Ordinary Differential Equation (O.D.E.)

Content

- Ordinary Differential Equation
  - Taylor's series
  - Picard's Method
  - Euler's Method

Numerical Solution by Ordinary Differential Equations-

- Most of the Numerical methods used to solve ODE are based directly (or indirectly) on truncated Taylor series expansion
- Taylor Series method
- Picard's Method
- Euler's method

Numerical Solution by Ordinary Differential Equations-

The solution of ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x.

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Solution of Ordinary Differential Equations by Taylor's series-

Consider the O.D.E

$$y' = \frac{dy}{dx} = f(x, y) \qquad \text{.......(1)}$$

With initial condition $y(x_0) = y_0$

Solution of Ordinary Differential Equations by Taylor's series-

If y(x) is the exact solution of eq(1) then the Taylor's series From y(x) around $x=x_i$ is given by

$$y(x) = y_i + \frac{(x-x_i)y_i^{'}}{1!} + \frac{(x-x_i)^2}{2!}y_i^{''} + \ldots\ldots\frac{(x-x_i)^n}{n!}y_i^n$$

Now, consider the interval $[x_i, x_{i+1}]$. The length of the interval is $h = x_{i+1} - x_i$.
Substituting , $x = x_{i+1}$

Solution of Ordinary Differential Equations by Taylor's series-

$$y(x) = y_i + \frac{(x-x_i)y_i^{'}}{1!} + \frac{(x-x_i)^2}{2!} y_i^{''} + \ldots\ldots \frac{(x-x_i)^n}{n!} y_i^n \qquad \ldots..(1)$$

Substituting , $x = x_{i+1}$ in eq (1)

$$y(x_{i+1}) = y_i + \frac{hy_i^{'}}{1!} + \frac{h^2}{2!} y_i^{''} + \ldots\ldots \frac{h^n}{n!} y_i^n \qquad \ldots..(2)$$

Using the definition of the order, we say that the Taylor series method of eq (2) is of order n

Example:

Consider the initial value problem $y' = x(y + 1)$, $y(0) = 1$. Compute $y(0.2)$ with $h = 0.1$ using Taylor series method of order two.

We have $y' = f(x, y) = x(y + 1)$, $x_0 = 0$, $y_0 = 1$, $h=0.1$

$$y' = \frac{dy}{dx} = f(x, y) \qquad \text{…….(1)}$$

With initial condition $y(0) = 1$

Solution

Taylor series second order method.

$$y(x_{i+1}) = y_i + \frac{hy_i^{'}}{1!} + \frac{h^2}{2!} y_i^{''}$$

$$y(x_{i+1}) = y_i + 0.1y_i^{'} + \frac{0.1^2}{2!} y_i^{''}$$

$$y(x_{i+1}) = y_i + 0.1y_i^{'} + 0.005y_i^{''}$$

We have, y’=x(y+1)

y’’=xy’+y+1

$x_0 = 0, y_0 = 1,$

$y'_0 = 0$ and $y_0'' = 0+1+1 = 2$

Solution (Cont..)

Taylor series second order method.

With $x_0 = 0$, $y_0 = 1$, we get
$y'_0 = 0$ and $y_0'' = 0+1+1 = 2$

$y(0.1) \approx y_1 = y_0 + 0.1y'_0 + 0.005\, y''_0$
$= 1 + 0 + 0.005\,[2] = 1.01.$

With $x_1 = 0.1$, $y_1 = 1.01$, we get
$y_1' = 0.1(1.01 + 1) = 0.201$
$y_1'' = x_1 y_1' + y_1 + 1 = (0.1)(0.201) + 1.01 + 1 = 2.0301.$
$Y(0.2) = y_1 + 0.1\, y_1' + 0.005 y_1''$
$= 1.01 + 0.1\,(0.201) + 0.005(2.0301)] = 1.04025.$

Picard's method of successive approximation-

Consider the O.D.E

$$y' = \frac{dy}{dx} = f(x, y) \qquad \text{.......(1)}$$

With initial condition $y(x_0) = y_0$

Picard's method of successive approximation–

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)dx$$

$$y - y_0 = \int_{x_0}^{x} f(x,y)dx$$

$$y = y_0 + \int_{x_0}^{x} f(x,y)dx$$

For the first approximation, replace y by y_0 in f(x,y)

$$y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)dx$$

Picard's method of successive approximation–

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

For the second approximation, replace y_0 by y_1 in $f(x,y_0)$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

For the third approximation, replace y_1 by y_2 in $f(x,y_1)$

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

Similary, For the n^{th} approximation, replace y_{n-2} by y_{n-1} in $f(x,y_{n-2})$

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx$$

Example:
Find an approximate value of y using Picard's method, if $dy/dx=x-y^2$, given x=0, y=1

Given $x_0=0$, $y_0=1$, $f(x,y)=x-y^2$

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$$

For the first approximation,

$$y_1 = y_0 + \int_0^x (x - y_0^2)dx$$

$$y_1 = y_0 + \int_0^x (x-1)dx$$

$$y_1 = 1 + \frac{x^2}{2} - x$$

Example:
Find an approximate value of y using Picard's method, if $dy/dx=x-y^2$, given x=0, y=0

Given $x_0=0$, $y_0=1$, $f(x,y)=x-y^2$

For the second approximation, replace y_0 by y_1 in $f(x,y_0)$

$$y_2 = 1+\int_0^x (x-y_1^2)\,dx$$

$$y_2 = 1+\int_0^x (x-\left(1+\frac{x^2}{2}-x\right)^2)\,dx$$

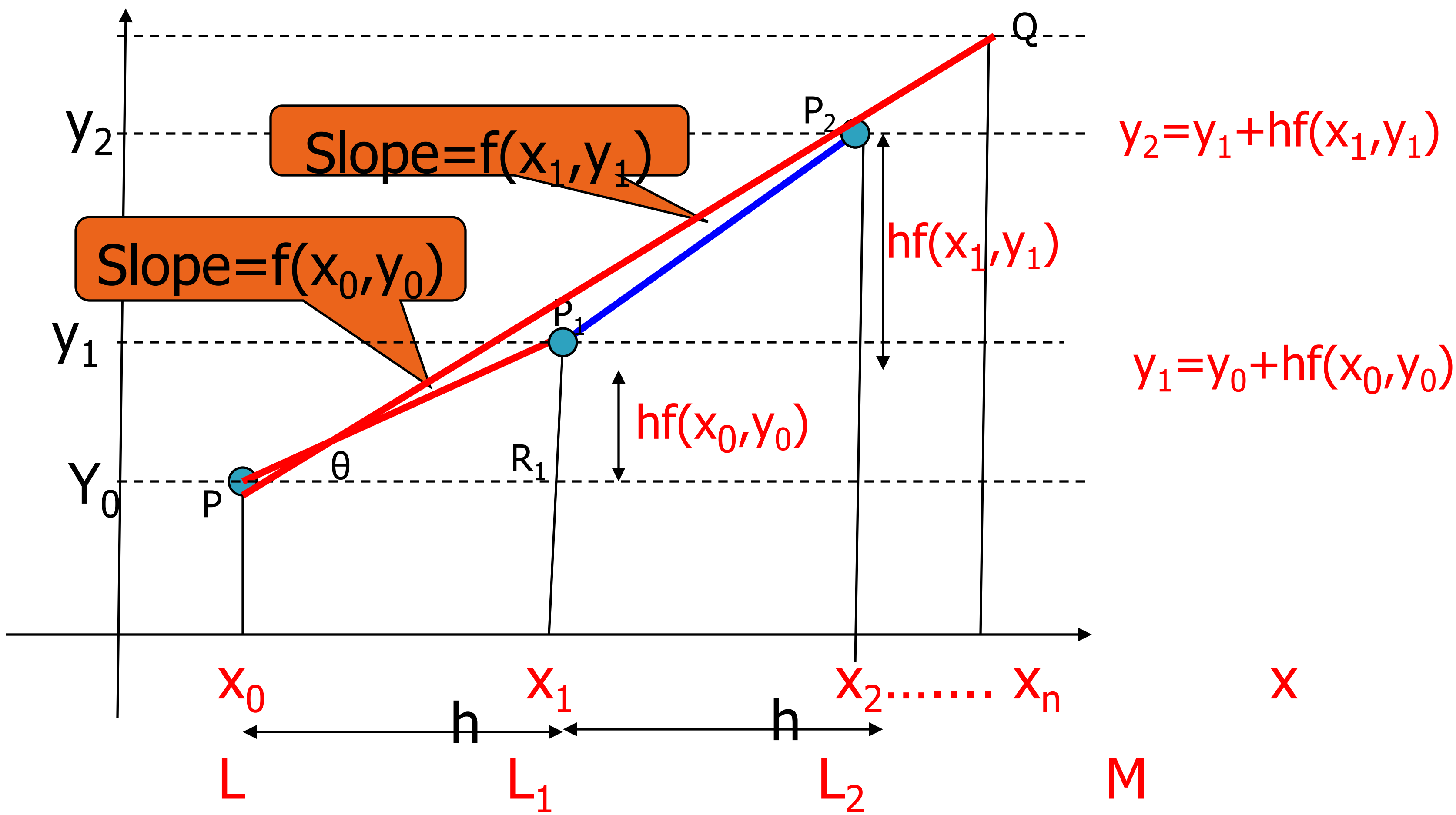
$$y_2 = 1+\int_0^x (x-\left(1+x^2+\frac{x^4}{4}-2x-x^3+x^2\right))\,dx$$

$$y_2 = 1-x+\frac{3x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}-\frac{x^5}{20}$$

$$y(0.1) = 1-0.1+\frac{3(0.1)^2}{2}-\frac{2(0.1)^3}{3}+\frac{(0.1)^4}{4}-\frac{(0.1)^5}{20}$$

Euler’s Method

Interpretation of Euler Method



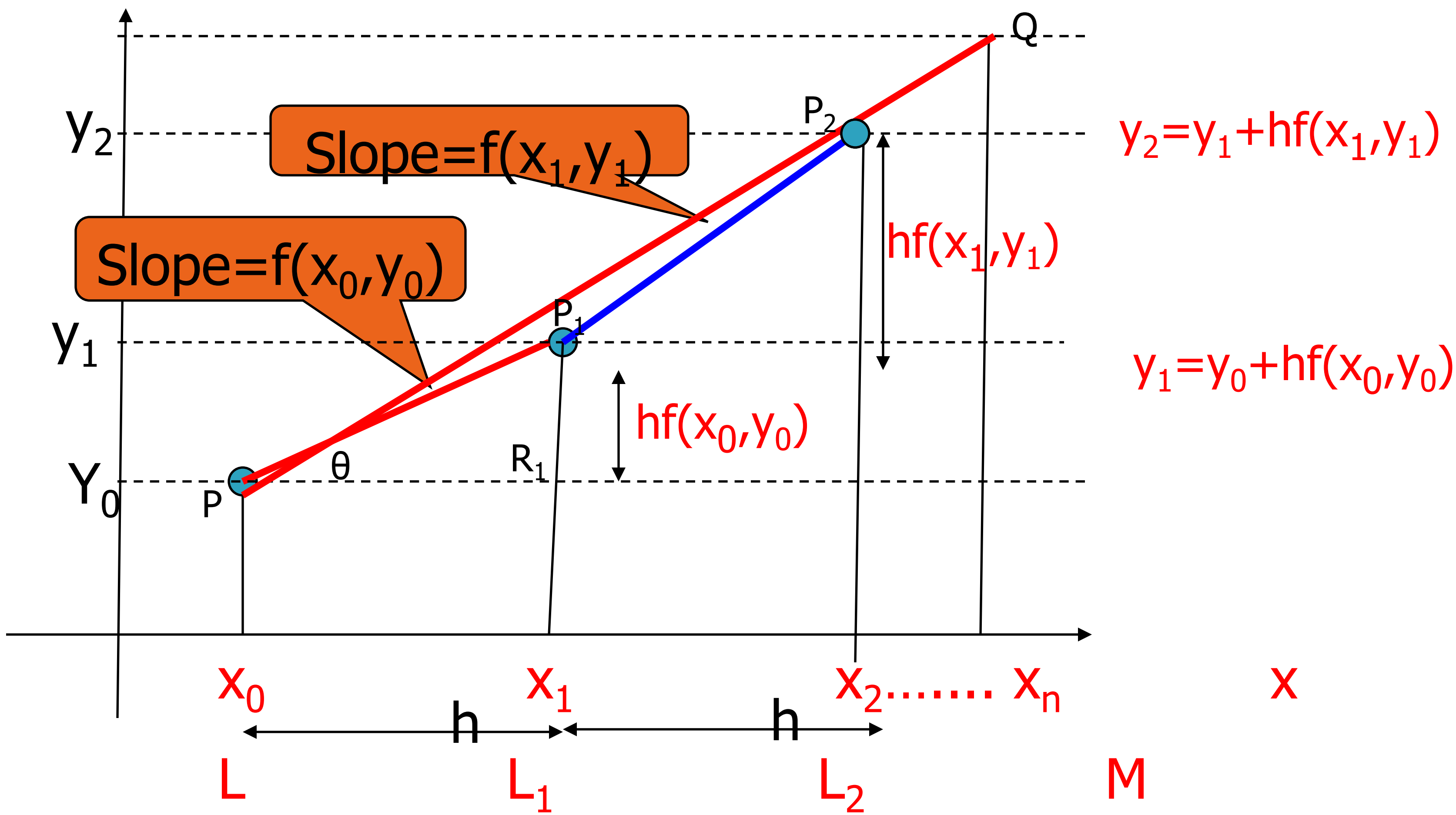
Euler's Method

Consider the O.D.E $y' = \frac{dy}{dx} = f(x, y)$ ....(1)

such that $y(x_0) = y_0$

Let us divide LM into n sub intevals each of width h L_1, L_2, …..L_n
In the interval $LL_{1,}$ we approximate the curve by the tangent at P.
If the ordinate through L, meets this tangent in $P_1(x_0+h, y_1)$, then
$y_1=L_1P_1$
$y_1=L_1R_1+R_1P_1$
$=y_0+PR_1 \tan\theta$
$y_1 =y_0+h \tan\theta$
$y_1 =y_0+h (dy/dx)_p$
$y_1 =y_0+h f(x_0,y_0)$

Interpretation of Euler Method



Euler's Method

Let P, Q be the curve of eq (1) through P_1 and let its tangent at P_1 Meet the ordinate through L_2 in P_2 (x_0+2h, y_2)

$$y_2 = y_1 + h\, f(x_1, y_1)$$

Repeating this process n times, we finally reach on the approximation of MQ

$$y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})$$

Example:

Consider the initial value problem $y' = x(y + 1)$, $y(0) = 1$. Compute $y(0.2)$ with $h = 0.1$ using Euler method.

We have $y' = f(x, y) = x(y + 1)$, $x_0 = 0$, $y_0 = 1$, $h=0.1$

$$y' = \frac{dy}{dx} = f(x, y) \qquad \ldots\ldots.(1)$$

With initial condition $y(0) = 1$

Solution

By Euler's method.

$$y(x_n) = y_{n-1} + hy'_{n-1}$$

$$y(x_n) = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

$$y(x_n) = y_{n-1} + 0.1x_{n-1}(y_{n-1} + 1)$$

$$with, x_0 = 0, y_0 = 1$$

$$\mathrm{y}(0.1) = \mathrm{y}_1 = \mathrm{y}_0 + 0.1[\mathrm{x}_0 (\mathrm{y}_0 + 1)]$$

$$\mathrm{y}(0.1) = \mathrm{y}_1 = 1 + 0.1[0] = 1.0.$$

$$with, x_1 = 0.1, y_1 = 1$$

$$\mathrm{y}(0.2) = \mathrm{y}_2 = \mathrm{y}_1 + 0.1[\mathrm{x}_0 (\mathrm{y}_0 + 1)]$$

$$\mathrm{y}(0.2) = 1.0 + 0.1[(0.1)(2)] = 1.02.$$

Practice Problems

1. *Find y at x* = 0.1 *and x* = 0.2 *correct to three decimal places, given*
   $y' - 2y = 3e^x$, $y(0) = 0$
2. Use Taylor series method of order four to solve
   $y' = x^2 + y^2$, $y(0) = 1$
   for $x \in [0, 0.4]$ with $h = 0.2$

3. Find an approximation to $y(1.6)$, for the initial value problem
   $y' = x + y^2$, $y(1) = 1$
   using the Euler method with $h = 0.1$ and $h = 0.2$.
4. Given the initial value problem,
   $y' = 2x + \cos y$, $y(0) = 1$
   show that it is sufficient to use Euler method with step length $h = 0.2$ to compute $y(0.2)$ with an error less than 0.05.

Suggested books

1. Numerical Methods by **S.R.K Lyenger & R.K. Jain.**

2. Introductory methods of Numerical analysis by **S.S. Sastry**.

Thank you