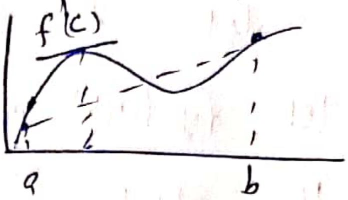
## Mean Value Theorem:

If $f: D \to \mathbb{R}$ is continuous in $[a,b]$ and differentiable in $(a,b)$ then $\exists$ at least one point $c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

In otherwords: Lagrange Mean Value Theorem




If $f: D \to \mathbb{R}$ is continuous on $[a, a+h]$ and differentiable on $(a, a+h)$ then $\exists$ a $\theta \in (0,1)$ such that

$$f(a+h) - f(a) = h\, f'(a+\theta h).$$

$a + \theta h$

$\theta = 0$

$\Rightarrow a$

$\theta = 1$

$\Rightarrow a+h$

$\dfrac{f(a+h)-f(a)}{h}$

$f'(a)$

$\approx f'(a+\theta h)$

## Mean-Value Theorem for function of Several Variables

If $f_x$ exists throughout a neighbourhood of a point $(a,b)$ and $f_y(a,b)$ exists then for any point $(a+h, b+k)$ of this nbhd

$$f(a+h, b+k) - f(a,b)$$
$$= h\, f_x(a+\theta h, b+k) + k\,[f_y(a,b) + \eta]$$

where $0 < \theta < 1$

and $\eta$ is a function of $k$ and $\eta \to 0$ as $k \to 0$.

Now,

$$f(a+h, b+k) - f(a,b) = f(a+h, b+k) - f(a, b+k) + f(a, b+k) - f(a,b)$$

Since $f_x$ exists in a nbhd of $(a,b)$ $\Rightarrow$ by Lagrange mean value theorem

$$f(a+h, b+k) - f(a, b+k) = h\, f_x(a+\theta h, b+k), \quad 0<\theta<1 \qquad (1)$$

Also, $f_y$ exists $\Rightarrow \displaystyle\lim_{k \to 0} \frac{f(a, b+k) - f(a,b)}{k} = f_y(a,b)$ (2)

$$\Rightarrow f(a, b+k) - f(a,b) = k\,[f_y(a,b) + \eta].$$

where $\eta(k) \to 0$ as $k \to 0$. (3)

$\therefore$ from (1), (2) & (3), the required result is obtained.

## Sufficient condition for continuity

A sufficient condition that a function $f$ be continuous at $(a,b)$ is that one of the partial derivatives exists and is bounded in a neighbourhood of $(a,b)$ and that the other exists at $(a,b)$.

Proof: let $f_x$ exists and be bounded in nhbd of $(a,b)$ and let $f_y(a,b)$ exists, then for any point $(a+h, b+k)$ of this nhbd.

$$f(a+h, b+k) - f(a,b) = h f_x(a+\theta h, b+k) + k[f_y(a,b) + \eta]$$

where: $0 < \theta < 1$, $\eta(k) \to 0$ as $k \to 0$

Using $\lim_{(h,k)\to 0}$

since $f_x(a+\theta h, b+k)$ is bounded, we have

$$\lim_{(h,k)\to(0,0)} f(a+h, b+k) = f(a,b)$$

$\therefore$ $f$ is continuous at $(a,b)$.

Hence the result.