B.Sc. Mathematics – 2nd Semester

MTB 202 – Statics and Dynamics

by

Dr. Krishnendu Bhattacharyya

Department of Mathematics,

Institute of Science, Banaras Hindu University

Part – III

General Motion in Two Dimensions

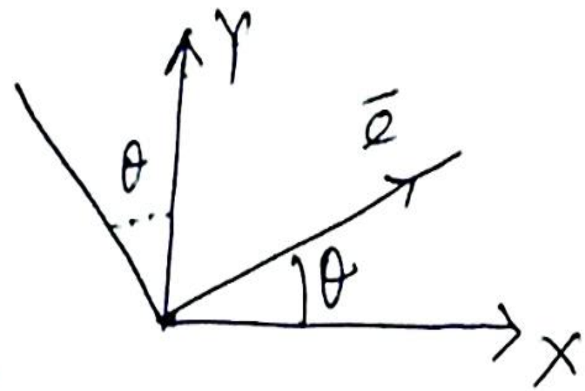
Rotation of a vector

Let a vector $\vec{A}$ be defined in a plane OXY. Let it be rotating about its tip at the origin. Let at any moment (time) its direction makes θ angle with the x-axis OX. Let $\vec{e}$ be unit vector along the direction of $\vec{A}$. The vector changes due to change in its magnitude as well as its direction.

The vector is defined as $\vec{A} = A\vec{e}$.

Now its rate of change with respect to θ may be expressed as

$$\frac{d\vec{A}}{d\theta} = A\frac{d\vec{e}}{d\theta} + \vec{e}\,\frac{dA}{d\theta}, \qquad (1)$$



where on right hand side first term expresses change in the direction, while second term expresses change in its magnitude.

Taking dot product of (1) with the unit vector $\vec{e}$ we have

$$\vec{e}.\frac{d\vec{A}}{d\theta}=A\left(\vec{e}\,\frac{d\vec{e}}{d\theta}\right)+\frac{dA}{d\theta} \qquad (2)$$

Now, $\dfrac{dA}{d\theta}=\dfrac{1}{2A}\dfrac{dA^2}{d\theta}=\dfrac{1}{2A}\dfrac{d}{d\theta}\left(\vec{A}.\vec{A}\right)=\dfrac{1}{2A}\left[\vec{A}.\dfrac{d\vec{A}}{d\theta}+\dfrac{d\vec{A}}{d\theta}.\vec{A}\right]$

$$\text{i.e., } \frac{dA}{d\theta}=\frac{\vec{A}}{A}.\frac{d\vec{A}}{d\theta}=\vec{e}.\frac{d\vec{A}}{d\theta} \qquad (3)$$

Then from (2) and (3) we have

$$\vec{e}.\frac{d\vec{e}}{d\theta}=0 \text{ as } A\neq 0 \tag{4}$$

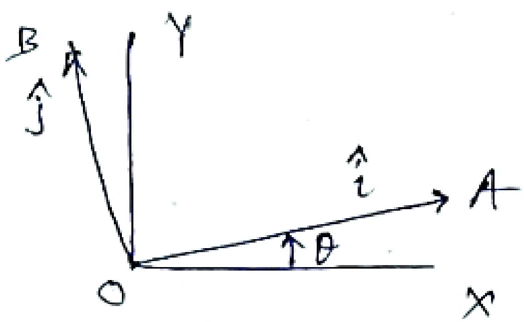
Now we can conclude following two results:

(1) The magnitude of an unit vector rotating in two dimension about its initial point does not change with respect to θ.

(2) If $\vec{e}$ be a unit vector, then $\dfrac{d\vec{e}}{d\theta}$ is another unit vector, but perpendicular to $\vec{e}$.

Now let us apply above results on two mutually perpendicular unit vectors $\hat{i}$ and $\hat{j}$.

Let $\hat{i}$ be along *OA* and $\hat{j}$ be along *OB*. Plane *OAB* be rotating with respect to a fixed plane *OXY*.



Let at any moment (time), *OA* makes angle θ with *OX*, then $\dfrac{d\hat{i}}{d\theta}$ will represent a unit vector perpendicular to *OA*, i.e. along *OB*. Further $\dfrac{d\hat{j}}{d\theta}$ will represent a unit vector along *OA*.

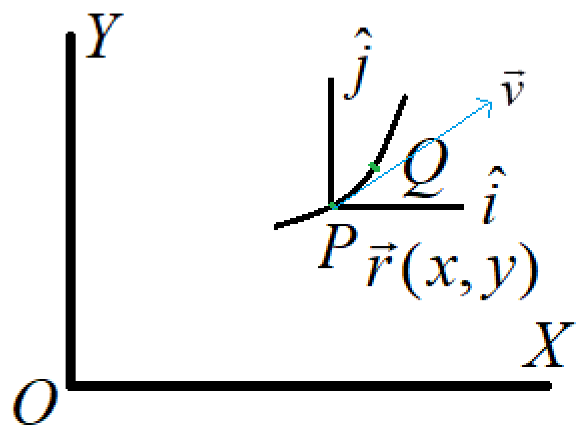
Thus we have $\dfrac{d\hat{i}}{d\theta} = \hat{j}$ and $\dfrac{d\vec{j}}{d\theta} = -\hat{i}$,

where negative sign occurs because the sense of change in $\hat{j}$ will be opposite to that in $\hat{i}$.

Cartesian Components of Motion

Let OXY be a Cartesian plane a fixed frame of reference. Let a particle describe a path in the plane. Let us consider two positions on the path P and Q very close to each other. Two perpendicular directions at both positions are parallel to x-axis and y-axis respectively.

Thus as the position changes on the path, the two perpendicular directions do not change.



Let position vector of P be $\vec{r}(x, y)$. If $\hat{i}$ and $\hat{j}$ are unit vectors along x-axis and y-axis, respectively, then we may write $\vec{r} = x\hat{i} + y\hat{j}$

Now, the rate of change with respect to time t with reference to fixed frame will be the linear velocity $\vec{v}$ of the particle and it is given by

$$\dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + x\dot{\hat{i}} + y\dot{\hat{j}} \ .$$

But, in Cartesian system unit vector does not change. So we have $\dot{\hat{i}} = \dot{\hat{j}} = 0$. Thus we have $\dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} = \vec{v}$

Now, if α be angle made by the velocity vector with the positive x-axis, v is the magnitude of the velocity then

$$v^2 = \dot{x}^2 + \dot{y}^2 \text{ and } \dot{x} = v\cos\alpha, \dot{y} = v\sin\alpha \text{, i.e., } \tan\alpha = \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}.$$

Similarly, if $\vec{a}$ be acceleration with $|\vec{a}| = a$ and β be the angle between the acceleration vector and positive x-axis, then

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j},\ a^2 = \ddot{x}^2 + \ddot{y}^2 \text{ and } \tan\beta = \frac{\ddot{y}}{\ddot{x}}.$$

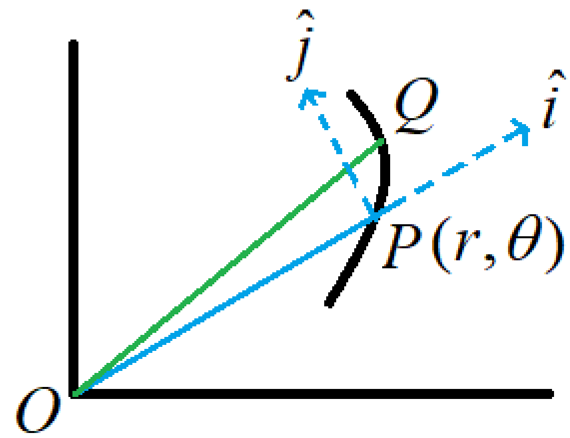
So, it is clear from above that the direction of the velocity and acceleration at any point on the path are different.

Radial and Transverse Components (Polar Components)

Two perpendicular directions in polar system of a two dimensional frame of reference are radial and transverse directions (cross radial direction).

Let the position of a moving particle at time on its path be $P(r,\theta)$.

Further, let unit vector along the radial direction OP be $\hat{i}$ and that along transverse direction at P along θ increasing direction be $\hat{j}$.



It may be noted that the radial directions at two very close positions P and Q are not parallel. It means that the radial direction $\hat{i}$ and the transverse direction $\hat{j}$ change with θ.

So we have $\dfrac{d\hat{i}}{d\theta} = \hat{j}$ and $\dfrac{d\hat{j}}{d\theta} = -\hat{i}$.

Now, the position vector at P be $\vec{r} = r\hat{i}$, where $|\vec{r}| = r$.

Now, its rate of change with time with reference to fixed frame will give the linear velocity of the particle at P along tangent, i.e.,

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = \dot{r}\hat{i} + r\frac{d\hat{i}}{dt} = \dot{r}\hat{i} + r\frac{d\hat{i}}{d\theta}\frac{d\theta}{dt}$$

$$\vec{v} = \dot{r}\hat{i} + r\dot{\theta}\hat{j} \qquad (1)$$

So, the two velocity components at $P(r,\theta)$ are

(i) Radial velocity $\dot{r}$ along the radial direction $(\hat{i})$.

(ii) Transverse velocity $r\dot{\theta}$ along the transverse direction $(\hat{j})$.

If the velocity vector makes an angle α with the radial direction then

$$\tan\alpha = \frac{r\dot{\theta}}{\dot{r}} = r\frac{d\theta}{dr} \text{ and } v^2 = \dot{r}^2 + (r\dot{\theta})^2, \text{ where } |\vec{v}| = v.$$

Differentiating (1) with respect to time with reference to fixed frame we get

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\hat{i} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{j} + \dot{r}\frac{d\hat{i}}{dt} + r\dot{\theta}\frac{d\hat{j}}{dt}$$

$$\text{i.e., } \vec{a} = \ddot{r}\hat{i} + \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{j} + \dot{r}\frac{d\hat{i}}{d\theta}\frac{d\theta}{dt} + r\dot{\theta}\frac{d\hat{j}}{d\theta}\frac{d\theta}{dt}$$

$$\text{i.e., } \vec{a} = \ddot{r}\hat{i} + \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{j} + \dot{r}\dot{\theta}\hat{j} + r\dot{\theta}^2\left(-\hat{i}\right)$$

$$\text{i.e., } \vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{i} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{j}$$

Thus radial and transverse acceleration components are

$$\ddot{r} - r\dot{\theta}^2 \text{ and } 2\dot{r}\dot{\theta} + r\ddot{\theta}, \text{ i.e., } \ddot{r} - r\dot{\theta}^2 \text{ and } \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right).$$

Also we have $a^2 = \left(\ddot{r} - r\dot{\theta}^2\right)^2 + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)^2$ and $\tan\beta = \dfrac{2\dot{r}\dot{\theta} + r\ddot{\theta}}{\ddot{r} - r\dot{\theta}^2}$,

where $a = |\vec{a}|$ and β is the angle which the direction of $\vec{a}$ makes with the radial direction.

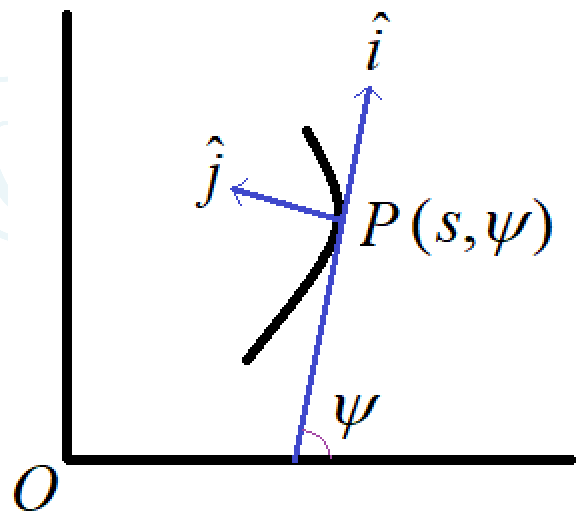
<u>Tangential and Normal Components (Intrinsic Components)</u>

Two perpendicular direction in intrinsic systems are tangential direction, i.e., arc length increasing direction and normal direction.

Let a particle be describing a plane curve and at any time t $P(s,\psi)$ be position of it. Let $\hat{i}$ and $\hat{j}$ be the unit vectors along tangential and normal directions. The two perpendicular directions change when the particle moves to a neighboring point Q, i.e., those changes with ψ.

Hence, we have $\dfrac{d\hat{i}}{d\psi} = \hat{j}$ and $\dfrac{d\hat{j}}{d\psi} = -\hat{i}$.

Now, the linear velocity at P will be the rate of change of arc length w.r.t. time and it is $\dot{s}=\dfrac{ds}{dt}$ along the tangent.



No normal velocity component is present.

Thus, the velocity (linear velocity) $\vec{v}$ is at P may be written as

$$\vec{v}=\dot{s}\hat{i}\text{, i.e., }|\vec{v}|=v=\dot{s}.$$

So, the acceleration of the particle will be

$$\vec{a}=\frac{d\vec{v}}{dt}=\ddot{s}\hat{i}+\dot{s}\frac{d\hat{i}}{dt}=\ddot{s}\hat{i}+\dot{s}\frac{d\hat{i}}{d\psi}\frac{d\psi}{ds}\frac{ds}{dt}$$

$$\text{i.e., } \vec{a} = \ddot{s}\hat{i} + \dot{s}\hat{j}\frac{1}{\rho}\dot{s} = \ddot{s}\hat{i} + \frac{v^2}{\rho}\hat{j},$$

where $\rho = \dfrac{ds}{d\psi}$ is the radius of curvature of the curve at P and it may be obtained from intrinsic equation $s = f(\psi)$ of the curve.

Thus, the tangential and normal accelerations are $\ddot{s}$ and $\dfrac{v^2}{\rho}$.

If the acceleration makes angle α with the tangential direction and $|\vec{a}| = a$ then $a^2 = \ddot{s}^2 + \left(\dfrac{v^2}{\rho}\right)^2$ and $\tan\alpha = \dfrac{v^2/\rho}{\ddot{s}}$.

Example: If the radial and transverse velocities of a point are always proportional to each other and this holds for acceleration also, then prove that its velocity will vary as some power of the radius vector.

Solution: It is given that

$$\dot{r} \propto r\dot{\theta} \text{ and } \ddot{r} - r\dot{\theta}^2 \propto \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$$

$$\Rightarrow \dot{r} = \lambda r\dot{\theta} \;\ldots\ldots\; (1) \quad \text{and} \quad \ddot{r} - r\dot{\theta}^2 = \frac{\mu}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$$

$$\ddot{r} - r\dot{\theta}^2 = \mu\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) \;\ldots\ldots\; (2)$$

Let v be the velocity. Then $v=\sqrt{\dot{r}^2+\left(r\dot{\theta}\right)^2}=\sqrt{\dot{r}^2+\frac{\dot{r}^2}{\lambda^2}}\;\left[\text{as } r\dot{\theta}=\frac{\dot{r}}{\lambda}\right]$,

i.e., $v=\left(1+\frac{1}{\lambda^2}\right)^{1/2}\dot{r}$.

Now from (2) we have

$$\ddot{r}-r\left(\frac{\dot{r}}{\lambda r}\right)^2=\frac{\mu}{r}\frac{d}{dt}\left(r^2\frac{\dot{r}}{\lambda r}\right)\Rightarrow\ddot{r}-\frac{\dot{r}^2}{\lambda^2 r}=\frac{\mu}{\lambda r}\frac{d}{dt}\left(r\dot{r}\right)=\frac{\mu}{\lambda r}\left(r\ddot{r}+\dot{r}^2\right)$$

$$\Rightarrow\ddot{r}-\frac{\dot{r}^2}{\lambda^2 r}=\frac{\mu}{\lambda}\ddot{r}+\frac{\mu}{\lambda}\frac{\dot{r}^2}{r}\Rightarrow\ddot{r}-\frac{\mu}{\lambda}\ddot{r}=\frac{\dot{r}^2}{\lambda^2 r}+\frac{\mu}{\lambda}\frac{r^2}{r}$$

$$\Rightarrow \ddot{r}\left(1-\frac{\mu}{\lambda}\right)=\left(\frac{1}{\lambda^2}+\frac{\mu}{\lambda}\right)\frac{\dot{r}^2}{r}$$

$$\Rightarrow A\ddot{r}=B\frac{\dot{r}^2}{r},\ \text{ where } A=1-\frac{\mu}{\lambda}, B=\frac{1}{\lambda^2}+\frac{\mu}{\lambda}$$

$$\Rightarrow \ddot{r}=C\frac{\dot{r}^2}{r}\ \left(\text{where } C=\frac{B}{A}\right)\Rightarrow \frac{\ddot{r}}{\dot{r}}=C\frac{\dot{r}}{r}$$

Integrating we have $\log\dot{r}=C\log r+\log D \Rightarrow \dot{r}=Dr^C$

Therefore, $v=\left(1+\frac{1}{\lambda^2}\right)^{1/2} Dr^C$.

Thus, the velocity is proportional to any arbitrary power of r.

<u>Example</u>: The velocities of a particle along and perpendicular to radius vector from a fixed origin are λr^2 and $\mu\theta^2$. Show that the equation to the path is $\dfrac{\lambda}{\theta}=\dfrac{\mu}{2r}+c$, and the components of acceleration are

$$2\lambda^2 r^3-\mu^2\frac{\theta^4}{r} \text{ and } \lambda\mu r\theta^2+2\mu^2\frac{\theta^3}{r}.$$

<u>Solution</u>: It is given that

$$\dot{r}=\lambda r^2 \;\ldots\ldots\; (1) \quad \text{and} \quad r\dot{\theta}=\mu\theta^2 \;\ldots\ldots\; (2)$$

Dividing (1) by (2) we have $\dfrac{\frac{dr}{dt}}{r\,\frac{d\theta}{dt}}=\dfrac{\lambda r^2}{\mu\theta^2}$

$$\Rightarrow \mu\frac{dr}{r^3}=\lambda\frac{d\theta}{\theta^2} \Rightarrow -\frac{\mu}{2r^2}=-\frac{\lambda}{\theta}+c \quad \text{(where } c \text{ is constant)}$$

$$\Rightarrow \frac{\lambda}{\theta}=\frac{\mu}{2r^2}+c$$

We have

Radial acceleration $=\ddot{r}-r\dot{\theta}^2$ and Transverse acceleration $=2\dot{r}\dot{\theta}+r\ddot{\theta}$.

From (2) we have $r\dot{\theta}=\mu\dot{\theta}^2$

Differentiating w.r.t. t we have $\dot{r}\dot{\theta}+r\ddot{\theta}=2\mu\theta\dot{\theta}$.

Therefore $2\dot{r}\dot{\theta}+r\ddot{\theta}=2\mu\theta\dot{\theta}+\dot{r}\dot{\theta}\ =2\mu\theta\dfrac{\mu\theta^2}{r}+\lambda r^2\dfrac{\mu\theta^2}{r}\ =\lambda\mu r\theta^2+2\mu^2\dfrac{\theta^3}{r}$,

which is the transverse acceleration.

From (1) we have

$$\dot{r}=\lambda r^2 \Rightarrow \ddot{r}=2\lambda r\dot{r}=2\lambda r\lambda r^2 \Rightarrow \ddot{r}=2\lambda^2 r^3.$$

Therefore $\ddot{r}-r\dot{\theta}^2=2\lambda^2 r^3-r\left(\dfrac{\mu\theta^2}{r}\right)=2\lambda^2 r^3-\mu^2\dfrac{\theta^4}{r}$, which is the radial acceleration.

Example: Prove that the path of a point which possesses two constant velocities one along a fixed direction and the other perpendicular to the radius vector drawn from a fixed point is a conic.

Example: A particle moves along circle $r = 2a\cos\theta$ in such a way that its acceleration towards the origin is always zero. Prove that

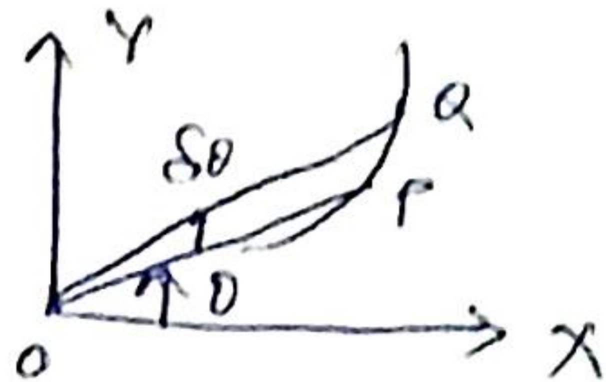
$$\frac{d^2\theta}{dt^2} = -2\cot\theta.\dot{\theta}^2.$$

Angular Velocity and Acceleration

The angular velocity of a point P about another point O is the rate of change of the angle which or makes with some fixed direction.

The angular velocity of P about O

$$= \lim_{\delta t \to 0} \frac{\delta \theta}{\delta t} = \frac{d\theta}{dt} = \dot{\theta}.$$



Similar, angular acceleration is

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2} = \ddot{\theta}.$$

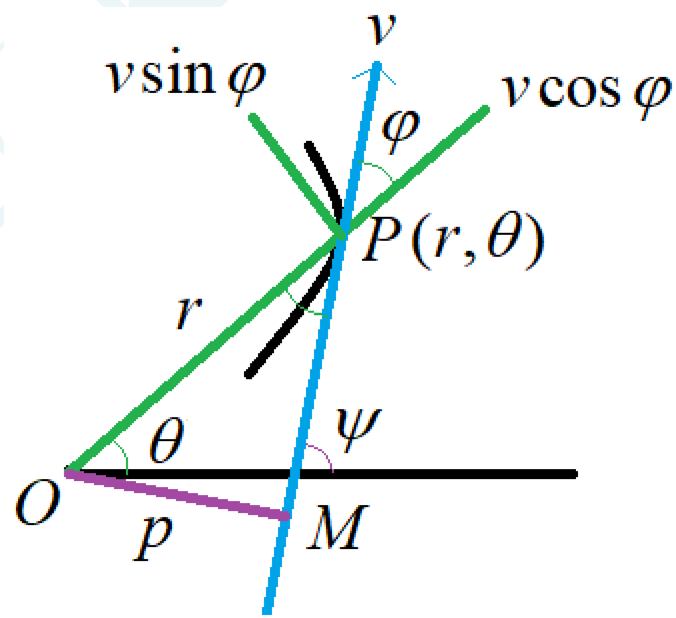
Rotation between angular and linear velocities

Let v be the velocity of the particle at $P(r,\theta)$ along the tangent.

So we have

$\dot{r} = v\cos\varphi$ …… (1)

$r\dot{\theta} = v\sin\varphi$ …… (2)



We also have from ΔOPM

$\sin\varphi = \dfrac{p}{r}$ ……. (3)

From (1) and (3) we get

$$\dot{\theta} = \frac{v \sin\varphi}{r} = \frac{vp}{r^2}.$$

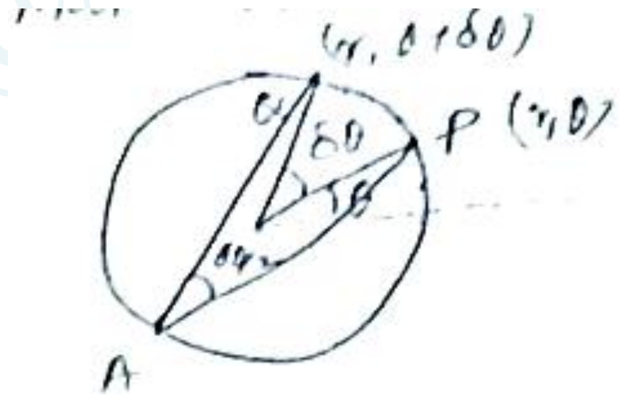
For circular motion it should be (with centre as pole)

$$\dot{\theta} = \frac{v}{a}.$$

Example: If a point moves along a circle, prove that its angular velocity about any point on the circle is half of that about the centre.

Solution:

Let the angle of any point P be θ and that of a neighbouring point Q be $\theta+\delta\theta$. Therefore, the change in angle due to the movement of the point from P to Q be $\delta\theta$ with respect to the centre.



Let A be any point on the circle. Then the change in angle of the point due to the movement from P to Q with respect to A is $\dfrac{\delta\theta}{2}$.

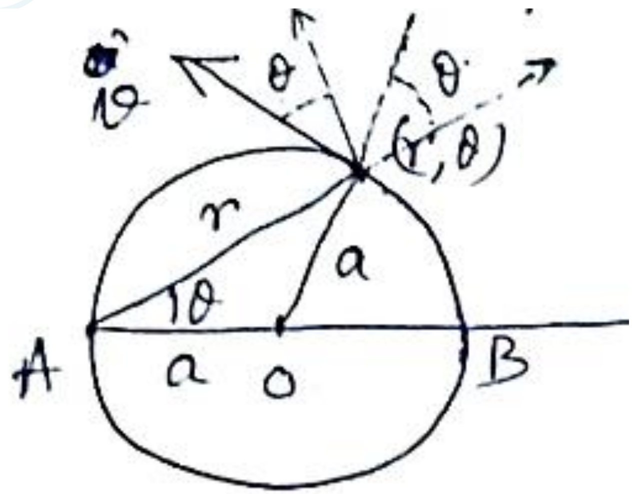
Now, the angular velocity about the centre is $\lim\limits_{\delta t\to 0}\dfrac{\delta\theta}{\delta t}=\dot{\theta}$ and that of about A any point on the circle is $\lim\limits_{\delta t\to 0}\dfrac{1}{2}\dfrac{\delta\theta}{\delta t}=\dfrac{1}{2}\dot{\theta}$.

<u>Example</u>: A point describes a circle of radius a with a uniform speed v; show that the radial and transverse acceleration are $-\dfrac{v^2}{a}\cos\theta$ and

$-\frac{v^2}{a}\sin\theta$ if a diameter is taken as initial line and end of the diameter as pole.

Solution: The equation of the circle whose or diameter is initial line and one end of that diameter is pole, is given by $r = 2a\cos\theta$ …. (1)

It is given that a point is moving on circle given by (1) with a uniform speed *v*.



Therefore the radial velocity $\dot{r} = -v\sin\theta$

And transverse velocity $r\dot{\theta} = v\cos\theta$

$\therefore \ddot{r} = -v\cos\theta\,\dot{\theta}$ and $\dot{\theta} = \dfrac{v\cos\theta}{r} = \dfrac{v}{2a}$

Now, we know that radial acceleration $= \ddot{r} - r\dot{\theta}^2$

and transverse acceleration $= 2\dot{r}\dot{\theta} + r\ddot{\theta}$

So, radial acceleration $= -v\cos\theta\dfrac{v}{2a} - \left(\dfrac{v}{2a}\right)^2 2a\cos\theta$

$$= -\frac{v^2}{2a}\cos\theta - \frac{v^2}{2a}\cos\theta = -\frac{v^2}{a}\cos\theta$$

and transverse acceleration $= 2\dot{r}\dot{\theta} + r\ddot{\theta} = -2v\sin\theta\dfrac{v}{2a} + 0 \;\left[\text{as } \ddot{\theta}=0\right]$

$$= -\frac{v^2}{a}\sin\theta.$$

<u>Example</u>: A point describes uniformly a given straight line, show that its angular velocity about a fixed point varies inversely as the square of its distance from the fixed point.

Example: A point P describes a curve with a constant velocity and the radius vector joining P to a fixed point O has an angular velocity which is inversely proportional to OP; show that the curve is an equiangular spiral with O as pole and that the acceleration of P along the normal varies inversely as OP.

Example: If the velocity of a point moving in a plane curve varies as the radius of curvature, show that the direction of motion revolves with constant velocity. Also if the angular velocity of the moving point about a fixed origin be constant, show that if transverse acceleration varies as its radial velocity.

Example: A point moves in plane curve so that its tangential acceleration is constant and the magnitudes of tangential velocity and the normal acceleration arc in a constant ratio. Show that the intrinsic equation of the path is of the form $s = A\psi^2 + B\psi + C$.

Solution: We have $\ddot{s} = k_1$ ……… (1)

and $$\frac{\dot{s}}{v^2/\rho} = k_2 \Rightarrow \frac{v\rho}{v^2} = k_2 \Rightarrow \rho = k_2 v \text{ …… (2)}$$

From (1) we have $$\ddot{s} = \frac{dv}{dt} = v\frac{dv}{ds} = k_1$$

$$\Rightarrow vdv = k_1 ds$$

Integrating we get

$$\frac{v^2}{2} = k_1 s + k_3 \Rightarrow v = \sqrt{2k_1 s + k_3}$$

Now from (2) we have

$$\frac{ds}{d\psi} = k_2\sqrt{2k_1 s + k_3} \Rightarrow \frac{ds}{\sqrt{2k_1 s + k_3}} = k_2\psi$$

Integrating we have

$$\frac{\sqrt{2k_1 s + k_3}}{2k_1.k_2} = k_2\psi + k_4$$

$$\Rightarrow \sqrt{2k_1 s + k_3} = k_1 k_2 \psi + k_1 k_4$$

$$\Rightarrow 2k_1 s + k_3 = k_1^2 k_2^2 \psi^2 + 2k_1^2 k_2 k_4 \psi + k_1^2 k_4^2$$

$$\Rightarrow s = \frac{k_1 k_2^2}{2}\psi^2 + 2k_1 k_2 k_4 \psi + \frac{k_1^2 k_4^2 - k_3}{2k_1}$$

Thus $s = A\psi^2 + B\psi + C$ is the intrinsic equation of the path. (Hence the result)

Example: A point moves in a curve so that its tangential and normal acceleration are equal and the tangent rotates with constant angular velocity. Show that the intrinsic equation of the path is of the form

$$s = Ae^{\psi} + B.$$

Solution: We have $\ddot{s} = \dfrac{v^2}{\rho}$ …….. (1) and $\dfrac{d\psi}{dt} = k_1 = \text{constant}$ ……. (2)

From (1) we get $v\dfrac{dv}{ds} = \dfrac{v^2}{\rho} \Rightarrow \dfrac{dv}{ds} = v\dfrac{d\psi}{ds} \Rightarrow \dfrac{dv}{v} = d\psi$

Integrating, we obtain

$\log v = \psi + \log k_2 \quad (k_2 \text{ is a constant})$

$$\Rightarrow v = k_2 e^{\psi} \Rightarrow \frac{ds}{dt} = \frac{ds}{d\psi}\frac{d\psi}{dt} = k_2 e^{\psi}$$

$$\Rightarrow \frac{ds}{d\psi}k_1 = k_2 e^{\psi} \Rightarrow ds = \frac{k_2}{k_1}e^{\psi}d\psi$$

Integrating we get

$$s = \frac{k_2}{k_1} e^{\psi} + k_3 \quad (k_3 \text{ is a constant}).$$

Hence the intrinsic equation of the path is of the form $s = Ae^{\psi} + B$, where A and B are constants.

Example: A particle describes a curve (for which s and ψ vanish simultaneously) with uniform v. If the acceleration at any point s be $\frac{v^2 c}{s^2 + c^2}$, prove that curve is a catenary.

Solution: Since v is constant, then $\ddot{s} = 0$

Then the acceleration given by $\sqrt{(\ddot{s})^2 + \left(\dfrac{v^2}{\rho}\right)^2} = \dfrac{v^2}{\rho}$

Therefore, we have

$$\frac{v^2}{\rho} = \frac{v^2 c}{s^2 + c^2} \Rightarrow \frac{d\psi}{ds} = \frac{c}{s^2 + c^2} \Rightarrow d\psi = \frac{c\,ds}{s^2 + c^2}$$

Integrating, we get

$$\psi = c\frac{1}{c}\tan^{-1}\left(\frac{s}{c}\right) + k \quad \text{i.e., } \psi = \tan^{-1}\left(\frac{s}{c}\right) + k.$$

Now, at $s = 0, \psi = 0 \Rightarrow k = 0$. So, $s = c\tan\psi$

Thus, $\psi = \tan^{-1}\left(\frac{s}{c}\right)$ is the equation of path, which is a catenary. (Proved)

<u>Example</u>: A small bead slides with constant speed v on a smooth wire in the shape of a cardioid $r = a(1+\cos\theta)$. Show that the value of $\dot{\theta}$ is $\left(v\sec\frac{\theta}{2}\right)\Big/2a$ and that the radial component of the acceleration is constant.