Let $z = f(x, y)$ be a function of two independent variables x and y, defined in a domain N and let it be differentiable at a point (x, y) of the domain. The first differential of z at (x, y), denoted by dz is given by

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad \text{...(1)}$$

If dx and dy are regarded as constants and if $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiable at (x, y) then dz is a function of x and y and is itself differentiable at (x, y). The differential of dz, called the *second differential* of z, is denoted by d^2z and is calculated in the same way as the first.

$$\therefore \qquad d^2z = d(dz) = d\left(\frac{\partial z}{\partial x}\right)dx + d\left(\frac{\partial z}{\partial y}\right)dy \qquad ...(2)$$

Replacing z by $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in (1), we get

$$d\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}dx + \frac{\partial^2 z}{\partial y \partial x}dy$$

$$d\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}dx + \frac{\partial^2 z}{\partial y^2}dy$$

Also by Young's theorem, since $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiable, we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

$$\therefore \qquad d^2z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}dy^2 \qquad ...(3)$$

where, of course, $dx^2 = dx \cdot dx = (dx)^2$, $dy^2 = (dy)^2$

In abbreviated notation, it may be written as

$$d^2z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 z \qquad ...(4)$$

Again d^2z is differentiable at (x, y) if all the second order partial derivatives $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ are differentiable at (x, y). This condition also ensures the legitimacy of inverting the order of the partial derivatives with respect to x and with respect to y, and so

$$d^3z = \frac{\partial^3 z}{\partial x^3}dx^3 + 3\frac{\partial^3 z}{\partial x^2 \partial y}dx^2\,dy + 3\frac{\partial^3 z}{\partial x \partial y^2}dx\,dy^2 + \frac{\partial^3 z}{\partial y^3}\partial y^3 = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^3 z \qquad ...(5)$$

Proceeding in the manner, we can define the successive differentials $d^4z, d^5z,...$ Thus the differential of nth order, d^nz exists if $d^{n-1}z$ is differentiable, which implies that all the partial derivatives of the $(n-1)$th order are differentiable. This condition also ensures the legitimacy of inverting the order of the partial derivatives with respect to x and with respect to y in the partial derivatives of order n. Thus it may be shown by Mathematical induction that

$$d^n z = \frac{\partial^n z}{\partial x^n} dx^n + n\frac{\partial^n z}{\partial x^{n-1}\partial y} dx^{n-1} dy + \frac{n(n-1)}{2!} \frac{\partial^n z}{\partial x^{n-2}\partial y^2} dx^{n-2} dy^2 + \ldots + \frac{\partial^n z}{\partial y^n} dy^n$$

$$= \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^n z.$$

Note: In the above discussion, x and y are *Independent Variables* and so dx and dy may be treated as constants. The reason for this being so is that the differentials of independent variables are the arbitrary increments of these variables, $dx = \delta x$, $dy = \delta y$.

7. FUNCTIONS OF FUNCTIONS

So far we have considered functions of the form

$$z = f(x, y, \ldots)$$

where the variables $x, y, \ldots$ are the independent variables. We now consider functions

$$z = f(x, y, \ldots)$$

where $x, y, \ldots$ are not independent variables, but are themselves functions of other independent variables $u, v, \ldots$, so that

$$x = g(u, v, \ldots) \text{ and } y = h(u, v, \ldots)$$

To fix the ideas, we consider only two variables x and y as functions of two independent variables u and v. The method of proof is, however, general.

Theorem 5. *If $z = f(x, y)$ is a differentiable function of x, y and $x = g(u, v)$, $y = h(u, v)$ are themselves differentiable functions of the independent variables u, v, then z is a differentiable function of u, v and*

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

just as though x, y were the independent variables.

Let (u, v), $(u + \delta u, v + \delta v)$ be two neighbouring points of the domain of definition of x and y, and (x, y), $(x + \delta x, y + \delta y)$ the corresponding points of the domain of definition of z, so that

$$\delta x = g(u + \delta u, v + \delta v) - g(u, v)$$

$$\delta y = h(u + \delta u, v + \delta v) - h(u, v)$$

The differentiability, and hence the continuity of g and h imply that

$$\delta x \to 0, \delta y \to 0, \text{ as } (\delta u, \delta v) \to (0, 0) \qquad \ldots(1)$$

Again, since g and h are differentiable function of u and v,

$$\delta x = g_u \delta u + g_v \delta v + \phi_1 \delta u + \psi_1 \delta v \qquad \ldots(2)$$

$$\delta y = h_u \delta u + h_v \delta v + \phi_2 \delta u + \psi_2 \delta v,$$

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where $\phi_1, \phi_2, \psi_1, \psi_2$ are functions of δu, δv, and tend to zero as,

$$(\delta u, \delta v) \to (0, 0).$$

Also, $dx = g_u\, du + g_v\, dv$, $dy = h_u\, du + h_v\, dv$.

Also, since f is a differentiable function of x, y, we have

$$\delta z = f_x\, \delta x + f_y\, \delta y + \phi_3\, \delta x + \psi_3\, \delta y, \qquad \text{...(3)}$$

where ϕ_3, ψ_3 are functions of $\delta x, \delta y$, and tend to zero as $(\delta x, \delta y) \to (0, 0)$.

From equations (1), (2), and (3) we get

$$\delta z = (f_x g_u + f_y h_u)\, \delta u + (f_x g_v + f_y h_v)\, \delta v + F_1\, \delta u + F_2\, \delta v$$

where
$$F_1 = f_x\phi_1 + f_y\phi_2 + \phi_3 g_u + \phi_3\phi_1 + \psi_3 h_u + \psi_3\phi_2$$
$$F_2 = f_x\psi_1 + f_y\psi_2 + \phi_3 g_v + \phi_3\psi_1 + \psi_3 h_v + \psi_3\psi_2$$

Since the coefficients F_1 and F_2 of $\delta u, \delta v$ tend to zero as $(\delta u, \delta v) \to (0, 0)$, therefore z is a *differentiable function of* u, v and

$$\begin{aligned} dz &= (f_x g_u + f_y h_u)\, du + (f_x g_v + f_y h_v)\, dv \\ &= f_x(g_u\, du + g_v\, dv) + f_y(h_u\, du + h_v\, dv) \\ &= f_x\, dx + f_y\, dy \end{aligned}$$

$$\therefore \quad dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

Remark: The theorem establishes a fact of fundamental importance that *the first differential of a function is expressed always by the same formula, whether the variables concerned are independent or whether they are themselves functions of other independent variables.*

Note: The differential dz is sometimes referred to as the *total differential.*

7.1 Differentials of Higher Order of a Function of Functions

If $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ are differentiable functions of x, y so that they are also differentiable functions of u, v, and dx, dy are differentiable functions of u, v, then from the preceding theorem we have

$$d^2 z = d(dz) = d\left(\frac{\partial z}{\partial x}\right) dx + \frac{\partial z}{\partial x} d^2 x + d\left(\frac{\partial z}{\partial y}\right) dy + \frac{\partial z}{\partial y} d^2 y \qquad \text{...(1)}$$

and on comparison with (2) and (3) of § 6, we see that

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2 + \frac{\partial z}{\partial x} d^2 x + \frac{\partial z}{\partial y} d^2 y \qquad \text{...(2)}$$

$$= \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^2 z + \frac{\partial z}{\partial x} d^2 x + \frac{\partial z}{\partial x} d^2 y$$

The differentials of higher orders can be written in the same manner, but their formation becomes more and more complicated and lengthy, and no simple general formula for $d^n z$ can be given.

The introduction of more than two *intermediary variables** causes no fresh difficulty. Thus, when $z = f(x_1, x_2, x_3)$ and x_1, x_2, x_3 are not the independent variables,

$$d^2 z = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \frac{\partial}{\partial x_3}dx_3\right)^2 z + \frac{\partial z}{\partial x_1}d^2x_1 + \frac{\partial z}{\partial x_2}d^2x_2 + \frac{\partial z}{\partial x_3}d^2x_3$$

Note: If x, y are linear functions of independent variables u and v, *i.e.*, x and y are of the form $x = a + bu + cv$, $y = a' + b'u + c'v$ then dx and dy are constants, and so d^2x, d^2y and all higher differentials of x and y are zero, and therefore

$$d^n z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n z,$$

the form being same as for independent x and y.

7.2 The Derivation of Composite Functions (*The chain rule*)

From the preceding theorem we deduce two important results:

I. *If*

(*i*) *x, y be differentiable functions of a single variable, and*

(*ii*) *z is differentiable function of x and y,*

then z possesses continuous derivative with respect to t, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Because of (*i*),

$$dx = \frac{dx}{dt}\cdot dt, \text{ and } dy = \frac{dy}{dt}\cdot dt$$

Since z is a differentiable function of x and y, and x, y are differentiable functions of t, we deduce from § 7, that z is a differentiable function of t.

$$\therefore \quad dz = \frac{dz}{dt}\cdot dt \qquad \text{...(}$$

Also

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{\partial z}{\partial x}\frac{dx}{dt}dt + \frac{\partial z}{\partial y}\frac{dy}{dt}dt$$

From equations (1) and (2), ...(3)

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

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* Variables like x, y which are functions of independent variables u, v are called *intermediary variables*.

Again because of conditions (i) and (ii), $\dfrac{dz}{dt}$ is a continuous function of t.

Corollary. *If $z = f(x, y)$ possesses nth order partial derivatives, and x, y are linear functions of a single variable t, i.e., $x = a + ht$, $y = b + kt$, where a, b, h, k are constants, then*

$$\frac{d^n z}{dt^n} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n z$$

Now

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)z \qquad \text{...(1)}$$

Replacing z by $\left(h\dfrac{\partial z}{\partial x} + k\dfrac{\partial z}{\partial y}\right)$ in (1), we get

$$\frac{d^2 z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right) = h\frac{\partial}{\partial x}\left(h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y}\right) + k\frac{\partial}{\partial y}\left(h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y}\right)$$

$$= h^2\frac{\partial^2 z}{\partial x^2} + 2hk\frac{\partial^2 z}{\partial x \partial y} + k^2\frac{\partial^2 z}{\partial y^2} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 z$$

By induction, we may obtain the required expression for $\dfrac{d^n z}{dt^n}$.

II. *If*

(i) *x, y are differentiable functions of two independent variables u and v, and*

(ii) *z is a differentiable function of x and y,*

then z possesses continuous partial derivatives with respect to u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \textit{ and } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Because of (i)

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \\ dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \end{aligned}\right\} \qquad \text{...(1)}$$

Since z is a differentiable function of x and y and x, y are differentiable functions of u and v, we deduce from § 7, that z is a differentiable function of u, and v, and

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \qquad \text{...(2)}$$

Also
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{\partial z}{\partial x}\left(\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$$
$$= \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right)du + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right)dv \quad ...(3)$$

Hence, from equations (2) and (3), we get
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \text{ and } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Again, because of conditions (*i*) and (*ii*) we see that $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ are continuous functions of u, v.

Note: In (1) when x is a function of a single variable t, we have $dx = \frac{dx}{dt}dt$, so that the derivative $\frac{dx}{dt}$ appears as the coefficient of a differential and that is precisely the reason why the derivative is also called the *differential coefficient*.

Example 19. If $z = e^{xy^2}$, $x = t\cos t, y = t\sin t$, compute $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$.

■ $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^2 e^{xy^2})(\cos t - t\sin t) + (2xye^{xy^2})(\sin t + t\cos t)$$

At $t = \frac{\pi}{2} \Rightarrow x = 0, y = \frac{\pi}{2}$.

$$\therefore \qquad \left[\frac{dz}{dt}\right]_{t=\pi/2} = \frac{\pi^2}{4}\left(-\frac{\pi}{2}\right) = -\frac{\pi^3}{8}.$$

Example 20. If $z = x^3 - xy + y^3$, $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta}$.

■ $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (3x^2 - y)\cos\theta + (3y^2 - x)\sin\theta$$
$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = (3x^2 - y)(-r\sin\theta) + (3y^2 - x)\, r\cos\theta.$$

Example 21. Show that $z = f(x^2 y)$, where f is differentiable, satisfies
$$x\left(\frac{\partial z}{\partial x}\right) = 2y\left(\frac{\partial z}{\partial y}\right).$$

■ Let $x^2 y = u$, so that $z = f(u)$. Thus

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = f'(u) \cdot 2xy$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = f'(u) \cdot x^2$$

$\therefore$
$$x\frac{\partial z}{\partial x} = f'(u)\, 2x^2 y = 2y\frac{\partial z}{\partial y}$$

Aliter. $dz = f'(u)\,du = f'(x^2 y)\,(2xy\,dx + x^2\,dy)$

Also $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial x}dy$

Then $\frac{\partial z}{\partial x} = 2xyf'(x^2 y),\ \frac{\partial z}{\partial y} = x^2 f'(x^2 y)$

The result now follows as above.

Example 22. If for all values of the parameter λ, and for some constant n, $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ (F is then called a *homogeneous function* of degree n), identically where F is assumed differentiable, prove that $x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = nF$. Hence show that, for $F(x, y) = x^4 y^2 \sin^{-1}\frac{y}{x}$,

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 6F.$$

■ Let $\lambda x = u$, $\lambda y = v$. Then

$$F(u, v) = \lambda^n F(x, y) \qquad \text{...(1)}$$

The derivative w.r.t. λ of the left side of (1) is

$$\frac{\partial F}{\partial \lambda} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial \lambda} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial \lambda} = x\frac{\partial F}{\partial u} + y\frac{\partial F}{\partial v}$$

The derivative w.r.t. λ of the right side of (1) is $n\lambda^{n-1} F(x, y)$. Then

$$x\frac{\partial F}{\partial u} + y\frac{\partial F}{\partial v} = n\lambda^{n-1} F$$

The result follows for $\lambda = 1$, then $u = x$, $v = y$.

Again, since $F(\lambda x, \lambda y) = (\lambda x)^4 (\lambda y)^2 \sin^{-1} y/x = \lambda^6 F(x, y)$, the result follows for $n = 6$. That it is so, can also be shown by direct differentiation.

Example 23. If z is given as a function of two independent variables x and y, change the variables so that x becomes the function, and z and y the independent variables, and express the first and second order partial derivatives of x with respect to z and y in terms of the derivatives of z with respect to x and y.