5.2 Sufficient Conditions for the Equality of f_{xy} and f_{yx}

As was said earlier there is no *a priori* reason why f_{xy} and f_{yx} should always be equal. We now give two theorems the object of which is to set out precisely under what conditions it is safe to assume that $f_{xy} = f_{yx}$ at a point, *i.e.*, **sufficient conditions** for the equality of f_{xy} and f_{yx}.

Theorem 3. ***Young's theorem.*** *If f_x and f_y are both differentiable at a point (a, b) of the domain of definition of a function f, then*

$$f_{xy}(a, b) = f_{yx}(a, b)$$

The differentiability of f_x and f_y at (a, b) implies that they exist in a certain neighbourhood of (a, b) and that all the second order partial derivatives $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ exist at (a, b).

We prove the theorem by taking equal increment h both for x and y and calculating $\phi(h, h)$ in two different ways.

Let $(a+h, b+h)$ be a point of this neighbourhood. Consider

$$\phi(h, h) = f(a+h, b+h) - f(a+h, b) - f(a, b+h) + f(a, b)$$
$$G(x) = f(x, b+h) - f(x, b)$$

so that

$$\phi(h, h) = G(a+h) - G(a) \quad \text{...(1)}$$

Since f_x exists in a neighbourhood of (a, b), the function $G(x)$ is derivable in $]a, a+h[$ and therefore by Lagrange's mean value theorem, we get from (1),

$$\phi(h, h) = hG'(a+\theta h), \quad 0 < \theta < 1$$
$$= h\{f_x(a+\theta h, b+h) - f_x(a+\theta h, b)\} \quad \text{...(2)}$$

Again, since f_x is differentiable at (a, b), we have

$$f_x(a+\theta h, b+h) - f_x(a, b) = \theta h f_{xx}(a, b) + h f_{yx}(a, b) + \theta h \phi_1(h, h) + h\psi_1(h, h) \quad \text{...(3)}$$

and

$$f_x(a+\theta h, b) - f_x(a, b) = \theta h f_{xx}(a, b) + \theta h \phi_2(h, h) \quad \text{...(4)}$$

where ϕ_1, ψ_1, ϕ_2 all tend to zero as $h \to 0$.

From equations (2), (3), and (4), we get

$$\phi(h, h)/h^2 = f_{yx}(a, b) + \theta\phi_1(h, h) + \psi_1(h, h) - \theta\phi_2(h, h) \quad \text{...(5)}$$

By a similar argument, on considering

$$H(y) = f(a+h, y) - f(a, y)$$

we can show that

$$\phi(h, h)/h^2 = f_{xy}(a, b) + \phi_3(h, h) + \theta'\psi_2(h, h) - \theta'\psi_3(h, h) \quad \text{...(6)}$$

where ϕ_3, ψ_2, ψ_3 all tend to zero as $h \to 0$.

On taking the limit as $h \to 0$, we obtain from equations (5) and (6)

$$\lim_{h \to 0} \frac{\phi(h, h)}{h^2} = f_{xy}(a, b) = f_{yx}(a, b).$$

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Theorem 4. ***Schwarz's theorem.*** *If f_y exists in a certain neighbourhood of a point (a, b) of the domain of definition of a function f, and f_{yx} is continuous at (a, b), then $f_{xy}(a, b)$ exists and is equal to $f_{yx}(a, b)$.*

Under the given conditions, f_x, f_y, and f_{yx} all exist in a certain neighbourhood of (a, b). Let $(a + h, b + k)$ be a point of this neighbourhood.

Consider

$$\phi(h, k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$$
$$G(x) = f(x, b+k) - f(x, b)$$

so that

$$\phi(h, k) = G(a+h) - G(a) \quad \text{...(1)}$$

Since f_x exists in a neighbourhood of (a, b), the function $G(x)$ is derivable in $]a, a+h[$, and therefore by Lagrange's mean value theorem, we get from (1)

$$\phi(h, k) = hG'(a+\theta h), \quad 0 < \theta < 1$$
$$= h\{f_x(a+\theta h, b+k) - f_x(a+\theta h, b)\} \quad \text{...(2)}$$

Again, since f_{yx} exists in a neighbourhood of (a, b), the function f_x is derivable with respect to y in $]b, b+k[$, and therefore by Lagrange's mean value theorem, we get from (2)

$$\phi(h, k) = hkf_{yx}(a+\theta h, b+\theta' k), \quad 0 < \theta' < 1$$

or

$$\frac{1}{h}\left\{\frac{f(a+h, b+k) - f(a+h, b)}{k} - \frac{f(a, b+k) - f(a, b)}{k}\right\} = f_{yx}(a+\theta h, b+\theta' k)$$

Proceeding to limits when $k \to 0$, since f_y and f_{yx} exist in a neighbourhood of (a, b), we get

$$\frac{f_y(a+h, b) - f_y(a, b)}{h} = \lim_{k \to 0} f_{yx}(a+\theta h, b+\theta' k)$$

Again, taking limits as $h \to 0$, since f_{yx} is continuous at (a, b), we get

$$f_{xy}(a, b) = \lim_{h \to 0} \lim_{k \to 0} f_{yx}(a+\theta h, b+\theta' k) = f_{yx}(a, b)$$

(See ...

Example 17. Show that for the function

$$f(x, y) = \begin{cases} \dfrac{x^2 y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$f_{xy}(0, 0) = f_{yx}(0, 0)$, even though the conditions of Schwarz's theorem and also of Young's theorem ar[e] not satisfied.

Now

$$f_x(0, 0) = \lim_{x \to 0} \frac{f(x, 0) - f(0, 0)}{x} = 0$$

Similarly, $f_y(0, 0) = 0$.

Also, for $(x, y) \neq (0, 0)$,

$$f_x(x, y) = \frac{(x^2 + y^2) \cdot 2xy^2 - x^2y^2 \cdot 2x}{(x^2 + y^2)^2} = \frac{2xy^4}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{2x^4y}{(x^2 + y^2)^2}$$

Again

$$f_{yx}(0, 0) = \lim_{y \to 0} \frac{f_x(0, y) - f_x(0, 0)}{y} = 0$$

and

$f_{xy}(0, 0) = 0$, so that $f_{xy}(0, 0) = f_{yx}(0, 0)$

For $(x, y) \neq (0, 0)$, we have

$$f_{yx}(x, y) = \frac{8xy^3(x^2 + y^2)^2 - 2xy^4 \cdot 4y(x^2 + y^2)}{(x^2 + y^2)^4} = \frac{8x^3y^3}{(x^2 + y^2)^3}$$

and it may be easily shown (by putting $y = mx$) that

$$\lim_{(x, y) \to (0, 0)} f_{yx}(x, y) \neq 0 = f_{yx}(0, 0)$$

so that f_{yx} is not continuous at $(0, 0)$, *i.e.*, the conditions of Schwarz's theorem are not satisfied.

Let us now show that the conditions of Young's theorem are also not satisfied.

Now

$$f_{xx}(0, 0) = \lim_{x \to 0} \frac{f_x(x, 0) - f_x(0, 0)}{x} = 0$$

Also f_x is differentiable at $(0, 0)$ if

$$f_x(h, k) - f_x(0, 0) = f_{xx}(0, 0) \cdot h + f_{yx}(0, 0) \cdot k + h\phi + k\psi$$

or

$$\frac{2hk^4}{(h^2 + k^2)^2} = h\phi + k\psi$$

where ϕ, ψ tend to zero as $(h, k) \to (0, 0)$.

Putting $h = \rho \cos\theta$ and $k = \rho \sin\theta$, and dividing by ρ, we get

$$2 \cos\theta \sin^4\theta = \cos\theta \cdot \phi + \sin\theta\psi$$

and $(h, k) \to (0, 0)$ is same thing as $\rho \to 0$ and θ is arbitrary. Thus proceeding to limits, we get

$$2 \cos\theta \sin^4\theta = 0$$

which is impossible for arbitrary θ.

$\Rightarrow \quad f_x$ is not differentiable at (0, 0)

Similarly, it may be shown that f_y is not differentiable at (0, 0).

Thus the conditions of Young's theorem are also not satisfied but, as shown above,

$$f_{xy}(0, 0) = f_{yx}(0, 0).$$

Example 18. Show that the function

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$$

$$f(0, 0) = 0$$

does not satisfy the conditions of Schwarz's theorem and

$$f_{xy}(0, 0) \neq f_{yx}(0, 0)$$

■ It may be shown, as in example 15, that

$$f_{xy}(0, 0) = 1, \; f_{yx}(0, 0) = -1$$

so that

$$f_{xy}(0, 0) \neq f_{yx}(0, 0)$$

Now, for $(x, y) \neq (0, 0)$ we have

$$f_x(x, y) = \frac{(x^2 + y^2)y(3x^2 - y^2) - 2x^2y(x^2 - y^2)}{(x^2 + y^2)^2} = \frac{y\{x^4 + 4x^2y^2 - y^4\}}{(x^2 + y^2)^2}$$

$$\therefore \quad f_{yx}(x, y) = \frac{(x^2 + y^2)^2\{x^4 + 12x^2y^2 - 5y^4\} - 4y^2(x^2 + y^2)\{x^4 + 4x^2y^2 - y^4\}}{(x^2 + y^2)^4}$$

$$= \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3}.$$

By putting $y = mx$ or $x = r\cos\theta, \; y = r\sin\theta$, it may be shown that

$$\lim_{(x, y) \to (0, 0)} f_{yx}(x, y) \neq -1 = f_{yx}(0, 0).$$

Thus f_{yx} is not continuous at (0, 0).

It may similarly be shown that f_{xy} is also not continuous at (0, 0).

Thus, the conditions of Schwarz's theorem are not satisfied.