B.Sc. Mathematics – 2nd Semester

MTB 202 – Statics and Dynamics

by

Dr. Krishnendu Bhattacharyya

Department of Mathematics,

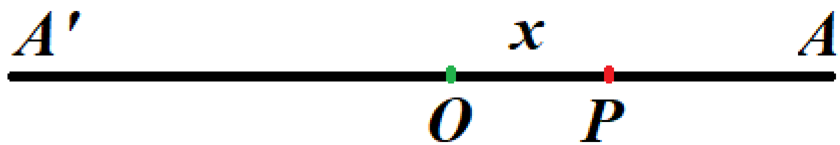
Institute of Science, Banaras Hindu University

Part – IV

Simple Harmonic Motion

A particle is said to execute Simple Harmonic motion if it moves in a straight line such that its acceleration is always directed towards a fixed point on it and is proportional to the distance from the fixed point.

Let O be a fixed point on the line $A'OA$ and let P be the position of the particle at any time t such that $OP = x$ and the velocity at P be v in the direction OA.



So, the acceleration at P is μx, where μ is a positive constant.

Since the acceleration is always directed towards a fixed point *O*, i.e. the acceleration is in the direction opposite to that in which x increases, so, the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\mu x \qquad (1)$$

Or, $$v\frac{dv}{dx} = -\mu x$$

Or, $$vdv = -\mu x dx$$

Integrating we get

$$\frac{v^2}{2} = -\frac{\mu x^2}{2} + c_1 \qquad (2)$$

Since the particle is moving in the direction OA and the acceleration is given to be directed towards the fixed point O, i.e., in the opposite direction, so the particle must come into rest at some point on the line.

Let the point be A such that $OA = a$.

Then putting at $x = a, v = 0$ in (2), we get $c_1 = \dfrac{\mu a^2}{2}$

Hence from (2) we have

$$v^2 = \mu\left(a^2 - x^2\right) \text{ Or, } v = \pm\sqrt{\mu}\sqrt{\left(a^2 - x^2\right)} \qquad (3)$$

This is the expression of the velocity of the particle at P for any displacement x from O.

<u>Case I</u>: Let the time t be measured from the instant when the particle is at O.

In this case, as t increases x also increases and hence $\frac{dx}{dt}$ is positive.

Therefore, from (3), we get

$$v=\sqrt{\mu}\sqrt{\left(a^2-x^2\right)} \text{ Or, } \frac{dx}{dt}=\sqrt{\mu}\sqrt{\left(a^2-x^2\right)} \text{ Or, } \frac{dx}{\sqrt{\left(a^2-x^2\right)}}=\sqrt{\mu}dt$$

Integrating, we get $\sin^{-1}\left(\frac{x}{a}\right)=\sqrt{\mu}t+c_2$, (4) where c_2 is arbitrary constant.

Or $x = a\sin\left(\sqrt{\mu}t + c_2\right)$

Initially, $x = 0$ at $t = 0$. So, using this in (4) we get $c_2 = 0$

Hence, we have $x = a\sin\left(\sqrt{\mu}t\right)$ (5)

Thus (5) gives the position of the particle at any time *t*, when *t* is measured from the fixed point *O*.

As the particle proceeds towards *A* and the acceleration being towards *O*, the velocity goes on decreasing as *x* increases and hence it must vanish at *A* such that $OA = a$. So the particle is at rest for an instant at *A*. Then since the acceleration is directed towards O, the particle moves in the direction AO with a velocity which increases as x decrease and it is maximum at O. Due to this velocity the particle proceeds further to the opposite side A, i.e., towards A'. The velocity again goes on decreasing since the acceleration is directed towards O and the particle comes to rest at A'. Again it moves towards O and thus it is continuously moving between A and A'.

The nature of the motion is therefore oscillatory.

Case II: Let the time t be measured from the instant when the particle is at A (the extreme point or position of rest).

Since x is always measured from the fixed point O and in this case the particle moves towards O, i.e. in the direction AO, so, x decreases as t increases. Therefore, $\frac{dx}{dt}$ is negative.

Hence from (3) we get $\frac{dx}{dt} = -\sqrt{\mu}\sqrt{(a^2 - x^2)}$ Or, $-\frac{dx}{\sqrt{(a^2 - x^2)}} = \sqrt{\mu}dt$

Integrating we get $\cos^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu}t + c_3$ (6)

where c_3 is an arbitrary constant.

Initially at *A*, $x = a, t = 0$. Using this in (6) we get $c_3 = 0$

Hence from (6) we have $x = a\cos\left(\sqrt{\mu}t\right)$ (7)

Thus (7) gives the position of the particle at any time *t*, when *t* is measured from the extreme position.

Amplitude:

The maximum displacement of the particle on either side of the centre of oscillation is called the amplitude of the motion.

Here the amplitude of the simple harmonic motion(SHM) is a.

Periodic Motion:

A particle is said to have periodic motion if it moves in such a manner that it occupies the same position after a certain fixed interval of time and moves in the same direction with same velocity.

In SHM, we have

$$x = a\cos\left(\sqrt{\mu}t\right), \qquad \text{(i)}$$

where t is measured from the position of rest.

Or, $x = a\cos\left(2\pi + \sqrt{\mu}t\right) = a\cos\left\{\sqrt{\mu}\left(t + \frac{2\pi}{\sqrt{\mu}}\right)\right\}$

where a is the amplitude of the motion and x is measured from the mean position.

Now, differentiating both sides of (i) w.r.t t we have

$\frac{dx}{dt} = -a\sqrt{\mu}\sin\left(\sqrt{\mu}t\right)$, Or, $v = -a\sqrt{\mu}\sin\left(2\pi + \sqrt{\mu}t\right)$

So, $v = -a\sqrt{\mu}\sin\left\{\sqrt{\mu}\left(t + \frac{2\pi}{\sqrt{\mu}}\right)\right\}$

Thus, the values of x and v are repeated when t increased by $\frac{2\pi}{\sqrt{\mu}}$.

Hence after every interval of $\frac{2\pi}{\sqrt{\mu}}$, we have the same position and same velocity in the same direction and consequently, SHM is periodic, the period being $\frac{2\pi}{\sqrt{\mu}}$, which is independent of the amplitude a.

Period:

The time required for a complete oscillation is called the periodic time or the period of the oscillation or simply period.

Frequency:

The number of complete oscillations in one second is called frequency.

Thus if n be the frequency and T be the periodic time, then $nT = 1$ or $n = \frac{1}{T}$. So for S.H.M the frequency $n = \frac{\sqrt{\mu}}{2\pi}$

Phase and Epoch:

Re-writing the equation of SHM we have $\frac{d^2x}{dt^2}+\mu x=0$ which is a linear equation with constant coefficient and its general solution is

$$x = a\cos\left(\sqrt{\mu}t+\varepsilon\right) \tag{ii}$$

The constant ε is called the starting phase or the epoch of the motion and the quantity $\left(\sqrt{\mu}t+\varepsilon\right)$ called the argument of the motion.

The phase at any time t of a S.H.M is the time that has elapsed since the particle was at its maximum distance in the positive direction.

From (ii), we get x is maximum when $\cos\left(\sqrt{\mu}t+\varepsilon\right)$ is maximum, i.e. when $\cos\left(\sqrt{\mu}t+\varepsilon\right)=1$

Clearly, if x is maximum at time t_0, then $\cos\left(\sqrt{\mu}t_0+\varepsilon\right)=1$

So, $\sqrt{\mu}t_0+\varepsilon=0 \Rightarrow t_0=-\dfrac{\varepsilon}{\sqrt{\mu}}$.

Phase at $t=t-t_0=t+\dfrac{\varepsilon}{\sqrt{\mu}}=\dfrac{\sqrt{\mu}t+\varepsilon}{\sqrt{\mu}}=\dfrac{\text{argument}}{\sqrt{\mu}}$

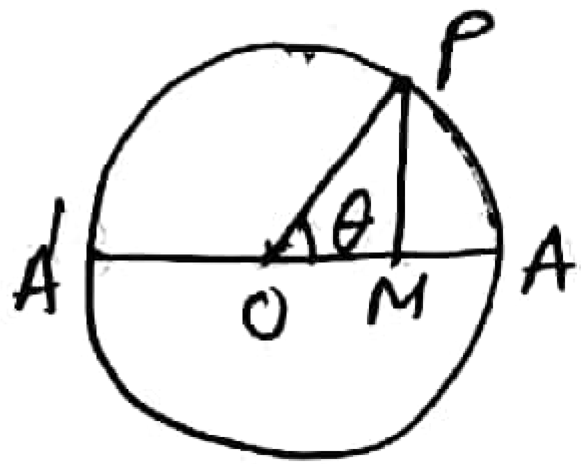
A geometrical representation of the S.H.M:

Let a particle P move on a circle with constant angular velocity ω and let M be the foot of the perpendicular from P on any diameter $A'OA$. If a be the radius of the circle, the only acceleration of P is $\omega^2 a$ towards O.

If $\angle AOP = \theta$ and $OM = x$, the component of this acceleration along OA is $\omega^2 a \cos\theta = \omega^2 a \dfrac{x}{a} = \omega^2 x$ towards O.

Hence the equation of the motion of the point M is $\ddot{x} = -\omega^2 x$

This is a S.H.M.



Thus, if a particle describes a circle with constant angular velocity the foot of the perpendicular from it on any diameter executes a simple harmonic motion.

Note 1: Acceleration of a particle at a distance x from the fixed point O is μx.

Note 2: The acceleration is maximum at the extreme position, i.e. when $x = a$ and thus $f_{\max} = \mu a$

Note 3: The velocity is maximum at the mean position, i.e., when $x = 0$. Thus $v_{\max} = \sqrt{\mu} a$

<u>Time period in terms of velocities and accelerations at two positions</u>:

Let accelerations and velocities at two positions x_1 and x_2 from the centre be a_1, a_2 and v_1, v_2 respectively.

Thus, we have $a_1 = -\mu x_1, a_2 = -\mu x_2$

$$\text{and } v_1^2 = \mu\left(a^2 - x_1^2\right), v_2^2 = \mu\left(a^2 - x_2^2\right)$$

Then we can write

$$a_1 - a_2 = \mu\left(x_2 - x_1\right),\ v_1^2 - v_2^2 = \mu\left(x_2^2 - x_1^2\right) \text{ and } a_2^2 - a_1^2 = \mu^2\left(x_2^2 - x_1^2\right)$$

Therefore, $\mu = \dfrac{a_2^2 - a_1^2}{v_1^2 - v_2^2}$.

Hence the time period T can be written as $T = 2\pi\sqrt{\dfrac{v_1^2 - v_2^2}{a_2^2 - a_1^2}}$.

We also have $T = 2\pi\sqrt{\dfrac{x_2 - x_1}{a_1 - a_2}}$ and $T = 2\pi\sqrt{\dfrac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$.

<u>Amplitude in terms of velocities and acceleration at two positions</u>:

From above we have $a^2 = \dfrac{1}{\mu}\left(v_1^2 + \mu x_1^2\right)$

$$\Rightarrow a^2 = \frac{1}{\mu}\left(v_1^2 + \frac{a_1^2}{\mu}\right) = \frac{1}{\mu^2}\left(v_1^2\mu + a_1^2\right), \qquad \left[\text{as } x_1^2 = \frac{a_1^2}{\mu^2}\right]$$

Then $a^2 = \dfrac{\left(v_1^2 - v_2^2\right)^2}{\left(a_2^2 - a_1^2\right)^2}\left(v_1^2 \dfrac{\left(a_2^2 - a_1^2\right)}{v_1^2 - v_2^2} + a_1^2 \right)$

$$= \frac{\left(v_1^2 - v_2^2\right)}{\left(a_2^2 - a_1^2\right)^2}\left(v_1^2 a_2^2 - v_1^2 a_1^2 + v_1^2 a_1^2 - v_2^2 a_1^2\right)$$

$$= \frac{\left(v_1^2 - v_2^2\right)}{\left(a_2^2 - a_1^2\right)^2}\left(v_1^2 a_2^2 - v_2^2 a_1^2\right)$$

Therefore, $a = \dfrac{1}{\left(a_2^2 - a_1^2\right)}\left\{\left(v_1^2 - v_2^2\right)\left(v_1^2 a_2^2 - v_2^2 a_1^2\right)\right\}^{1/2}.$

Velocity and position in terms of time period:

We have $v^2=\mu\left(a^2-x^2\right)$ and $x=a\cos\sqrt{\mu}t$ with $\sqrt{\mu}=\frac{2\pi}{T}$

$\therefore v^2=\frac{4\pi^2}{T^2}\left(a^2-x^2\right)$ and $x=a\cos\frac{2\pi}{T}t$

Thus the maximum velocity $v_{\max}=\frac{2\pi}{T}a$.

Time between two positions of SHM:

Let at times t_1 and t_2 the particles at position x_1 and x_2 respectively.

We have $v = -\sqrt{\mu}\sqrt{a^2 - x^2}$ i.e., $\dfrac{dx}{dt} = -\sqrt{\mu}\sqrt{a^2 - x^2}$

i.e., $\sqrt{\mu}dt = -\dfrac{dx}{\sqrt{a^2 - x^2}}$

Now integrating the above differential equation from t_1 to t_2 in the left side and from x_1 to x_2 in the right side we get

$$\sqrt{\mu}\int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} -\frac{dx}{\sqrt{a^2 - x^2}}$$

$$\Rightarrow t_2 - t_1 = \frac{1}{\sqrt{\mu}}\left(\cos^{-1}\frac{x_2}{a} - \cos^{-1}\frac{x_1}{a}\right)$$

If the time is t^* moves from $x = b$ to the end point in the same side (while moving towards the end)

$$t^* = \frac{1}{\sqrt{\mu}}\cos^{-1}\frac{b}{a} = \frac{1}{\sqrt{\mu}}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a} == \frac{1}{\sqrt{\mu}}\tan^{-1}\frac{\sqrt{a^2-b^2}}{b}.$$

<u>Time period and maximum velocity in case positions of two zero velocities are on the same side of the origin</u>:

Let both such positions lie on one side of the origin O at a distance $OA' = a$ and $OA = b > a$ respectively.

Then the centre C of the S.H.M lies between the two at a distance $\dfrac{a+b}{2}$ from origin and at a distance $\dfrac{b-a}{2}$ from both positions A and A'.

Therefore the amplitude of the SHM is $\dfrac{b-a}{2}$.

$$\therefore v_{\max} = \sqrt{\mu}\,\frac{b-a}{2} \text{ and } T = \frac{\pi(b-a)}{v_{\max}}.$$

The equation of S.H.M will be

$$\left[\frac{d^2}{dt^2}\left(x-\frac{b+a}{2}\right) = -\mu\left(x-\frac{b+a}{2}\right)\right]$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\mu\left(x-\frac{a+b}{2}\right).$$

Average velocity and acceleration in SHM:

Average velocity $=\frac{4}{T}\int_0^{T/4} v\,dt$ and average acceleration $=\frac{4}{T}\int_0^{T/4} f\,dt$

where T is period of SHM and $x=a\cos\left(\sqrt{\mu}t\right)$, $v=-a\sqrt{\mu}\sin\left(\sqrt{\mu}t\right)$,

$f=-a\mu\cos\left(\sqrt{\mu}t\right)$

Example: A point in a straight line with SHM has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2. Show that the period of the motion is $2\pi\frac{\sqrt{x_1^2-x_2^2}}{v_2^2-v_1^2}$.

Example: A point executes SHM such that in two of its positions the velocities are u, v and the corresponding accelerations are α, β show that the distance between then position is $\dfrac{v^2-u^2}{\alpha+\beta}$ and the amplitude of the motion is $\dfrac{\left[\left(v^2-u^2\right)\left(\alpha^2v^2-\beta^2u^2\right)\right]^{1/2}}{\beta^2-\alpha^2}$.

Solution: We have $\alpha=-\mu x_1, \beta=-\mu x_2$ where x_1 and x_2 are positions for the accelerations α and β and the equation of SHM is $\dfrac{d^2x}{dt^2}=-\mu x$.

Also $u^2 = \mu\left(a^2 - x_1^2\right), v^2 = \mu\left(a^2 - x_2^2\right)$

$\therefore v^2 - u^2 = \mu\left(x_1^2 - x_2^2\right)$....... (1) and $\alpha + \beta = -\mu\left(x_1 + x_2\right)$ .... (2)

Dividing (1) by (2) we get

$\left(x_2 - x_1\right) = \dfrac{v^2 - u^2}{\alpha + \beta}$ (Part 1 proved)

We also have, $\beta^2 - \alpha^2 = \mu^2\left(x_2^2 - x_1^2\right)$ .... (3)

$$u^2 - v^2 = \mu\left(x_2^2 - x_1^2\right) \qquad \text{........} \quad (4)$$

So, $\mu = \dfrac{\beta^2 - \alpha^2}{u^2 - v^2}$.

From above $a^2 = \frac{u^2}{\mu} + x_1^2 = \frac{u^2}{\mu} + \frac{\alpha^2}{\mu^2} = \frac{1}{\mu^2}\left(\mu u^2 + \alpha^2\right)$

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Example: Show that in a SHM the average speed and the average acceleration (in magnitude) are obtain by multiplying their maximum values by 0.637.

Solution: Let the position of the particle executing SHM be $x = a\cos\left(\sqrt{\mu}t\right)$, where a is the amplitude of the SHM, μ is constant and the time period be T. Then $T = \dfrac{2\pi}{\sqrt{\mu}}$. Let v and f be velocity and acceleration at time t.

We have average velocity $= \dfrac{4}{T}\int_0^{T/4} v\,dt$

$$=\frac{4}{T}\int_0^{T/4}\left\{-a\sqrt{\mu}\sin\left(\sqrt{\mu}t\right)\right\}dt=\frac{4a}{T}\left[-\cos\left(\sqrt{\mu}t\right)\right]_0^{T/4}=\frac{4a}{T}\left[-\cos\left(\sqrt{\mu}\frac{T}{4}\right)\right]$$

$$=-\frac{4a}{T}.$$

Average acceleration $=\frac{4}{T}\int_0^{T/4} f dt$

$$=\frac{4}{T}\int_0^{T/4}\left\{-a\mu\cos\left(\sqrt{\mu}t\right)\right\}dt=-\frac{4a\sqrt{\mu}}{T}\left[\sin\left(\sqrt{\mu}t\right)\right]_0^{T/4}$$

$$=-\frac{4a\sqrt{\mu}}{T}\sin\left(\sqrt{\mu}\frac{T}{4}\right)=-\frac{4a\frac{2\pi}{T}}{T}\sin\left(\frac{2\pi}{T}\frac{T}{4}\right)=-\frac{4a2\pi}{T^2}=-\frac{8a\pi}{T^2}.$$

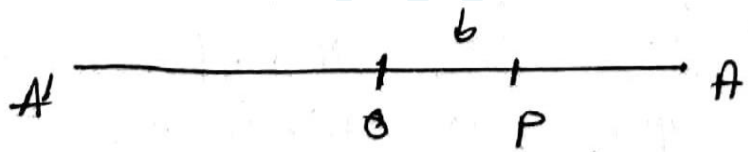
We know, $v_{\max} = \sqrt{\mu}a = \frac{2\pi}{T}a$ and $f_{\max} = \mu a = \frac{4\pi^2}{T^2}a$.

So, $\frac{v_{ave}}{v_{\max}} = \frac{2}{\pi} = 0.637$ (approx.) and $\frac{f_{ave}}{f_{\max}} = \frac{2}{\pi} = 0.637$.

<u>Example</u>: A body moving in straight line *OAB* with S.H.M has zero velocity when at points *A* and *B* whose distance from *O* are *a* and *b* respectively and has a velocity v when half-way between them show that the complete period is $\frac{\pi(b-a)}{v}$.

Example: A particle is performing SHM of period T about a centre O and it passes through a point P ($OP=b$) with velocity v in the direction OP. Prove that the time which elapses before it returns to P is $\frac{T}{\pi}\tan^{-1}\frac{vT}{2\pi b}$

Solution:



$$v=\frac{dx}{dt}=-\sqrt{\mu}\sqrt{a^2-x^2}\ \ldots\ldots\ldots\ (1)$$

$$\Rightarrow -\frac{dx}{\sqrt{a^2-x^2}}=\sqrt{\mu}dt$$

Integrating from $x=b$ to $x=a$ we have

$$\int_b^a -\frac{dx}{\sqrt{a^2-x^2}} = \sqrt{\mu}\int_{t_1}^{t_2} dt,$$

where t_1 is the time at $x=b$ and t_2 is the time at $x=a$

So, $\left(t_2-t_1\right)=\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{b}{a}=\frac{1}{\sqrt{\mu}}\tan^{-1}\frac{\sqrt{a^2-b^2}}{b}$

From (1) we have for $x=b \quad v=-\sqrt{\mu}\sqrt{a^2-b^2}$.

Also we have $\sqrt{\mu}=\frac{2\pi}{T}$ as T is the period.

$$\therefore (t_2 - t_1) = \frac{T}{2\pi}\tan^{-1}\left(-\frac{vT}{2\pi b}\right)$$

So the time from $x = b$ to $x = a$ is

$$\frac{T}{2\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right).$$

Example: A particle is moving with SHM and while making an excursion from one position of rest to the other its distance from the middle point of its path at three consecutive seconds are observed to be x_1, x_2, x_3. Prove that the time of a complete revolution is $2\pi \Big/ \cos^{-1}\left(\dfrac{x_1 + x_3}{2x_2}\right)$

Example: A particle starts from rest under an acceleration $k^2 x$ directed towards a fixed point and after time t another particle starts from the same position under the same acceleration. Show that the particles will collide at time $\dfrac{\pi}{k} + \dfrac{t}{2}$ after the start of the first particle provided $t < \dfrac{2\pi}{k}$.