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B. Sc. (Hons) Semester III Open Book Examination 2 0 2 1 – 2 2

MATHEMATICS

Paper No. MTB-301 : Algebra–I

Time : 4 hours 30 minutes *Full Marks* : 70

Instructions :

(i) The Question Paper contains 08 questions out of which you are required to answer any 04 questions. The question paper is of 70 Marks with each question carrying 17·5 Marks.

प्रश्नपत्र में 8 प्रश्न पूँछे गये हैं जिनमें से 4 प्रश्नों का उत्तर देना है। प्रश्नपत्र 70 अंकों का है, जिसमें प्रत्येक प्रश्न 17·5 अंक का है।

(ii) The total duration of the examination will be 4·30 Hours (Four Hours and Thirty Minutes), which includes the time for downloading the question paper from the portal, writing the answers by hand and uploading the hand-written answer sheets on the portal.

परीक्षा का कुल समय 4·30 घंटे का है जिसमें प्रश्नपत्र को पोर्टल से डाउनलोड करके पुनः प्रश्नों का हस्तलिखित उत्तर पोर्टल पर अपलोड करना है।

(iii) For the students with benchmark disability as per Persons with Disability Act, the total duration of examination shall be 6 Hours (Six Hours) to complete the examination process, which includes the time for downloading the question paper from the portal, writing the answers by hand and uploading the hand-written answer sheets on the portal.

दिब्यांग छात्रों के लिये परीक्षा का समय 6 घंटे निर्धारित है जिसमें प्रश्नपत्र को पोर्टल से डाउनलोड करना एवं हस्तलिखित उत्तर को पोर्टल पर अपलोड करना है।

(iv) Answers should be hand-written on plain white A4 size paper using black or blue pen. Each question can be answered in up to 350 words on 3 (Three) plain A4 size paper (only one side to be used).

प्रश्नों का हस्तलिखित उत्तर सादे सफेद A4 साइज के पन्ने पर काले अथवा नीले कलम से लिखा होना चाहिये। प्रत्येक प्रश्न का उत्तर 350 शब्दों तक तीन सादे पृष्ठ A4 साइज में होना चाहिये। प्रश्नों के उत्तर के लिए केवल एक तरफ के पृष्ठ का ही उपयोग किया जाना चाहिये।

(v) Answers to each question should start from a fresh page. All pages are required to be numbered. You should write your Course Name, Semester, Examination Roll Number, Paper Code, Paper Title, Date and Time of Examination on the first sheet used for answers.

प्रत्येक प्रश्न का उत्तर नये पृष्ठ से शुरू करना है। सभी पृष्ठों को पृष्ठांकित करना है। छात्र को प्रथम पृष्ठ पर प्रश्नपत्र का विषय, सेमेस्टर, परीक्षा अनुक्रमांक, प्रश्नपत्र कोड, प्रश्नपत्र का शीर्षक, दिनांक एवं समय लिखना है।

Q.1. *(a)* Consider a system of equation of the form $AX = b$, where

$$A = (a_{ij})_{m\times n} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

with $b \neq 0$.

Describe the following :

(i) When is the system consistent?

(ii) When is the system inconsistent?

In each of the above cases, describe the types of solutions obtained. 8

(b) Investigate for what values of a and b the system of equations $x + y + z = 7,\ 2y + 3z = 9,\ 2x + 2y + az = b$ have *(i)* no solution, *(ii)* a unique solution and *(iii)* in infinite no. of solutions. 9½

Q.2. *(a)* Show that every square matrix can be uniquely expressed as the sum $P + iQ$, where P and Q are Hermitian matrices. 9

(b) Prove that the matrix

$$B = \begin{pmatrix} 0 & 1+i & 2-i \\ -1+i & i & 3+7i \\ -2-i & -3+7i & -3i \end{pmatrix}$$

is skew Hermitian. 8½

Q.3. *(a)* Consider the following permutation :

$$p\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 7 & 5 & 4 & 1 & 9 & 6 & 8 \end{pmatrix}$$

Express p as a product of disjoint cycles. Further express it as a product of 2-cycles. Determine whether p is even or odd. 9½ *(b)* State and prove Cayley's theorem for groups. 8

Q.4. *(a)* Prove that a group of order 9 is Abelian. 7½

(b) Let $H = \{ xyx^{-1}y^{-1} \mid x, y \in G\}$. Thus H is a subset of the group G. Show that H generates a subgroup $\langle H \rangle$ of G. Also show that $\langle H \rangle$ is a normal subgroup of G. Verify whether $\frac{G}{\langle H \rangle}$ is Abelian. 10

Q.5. *(a)* If G is a group and N a normal subgroup of G, then show that $\frac{G}{N}$ is a group. 8

(b) Let H be a subgroup of a group G and let $N(H) = \{ g \in G \mid gHg^{-1} = H \}$. Prove that $N(H)$ is a subgroup of G and that H is normal in $N(H)$. 9½

Q.6. If H and K are finite subgroups of a group G of orders $o(H)$ and $o(K)$ respectively, then show that—

$$o(HK) = \frac{o(H) \cdot o(K)}{o(H \cap K)}$$

Under what conditions is HK also a subgroup of G ? Justify your answer. 17½

Q.7. *(a)* If p is a prime number and a is any integer, then show that $a^p \equiv a \bmod p$. Choose two suitable pairs (a, p) for which the above relation is satisfied. 5+2½

(b) Show with details that if G is a finite group and $a \in G$, then $o(a) \mid o(G)$. 10

Q.8. *(a)* If $o(G)$ is pq, where G is a group and p and q are distinct prime numbers. Suppose G has a normal subgroup of order p and a normal subgroup of order q, then prove that G is cyclic. 9

(b) Find all the normal subgroups in S_4. 8½

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