Date
Page

Taylor's Theorem for a function of two variables-
If $f(x,y)$ is a function which possesses continuous partial derivatives of any order then

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots \quad *$$

Proof- By Taylor's Theorem for a function of single variable,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots + \frac{h^n}{n!} f^n(x) + \cdots$$

$$f(x+h) = f(x) + h\frac{d}{dx}f(x) + \frac{h^2}{2!}\frac{d^2 f(x)}{dx^2} + \frac{h^3}{3!}\frac{d^3 f(x)}{dx^3} + \cdots + \frac{h^n}{n!}\frac{d^n f(x)}{dx^n} + \cdots \quad (1)$$

By Taylor's Theorem for a function of single variable x,

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial}{\partial x}f(x,y+k) + \frac{h^2}{2!}\frac{\partial^2 f(x,y+k)}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3}{\partial x^3}f(x,y+k) + \cdots + \frac{h^n}{n!}\frac{\partial^n}{\partial x^n}f(x,y+k) + \cdots \quad (2)$$

By (1)

i.e. $f(x+h, y+k)$ has been expanded in powers of h.
In equation (2) expanding $f(x,y+k)$ in powers of k by again Taylor's Theorem for a function of single variable y,

$$f(x+h, y+k) = \Big\{ f(x,y) + k\frac{\partial}{\partial y}f(x,y) + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \frac{k^3}{3!}\frac{\partial^3 f(x,y)}{\partial y^3}$$

Date
Page

$$+\frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(x,y)\right.$$ Rem

$$\left.+\cdots\right\}+\frac{h^3}{3!}\frac{\partial^3}{\partial x^3}\left\{f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)\right.$$

$$\left.+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(x,y)+\cdots\right\}+\cdots+\frac{h^n}{n!}\frac{\partial^n}{\partial x^n}\left\{f(x,y)\right.$$

$$\left.+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\cdots\right\}+\cdots$$

$$\therefore f(x+h,y+k)=f(x,y)+\left(h\frac{\partial}{\partial x}f(x,y)+k\frac{\partial}{\partial y}f(x,y)\right)+$$

$$\frac{1}{2!}\left[h^2\frac{\partial^2 f(x,y)}{\partial x^2}+2hk\frac{\partial^2 f(x,y)}{\partial x\partial y}+k^2\frac{\partial^2 f(x,y)}{\partial y^2}\right]$$

$$+\frac{1}{3!}\left[h^3\frac{\partial^3 f(x,y)}{\partial x^3}+3h^2k\frac{\partial^3 f(x,y)}{\partial x^2\partial y}+3k^2h\frac{\partial^3 f(x,y)}{\partial x\partial y^2}+k^3\frac{\partial^3 f(x,y)}{\partial y^3}\right]+$$

$$\cancel{\therefore f(x+h,y+k)=f(x,y)+h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}+\frac{1}{2!}\Big\{}$$

$$\therefore f(x+h,y+k)=f(x,y)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x,y)+\frac{1}{2!}\left\{h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x,y)$$

$$+\frac{1}{3!}\left\{h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(x,y)+\cdots$$

$$+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^n f(x,y)+\cdots$$

Note $$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)$$ Ex②

$$=h\frac{\partial}{\partial x}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)+k\frac{\partial}{\partial y}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)$$

$$=h^2\frac{\partial^2}{\partial x^2}+2hk\frac{\partial^2}{\partial x\partial y}+k^2\frac{\partial^2}{\partial y^2}$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x,y)=h^2\frac{\partial^2}{\partial x^2}f(x,y)+2hk\frac{\partial^2 f(x,y)}{\partial x\partial y}$$

$$+k^2\frac{\partial^2}{\partial y^2}f(x,y)$$

Date
Page

Remark ① Let f be a function of n variable $x_1, x_2, \ldots, x_n$.

Taylor's Theorem for a function of n variable is

$f(x_1+h_1, x_2+h_2, x_3+h_3, \ldots, x_n+h_n) = f(x_1, x_2, \ldots x_n)$

$+\left(h_1\frac{\partial}{\partial x_1}+h_2\frac{\partial}{\partial x_2}+\cdots+h_n\frac{\partial}{\partial x_n}\right)f(x_1,x_2 \ldots x_n) + \frac{1}{2!}\left(h_1\frac{\partial}{\partial x_1}\right.$

$\left.+h_2\frac{\partial}{\partial x_2}+\cdots+h_n\frac{\partial}{\partial x_n}\right)^2 f(x_1, x_2, \ldots, x_n) + \cdots$

$+\frac{1}{n!}\left(h_1\frac{\partial}{\partial x_1}+h_2\frac{\partial}{\partial x_2}+\cdots h_n\frac{\partial}{\partial x_n}\right)^n f(x_1, x_2, \ldots)$

$+ \cdots$

② $f(x,y) = f(a+x-a,\ b+y-b)$

$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$

$+\frac{1}{2!}\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b)$

$+ (y-b)^2 f_{yy}(a,b)\} + \cdots$

The expression has been obtained by replac x by a, h by $x-a$, y by b, k by $(y-b)$ in eq 4 *

Ex ⑤ Expand $f(x,y) = e^{xy}$ in powers of $(x-1)$ and $(y-1)$

or Expand $f(x,y) = e^{xy}$ about $(1,1)$

A $f(x,y) = f(1+x-1, 1+y-1)$

$= f(1,1) + \{(x-1)f_x(1,1) + (y-1)f_y(1,1)\}$

$+\frac{1}{2!}\{(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(\ldots$

$+\frac{1}{3}\{(x-1)^3 f_{xxx}(1,1) + 3(x-1)^2(y-1)f_{xxy} + 3(x-1)(y-1)^2 f_{xyy}(1,\ldots$

$+ (y-1)^3 f_{yyy}(1,1)\} + \cdots$ ④

$$f(x,y) = e^{xy}$$

$$f_x(x,y) = y\,e^{xy} \quad , \quad f_{xx} = y^2 e^{xy}$$

$$f_y(x,y) = x\,e^{xy} \quad , \quad f_{yy} = x^2 e^{xy}$$

$$f_{xy} = e^{xy} + x e^{xy} . y$$

$$= (1+xy)\, e^{xy}$$

$$f_{xxx}(x,y) = y^3 e^{xy} \quad , \quad f_{xyy} = x e^{xy} + (1+xy)\, x e^{xy}$$

$$f_{yyy}(x,y) = x^3 e^{xy} \qquad = e^{xy}(x+x+x^2y)$$

$$f_{xxy} = 2y e^{xy} + y^2 e^{xy} . x \qquad = (2x+x^2y)\, e^{xy}$$

$$= (xy^2+2y)\, e^{xy}.$$

$$f(1,1) = e \quad , \quad f_{xx} = e$$

$$f_x(1,1) = e \quad , \quad f_{yy} = e$$

$$f_{xy} = 2e \quad , \quad f_{xxx} = e \, , \quad f_{yyy} = e$$

$$f_{xyy} = 3e \quad , \quad f_{xxy} = 3e \quad \text{--- (3)}$$

By (1) and (3),

$$f(x,y) = e^{xy} = e + \{(x-1)e + (y-1)e\}$$
$$+ \frac{1}{2!}\{(x-1)^2 e + 2(x-1)(y-1)\times 2e + (y-1)^2 e\}$$
$$+ \frac{1}{3!}\{(x-1)^3 e + 3.(x-1)^2(y-1)3e + 3(x-1)(y-1)^2 3e$$
$$+ (y-1)^3 e\} + \text{---}$$

$$= e\Big[1 + \{(x-1) + (y-1)\} + \frac{1}{2!}\{(x-1)^2 + 4(x-1)(y-1) + (y-1)^2\} + \frac{1}{3}\{(x-1)^3 + 9(x-1)^2(y-1) + 9(x-1)(y-1) + (y-1)^3$$

$$\Big]$$

Ex-2 Expand the following functions.

$f(x,y) = e^{x}\cos y$ about $(1, \pi/2)$

Ex-3 Expand $f(x,y) = e^{x}\sin y$ in powers of $(x-1)$ and $(y-\pi/2)$

Ex-4 expand $f(x,y) = \tan^{-1}(y/x)$ in powers of $(x-1)$ & $(y-1)$

Ex-4 Expand $f(x,y) = \tan^{-1}(y/x)$ in powers of $(x-1)$ & $y-1$ and Find the appoximate value of $\tan^{-1}\left(\frac{0.9}{0.1}\right)$

Ex-5 expand $f(x,y) = \tan^{-1}xy$ in powers of $(x-1)$ & $(y-1)$ and Find the appoximate value of $\tan^{-1}(0.9 \times 1.1)$ $(1,1)$

Solution (6)

$$f(x,y) = f(1+x-1, 1+y-1)$$

$$= f(1,1) + \frac{1}{1!}\{(x-1)f_x(1,1) + (y-1)f_y(1,1)\}$$

$$+ \frac{1}{2!}\{(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1)$$

$$+ (y-1)^2 f_{yy}(1,1)\} + \cdots \quad (1)$$

$$f(x,y) = \tan^{-1}(xy), \quad f(1,1) = \pi/4$$

$$f_x(x,y) = \frac{y}{1+x^2y^2}, \quad f_x(1,1) = \frac{1}{2}$$

$$f_y(x,y) = \frac{x}{1+x^2y^2}, \quad f_y(1,1) = \frac{1}{2}$$

$$f_{xx}(x,y) = \frac{-y \times 2xy^2}{(1+x^2y^2)^2} = \frac{-2xy^3}{(1+x^2y^2)^2}, \quad f_{xx} = -\frac{1}{2}$$

$$f_{yy}(x,y) = \frac{-x \cdot 2x^2y}{(1+x^2y^2)} = -\frac{2x^3y}{(1+x^2y^2)^2} = -\frac{1}{2}$$

$$f_{xy}(x,y) = \frac{(1+x^2y^2)\cdot 1 - x\cdot 2xy^2}{(1+x^2y^2)^2} = \frac{1-x^2y^2}{(1+x^2y^2)^2}$$

$$f_{xy}(1,1) = 0 \quad (2)$$

By (1) and (2),

$$f(x,y) = \tan^{-1}(xy) = \frac{\pi}{4} + \{(x-1)\times\tfrac{1}{2} + (y-1)\tfrac{1}{2}\}$$

* for x ... we get x ~~f~~

Date
Page

$$\tan^{-1}(xy) = \frac{\pi}{4} + \frac{1}{2}\{(x-1)+(y-1)\} - \frac{1}{4}\{(x-1)^2+(y-1)^2\} + \cdots$$

Put $x = 0.9$, $y = 1.1$ in above expression.

$$\tan^{-1}(0.9\times 1.1) = \frac{\pi}{4} + \frac{1}{2}\{(0.9-1)+(1.1-1)\} - \frac{1}{4}\{(0.9-1)^2+(1.1-1)^2\}$$

$$\simeq \frac{\pi}{4} + \frac{1}{2}\cancel{[(-0.1)+0.1]} - \frac{1}{4}[(0.1)^2+(0.1)^2]$$

$$= \frac{\pi}{4} - \frac{1}{4}\times 2(0.1)^2$$

$$= \frac{\pi}{4} - \frac{1}{2}\times 0.01 = 0.785-.5$$

$$= 0.780 \text{ (approximately)}$$

$f(x,y) = V$ $\tan^{-1}(0.9\times 0.1) = f(1-(0.1), 1+0.1)$

$f(1+x-1, 1+y-1)$

$$= \frac{\pi}{4} + \frac{1}{2}[-0.1+0.1] - \frac{1}{4}\{(-0.1)^2+(0.1)^2\} + \cdots \text{ by (1)}$$

$$\tan^{-1}(0.9\times 0.1) = \tan^{-1}1 + 0 - \frac{1}{4}[2\times(0.1)^2]$$

$$\cancel{f(x,y)=} \; f(1+x-1, 1+y-1) = f(0.9\times 0.1) = f(1-0.1, 1+0.1)$$

$$= f(1,1) + \{(-0.1) f_x + 0.1 f_y(1,1)\} + \frac{1}{2!}\{(0.1)^2 f_{xx}(1,1)$$

$$+ 2(-0.1\times 0.1) f_{xy}(1,1) + (0.1)^2 f_{yy}(1,1)\} + \cdots \text{ by (1)}$$

$$= \frac{\pi}{4} + 0 - \frac{1}{4}\times 2\times(0.1)^2$$

$$= \frac{3.14}{4} - \frac{1}{2}\times 0.01$$

$$= 0.785 - 0.005$$

$$= 0.780 \text{ (approximately)} \;*$$

$+ \cdots$ (4) Note. By Taylor's Theorem for a function of variables find the appoximate valu

$\tan^{-1}(0.8 \times 1.3)$.

$\tan^{-1}[(0.8)(1.3)] = \tan^{-1}[(1-0.2)(1+0.3)]$

$(1.1-1)^2 ?+$

expand $\tan^{-1}(xy)$ in powers of $(x-1)$ & $(y-1)$

Ex-2 Expand the following function

$f(x,y) = e^x \cos y$ about $(1, \pi/2)$

1)

$$f(x,y) = f\left(1+x-1, \frac{\pi}{2} + y - \pi/2\right)$$

$$= f\left(1, \frac{\pi}{2}\right) + (x-1) f_x(1, \pi/2) + (y-\pi/2) f_y(1, \pi/2)$$

$$+ \frac{1}{2!}\left\{(x-1)^2 f_{xx}(1,\pi/2) + 2(x-1)(y-\pi/2) f_{xy}(1,\pi/2) + (y-\tfrac{\pi}{2})^2 + \right.$$

$$+ \frac{1}{3!}\left[(x-1)^3 f_{xxx}(1,\pi/2) + 3(x-1)^2(y-\pi/2) f_{xxy}(1,\pi/2) + 3(x-1)(y-\pi/2)^2 f_{xyy}(1,\pi/2) + (y-\pi)^3 f_{yyy}(1,\pi/2)\right]$$

$+ \cdots$ (1)

$f(x,y) = e^x \cos y$,

$f_x = e^x \cos y$, $f_y = -e^x \sin y$

$f_{xx} = e^x \cos y$, $f_{yy} = -e^x \sin y$

$f_{xxx} = e^x \cos y$ $f_{yyy} = e^x \cos y$

$f_{xy} = -e^x \sin y$ — (2)

1)

$f_{xyy} = -e^x \cos y$, $f_{xxy} = -e^x \sin y$.

By (2)

$f_{xx}(1,1)$

$f(1, \pi/2) = 0$

$f_x(1, \pi/2) = 0$

) + —

By (1)

$$\left.\begin{aligned}
f_{xx}(1, \pi/2) &= 0 \\
f_y &= -e \\
f_{yy} &= -e \\
f_{xy} &= -e
\end{aligned}\right\} \quad \text{— (3)}$$

$f_{xxx} = 0$

$f_{yyy} = 0$

$f_{xxy} = -e$

Date
Page

$$f(x,y) = \tan^{-1}(y/x)$$

$$f_x(x,y) = \frac{1}{1+(y/x)^2} \times -\frac{y}{x^2} = \frac{x^2}{x^2+y^2} \times \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$f_y(x,y) = \frac{1}{1+(y/x)^2} \times \frac{1}{x} = \frac{x^2}{x^2+y^2} \times \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$f_{xx}(x,y) = \frac{-y}{(x^2+y^2)^2} \cdot 2x = \frac{+2xy}{(x^2+y^2)^2}$$

$$f_{yy} = \frac{-x}{(x^2+y^2)^2} \times 2y = \frac{-2xy}{(x^2+y^2)^2}$$

$$f_{xy}(x,y) = \frac{(x^2+y^2)\times(-1) - (\cancel{x^2}y)\times 2y}{(x^2+y^2)^2}$$

$$= \frac{-x^2 - y^2 + 2y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2} \quad \text{—— (2)}$$

By (2)

$$f(1,1) = \frac{\pi}{4}$$
$$f_x(1,1) = -\frac{1}{2}$$
$$f_y(1,1) = \frac{1}{2} \quad \text{—— (3)}$$
$$f_{xx}(1,1) = \frac{1}{2}$$
$$f_{yy}(1,1) = -\frac{1}{2}$$
$$f_{xy}(1,1) = 0$$

By (1) and (3), we have

$$f(x,y) = \tan^{-1}(y/x) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$$

$$+ \frac{1}{2}\{ \tfrac{1}{2}(x-1)^2 - \tfrac{1}{2}(y-1)^2 \} + \ldots$$

By ①

$f(x,y) = f(0.9, 0.1) = f(1-0.1, 1-0.9)$

By ① $\tan^{-1}\left(\frac{0.9}{0.1}\right) = f(1,1) + (x-1) f_x(1,1) + (y-1) f_y(1,1)$

$+ \frac{1}{2!}\{ (x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1) f_{xy}(1,1)$

$+ (y-1)^2 f_{yy}(1,1)\} + \text{---}$

$= \frac{\pi}{4} + (0.1) \times \left(-\frac{1}{2}\right) + (-0.9) \times \frac{1}{2}$

$+ \frac{1}{2!}\left[(-0.1)^2 \times \frac{1}{2} + 2 \times (-0.1) \times (-0.9) \times 0\right.$

$\left. + (-0.9)^2 \times \left(-\frac{1}{2}\right)\right] + \text{---}$

$= 0.785 + \frac{0.1}{2} - \frac{0.9}{2} + \frac{1}{2!}\left[\frac{0.01}{2} - \frac{0.81}{2}\right]$

$= 0.785 + 0.05 - 0.45 + \frac{1}{2!}[-0.40]$

$= 0.790 - 0.45 - 0.20$

$= 0.725$ (approximately)