Homogeneous Equation –

Consider a function $f(x,y)$ of two variables x and y, as $f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_r x^{n-r} y^r + a_{n-1} x y^{n-1} + a_n y^n$,

where, a_i, $i = 0, 1, 2, \dots n$ is constant. The degree of each term is n. Thus $f(x,y)$ is called a homogeneous function in x and y of degree n.

$$f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$$

$$= x^n \left[a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(\frac{y}{x}\right)^2 + \dots + \left(\frac{y}{x}\right)^n \right]$$

Similarly,

$$f(x,y) = y^n \left[a_0 \left(\frac{x}{y}\right)^n + a_1 \left(\frac{x}{y}\right)^{n-1} + \dots + a_n \left(\frac{x}{y}\right)^0 \right]$$

$$\Rightarrow f(x,y) = x^n F\left(\frac{y}{x}\right), \text{ where, } F\left(\frac{y}{x}\right) = a_0 + a_1\left(\frac{y}{x}\right) + \dots + a_n\left(\frac{y}{x}\right)^n$$

If $f(x,y) = x^n F\left(\frac{y}{x}\right)$, where $F\left(\frac{y}{x}\right)$ is a function of $\left(\frac{y}{x}\right)$,

then the function $f(x,y)$ is homogeneous in x ~~and~~ and y of degree n.

Examples → 1.- $f(x,y) = x^2 \sin\left(\frac{y}{x}\right)$ is a homogeneous function in x and y of degree 2.

Solⁿ -

$$f(x,y) = x^2\left[\frac{y}{x} - \frac{(y/x)^3}{3!} + \frac{(y/x)^5}{5!} - \frac{(y/x)^7}{7!} - \cdots\right]$$

$$= xy - \frac{y^3/x}{3!} + \frac{y^5/x^3}{5!} - \cdots$$

Since, each term has same degree.

So, $f(x,y)$ is a homogeneous equation hence proved

Consider a function $f(x_1, x_2, \ldots x_m)$ of m variables $x_1, x_2, x_3 \ldots x_m$ of degree n.

$$f(x_1, x_2, \ldots x_m) = x_1^n F\left[\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots \frac{x_m}{x_1}\right]$$

$$= x_2^n F\left[\frac{x_1}{x_2}, \frac{x_3}{x_2}, \ldots \frac{x_m}{x_2}\right]$$

$$= x_r^n F\underbrace{\left[\frac{x_1}{x_r}, \frac{x_3}{x_r}, \ldots, \frac{x_m}{x_r}\right]}_{(m-1) \text{ variables}}$$

Euler's theorem for Homogeneous function—

Theorem:-

let $f(x,y)$ be a Homogenous function in two variables of x and y of degree n.

then,

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf}$$

<u>Proof</u>:— consider $f(x,y) = x^n f\left(\frac{y}{x}\right)$

Taking partial derivative of f w.r.t x,

we get,

$$\frac{\partial f}{\partial x} = f_x(x,y) = nx^{n-1} f(y/x) + x^n f'(y/x) \cdot \frac{-y}{x^2}$$

$$= nx^{n-1} f(y/x) - x^{n-2} y f'(y/x)$$

similarly, $\frac{\partial f}{\partial y} = x^n f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$

$$= x^{n-1} f'\left(\frac{y}{x}\right)$$

Thus, we have,

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$

$= nx^n f(y/x) - \cancel{x^{n-1} y f'(y/x)} + \cancel{x^{n-1} y f'(y/x)}$

$= n x^n f(y/x)$

$= nf$ ~~hence proved~~

$=$ R.H.S

(Assignment Add)

Theorem(2):- let $f(x_1, x_2 \ldots x_n)$ be a Homogeneous function ~~for~~ in m variables x_i f $i = 1, 2, \ldots m$ of degree n.

Thus,

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \ldots + x_m \frac{\partial f}{\partial x_m} = nf$$

i.e.

$$\sum_{i=1}^{m} x_i \frac{\partial f}{\partial x_i} = nf$$

Proof→ consider $f(x, y) = x_1^n F\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots \frac{x_m}{x_1}\right)$

Taking partial derivative, w.r.t x_1,

we get,

$$\frac{\partial f}{\partial x_1} = x_1^n f'\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots \frac{x_m}{x_1}\right)\left(\frac{-x_2}{x_1^2}, \frac{-x_3}{x_1^2}, \ldots \frac{-x_m}{x_1^2}\right)$$

$$+ n x_1^{n-1} f\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots \frac{x_m}{x_1}\right)$$

∂

Q.1. Verify Euler's theorem for the following -

(1) $f(x,y) = ax^2 + 2hxy + by^2$

(2) $f(x,y,z) = ayz + bzx + cxy$

Q.2. if $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then show

that - $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

Solutions.

1. (i) Given - $f(x,y) = ax^2 + 2hxy + by^2$

Since, degree of each term $= 2$

$\Rightarrow n = 2$

$\Rightarrow f(x,y)$ is homogeneous function.

Now,

$$\frac{\partial f}{\partial x} = 2ax + 2hy$$

and,

$$\frac{\partial f}{\partial y} = 2hx + 2by$$

So,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

$$= 2ax^2 + 2hxy + 2hxy + 2by^2$$

$$= 2(ax^2 + by^2 + 2hxy)$$

$$= 2f(x,y) \quad \text{hence proved}$$

$=$

If $u = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$ then show that,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

Sol^n - given → $u = x\phi(y/x) + \psi(y/x)$

let $v = x\phi(y/x)$

and, $w = \psi(y/x)$,

then, $u = (v+w)$,

So, v is a homogeneous function in x and y of degree 1

and, w is a homogeneous function in x and y of degree 0.

Now, $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \cdot \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

$$\Rightarrow x^2\left(\frac{\partial^2 (v+w)}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 (v+w)}{\partial x \cdot \partial y}\right) + y^2\frac{\partial^2 (v+w)}{\partial y^2}$$

$$= x^2\frac{\partial^2 v}{\partial x^2} + x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x . \partial y} + 2xy\frac{\partial^2 w}{\partial x \cdot \partial y} + y^2\frac{\partial^2 v}{\partial y^2} + y^2\frac{\partial^2 w}{\partial y^2}$$

$$= \left(x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \cdot \partial y} + y^2\frac{\partial^2 v}{\partial y^2}\right) + \left(x^2\frac{\partial^2 w}{\partial x^2} + y^2\frac{\partial^2 w}{\partial y^2} + 2xy\frac{\partial^2 w}{\partial x \cdot \partial y}\right)$$

$= 1(1-1)\,v + 0(0-1)\,w$

$= 0 + 0$

$= 0$

$=$ R.H.S hence proved

Q. If $z = x^m \phi(y/x) + x^n \psi(y/x)$, then show that,

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

Sol^n: Let $v = x^m \phi(y/x)$ and $w = x^n \psi(y/x)$

here, v is a homogeneous function in x and y of degree m.

and, w is a homogeneous function in x and y of degree n.

Now, $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \cdot \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + mnz$

$$= x^2 \frac{\partial^2 (v+w)}{\partial x^2} + 2xy \frac{\partial^2 (v+w)}{\partial x \cdot \partial y} + y^2 \frac{\partial^2 (v+w)}{\partial y^2} + mn(v+w)$$

$$= \left(x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \cdot \partial y} + y^2 \frac{\partial^2 v}{\partial y^2}\right) + \left(x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x \cdot \partial y} + y^2 \frac{\partial^2 w}{\partial y^2}\right) + mn(v+w)$$

$= (m-1)m\,v + (n-1)n\,w + mn(v+w)$

$= m(m+n-1)v + n(m+n-1)w$

$= (m+n-1)(mv + nw)$

$$= (m+n-1)(mv+nw)$$

$$= (m+n-1)\left(m.x^m \phi^m(y/x) + n.x^n \psi(y/x) - - - -\right)$$

and,

$$= (m+n-1)(mv+nw) \quad (\text{Using Euler's theorem})$$

$$= (m+n-1)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}\right)$$

$$= (m+n-1)\left(x\frac{\partial (v+w)}{\partial x} + y\frac{\partial (v+w)}{\partial y}\right)$$

$$= (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

$= R.H.S$ hence proved

Euler's Theorem Problems

Q1. If $u = \log\left(\frac{x^2+y^2}{\sqrt{x}+\sqrt{y}}\right)$ then prove that

(a) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}$

(b) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{3}{2}$

Soln: Given

$$u = \log\left(\frac{x^2+y^2}{\sqrt{x}+\sqrt{y}}\right)$$

$$\Rightarrow e^u = \frac{x^2+y^2}{\sqrt{x}+\sqrt{y}} = \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)}$$

$$= x^{3/2}\left(\frac{1+\left(\frac{y}{x}\right)^2}{1+\left(\frac{y}{x}\right)^{\frac{1}{2}}}\right)$$

This is the Homogeneous eq-n of degree $n = \frac{3}{2}$.

Hence, using Euler's 1st order eqn

$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu\right)$, we have

$$x \cdot \frac{\partial e^u}{\partial x} + y\frac{\partial e^u}{\partial y} = \frac{3}{2} \cdot e^u$$

$$\Rightarrow e^u x\frac{\partial u}{\partial x} + y \cdot e^u\frac{\partial u}{\partial y} = \frac{3}{2} \cdot e^u$$

$$\Rightarrow \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}} \quad \text{——— (1)}$$

Proved.

Homogenous function

$u = x^n f\left(\frac{y}{x}\right)$

Euler's 1st order

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

Euler's 2nd order

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

Partial derivative eq-n (1) w.r.t. x

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = 0 \Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x \partial y} = 0 \quad \text{——— (2)}$$

Multiplying with x

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P.d. eqn ① w.r.t. y

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 0$$

multiplying y both sides

$$xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 0 \quad \text{--- ③}$$

Adding ② and ③

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + \boxed{x u_x + y u_y} = 0$$

$$\therefore\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{3}{2}$$

Q2. If $u = (x^2+y^2)^{1/3}$ show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{2}{9}u$

Soln: Given eqn is $u = (x^2+y^2)^{1/3}$

$$= (x^2)^{1/3}\left(1+\frac{y^2}{x^2}\right)^{1/3}$$

$$= x^{2/3} f\left(\frac{y}{x}\right)$$

This is Homogeneous fn with degree $n = \frac{2}{3}$.

By Euler's 1st order eqn

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = \frac{2}{3}u \quad \text{--- ①}$$

P.d. eq ① w.r.t. x and y and adding

$$\left.\begin{aligned} x\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} &= \frac{2}{3}\frac{\partial u}{\partial x} \\ xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} &= \frac{2}{3}\frac{\partial u}{\partial y} \end{aligned}\right\}$$

$$\Rightarrow x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + \boxed{x u_x + y u_y} = \frac{2}{3}x u_x + \frac{2}{3}y u_y$$

$$\Rightarrow x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \left(\frac{2}{3}-1\right)(x u_x + y u_y)$$

$$= -\frac{1}{3}\cdot\frac{2}{3}u = -\frac{2}{9}u.$$

Q3. If $u = \log(x^4 + x^2y^2 + y^4)$, then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + 4 = 0.$$

Soln: Given eqn is $u = \log(x^4 + x^2y^2 + y^4)$

$$\Rightarrow e^u = x^4\left(1 + \frac{y^2}{x^2} + \frac{y^4}{x^4}\right)$$

$$z = e^u = x^4 f\left(\frac{y}{x}\right) \quad \text{—— ①}$$

eqn ① is homogeneous with degree $n = 4$.

By Euler's 1st order eq.n.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

$$\Rightarrow x e^u \frac{\partial u}{\partial x} + y e^u \frac{\partial u}{\partial y} = 4 \cdot e^u$$

$$\Rightarrow \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4} \quad \text{—— (II)}$$

taking p. d. of eq-n (II) w.r.t. x & y and adding

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial x y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

(from eq-n (II))

$$\therefore x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + 4 = 0.$$

proved.

Q.4. If $u = \tan^{-1}\left(\dfrac{x^{1/2} + y^{1/2}}{x^{1/3} - y^{1/3}}\right)$ then prove that

ⓐ $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{12}\sin 2u.$

ⓑ $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{12}\sin 2u\left(\dfrac{1}{6}\cos 2u - 1\right)$

Soln: Given eq-n is $u = \tan^{-1}\left(\dfrac{x^{1/2} + y^{1/2}}{x^{1/3} - y^{1/3}}\right)$

$$\tan u = x^{1/2} \cdot x^{-1/3}\left(\frac{1 + (y/x)^{1/2}}{1 - (y/x)^{1/3}}\right)$$

$$\therefore \tan u = x^{1/6} f\left(\frac{y}{x}\right) \quad \text{——— ①}$$

eq-n ① is a homogeneous eqn with degree $n = \frac{1}{6}$

$\therefore$ By Euler's 1st order eqn

$$x\frac{\partial \tan u}{\partial x} + y\frac{\partial}{\partial y}\tan u = \frac{1}{6}\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x}\sec^2 u + y\sec^2 u\frac{\partial u}{\partial y} = \frac{1}{6}\tan u$$

$$\Rightarrow x u_x + y u_y = \frac{1}{6}\frac{\tan u}{\sec^2 u} = \frac{1}{6}\sin u\cos u$$

$$= \frac{1}{6}\cdot\frac{\sin 2u}{2} = \frac{1}{12}\sin 2u$$

for ⓑ do yourself.

Q5. If $u = \tan^{-1}\left(\frac{y^2}{x}\right)$, then find the value of

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}.$$

Q6. If $u = \sin^{-1}\left(\dfrac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}}\right)$ | $\frac{\sin 2u}{2}(\cos 2u - 1)$.

then prove that

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \frac{1}{144} \tan u \left(\tan^2 u - 11\right)$$

Q7. If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$

find the value of

(a) $x u_x + y u_y$ —— $\left(\frac{5}{4}\sin 2u\right)$

(b) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$.

$\rightarrow \left(\frac{25}{16}\sin 4u - \frac{5}{4}\sin 2u\right)$

Q8. If $u = \operatorname{cosec}^{-1}\left(\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$

then prove that

$$x^2 u_{xx} + 2xy\, u_{yy} + y^2 u_{yy}$$

$$= \frac{\tan u}{12}\left[\frac{13}{12} + \frac{\tan^2 u}{12}\right].$$