B.Sc. Mathematics – 2nd Semester

MTB 202 – Statics and Dynamics

by

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Part – V

Central Orbit and Kepler's Laws

Let us suppose that the force F acting on a particle of mass m has the following characteristics –

(i) It is always directed towards a fixed point.

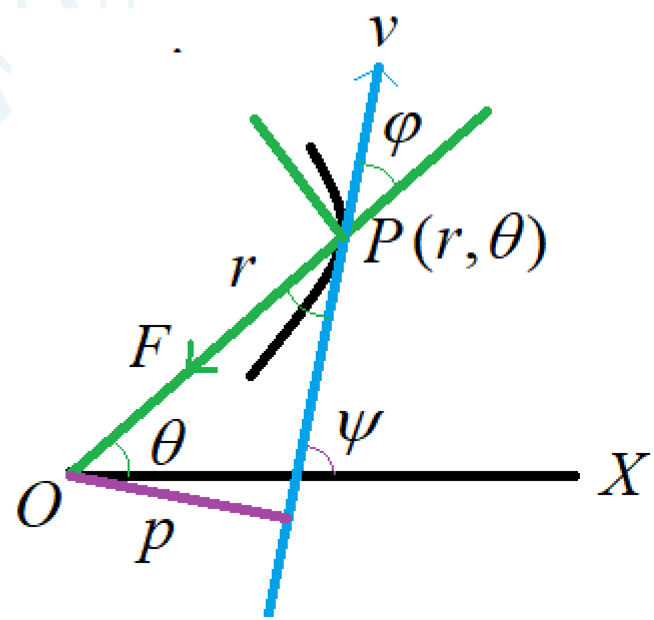
(ii) The magnitude of the force F is a force of distance r of the particle from the fixed point.

Such a force is called central force and the fixed point is called the centre of force. The path is described by the particle is called a central orbit.

Equation of Motion (for a Central Orbit):

Let O be the centre of force. Let OX be a fixed line through O. Let P be the position of the particle at any time t. With O as pole and OX as initial line let (r,θ) be the coordinates of P.

Let F be the central force per unit mass acting along the line PO.



There is no force acting along the cross radial direction. Hence the equation of motion of the particle along radial and transverse directions are

$$m\left\{\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right\}=-mF\text{, i.e., }\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=-F, \qquad (1)$$

$$m\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0\text{, i.e., }\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0, \qquad (2)$$

where m is the mass of the particle.

Integrating (2) we have

$$r^2\frac{d\theta}{dt}=\text{constant}=h\ \text{(say)} \qquad (3)$$

Case-I: Equation in r and θ

We have $\dot{r} = \frac{dr}{d\theta}.\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta}$

$$\ddot{r} = \frac{d}{dt}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right) = \frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)\dot{\theta}$$

$$= \frac{h}{r^2}\left[\frac{d}{dr}\left(\frac{h}{r^2}\right)\left(\frac{dr}{d\theta}\right)^2 + \frac{h}{r^2}\frac{d^2r}{d\theta^2}\right] = -\frac{2h^2}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{h^2}{r^4}\frac{d^2r}{d\theta^2}$$

Then from (1) we have

$$\frac{h^2}{r^4}\frac{d^2r}{d\theta^2} - \frac{2h^2}{r^5}\left(\frac{dr}{d\theta}\right)^2 - r\frac{h^2}{r^4} = -F$$

$$\Rightarrow h^2 r\frac{d^2r}{d\theta^2}-2h^2\left(\frac{dr}{d\theta}\right)^2-r^2h^2=-r^5F \qquad (4)$$

<u>Case-II: Equation in u and θ</u>

Let $u=\dfrac{1}{r}$, i.e., $r=\dfrac{1}{u}$,

Then from (3) we get $\dot{\theta}=hu^2$ (5)

Now $\dot{r}=\dfrac{dr}{d\theta}\dfrac{d\theta}{dt}=\dfrac{d}{d\theta}\left(\dfrac{1}{u}\right)\dfrac{d\theta}{dt}=-\dfrac{1}{u^2}\dfrac{du}{d\theta}\dfrac{d\theta}{dt}=-\dfrac{1}{u^2}\dfrac{du}{d\theta}hu^2=-h\dfrac{du}{d\theta}$

$$\ddot{r}=-h\frac{d}{dt}\left(\frac{du}{d\theta}\right)=-h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}=-h\frac{d^2u}{d\theta^2}hu^2=-h^2u^2\frac{d^2u}{d\theta^2}$$

Then from (1) we get

$$-h^2u^2\frac{d^2u}{d\theta^2}-\frac{1}{u}h^2u^4=-F\Rightarrow h^2u^2\frac{d^2u}{d\theta^2}+h^2u^3=F$$

$$\Rightarrow u^2\left(u+\frac{d^2u}{d\theta^2}\right)=\frac{F}{h^2},\ \text{Or,}\ u+\frac{d^2u}{d\theta^2}=\frac{F}{h^2u^2} \tag{6}$$

This equation of motion of central orbit is used in practical purpose.

Pedal Equation of Central Orbit:

Let the perpendicular from O on the tangent on the orbit at P be ON and let $ON=p$

Then $p = r\sin\phi$ and $\tan\phi = r\dfrac{d\theta}{dr}$

So, $$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}\left(1+\cot^2\phi\right) = \frac{1}{r^2}\left(1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right)$$

i.e., $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

We know $r=\dfrac{1}{u}$ then $\dfrac{dr}{d\theta}=-\dfrac{1}{u^2}\dfrac{du}{d\theta}$.

Then we have $\dfrac{1}{p^2}=u^2+\left(\dfrac{du}{d\theta}\right)^2$.

Now, differentiating both sides with respect to θ we get

$$-\frac{2}{p^3}\frac{dp}{dr}\frac{dr}{d\theta}=2u\frac{du}{d\theta}+2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}$$

$$\Rightarrow -\frac{2}{p^3}\frac{dp}{dr}\left(-\frac{1}{u^2}\frac{du}{d\theta}\right)=2\frac{du}{d\theta}\left(u+\frac{d^2u}{d\theta^2}\right)$$

$$\Rightarrow \frac{1}{p^3}\frac{dp}{dr} = u^2\left(u + \frac{d^2u}{d\theta^2}\right) \Rightarrow \frac{1}{p^3}\frac{dp}{dr} = \frac{F}{h^2}$$

$$\Rightarrow \frac{h^2}{p^3}\frac{dp}{dr} = F \tag{7}$$

Linear velocity at a position:

We know that the angular velocity of a particle is given by $\dot{\theta} = \frac{vp}{r^2}$

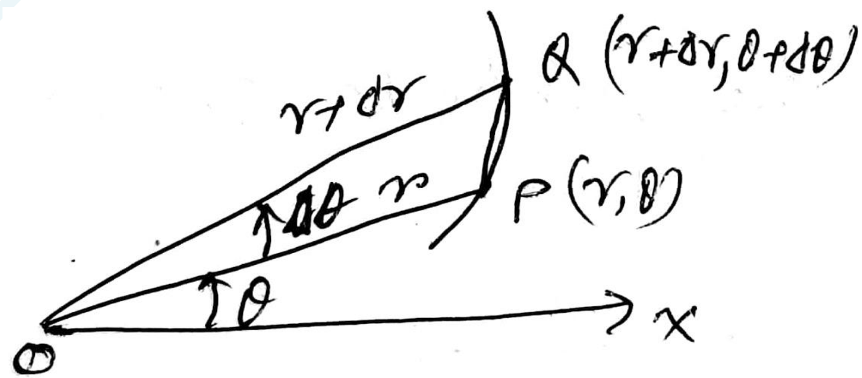
$\Rightarrow \frac{h}{r^2} = \frac{vp}{r^2} \Rightarrow v = \frac{h}{p} \Rightarrow v \propto \frac{1}{p}$. We also have $v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$.

Significance of *h*:

In a central orbit the sectorial area traced out by the radius vector through the centre of force to the particle per unit time is constant and equal to $\frac{1}{2}h$. (The rate of description of the sectorial area)

Proof: Let $P(r,\theta)$ and $Q(r+\Delta r,\theta+\Delta\theta)$ be the positions of the moving particle on the central orbit at time t and $t+\Delta t$ respectively.

The radius vector OP has described the sectorial area OPQ



in time Δt. As $\Delta t \to 0, \Delta\theta \to 0$ and $\Delta r \to 0$ and arc $PQ \to$ chord PQ.

Then the rate of description of the sectorial area

$$= \lim_{\Delta t \to 0} \frac{\text{area of the sector OPQ}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\text{area of the triangle OPQ}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{1}{2} r\left(r + \Delta r\right) \sin\Delta\theta}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{1}{2} r\left(r + \Delta r\right) \frac{\sin\Delta\theta}{\Delta\theta} \frac{\Delta\theta}{\Delta t}\right)$$

$$= \frac{1}{2} \lim_{\Delta t \to 0} r\left(r + \Delta r\right) \lim_{\Delta\theta \to 0} \frac{\sin\Delta\theta}{\Delta\theta} \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{1}{2} r\left(r + 0\right).1.\frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h$$

The rate of describing the sectorial area by the radius vector to the particle moving under a centre force is called areal velocity.

Therefore, $h = 2\times$ rate of describing the sectorial area

$$= 2\times \text{ areal velocity}$$

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Angular momentum of the particle is the moment of momentum of the particle about the pole (or origin).

The radial and transverse components of momentum are $m\dot{r}$ and $mr\dot{\theta}$.

Therefore, angular momentum = moment of momentum

$$= r.mr\dot{\theta} = mr^2\dot{\theta} = mh$$

Thus, $h =$ Angular momentum per unit mass

Hence, the angular momentum of a particle moving under central force is constant.

<u>Some cases when polar equation of the central orbit is not given</u>:

(I) <u>Law of force is given</u>

<u>Suppose</u> $F \propto \dfrac{1}{r^7} \Rightarrow u + \dfrac{d^2u}{d\theta^2} \propto u^5$

Integrating, $u^2 + \left(\dfrac{du}{d\theta}\right)^2 \propto u^6 \Rightarrow v^2 \propto u^6 \Rightarrow v \propto u^3$.

So, $p \propto r^3 \Rightarrow p = ar^3$.

(II) *Law of velocity is given*

Suppose $v \propto \frac{1}{r^3} \Rightarrow u^2 + \left(\frac{du}{d\theta}\right)^2 \propto u^6$

Differentiate w.r.t. θ

$$\Rightarrow u + \frac{d^2u}{d\theta^2} \propto u^5 \Rightarrow \frac{F}{u^2} \propto u^5 \Rightarrow F \propto u^5 \Rightarrow F \propto \frac{1}{r^7}$$

Also, $p \propto r^3 \Rightarrow p = ar^3$.

(III) *Pedal Equation is given*

Suppose $a^2 p = r^3 \Rightarrow p \propto r^3 \Rightarrow v \propto u^3$ and so on.

Polar Equation of the central orbit is given:

1) Elliptic Orbit:

Let the central orbit be an ellipse whose equation is

$$\frac{l}{r}=1+e\cos\theta,$$

where $l=\frac{b^2}{a}$ is the semi-latus rectum, a is the semi-major axis, b is the semi-minor axis and $b^2=a^2\left(1-e^2\right)$, i.e., $e^2=1-\frac{b^2}{a^2},\left(e<1\right)$ is eccentricity.

We have $u = \dfrac{1+ecos\theta}{l}$.

So, $\dfrac{du}{d\theta} = -\dfrac{e\sin\theta}{l}$ and $\dfrac{d^2u}{d\theta^2} = -\dfrac{e\cos\theta}{l}$.

From equation of central orbit, we have

$$\frac{F}{h^2u^2} = \frac{1}{l} \Rightarrow F = \frac{h^2}{l}u^2 = \frac{\mu}{r^2} \text{ where } \mu = \frac{h^2}{l}, \qquad \left[u + \frac{d^2u}{d\theta^2} = \frac{F}{h^2u^2}\right]$$

where h is angular momentum per unit mass.

Thus, the central force for an elliptic varies inversely as the square of radius vector.

We have $v^2 = h^2\left[u^2 + \left(\dfrac{du}{d\theta}\right)^2\right]$

$$v^2 = \mu l\left[\left(\frac{1+e\cos\theta}{l}\right)^2 + \left(-\frac{e\sin\theta}{l}\right)^2\right]$$

$$= \frac{\mu l}{l^2}\left[1+2e\cos\theta + e^2\cos^2\theta + e^2\sin^2\theta\right]$$

$$= \frac{\mu}{l}\left[1+2e\cos\theta + e^2\right]$$

$$= \frac{\mu}{l}\left[1+2e\cos\theta + 1 - \frac{b^2}{a^2}\right]$$

$$= \mu\left[\frac{2(1+e\cos\theta)}{l} - \frac{b^2}{a^2 l}\right] = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

$\Rightarrow v^2 < \dfrac{2\mu}{r}$ and the linear velocity varies inversely to square root of radius vector of that position.

We have $v^2 = \dfrac{h^2}{p^2} = \mu\left[\dfrac{2}{r} - \dfrac{1}{a}\right]$

$$\Rightarrow \frac{\mu l}{p^2} = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

$$\Rightarrow \frac{b^2/a}{p^2} = \frac{2}{r} - \frac{1}{a} \Rightarrow \frac{b^2}{p^2} = \frac{2a}{r} - 1.$$

This is the pedal equation of the ellipse.

2) *Hyperbolic orbit:*

$$\frac{l}{r} = 1 + e\cos\theta, e > 1, \ l = \frac{b^2}{a}, \ e^2 = 1 + \frac{b^2}{a^2}$$

Similarly, law of force $F \propto \frac{1}{r^2} = \frac{\mu}{r^2}, h^2 = \mu l$

The velocity $v^2 = \mu\left(\frac{2}{r}+\frac{1}{a}\right) > \frac{2\mu}{r}$ and the pedal equation $\frac{b^2}{p^2} = \frac{2a}{r}+1$.

3) ***Parabolic orbit:***

$\frac{l}{r}=1+e\cos\theta$, $e=1$, $l=2a$. So the equation becomes $\frac{2a}{r}=1+\cos\theta$.

Law of force: $F=\frac{\mu}{r^2}$, with $h^2=\mu l=2a\mu$

So, the velocity: $v^2=\frac{2\mu}{r}$

The pedal equation: $\frac{h^2}{p^2}=\frac{2\mu}{r} \Rightarrow \frac{2a\mu}{p^2}=\frac{2\mu}{r} \Rightarrow p^2=ar$

So, for $v^2 < \dfrac{2\mu}{r}$, central orbit is ellipse

$v^2 > \dfrac{2\mu}{r}$, central orbit is hyperbola

$v^2 = \dfrac{2\mu}{r}$, central orbit is parabola

where $\mu = \dfrac{h^2}{l}$.

4) The curve is $r^n = a^n \cos n\theta$, where a is a parameter of the curve:

Putting $u = \frac{1}{r}$ i.e. $r = \frac{1}{u}$ we have, $1 = a^n u^n \cos n\theta$

Taking logarithm and then differentiating w.r.t θ we get

$$n \log u + \log \cos n\theta + n \log a = 0$$

$$\frac{n}{u}\frac{du}{d\theta} - n \tan n\theta = 0 \Rightarrow \frac{du}{d\theta} = u \tan n\theta$$

So, $\dfrac{d^2u}{d\theta^2} = u\left(\tan^2 n\theta + n \sec^2 n\theta\right)$.

Then from the equation of central orbit we get

$$F = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2\left[u + u\left(\tan^2 n\theta + n\sec^2 n\theta\right)\right]$$

$$= h^2u^3\left[1 + \tan^2 n\theta + n\sec^2 n\theta\right]$$

$$= h^2u^3(n+1)\left(a^n u^n\right)^2$$

$$= h^2a^{2n}(n+1)u^{2n+3} = \frac{(n+1)h^2a^{2n}}{r^{2n+3}}$$

Therefore, law of force $F \propto \frac{1}{r^{2n+3}}$, where $\mu = (n+1)h^2a^{2n}$.

Again $v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$

$$= h^2\left(u^2 + u^2 \tan^2 n\theta\right)$$

$$= h^2 u^2 \sec^2 n\theta = h^2 u^2 \left(a^n u^n\right)^2$$

$$= h^2 a^{2n} u^{2n+2} = \frac{h^2 a^{2n}}{r^{2n+2}}$$

Law of velocity is $v = \frac{ha^n}{r^{n+1}} \Rightarrow v \propto \frac{1}{r^{n+1}}$

Finally, the pedal equation is $\dfrac{h}{p}=\dfrac{ha^n}{r^{n+1}} \Rightarrow a^n p = r^{n+1}$.

Now, we see the cases where $n=1,-1,2,-2,-\dfrac{1}{2},\dfrac{1}{2},........$

<u>Case I:</u> $n=1$

$r = a\cos\theta \left[x^2+y^2=ax, \text{ a circle}\right]$

So, law of force $F \propto \dfrac{1}{r^5}$, i.e., $F=\dfrac{\mu}{r^5}$, $\mu = 2h^2a^2$

The velocity, $v = hau^2$, or $v \propto \dfrac{1}{r^2}$

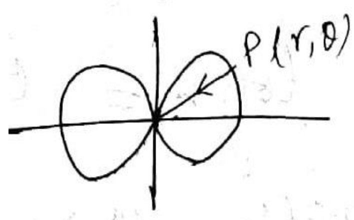
The pedal equation, $ap = r^2$.

Case II: $n = -1$, then $\frac{1}{r} = \frac{1}{a}\cos\theta$ or $a = r\cos\theta$ or $x = a$

$\mu = 0, F = 0$

Case III: $n = 2 \Rightarrow r^2 = a^2\cos 2\theta$ $\left[(x^2+y^2)^2 = a^2(x^2-y^2), \text{ a Lemniscate}\right]$

$F = 3h^2a^4u^7$, or $F = \frac{\mu}{r^7} \Rightarrow F \propto \frac{1}{r^7}$, $\mu = 3h^2a^4$



Velocity: $v = a^2hu^3 \Rightarrow v \propto \frac{1}{r^3}$

Pedal equation: $a^2p = r^3$

Case IV: $n=-2$, a rectangular hyperbola.

$$\frac{1}{r^2}=\frac{1}{a^2}\cos 2\theta \Rightarrow a^2=r^2\cos 2\theta \qquad \left[x^2-y^2=a^2\right]$$

$$F=-h^2a^{-4}u^{-1}=-h^2a^{-4}r \Rightarrow F\propto r,\ \mu=-h^2a^{-4}$$

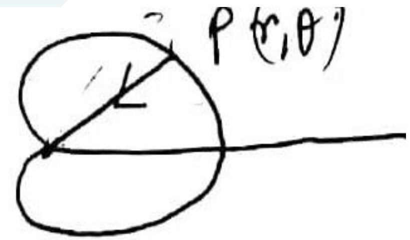
$$v=h^2a^{-4}u^{-1}=ha^{-2}r \Rightarrow v\propto r$$

Pedal equation: $pr=a^2$.

Case V: $n=-\frac{1}{2}$, $\frac{1}{\sqrt{r}}=\frac{\cos\left(\theta/2\right)}{\sqrt{a}} \Rightarrow \frac{2a}{r}=1+\cos\theta$, a parabola.

Case VI: $n=\frac{1}{2},\ \sqrt{r}=\sqrt{a}\cos\left(\theta/2\right)\Rightarrow r=\frac{a}{2}(1+\cos\theta)$, a cardioid .

$$F=\frac{3}{2}h^2au^4\Rightarrow F\propto\frac{1}{r^4},\ \mu=\frac{3}{2}h^2a$$



$$v=h\sqrt{a}u^{3/2}\Rightarrow v\propto\frac{1}{r^{3/2}}$$

Pedal equation: $\sqrt{a}\,p=r^{3/2}\Rightarrow ap^2=r^3$.

5) *<u>Equiangular spiral:</u>*

$$r = ae^{\theta \cot \alpha}, \quad a, \alpha \text{ being constants.}$$

So, $\log r = \log a + \theta \cot \alpha$

Differentiating w.r.t θ we get

$$\frac{1}{r}\frac{dr}{d\theta} = \cot \alpha \Rightarrow \cot \phi = \cot \alpha \Rightarrow \phi = \alpha \quad [\text{as } \tan \phi = r\frac{d\theta}{dr}]$$

$\therefore\ p = r \sin \phi = r \sin \alpha$, which is the pedal equation of the spiral.

The differential equation of central orbit in pedal form is $\dfrac{h^2}{p^3}\dfrac{dp}{dr} = F$.

Therefore, $F = \dfrac{h^2}{p^3}\sin\alpha \quad \left[\text{as } \dfrac{dp}{dr} = \sin\alpha\right]$

$$= \frac{h^2 \sin\alpha}{r^3 \sin^3\alpha} = \frac{h^2 \operatorname{cosec}^2\alpha}{r^3}$$

$\Rightarrow F \propto \dfrac{1}{r^3}$, where $\mu = h^2 \operatorname{cosec}^2\alpha$.

$$v = \frac{h}{p} = \frac{h}{r\sin\alpha} = \frac{h \operatorname{cosec}\alpha}{r} \Rightarrow v \propto \frac{1}{r}.$$

Kepler's Laws of Motion:

Kepler, an astronomer (contemporary to Galileo) gave three laws of the motion of the planets round the sum.

The three laws are –

(i) Each planet describes an elliptic orbit round the sun. Sun lies at one of its two foci.

(ii) The area described by the radius vector drawn from the planet to the sun, in the same orbit, is proportional to the time of describing it.

(iii) The square of the periodic time of the planet is proportional to the cube of the semi-major axis of its orbit.

Here the planet describes an elliptic central orbit with sum as centre of force. So, central force varies inversely to the square of the distance between the sun and the planet. Thus if F is the central force on the planet and r is the distance between the sun and the planet at any time t then

$$F=\frac{\mu}{r^2} \text{ with } \mu=\frac{h^2}{l}=\text{constant},$$

where l is the semi latus radius and h is the angular momentum per unit mass of the planet or twice of the sectorial area per unit time traced by the planet.

For the verification of third law, let a and b be the semi major and semi minor axis of the orbit. Then the area enclosed by the orbit is πab.

If T is the time period, then we have

$$T = \frac{\pi ab}{h/2} = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{\mu l}} \frac{2\pi ab}{\sqrt{\mu b^2/a}}$$

i.e., $T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} \Rightarrow T^2 = \frac{4\pi^2}{\mu} a^3$. So, $T^2 \propto a^3$.

Validity of Third Law:

The third law is only true for any planet when the mass of the planet is negligible in comparison with that of the sun.

Proof: Let a planet of mass m revolves around the sun; let mass of the sun be M and G be the constant of gravitation.

The force of attraction between the two is $G\frac{m\times M}{r^2}$, r is the distance between the sun and the planet at any instance. Let g be the acceleration due to gravity of the planet, then $mg = G\frac{m\times M}{r^2}$.

So the acceleration of the planet is $g = G\dfrac{M}{r^2}$ towards the sun. Similarly, if g' be that of the sun then $g' = G\dfrac{m}{r^2}$ towards the planet. Let us imposed an acceleration g' on both the planet and the sun in the direction from the planet to the sun. This will make zero acceleration from the sun towards the planet and the resultant acceleration of the planet towards the sun will become $(g+g') = G\dfrac{m+M}{r^2}$

Now it will be a central force under which the planet is orbiting the sun and so $\mu = (m+m)G$.

The angular momentum and the period will be

$$\sqrt{\mu l} \quad \text{and} \quad T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}, \ \text{i.e.,} \ \sqrt{(m+M)Ga(1-e^2)} \quad \text{and} \quad \frac{2\pi}{\sqrt{(m+M)G}} a^{3/2}$$

respectively.

Now, let us consider two such planets. If T_1 and T_2 be time periods, m_1 and m_2 be the masses and a_1 and a_2 be the semi-major axes of the paths of the two planets, then

$$T_1 = \frac{2\pi}{\sqrt{(M+m_1)G}} a_1^{3/2}, T_2 = \frac{2\pi}{\sqrt{(M+m_2)G}} a_2^{3/2}$$

Or, $\dfrac{M+m_1}{M+m_2}\dfrac{T_1^2}{T_2^2}=\dfrac{a_1^3}{a_2^3}$

Now if Kepler's law is true, we have $\dfrac{M+m_1}{M+m_2}=1$ which is possible only when m_1 and m_2 are negligible when compared to M, i.e. $M+m_1 \approx M$ and $M+m_2 \approx M$. Hence the result is proved.

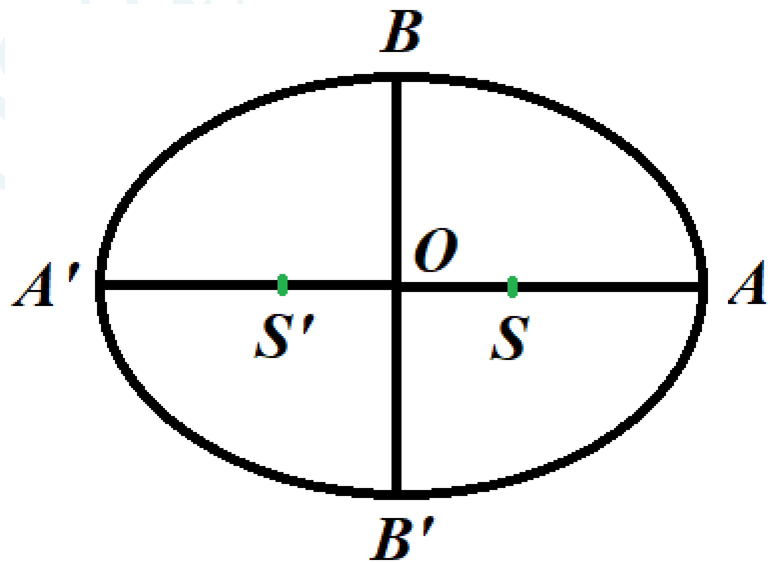
<u>Linear velocity and angular velocity at the ends of major axis and minor axis</u>:

We know that in elliptic orbit $\frac{l}{r} = 1 + e\cos\theta$ the linear velocity at any position (r,θ) is given by $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$ and the angular velocity is given by $\dot{\theta} = \frac{h}{r^2}$.

We also have $T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$ with $\mu = \frac{h^2}{l}$.

Now, let V_A and $V_{A'}$ be the velocities at the nearest and the farthest positions of a planet from the sun and those are the end points major axis of the elliptic orbit. Then the radius vectors at A and A' are $r_1 = a - ae$ and $r_2 = a + ae$.

Then we have



$$V_A^2 = \mu\left(\frac{2}{a-ae} - \frac{1}{a}\right) = \frac{\mu}{a}\frac{1+e}{1-e}$$

and $V_{A'}^2 = \mu\left(\frac{2}{a+ae} - \frac{1}{a}\right) = \frac{\mu}{a}\frac{1-e}{1+e}$

So, we have $V_A^2 : V_{A'}^2 = \dfrac{1+e}{1-e}$.

Similarly, when the planet is at one extremity of the minor axis B of its orbit, then the radius vector at B is given by

$$r_B^2 = b^2 + a^2e^2 = a^2 \Rightarrow r_B = a \;\left[\because b^2 = a^2 - a^2e^2\right].$$

So, the linear velocity at this position will be

$$V_B^2 = \mu\left[\frac{2}{a} - \frac{1}{a}\right] = \frac{\mu}{a} = \frac{1}{2}\left(\frac{2\mu}{a}\right)$$

Thus velocity at B', the other extremity of minor axis is the same.

Let the angular velocity at A and A' be ω_A and $\omega_{A'}$.

Then we have

$$\omega_A = \frac{h}{(a-ae)^2} = \frac{\sqrt{\mu l}}{a^2(1-e)^2} = \frac{\sqrt{\mu a(1-e^2)}}{a^2(1-e)^2} = \frac{\sqrt{\mu}}{a^{3/2}}\frac{(1+e)^{1/2}}{(1-e)^{3/2}} = \frac{2\pi}{T}\frac{(1+e)^{1/2}}{(1-e)^{3/2}}$$

$$\omega_{A'} = \frac{h}{(a+ae)^2} = \frac{\sqrt{\mu l}}{a^2(1+e)^2} = \frac{2\pi}{T}\frac{(1-e)^{1/2}}{(1+e)^{3/2}}$$

So, we have $\omega_A : \omega_{A'} = (1+e)^2 : (1-e)^2$.

Now, if ω_B and $\omega_{B'}$ is angular velocity at B and B' then we have

$$\omega_B = \omega_{B'} = \frac{\sqrt{\mu l}}{a^2} = \frac{\sqrt{\mu a\left(1-e^2\right)}}{a^2} = \frac{2\pi}{T}\sqrt{1-e^2}\,.$$