Ordinary Differential Equation (O.D.E.)

Content

- Ordinary Differential Equation
  - Euler's Method
  - Euler Modified Method
  - Predictor Corrector Method
  - Runge Kutta Method

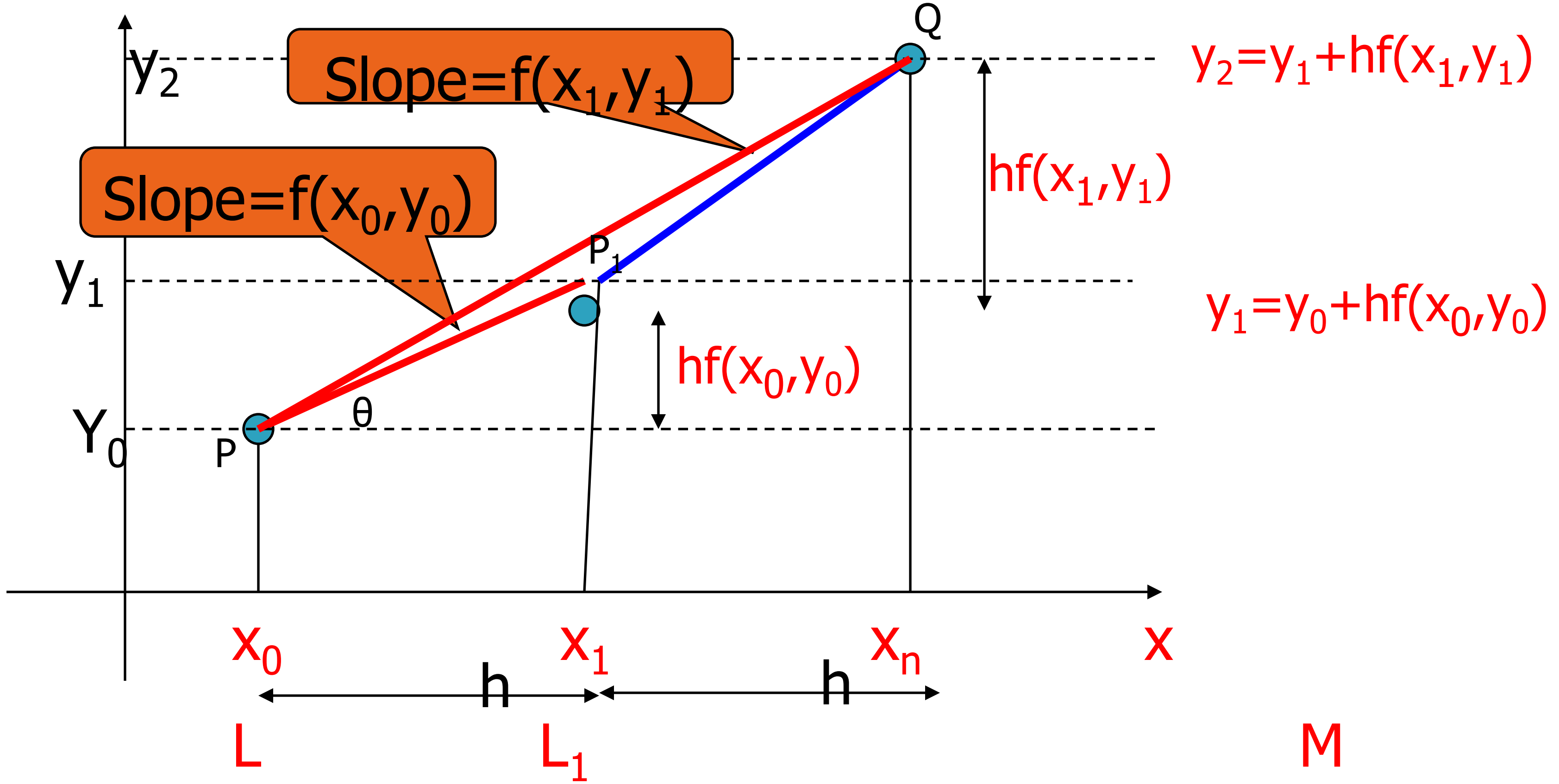
Euler's Method

Consider the O.D.E $\quad y' = \dfrac{dy}{dx} = f(x, y) \quad$ ......(1)

such that $\quad y(x_0) = y_0$

Let us divide LM into n sub intevals each of width h L_1, L_2, ...L_n
In the interval $LL_{1,}$ we approximate the curve by the tangent at P.
If the ordinate through L, meets this tangent in $P_1(x_0+h,y_1)$, then
$y_1=L_1P_1$
$y_1=L_1R_1+R_1P_1$
$\quad =y_0+PR_1 \tan\theta$
$y_1 =y_0+h \tan\theta$
$y_1 =y_0+h (dy/dx)_p$
$y_1 =y_0+h f(x_0,y_0)$

Interpretation of Euler Method



Euler’s Modified Method

Euler's Modified Method

Consider the O.D.E $$y' = \frac{dy}{dx} = f(x, y) \qquad \text{......(1)}$$

such that $y(x_0) = y_0$

Euler's Method, $y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})$

Euler's Modified Method, $(y_n)^{(m)} = y_{n-1} + (h/2)[\, f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{m-1})]$

Euler's Modified Method

Euler's Method, $y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})$ .....(1)

Euler's Modified Method,

$$(y_n)^{(m)} = y_{n-1} + (h/2)[\, f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{m-1})] \quad ......(2)$$

Suppose we have to calculate y at x=0.2

So first we will calculate the value of y(0.2) using Euler's Method.

Further, we will modify it using Euler modified method.

In eq(2), m is modified iteration.

Example:

Consider the initial value problem $y' = (x+y^2)$, $y(0) = 1$. Compute $y(0.2)$ with $h = 0.1$ using Euler's Modified method.

We have $y' = f(x, y) = (x+y^2)$,

With $x_0 = 0, y_0 = 1$

And $h=0.1$

Solution

By Euler's method,

First iteration

$$with, x_0 = 0, y_0 = 1, h = 0.1$$

$$x_1 = 0.1$$

$$first..iteration$$

$$y(x_n) = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

$$y(x_1) = y_0 + 0.1(x_0 + y_o^2)$$

$$\mathrm{y}(0.1) = \mathrm{y}_1 = \mathrm{y}_0 + 0.1(x_0 + y_o^2)$$

$$\mathrm{y}(0.1) = \mathrm{y}_1 = 1 + 0.1[0 + 1] = 1.1.$$

$$with, x_1 = 0.1, y_1 = 1.1$$

Solution

Euler's Modified method.

$$y^{m}(x_n) = y_{n-1} + \frac{h}{2}[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{m-1})$$

Now, by Euler's Modified , put n=1 and m=1

$$y^{1}(0.1) = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{0})$$

$$y^{1}(0.1) = y_0 + \frac{0.1}{2}[(x_0 + y_0^{2}) + (x_1 + (y_1^{0})^{2})$$

$$y^{1}(0.1) = 1 + \frac{0.1}{2}[(0 + 1^{2}) + (0.1 + (1.1)^{2})$$

$$y^{1}(0.1) = 1.1155$$

Solution

Second modification

$$y^{m}(x_n) = y_{n-1} + \frac{h}{2}[\,f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{m-1})$$

Now, by Euler's Modified , put n=1 and m=2

$$y^{2}(0.1) = y_0 + \frac{h}{2}[\,f(x_0, y_0) + f(x_1, y_1^{1})$$

$$y^{2}(0.1) = y_0 + \frac{0.1}{2}[(x_0 + y_0^{2}) + (x_1 + (y_1^{1})^{2})$$

$$y^{2}(0.1) = 1 + \frac{0.1}{2}[(0 + 1^{2}) + (0.1 + (1.1155)^{2})$$

$$y^{2}(0.1) = ??$$

Milne's Predictor Corrector method

Consider the O.D.E $y' = \frac{dy}{dx} = f(x, y)$ ....(1)

such that $y(x_0) = y_0$

To find y at given x.

This method is made of two separate methods called predictor and corrector.

Milne's Predictor Corrector method (Cont..)

Consider the O.D.E $\quad y' = \dfrac{dy}{dx} = f(x, y) \quad$ ....(1)

such that $\quad y(x_0) = y_0$

To find y at given x.

Note: use atleast 4 iterations to reach x.

Milne's Predictor Corrector method (Cont..)

Step-1 Consider the data points, x0, x1=x0+h, x2=x0+2h,……,

Calculate the value of y1=y(x1), $y_2=y(x_2)$, ….using any one of the following methods:

Euler's ,method, Euler's modified method, Taylor's method, Picard's method. This will give y=f(x,y)

Step-2 Now find the value of

$y'_1=f(x_1,y_1)$, $y'_2=f(x_2,y_2)$, $y'_3=f(x_3,y_3)$

Step-3 By Milne's Predictor formula

$y_4=y_0+(4h/3)\,(2y_1'-y_2'+2y_3')$

find $y_4'=f(x_4,y_4)$

Step-4 By Milne's Corrector formula

$y_4=y_2+(h/3)\,(y_2'+4y_3'+y_4')$

Milne's Predictor Corrector method (Cont..)

Now repeat this step 3 & 4 again to find y_5.
To calculate y_5
Find y_4',

Step-3 By Milne's Predictor formula

$$y_5=y_1+(4h/3)\,(2y_2'-y_3'+2y_4')$$

find $y_5'=f(x_5,y_5)$

Step-4 By Milne's Corrector formula

$$y_5=y_3+(h/3)\,(y_3'+4y_4'+y_5')$$

Example:

Consider the initial value problem $y' = x-y^2$, for x=0 to 1 with initial condition, $y(0) = 0$, using Milne's Predictor-Corrector method.

We have $y' = f(x, y) = x - y^2$, $x_0 = 0$, $y_0 = 0$

Take , h=1/5=0.2

$$y' = \frac{dy}{dx} = x - y^2$$

$$y(0) = 1$$

Solution

By Picard's method.

$$y_n = y_0 + \int_{x_0}^{x_n} f(x, y_{n-1})dx$$

$$y_1 = y_0 + \int_{0}^{0.2} f(x, y_0)dx$$

$$y_1 = 0 + \int_{0}^{0.2} (x - y_0^2)dx$$

$$y_1 = 0 + \int_{0}^{0.2} (x - 0)dx$$

$$y_1 = \frac{x^2}{2}$$

$$y_2 = y_0 + \int_{0}^{x} f(x, y_1)dx$$

$$y_2 = 0 + \int_{0}^{x} (x - y_1^2)dx$$

Solution

By Picard's method.

$$y_2 = 0 + \int_0^x \left(x - \left(\frac{x^2}{2} \right)^2 \right) dx$$

$$y_2 = 0 + \int_0^x \left(x - \frac{x^4}{4} \right) dx$$

$$y_2 = \frac{x^2}{2} - \frac{x^5}{20}$$

$$so, y = \frac{x^2}{2} - \frac{x^5}{20}$$

$$x_1 = 0.2, y_1 = \frac{0.2^2}{2} - \frac{0.2^5}{20} = 0.02$$

$$x_2 = 0.4, y_2 = 0.0795, x_3 = 0.6, y_3 = 0.176$$

Solution

$$x_1 = 0.2, y_1 = 0.02$$

$$y_1' = f(x_1, y_1)$$

$$y_1' = (x_1 - y_1^2) = 0.2 - (0.02)^2 = 0.1996$$

$$x_2 = 0.4, y_2 = 0.0795$$

$$y_2' = x_2 - y_2^2 = 0.3937$$

$$x_3 = 0.6, y_3 = 0.176$$

$$y_3' = x_3 - y_3^2 = 0.5690$$

Milne's Predictor Corrector method (Cont..)

By Milne's Predictor formula

$$y_4=y_0+(4h/3)\,(2y_1'-y_2'+2y_3')$$

$$= 0.3049$$

find $y_4'=f(x_4,y_4)=x_4-(y_4)^2=0.8-(0.3049)^2=0.707$

By Milne's Corrector formula

$$y_4=y_2+(h/3)\,(y_2'+4y_3'+y_4')$$

$$=0.0795+(0.2/3)(0.3937+4*0.5690+0.707)$$

$$=0.3043$$

Milne's Predictor Corrector method (Cont..)

At x_4=0.8, y_4=0.3043

Now, y_4'=f(x4,y4)=x_4-(y_4)^2 =0.7074

By Milne's Predictor formula

$y_5=y_1+(4h/3)\,(2y_2'-y_3'+2y_4')$

$= 0.4554$

find $y_5'=f(x_5,y_5)=x_5-(y_5)^2=1-(0.4554)^2=0.7926$

By Milne's Corrector formula

$y_5=y_3+(h/3)\,(y_3'+4y_4'+y_5')$

$=0.176+(0.2/3)(0.5690+4*0.7074+0.7926)$

$=0.4554$

Example:

Consider the initial value problem $y' = x+y$, with initial condition, $y(0) = 1$, $x=0.20$, $x=0.30$. solve it using Milne's Predictor-Corrector method.

We have $y'=f(x, y) = x+y$, $x_0 = 0$, $y_0 =1$

Take , $h=0.05$

$$y(0) = 1$$

Solution

$$x_1 = 0.05, y_1 = 1.0525$$

$$y_1' = f(x_1, y_1)$$

$$y_1' = (x_1 + y_1) = 0.05 + 1.0525 = 1.1025$$

$$x_2 = 0.10, y_2 = 1.1103$$

$$y_2' = x_2 + y_2 = 1.2103$$

$$x_3 = 0.15, y_3 = 1.1736$$

$$y_3' = x_3 + y_3 = 1.3236$$

Milne's Predictor Corrector method (Cont..)

By Milne's Predictor formula

$$y_{n+1}=y_{n-3}+(4h/3)\,(2y_{n-2}'-y_{n-1}'+2y_n')$$

$n=3$, $h=0.05$

$$y_4=y_0+(4h/3)\,(2y_1'-y_2'+2y_3')$$
$$= 1.2428$$

find $y_4'=f(x_4,y_4)=x_4+y_4=1.4428$

By Milne's Corrector formula

$$y_4=y_2+(h/3)\,(y_2'+4y_3'+y_4')$$
$$=1.1103+(0.05/3)(1.2103+4*5.2944+1.4428)$$
$$=1.2428 \text{ (which is same as predicted value)}$$

Milne's Predictor Corrector method (Cont..)

At x_4=0.20, y_4=1.2428

Now, y_4'=f(x4,y4)=x_4+y_4 =1.4428

By Milne's Predictor formula

$$y_5=y_1+(4h/3)\,(2y_2'-y_3'+2y_4')$$

$$= 1.0525+(4*0.05/3)(2.4206-1.3236+2.8856)$$

find $y_5'=f(x_5,y_5)=x_5+y_5=1.568$

By Milne's Corrector formula

at x_5=0.25 $\quad y_5=y_3+(h/3)\,(y_3'+4y_4'+y_5')$

$$=1.1736+(0.05/3)(1.3236+5.7712+1.568)$$

$$=1.3180$$

Runge Kutta method

Runge Kutta method

Consider the O.D.E $y' = \frac{dy}{dx} = f(x, y)$ ....(1)

such that $y(x_0) = y_0$

To find y at given x.

Runge Kutta method

Consider the O.D.E $y' = \frac{dy}{dx} = f(x, y)$ …….(1)

such that $y(x_0) = y_0$, To find y at given x.

$Y_{n+1}=y_n+1/6(k_1+2k_2+2k_3+k_4)$
For $x_{n+1}=x_n+h$, n=0, 1, 2, 3, ……

Where,

$k_1=hf(x_n,y_n)$
$k_2=hf(x_n+h/2, y_n +k_1/2)$
$k_3=h f(x_n+h/2, y_n +k_2/2)$
$k_4=h f(x_n+h, y_n +k_3)$

Example

Consider the O.D.E $y' = \frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ …….(1)

such that $y(0) = 1$, find y at x=0.2 and 0.4.

We have, $$y' = \frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$$

Taking h=0.2, $x_0=0$, $x_1=x_0+h=0+0.2=0.2$, x2=0.4

By R.K method, put n=0

$Y_{n+1}=y_n+1/6(k_1+2k_2+2k_3+k_4)$, for $x_{n+1}=x_n+h$, n=0, 1, 2, 3, ……

Where,

$k_1=hf(x_n,y_n)$

$k_2=hf(x_n+h/2, y_n +k_1/2)$

$k_3=h f(x_n+h/2, y_n +k_2/2)$

$k_4=h f(x_n+h, y_n +k_3)$

Example

Consider the O.D.E $$y' = \frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$$

such that $y(0) = 1$, find y at x=0.2 and 0.4.

We have, $$y' = \frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$$

Taking h=0.2, $x_0=0$, $x_1=x_0+h=0+0.2=0.2$, x2=0.4

By R.K method, put n=0

$y_1=y_0+1/6(k_1+2k_2+2k_3+k_4)$, …….(1)

for $x_1=0.2$

Where,

$k_1=hf(x_0,y_0)$ = 0.2 f(0, 1) = 0.2

$k_2=hf(x_0+h/2, y_0 +k_1/2)$= 0.2 f(0.1, 1.1) = 0.19672

$k_3=h f(x_0+h/2, y_0 +k_2/2)$= 0.2f(0.1, 1+ 0.19672/2) = 0.1967

$k_4=h f(x_0+h, y_0 +k_3)$ = 0.2f(0.2, 1+ 0.1967) = 0.1891

Solution (Cont..)

$y_1=y_0+1/6(k_1+2k_2+2k_3+k_4)$, …….(1)

for $x_1=0.2$

Where,

$k_1=hf(x_0,y_0) = 0.2\ f(0, 1) = 0.2$

$k_2=hf(x_0+h/2,\ y_0+k_1/2) = 0.2\ f(0.1, 1.1) = 0.19672$

$k_3=h\ f(x_0+h/2,\ y_0+k_2/2) = 0.2f(0.1, 1+ 0.19672/2) = 0.1967$

$k_4=h\ f(x_0+h,\ y_0+k_3) = 0.2f(0.2, 1+ 0.1967) = 0.1891$

Use these values in eq. (1)

$y_1=y_0+1/6(k_1+2k_2+2k_3+k_4)$

$= 1+1/6(0.2+2*0.19672+2*0.1967+0.1891) = 1.19599$

Solution (Cont..)

By R.K method, put n=1

$y_2=y_1+1/6(k_1+2k_2+2k_3+k_4)$, …….(1)

for $x_2=0.4$

Where,

$k_1=hf(x_1,y_1) = 0.2\ f(0.2, 1.196) = 0.1891$

$k_2=hf(x_1+h/2, y_1+k_1/2)= 0.2\ f(0.2+0.2/2, 1.196+0.1891/2) = 0.1795$

$k_3=h\ f(x_1+h/2, y_1+k_2/2)= 0.2f(0.2+0.2/2, 1.196+ 0.1795/2) = 0.1793$

$k_4=h\ f(x_1+h, y_1+k_3) = 0.2f(0.2+0.2, 1.196+ 0.1793) = 0.1688$

Use these values in eq. (1)

$y_2=y_1+1/6(k_1+2k_2+2k_3+k_4)$

$= 1.196+1/6(0.1891+2* 0.1795+2* 0.1793+ 0.1688) =?$

Suggested books

1. Numerical Methods by **S.R.K Lyenger & R.K. Jain.**

2. Numerical Analysis by **Richard L. Burden.**

3. Introductory methods of Numerical analysis by **S.S. Sastry**.

Thank you