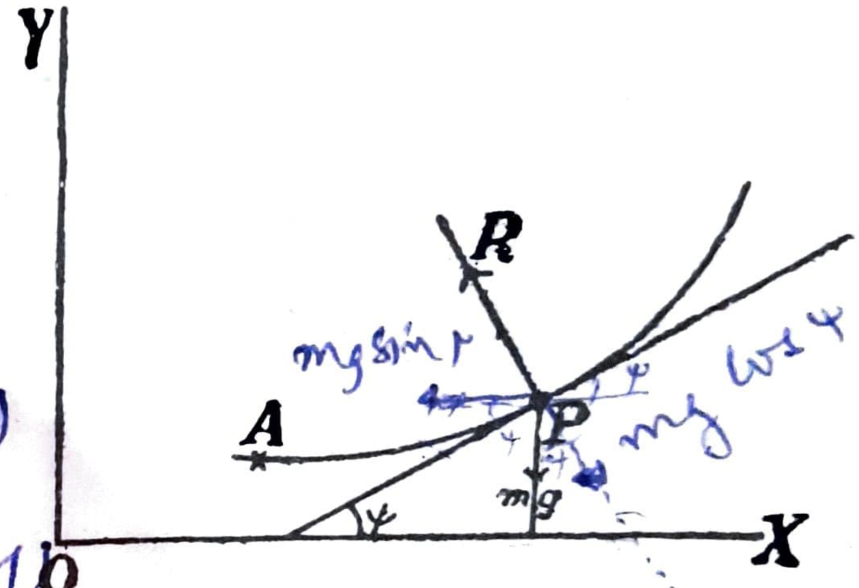
6·4. Motion on a smooth curve in a vertical plane.

A heavy particle is made to move on a smooth curve in a vertical plane; to discuss the motion.



The external forces are the weight mg of the particle downwards and the normal reaction R of the curve. If P be the particle at time t, the equations of motion (tangential and normal) are

$$mv\frac{dv}{ds}=-mg\sin\psi$$ (tangent)

$$m\frac{v^2}{\rho}=R-mg\cos\psi$$ (normal)

if s be measured from a fixed point A down the curve below P, ρ the radius of curvature of the curve at P.

Since $\sin\psi=\frac{dy}{ds}$ we have from the first equation

$$mv\frac{dv}{ds}=-mg\frac{dy}{ds}$$

Integrating $\frac{1}{2}mv^2=-mgy+C$ where C is a constant. If the particle be projected from a point where $y=y_1$ with a velocity u then

$$\tfrac{1}{2}mu^2=-mg\,y_1+C$$

Hence $\frac{1}{2}mv^2-\frac{1}{2}mu^2=-mg(y-y_1)=-mgh$.

where h is the height of P above the point of projection. This is in fact the equation of energy and follows at once from the principle of energy namely the change in kinetic energy is equal to the work done.

Thus for upward projection

$$v^2=u^2-2gh$$

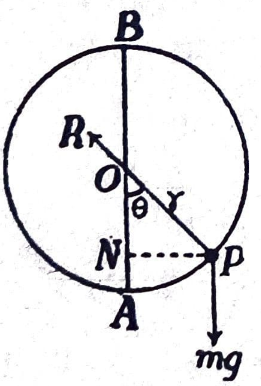
and for downward projection $v^2=u^2+2gh$

where h is the vertical distance between the two points. These equations hold good whatever be the paths followed. The second equation gives R, the reaction on the particle. The particle will leave the curve when $R=0$.

6·5. Motion on the inside of a smooth vertical circle.

A particle is projected from the lowest point with velocity u and moves along the inside of a smooth vertical circle ; to discuss the motion.

Let A and B be the lowest and highest points of the circle, P the position of the particle at time t where $\angle AOP=\theta$, O being the centre of the circle. Let v be the velocity at P and PN perpendicular to AB.



By the equation of energy

$$v^2=u^2-2g\,AN$$
$$=u^2-2gr\,(1-\cos\theta)$$

where $r=$ radius of the circle.

Equation of motion along the normal is

$$\frac{mv^2}{r}=R-mg\cos\theta.$$ —(from normal comp of force)

$$\therefore\ R=m\left(\frac{v^2}{r}+g\cos\theta\right)=m\left[\frac{u^2}{r}-2g(1-\cos\theta)+g\cos\theta\right]$$

$$=m\left[\frac{u^2}{r}-2g+3g\cos\theta\right].$$

v vanishes when $u^2=2gr(1-\cos\theta)$ or $\cos\theta=1-\dfrac{u^2}{2gr}$ ...(1)

R vanishes when $\dfrac{u^2}{r}=2g-3g\cos\theta$ or $\cos\theta=\dfrac{1}{3}\left(2-\dfrac{u^2}{gr}\right)$ ...(2)

If v vanishes before R vanishes, *i.e.*, if θ from (1) is less than θ from (2) *i.e.*, if $\cos\theta$ from (1) is greater than $\cos\theta$ from (2) then

$$1-\frac{u^2}{2gr}>\frac{1}{3}\left(2-\frac{u^2}{gr}\right)$$

i.e., $$3(2gr-u^2)>2(2gr-u^2)$$

i.e., $$u^2<2gr.$$

If $u^2=2gr$, v and R will vanish simultaneously and the particle will rise up to the horizontal diameter of the circle.

Therefore when $u^2<2gr$, v will vanish in some point in the first quadrant, R will not vanish, therefore, the particle will stop there and will come down, pass through the lowest point and will go up in the fourth quadrant and rise up to the same height and will then come down and so on. Thus the particle will oscillate about the lowest point. When $u^2=2gr$, the particle will go up to the end of the horizontal diameter and will oscillate through a quadrant on each side of the vertical.

If R vanishes before v vanishes, we will have

$$1-\frac{u^2}{2gr}<\frac{1}{3}\left(2-\frac{u^2}{gr}\right) \qquad \therefore\ u^2>2gr.$$

The particle will rise above the horizontal diameter.

The particle will go right round the circle if both v and R do not vanish at the highest point when $\theta=\pi$, *i.e.*, if $u^2>4gr$ and if $u^2>5gr$ *i.e.*, if $v^2>5gr$, the particle will describe the complete circle.

Thus if $u^2>2gr$ but $<5gr$, the particle will go above the horizontal diameter but will not go up to the highest point *i.e.*, the particle will leave the circle at some point in the second quadrant and will afterwards move in a parabola.

Thus we obtain the following results :—

(i) *when $u^2<2gr$, the particle oscillates on each side of the lowest point,*

(ii) *when $u^2>2gr$ but $<5gr$, the particle leaves the circle and describes a parabola.*

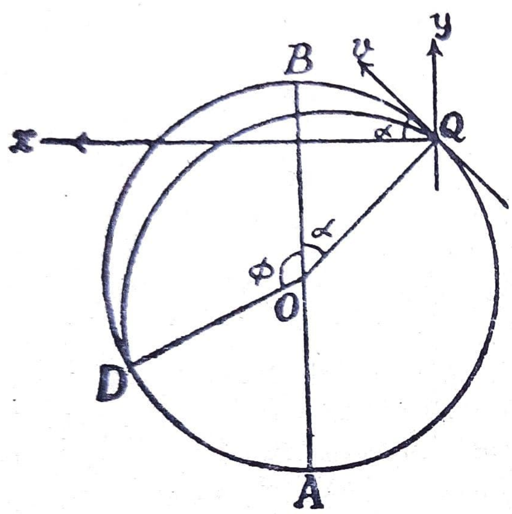
(iii) *when $u^2 > 5gr$, the particle makes complete revolutions.*

The reaction at A is

$$R_0 = m\left[\frac{u^2}{r} - 2g + 3g\right] = m\left(\frac{u^2}{r} + g\right)$$

Hence when $u^2 > 5gr$, $R_0 > 6mg$, *i.e., the pressure at the lowest point must be greater than 6 mg, if the particle is to go right round the circle.*

In case (*ii*) we will have to find the point when the particle after leaving the circle cuts it again.



Let the particle leave the circle at some point Q where $R=0$

i.e., $$\cos\theta = \frac{1}{3}\left(2 - \frac{u^2}{gr}\right)$$

If $\angle QOB = \alpha$, $\cos\alpha = \frac{1}{3}\left(\frac{u^2}{gr} - 2\right)$

If v be the velocity at Q

$$v^2 = u^2 - 2gr\left\{1 - \frac{1}{3}\left(2 - \frac{u^2}{gr}\right)\right\}$$

$$= (u^2 - 2gr) - \tfrac{2}{3}(u^2 - 2gr)$$

$$v^2 = \frac{u^2 - 2gr}{3} = gr\cos\alpha.$$

Latus rectum of the parabola it describes after leaving is given by

$$\boxed{\frac{2v^2\cos^2\alpha}{g}} = 2r\cos^3\alpha = \frac{2(u^2 - 2gr)^3}{27\,g^3\,r^2}$$

Take Q as origin, the vertical upwards as y-axis and the horizontal to the left as x-axis. Then the equation to the parabolic path of the particle after leaving at Q is

$$y = x\tan\alpha - \tfrac{1}{2}g\,\frac{x^2}{v^2\cos^2\alpha}$$

$$= x\tan\alpha - \frac{x^2}{2r\cos^3\alpha}.$$

The equation to the circle with Q as origin is

$$(x - r\sin\alpha)^2 + (y + r\cos\alpha)^2 = r^2$$

i.e., $$y = x\tan\alpha - \frac{x^2 + y^2}{2r\cos\alpha}.$$

The point D where the two curves cut again be (x', y') so that

$$\frac{x'^2}{2r\cos^3\alpha}=\frac{x'^2+y'^2}{2r\cos\alpha}$$

or

$$x'^2\sin^2\alpha=y'^2\cos^2\alpha$$

$\therefore$

$$y'=\pm x'\tan\alpha$$

when $y'=x'\tan\alpha$, we get $x'=0=y'$, *i.e.*, the point Q.

when $y'=-x'\tan\alpha$, we get $x'=4r\cos^2\alpha\sin\alpha$.

$$y'=-4r\cos\alpha\sin^2\alpha.$$

Let angle $BOD=\phi$, then $\tan\phi=\dfrac{x'-r\sin\alpha}{y'-(-r\cos\alpha)}$

$$=\frac{4r\cos^2\alpha\sin\alpha-r\sin\alpha}{-4r\cos\alpha\sin^2\alpha+r\cos\alpha}$$

$$=\tan\alpha.\frac{4\cos^2\alpha-1}{1-4\sin^2\alpha}$$

$$=\tan\alpha.\frac{4-(1+\tan^2\alpha)}{1+\tan^2\alpha-4\tan^2\alpha}$$

$$=\tan\alpha.\frac{3-\tan^2\alpha}{1-3\tan^2 a}=\tan 3a$$

$\therefore$

$$\phi=3\alpha$$

Hence arc $BD=3$ arc BQ.

Height reached by the particle above the horizontal line through Q

$$=\frac{v^2\sin^2\alpha}{2g}=\frac{gr\cos\alpha\sin^2\alpha}{2g}=\tfrac{1}{2}r\cos\alpha\sin^2\alpha$$

$\therefore$ Height above $A=\frac{1}{2}r\cos\alpha\sin^2\alpha+r\cos\alpha+r.$

To find at what point the particle crosses the vertical diameter, we see that the equation to OB is

$$x=r\sin\alpha$$

Putting $x=r\sin\alpha$ in the equation to the parabola

$$y=x\tan\alpha-\frac{x^2}{2r\cos^3\alpha}$$

we get
$$y=r\left[\frac{\sin^2\alpha}{\cos\alpha}-\frac{\sin^2\alpha}{2\cos^3\alpha}\right]$$

hence the height of the point above A

$$r+r\cos\alpha+y=r+r\cos\alpha+r\left[\frac{\sin^2\alpha}{\cos\alpha}-\frac{\sin^2\alpha}{2\cos^3\alpha}\right]$$

$$=r\,\frac{2\cos^3\alpha+2\cos^4\alpha+2\sin^2\alpha\cos^2\alpha-\sin^2\alpha}{2\cos^3\alpha}$$

$$=\frac{r}{2\cos^3\alpha}\left\{2\cos^3\alpha\,(1+\cos\alpha)+\sin^2\alpha\,(2\cos^2\alpha-1)\right\}$$

$$=\frac{r(1+\cos\alpha)}{2\cos^3\alpha}\left\{2\cos^3\alpha+(1-\cos\alpha)\,(2\cos^2\alpha-1)\right\}$$

$$=\frac{r(1+\cos\alpha)^2}{2\cos^3\alpha}\,(2\cos\alpha-1).$$

Cor. In the case of a bead moving in a circular wire or a particle moving inside a circular tube, the condition for a complete revolution is that $v>0$ when $\theta=\pi$, *i.e.*, $u^2>4gr$.

Ex. 5. *A particle is free to move on a smooth vertical circular wire of radius r. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time*

$$\sqrt{\frac{r}{g}}\log(\sqrt{5}+\sqrt{6}).$$

Here $u^2=4gr$

$\therefore$ $v^2=u^2-2gr(1-\cos\theta)=2gr(1+\cos\theta)$

$=4gr\cos^2\theta/2$

$$R=\frac{mv^2}{r}+mg\cos\theta$$

$\therefore$ where $R=0,\ v^2=-gr\cos\theta$

If at $\theta=\theta_1,\ R=0$ then $-gr\cos\theta_1=2gr(1+\cos\theta_1)$

$\therefore$ $\cos\theta_1=-\frac{2}{3}$

Also $v=2\sqrt{gr}\cos\theta/2$

Hence $r\dot\theta=2\sqrt{gr}\cos\theta/2$

or $$\sqrt{\frac{g}{r}}\,t=\int_0^{\theta_1}\frac{d\theta}{2\cos\theta/2}$$

$$=\log\left(\tan\theta/2+\sec\theta/2\right]_0^{\theta_1}$$

$$=\log\left(\tan\frac{\theta_1}{2}+\sec\frac{\theta_1}{2}\right)$$

Now $$\cos\theta_1=-\frac{2}{3}\quad\therefore\quad 2\cos^2\theta_1/2-1=-\frac{2}{3}$$

$\therefore$ $$\cos^2\frac{\theta_1}{2}=\frac{1}{6}\quad\therefore\ \sec\frac{\theta_1}{2}=\sqrt{6},\ \tan\frac{\theta_1}{2}=\sqrt{5}$$

Hence $$t=\sqrt{\frac{r}{g}}\log(\sqrt{5}+\sqrt{6}).$$

Ex. 6. *Find the velocity with which a particle must be projected along the interior of a smooth vertical hoop of radius r, from the lowest point in order that it may leave the hoop at an angular distance of 30° from the vertical. Show that it will strike the hoop again at an extremity of the horizontal diameter.*

Show also that if velocity of projection be $\sqrt{\frac{7}{2}gr}$, the particle will leave the hoop and return to the lowest point.

The particle leaves the circle at an angle α from the upward drawn vertical where $\cos\alpha=\frac{1}{3}\left(\frac{u^2}{gr}-2\right)$

Hence $\alpha=30$, $\therefore$ $$\frac{\sqrt{3}}{2}=\frac{1}{3}\left(\frac{u^2}{gr}-2\right)$$

$\therefore$ $$u^2=\tfrac{1}{2}gr(3\sqrt{3}+4).$$

Since $\angle BOQ=30°$ $\quad\therefore\quad \angle BOD=3\cdot 30°=90°$ *i.e.*, D is at the extremity of the horizontal diameter.

Where $u^2=\frac{7}{2}gr$, $\cos\alpha=\frac{1}{3}(\frac{7}{2}-2)=\frac{1}{2}$ $\quad\therefore\quad \alpha=60$

i.e., $\angle BOQ=60°$ Hence $\angle BOD=180°$ *i.e.*, D coincides with A.

Ex. 7. *A particle is projected along the inside of a smooth vertical circle of radius r, from the lowest point. Show that the velocity of projection so that after leaving the circle the particle may pass* ... $\sqrt{\frac{1}{2}gr(\sqrt{3}+1)}$.