Lagranges multipliers method

Suppose we would like to find maxima or minima of a f^n f which is defined in some open set $D \subseteq \mathbb{R}^2$, subject to the condition $\phi(x,y)=0$, where ϕ is also defined in D. Suppose f attains maxima at a point $P_0=(x_0,y_0)$ as (x,y) varies over the set $S_f := \{(x,y) : \phi(x,y)=0\}$.

Let us also assume that f_x, f_y, ϕ_x, ϕ_y exist in a nbd of P_0 and there exists a f^n $y=g(x)$ defined in a nbd of x_0 such that $\phi(x, g(x))=0$ $\forall x$ taken from that nbd of x_0. Then $u=f(x,y)$ is a f^n of a single variable $x \in$ Nbd of that x_0. Thus, u attains maximum at P_0 so that derivative of u w.r.t. x is <u>zero</u> at x_0.

Thus we have

$$\frac{du}{dx} = f_x + f_y\, y' = 0,\quad \phi_x + \phi_y\, y' = 0 \text{ at } P_0.$$

Hence we must have

$$(f_x + f_y\, y') + \lambda(\phi_x + \phi_y\, y') = 0 \text{ at } P_0 \ \forall \lambda \in \mathbb{R}.$$

i.e. $(f_x + \lambda\phi_x) + (f_y + \lambda\phi_y)\, y' = 0$ at u_0 $\forall \lambda \in \mathbb{R}$.

Choosing λ_0 such that $f_y + \lambda_0\phi_y = 0$ at (x_0,y_0), we obtain $\phi=0$, ~~f_x+f~~ $f_x + \lambda\phi_x = 0$, $f_y + \lambda\phi_y = 0$ for $(\lambda, x, y) = (\lambda_0, x_0, y_0)$. $\longrightarrow$ (1)

Note that, writing

$$F(x,y,\lambda) = f(x,y) + \lambda\varphi(x,y),$$ the

condition ① is same as

$$\varphi = 0,\ F_x = 0 = F_y \text{ at } (\lambda_0, x_0, y_0).$$

The parameter λ above is called the Lagrange multiplier, and the method using Lagrange multiplier is the procedure of finding (λ, x, y) such that $\varphi = 0$, $F_x = 0 = F_y$ so that the required point at which f attains an extremum in S_f is one among these points.

Ex ① Among all rectangles with given perimeter l, let us find the one having maximum area.

Then the problem is to find the point (x_0, y_0) at which the fn $f(x,y) = xy$ attains maximum subject to the constraint

$$\varphi(x,y) = 2(x+y) - l.$$

Then $f_x + \lambda\varphi_x = y + 2\lambda = 0$, $f_y + \lambda\varphi_y = 0 = x + 2y$

Thus $x = -2\lambda = y$, $l = 2(x+y) = 4x$

So that $x = y = l/4$.

Ex (2) We show that among all the rectangular parallelepiped inscribed in given sphere, cube has the maximum volume.

Let x, y, z be the sides of the parallelepiped. Then we must have $x^2+y^2+z^2=d^2$, d is the diameter of the sphere.

So, we must find the maximum of the fn $f(x,y,z) = xyz$ subject to $\phi(x,y,z) = x^2+y^2+z^2-d^2$

Thus

$$f_x + \lambda \phi_x = yz + \lambda 2x = 0 \longrightarrow (1)$$

$$f_y + \lambda \phi_y = xz + \lambda 2y = 0 \longrightarrow (2)$$

$$f_z + \lambda \phi_z = xy + \lambda 2z = 0 \longrightarrow (3)$$

From (1), (2) & (3), we see that $x(f_x + \lambda\phi_x) + y(f_y + \lambda\phi_y) + z(f_z + \lambda\phi_z) = 0$

i.e. $3xyz + 2\lambda(x^2+y^2+z^2) = 0$

$$\therefore 3xyz = -2\lambda d^2 \therefore 2\lambda = -\frac{3xyz}{d^2}$$

$\therefore$ From (1), $yz + \lambda \cdot 2x = 0$

$$\Rightarrow \cancel{yz + \lambda \cdot 2} \; yz - x \cdot \frac{3xyz}{d^2} = 0$$

$$\Rightarrow yz\left(1 - \frac{3x^2}{d^2}\right) = 0$$

$$\therefore 1 - \frac{3x^2}{d^2} = 0 \Rightarrow x = \frac{d}{\sqrt{3}}$$

Similarly $y = z = \frac{d}{\sqrt{3}}$.

H.W Find maximum of $f(x,y,z) = x^2y^2z^2$ subject to $x^2+y^2+z^2=1$

* Find the parallelepiped of maximum volume with a given surface are $f = xyz$, $\phi = 2(xy+yz+zx) - A$.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

put $x = \sin^2\theta$

$$B(m,n) = \int_0^{\pi/2} (\sin\theta)^{2m-2} (\cos\theta)^{2n-2}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$