1)

# MID SEM


SURAJ KUMAR YADAV
20220PHY014
PK-44
31/03/22
MTB 402


2) a) We know, $\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$

So, $\sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots$

Taking laplace transform both sides:-

$$\mathcal{L}\{\sin\sqrt{t}\} = \mathcal{L}\{t^{1/2}\} - \frac{1}{3!}\mathcal{L}\{t^{3/2}\} + \frac{1}{5!}\mathcal{L}\{t^{5/2}\} - \frac{1}{7!}\mathcal{L}\{t^{7/2}\} + \cdots$$

$$= \frac{\Gamma(\frac{3}{2})}{s^{3/2}} - \frac{\Gamma(5/2)}{3!\,s^{5/2}} + \frac{\Gamma(7/2)}{5!\,s^{7/2}} - \frac{\Gamma(9/2)}{7!\,s^{9/2}} + \cdots$$

$$= \frac{1}{s^{3/2}}\left[\frac{\frac{1}{2}\sqrt{\pi}}{1} - \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{3.2.1.s} + \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{5.4.3.2.1.s^2} - \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{7.6.5.4.3.2.1 s^3} + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{(4s)^2 2!} - \frac{1}{(4s)^3 3!} + \cdots\right]$$

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}\left(e^{-1/4s}\right)$$

Using derivative property $\Rightarrow$ Let $f(t) = \sin\sqrt{t}$,

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

$$= s\,\mathcal{L}\{\sin(\sqrt{t})\} - \sin 0$$

$$\frac{1}{2}\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = s\frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s}$$

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\,e^{-1/4s}$$

b) We know, $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

Using divide by 't' property

$$\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \int_s^\infty\left(\frac{a}{s^2+a^2}\right)ds$$

2)

$$\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \left[\frac{a}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]_{s}^{\infty}$$

$$= \underset{s\to\infty}{Lt}\tan^{-1}\left(\frac{s}{a}\right) - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

$$= \tan^{-1}\left(\frac{a}{s}\right)$$

For $\mathcal{L}\left\{\frac{\cos at}{t}\right\}$ existence :-

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\left\{\frac{\cos at}{t}\right\} = \frac{2}{2}\cdot\int_s^\infty \frac{s}{s^2+a^2}\,ds = \frac{1}{2}\left[\log(s^2+a^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\underset{s\to\infty}{Lt}\log(s^2+a^2) - \frac{1}{2}\log(s^2+a^2)\right]$$

since $\underset{s\to\infty}{Lt}\log(s^2+a^2)$ is tending to infinity

So, $\mathcal{L}\left\{\frac{\cos at}{t}\right\}$ does not exist.

2) a) $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 = f(t)$$

$$\text{So, } \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} = f(t-1)\,u(t-1)$$

$$= 1\cdot u(t-1)$$

$$= \begin{cases} 1, & t>1 \\ 0, & t<1 \end{cases}$$

b) $\mathcal{L}^{-1}\left\{\frac{-1-3s}{(1-s)(1+s^2)}\right\} = \mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$

Here, $f(s) = 3s+1$

& $h(s) = (s-1)(s^2+1) = (s-1)(s^2-i^2) = (s-1)(s-i)(s+i)$

~~$\therefore h'(s) = 3s^2$~~

$$h(s) = s^3 - s^2 + s - 1$$

$\therefore h'(s) = 3s^2 - 2s + 1$ & $\alpha_1 = 1,\ \alpha_2 = -i,\ \alpha_3 = i$

$$\therefore \mathcal{L}^{-1}\left\{ \frac{3s+1}{(s-1)(s^2+1)} \right\}$$

$$= \frac{F(1)}{h'(1)} e^{t} + \frac{F(-i)}{h'(-i)} e^{-it} + \frac{F(i)}{h'(i)} e^{it}$$

$$= \frac{4e^t}{2} + \frac{-3i+1}{-2+2i} e^{-it} + \frac{3i+1}{-(2+2i)}$$

$$= 2e^t + \frac{(3i-1)(1+i)}{2(1-i)(1+i)} e^{-it} - \frac{(3i+1)(1-i)}{2(1+i)(1-i)} e^{it}$$

$$= 2e^t + \frac{1}{2}(i-2)e^{-it} - \frac{1}{2}(i+2)e^{it}$$

$$= 2e^t + \frac{i}{2}e^{-it} - e^{-it} - \frac{1}{2} i e^{it} - e^{it}$$

$$= 2e^t - \frac{i}{2}\left(e^{it} - e^{-it}\right) - \left(e^{it} + e^{-it}\right)$$

$$= 2e^t - \frac{i}{\cancel{2}} \cancel{2} i \sin t - 2\cos t$$

$$= 2e^t + \sin t - 2\cos t$$

4)

4) $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}(s+a)}\right\}$

We know $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{t^{-1/2}}{\Gamma(\frac{1}{2})} = \frac{1}{\sqrt{\pi t}}$

& $\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}(s+a)}\right\} = \int_0^t \frac{1}{\sqrt{\pi}} u^{-1/2} e^{a(t-u)} du$$

$$= \frac{e^{+at}}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{u}} - e^{au} du$$

Let $au = +x^2$ $\begin{cases} au = t \to x = \sqrt{at} \\ au = 0 \to x = 0 \end{cases}$

$a\,du = +2x\,dx$

$$= \frac{-e^{+at}}{\sqrt{\pi}} \int_0^{\sqrt{at}} \frac{\sqrt{a}}{x} \bar{e}^{x^2} \frac{2x\,dx}{a}$$

$$= \frac{-e^{+at}}{\sqrt{a}} \times \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{at}} e^{-x^2} dx$$

$$= -\frac{e^{+at}}{a} \operatorname{erf}(\sqrt{at})$$

5)

5) $y''(t) + y(t) = t$ — ① , $y'(0) = 1$, $y(\pi) = 0$

Let $\boxed{y(0) = k}$

Taking Laplace transform both sides :-

$\mathcal{L}\{y''(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{t\}$

$s^2 Y - sK - 1 + Y = \frac{1}{s^2}$

$Y(1+s^2) = sK + 1 + \frac{1}{s^2}$

$Y = \frac{Ks}{1+s^2} + \frac{1}{1+s^2} + \frac{1}{s^2(1+s^2)}$ — ①

Now, $\mathcal{L}^{-1}\left\{\frac{Ks}{s^2+1}\right\} = K\cos t$ — ②

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ — ③

~~Using divide b~~

$\mathcal{L}^{-1}\left\{\frac{1}{s^2(1+s^2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= t - \sin t$ — ④

Now, taking Laplace Inverse transform of ① :-

$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{Ks}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{1+s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2(1+s^2)}\right\}$

Using ②, ③ & ④ :-

$y(t) = K\cos t + \sin t + t - \sin t$

$y(t) = K\cos t + t$ — ⑤

given $y(\pi) = 0$

$0 = K(-1) + \pi \Rightarrow \boxed{\pi = K}$ — ⑥

using ⑥ in ⑤ :

$\boxed{y(t) = \pi\cos t + t}$ ⇒ req. solⁿ

6)

1) Existence Theorem of Laplace theorem:

If $f(t)$ is piecewise continuous on the interval $[0,\infty)$ & of exponential order, then $\mathcal{L}\{f(t)\}$ exists for $s > a$

Proof

By additive interval property of definite integrals,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

$$= I_1 + I_2$$

Here, $I_1$ exists, because it can be written as a sum of integrals. On which $e^{-st} f(t)$ is continuous (finite integration)

→ Now, $f$ is of exponential order, so there exist const. $a, m > 0$ & $T > 0$ such that $|f(t)| \leq m e^{at}$ $\forall t > T$

which gives

$$|I_2| \leq \int_T^\infty |e^{-st} f(t)|\,dt < m \int_T^\infty e^{-st} |f(t)|\,dt$$

$$|I_2| \leq \int_T^\infty |e^{-st} f(t)|\,dt \leq m \int_T^\infty e^{-st} e^{at}\,dt$$

$$\leq m \int_T^\infty e^{-(s-a)t}\,dt$$

$$\leq m \frac{e^{-(s-a)T}}{(s-a)} \quad \text{for } s > a$$

Since, $\int_T^\infty e^{-st} f(t)\,dt$ converges, therefore the integral $\int_T^\infty |e^{-st} f(t)|\,dt$ converges by comparison test for improper integrals. This implies that $I_2$ exist for $s > a$ since both $I_1$ & $I_2$ exist. ∴ $\mathcal{L}\{f(t)\}$ exist for $s > a$.

⇒ Above condⁿ is sufficient but not necessary

Eg: $f(t) = \frac{1}{\sqrt{t}}$

As $t \to 0$, $F(t) \to \infty$, Hence, $f(t)$ is not piecewise continuous on every finite interval in range $t \geq 0$.

Now, by def$^n$,

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int_0^\infty e^{-st}\left(\frac{1}{\sqrt{t}}\right)dt = \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}\,dx ,$$

Put $\sqrt{st} = x$ & $\frac{dt}{\sqrt{t}} = \frac{2\,dx}{\sqrt{s}}$ , $s>0$

$$= \frac{2}{\sqrt{s}}\,\frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{s}} , \quad s>0 \quad \text{as } \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Hence, $\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}$ exist for $s>0$ even if $\frac{1}{\sqrt{t}}$ is not piecewise continuous in range $t \geq 0$.