23. Simple Harmonic Motion. (91), (93)

A particle is said to execute Simple Harmonic Motion if it moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point.

Let O be the fixed point on a line BOA and P be the position of the particle at time t where $OP = x$, so that the acceleration of the particle in the sense OP is $\ddot{x}$.

Now the given acceleration is towards O and is proportional to x. Let it be μx, where μ is constant.

Since $\ddot{x}$ is in the direction of OP produced and μx is towards O, the equation of motion is

$$\ddot{x} = -\mu x.$$

$\frac{d^2x}{dt^2} \propto x \Rightarrow \frac{d}{dx}\left(\frac{dx}{dt}\right) = -\mu x$

Taking $v\frac{dv}{dx}$ instead of $\ddot{x}$; we can write the above equation as

$v\frac{dx}{dt} = -\mu x$

$$v\frac{dv}{dx} = -\mu x. \qquad \ldots(1)$$

$v\,dx = -\mu x\,dt$

Integrating with respect to x, we get

$\frac{v^2}{2} = -\mu\frac{x^2}{2} + C_1$ — (2)

$$\frac{v^2}{2} = -\mu\frac{x^2}{2} + \frac{C}{2} \quad \text{where } C \text{ is a constant}$$

or

$$v^2 = -\mu x^2 + C.$$

If A be the extreme position of the particle *i.e.*, it is at rest at A

i.e., when $x = a$, $v = 0$ where $OA = a$, we get

$$0 = -\mu a^2 + C \qquad \therefore \quad C = \mu a^2$$

$C_1 = \frac{\mu a^2}{2}$

$\frac{v^2}{2} = -\frac{\mu x^2}{2} + \frac{1}{2}\mu a^2 \Rightarrow v^2 = -\mu x^2 + \mu a^2$

$v^2 = \mu(a^2 - x^2)$

$$v^2=\mu(a^2-x^2), \quad \ldots(2)$$

Hence

$$v=\pm\sqrt{\mu}.\sqrt{a^2-x^2}.$$

i.e., If the particle moves from A towards O, v is negative

$$\dot{x}=v=-\sqrt{\mu}\sqrt{a^2-x^2}$$

Hence

$$\frac{dx}{dt}=-\sqrt{\mu}\sqrt{a^2-x^2}$$

or

$$\sqrt{\mu}.dt=-\frac{dx}{\sqrt{a^2-x^2}}.$$

or

Integrating we get $\sqrt{\mu}.t=\cos^{-1}\dfrac{x}{a}+C_1$ where C_1 is a constant

Initially at A, $t=0$, $x=a$ *i.e.*, the particle started from A,

then

$$0=\cos^{-1}1+C_1 \quad \therefore \quad C_1=0.$$

Hence

$$\sqrt{\mu}.t=\cos^{-1}\frac{x}{a}$$

or

$$\boxed{x=a\cos\sqrt{\mu}.t.} \quad \ldots(3)$$

If the particle moves from O towards A, v is positive

so that

$$\dot{x}=\sqrt{\mu}.\sqrt{a^2-x^2}$$

or

$$\sqrt{\mu}.dt=\frac{dx}{\sqrt{a^2-x^2}}.$$

Integrating, we get, $\sqrt{\mu}.t=\sin^{-1}\dfrac{x}{a}+C_2$ where C_2 is a constant

If the particle starts from O, $t=0$, $x=0$, $0=\sin^{-1}0+C_2$

$\therefore$

$$C_2=0,$$

so that

$$\boxed{x=a\sin\sqrt{\mu}.t.} \quad \ldots(4)$$

Thus the solution of (1) is $x=a\cos\sqrt{\mu}.t$ or $x=a\sin\sqrt{\mu}.t$ according as the starting point is A or O.

From (2), $\quad v=0$ when $x=\pm a$.

Thus if B is a point on the other side of O such that $OB=OA=a$, the particle comes to rest also at B. When $x=0$, $v=\pm\sqrt{\mu}a$ *i.e.*, at O, the velocity is $\sqrt{\mu}a$.

Consider the solution $x=a\cos\sqrt{\mu}.t$.

The motion starts from A under an attraction towards O.

When the particle reaches O, $x=0$. $\therefore$ $\cos\sqrt{\mu}.t=0$ $\therefore$ $\sqrt{\mu}t=\dfrac{\pi}{2}$

i.e., $t=\dfrac{\pi}{2\sqrt{\mu}}$ is the time required in moving from A to O.

As the particle reaches O, the attraction ceases but the particle has a velocity $\sqrt{\mu}.a$ towards the negative side of O hence the particle passes O and moves towards the negative side. As soon as the particle comes to the left side of O, attraction changes direction and becomes towards O ; hence the velocity will go on decreasing as the particle moves towards the left, till at B, the velocity becomes zero so that the particle stops. But the particle is being attracted towards O hence starts moving towards O and reaches O with a velocity $\sqrt{\mu}.a$, due to which it passes O and moves towards A and again stops at A where its velocity becomes zero. The motion is then repeated. Thus the motion is from A to B and back to A and so on. The motion is *oscillatory* Time from O to B is equal to that from A to O hence the **period** *i.e.*, the time from A to B and back to A is $4.\dfrac{\pi}{2\sqrt{\mu}}=\dfrac{2\pi}{\sqrt{\mu}}$. The distance $a\,(=OA)$ *i.e.*, the distance of the centre from one of the positions of rest is called the **amplitude**.

Thus the period which is equal to $\dfrac{2\pi}{\sqrt{\mu}}$ is independent of the amplitude *i.e.*, whatever be the amplitude the period is the same. Thus the simple harmonic motion is oscillatory and periodic, the period being independent of amplitude.

The **frequency** is the number of complete oscillations in one second, so that if n be the frequency and T the periodic time,

$$\boxed{n=\frac{1}{T}=\frac{\sqrt{\mu}}{2\pi}.}$$

The equation (1), namely $\ddot{x}=-\mu x$, can be solved as a differential equation. The most general solution of this equation is

$$x=A\cos\sqrt{\mu}.t+B\sin\sqrt{\mu}.t \qquad ...(5)$$

A, B are constants to be determined from initial conditions. In the first case when the motion starts from A, the initial conditions are $t=0$, $x=a$, $\dot{x}=0$.

Now $\qquad t=0,\ x=a$ gives $a=A$.

Differentiating (5), $\dot{x}=-A\sqrt{\mu}\sin\sqrt{\mu}.t+B\sqrt{\mu}\cos\sqrt{\mu}.t \qquad ...(6)$

The condition $t=0$, $\dot{x}=0$ gives $0=0+B\sqrt{\mu} \qquad \therefore \quad B=0.$

Hence the solution is $x=a\cos\sqrt{\mu}t$.

In the second case when the motion starts from O, the first condition is $t=0$, $x=0$. $\qquad \therefore \qquad 0=A,\ A=0.$

Hence $\qquad x=B\sin\sqrt{\mu}t.$

To determine B, we must know the velocity of projection from O.

Let us take the case of a particle, projected from A with velocity V along OA produced, so that the initial conditions are $t=0$, $x=a$, $\dot{x}=V$.

Hence from (5) and (6), we get

$$a=A$$

$$V=B\sqrt{\mu}. \qquad \therefore \quad B=\frac{V}{\sqrt{\mu}}$$

Hence the solution is $x=a \cos \sqrt{\mu}.t+\dfrac{V}{\sqrt{\mu}} \sin \sqrt{\mu}.t.$

Also the general solution of (1) can be written as

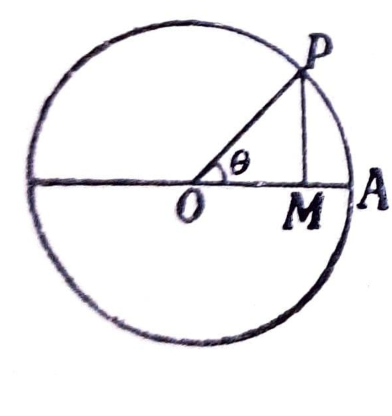
$$x=a \cos (\sqrt{\mu}t+\xi)$$

This is periodic with period $\dfrac{2\pi}{\sqrt{\mu}}$.

The quantity ξ is called the **epoch**, the angle $\sqrt{\mu}.t+\xi$ is called the **argument**. The particle is at its maximum distance at time t_0 where $\sqrt{\mu}t_0+\xi=0$ *i.e.*, $t_0=-\dfrac{\xi}{\sqrt{\mu}}$. Hence the time that has elapsed since the particle was at its maximum distance is equal to $t-t_0=t+\dfrac{\xi}{\sqrt{\mu}}=\dfrac{\sqrt{\mu}t+\xi}{\sqrt{\mu}}$. This is the **phase** at time t.

A geometrical representation of the S.H.M.

Let a particle P move on a circle with constant angular velocity ω and let M be the foot of the perpendicular from P on any diameter OA. If a be the radius of the circle, the only acceleration of P is $\omega^2 a$ towards O.



If $\angle AOP=\theta$ and $OM=x$, the component of this acceleration along OA

$$=\omega^2 a. \cos \theta=\omega^2 a. \frac{x}{a}=\omega^2 x \text{ towards } O.$$

Hence the equation of motion of the point M is

$$\boxed{\ddot{x}=-\omega^2 x.}$$

This is $S.H.M.$

Thus *if a particle describes a circle with constant angular velocity the foot of the perpendicular from it on any diameter executes a simple harmonic motion.*