2. CONTINUITY

A function f is said to be *continuous* at a point (a, b) of its domain of definition, if

$$\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b)$$

In other words, a function f is said to be *continuous* at a point (a, b) of its domain of definition if for $\varepsilon > 0$, there exists a neighbourhood N of (a, b) such that

$$|f(x, y) - f(a, b)| < \varepsilon, \quad \text{for all } (x, y) \in N$$

Note: The definition of continuity of a function f at a point (a, b) requires that besides (a, b), f is defined in a certain neighbourhood of (a, b) and moreover the limit of f when $(x, y) \to (a, b)$ exists and equals to the value $f(a, b)$.

A function which is not continuous at a point is said to be *discontinuous* there at.

Remark: A point to be particularly noticed is that if a function of more than one variable is continuous at a point, it is continuous at that point when considered as a function of a single variable. To be more specific if a function f of two variables x, y is continuous at (a, b) then $f(x, b)$ is a continuous function of x at $x = a$ and $f(a, y)$ that of y at $y = b$.

The converse however is not true, *i.e.*, a function may be a continuous function of one variable when the others remain constant and yet not be a continuous function of all the variables.

For instance, consider a function f, where

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y}, & (x, y) \neq (0, 0) \\ 0, & \text{at } (0, 0) \end{cases}$$

The function is not continuous at $(0, 0)$ for $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$ does not exist. But

$$\lim_{x \to 0} f(x, 0) = 0 = f(0, 0), \text{ and } \lim_{y \to 0} f(0, y) = 0 = f(0, 0)$$

so that f is continuous at $(0, 0)$, when considered as a function of a single variable x or that of y.

A function is said to be continuous in a region if it is continuous at every point of the same.

As in limits, it can be easily proved that the sum, difference, product and quotient (provided the denominator does not vanish) of two continuous functions are also continuous.

The theorems on continuity for functions of a single variable can be easily extended to functions of several variables; the proofs for some of them, except for verbal changes, are the same while for others the method is not quite the same. However, within the scope of the present work, it is not possible to discuss all of them here.

Example 7. Investigate the continuity at $(0, 0)$ of

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

■ Since $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$ does not exist, therefore the function is not continuous at $(0, 0)$.

Example 8. Investigate for continuity at $(1, 2)$

$$f(x, y) = \begin{cases} x^2 + 2y, & (x, y) \neq (1, 2) \\ 0, & (x, y) = (1, 2) \end{cases}$$

■ Here

$$\lim_{(x, y) \to (1, 2)} f(x, y) = 5 \neq f(1, 2).$$

Hence, the function is not continuous at $(1, 2)$.

The point $(1, 2)$ is a *point of discontinuity* of the function.

However, if the function has the value 5 at $(1, 2)$, it was then continuous at the point.

Remark: If, as in the above example, it is possible to so redefine the value of the function at a point of discontinuity that the new function is continuous, we say that the point is a *removable discontinuity* of the original function.

Example 9. Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at the origin.

■ Let $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \quad \left|\frac{xy}{\sqrt{x^2 + y^2}}\right| = r\,|\cos\theta\,\sin\theta| \leq r = \sqrt{x^2 + y^2} < \varepsilon,$$

if

$$x^2 < \frac{\varepsilon^2}{2},\ y^2 < \frac{\varepsilon^2}{2}$$

or, if

$$|x| < \frac{\varepsilon}{\sqrt{2}},\ |y| < \frac{\varepsilon}{\sqrt{2}}$$

Thus

$$\left|\frac{xy}{\sqrt{x^2 + y^2}} - 0\right| < \varepsilon, \text{ when } |x| < \frac{\varepsilon}{\sqrt{2}}, |y| < \frac{\varepsilon}{\sqrt{2}}$$

$$\Rightarrow \quad \lim_{(x, y)\to(0, 0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0$$

$$\therefore \quad \lim_{(x, y)\to(0, 0)} f(x, y) = f(0, 0)$$

Hence, f is continuous at (0, 0).

EXERCISE

1. Show that the following functions are discontinuous at the origin:

(i) $f(x, y) = \begin{cases} \dfrac{1}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(ii) $f(x, y) = \dfrac{x^4 - y^4}{x^4 + y^4}$, $(x, y) \neq (0, 0)$, $f(0, 0) = 0$

(iii) $f(x, y) = \dfrac{(x^2 y^2)}{(x^4 + y^4)}$, $(x, y) \neq (0, 0)$, $f(0, 0) = 0$

2. Show that the following functions are continuous at the origin:

(i) $f(x, y) = \dfrac{x^2 y^2}{(x^2 + y^2)}$, $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.

(ii) $f(x, y) = \begin{cases} \dfrac{x^3 y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

3. Show that the following functions are discontinuous at (0, 0).

(i) $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(ii) $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$

(iii) $f(x, y) = \dfrac{xy^3}{x^2 + y^6}$, $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.

4. Discuss the following functions for continuity at (0, 0).

(i) $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^3 + y^3}, & x^2 + y^2 \neq 0 \\ 0, & x + y = 0 \end{cases}$

(ii) $f(x, y) = \begin{cases} 2xy\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(iii) $f(x, y) = \begin{cases} 0, & (x, y) = (2y, y) \\ \exp\{|x - 2y|/(x^2 - 4xy + 4y^2)\}, & (x, y) \neq (2y, y). \end{cases}$

5. Show that f has a removable discontinuity at (2, 3):

$$f(x, y) = \begin{cases} 3xy, & (x, y) \neq (2, 3) \\ 6, & (x, y) = (2, 3) \end{cases}$$

Suitably redefine the function to make it continuous.

6. Show that the function f is continuous at the origin, where

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

7. Can the given functions be appropriately defined at (0, 0) in order to be continuous there?

(*i*) $f(x, y) = |x|^y$, (*ii*) $f(x, y) = \sin\dfrac{x}{y}$,

(*iii*) $f(x, y) = \dfrac{x^3 + y^3}{x^2 + y^2}$, (*iv*) $f(x, y) = x^2 \log(x^2 + y^2)$.

3. PARTIAL DERIVATIVES

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant is called the *partial derivative* of the function with respect to the variable. Partial derivative of $f(x, y)$ with respect to x is generally denoted by $\partial f/\partial x$ or f_x or $f_x(x, y)$, while those with respect to y are denoted by $\partial f/\partial y$ or f_y or $f_y(x, y)$.

$$\therefore \quad \frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

and

$$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

when these limits exist.

The partial derivatives at a particular point (a, b) are often denoted by

$$\left[\frac{\partial f}{\partial x}\right]_{(a, b)}, \quad \frac{\partial f(a, b)}{\partial x} \quad \text{or} \quad f_x(a, b)$$

and

$$\left[\frac{\partial f}{\partial y}\right]_{(a, b)}, \quad \frac{\partial f(a, b)}{\partial y} \quad \text{or} \quad f_y(a, b)$$

$$\therefore \quad f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

$$f_y(a, b) = \lim_{k \to 0} \frac{f(a, b + k) - f(a, b)}{k}$$

in case the limit exists.

Example 10. If $f(x, y) = 2x^2 - xy + 2y^2$, then find $\partial f/\partial x$ and $\partial f/\partial y$ at the point (1, 2).

▪ Now

$$\frac{\partial f}{\partial x} = 4x - y = 2, \text{ at } (1, 2)$$

$$\frac{\partial f}{\partial y} = -x + 4y = 7, \text{ at } (1, 2)$$

CHAPTER 15