B.Sc. Mathematics – 2nd Semester

MTB 202 – Statics and Dynamics

by

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Part – II

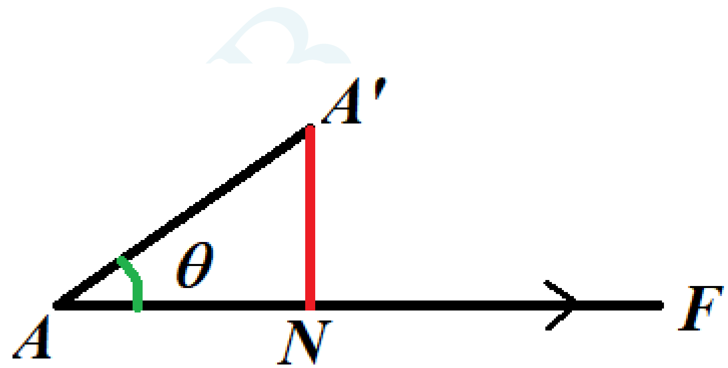
Virtual Work

In statics we always have a body or a system of bodies in equilibrium under the action of given forces. The general method of solving problems is write down the equation of conditions of equilibrium, i.e., equations of resolution and equation of moment. But, a great disadvantage is that such equations contain unknown forces, like reaction, thrust, tension etc which should be eliminated.

The principle of virtual work affords us a method of writing down equations involving no such unknown forces or, involving only such unknown forces which we want to find out. It is a powerful method and may be used with great advantage.

<u>Work done by a force</u>:

If a force $\vec{F}$ acts at a point of a body and the point of application A' under goes a displacement AA' then the product $F.AA'\cos\theta$ is called the work



done by the force $\vec{F}$ in the displacement AA', where θ is the angle between the direction of force and displacement AA' and $\left|\vec{F}\right| = F$.

Let $A'N$ be drawn perpendicular to the line of action of $\vec{F}$. Then AN is the orthogonal projection of AA' on the line of action of $\vec{F}$ and is equal to $AA'\cos\theta$ and the work done is $F.AN$. So, the work done by a force in any displacement is equal to the product of the force and orthogonal projection of the displacement on the line of action of the force. AN is called the displacement in the direction of force.

Again $F\,AA'\cos\theta = AA'.F\cos\theta$. So, the work done in any displacement is also equal to the product of the displacement and the resolve part of the force in the direction of the displacement. No work is done when the displacement is perpendicular to the force.

It may be noted that the work done by the force $\vec{F}$ in a displacement vector $\overrightarrow{AA'}$ is the scalar quantity $\vec{F}\cdot\overrightarrow{AA'}$.

Virtual Work:

When the point of application A of a force $\vec{F}$ does not actually move to A', but it is imagined to move A' then the displacement AA' is called a virtual displacement of the point A and the corresponding work done by the force $\vec{F}$ on the assumption that the point of application has undergone the displacement AA' is called the virtual work done by the force $\vec{F}$ in the virtual displacement AA'.

It is of course measured by $F\,AA'\cos\theta$ or $F.AN$ or $AA'\times$ resolved part of $\vec{F}$ in the direction of AA'.

Static bodies are at rest, so under the action of forces those do not actually move. All displacements of the points of application of the forces are therefore hypothetical, i.e., imagined and the works calculated are the works which would be done if the displacements were actually made, that is why the displacements and the works done are called Virtual.

Theorem: When any number of forces act on a particle and the particle under goes a small displacement then the total work done by all the forces is equal to the work done by their resultant.

Principle of Virtual work for a system of coplanar forces acting on a particle:

The necessary and sufficient condition that a particle acted upon a number of coplanar forces, be in equilibrium is that the algebraic sum of the virtual works done by the forces in a small displacement, consistent with the geometrical conditions of the system, is zero.

Principle of virtual work for a system of a coplanar forces acting at different points of rigid body:

The necessary and sufficient condition that a rigid body acted upon by a number of coplanar forces, be in equilibrium, is that the algebraic sum of the virtual works done by the forces in a small displacement, consistent with the geometrical condition of the system, is zero.

Proof: Let us take Ox, Oy as the axes of the coordinates in the plane of the forces.

The most general virtual displacement of the body, the plane of the forces remaining unaltered is a rotation about O together with a translation.

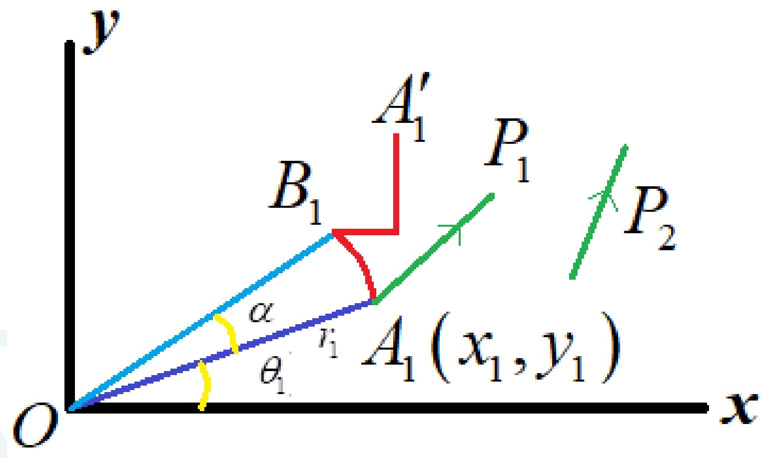
Let the body be given a rotation about O through any small angle α and any small translation whose components parallel to OX and OY are a, b.

Let P_1, P_2, P_3 …….. be the forces acting at the points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, $A_3(x_3, y_3)$ ….. of the body and let its components parallel to axes be X_1, Y_1 ; X_2, Y_2 ; X_3, Y_3 ;….. respectively.

Again, let the polar coordinates of $A_1, A_2, A_3, \ldots\ldots$ be (r_1, θ_1), (r_2, θ_2), (r_3, θ_3) … respectively.

Then $x_1 = r_1 \cos\theta_1$, $y_1 = r_1 \sin\theta_1$

Due to the rotation A_1 and B_1 whose polar coordinates are $(r_1, \theta_1 + \alpha)$. So its Cartesian coordinates are $\left(r_1 \cos(\theta_1 + \alpha), r_1 \sin(\theta_1 + \alpha)\right)$



The point A_1' is obtained by a translation of point B_1' whose components are a, b parallel to OX and OY respectively.

Therefore, the coordinates of A_1' are

$$\left(r_1 \cos(\theta_1 + \alpha) + a,\ r_1 \sin(\theta_1 + \alpha) + b\right).$$

If δx_1 and δy_1 are the changes in the coordinates of A_1 due to its displacement from A_1 to A_1' then

$$\delta x_1 = r_1 \cos(\theta_1 + \alpha) + a - x_1$$

$$= r_1 \cos\theta_1 \cos\alpha - r_1 \sin\theta_1 \sin\alpha + a - x_1$$

$$= x_1 \cos\alpha - y_1 \sin\alpha + a - x_1$$

$$= x_1 - y_1\alpha + a - x_1$$

$$= a - \alpha y_1$$

[Writing $\alpha = \sin\alpha$ and $1 = \cos\alpha$ neglecting higher powers of α which are small quantities as α is small.]

$$\delta y_1 = r_1 \sin\left(\theta_1 + \alpha\right) + b - y_1$$

$$= r_1 \sin\theta_1 \cos\alpha + r_1 \cos\theta_1 \sin\alpha + b - y_1$$

$$= y_1 \cos\alpha + x_1 \sin\alpha + b - y_1$$

$$= y_1 + x_1\alpha + b - y_1$$

$$= b + \alpha x_1$$

Therefore, the virtual work done by force P_1 is $X_1\delta x_1 + Y_1\delta y_1$

$$= X_1\left(a - \alpha y_1\right) + Y_1\left(b + \alpha x_1\right)$$

$$= aX_1 + bY_1 + \alpha\left(x_1Y_1 - y_1X_1\right)$$

Similarly, we can obtain the virtual work of the forces P_2, P_3 ,……. Where a, b and α are same for each forces.

Hence the (algebraic sum of the) virtual work done by all the forces is equal to $\sum aX_1 + \sum bY_1 + \sum \alpha\left(x_1Y_1 - y_1X_1\right)$

$$= a\sum X_1 + b\sum Y_1 + \alpha\sum\left(x_1Y_1 - y_1X_1\right) \qquad (1)$$

Now let the system be in equilibrium then we have $\sum X_1 = 0, \sum Y_1 = 0, \sum\left(x_1Y_1 - y_1X_1\right) = 0$

Hence from (1) it follows that the algebraic sum of the virtual works done by the system of forces is zero. (Thus the necessary part is proved)

Conversely, let the algebraic sum of the virtual work done by the system of forces is zero for every arbitrary small displacement in the plane of forces.

Then we have from (1)

$$a\sum X_1 + b\sum Y_1 + \alpha\sum\left(x_1Y_1 - y_1X_1\right) = 0 \qquad (2)$$

for all arbitrary small values of a, b and α

Let us consider a displacement such that the body is moved only through a displacement parallel to x-axis.

Then $a \neq 0, b = 0, \alpha = 0$ and from (2) we have

$$a\sum X_1 = 0 \Rightarrow \sum X_1 = 0 \text{ as } a \neq 0.$$

Similarly, choosing a displacement parallel to the *y*-axis it can be shown that $\sum Y_1 = 0$

Finally, let the displacement be one of a simple rotation.

Then $a = 0, b = 0, \alpha \neq 0$ and (2) gives

$$\alpha \sum \left(x_1 Y_1 - y_1 X_1 \right) = 0 \Rightarrow \sum \left(x_1 Y_1 - y_1 X_1 \right) = 0 \text{ as } \alpha \neq 0$$

Then $\sum X_1 = 0, \sum Y_1 = 0$ and $\sum \left(x_1 Y_1 - y_1 X_1 \right) = 0,$ which implies that the system is in equilibrium. (Hence the sufficient part is proved)

Unwanted Forces:

If we want to get rid of a certain unwanted force, we imagine a displacement (consistent with the geometrical condition of the system) at the right angle to the line of action of that force.

Tension and Thrust:

The tension in a string connecting two particles (or bodies) has a tendency to bring the particles (or bodies) together. Therefore, if a string be considered as divided at any point P of it into two parts AP and PB, the tension at P in the parts AP and PB are directed respectively towards B and A.

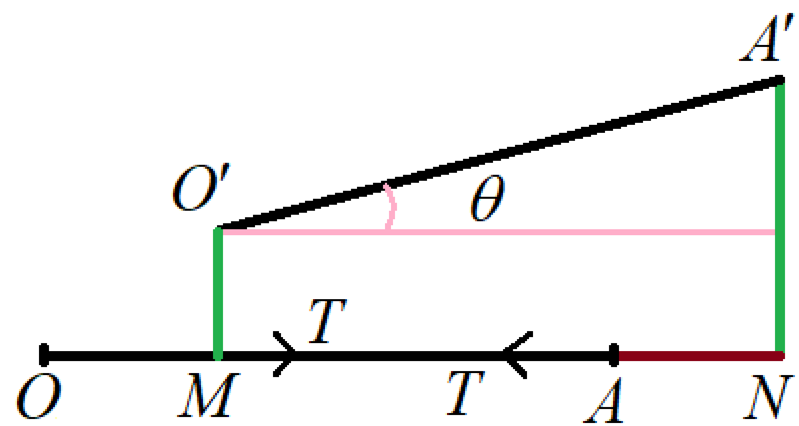
Just the reverse is the case with thrust applied by a rod which keeps the two particles (or bodies) at its ends from coming together. Consequently, the thrust at C point in the two parts AP and PB, say, in which a rod is divided at P, are directed towards A and B, respectively.

Forces which do not appear in the equation of virtual work:

There are certain forces which have no contribution to the virtual work i.e. the virtual works done by them are zero. Hence these forces may be omitted in forming the equation of virtual work. We state some usually occurring such forces.

(1) The tension of an inextensible string:

Let OA be such a string whose tension is T. In the displaced position, $O'A'$ (let) is the string and we draw perpendicular $O'M$ and $A'N$.



Hence the virtual work of the tension.

$$=T.OM-T.AN \;=T\left(OM-AN\right)=T\left(OA-MN\right)$$

$$[OM-AN=\left(OA-MA\right)-\left(MN-MA\right)=OA-MN\,]$$

$$=T.OA\left(1-\cos\theta\right)$$

$=T.OA\left(1-1\right)=0$ (to the first order of small quantities)

(2) The tension thrust in a rod whose length remains unaltered:

(3) Forces of action and reaction between two particles when the distance between them remains in variables:

Similar to inextensible string.

(4) The reaction R at any smooth surface with which the body is in contact:

If the surface is smooth, the reaction R is normal to the surface at the point contact P, so that if P moves to a neighboring point P', then PP' in at right angle to the force. Its virtual work is thus zero.

If the surface is rough then the work done by friction F viz, $F\left(-PP'\right)$ come in the equation since it is not, in general, zero.

(5) The reaction at any point P with a fixed surface on which the body rolls without sliding:

For the point of contact P of the body is for the moment at rest and so its displacement is zero.

The normal reaction at P and the friction at P have no displacement and hence zero virtual work.

It is evident that whether a force will appear in the equation of virtual work or not depends not only on the force but also on the displacement given.

Forces which appear in the equation of virtual work

Every force which does not work due to the displacement imagined will appear in the equation of virtual work.

(1) The tensions at the ends of a string or rod when the displacement alters its length:

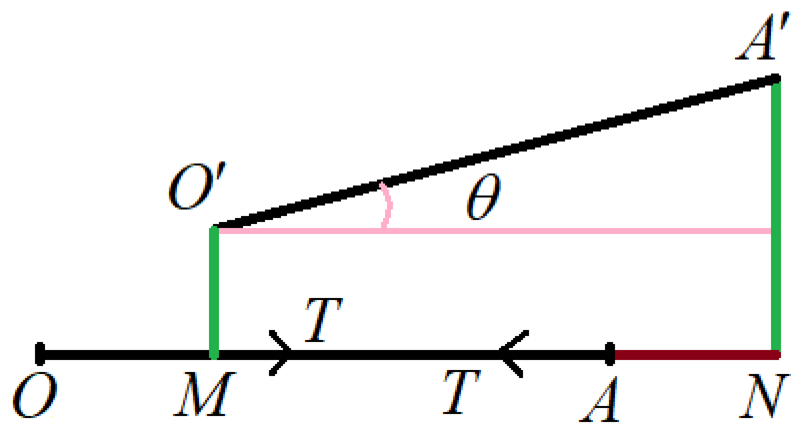
If T is the equal tensions and l is the length of the string or rod, then the work done is $-T\delta l$

Proof: Let OA be such a string whose tension is T. In the displaced position let $O'A'$ be the string and we draw perpendicular $O'M$ and $A'N$ on OA.

Let us suppose that the length of string or rod l be not constant.

Let in the displaced position $O'A' = OA + \delta(OA) = l + \delta l$

Then the total work done by the tension T at O and T at A is equal to



$T.OM + (-T)AN$

$= T(OA - MA) - T(MN - MA)$

$= T(OA - MN)$

$= T(OA - O'A'\cos\theta) = T(OA - O'A')$ $[\cos\theta \cong 1 \text{ as } \theta \text{ is small}]$

$$= T\left(l - l - \delta l\right) = -T\delta l$$

$= -$ Tension in the string or rod $\times$ increment in the length of the string or rod.

(2) The thrust at the ends of a rod when displacement alters its length:

If T be the equal thrust and l be the length of the rod, work done is $T\delta l$

<u>**Proof**</u>**:** Same as above

(3) Weight of a body or of a system of bodies:

If W be the weight and y be the depth of the centre of gravity of the body or of the system of bodies below some fixed plane, then work done by the weight due to a small displacement δy be $W\delta y$, but if y is the height of the centre of gravity above some fixed plane then the work done is $-W\delta y$. For the distance of the centre of gravity where the weight is measured positively in a direction opposite to the direction in which the weight acts.

Example: Four equal heavy uniform rods are freely joined so as to form a rhombus which is freely suspended by one angular point and the middle points of the two upper rods are connected by a light rod, so that the rhombus cannot collapse. Prove that the thrust of this rod is $4W\tan\alpha$, where W is the weight of each rod and 2α is the angle of the rhombus at the point of suspension.

Solution:

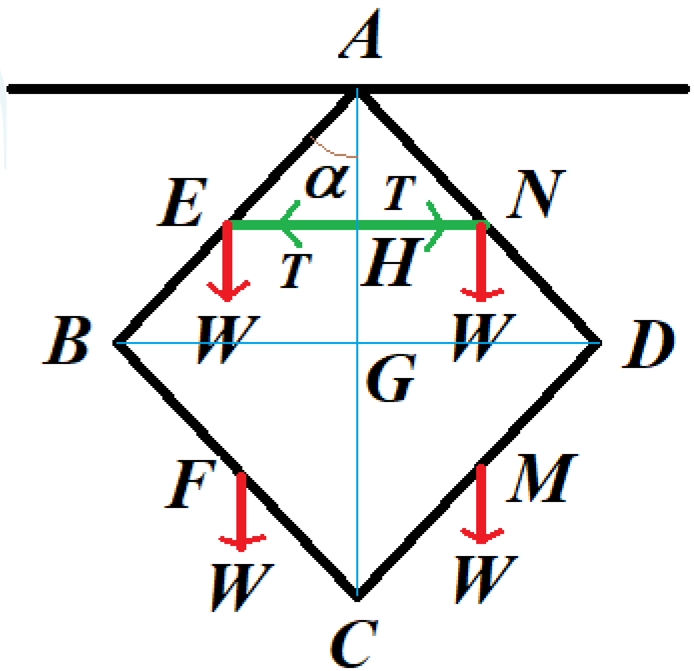
Let $ABCD$ be the rhombus. Again, let E, F, M, N be the midpoint of AB, BC, CD, DA respectively. Let the length of each rod be a.

Since $ABCD$ is a rhombus we have G as the midpoint of AC and BD and also G is the centre of gravity of the system.

So the force $4W$ acts at G downwards with W being the weight of each rod.

Let T be the thrust of the rod.

Let the system be given a virtual displacement so that the angle α changes.



Then the equation of virtual work is

$$4W\delta\left(AG\right)+T\delta\left(EN\right)=0 \qquad (1)$$

Now from right angle triangle ΔABG we have

$$\cos\alpha=\frac{AB}{AG}=\frac{a}{AG}\Rightarrow AG=a\cos\alpha\,.$$

From ΔAEH we get $EH=\frac{a}{2}\sin\alpha$. So, $EN=2EH=a\sin\alpha$.

Therefore, From (1) we have

$$4W\delta\left(a\cos\alpha\right)+T\delta\left(a\sin\alpha\right)=0 \Rightarrow -4Wa\sin\alpha\delta\alpha+Ta\cos\alpha\delta\alpha=0$$

$$\Rightarrow a\left(T\cos\alpha-4W\sin\alpha\right)\delta\alpha=0$$

$$\Rightarrow T\cos\alpha-4W\sin\alpha=0 \quad [\text{as } \delta\alpha\neq 0,\, a\neq 0] \quad \Rightarrow T=4W\tan\alpha\,.$$

Example: Four equally uniform rods are jointed to form a rhombus $ABCD$ which is placed in a vertical plane with AC vertical and A resting on horizontal plane. The rhombus is kept in the position in which $\angle BAC$ is θ by a light string joining B and D, show that its tension is $2w\tan\theta$, where $w$is the weight of the rod.

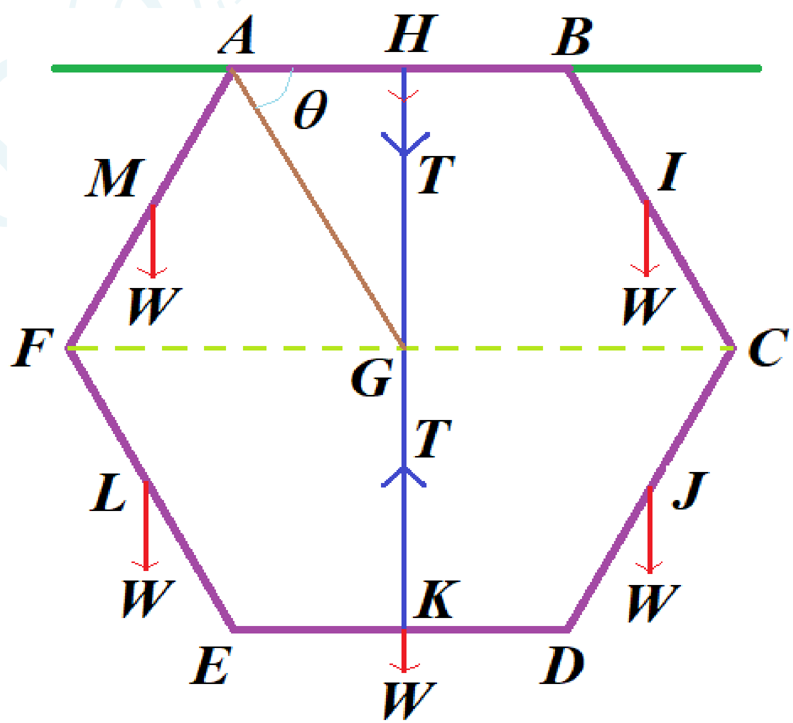
Example: Four equal uniform rods each of weight W, are jointed to form a rhombus $ABCD$, which is placed in a vertical plane with AC vertical and A resting on a horizontal plane. The rhombus is kept in the position in which $\angle BAC = \theta$ by a light string joining B and D. Find the tension of the string.

Example: Four uniform rods are freely jointed at their extremities and form a parallelogram $ABCD$ which is suspended by the joint A and kept in shape by a string AC. Prove that the tension of the string is equal to half the whole weight.

Example: Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are joined by a string. Prove that the tension of the string is $3W$.

Solution: Let H, I, J, K, L, M be the midpoints of AB, BC, CD, DE, EF and FA respectively.

Let T be the tension of the string.



Let at the position of equilibrium $\angle FAB = 2\theta$ and let G be the centre of gravity of system. Now the force $6W$ acts at G downwards and the depth of G from the fixed rod HG. Also the length of string is HK=2HG.

Now we give a small virtual displacement so that the angle θ changes.

Then the equation of virtual work is

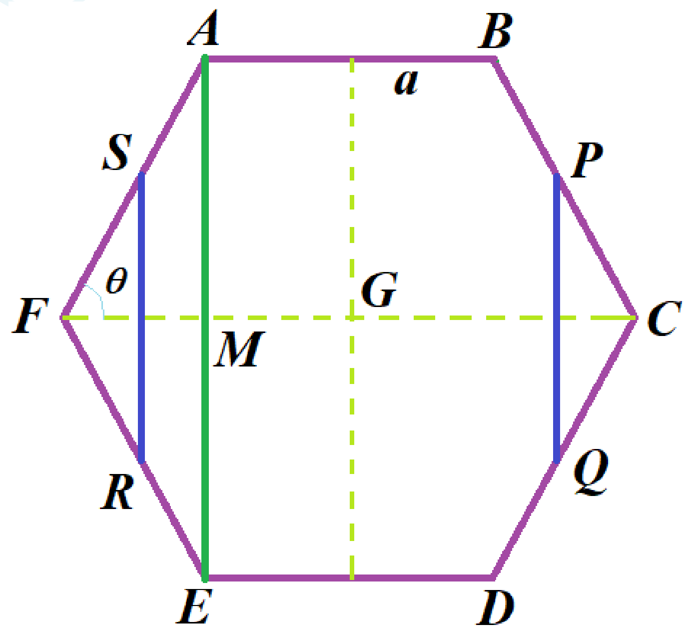
$$6W\delta(HG) - T\delta(2HG) = 0$$

$$\Rightarrow (6W - 2T)\delta(HG) = 0$$

$$\Rightarrow 2T = 6W \text{ as } \delta(HG) \neq 0$$

$$\Rightarrow T = 3W \text{ (Proved)}$$

Example: Six equal bars are freely jointed at their extremities forming a regular hexagon *ABCDEF* which is kept in shape by vertical strings jointed in the middle points of *BC*, *CD* and *AF*, *FE* respectively and *AB* is fixed horizontally. Show that the tension of each string is three times the weight of a bar.



Solution: Let W be the weight and a be the length of each rod. Let P, Q, R, S be the mid points of BC, CD, EF and FA respectively.

Let T be the tension of each of the string PQ and RS.

Let at equilibrium position $\angle AFE = 2\theta$

So $\angle AFG = \angle EFG = \theta$, G being the centre of gravity of the system.

Now, from ΔAFM we have

$$AM = AF\sin\theta = a\sin\theta.$$

And from ΔAEF we have

$$SR = \frac{1}{2}AE = AM = a\sin\theta.$$

Now, the depth of a G from the horizontal bar AB is $AM = a\sin\theta$

Now, the force $6W$ acts at G downwards. We give a small virtual displacement so that the angle θ alters.

Then the equation of virtual work is

$$6W\delta(a\sin\theta) - 2T(a\sin\theta) = 0$$

$$\Rightarrow 6W - 2T = 0 \;[\text{as } a\sin\theta \neq 0] \quad \Rightarrow T = 3W.$$

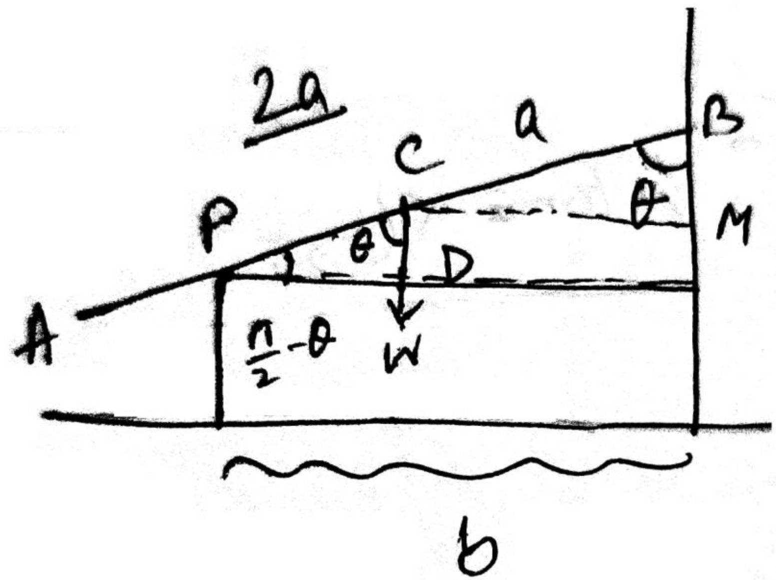
Hence, the tension of each string is three times the weight of a bar.

<u>Example</u>: Six equal heavy beams are free by jointed at their ends to form a hexagon and are placed in a vertical plane with one beam resting on a horizontal plane; the middle points of the two upper slant beams, which are inclined at an angle θ to the horizon, are connected by a light rod. Find its tension in terms of W and θ, where W is the weight of each beam.

<u>Example</u>: A regular hexagon $ABCDEF$ is composed of six equal heavy rods freely jointed together and two opposite angles C and F are connected by a string, which is horizontal, AB being contact with a horizontal plane. A weight W is placed at the middle point DE. If w be the weight of each rod. Show that the tension of the string is $\dfrac{3w+W}{\sqrt{3}}$.

Example: A uniform rod of length $2a$ rests in equilibrium against a smooth vertical wall and upon a smooth peg distant b from the wall. Show by the principle of virtual work that, in the position of equilibrium, the rod is inclined to the wall at an angle $\sin^{-1}(b/a)^{\frac{1}{3}}$.

Solution:



The equation of virtual work is

$$-W\delta(CD)=0$$

We have, $CM = a\sin\theta$

Therefore, $PD = b - a\sin\theta$

Now $CD = PD\cot\theta \;= \left(b - a\sin\theta\right)\cot\theta$

So we have $W\delta\left[\left(b - a\sin\theta\right)\cot\theta\right] = 0$

$$\Rightarrow W\delta\theta\left\{-a\cos\theta\cot\theta - \left(b - a\sin\theta\right)\operatorname{cosec}^2\theta\right\} = 0$$

$$\Rightarrow -a\cos\theta\cot\theta - \left(b - a\sin\theta\right)\operatorname{cosec}^2\theta = 0$$

$$\Rightarrow a\frac{\cos^2\theta}{\sin\theta} + \left(b - a\sin\theta\right)\frac{1}{\sin^2\theta} = 0$$

$$\Rightarrow a\left(1 - \sin^2\theta\right)\sin\theta + \left(b - a\sin\theta\right) = 0$$

$$\Rightarrow a\sin\theta - a\sin^3\theta + b - a\sin\theta = 0$$

$$\Rightarrow \sin^3\theta = b/a \quad \Rightarrow \theta = \sin^{-1}(b/a)^{\frac{1}{3}}.$$

Example: A heavy uniform rod of length $2a$, rests with its ends in constant with two smooth inclined plane, of inclination α and β to the horizon. If θ be the inclination of the rod to the horizon prove, by the principle of virtual work, that

$$\tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta).$$

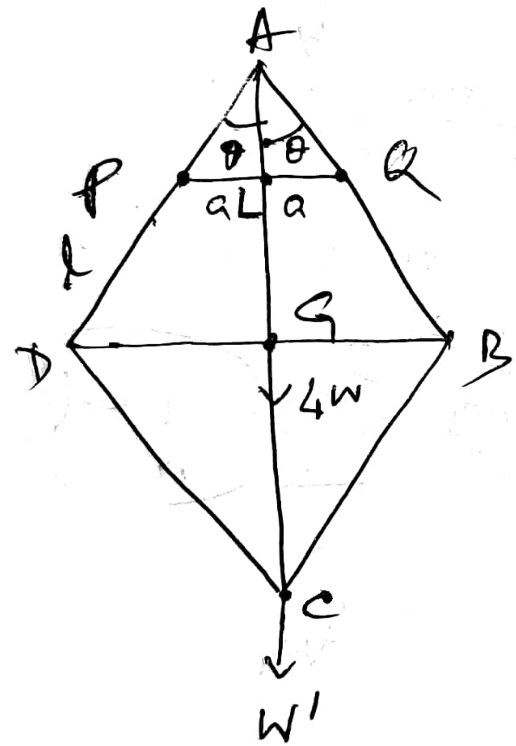
Example: $ABCD$ is a rhombus formed with four rods each of length l and negligible weight joined by smooth hinges. A weight W is attached to the lowest hinge C and the frame rests on two smooth pages in horizontal line in contact with the rods AB and AD. B and D are horizontal line and are joined by a string. If the distance of the pegs is $2c$ and the angle at A is 2α, show that the tension in the string is $W\tan\alpha\left(\frac{c}{2l}\cos ec^3\alpha-1\right)$.

Example: A rhombus is formed of four rods each of weight W and length l with smooth joints. It rests symmetrically with its upper sides in contact with two smooth pegs at the same level and at a distance $2a$ apart. A weight W' is hung to the lowest point. If the sides of the rhombus make an angle θ with the vertical, prove that $\sin^3\theta=\frac{a(4W+W')}{l(4W+2W')}$.

Solution: We have equation of virtual work is

$$4W\delta(LG)+W'\delta(LC)=0 \qquad (1)$$

$LG=l\cos\theta-a\cot\theta$ and $LC=2l\cos\theta-a\cot\theta$.



From (1) we have

$$4W\delta(l\cos\theta-a\cot\theta)+W'\delta(2l\cos\theta-a\cot\theta)=0$$

$$\Rightarrow 4W\left(-l\sin\theta+a\operatorname{cosec}^2\theta\right)$$

$$+W'\left(-2l\sin\theta+a\operatorname{cosec}^2\theta\right)=0$$

$$\Rightarrow -l(2W'+4W)\sin\theta+a(4W+W')\operatorname{cosec}^2\theta=0$$

$$\Rightarrow \sin\theta(4W+2W')l=\operatorname{cosec}^2\theta(4W+W')a$$

$$\Rightarrow \sin^3\theta = \frac{a(4W + W')}{l(4W + 2W')}.$$

Example: A square of side 2a is placed with its plane vertical, between two smooth pegs which one in a horizontal line and at a distance c apart. Show that it will be in equilibrium when the inclination of one of its edges to the horizon is either $\frac{\pi}{4}$ or $\frac{1}{2}\sin^{-1}\frac{a^2 - c^2}{c^2}$.

Hooke's Law

If the natural and expended length of an elastic string are a and l and the modulus of elasticity is λ then by Hooke's law the tension of the string, T is given by $T = \lambda \dfrac{l-a}{a}$

Example: A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi-vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle of radius

$$a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right).$$

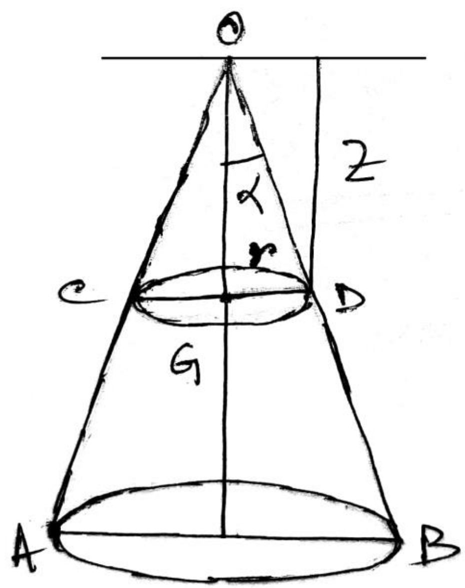
<u>Solution</u>: Let ABO be the smooth cone.

Let at the position of equilibrium. Let CD be the elastic string with centre at G and T be the tension on the string.

Let r be the radius of the string and z be the depth of the centre below O, the vertex of the cone.

Then $r = z\tan\alpha$.

Therefore, the length of the string $l = 2\pi r = 2\pi z\tan\alpha$



Here the force W (the weight of the string) acts at G downwards and the tension of the string work.

We now give a small virtual displacement in which the string moves vertically downwards parallel to itself remaining with contact of the cone.

The equation of virtual work is

$$W\delta z - T\delta l = 0 \Rightarrow W\delta z - T\delta\left(2\pi z\tan\alpha\right) = 0$$

$$\Rightarrow W\delta z - T2\pi\tan\alpha\delta z = 0 \Rightarrow W - T2\pi\tan\alpha = 0$$

$$\Rightarrow T = \frac{W}{2\pi}\cot\alpha \qquad (1)$$

Also by Hooke's law we have

$$T=\lambda\frac{2\pi r-2\pi a}{2\pi a}=\lambda\frac{r-a}{a} \qquad (2)$$

From (1) and (2) we get

$$\lambda\frac{r-a}{a}=\frac{W}{2\pi}\cot\alpha \Rightarrow r-a=\frac{aW}{2\pi\lambda}\cot\alpha$$

$$\Rightarrow r=a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right).$$

<u>Example</u>: One end of a uniform rod *AB* of length *a* and weight *W*, is attached by a frictionless joint to a smooth vertical wall and the other end *B* is smoothly jointed to an equal rod *BC*. The middle points of the rod, are jointed by an elastic string, of natural length a and modulus of elasticity $4W$. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A and the angle between the rods is $2\sin^{-1}\left(\frac{3}{4}\right)$.