Example: Recall a function $f: X \to Y$ is one-one iff $\forall x, y \in X$ $f(x) = f(y) \Rightarrow x = y$,

'f' is said to be onto iff $\forall y \in Y, \exists x \in X$ s.t. $f(x) = y$.

———:

let X be a non-empty set. let $S(X)$ denote the collection of all bijections defined on X; i.e.

$$S(X) := \{ f: X \to X ;\ f \text{ is 1-1 and onto} \}.$$

Claim, $S(X)$ is a group w.r.t. the function composition.

(1) $S(X)$ is non-empty, because Identity map, $Id \in S(X)$.

(2) Let $f, g \in S(X)$. We show that $f \circ g \in S(X)$.

(i) $f \circ g$ is one-one.

$$f \circ g(x) = f \circ g(y)$$
$$\Rightarrow f(g(x)) = f(g(y))$$
$$\Rightarrow g(x) = g(y)$$
$$\Rightarrow x = y.$$

(ii) $f \circ g$ is onto.

let $y \in X$

Choose $x = g^{-1} \circ f^{-1}(y)$

Then $f \circ g(x) = f \circ g(g^{-1} \circ f^{-1}(y)) = y$.

(2) Hence, function composition '$\circ$' is indeed a binary operation on X.

(3) Clearly, $\forall f, g, h \in S(X)$

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

(4) Identity map play the role of identity element.

(5) Since ~~Each~~ each $f \in S(X)$ is a bijection.

Hence f^{-1} ~~plays~~ is the inverse of $f \in S(X)$.

Hence $(S(X), \circ)$ is a group.

$(S(X), \circ)$ is called a permutation group.

Now assume $X = \{1\}$.

$S(X) = ?$

Let $X = \{1, 2, 3\}$.

$S(X) = ?$

$S(X) = \{f_1, f_2, f_3, f_4, f_5, f_6\}$.

— $f_1(1) = 1$ $f_1(2) = 2$ $f_1(3) = 3$ ~~$f_1(4) = 4$ $f_1(5) = 5$, $f_1(6) = 6$.~~

— $f_2(1) = 2$, $f_2(2) = 1$, $f_2(3) = 3$, ~~$f_2(4) = 4$, $f_2(5) = 5$, $f_2(6) = 6$~~

$\vdots$

$f_6 = ?$ - -

Let $X = \{1, 2, \ldots, n\}$. Then no. of elements in $S(X)$,

$|S(X)| = \cancel{6}\, n!$

Another representation of $S(X)$. Let $X = \{1, 2, \ldots, n\}$.

Let $f \in S(X)$. Then 'f' can be re-written as

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ f(1) & f(2) & f(3) & & f(n) \end{pmatrix}.$$

<u>Note that</u> in the representation of 'f', no number in $\{1, 2, \ldots, n\}$ is repeated in upper and lower row.

<u>Example</u>: $S(X) = ?$ $\quad X = \{1, 2, 3\}$.

$$f_1 \to \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$f_2 \to \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$f_3 \to \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad f(1) = 3,\ f(2) = 2,\ f(3) = 1$$

$$f_4 \to \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad f(1) = 1,\ f(2) = 3,\ f(3) = 2$$

$$f_5 \to \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$f_6 \to \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

<u>Thm</u>: No. of elements in $S(X)$ for $X = \{1, 2, \ldots, n\}$ is $n!$.

<u>Pf</u>: For a bijection f,

$f(1)$ has n choices.	No. of possible 'f'
$f(2)$ has $n-1$ choices	$n \times n-1 \times \cdots \times 1$
$f(3)$ has $n-2$ ch	$= \underline{n!}$
$f(n)$ has 1 ch	