Ex I (B)

Q1 If the radial and transverse velocities of a point are always proportional to each other and this hold for acceleration also, prove that its velocity will vary as some power of the radius vector.

proof Sol It is given that radial velocity $\propto$ transverse velocity

$$\frac{dr}{dt} \propto r\frac{d\theta}{dt} \Rightarrow \frac{dr}{dt} = K\left(r\frac{d\theta}{dt}\right) \Rightarrow \boxed{r\frac{d\theta}{dt} = \frac{1}{K}\frac{dr}{dt}}$$

$$\Rightarrow \frac{dr}{d\theta} = Kr \Rightarrow \frac{dr}{r} = K\,d\theta$$

Integrating both sides $\int \frac{dr}{r} = \int K\,d\theta \Rightarrow \log r = K\theta + A$

$$r = e^{K\theta}\cdot e^{A} \Rightarrow \boxed{r = Ge^{K\theta}} \rightarrow \text{equiangular spiral} \quad \boxed{G = e^{A}}$$

It is given that radial acceleration $\propto$ the transverse acceleration

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \propto \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

$$\Rightarrow \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{\lambda}{r}\frac{d}{dt}\left(r\cdot r\frac{d\theta}{dt}\right)$$

$$\Rightarrow \frac{d^2r}{dt^2} - r\left\{\frac{1}{Kr}\frac{dr}{dt}\right\}^2 = \frac{\lambda}{r}\frac{d}{dt}\left\{r\frac{1}{K}\frac{dr}{dt}\right\}$$

$$\frac{d^2r}{dt^2} - \frac{1}{K^2 r}\left(\frac{dr}{dt}\right)^2 = \frac{\lambda}{Kr}\left\{r\frac{d^2r}{dt^2} + \left(\frac{dr}{dt}\right)^2\right\} = \frac{\lambda}{K}\frac{d^2r}{dt^2} + \frac{\lambda}{Kr}\left(\frac{dr}{dt}\right)^2$$

$$\left\{1 - \frac{\lambda}{K}\right\}\frac{d^2r}{dt^2} = \left\{\frac{1}{K^2} + \frac{\lambda}{K}\right\}\frac{1}{r}\left(\frac{dr}{dt}\right)^2$$

$\left\{\text{let } \frac{1+\lambda K}{K(K-\lambda)} = n\right\}$ Const

$$\frac{d^2r}{dt^2} = \frac{(1+\lambda K)}{K^2} \times \frac{K}{(K-\lambda)}\frac{1}{r}\left(\frac{dr}{dt}\right)^2 = \frac{1+\lambda K}{K(K-\lambda)}\cdot\frac{1}{r}\left(\frac{dr}{dt}\right)^2$$

$$\int \frac{\frac{d^2r}{dt^2}}{\left(\frac{dr}{dt}\right)}dt = \int \frac{n}{r}\frac{dr}{dt}dt \Rightarrow \log\left(\frac{dr}{dt}\right) = n\log r + \log c$$

$$\Rightarrow \boxed{\frac{dr}{dt} = cr^n} \quad \dot{r} = cr^n \rightarrow \boxed{v \propto r^n}$$

Resultant velocity $= \sqrt{\dot{r}^2 + (r\dot{\theta})^2} = \sqrt{\dot{r}^2 + \left(\frac{1}{K}\dot{r}\right)^2} = \sqrt{\frac{K^2+1}{K}}\,\dot{r} = \frac{\sqrt{K^2+1}}{K}cr^n$

Q2 The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$. Show that the equation to the path is

$$\frac{\lambda}{\theta} = \frac{\mu}{2r^2} + c$$

and the components of acceleration are

$$2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r} \quad \text{and} \quad \lambda\mu r\theta^2 + 2\mu^2\frac{\theta^3}{r}$$

Sol radial velocity $= \frac{dr}{dt} = \lambda r^2$ — (1), transverse velocity $= r\frac{d\theta}{dt} = \mu\theta^2$ — (2)

Dividing eq (1) by (2) $\frac{1}{r}\frac{dr}{d\theta} = \frac{\lambda r^2}{\mu\theta^2} \Rightarrow \frac{\mu}{\lambda}\frac{dr}{r^3} = \frac{d\theta}{\theta^2}$

Integrating: $\int \frac{\mu}{\lambda}\frac{dr}{r^3} = \int\frac{d\theta}{\theta^2} \Rightarrow -\frac{\mu}{2\lambda r^2} = -\frac{1}{\theta} + c_1$

$\Rightarrow + \frac{\mu}{2r^2} = +\frac{\lambda}{\theta} - c_1\lambda$

This is the Equation of the path of the particle. $\boxed{\frac{\lambda}{\theta} = \frac{\mu}{2r^2} + c}$ $\boxed{c = c_1\lambda}$

Now, radial acceleration $= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

$= \frac{d}{dt}\left(\frac{dr}{dt}\right) - \frac{1}{r}\left\{r\frac{d\theta}{dt}\right\}^2$

$= \frac{d}{dt}(\lambda r^2) - \frac{1}{r}\{\mu\theta^2\}^2 = 2\lambda r\frac{dr}{dt} - \frac{\mu^2\theta^4}{r}$

$= 2\lambda r(\lambda r^2) - \frac{\mu^2\theta^4}{r} = \boxed{2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r}}$

Again Transverse acceleration $= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$

$= \frac{1}{r}\frac{d}{dt}\left\{r\cdot\left(r\frac{d\theta}{dt}\right)\right\} = \frac{1}{r}\frac{d}{dt}\left\{r\mu\theta^2\right\} = \frac{\mu}{r}\left\{r2\theta\dot\theta + \theta^2\dot r\right\}$

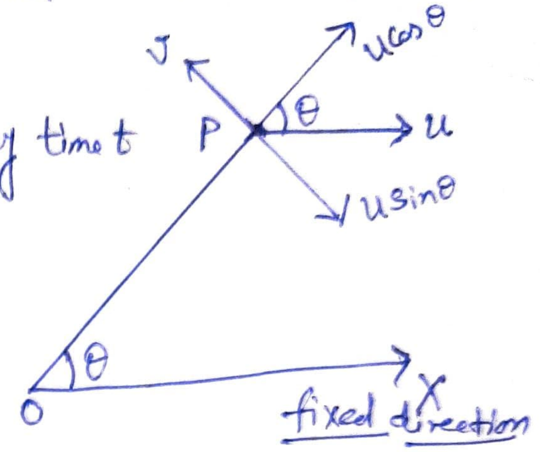
$= \frac{\mu}{r}\left\{2\theta\left(r\frac{d\theta}{dt}\right) + \theta^2\left(\frac{dr}{dt}\right)\right\} = \frac{\mu}{r}\left\{2\theta\cdot\mu\theta^2 + \theta^2(\lambda r^2)\right\}$

$= \frac{\mu}{r}\left\{2\mu\theta^3 + \lambda r^2\theta^2\right\} = \lambda\mu r\theta^2 + 2\mu^2\frac{\theta^3}{r}$

Ans.

Q3 Prove that the path of a point which possesses two constant velocities, one along a fixed direction and the other perpendicular to the radius vector drawn from a fixed point, is a conic section.

Sol. Take the fixed point O as pole and the fixed direction as the initial line OX.



Let $P(r, \theta)$ be the position of the particle at any time t then at point P possesses two constant velocities

(i) along a fixed direction is u.

(ii) perpendicular to OP is v.

Resolving the velocities of P along and perpendicular to the radius vector OP,

radial velocity $= \dfrac{dr}{dt} = u\cos\theta$ — ①

transverse velocity $= r\dfrac{d\theta}{dt} = v - u\sin\theta$ — ②

Dividing ①/② $\quad \dfrac{dr}{r\,d\theta} = \dfrac{u\cos\theta}{v - u\sin\theta} \Rightarrow \dfrac{dr}{r} = \dfrac{u\cos\theta}{v - u\sin\theta}\,d\theta \qquad v - u\sin\theta = t$

Integrating $\displaystyle\int \frac{dr}{r} = \int \frac{u\cos\theta}{v - u\sin\theta}\,d\theta \Rightarrow \log r = \int \frac{-dt}{t} + \log c \qquad -u\cos\theta\,d\theta = dt$

$\log r = -\log(v - u\sin\theta) + \log c \Rightarrow \log c - \log r = \log(v - u\sin\theta)$

$\log\left(\dfrac{c}{r}\right) = \log(v - u\sin\theta) = \dfrac{c}{r} = v - u\sin\theta$

$\dfrac{c}{r} = v + u\cos(\theta + \pi/2)$

$$\Rightarrow \quad \frac{(c/v)}{r} = 1 + \left(\frac{u}{v}\right)\cos(\theta + \pi/2)$$

This equation is of the form $\longrightarrow$ $\boxed{\dfrac{l}{r} = 1 + e\cos\theta}$

Which is a conic, whose focus is the pole O and eccentricity is $\left(\dfrac{u}{v}\right)$

Q4. A particle moves along a circle $r = 2a\cos\theta$ in such a way that its acceleration towards the origin is always zero. Prove that

$$\frac{d^2\theta}{dt^2} = -2\cot\theta\,\dot{\theta}^2$$

Proof: The equation of the path is $r = 2a\cos\theta$ — (1)

Acceleration towards the origin i.e. radial acceleration is zero

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0 \quad \text{— (2)}$$

$$\frac{dr}{dt} = 2a(-\sin\theta)\dot{\theta}$$

$$\frac{d^2r}{dt^2} = -2a\left\{\sin\theta\,\ddot{\theta} + \dot{\theta}\cos\theta\,\dot{\theta}\right\}$$

$$= -2a\sin\theta\,\ddot{\theta} - 2a\cos\theta\,\dot{\theta}^2$$

Putting these value in eq (2)

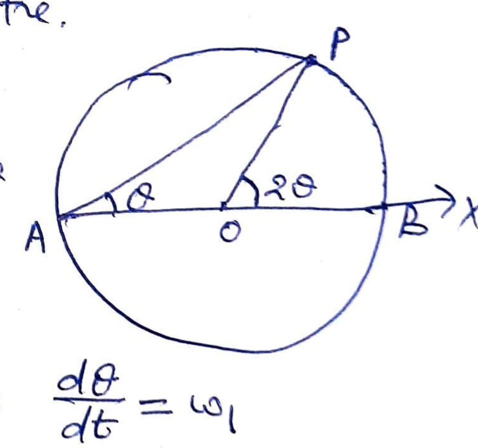
$$-2a\sin\theta\frac{d^2\theta}{dt^2} - 2a\cos\theta\left(\frac{d\theta}{dt}\right)^2 - 2a\cos\theta\left(\frac{d\theta}{dt}\right)^2 = 0$$

$$\sin\theta\frac{d^2\theta}{dt^2} + 2\cos\theta\left(\frac{d\theta}{dt}\right)^2 = 0$$

$$\frac{d^2\theta}{dt^2} = -2\frac{\cos\theta}{\sin\theta}\left(\frac{d\theta}{dt}\right)^2 \Rightarrow \boxed{\frac{d^2\theta}{dt^2} = -2\cot\theta\,.\,\dot{\theta}^2}$$

EX I(c)

Q1 If a point moves along a circle, prove that its angular velocity about any point on the circle is half of that about the centre.



Let P be the point on circle which moves along a circle. Let A be the point on circle and taking AOB the diameter of circle as fixed axis and A as pole then $\angle PAO = \theta$

Angular velocity of P with respect to A $= \frac{d\theta}{dt} = \omega_1$

Let O is centre of circle, $\angle POB = 2\theta$.

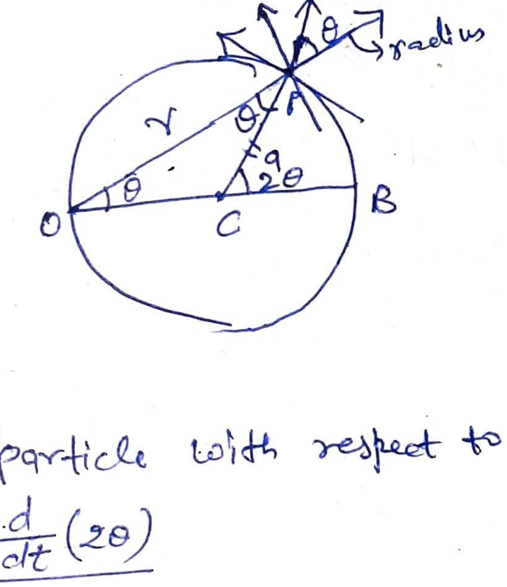
Angular velocity of P with respect to O $= \frac{d}{dt}(2\theta) = 2\frac{d\theta}{dt} = \omega_2$

$$\Rightarrow \omega_2 = 2\omega_1 \Rightarrow \boxed{\underset{\text{any point}}{\omega_1} = \frac{1}{2}\underset{\text{centre}}{\omega_2}}$$

Proved

Ex I(c)

Q2 A point describes a circle of radius a with a uniform speed v; Show that the radial and transverse acceleration are $-\frac{v^2}{a}\cos\theta$ and $-\frac{v^2}{a}\sin\theta$. If a diameter is taken as initial line and one end of the diameter as pole.



Let C is centre of circle and OCB is diameter of circle and O is pole and end of diameter. Let $OP = r$, $\angle POC = \theta$

~~Let~~ C be the centre, so $\angle PCB = 2\theta$

The radial and transverse velocity of particle with respect to circle C is $\underline{\dot{r}}$ and $\underline{r\frac{d}{dt}(2\theta)}$

But on the circle at point P, $r = a$

radial velocity $\dot{r} = 0$, transverse velocity $= 2a\dot{\theta} = v$

$\dot{\theta}$ is angular velocity w.r.to O. $\boxed{\dot{\theta} = \frac{v}{2a}}$

radial acceleration $= \ddot{r} - r\dot{\theta}^2$; transverse acceleration

with respect to Centre C. $(r = a)$

$\dot{r} = 0$, $\ddot{r} = 0$

$= -a\left\{\frac{d}{dt}(2\theta)\right\}^2$

$= -4a\dot{\theta}^2 \Rightarrow -4a\left\{\frac{v}{2a}\right\}^2$

$= -4a\frac{v^2}{4a^2} = \underline{-\frac{v^2}{a}}$

$= \frac{1}{r}\frac{d}{dt}(r^2 2\dot{\theta})$

$= \frac{1}{a}\frac{d}{dt}(a^2 2\dot{\theta})$

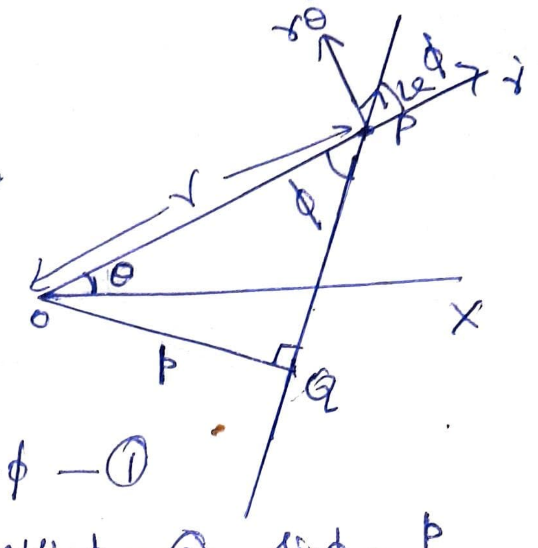
$= 2a\frac{d}{dt}\left\{\frac{v}{2a}\right\} = 0$

$\underline{v \text{ is uniform}}$

Component of radial Acceleration along to radius vector OP is $= -\frac{v^2}{a}\cos\theta$ = radial velocity at P along OP

transver velocity $= -\frac{v^2}{a}\sin\theta$ = Perpendicular to OP.

Q3. A point describes uniformly a given straight line; show that its angular velocity about a fixed point varies inversely as the square of its distance from the fixed point.



Let particle P moves on a straight line with velocity v. v = uniform velocity

let draw a perpendicular from pole O to straight line; $OQ = p$

radial velocity at P is $\dot{r} = v\cos\phi$ — ①

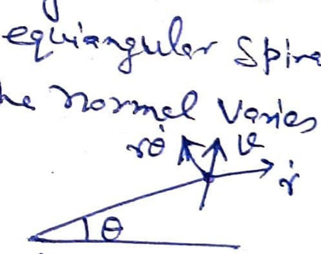
transverse velocity at P is $r\dot{\theta} = v\sin\phi$ — ② $\sin\phi = \frac{p}{r}$

Putting $\sin\phi = \frac{p}{r}$ in eq ② $r\dot{\theta} = v\frac{p}{r} \Rightarrow \dot{\theta} = \frac{vp}{r^2}$

As straight line $\Rightarrow$ then OQ is always constant $OQ = p$. v is constant given.

$\Rightarrow \boxed{\dot{\theta} \propto \frac{1}{r^2}}$ r is distance from the fixed point O.

Q 4. A point P describes a curve with a constant velocity and the radius vector joining P to a fixed point O has an angular velocity which is inversely proportional to OP; show that the curve is an equiangular spiral with O as pole and that the acceleration of P along the normal varies inversely as OP.



Proof v is constant velocity at P. $\frac{d\theta}{dt} \propto \frac{1}{r}$

$\frac{d\theta}{dt} = \frac{K}{r}$. $v = \sqrt{\dot{r}^2 + (r\dot{\theta})^2} \Rightarrow v^2 = \dot{r}^2 + \left(r\frac{K}{r}\right)^2 = \dot{r}^2 + K^2$

$\dot{r} = \sqrt{v^2 - K^2} = \text{Constant} = \lambda \Rightarrow \frac{dr}{dt} = \lambda$, $\frac{d\theta}{dt} = \frac{K}{r}$

$\frac{dr}{d\theta} = \frac{\lambda \times r}{K} \Rightarrow \int \frac{dr}{r} = \int \frac{\lambda}{K} d\theta = \log r = \left(\frac{\lambda}{K}\right)\theta + \log c$ $\boxed{\frac{\lambda}{K} = \alpha}$

$\log(r/c) = \alpha\theta \Rightarrow \boxed{r = ce^{\alpha\theta}}$ → equation of equiangular spiral.

Transverse ~~velocity~~ acceleration $\perp$ to radius vector

$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{K}{r}\right) = \frac{1}{r}\frac{d}{dt}(Kr) = \frac{K}{r}\frac{dr}{dt} = \frac{K}{r}\sqrt{v^2 - K^2}$

Transver acceleration = normal to OP $\propto \frac{1}{r}$ $\{\because K\sqrt{v^2-K^2} = \text{Const}\}$