# Eigen Value Problem

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# The Eigen value Problem

Let A be a square matrix of order n with element $a_{ij}$, we have to find a column vector $X$ and a constant $\lambda$
s.t.

$$AX = \lambda X \qquad \ldots\ldots\ldots\ldots(1)$$

$$\Longrightarrow \quad |A - \lambda I| X = 0 \qquad \ldots\ldots\ldots\ldots(2)$$

on expansion, it gives a polynomial of $n^{th}$ degree in $\lambda$, which is called Characteristic equation of the matrix A, its roots $\lambda_i$ (i=1,2,3…..n) are called Eigen values or latent roots.

Corresponding to each Eigen values from eq. (2) there exists a non zero solution

$$x = [x_1, x_2 \ldots\ldots x_n]$$

Which is known as Eigen vector

# Computing Eigenvalues

Since *X* is required to be nonzero, the eigenvalues must satisfy

$$\det(A - \lambda I) = 0$$

which is called the *characteristic equation*.

Solving it for values of $\lambda$ gives the eigenvalues of matrix A.

# Example consider a 2 × 2 matrix

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \qquad \text{so } A - \lambda I = \begin{bmatrix} 1 - \lambda & -2 \\ 3 & -4 - \lambda \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (1 - \lambda)(-4 - \lambda) - (3)(-2) \\ &= \lambda^2 + 3\lambda + 2 \end{aligned}$$

Set $\lambda^2 + 3\lambda + 2$ to 0

Then = $\lambda$ = (-3 +/- sqrt(9-8))/2

So the two values of $\lambda$ are -1 and -2.

# Finding the Eigenvectors

Once you have the eigenvalues, you can plug them into the equation $A\mathbf{x} = \lambda\mathbf{x}$ to find the corresponding sets of eigenvectors $\mathbf{x}$.

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{so} \quad \begin{aligned} x_1 - 2x_2 &= -x_1 \\ 3x_1 - 4x_2 &= -x_2 \end{aligned}$$

$$\boxed{\begin{aligned} &(1) \quad 2x_1 - 2x_2 = 0 \\ &(2) \quad 3x_1 - 3x_2 = 0 \end{aligned}}$$

These equations are not independent. If you multiply (2) by 2/3, you get (1).

The simplest form of (1) and (2) is $x_1 - x_2 = 0$, or just $x_1 = x_2$.

Since $x_1 = x_2$, we can represent all eigenvectors for eigenvalue -1 as multiples of a simple basis vector:

$E = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where t is a parameter.

So $[1\ 1]^T$, $[3\ 3]^T$, $[100\ 100]^T$ are all possible eigenvectors for eigenvalue -1.

For the second eigenvalue (-2) we get

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{so} \quad \begin{aligned} x_1 - 2x_2 &= -2x_1 \\ 3x_1 - 4x_2 &= -2x_2 \end{aligned}$$

$$\boxed{\begin{aligned} &(1)\quad 3x_1 - 2x_2 = 0 \\ &(2)\quad 3x_1 - 2x_2 = 0 \end{aligned}}$$

so eigenvectors are of the form $t \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

# Another observation we will use:

For 2 x 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ,

$\lambda_1 + \lambda_2 = a + d$, which is called trace(A)
and
$\lambda_1\lambda_2 = ad - bc$, which is called det(A).

Finally, zero is an eigenvalue of A if and only if
A is singular and det(A) = 0.

# Power Method:

The method for finding the largest eigen value in magnitude and the corresponding eigen vector of the eigen value problem $\mathbf{Ax} = \lambda\, \mathbf{x}$, is called the power method.

It is used to calculate largest eigen value in magnitude of a given matrix and the corresponding eigen vector.

# Power Method (Cont..)

It is an iterative method implemented using an initial starting vector X. The starting vector can be arbitrary if no suitable approximation is available.

Let ***x*** be a column vector, which is as near the solutions as possible and evaluate $\mathbf{Ax^{(0)}}$, which is written as $\mathbf{\lambda^{(1)}x^{(1)}}$

so the first eigen value and the corresponding eigen vector

$$\mathbf{Ax^{(0)} = \lambda^{(1)}x^{(1)}}$$

Similarly, the second eigen value and the corresponding eigen vector

$$\mathbf{Ax^{(1)} = \lambda^{(2)}x^{(2)}}$$

Repeat this process till $|\mathbf{x^n - x^{n-1}}|$ is negligible.

$\lambda^n$ will be the largest Eigen value and $x^n$ will be the largest eigen vector.

This iterative procedure for finding the dominated eigen value of a matrix is known as Rayleigh's power method.

Example: Find the largest eigen value and vector of the following matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Let the initial eigen vector be $x^{(0)} = [0,1,0]'$

$$Ax^{(0)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Ax^{(0)} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

So first eigen value is 2 and eigen vector=[1,0.5,0]'

$\lambda^{(1)}$ **=2 and** $x^{(1)}$**= [1,0.5,0]'**

**Second iteration**

$$Ax^{(1)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$Ax^{(1)} = \begin{bmatrix} 2 \\ 2.5 \\ 0 \end{bmatrix} = 2.5 \begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda^{(2)}$ **=2.5 and** $x^{(2)}$**= [0.8,1,0]'**

**Third iteration**

$$Ax^{(2)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix}$$

$$Ax^{(2)} = \begin{bmatrix} 2.8 \\ 2.6 \\ 0 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ 0.93 \\ 0 \end{bmatrix}$$

$\lambda^{(3)}$ **=2.8 and** $x^{(3)}$**= [1,0.93,0]'**

## Fourth iteration

$$Ax^{(3)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.93 \\ 0 \end{bmatrix}$$

$$Ax^{(3)} = \begin{bmatrix} 2.86 \\ 2.93 \\ 0 \end{bmatrix} = 2.93 \begin{bmatrix} 0.98 \\ 1 \\ 0 \end{bmatrix}$$

**$\lambda^{(4)}$ =2.93 and $x^{(4)}$= [0.98,1,0]'**

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$$x^{(7)} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

**$\lambda^{(7)}$ =3 and $x^{(7)}$= [1,1,0]'**

# The Cayley Hamilton Theorem

This illustrates the **Cayley-Hamilton** theorem:
A square matrix satisfies its own characteristic equation.

We saw that the matrix $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ has characteristic equation $\lambda^2 - 5\lambda + 6 = 0$.

$$\mathbf{M}^2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix}$$

and $\mathbf{M}^2 - 5\mathbf{M} + 6\mathbf{I} = \begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix} - 5\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -1-5+6 & 5-5 \\ -10+10 & 14-20+6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

Use the Cayley-Hamilton theorem to find $\mathbf{M}^6$ if $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

Characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$

$\mathbf{M}^2 - 5\mathbf{M} + 6\mathbf{I} = 0$

$$
\begin{aligned}
\Rightarrow \quad & \mathbf{M}^2 = 5\mathbf{M} - 6\mathbf{I} \\
\Rightarrow \quad & \mathbf{M}^4 = (5\mathbf{M} - 6\mathbf{I})^2 \\
& = 25\mathbf{M}^2 - 60\mathbf{M} + 36\mathbf{I} \\
& = 25(5\mathbf{M} - 6\mathbf{I}) - 60\mathbf{M} + 36\mathbf{I} \\
& = 65\mathbf{M} - 114\mathbf{I} \\
& M^6 = M^4 \times M^2 \\
& = (65\mathbf{M} - 114\mathbf{I})(5\mathbf{M} - 6\mathbf{I}) \\
& = 325\mathbf{M}^2 - 960\mathbf{M} + 684\mathbf{I} \\
& = 325(5\mathbf{M} - 6\mathbf{I}) - 960\mathbf{M} + 684\mathbf{I} \\
& = 665\mathbf{M} - 1266\mathbf{I} \\
& = 665\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - 1266\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -601 & 665 \\ -1330 & 1394 \end{bmatrix}
\end{aligned}
$$

Practice Problems

1. Find the Eigen values and vectors of the following matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

2. Find the largest Eigen values and corresponding vector of the following matrix using power method.

$$A = \begin{bmatrix} 10 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 10 \end{bmatrix}$$

# Suggested books

1. Numerical Methods by **S.R.K Lyenger & R.K. Jain.**

2. Introductory methods of Numerical analysis by **S.S. Sastry**.

Thank you