MID - SEM

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1) Given PDE: $(p^2+q^2)x = pz$

Let $f = (p^2+q^2)x - pz$ —— ①

Now,

$f_x = p^2+q^2$, $f_y = 0$, $f_z = -p$, $f_p = 2px - z$, $f_q = 2qx$ —— ②

We know charpit's Auxiliary eq^n is given by

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \text{—— ③}$$

Using ② in ③ :-

$$\underset{\text{I}}{\frac{dp}{p^2+q^2-p^2}} = \underset{\text{II}}{\frac{dq}{0+q(-p)}} = \underset{\text{III}}{\frac{dz}{-p(2px-z)-q(2qx)}} = \underset{\text{IV}}{\frac{dx}{-2px+z}} = \underset{\text{V}}{\frac{dy}{-2qx}} \quad \text{—— ④}$$

Taking I & II of ④ :-

$$\frac{dp}{q^2} = \frac{dq}{-pq} \Rightarrow p\,dp + q\,dq = 0$$

On integrating,

$$\frac{p^2}{2} + \frac{q^2}{2} = c_1' \Rightarrow p^2+q^2 = 2c_1' = c_1^2 \text{ (let)} \quad \text{—— ⑤}$$

Using ⑤ in ① :-

$$c_1^2 x = pz \Rightarrow \boxed{p = \frac{c_1^2 x}{z}} \quad \text{—— ⑥}$$

Using ⑥ in ⑤ :-

$\frac{c_1^4 x^2}{z^2} + q^2 = c_1^2 \Rightarrow q^2 = c_1^2 - \frac{c_1^4 x^2}{z^2}$

$$\boxed{q = \pm \frac{c_1}{z}\sqrt{z^2 - c_1^2 x^2}} \quad \text{— (7)}$$

Now, solⁿ of PDE is given by

$$dz = p\,dx + q\,dy$$

$$dz = \frac{c_1^2 x\,dx}{z} + \frac{c_1}{z}\sqrt{z^2 - c_1^2 x^2}\,dy$$

$$z\,dz - c_1^2 x\,dx = \sqrt{z^2 - c_1^2 x^2}\,dy$$

$$\frac{z\,dz - c_1^2 x\,dx}{\sqrt{z^2 - c_1^2 x^2}} = c_1 dy$$

let $z^2 - c_1^2 x^2 = t$

$$2z\,dz - 2c_1^2 x\,dx = dt$$

$$\frac{dt}{2\sqrt{t}} = c_1 dy$$

On integrating

$$\sqrt{t} = c_1 y + c_2$$

$$\sqrt{z^2 - c_1^2 x^2} = c_1 y + c_2$$

$$\boxed{z^2 = (c_1 y + c_2)^2 + c_1^2 x^2} \Rightarrow \boxed{\text{Req. sol}^n} \quad (8)$$

Now, it pass through $x = 0$, $z^2 = 4y$

Put $x = 0 \Rightarrow z^2 = (c_1 y + c_2)^2 + c_1^2(0)$

$$z^2 = (c_1 y + c_2)^2 \quad \text{--- (9)}$$

Put $z^2 = 4y$: $\quad 4y = (c_1 y + c_2)^2 + c_1^2 x^2 \quad \text{--- (10)}$

Using (9) in (10) :-

$$4y = z^2 + c_1^2 x^2$$

$$c_1 = \frac{\sqrt{4y - z^2}}{x} \quad \text{--- (11)}$$

Using (11) in (9) :-

$$z^2 = \left(\frac{y\sqrt{4y - z^2}}{x} + c_2\right)^2$$

$$z = \frac{4y\sqrt{4y - z^2}}{x} + c_2 \Rightarrow c_2 = \frac{2x - 4y\sqrt{4y - z^2}}{x} \quad \text{--- (12)}$$

Using (11) & (12) in (8) :-

$$z^2 = \left(\frac{y\sqrt{4y - z^2}}{x} + \frac{2x - 4y\sqrt{4y - z^2}}{x}\right)^2 + \frac{4y - z^2}{\cancel{x^2}} \cancel{x^2}$$

$$z^2 = z + 4y - z^2$$

$$\boxed{2z^2 - z = 4y} \rightarrow \boxed{\text{req. sol}^n}$$

3) $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} + 2\frac{\partial^3 z}{\partial y^3} = e^{x+y}$

The given PDE can be rewritten as:-

$$\left(D^3 - 2D^2D' - DD'^2 + 2D'^3\right)z = e^{x+y}$$

<u>A.E:</u> $m^3 - 2m^2 - m + 2 = 0$

$m^2(m-2) - 1(m-2) = 0$

$(m^2-1)(m-2) = 0$

$m = -1, 1, 2$

$\therefore$ C.F $= \phi_1(y+2x) + \phi_2(y-x) + \phi_3(y+x)$

$$P.I = \frac{e^{x+y}}{D^3 - 2D^2D' - DD'^2 + 2D'^3}$$

$3D^2 - 4DD' - D'^2$

Put $D = 1, D' = 1$

Denominator $= 0$

$$P.I = \frac{x}{3D^2 - 4DD' - D'^2} e^{x+y} = -\frac{x}{2} e^{x+y}$$

$\therefore$ Complete soln is given by

$z = C.F + P.I.$

$z = \phi_1(y+2x) + \phi_2(y-x) + \phi_3(y+x) \pm \frac{x e^{x+y}}{2}$

2) $f = z(1-q^2) - 2(px + qy) = 0$ —— (1)

~~Lagrange's~~ Auxiliary eq^n is given by :-

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \text{—— (2)}$$

from (1) :-

$f_x = -2p$, $f_y = -2q$, $f_z = 1-q^2$, $f_p = -2x$

$f_q = -2qz - 2y$

Using above values in (2) :- —— (3)

$$\underset{(I)}{\frac{dp}{-2p+p(1-q^2)}} = \underset{(II)}{\frac{dq}{-2q+q(1-q^2)}} = \underset{(III)}{\frac{dz}{-p(-2x)-q(-2qz-2y)}} = \underset{(IV)}{\frac{dx}{2x}} = \underset{(V)}{\frac{dy}{2qz+2y}}$$

Taking (II) & (III) of (3) :-

$$\frac{dq}{q(-1-q^2)} = \frac{dz}{2px + 2q^2z + 2qy} = \frac{dz}{2(px+qy) + 2q^2z}$$

from (1) :-

$$= \frac{dz}{z(1-q^2) + 2q^2z}$$

$$\frac{dq}{-q(1+q^2)} = \frac{dz}{z(1-q^2+2q^2)} = \frac{dz}{z(1+q^2)}$$

$$\frac{dq}{-q} = \frac{dz}{z}$$

On integrating

$-\log q = \log z - \log c_1$

$$\boxed{q = \frac{c_1}{z}} \quad \text{—— (4)}$$

Using (4) in (1) :

$$z\left(1-\frac{c_1^2}{z^2}\right)-2\left(px+\frac{c_1}{z}y\right)=0$$

$$z-\frac{c_1^2}{z}-2px-\frac{2c_1 y}{z}=0$$

$$2px=z-\frac{c_1^2}{z}-\frac{2c_1 y}{z}$$

$$2px=\frac{z^2-c_1^2}{z}-\frac{2c_1 y}{z}$$

$$\boxed{p=\frac{z^2-c_1^2}{2xz}-\frac{c_1 y}{xz}} \quad \text{—— (5)}$$

Complete solⁿ is given by

$$dz=p\,dx+q\,dy$$

$$dz=\left(\frac{z^2-c_1^2}{2xz}-\frac{c_1 y}{xz}\right)dx+\left(\frac{c_1}{z}\right)dy$$

$$z\,dz=\left(\frac{z^2-c_1^2}{2}-c_1 y\right)\frac{1}{x}\,dx+c_1\,dy$$

$$\frac{z\,dz-c_1\,dy}{\frac{z^2-c_1^2}{2}-c_1 y}=\frac{1}{x}\,dx$$

Let $\frac{z^2-c_1^2}{2}-c_1 y=t$

$$\frac{2z\,dz}{2}-c_1\,dy=dt$$

$$\Rightarrow \frac{dt}{t}=\frac{dx}{x}$$

On integrating

$\log t = \log x + \log c_2$

$t = x c_2$

$\frac{z^2 - c_1^2}{2} - c_1 y = x c_2$

$z^2 - c_1^2 - 2c_1 y = 2x c_2$

$\boxed{z^2 = 2c_1 y + 2c_2 x + c_1^2}$ $\Rightarrow$ Complete integral — (6)

Given it pass through $x = 1$, $y = hs + k$, $z = s$, s is parameter — (7)

Using (7) in (6),

$s^2 = 2c_1(hs + k) + 2c_2 + c_1^2$ — (8)

Diff. (8) w.r.t s

$2s = 2c_1 h \Rightarrow s = c_1 h$ — (9)

Eliminating s from (9) & (8)

$2c_2 = -c_1^2(1 + h^2) - 2c_1 k$ — (10)

Substituting from (10) in (6)

$z^2 = c_1^2 + 2c_1 y - x[c_1^2(1+h^2) + 2c_1 k]$ — (11)

The envelope of (11) is

$z^2[(1+h^2)x - 1] = (y - kx)^2$

& req. surface is

$$z = \frac{y - kx}{\sqrt{(1+h^2)x - 1}}$$

4) $z = f(x-z) + g(x+y)$

Diff. z partially w.r.t x & y :-

$$\frac{\partial z}{\partial x} = p = \frac{\partial f}{\partial x}\left(1 - \frac{\partial z}{\partial x}\right) + \frac{\partial g}{\partial x}$$

$$p = \frac{\partial f}{\partial x}(1-p) + \frac{\partial g}{\partial x}$$

Let $g' = \frac{\partial g}{\partial x}$ & $f' = \frac{\partial f}{\partial x}$

$$p = f'(1-p) + g'$$

$$p(1+f') = f' + g' \quad \text{—— } (1)$$

Similarly for y :-

$$q(1+f') = g' \quad \text{—— } (2)$$

Now partially diff. p w.r.t x

$$\frac{\partial p}{\partial x}(1+p') + p\left(f''(1-p)\right\} = f''(1-p) + g''$$

where $\frac{\partial p}{\partial x} = \frac{\partial^2 z}{\partial x^2}$

$$\frac{\partial^2 z}{\partial x^2}(1+f') = g'' + (1-p)^2 \quad \text{—— } (3)$$

Partially diff. (2) w.r.t y & x

$$\frac{\partial q}{\partial y}(1+f') + q\left(f''(-q)\right) = g''$$

$$(1+f')\frac{\partial^2 z}{\partial y^2} = q^2 f'' + g'' \quad — (4)$$

Similarly, $\frac{\partial^2 z}{\partial x \partial y}(1+f') = (p-1)q f'' + g'' \quad — (5)$

Now, multiplying (3) with q & (4) with $(1-p)$,

$$q(1+f')\frac{\partial^2 z}{\partial x^2} = q g'' + q(1-p)^2 f'' \quad — (6)$$

$$(1-p)(1+f')\frac{\partial^2 z}{\partial y^2} = (1-p)q^2 f'' + (1-p)g'' \quad — (7)$$

Subtracting (7) from (6)

$$\left(q\frac{\partial^2 z}{\partial x^2} - (1-p)\frac{\partial^2 z}{\partial y^2}\right)(1+f') = (1-p-q)(-g'' + q(1-p)f'') \quad — (8)$$

Now, multiplying (5) with $(1-p-q)$

$$(1-p-q)(1+f')\frac{\partial^2 z}{\partial x \partial y} = (1-p-q)(g'' - (1-p)qf'') \quad — (9)$$

Adding (8) & (9) ↳

$$q\frac{\partial^2 z}{\partial x^2} + (1-p-q)\frac{\partial^2 z}{\partial x \partial y} - (1-p)\frac{\partial^2 z}{\partial y^2} = 0$$