Virtual Work

9·1. Position of equilibrium. We now come to a very powerful method of attacking problems on equilibrium. We begin by considering a heavy particle on a smooth curve, as shown in the figure (Fig 82).

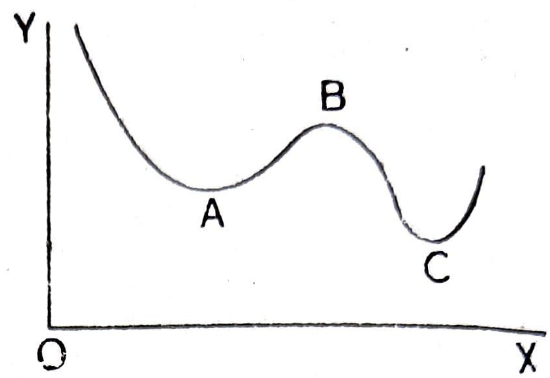

Fig. 82

If the axis of y is vertical it is obvious that the particle can rest in equilibrium at the points A, B or C. These are points of maxima or minima on the curve, i.e., points for which $dy/dx=0$.

The physical interpretation of the positions of equilibrium may be given as follows. If a small displacement be given along the curve to the particle from a position of equilibrium the work done is zero. For if W be the weight of the particle, the work done is $W\ \delta y$, i e., $W \dfrac{dy}{dx}\ \delta x$ (to a first approximation). But the positions of equilibrium are given by $dy/dx=0$.

It follows that $W\ \delta y=0$.

It should be noted that δy is equal to $(dy/dx)\delta x$ to a first approximation only. For, if $y=f(x)$ be the equation to the curve,

$$y+\delta y=f(x+\delta x)=f(x)+\delta x\, f'(x)+\frac{(\delta x)^2}{2\,!}\, f''(x)+\dots,$$

i.e.

$$\delta y=\delta x\,\frac{dy}{dx}+\frac{(\delta x)^2}{2\,!}\,\frac{d^2y}{dx^2}+\dots.$$

Many problems on equilibrium in Statics can be reduced to the above type. As an illustration consider the following problem.

9·2. Method of virtual work. The method of virtual work is as follows. In any given problem, we imagine the body to be displaced a little and then find the work done during the displacement. The condition of equilibrium is obtained by equating to zero the toal sum of the works done.

Since the body is not actually displaced, the work done is called *virtual work*. The virtual work that is calculated is the amount of work that would have been done if the displacements had actually been made.

We shall now formally enunciate the principle of virtual work and establish the same. S nce the proof is simpler for the case of a particle acted upon by a number of forces, we consider this first. In § 9·5, the case of rigid body will be dealt with.

9·3. Principle of virtual work for a system of coplanar forces acting on a particle. *The necessary and sufficient condition that a particle acted upon by a number of coplanar forces, be in equilibrium is that the sum of the virtual works done by the forces in a small displacement, consistent with the geometrical conditions of the system, is zero.*

Kanpur, B.A. 1984, *B.Sc.* 1985; *Gorakhpur*, 1983; *Bundelkhand* 1984]

Let any number of forces F_1, F_2, F_3, ... act on the particle whose actual position is given by O (Fig. 83).

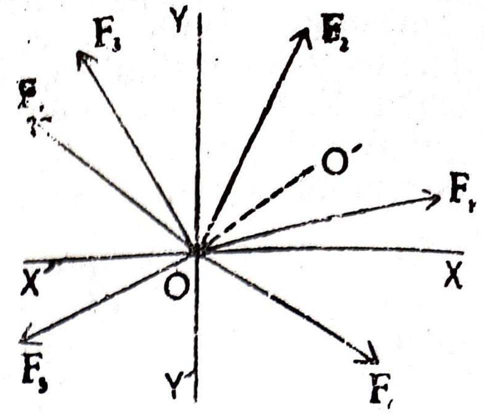

Fig. 83

Through O draw two rectangular axes OX and OY. Let the components of the force F_1 along OX and OY be X_1 and Y_1, of the force F_2, X_2 and Y_2, and so on.

We shall first show the necessity of the condition. In other words, we shall show that if a system of coplanar forces acting on a particle be in equilibrium and the particle undergoes a small displacement consistent with the geometrical conditions of the system, the algebraic sum of the virtual works done by the different forces is zero.

Suppose that OO' represents an arbitrary small displacement and that the coordinates of O' are $(\alpha, \beta,)$. Since, by § 7·2, the work done by a force is equal to the algebraic sum of the works done by its components, the virtual work of the force F_1 is

$$X_1\alpha+Y_1\beta.$$

Similarly the virtual works of the other forces F_2, F_3, etc. are

$$X_2\alpha+Y_2\beta,\ X_3\alpha+Y_3\beta, \text{ etc.}$$

Hence the algebraic sum of the virtual works done by the different forces is

$$(X_1\alpha+Y_1\beta)+(X_2\alpha+Y_2\beta)+(X_3\alpha+Y_3\beta)+\ldots,$$

i.e.

$$\alpha\Sigma X_1+\beta\Sigma Y_1.$$

Now the particle is in equilibrium under the action of the forces F_1, F_2, F_3, ... so that $\Sigma X_1=0$ and $\Sigma Y_1=0$ (see § 2·6).

Therefore

$$\alpha\Sigma X_1+\beta\Sigma Y_1=0.$$

Thus the condition that the algebraic sum of the virtual works done by the different forces is zero is necessary.

To establish the sufficiency of the condition we have to show that if the algebraic sum of the virtual works done by a system of coplanar forces acting on a particle be zero for all arbitrary small displacements, the particle is in equilibrium. Since for different displacements we will get different values for α and β, it is given here that for all small values of α and β,

$$\alpha\Sigma X_1+\beta\Sigma Y_1=0. \qquad \ldots \quad (1)$$

Now α and β are independent of each other. Suppose that α' and β are the projections of a new small displacement on the axes of x and y. Then we shall have

$$\alpha' \Sigma X_1 + \beta \Sigma Y_1 = 0. \quad \ldots \quad (2)$$

Subtracting (2) from (1), we get $(\alpha - \alpha')\Sigma X_1 = 0$.

But $\alpha - \alpha'$ is not zero, therefore $\Sigma X_1 = 0$.

In the same way by taking a displacement which has for its component displacements α, β' along the coordinate axes, we can show that $\Sigma Y_1 = 0$. Hence $\Sigma X_1 = 0$ and $\Sigma Y_1 = 0$ and the particle is in equilibrium.

The equation (1) is known as *the equation of virtual work*.

Suppose that a force F acts on a