Numerical Computing

Root Finding Methods

1. *Bisection* Method

2. *Regula Falsi* method

3. *Secant* Method

4. *Newton Raphson* method

Introduction

- Bisection Method: Bisection Method is a numerical method in Mathematics to find a root of a given *function*

- Objective is to find a solution of f$(x) = 0$

where, f is a polynomial or a transcendental function, given explicitly.

- Exact solutions are not possible for most equations.

- A number x **± ε**, (**ε** > 0) is an approximate solution of the equation if there is a solution in the interval.

Introduction (cont.)

- *Root* of a function:

  Root of a function *f(x)* is a value *a* such that:

$$f(a) = 0$$

Introduction (cont.)

- Example:

Function: $\mathbf{f(x) = x^2 - 4}$

Roots: $x = -2$, $x = 2$

Because:

$$f(-2) = (-2)^2 - 4 = 4 - 4 = 0$$

$$f(2) = (2)^2 - 4 = 4 - 4 = 0$$

A Mathematical Property

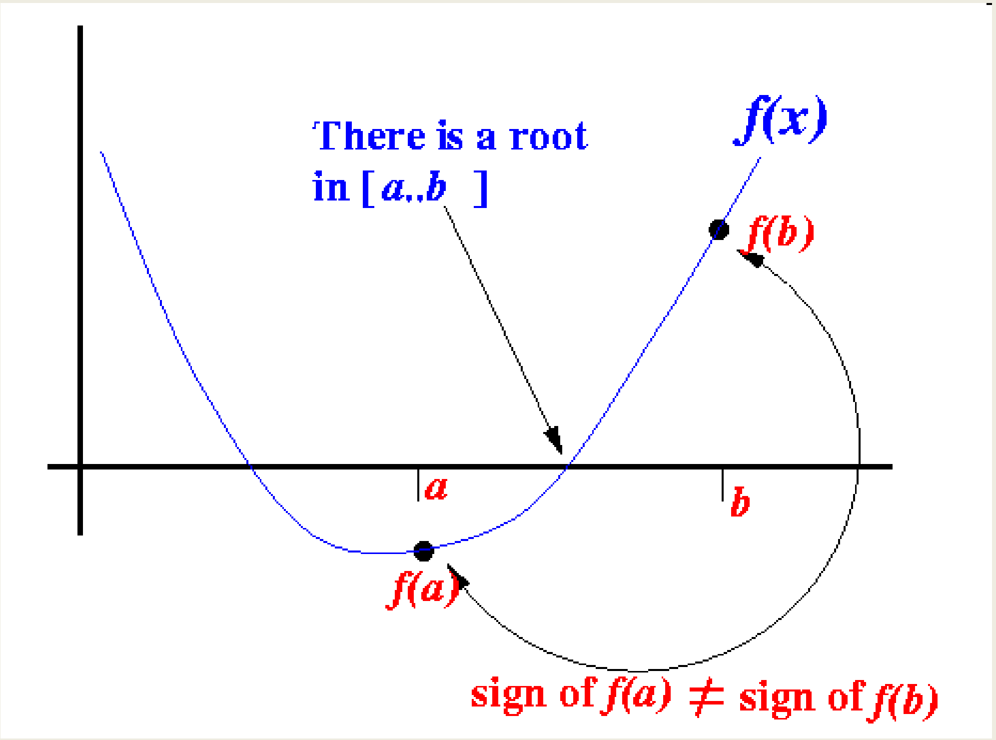
Well-known Mathematical Property:

If a function $f(x)$ is continuous on the interval $[a, b]$ and sign of $f(a) \neq$ sign of $f(b)$, then

There is a value $c \in [a, b]$ such that: $f(c) = 0$ i.e., there is a root c in the interval $[a, b]$

A Mathematical Property (cont.)

- Example:



The *Bisection* Method

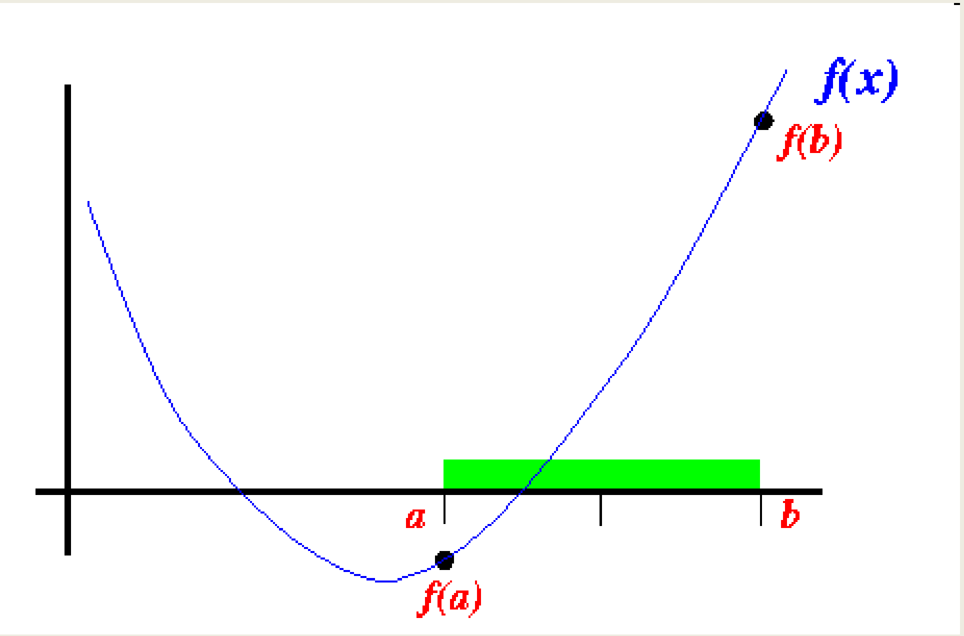
- The Bisection Method is a *successive* approximation method that narrows down an interval that contains a root of the function *f(x)*
- The Bisection Method is *given* an initial interval [*a*..*b*] that contains a root (We can use the property sign of *f(a)* ≠ sign of *f(b)* to find such an initial interval)
- The Bisection Method will *cut the interval* into 2 halves and check which half interval contains a root of the function
- The Bisection Method will keep *cut the interval* in halves until the resulting interval is extremely small

  The root is then *approximately equal* to *any value* in the final (very small) interval.

The *Bisection* Method (cont.)

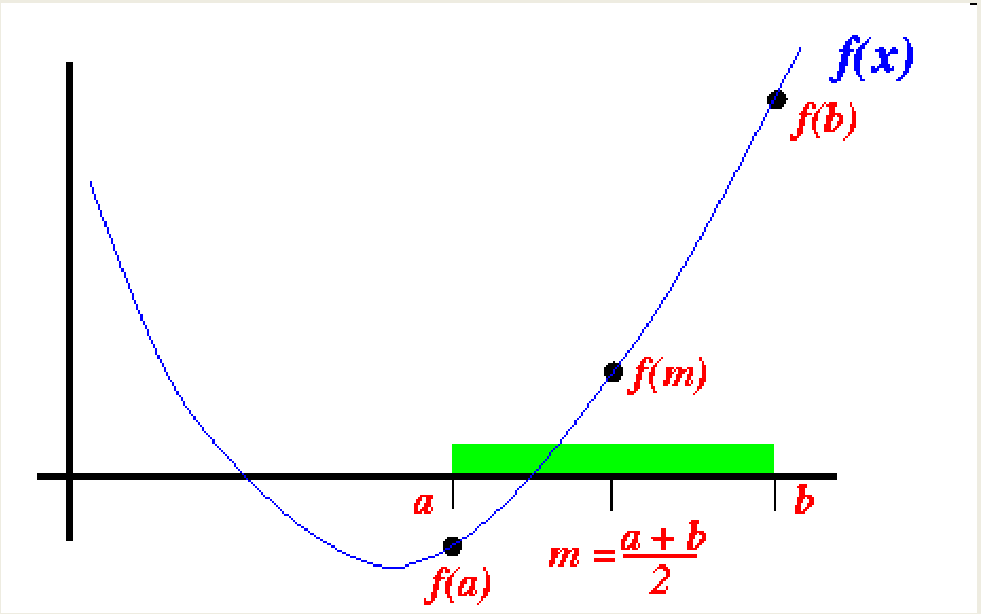
- Example:

• Suppose the interval [a..b] is as follows:



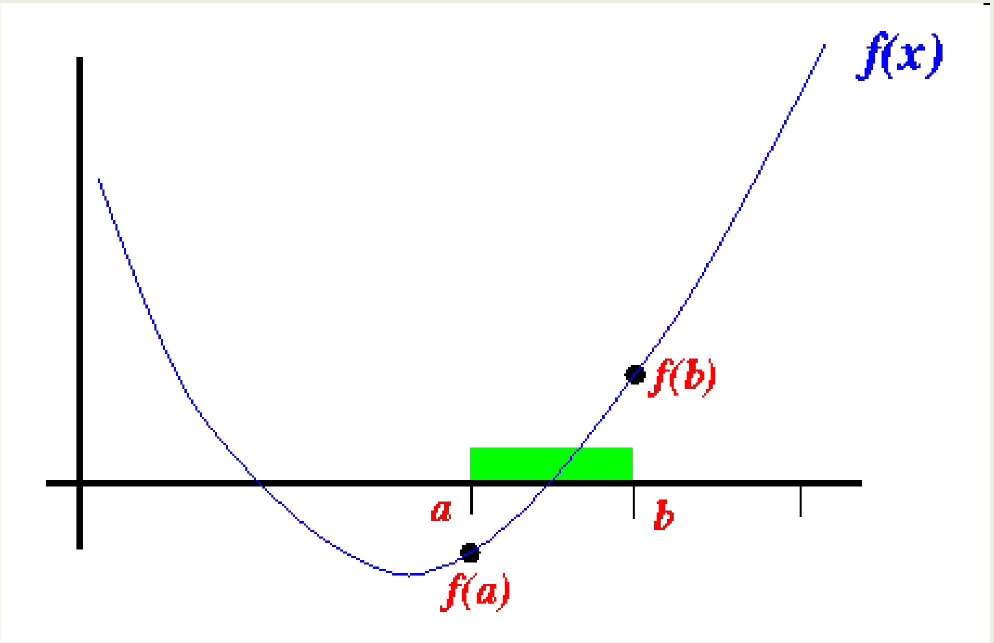
The *Bisection* Method (cont.)

- We cut the interval [*a*..*b*] in the middle: $m = (a+b)/2$



The *Bisection* Method (cont.)

- Because sign of *f(m)* ≠ sign of *f(a)* , we *proceed* with the search in the *new* interval [*a*..*b*]:



The *Bisection* Method (cont.)

We can use this statement to change to the new interval:

```
b = m;
```

The *Bisection* Method (cont.)

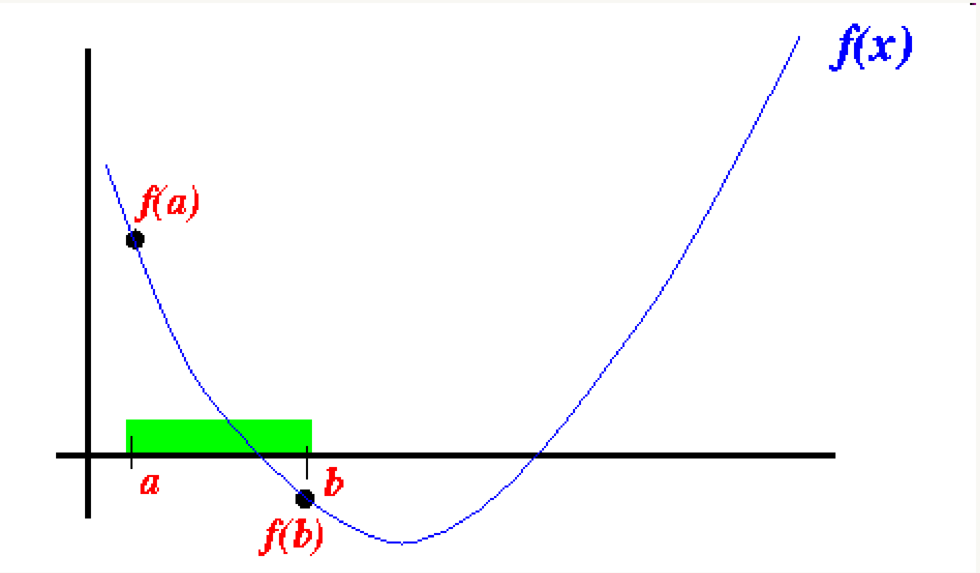
- In the above example, we have changed the end point *b* to obtain a smaller interval that still contains a root

  In other cases, we may need to changed the end point a to obtain a smaller interval that still contains a root

The *Bisection* Method (cont.)

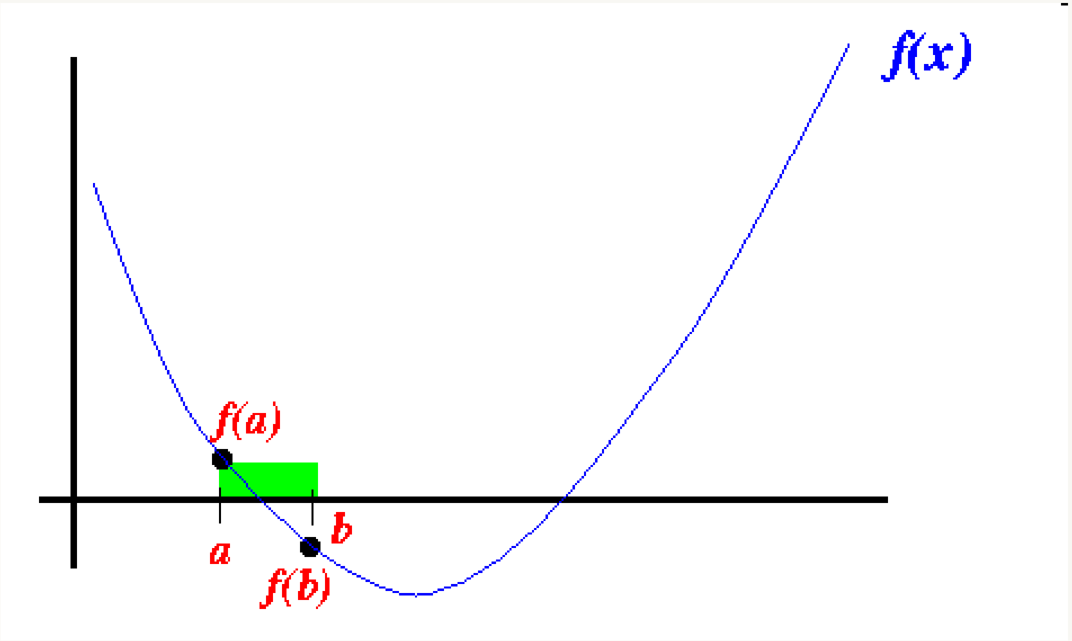
- Here is an example where you have to change the end point *a*:

• Initial interval [*a*..*b*]:



The *Bisection* Method (cont.)

- After cutting the interval in half, the root is contained in the right-half, so we have to change the end point *a*:



The *Bisection* Method (cont.)

- Rough description (pseudo code) of the Bisection Method:

```
  Given: interval [a, b] such that: sign of f(a) ≠ sign of f(b)


  repeat (until the interval [a, b] is "very small")
  {
          a+b
     m = -----;      // m = midpoint of interval [a, b]
           2

     if ( sign of f(m) ≠ sign of f(b) )
     {
        use interval [m, b]  in the next iteration
```

The *Bisection* Method (cont.)

```
(i.e.: replace a with m)
     }
     else
     {
        use interval [a..m]  in the next iteration
 (i.e.: replace b with m)
     }
  }

  Approximate root = (a+b)/2;    (any point between [a..b] will do
                                 because the interval [a..b] is very small)
```

Calculation of Number of iteration

repeat (until the interval [a, b] is "very small say ε)

$$\frac{|b-a|}{2^n} \le \varepsilon$$

$$|b-a| \le \varepsilon \,.\, 2^n$$

$$2^n \ge \frac{|b-a|}{\varepsilon}$$

$$n \log_e(2) \ge \log(|b-a| / \varepsilon)$$

$$n \ge \log(|b-a| / \varepsilon) \;/\; \log_e(2)$$

Example: if $|b-a| = 1$ and $\varepsilon = 0.001$

$$n \ge \log_e(1/0.001) \;/\; \log_e(2) \qquad = \log_e(1000) / \log_e(2) = 10 \text{ (approx)}$$

Advantages

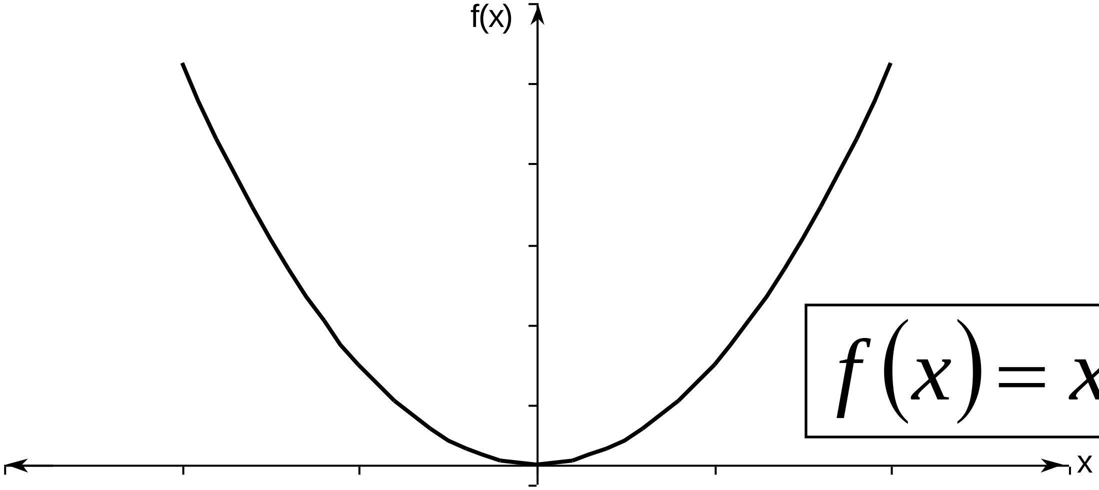
- Always convergent
- The root bracket gets halved with each iteration - guaranteed.

Drawbacks

- Slow convergence
- If one of the initial guesses is close to the root, the convergence is slower

Drawbacks (continued)

- If a function f(x) is such that it just touches the x-axis it will be unable to find the lower and upper guesses.



$$f(x) = x^2$$

Find a real root of equation $f(x)=x^3-4x-9=0$ using bisection method correct to three decimal places

Let a=2 and b=3

$f(a)=-9$ (-ve) $f(b)=6$ (+ve)

As $f(a) \times f(b)<0$

So first approximation $x0= (a+b)/2$

$=(2+3)/2$

$=2.5$

Now, $f(2.5)= (2.5)^3-4 \times 2.5- 9=-3.375$ (-ve)

So root lies between 2.5 and 3

Solution Cont…

Second approximation

$x1=(2.5+3)/2 \qquad = \qquad 2.75$

$f(x1)= 2.75^3-4 \times 2\ .75-9 \qquad = 0.7969$ (+ve)

So root lies between 2.5 and 2.75

third approximation, $\qquad x2=(2.5+2.75)/2 = 2.625$,

Compute $f(x2) =-1.41121$ (-ve)

$x_3 = (2.75+2.625)/2 = 2.6875$

repeat this procedure till the desired result is obtained.

$x_4 = 2.71875$, $x_5 = 2.70313$

$x_6 = 2.71094$, $x_7 = 2.70703$

$x_8 = 2.70508$, $x_9 = 2.70605$

$x_{10} = 2.70654$, $x_{11} = 2.70642$

As, $|x_{11}-x_{10}|=|2.70642-2.70654|=0.00012$

The computed result is correct to 3 decimal places.

Find real root of equation $f(x)=x^3-x-1=0$

Let a=1 and b=2

$f(a)=-1$ (-ve) $f(b)=5$ (+ve)

As $f(a) \times f(b)<0$

So first approximation $x0= (a+b)/2$

$=(1+2)/2$

$=1.5$

Now, $f(1.5)= (1.5)^3-1.5-1=$(+ve)

So root lies between 1 ans 1.5

$x1=(1+1.5)/2 \quad = \quad 1.25$

$f(x1)= 1.25^3-1.25-1 = -ve$

So root lies between 1.25 and 1.5

Second approximation, $x2=(1.25+1.5)/2$,

Compute $f(x2)$, and repeat this procedure till the desired result is obtained.

x3=1.3125

x4=1.34375

x5=1.328125

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Practice problem

1. Find the real root of the equation $x^3-2x-5=0$, correct to 2 decimal places.

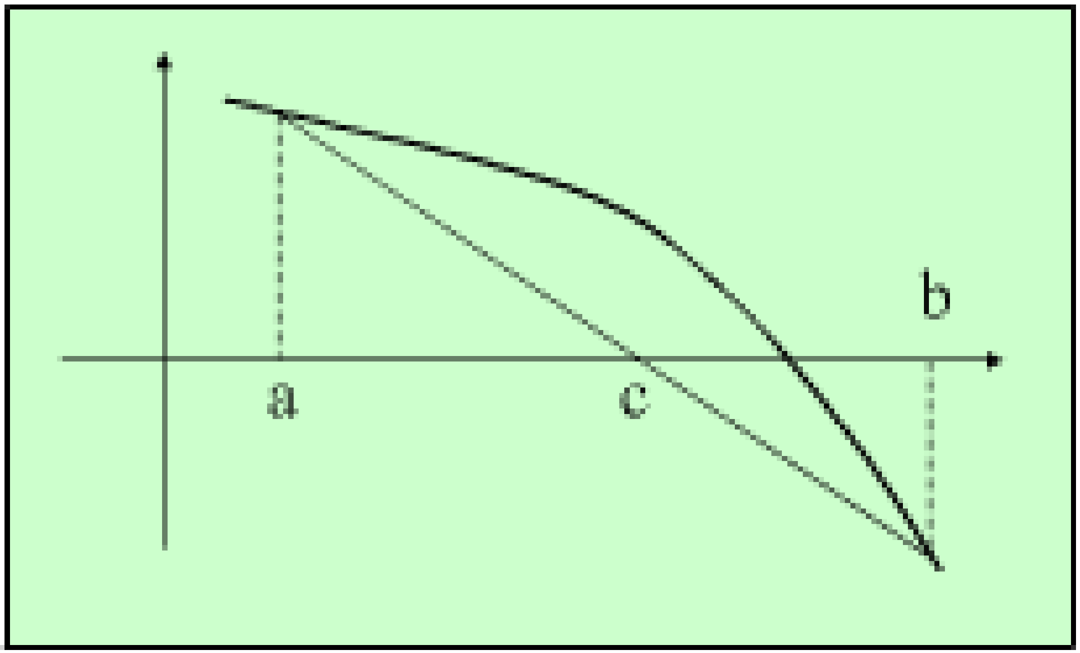
2. Find the real root of the function $f(x)=xe^x-1$correct to three decimal places, which lies between 0 and 1.

The False-Position Method (Regula-Falsi)

- To refine the bisection method, we can choose a 'false-position' instead of the midpoint.

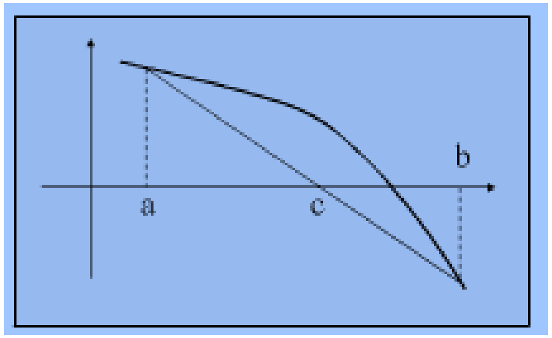
- The false-position is defined as the x position where a line connecting the two boundary points crosses the axis

The False-Position Method (Regula-Falsi) Cont..

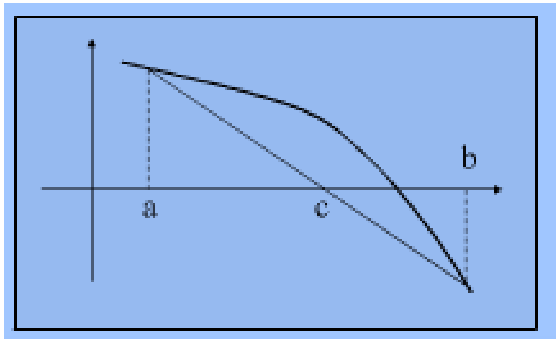


- Choose two points a and b such that f(a) and f(b) are of opposite sign i.e $f(a) \times f(b) < 0$

- So root must lie between these two points.



- The equation of chord joining two points [a, f(a)] and [b, f(b)] is given by:



$$y-y1 = \frac{(y2-y1)}{(x2-x1)}(x-x1) \quad \ldots\ldots..(1)$$

$$y-f(a) = \frac{f(b)-f(a)}{b-a}(x-a) \qquad \ldots\ldots\ldots\ldots..(2)$$

Note: This method consists in replacing the part of the curve between the points [a, f(a)] and [b, f(b)] by means of the chord joining these points.

take the point of intersection of the chord with x axis as an approximation to the root.

→ y=0

$$0-f(a) = \frac{f(b)-f(a)}{b-a}(x-a) \qquad \text{...............(3)}$$

$$x = a - \frac{f(a)(b-a)}{f(b)-f(a)} \qquad \text{....................(4)}$$

Let x1 is the first approximation

$$x1 = \frac{a\,f(b)-bf(a)}{f(b)-f(a)} \qquad \text{.....................(5)}$$

- If f(x1)=0, then it is the required root, else if f(x1) and f(a) are of opposite signs, then the root must lie between a and x1 and we replace b by x1.
- Otherwise replace a by x1.

- Note: the procedure is repeated till the root is obtained to the desired accuracy.

Practice problem

Find a real root of equation $f(x)=x^3-4x-9=0$ using Regula Falsi method correct to three decimal places

Suggested books

1. Numerical Methods by **S.R.K Lyenger & R.K. Jain.**

2. Introductory methods of Numerical analysis by **S.S. Sastry**.

Thank you