{\vec{A}}{A}.\frac{d\vec{A}}{d\phi} + \frac{\vec{A}}{A}.\frac{d\vec{A}}{d\phi}$$

$$\boxed{\frac{dA}{d\phi} = \hat{e}.\frac{d\vec{A}}{d\phi}}$$

Using this in eq (3)

$$\hat{e}.\frac{d\vec{A}}{d\phi} = \hat{e}.\frac{d\vec{A}}{d\phi} + A\left(\hat{e}.\frac{d\hat{e}}{d\phi}\right), \quad A \neq 0$$

$$\Rightarrow \hat{e}.\frac{d\hat{e}}{d\phi} = 0$$

(1) The change in magnitude of a unit vector does not exist, i.e. the magnitudes of both $\hat{e}$ and $\frac{d\hat{e}}{d\phi}$ are unity.

(2) $\frac{d\hat{e}}{d\phi} \perp \hat{e}$.

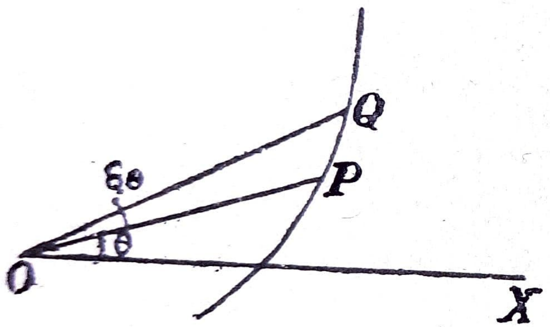
let $\hat{e}$ is replace by $\hat{i}$ then $\hat{i}.\frac{d\hat{i}}{d\phi} = 0 \Rightarrow \left|\frac{d\hat{i}}{d\phi}\right| = 1$

$\hat{j}$ $\quad \hat{j}.\frac{d\hat{j}}{d\phi} = 0 \Rightarrow \boxed{\frac{d\hat{j}}{d\phi} = -\hat{i}}$ $\quad \boxed{\text{So } \frac{d\hat{i}}{d\phi} = \hat{j}}$ $\quad \frac{d\hat{i}}{d\phi} \perp \hat{i}$

1·5. Angular Velocity and Acceleration.

Definition. The angular velocity of a point P about another point O is the rate of change of the angle which OP makes with some fixed direction.

A particle is moving in a plane curve. Take a line OX fixed in the plane of the curve as initial line and O as pole. Let P be the position of the particle at time t, being given by the angle $XOP = \theta$.



Let Q be the position at time $t+\delta t$, so that the angle described in time δt is $\delta\theta$.

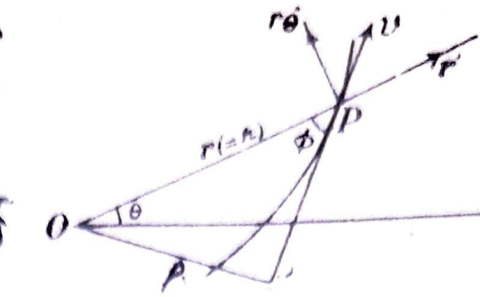
Thus the average angular velocity of P about O is $\dfrac{\delta\theta}{\delta t}$.

As δt becomes smaller and smaller, Q approaches P and $\dfrac{\delta\theta}{\delta t}$ becomes the rate of change of θ. This is angular velocity.

Thus angular velocity of P about $O = \lim\limits_{\delta t \to 0} \dfrac{\delta\theta}{\delta t} = \dfrac{d\theta}{dt} = \dot{\theta}$.

Similarly angular acceleration is $\dfrac{d}{dt}\left(\dfrac{d\theta}{dt}\right) = \dfrac{d^2\theta}{dt^2} = \ddot{\theta}$.

Let v be the velocity of the particle at P. It is along the tangent to the path at P.



Components of velocity of P along and perp. to OP are $\dot{r}$ and $r\dot{\theta}$. Resolving perp. to OP, we get

$$r\dot{\theta}=v\sin\phi$$

where ϕ is the angle the tangent of P makes with OP.

Thus $\dot{\theta}=\dfrac{v\sin\phi}{r}$

i.e., angular velocity of P about $O=\dfrac{\text{velocity of P resolved perp. to } OP}{OP}$

Also $\boxed{\dot{\theta}=\dfrac{v\sin\phi}{r}}$ and $\boxed{\sin\phi=\dfrac{p}{r}}$ where p is perp. from O to the tangent at P.

$\therefore$ $\boxed{\dot{\theta}=\dfrac{vp}{r^2}}$

Ex. 9. *A particle describes an equiangular spiral $r=ae^{\theta}$ in such a manner that its acceleration has no radial component. Prove that its angular velocity is constant and that the magnitude of the velocity and acceleration is each proportional to r.*

Here $\ddot{r}-r.\dot{\theta}^2=0$

From $r=ae^{\theta}$, $\dot{r}=ae^{\theta}\dot{\theta}=r\dot{\theta}$

$\ddot{r}=\dot{r}\dot{\theta}+r\ddot{\theta}=r\dot{\theta}^2+r\ddot{\theta}$

$\therefore$ $r\ddot{\theta}=\ddot{r}-r\dot{\theta}^2=0$ as given above.

Hence $\ddot{\theta}=0$

$\therefore$ $\dot{\theta}=\text{const.}=K$ say

Then $\dot{r}=Kr$ so that $v^2=\dot{r}^2+r^2\dot{\theta}^2=2K^2r^2$

$\therefore$ v varies as r

Since radial acc. is zero, only acc. is transverse

$$=2\dot{r}\dot{\theta}+r\ddot{\theta}=2\dot{r}\dot{\theta}=2K^2r$$

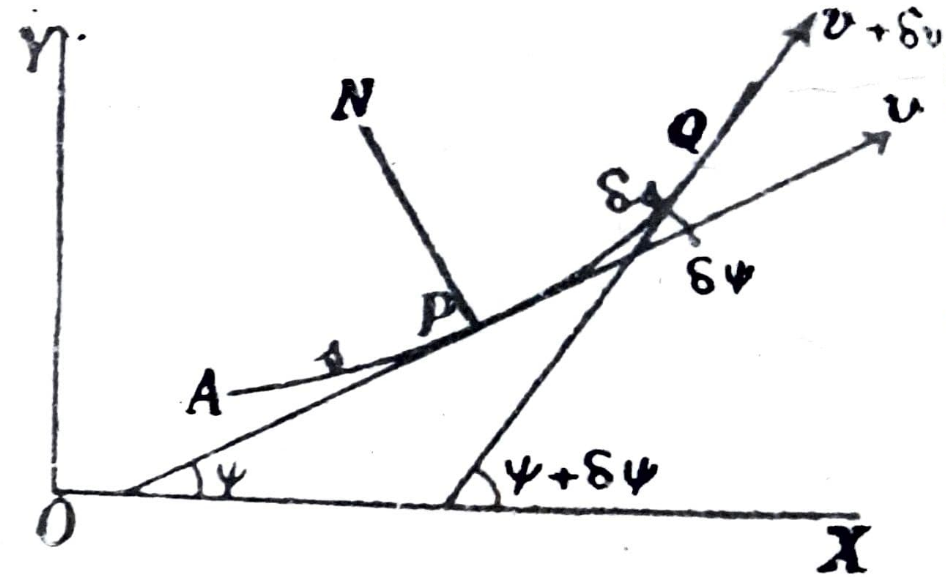
$\therefore$ acc. varies as r.

Ex. 10. *A rod moves with its ends on rectangular axes OX, OY. If x, y be a point P on the rod and if the angular velocity ω of the rod is constant, show that components of acceleration of P along the axes are $-x\omega^2$ and $-y\omega^2$ and the resultant acceleration is $OP.\omega^2$ towards O.*

If C be the middle point then $OC=CA=CB=a$, so that $\angle AOC=\theta=\angle OAC$.

A particle is moving in a plane curve ; to find components of its acceleration along the tangent and the normal to the curve at any instant.

Let A be a fixed point on the curve, and P be the position of the particle at time t where $AP=s$. Let v be the velocity of the particle at P, it being entirely along the tangent at P ; so that $v=\dot{s}$.



Let Q be the position of the particle at time $t+\delta t$ and $v+\delta v$ be the velocity there *i.e.*, along the tangent at Q.

Let the tangents at P and Q make angles ψ and $\psi+\delta\psi$ with a fixed line OX so that $\delta\psi$ is the angle between the tangents.

Thus the change of velocity along the tangent at P in times δt

$$=(v+\delta v)\cos\delta\psi-v$$

$$=(v+\delta v).\ 1-v, \text{ neglecting terms of second order}$$

$$=\delta v,$$

$\delta s \cos \psi$ $\frac{ds}{}$

and the change of velocity along the normal at P in time δt

$$=(v+\delta v)\sin\delta\psi-0$$

$$=(v+\delta v)\,\delta\psi, \qquad \text{neglecting second order terms.}$$

$$=v\,\delta\psi, \qquad \text{neglecting the other term.}$$

Thus tangential acceleration

$$=\lim_{\delta t\to 0}\frac{\text{change of velocity along the tangent in time } \delta t}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\delta v}{\delta t}=\frac{dv}{dt}=\frac{d^2s}{dt^2}$$

or $$=\frac{dv}{ds}\cdot\frac{ds}{dt}=v\frac{dv}{ds}.$$

tangential acceleration

Normal acceleration

$$=\lim_{\delta t\to 0}\frac{\text{change of velocity along the normal in time } \delta t}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{v\delta\psi}{\delta t}=\lim\frac{v\delta\psi}{\delta s}\cdot\frac{\delta s}{\delta t}$$

$$=\lim_{\delta s\to 0}\frac{v\delta\psi}{\delta s}\cdot\lim_{\delta t\to 0}\frac{\delta s}{\delta t}=\frac{v}{\rho}\cdot v=\frac{v^2}{\rho}$$

$v=\dot{s}$ normal acc $=\frac{\dot{s}^2}{\rho}$

where ρ is the radius of curvature at P.

Thus for a particle moving in a plane curve, component of acceleration along the tangent is $\frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$ or $v\frac{dv}{ds}$, in the sense in which s increases ; and the component of acceleration along the normal is $\frac{v^2}{\rho}$ in the inward sense.

Cor. For a particle moving in a circle of radius a, $s=a\theta$, so that tangential acc. $=\ddot{s}=a\ddot{\theta}$, in the sense θ increasing and normal acc. $=\frac{v^2}{\rho}=\frac{\dot{s}^2}{a}=\frac{a^2\dot{\theta}^2}{a}=a\dot{\theta}^2$, towards the centre.

EXAMPLES I (B)

1. If the radial and transverse velocities of a point are always proportional to each other and this holds for acceleration also, prove that its velocity will vary as some power of the radius vector.

Imp 2. The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$. Show that the equation to the path is

$$\frac{\lambda}{\theta}=\frac{\mu}{2r^2}+C,$$

and the components of accelerations are

$$2\lambda^2 r^3-\mu^2\frac{\theta^4}{r} \text{ and } \lambda\mu r\theta^2+2\mu^2\frac{\theta^3}{r}.$$

3. Prove that the path of a point which possesses two constant velocities, one along a fixed direction and the other perpendicular to the radius vector drawn from a fixed point, is a conic section.

4. A particle moves along a circle $r=2a\cos\theta$ in such a way that its acceleration towards the origin is always zero. Prove that

$$\frac{d^2\theta}{dt^2}=-2\cot\theta\,.\,\dot{\theta}^2.$$

1989-90

1. If a point moves along a circle, prove that its angular velocity about any point on the circle is half of that about the centre.

2. A point describes a circle of radius a with a uniform speed v; show that the radial and transverse accelerations are $-\frac{v^2}{a}\cos\theta$ and $-\frac{v^2}{a}\sin\theta$ if a diameter is taken as initial line and one end of the diameter as pole.

3. A point describes uniformly a given straight line; show that its angular velocity about a fixed point varies inversely as the square of its distance from the fixed point.

4. A point P describes a curve with a constant velocity and the radius vector joining P to a fixed point O has an angular velocity which is inversely proportional to OP; show that the curve is an equiangular spiral with O as pole and that the acceleration of P along the normal varies inversely as OP.

5. If the velocity of a point moving in a plane curve varies as the radius of curvature, show that the direction of motion revolves with constant angular velocity.

Also, if the angular velocity of the moving point about a fixed origin be constant, show that its transverse acceleration varies as its radial velocity.

6. A point moves in a plane curve so that its tangential acceleration is constant and the magnitudes of the tangential velocity and the normal acceleration are in a constant ratio. Show that the intrinsic equation of the path is of the form $s=A\psi^2+B\psi+C$.

7. A point moves in a curve so that its tangential and normal accelerations are equal and the tangent rotates with constant angular velocity. G-D3

Show that the intrinsic equation of the path is of the form $s=Ae^{\psi}+B$.

8. A particle describes a curve (for which s and ψ vanish simultaneously) with uniform speed v. If the acceleration at any point s be $\frac{v^2c}{s^2+c^2}$, prove that the curve is a catenary. G-D3

If X, T be the components of acc. parallel to initial line and cross-radially then resolving radially and cross radially, we get

$$X \cos \theta = \ddot{r} - \dot{\theta}^2$$

and $$T - X \sin \theta = \frac{1}{r} \cdot \frac{d}{dt}(r^2\dot{\theta}).\ \Big]$$ These give X and T.

EXAMPLES I (E)

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1. A wheel rolls uniformly on the ground without sliding, its centre describing a straight line. Show that the angular velocity of any point on the rim about the point of contact of the wheel with the ground is equal to the angular velocity of the wheel about its centre.

2. Two points are moving with uniform velocities u and v along perpendicular lines OX, OY, the motion being towards O. What $t=0$, they are at distances a, b, respectively from O. Calculate the angular velocity of the line joining them at time t and show that it is the greatest when $t = \dfrac{au+bv}{u^2+v^2}$.

✓ 3. A small bead slides with constant speed v on a smooth wire in the shape of the cardioid $r = a(1+\cos\theta)$. Show that the value of $\dot{\theta}$ is $(v \sec \theta/2)/2a$ and that the radial component of the acceleration is constant.

✓ 4. If the curve is the equi-angular spiral $r = ae^{\theta \cot \alpha}$ and if the radius vector to the particle has constant angular velocity, show that the resultant acceleration of the particle makes an angle 2α with the radius of vector and is of magnitude v^2/r where v is the speed of the particle.

5. Two particles P and Q, starting simultaneously from the same point O, move uniformly in a circle and in a straight line which touches the circle respectively, each with a speed u. Prove that the velocity of P relative

$\sqrt{a^2+b^2}$ ... time that he takes to cross the road then is $\frac{c}{u}\left(\frac{}{b} + \frac{}{a}\right)$

7. A point starts from the origin in the direction of the initial line with velocity f/ω, and moves with constant angular velocity ω about the origin, and with constant negative radial acceleration $-f$. Show that the rate of growth of the radial velocity is never positive but tends to the limit zero, and prove that the equation of the path is

$$\omega^2 r = f(1-e^{-\theta}).$$

8. A point is describing a circle of radius a in such a way that the tangential acceleration is k times the normal acceleration. If its speed at a certain point is u, prove that it will return to the same point after a time

$$\frac{a}{ku}(1-e^{-2\pi k}).$$

9. A point describes a plane curve in such a way that the resultant