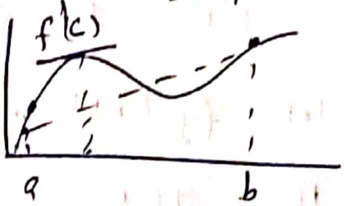
# Mean Value Theorem:

If $f: D \to \mathbb{R}$ is continuous in $[a,b]$ and differentiable in $(a,b)$ then $\exists$ at least one point $c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

In otherwords: Lagrange Mean Value Theorem




If $f: D \to \mathbb{R}$ is continuous on $[a, a+h]$ and differentiable on $(a, a+h)$ then $\exists$ a $\theta \in (0,1)$ such that-

$$f(a+h) - f(a) = h f'(a+\theta h).$$

## Mean - Value theorem for function of Several Variables

If $f_x$ exists throughout a neighbourhood of a point $(a,b)$ and $f_y(a,b)$ exists then for any point $(a+h, b+k)$ of this nbhd

$$f(a+h, b+k) - f(a,b)$$
$$= h f_x(a+\theta h, b+k) + k[f_y(a,b) + \eta]$$

where $0 < \theta < 1$

and $\eta$ is a function of $k$ and $\eta \to 0$ as $k \to 0$.

$a+\theta h$

$\theta = 0$

$\Rightarrow a$

$\theta = 1$

$\Rightarrow a+h$

$f'(a) = \dfrac{f(a+h)-f(a)}{h}$

$\approx f'(a+\theta h)$

Now,

$$f(a+h, b+k) - f(a,b) = f(a+h, b+k) - f(a, b+k) + f(a, b+k) - f(a,b)$$

Since $f_x$ exists in a nhbd of $(a,b)$ $\Rightarrow$ by Lagrange mean value theorem

$$f(a+h, b+k) - f(a, b+k) = h f_x(a+\theta h, b+k), \quad 0<\theta<1 \quad (1)$$

Also, $f_y$ exists $\Rightarrow \displaystyle\lim_{k\to 0} \frac{f(a, b+k) - f(a,b)}{k} = f_y(a,b) \quad (2)$

$$\Rightarrow f(a, b+k) - f(a,b) = k[f_y(a,b) + \eta].$$

where $\eta(k) \to 0$ as $k \to 0$. (3)

$\therefore$ from (1), (2) & (3), the required result is obtained.

## Sufficient Condition for Continuity

A sufficient condition that a function f be continuous at $(a,b)$ is that one of the partial derivatives exists and is bounded in a neighbourhood of $(a,b)$ and that the other exists at $(a,b)$.

Proof: Let $f_x$ exists and be bounded in nbhd of $(a,b)$ and let $f_y(a,b)$ exists, then for any point $(a+h, b+k)$ of this nbhd.

$$f(a+h, b+k) - f(a,b) = h f_x(a+\theta h, b+k) + k[f_y(a,b)+\eta]$$

where: $0<\theta<1$, $\eta(k) \to 0$ as $k \to 0$

Using $\lim_{(h,k)\to 0}$ since $f_x(a+\theta h, b+k)$ is bounded, we have

$$\lim_{(h,k)\to(0,0)} f(a+h, b+k) = f(a,b)$$

$\therefore$ f is continuous at $(a,b)$.

Hence the result.

# Differentiability

Let $f : D (\subseteq \mathbb{R}^2) \to \mathbb{R}$ be a function of two variables $x$ & $y$.

Let $(x, y)$, $(x + \delta x, y + \delta y)$ be two neighbouring points in the domain of definition of a function $f$. The change $\delta f$ is

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y)$$

The function $f$ is said to be differentiable at $(x, y)$ if the change $\delta f$ can be expressed in the form

$$\delta f = A\,\delta x + B\,\delta y + \delta x\,\phi(\delta x, \delta y) + \delta y\,\psi(\delta x, \delta y) \quad \text{— (1)}$$

where $A$ and $B$ are constants independent of $\delta x, \delta y$

and $\phi, \psi$ are functions of $\delta x, \delta y$ tending to $0$ as $\delta x, \delta y \to 0$ simultaneously.

The term $\underline{A\,\delta x + B\,\delta y}$ is called the differential of $f$ at $(x, y)$ and is denoted by $df$.

$$\boxed{df = A\,\delta x + B\,\delta y}$$

From (1) when $(\delta x, \delta y) \to (0, 0)$, we have.

$$f(x + \delta x, y + \delta y) - f(x, y) \to 0$$

or $f(x + \delta x, y + \delta y) \longrightarrow f(x, y)$

The function $f$ is continuous at $(x, y)$.

Thus, every differentiable function is continuous.

$$\boxed{\text{Differentiability}} \Rightarrow \boxed{\text{continuity}}$$

If $y$ remains constat i.e. $\delta y = 0$,

$$\delta f = A\,\delta x + \delta x\,\phi(\delta x, 0)$$

$$\Rightarrow \frac{\delta f}{\delta x} = A + \phi(\delta x, 0)$$

$$\boxed{\therefore \lim_{\delta x \to 0} \frac{\delta f}{\delta x} = A = \frac{\partial f}{\partial x}}$$

Similarly, $\boxed{\lim_{\delta y \to 0} \frac{\delta f}{\delta y} = B = \frac{\partial f}{\partial y}}$

Thus, the constants A and B are respectively the partial derivatives of f with respect to x and y.

Hence, $\boxed{\text{a function which is differentiable at a point posses the first order partial derivatives at that point.}}$

Converse is not true: function exists, continuous having partial derivatives, ~~may~~ but not differentiable at that point.

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$f$ is continuous at $(0,0)$, $f_x(0,0) = 1$, $f_y(0,0) = -1$.

But not differentiable

Differential of f at (x,y)

$$df = A\,\delta x + B\,\delta y. = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

taking $f = x \Rightarrow dx = \delta x$

$f = y \Rightarrow dy = \delta y$

Thus,
$$\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x\,dx + f_y\,dy}$$

Ex. Let
$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

put $x = r\cos\theta$, $y = r\sin\theta$

$$|f(x,y) - 0| = |r(\cos^3\theta - \sin^3\theta)| \leq 2|r| = 2\sqrt{x^2+y^2} < \varepsilon$$

$$\Rightarrow x^2 < \frac{\varepsilon^2}{8}, \quad y^2 < \frac{\varepsilon^2}{8}$$

$$\therefore |x-0| < \frac{\varepsilon}{2\sqrt{2}} = \delta$$

$$|y-0| < \frac{\varepsilon}{2\sqrt{2}} = \delta$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$$ $\Rightarrow$ function f is continuous at (0,0)

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1$$

$$f_y(0,0) = \lim_{k\to 0}\frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0}\frac{-k}{k} = -1.$$

partial derivatives exists at (0,0).

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$$df = f(h,k) - f(0,0) = Ah + Bk + h\phi + k\psi$$

when A and B are constants $(A=1,\ B=-1)$ and $\phi, \psi \to 0$ as $(h,k) \to (0,0)$.

putting $h = \rho\cos\theta$, $k = \rho\sin\theta$ and dividing by $\rho$,

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta + \phi\cos\theta + \psi\sin\theta$$

$$\boxed{f(a+h, b+k) - f(a,b) = \delta f}$$

Now, for $\theta = \tan^{-1}\left(\frac{h}{k}\right)$. $\rho \to 0$ implies that $(h,k) \to (0,0)$

Thus, we get the limit

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta \Rightarrow \text{Absurd.}$$

or

$$\cos\theta\sin\theta(\cos\theta - \sin\theta) = 0$$

which is not possible for arbitrary $\theta$.

Thus, the function is not differentiable at origin.

Ex. Show that $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x = y = 0 \end{cases}$

is not differentiable at $(0,0)$.

Ex. Show that the function $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x = y = 0 \end{cases}$

is continuous, posses partial derivatives

but is not differentiable at origin.

Ex 1. Show that $|x| + |y|$ is continuous but not differentiable at the origin.

Ex 2. Show that

$$f(x,y) = \begin{cases} x \sin \frac{1}{x} + y \sin \frac{1}{y}, & xy \neq 0 \\ x \sin \frac{1}{x}, & y=0, x \neq 0 \\ y \sin \frac{1}{y}, & x=0, y \neq 0 \\ 0, & x=0=y \end{cases}$$

is continuous but not differentiable at $(0,0)$.

Ex. 3. Discuss continuity, partial derivatives and differentiability of the following functions at $(0,0)$

a) $$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

b) $$f(x,y) = \begin{cases} y \sin \frac{1}{x}, & x \neq 0 \\ y & x=0 \end{cases}$$

c) $$f(x,y) = \begin{cases} x \sin \frac{1}{y}, & y \neq 0 \\ x, & y=0 \end{cases}$$