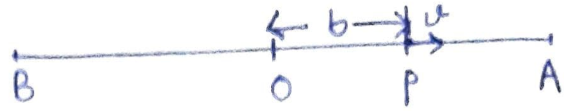
Ex II (G)

Q1 a particle is performing a S.H.M. of period T about a centre O and it passes through a point P ($OP = b$) with velocity v in the direction OP; prove that the time which elapses before its return to P is

$$\frac{T}{\pi}\tan^{-1}\left\{\frac{vT}{2\pi b}\right\}$$

Proof



Let the equation of the S.H.M. with Centre O as origin be $\frac{d^2x}{dt^2} = -\mu x$

time period $T = \frac{2\pi}{\sqrt{\mu}}$, let the amplitude be a. then $\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2)$ — ①

But given at $x = b$, $\frac{dx}{dt} = v$, then from Eq ①

$$\Rightarrow \quad v^2 = \mu(a^2 - b^2) \quad — ②$$

From P the particle comes to rest at A and then returns back to P. In S.H.M. the time from P to A is equal to the time from A to P.

the required time $= 2\cdot$ time from A to P

Now for the motion from A to P, we have

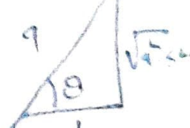
$$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{a^2-x^2} \quad \text{or} \quad dt = -\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{a^2-x^2}}$$

let t be the time from A to P, then at A $t = 0$, $x = a$ and at P $t = t$, $x = b$

$$\int_0^t dt = -\frac{1}{\sqrt{\mu}}\int_a^b \frac{dx}{\sqrt{a^2-x^2}} = +\frac{1}{\sqrt{\mu}}\left\{\cos^{-1}\left(\frac{x}{a}\right)\right\}_a^b = \frac{1}{\sqrt{\mu}}\left\{\cos^{-1}\left(\frac{b}{a}\right) - \underbrace{\cos^{-1}1}_{=0}\right\}$$

$$t = \frac{1}{\sqrt{\mu}}\cos^{-1}\left(\frac{b}{a}\right), \quad \text{the require time} = 2t = \frac{2}{\sqrt{\mu}}\cos^{-1}(b/a)$$

$$= \frac{2}{\sqrt{\mu}}\tan^{-1}\left\{\frac{\sqrt{a^2-b^2}}{b}\right\} = \frac{2}{\sqrt{\mu}}\tan^{-1}\left\{\frac{v}{\sqrt{\mu}\, b}\right\} \quad \text{from ②}$$



$T = \frac{2\pi}{\sqrt{\mu}}$

$\sqrt{\mu} = \frac{2\pi}{T}$

$$= \cancel{\frac{2T}{2\pi}\tan^{-1}\left\{\frac{v^2T^2}{4\pi^2 b}\right\}} = \frac{2\times T}{2\pi}\tan^{-1}\left\{\frac{vT}{2\pi b}\right\}$$

$$= \frac{T}{\pi}\tan^{-1}\left\{\frac{vT}{2\pi b}\right\}$$

Q2 A point in a straight line with S.H.M. has velocities v_1 and v_2 when its distance from the centre are x_1 and x_2. Show that the period of motion is

$$2\pi\sqrt{\frac{x_1^2-x_2^2}{V_2^2-V_1^2}}$$

let S.H.M. of equation is $\frac{d^2x}{dt^2} = -\mu x$

Integrating $\int 2\frac{dx}{dt}\left(\frac{d^2x}{dt^2}\right)dt = 2\int -\mu x \frac{dx}{dt}\,dt$

$$\left(\frac{dx}{dt}\right)^2 = -\mu x^2 + C$$

let at $x=a$, $\frac{dx}{dt}=0$, $\quad 0 = -\mu a^2 + C \Rightarrow C = \mu a^2$

$\left(\frac{dx}{dt}\right)^2 = \mu(a^2-x^2)$ | But $x = x_1$, $v = v_1$

$x = x_2$, $v = V_2$

$$v_1^2 = \mu(a^2-x_1^2)$$

$$\underline{-\,V_2^2 = {}_-\mu\,(a^2 \underset{+}{-} x_2^2)}$$

$$V_1^2-V_2^2 = \mu(x_2^2-x_1^2) \Rightarrow \mu = \frac{V_2^2-V_1^2}{x_1^2-x_2^2}$$

time period $T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi\sqrt{x_1^2-x_2^2}}{\sqrt{V_2^2-V_1^2}}$

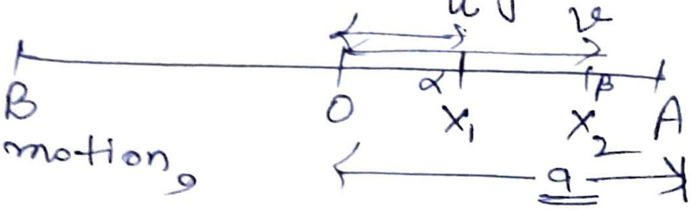
$$\boxed{T = 2\pi\sqrt{\frac{x_1^2-x_2^2}{V_2^2-V_1^2}}}$$

Q3 A point executes S.H.M. such that in two of its positions the velocities are u, v and the corresponding accelerations are α, β; Show that the distance between the position is $\frac{v^2-u^2}{\alpha+\beta}$, and the amplitude of the motion is

$$\frac{[(v^2-u^2)(\alpha^2v^2-\beta^2u^2)]^{1/2}}{\beta^2-\alpha^2}$$

Proof Let the equation of the S.H.M. with centre as origin be

$$\frac{d^2x}{dt^2}=-\mu x$$



If a be the amplitude of the motion, we have,

$$\left(\frac{dx}{dt}\right)^2=\mu(a^2-x^2),$$

where dx/dt is the velocity at a distance x from the centre.

Let x_1 and x_2 be the distance from the centre of the two positions where u and v are the velocities and α and β are the accelerations then

$$\alpha=\mu x_1 \quad \text{— (1)} \qquad\qquad u^2=\mu(a^2-x_1^2) \quad \text{— (3)}$$

$$\beta=\mu x_2 \quad \text{— (2)} \qquad\qquad v^2=\mu(a^2-x_2^2) \quad \text{— (4)}$$

$$\alpha+\beta=\mu(x_1+x_2) \quad \text{— (5)} \qquad u^2-v^2=\mu(x_2^2-x_1^2)=\mu(x_2+x_1)(x_2-x_1)$$

$$\Rightarrow \quad x_2-x_1=\frac{u^2-v^2}{(\alpha+\beta)}$$

$$\text{or} \quad x_1-x_2=\left(\frac{v^2-u^2}{\alpha+\beta}\right)$$

Now, we find amplitude $a=?$

Putting value of x_1, x_2 from Eq (1) & (2) to Eq (3) & (4)

$$u^2=\mu\left(a^2-\frac{\alpha^2}{\mu^2}\right)=\frac{1}{\mu}(a^2\mu^2-\alpha^2) \Rightarrow a^2\mu^2-u^2\mu-\alpha^2=0$$

$$v^2=\mu\left(a^2-\frac{\beta^2}{\mu^2}\right)=\frac{1}{\mu}(a^2\mu^2-\beta^2) \Rightarrow a^2\mu^2-v^2\mu-\beta^2=0$$

Solve by Cross multiplication

$$\frac{\mu^2}{u^2\beta^2-\alpha^2v^2}=\frac{\mu}{-a^2\alpha^2+\beta^2a^2}=\frac{1}{-a^2v^2+a^2u^2} \quad \Rightarrow \quad \mu=\frac{a^2(\beta^2-\alpha^2)}{a^2(u^2-v^2)}$$

$$\mu^2=\frac{u^2\beta^2-\alpha^2v^2}{a^2(u^2-v^2)}=\left\{\frac{\beta^2-\alpha^2}{u^2-v^2}\right\}^2 \Rightarrow \frac{u^2\beta^2-\alpha^2v^2}{a^2}=\frac{(\beta^2-\alpha^2)^2}{(u^2-v^2)}$$

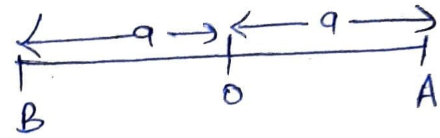
$$a^2=\frac{(u^2-v^2)(u^2\beta^2-\alpha^2v^2)}{(\beta^2-\alpha^2)^2} \Rightarrow a=\frac{\{(v^2-u^2)(\alpha^2v^2-u^2\beta^2)\}^{1/2}}{(\beta^2-\alpha^2)}$$

Q4. Show that in a S.H.M. the average speed and the average acceleration (in magnitude) are obtained by multiplying their maximum value by 0.637.

Proof

$$\text{average speed} = \frac{\text{Total distance}}{\text{Total time}} = \frac{4a}{T}$$

when a is amplitude of S.H.M.



~~Let~~ Let S.H.M. $\frac{d^2x}{dt^2} = -\mu x$ —— (1)

then $v^2 = \mu(a^2 - x^2)$ Maximum value

$V = \sqrt{\mu}\sqrt{a^2 - x^2}$

$\frac{dV}{dx} = \sqrt{\mu}\,\frac{(-\cancel{2}x)}{\cancel{2}\sqrt{a^2-x^2}}$ for maximum minima $\frac{dV}{dx} = 0$

$-\frac{x}{\sqrt{a^2-x^2}} = 0 \Rightarrow \boxed{x = 0}$ at $x = 0$ speed is maximum

$\boxed{V_{max} = \sqrt{\mu}\,a}$

$\frac{V_{avg}}{V_{max}} = \frac{4\cancel{a}}{T} \times \frac{1}{\sqrt{\mu}\cancel{a}} = \frac{\cancel{4}^{2}}{\cancel{2}\pi}\frac{\cancel{\sqrt{\mu}}}{\cancel{\sqrt{\mu}}} = \frac{2}{\pi}$ $\left[T = \frac{2\pi}{\sqrt{\mu}}\right.$

~~$V_{avg} = \frac{\pi}{2} V_{max}$~~ = $V_{avg} = \frac{2}{\pi} V_{max} = 0.636619 \approx 0.637\, V_{max}$

$$\text{Average Acceleration} = \frac{\int_{-a}^{+a} -\mu x\,dx}{2a} = -\mu\left(\frac{x^2}{2}\right.$$

~~Let~~ At point A, particle is at rest, the time taken by particle from A to O is $T/4$.

Total Acceleration in time $T/4$ period $= \int_0^{T/4} \ddot{x}\,dt = -\mu\int_0^{T/4} x\,dt$

$$\text{Average acceleration} = \frac{-\mu\int_0^{T/4} a\cos\sqrt{\mu}t\,dt}{T/4}$$ Since $\boxed{x = a\cos\sqrt{\mu}t}$

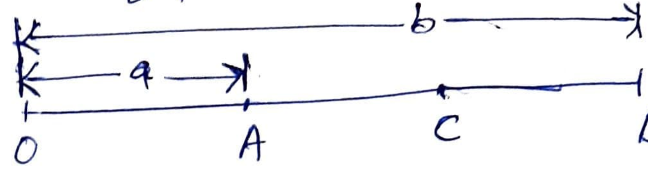
$= -\frac{4}{T}\mu a\left(\frac{\sin\sqrt{\mu}t}{\sqrt{\mu}}\right)_0^{T/4} = -\frac{4}{T}\mu a\frac{1}{\sqrt{\mu}}\left\{\sin\sqrt{\mu}\frac{T}{4} - 0\right\} = -\frac{4}{T}\sqrt{\mu}\,a\,\sin\frac{\cancel{\sqrt{\mu}}\,2\pi}{\cancel{2}4\cancel{\sqrt{\mu}}}$

$= -\frac{4}{T}\sqrt{\mu}\,a$ (taking Numeric Value) $\left|\ \frac{\alpha_{avg}}{\alpha_{max}} = \frac{\frac{4}{T}\sqrt{\mu}\cancel{a}}{\mu\cancel{a}} = \frac{4}{T}\frac{1}{\sqrt{\mu}} = \frac{\cancel{4}^{2}}{\cancel{2}\pi}\frac{\cancel{\sqrt{\mu}}}{\cancel{\sqrt{\mu}}} = \frac{2}{\pi}\right.$

$\Rightarrow \alpha_{avg} = \frac{2}{\pi}\alpha_{max} = 0.637\,\alpha_{max}$

Q5 A body moving in a straight line OAB with S.H.M. has zero velocity when at points A and B whose distances from O are a and b respectively and has a velocity v when half-way between them. Show that the complete period is $\dfrac{\pi(b-a)}{v}$.

Proof: In the figure, A and B are the position of instantaneous rest in a S.H.M.. Let C be the middle point of AB. Then C is the centre of motion



Also let $OA = a$, $OB = b$

$\Rightarrow$ The Amplitude of the motion of S.H.M. about point C.

$$= AC = OC - OA = (OB - BC) - OA = b - \tfrac{1}{2}AB - a$$

$$= (b-a) - \tfrac{1}{2}(OB - OA) = (b-a) - \tfrac{1}{2}(b-a) = \underline{\tfrac{1}{2}(b-a)}$$

$$AC = \frac{AB}{2} = \frac{1}{2}(OB - OA) = \frac{1}{2}(b-a)$$

Now, in a S.H.M. the velocity at the centre $= \sqrt{\mu} \times$ amplitude

$$= \sqrt{\mu}\,\tfrac{1}{2}(b-a)$$

But the velocity at the centre is given to v.

$$\Rightarrow \quad v = \sqrt{\mu}\,\tfrac{1}{2}(b-a)$$

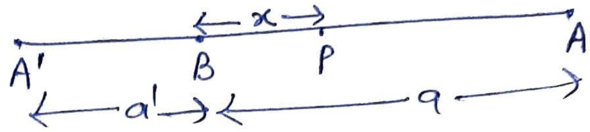
$$\sqrt{\mu} = \frac{2v}{(b-a)}$$

Hence time period $T = \dfrac{2\pi}{\sqrt{\mu}} = \dfrac{\cancel{2}\pi(b-a)}{\cancel{2}v} = \underline{\dfrac{\pi(b-a)}{v}}$ A

Q6 A particle rests in equilibrium under the attraction of two Centres of force which attract directly as the distance, their intensity being μ, μ'; the particle is slightly displaced towards one of them, Show that the time of a small oscillation is

$$\frac{2\pi}{\sqrt{\mu+\mu'}}$$

Proof



Suppose A and A' are two Centre of force, their intensities being μ and μ';

Let a particle of mass m be in equilibrium at point B under the attraction of these two Centres. if $A'B = a'$, $AB = a$

then the forces due to $A' = m\{\mu' a'\}$ } opposite to each other
the force due to $A = m\{\mu a\}$

at point B, mass m is in equilibrium.

$$\cancel{m}\{\mu' a'\} = \cancel{m}\{\mu a\} \Rightarrow \mu' a' = \mu a \quad \text{—— (1)}$$

Now, suppose the particle is slightly displaced towards A and let go. Let P be the position of the particle after time t, where $BP = x$

The attraction at P due to the Centre A is $m\mu AP = m\mu(a-x)$
the attraction at P due to the Centre A' is $m\mu A'P = m\mu(a'+x)$

Hence by Newton's Second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = m\mu(a-x) - m\mu'(a'+x) \quad (2)$$

$$\frac{d^2x}{dt^2} = \cancel{\mu\{a-x-a'-x\}} = \cancel{\mu a} - \mu x - \cancel{\mu' a'} - \mu' x$$

as $\mu a = \mu' a'$ by Eq (1)

$$\frac{d^2x}{dt^2} = -(\mu+\mu')x$$

Time Period $T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{\mu+\mu'}}$