23. Simple Harmonic Motion. (91), (93)

A particle is said to execute Simple Harmonic Motion if it moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point.

Let O be the fixed point on a line BOA and P be the position of the particle at time t where $OP=x$, so that the acceleration of the particle in the sense OP is $\ddot{x}$.

Now the given acceleration is towards O and is proportional to x. Let it be μx, where μ is constant.

Since $\ddot{x}$ is in the direction of OP produced and μx is towards O, the equation of motion is

$$\ddot{x} = -\mu x.$$

$\frac{d^2x}{dt^2} \propto x \Rightarrow \frac{d}{dx}\left(\frac{dx}{dt}\right) = -$

Taking $v\dfrac{dv}{dx}$ instead of $\ddot{x}$; we can write the above equation as

$$v\frac{dv}{dx} = -\mu x. \qquad \ldots(1)$$

$v\,dx = -\mu x\,dt$

$\frac{v^2}{2} = -\mu\frac{x^2}{2} + C_1$

Integrating with respect to x, we get

$$\frac{v^2}{2} = -\mu\frac{x^2}{2} + \frac{C}{2} \text{ where } C \text{ is a constant}$$

or

$$v^2 = -\mu x^2 + C.$$

If A be the extreme position of the particle *i.e.*, it is at rest at A

i.e., when $x=a$, $v=0$ where $OA=a$, we get

$$0 = -\mu a^2 + C \quad \therefore \quad C = \mu a^2$$

$C_1 = \frac{\mu a^2}{2}$

Hence $$v^2 = \mu(a^2 - x^2),$$

i.e., $$v = \pm\sqrt{\mu}.\sqrt{a^2 - x^2}. \quad \ldots(2)$$

If the particle moves from A towards O, v is negative

Hence $$\dot{x} = v = -\sqrt{\mu}\sqrt{a^2 - x^2}$$

or $$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{a^2 - x^2}$$

or $$\sqrt{\mu}.dt = -\frac{dx}{\sqrt{a^2 - x^2}}.$$

Integrating we get $\sqrt{\mu}.t = \cos^{-1}\frac{x}{a} + C_1$ where C_1 is a constant.

Initially at A, $t=0$, $x=a$ i.e., the particle started from A,

then $$0 = \cos^{-1} 1 + C_1 \quad \therefore \quad C_1 = 0.$$

Hence $$\sqrt{\mu}.t = \cos^{-1}\frac{x}{a}$$

or $$\boxed{x = a\cos\sqrt{\mu}.t.} \quad \ldots(3)$$

If the particle moves from O towards A, v is positive

so that $$\dot{x} = \sqrt{\mu}.\sqrt{a^2 - x^2}$$

or $$\sqrt{\mu}.dt = \frac{dx}{\sqrt{a^2 - x^2}}.$$

Integrating, we get, $\sqrt{\mu}.t = \sin^{-1}\frac{x}{a} + C_2$ where C_2 is a constant.

If the particle starts from O, $t=0$, $x=0$, $0 = \sin^{-1} 0 + C_2$

$\therefore$ $$C_2 = 0,$$

so that $$\boxed{x = a\sin\sqrt{\mu}.t.} \quad \ldots(4)$$

Thus the solution of (1) is $x = a\cos\sqrt{\mu}.t$ or $x = a\sin\sqrt{\mu}.t$ according as the starting point is A or O.

From (2), $$v = 0 \text{ when } x = \pm a.$$

Thus if B is a point on the other side of O such that $OB = OA = a$, the particle comes to rest also at B. When $x=0$, $v = \pm\sqrt{\mu}a$, i.e., at O, the velocity is $\sqrt{\mu}a$.

Consider the solution $x = a\cos\sqrt{\mu}\,t$.

The motion starts from A under an attraction towards O. When the particle reaches O, $x=0$. $\therefore$ $\cos\sqrt{\mu}.t = 0$ $\therefore$ $\sqrt{\mu}t = \frac{\pi}{2}$

i.e., $t = \frac{\pi}{2\sqrt{\mu}}$ is the time required in moving from A to O.

As the particle reaches O, the attraction ceases but the particle has a velocity $\sqrt{\mu}.a$ towards the negative side of O hence the particle passes O and moves towards the negative side. As soon as the particle comes to the left side of O, attraction changes direction and becomes towards O ; hence the velocity will go on decreasing as the particle moves towards the left, till at B, the velocity becomes zero so that the particle stops. But the particle is being attracted towards O hence starts moving towards O and reaches O with a velocity $\sqrt{\mu}.a$, due to which it passes O and moves towards A and again stops at A where its velocity becomes zero. The motion is then repeated. Thus the motion is from A to B and back to A and so on. The motion is *oscillatory* Time from O to B is equal to that from A to O hence the **period** *i.e.*, the time from A to B and back to A is $4.\dfrac{\pi}{2\sqrt{\mu}}=\dfrac{2\pi}{\sqrt{\mu}}$. The distance $a\,(=OA)$ *i.e.*, the distance of the centre from one of the positions of rest is called the **amplitude**.

Thus the period which is equal to $\dfrac{2\pi}{\sqrt{\mu}}$ is independent of the amplitude *i.e.*, whatever be the amplitude the period is the same. Thus the simple harmonic motion is oscillatory and periodic, the period being independent of amplitude.

The **frequency** is the number of complete oscillations in one second, so that if n be the frequency and T the periodic time,

$$\boxed{n=\frac{1}{T}=\frac{\sqrt{\mu}}{2\pi}.}$$

The equation (1), namely $\ddot{x}=-\mu x$, can be solved as a differential equation. The most general solution of this equation is

$$x=A\cos\sqrt{\mu}.t+B\sin\sqrt{\mu}.t \qquad \ldots(5)$$

A, B are constants to be determined from initial conditions. In the first case when the motion starts from A, the initial conditions are $t=0$, $x=a$, $\dot{x}=0$.

Now $\qquad t=0,\ x=a$ gives $a=A$.

Differentiating (5), $\dot{x}=-A\sqrt{\mu}\sin\sqrt{\mu}.t+B\sqrt{\mu}\cos\sqrt{\mu}.t \qquad \ldots(6)$

The condition $t=0$, $\dot{x}=0$ gives $0=0+B\sqrt{\mu} \qquad \therefore \quad B=0.$

Hence the solution is $x=a\cos\sqrt{\mu t}$.

In the second case when the motion starts from O, the first condition is $t=0$, $x=0$. $\qquad \therefore \quad 0=A,\ A=0.$

Hence $\qquad x=B\sin\sqrt{\mu t}.$

To determine B, we must know the velocity of projection from O.

Let us take the case of a particle, projected from A with velocity V along OA produced, so that the initial conditions are $t=0$, $x=a$, $\dot{x}=V$.

Hence from (5) and (6), we get

$$a=A$$

$$V=B\sqrt{\mu}. \qquad \therefore \quad B=\frac{V}{\sqrt{\mu}}$$

Hence the solution is $x=a\cos\sqrt{\mu}.t+\dfrac{V}{\sqrt{\mu}}\sin\sqrt{\mu}.t.$

Also the general solution of (1) can be written as

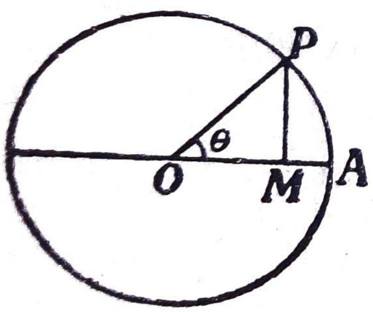
$$x=a\cos(\sqrt{\mu}t+\xi)$$

This is periodic with period $\dfrac{2\pi}{\sqrt{\mu}}$.

The quantity ξ is called the **epoch**, the angle $\sqrt{\mu}.t+\xi$ is called the **argument**. The particle is at its maximum distance at time t_0 where $\sqrt{\mu}t_0+\xi=0$ *i.e.*, $t_0=-\dfrac{\xi}{\sqrt{\mu}}$. Hence the time that has elapsed since the particle was at its maximum distance is equal to $t-t_0=t+\dfrac{\xi}{\sqrt{\mu}}=\dfrac{\sqrt{\mu}t+\xi}{\sqrt{\mu}}$. This is the **phase** at time t.

A geometrical representation of the S.H.M.

Let a particle P move on a circle with constant angular velocity ω and let M be the foot of the perpendicular from P on any diameter OA. If a be the radius of the circle, the only acceleration of P is $\omega^2 a$ towards O.



If $\angle AOP=\theta$ and $OM=x$, the component of this acceleration along OA

$$=\omega^2 a.\cos\theta=\omega^2 a.\frac{x}{a}=\omega^2 x \text{ towards } O.$$

Hence the equation of motion of the point M is

$$\boxed{\ddot{x}=-\omega^2 x.}$$

This is *S.H.M.*

Thus *if a particle describes a circle with constant angular velocity the foot of the perpendicular from it on any diameter executes a simple harmonic motion.*

Ex. 9. *A particle is moving with S.H.M. and while making an excursion from one position of rest to the other, its distances from the middle point of its path at three consecutive seconds are observed to be* x_1, x_2, x_3, *prove that the time of a complete revolution is*

$$2\pi/\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right).$$

From $$x=a\cos\sqrt{\mu}t$$

$$x_1=a\cos\sqrt{\mu}t,\ x_2=a\cos\sqrt{\mu}\,(t+1)$$

$$x_3=a\cos\sqrt{\mu}\,(t+2)$$

$\therefore$ $$x_1+x_3=2a\cos\sqrt{\mu}\,(t+1)\cos\sqrt{\mu}$$

$\therefore$ $$\frac{x_1+x_3}{2x_2}=\cos\sqrt{\mu}$$

$\therefore$ $$\sqrt{\mu}=\cos^{-1}\frac{x_1+x_3}{2x_2} \qquad T=\frac{2\pi}{\sqrt{\mu}}$$

Ex. 10. *A particle starts from rest under an acceleration K^2x directed towards a fixed point and after time t another particle starts from the same position under the same acceleration. Show that the particles will collide at time $\frac{\pi}{K}+\frac{t}{2}$ after the start of the first particle provided $t<\frac{2\pi}{K}$.*

$\frac{d^2x}{dt^2}=$ $$\ddot{x}=-K^2x \qquad \therefore \text{ period}=\frac{2\pi}{K}$$

The condition $t<\frac{2\pi}{K}$ indicates that the second starts before the first has made one complete oscillation. Let them meet after time t' of the start of the second

then $$a\cos K(t+t')=a\cos Kt'$$

$$\therefore\ K(t+t')=2\pi-Kt' \qquad \therefore\ t'=\frac{\pi}{K}-\frac{t}{2}.$$

Hence, $$t+t'=\frac{\pi}{K}+\frac{t}{2}.$$

Ex. 11. *A horizontal shelf is moved up and down with S.H.M. of period $\frac{1}{2}$ sec. What is the amplitude admissible in order that a weight placed on the shelf may not be jerked off?*

$T=$ $$\frac{2\pi}{\sqrt{\mu}}=\frac{1}{2} \qquad \therefore \qquad \mu=16\pi^2$$

Weight will be jerked off when the max. acc. of *S.H.M.* is greater than g and if it is not to be jerked off, max. acc. of *S.H.M.* must be g

i.e., $$\mu a=g \qquad \therefore\ a=\frac{g}{16\pi^2};$$

Ex. 12. *In a S.H.M. of amplitude a and period T, prove that*

$$\int_0^T v^2 dt = \frac{2\pi^2 a^2}{T}$$

$$x = a \cos \sqrt{\mu} t, \quad v = \dot{x} = -a\sqrt{\mu} \sin \sqrt{\mu} t,$$

$$T = \frac{2\pi}{\sqrt{\mu}}$$ $v = \frac{dx}{dt} = -a\sqrt{\mu} \sin \sqrt{\mu} t$

$$\int_0^T v^2 dt = a^2\mu \int_0^T \sin^2 \sqrt{\mu} t \, dt = a^2\mu \int_0^T \sin^2 \frac{2\pi t}{T} dt$$

$$= a^2\mu \int_0^{2\pi} \sin^2 z \, \frac{T}{2\pi} dz \qquad \text{Put } \frac{2\pi t}{T} = z$$ $dt = \frac{T}{2\pi} dz$

$$= \frac{a^2 \mu T}{2\pi} \cdot \pi = \frac{a^2 T}{2} \cdot \frac{4\pi^2}{T^2} = \frac{2\pi^2 a^2}{T}.$$ $\frac{2\pi}{T} dt = dz$

1989

EXAMPLES II (C)

1. A particle is performing a $S.H.M.$ of period T about a centre O and it passes through a point $P(OP=b)$ with velocity v in the direction OP; prove that the time which elapses before its return to P is

$$\frac{T}{\pi} \tan^{-1} \frac{vT}{2\pi b}.$$

O P

1990 2. A point in a straight line with $S.H.M.$ has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2. Show that the period of motion is

$$2\pi\sqrt{\frac{x_1^2-x_2^2}{v_2^2-v_1^2}}.$$

1990 3. A point executes $S.H.M.$ such that in two of its positions the velocities are u, v and the corresponding accelerations are α, β; show that the distance between the positions is $\dfrac{v^2-u^2}{\alpha+\beta}$, and the amplitude of the motion is

$$\frac{[(v^2-u^2)(\alpha^2v^2-\beta^2u^2)]^{\frac{1}{2}}}{\beta^2-\alpha^2}.$$

4. Show that in a $S.H.M.$ the average speed and the average acceleration (in magnitude) are obtained by multiplying their maximum values by 0·637.

1990 5. A body moving in a straight line OAB with $S.H.M.$ has zero velocity when at points A and B whose distances from O are a and b respectively and has a velocity v when half-way between them. Show that the complete period is

$$\frac{\pi(b-a)}{v}.$$

1990 6. A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their intensity being μ, μ'; the particle is slightly displaced towards one of them, show that the time of a small oscillation is

$$\frac{2\pi}{\sqrt{\mu+\mu'}}.$$

7. A particle P moves in a straight line OCB being attracted by a force