**Def**: (Rank of a matrix): Let $A$ be an $m \times n$ matrix. Let $B$ be a matrix which is in row-Echelon form and obtained from $A$ after applying finitely many elementary row operations.

**Then** Rank of $A$, denoted as $\text{Rank}(A)$, is defined as the number of non-zero rows in $B$.

**Example**: Determine the rank of following matrices.

(1) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, (2) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(1) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

$\xrightarrow[R_3 \to R_3 - R_2]{R_2 \to (-1) R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{2} R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\text{Rank}(A) = 3$.

(ii) Rank(A) = 2

**Example:** Find the rank of the following matrix exploring all possibilities for 'a'.

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & a & 1-a & a^2+1 \\ 1 & 2-a & -1 & -2a^2 \end{bmatrix}$$

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Let ~~us fi~~ reduce the above matrix into row-Echelon form.

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & a & 1-a & a^2+1 \\ 1 & 2-a & -1 & -2a^2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & a & 1-a & a^2+1 \\ 0 & -a & 0 & -2a^2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & a & 1-a & a^2+1 \\ 0 & 0 & 1-a & -a^2+1 \end{bmatrix}$$ (say this matrix is $B$).

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When the two lower rows of $B$ are identical then rank(A) = 2 (Note that for any value of $a$ second row is non-zero). Also if last row of $B$ is zero, then also rank of $A$ is 2

$\Rightarrow$ If $a=0$ then $\text{rank}(A)=2$, and

~~Obs~~ $\boxed{\text{Also, } \cancel{a^2+1 = -a^2+1}}$

~~If $a \neq 0$, then $\text{rank}(A) = 3$.~~

Now, ~~if~~ last row of $B$ is zero means,

$1-a=0$, $-a^2+1=0$

$\Rightarrow a=1$

So, if $a=1$ then $\text{rank}(A)=2$.

Hence if $a=0,1$ then $\text{rank}(A)=2$, otherwise $\text{rank}(A)=3$.

**Theorem** (without proof): Let a system $Ax=b$ of linear equation is given in $n$-variables. Then

(1) If $\text{rank}(A) = \text{rank}(A|b) = n$ then the system $Ax=b$ has a unique solution.

(2) If $\text{rank}(A) = \text{rank}(A|b) < n$ then

the system has infinitely many solutions.

(3) If rank(A) < rank(A|b), then the system $Ax = b$ has no solution.

Ex: For what value(s) of 'b' the following system of linear equations has a solution

$$3x + 4y + 4z = b,$$
$$x + 2y + 2z = 3$$
$$x + y + z = 1$$

Solve this system completely when it has a solution.

$$\begin{bmatrix} 3 & 4 & 4 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ 1 \end{bmatrix}$$

$$[A|b] : \left[\begin{array}{ccc|c} 3 & 4 & 4 & b \\ 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 \\ 3 & 4 & 4 & b \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & b-3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & b-3 \end{bmatrix}$$

$\xrightarrow{R_3 \to R_3 - R_2}$ $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & b-5 \end{bmatrix}$

So, the system $Ax = b$ ~~has~~ is consistent if

$b - 5 = 0 \implies b = 5$.

In this case, this system has infinitely many solutions, since $\text{rank}(A) = \text{rank}(A|b) = 2 < 3$.

Hence $[A|b]$ reduces to

$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$, Basic variables : $x, y$
free variables : $z$.

$x + y + z = 1$
$y + z = 2 \implies y = 2 - z$
$\implies x = -1$

$\implies$ For $t \in \mathbb{R}$, $(-1, 2-t, t)$ is a solution.

**Ex:** Which of the following system has non-trivial solution.

| | |
|---|---|
| $ax + 2y + 3z = 0$ | $2x + y - z = 0$ |
| $2y + 3z = 0$ | $x - 2y - 3z = 0$ |
| $x + 2y + 3z = 0$ | $3x + y - 2z = 0$ |

**Ex:** Find all possible values of $a, b, c, d \in \mathbb{C}$ s.t. the following system of linear equation have infinitely many solutions.

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$ax + by + cz = d$$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ a & b & c & d \end{array}\right] \xrightarrow[R_3 \to R_3 - aR_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & b-a & c-a & d-3a \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - (b-a)R_2} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & (c-a)-2(b-a) & (d-3a)-(b-a) \end{array}\right]$$

So, if $c - 2b + \cancel{a} = d - b + \cancel{a} - 3$, then $Ax = b$ has infinitely many solution.

2) If $c = d + b - 3$ then $Ax = b$ has infinitely many solution.

Ex: Determine all values of $b_i$ that make the following system consistent.

$$x + y - z = b_1$$
$$2y + z = b_2$$
$$y - z = b_3.$$

Ex: Determine the condition on $b_i$ so that the following system has no solution

$$2x + y + 7z = b_1$$
$$6x - 2y + 11z = b_2$$
$$2x - y + 3z = b_3.$$