Ex. 6. *Find the velocity with which a particle must be projected along the interior of a smooth vertical hoop of radius r, from the lowest point in order that it may leave the hoop at an angular distance of 30° from the vertical. Show that it will strike the hoop again at an extremity of the horizontal diameter.*

Show also that if velocity of projection be $\sqrt{\frac{7}{2}gr}$, the particle will leave the hoop and return to the lowest point.

The particle leaves the circle at an angle α from the upward drawn vertical where $\cos\alpha = \frac{1}{3}\left(\frac{u^2}{gr} - 2\right)$

Hence $\alpha = 30$, $\therefore \frac{\sqrt{3}}{2} = \frac{1}{3}\left(\frac{u^2}{gr} - 2\right)$.

$\therefore \qquad u^2 = \frac{1}{2}gr(3\sqrt{3}+4).$

Since $\angle BOQ = 30°$ $\quad \therefore \quad \angle BOD = 3 \cdot 30° = 90°$ *i.e.*, D is at the extremity of the horizontal diameter.

Where $u^2 = \frac{7}{2}gr$, $\cos\alpha = \frac{1}{3}(\frac{7}{2} - 2) = \frac{1}{2}$ $\quad \therefore \quad \alpha = 60$

i.e., $\angle BOQ = 60°$ Hence $\angle BOD = 180°$ *i.e.*, D coincides with A.

Ex. 7. *A particle is projected along the inside of a smooth vertical circle of radius r, from the lowest point. Show that the velocity of projection so that after leaving the circle the particle may pass through the centre is $\sqrt{\frac{1}{2}gr}[\sqrt{3}+1]$.*

The particle will leave the circle at some point Q at an angle α with the verticle with velocity v

where $$v^2=\frac{u^2-2gr}{3}=gr\cos\alpha$$

and $$\cos\alpha=\frac{1}{3}\left(\frac{u^2}{gr}-2\right)$$

With Q origin, the centre is the point $(r\sin\alpha, -r\cos\alpha)$ and by the question it lies on the parabola

$$y=x\tan\alpha-\frac{1}{2}\frac{gx^2}{v^2\cos^2\alpha}$$

i.e., $$y=x\tan\alpha-\frac{1}{2}\frac{gx^2}{gr\cos^3\alpha}.$$

Hence $$-r\cos\alpha=r\sin\alpha\tan\alpha-\frac{1}{2}\frac{r^2\sin^2\alpha}{r\cos^3\alpha}$$

i.e., $$\frac{\sin^2\alpha}{\cos\alpha}+\cos\alpha=\tfrac{1}{2}\frac{\sin^2\alpha}{\cos^3\alpha}\quad\text{or}\quad\tan^2\alpha=2$$

$\therefore$ $$\sec^2\alpha=3.\quad\text{Hence }\cos\alpha=\frac{1}{\sqrt{3}}$$

Therefore $$\frac{1}{\sqrt{3}}=\frac{1}{3}\left(\frac{u^2}{gr}-2\right)$$

$\therefore$ $$u^2=gr(2+\sqrt{3})$$

$\therefore$ $$u=\sqrt{\frac{gr}{2}}(\sqrt{3}+1).$$

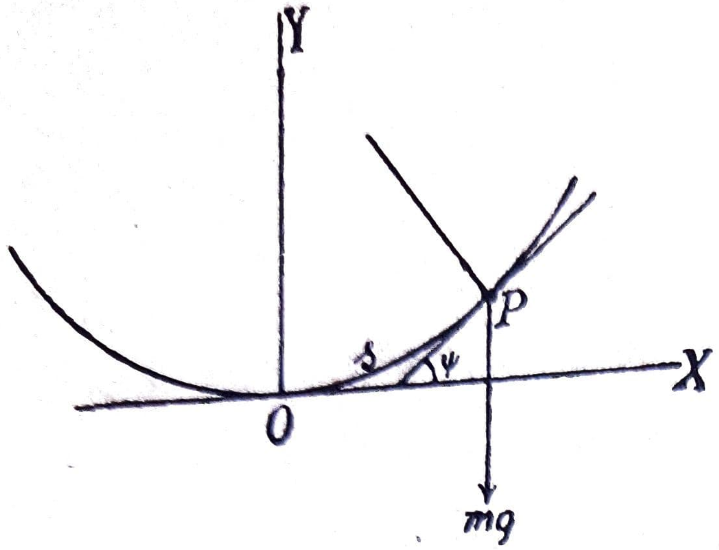
4. A heavy particle is suspended by a string from a fixed point and rotates in a vertical circle. Show that the sum of tension of the string when the particle is at the opposite ends of a diameter is the same for all diameters.

6. A heavy particle hangs from a point O by a string of length a. It is projected horizontally with a velocity v such that $v^2=(2+\sqrt{3})ag$. Show that the string becomes slack when it has described an angle

$$\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right).$$

7. A particle is hangi...

6·7. Cycloidal pendulum. We have seen that the motion of a simple pendulum is simple harmonic as long as the amplitude is small. For larger amplitudes the motion is not simple harmonic and consequently the period is different and cannot be said to be independent of the amplitude and variations in amplitude will cause irregularities in the time-keeping of a clock.



If a particle is constrained to move along the arc of smooth cycloid in a vertical plane, placed with vertex downwards and axis vertical, the tangential and normal equations of motion are

$$m\ddot{s}=-mg\sin\psi$$

and

$$\frac{m\dot{s}^2}{\rho}=R-mg\cos\psi$$

where s is measured from the vertex O, ψ the angle the tangent at any point P makes with the horizontal, R the reaction of the cycloid and ρ its radius of curvature at P. But the equation to the cycloid is $s=4a\sin\psi$.

Hence the first equation can be written as

$$\boxed{\ddot{s}=-\frac{g}{4a}s,}$$

which shows that the *motion is simple harmonic whatever be the amplitude and the time of a complete oscillation is* $2\pi\sqrt{\dfrac{4a}{g}}$. This property finds its application in the formation of clocks.

Let CAC' be a cycloidal arc and let CA' and $C'A'$ be the evolutes of CAC' which are also cycloids such that arc $C'A'=$ CA.

cycloidal arc CAC' and moves in true simple harmonic motion independent of the amplitude. The system is called a *Cycloidal Pendulum*.

Ex. 12, *A heavy particle slides down a smooth cycloid starting from rest at the cusp, the axis being vertical and vertex downwards. Prove that the magnitude of the acceleration of the particle is equal to g at every point of its path.*

Cycloid $s=4a\sin\psi$, $\rho=4a\cos\psi$

$$\ddot{s}=-g\sin\psi=-\frac{g}{4a}s$$

Integrating

$$\frac{\dot{s}^2}{2}=-\frac{g}{8a}s^2+C$$

when $\qquad s=4a,\ \dot{s}=0 \quad \therefore \quad C=2ag$

$$\therefore \quad \dot{s}^2=-\frac{g}{4a}s^2+4ag=4ag\cos^2\psi$$

If f be the acceleration

$$f^2=(\ddot{s})^2+\left(\frac{\dot{s}^2}{\rho}\right)^2=g^2\sin^2\psi+\left(\frac{4ag\cos^2\psi}{4a\cos\psi}\right)^2$$

$$=g^2\sin^2\psi+g^2\cos^2\psi=g^2$$

$$\therefore \quad f=g.$$

Ex. 13. *A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc ; show that the time of reaching the vertex is*

$$2\sqrt{\frac{a}{g}}\tan^{-1}\left(\frac{\sqrt{4ag}}{V}\right).$$

The equation of motion is $\ddot{s}=-\frac{g}{4a}s$.

Its solution is $s=A\cos\sqrt{\frac{g}{4a}}\,t+B\sin\sqrt{\frac{g}{4a}}\,t$.

Initially $t=0$, $s=4a$, $\dot{s}=-V$

Hence $4a=A$

also $\qquad \dot{s}=-A\sqrt{\frac{g}{4a}}\sin\sqrt{\frac{g}{4a}}\,t+B\sqrt{\frac{g}{4a}}\cos\sqrt{\frac{g}{4a}}\,t$

$$\therefore \quad -V=B\sqrt{\frac{g}{4a}}. \qquad \therefore\ B=-V\sqrt{\frac{4a}{g}}.$$

Ex 14

Hence $s=4a\cos\sqrt{\frac{g}{4a}}\,t-V\sqrt{\frac{4a}{g}}\sin\sqrt{\frac{g}{4a}}\,t$

when it comes to the vertex, $s=0$

$$\therefore \quad \tan\sqrt{\frac{g}{4a}}\,t=\frac{4a}{V\sqrt{\frac{4a}{g}}}=\frac{\sqrt{4ag}}{V}$$

$$\therefore \quad t=\sqrt{\frac{4a}{g}}\tan^{-1}\left(\frac{\sqrt{4ag}}{V}\right).$$

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Ex. 14. *A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest; prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.*

This equation of motion is $\ddot{s}=-\frac{g}{4a}s.$

$$\therefore \quad \text{period}=2\pi\sqrt{\frac{4a}{g}}=4\pi\sqrt{\frac{a}{g}}.$$

$$\therefore \quad \text{time from cusp to vertex}=\pi\sqrt{\frac{a}{g}}.$$

Integrating the above equation, we get

$$\dot{s}^2=-\frac{g}{4a}s^2+C \text{ where } C \text{ is a constant.}$$

Initially $s=4a$, $\dot{s}=0$ at the cusp $\quad\therefore C=4ag$

Hence $\dot{s}^2=4ag-\frac{g}{4a}s^2=\frac{g}{4a}(16a^2-s^2)$

$$\therefore \quad \sqrt{\frac{g}{4a}}\,dt=-\frac{ds}{\sqrt{16a^2-s^2}}$$

when $y=a$, we have since $s^2=8ay$ for a cycloid

$$s^2=8a^2 \qquad \therefore\ s=2a\sqrt{2}.$$

Hence time from cusp in falling through the first half of the vertical height is given by

$$\sqrt{\frac{g}{4a}}\,t=-\int_{4a}^{2a\sqrt{2}}\frac{ds}{\sqrt{16a^2-s^2}}=\int_{2a\sqrt{2}}^{4a}\frac{ds}{\sqrt{16a^2-s^2}}$$

$$=\left[\sin^{-1}\frac{s}{4a}\right]_{2a\sqrt{2}}^{4a}=\sin^{-1}1-\sin^{-1}\frac{1}{\sqrt{2}}$$

$$=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

$\therefore\qquad t=\dfrac{\pi}{2}\sqrt{\dfrac{a}{g}}=$ half the time from cusp to vertex.

Therefore, the remaining half of the vertical distance is also described in time $\dfrac{\pi}{2}\sqrt{\dfrac{a}{g}}$.

... ted from the vertex of a cycloid whose axis

Ex. 16. *Two particles are let drop from the cusp of a cycloid down the curve of an interval of time t ; prove that they will meet in a time*

$$2\pi\sqrt{\frac{a}{g}}+\frac{t}{2}.$$

Equation of motion $\qquad \ddot{s}=-\dfrac{g}{4a}s$

$\therefore$ Solution is $\qquad s=4a\cos\sqrt{\dfrac{g}{4a}}\,t$ since when $t=0$,

$s=4a$ and $\dot{s}=0$.

For the second particle $s=4a\cos\sqrt{\dfrac{g}{4a}}(t-t_1)$ if it starts after interval t_1.

Hence when they meet $\cos\sqrt{\dfrac{g}{4a}}\,t=\cos\sqrt{\dfrac{g}{4a}}\,(t-t_1)$.

$$\therefore \qquad \sqrt{\frac{g}{4a}}\,(t-t_1)=2\pi-\sqrt{\frac{g}{4a}}\,t$$

or
$$t-t_1=4\pi\sqrt{\frac{a}{g}}-t$$

or
$$2t=4\pi\sqrt{\frac{a}{g}}+t_1 \quad \therefore \quad t=2\pi\sqrt{\frac{a}{g}}+\frac{t_1}{2}.$$

EXAMPLES VI (E)

1. A particle falls down a cycloid starting from the cusp. Show that when it arrives at the vertex the pressure on the curve is equal to twice the weight of the particle.

2. When a particle falls from rest at the cusp down the arc of an erect cycloid, prove that it describes half the path to the lowest point in two-thirds the time to the lowest point.

3. A particle is moving on a smooth curve under gravity and its velocity varies as the distance (measured along the arc) from the highest point. Prove that the curve must be a cycloid.

4. If a particle slides down a smooth cycloid starting from a point whose arcual distance from the vertex is b, prove that its speed at any time t is $\dfrac{2\pi b}{T}\sin\dfrac{2\pi t}{T}$ where T is the time of complete oscillation of the particle.

5. A particle oscillates from cusp to cusp in a smooth cycloid whose axis is vertical and vertex lowest. Show that the velocity v at any point P is equal to the resolved part of the velocity V at the vertex along the tangent at P.

6. If a particle starts from rest at a given point of a cycloid with its axis vertical and vertex downwards, prove that it falls through $\frac{1}{n}$ of the vertical distance to the lowest point in the time $2\sqrt{\frac{a}{g}}\sin^{-1}\frac{1}{\sqrt{n}}$ where a is the radius of the generating circle.

7. Two particles connected by a fine string are constrained to move in a