System of Linear Equations

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Systems of Linear Equations

Suppose, we have system of linear equation

$$a_{11}x_1+a_{12}x_2+a_{13}x_3+\dots\dots\dots +a_{1n}x_n =b_1$$
$$a_{21}x_1+a_{22}x_2+a_{23}x_3+\dots\dots\dots +a_{2n}x_n =b_2$$
$$\vdots$$
$$a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\dots\dots\dots +a_{nn}x_n =b_n$$

Let $A=\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & & & & \\ . & & & & \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{bmatrix}$ $X=\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$ $b=\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}$

Ax=b

Factorization method or Decomposition or Triangularisation or LU Decomposition method

This method is based on the fact that a square matrix A can be factorized into the form LU,

$$A=LU$$

Where, L=lower triangular matrix and U=upper triangular matrix

Let $$A=\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & & & & \\ . & & & & \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{bmatrix}$$

LU Decomposition method

Let
$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & & & & \\ . & & & & \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 & 0 & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 & 0 & 0 \\ . & . & . & 0 & 0 \\ . & . & . & . & 0 \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & . & \ell_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & \ldots & u_{1n} \\ 0 & u_{22} & .. & .. & u_{2n} \\ 0 & 0 & u_{33} & \ldots & u_{3n} \\ 0 & 0 & 0 & .. & .. \\ 0 & 0 & 0 & .. & u_{nn} \end{bmatrix}$$

$$[A] = [L][U] = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

LU Decomposition method

The basic idea is to multiply the values at R.H.S. and calculate the value of unknowns.

Note:

A: n^2 elements

l: no. of unknowns: $1+2+3+\ldots.n=1/2(n(n+1))$

U : no. of unknowns: $1+2+3+\ldots.n=1/2(n(n+1))$

Total no. of unknowns on R.H.S=n^2+n

Note: when we multiply the values at R.H.S and compare with the elements of L.H.S.

The number of equations=n^2

→we are getting n arbitrary unknowns

LU Decomposition method

Choose either

1) $l_{ii} = 1$, $i = 1, 2, 3, \ldots\ldots n$, called Do-little method.

2) $u_{ii} = 1$, $i = 1, 2, 3, \ldots\ldots n$, called Crout's method.

Do-little's method

A Square matrix can be factorized as LU
Where, L=unit lower triangular matrix and U=upper triangular matrix
i.e. $l_{ii}=1$

$$[A]=[L][U]$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Do-little's method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

$$u_{11} = a_{11}, u_{12} = a_{12,} u_{13} = a_{13}$$

$$l_{21}u_{11} = a_{21}, l_{21}u_{12} + u_{22} = a_{22}, l_{21}u_{13} + u_{23} = a_{23}$$

$$l_{31}u_{11} = a_{31}, l_{31}u_{12} + l_{32}u_{22} = a_{32}, l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$$

Do-little's method

On solving, we get

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{a_{21}}{a_{11}}, l_{31} = \frac{a_{31}}{u_{11}} == \frac{a_{31}}{a_{11}}$$

$$u_{22} = a_{22} - l_{21}u_{12} = a_{22} - \frac{a_{21}}{a_{11}}a_{12}$$

$$u_{23} = a_{23} - l_{21}u_{13} = a_{23} - \frac{a_{21}}{a_{11}}a_{13}$$

$$l_{32}u_{22} = a_{32} - l_{31}u_{12} = a_{32} - \frac{a_{31}}{a_{11}}a_{12}$$

$$l_{32} = \frac{1}{u_{22}}\left(a_{32} - \frac{a_{31}}{a_{11}}a_{12}\right)$$

$$u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

Do-little's method

Consider the system of linear equations

$$a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots\ldots.. +a_{1n}x_n = b_1$$
$$a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots\ldots.. +a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\ldots\ldots.. +a_{nn}x_n = b_n$$

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & & & & \\ . & & & & \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}$$

Which can be written Ax=b

Do-little's method

Which can be written

$$Ax=b \qquad \ldots\ldots\ldots\ldots(1)$$

Let $\quad A=LU \qquad \ldots\ldots\ldots\ldots(2)$

$$LUx=b \qquad \ldots\ldots\ldots\ldots(3)$$

Let $\quad Ux=Y$ in eq. (3) $\qquad \ldots\ldots\ldots..(4)$

Now eq. (3) becomes

$$LY=b$$

As,

$$L=\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \qquad Y=\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \qquad b=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$Y_1=b_1$$

$$l_{21}Y_{1+}Y_{2=}b_2$$

$$l_{31}Y_{1+}l_{32}Y_{2+}Y_{3=}b_3$$

Do-little's method

Let $\quad Ux=Y \quad$ ………..(4)

$Y_1=b_1$

$l_{21}Y_1+Y_2=b_2$

$l_{31}Y_1+l_{32}Y_2+Y_3=b_3$

this can be solved using forward substitution, when Y is known,

Ux=Y becomes

$U_{11}x_1+U_{12}x_2+U_{13}x_3=Y_1$

$U_{22}x_2+U_{23}x_3=Y_2$

$U_{33}x_3=Y_3$

Which can be solved using backward substitution

Crout’s method

A Square matrix can be factorized as LU
Where, L=lower triangular matrix and U=unit upper triangular matrix
i.e, $U_{ii}=1$

$$[A]=[L][U]$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Crout's method (Cont…)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{11}u_{12} & \ell_{11}u_{13} \\ \ell_{21} & \ell_{21}u_{12} + \ell_{22} & \ell_{21}u_{13} + \ell_{22}u_{23} \\ \ell_{31} & \ell_{31}u_{12} + \ell_{32} & \ell_{31}u_{13} + \ell_{32}u_{23} + \ell_{33} \end{bmatrix}$$

Crout's method (Cont…)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{11}u_{12} & \ell_{11}u_{13} \\ \ell_{21} & \ell_{21}u_{12} + \ell_{22} & \ell_{21}u_{13} + \ell_{22}u_{23} \\ \ell_{31} & \ell_{31}u_{12} + \ell_{32} & \ell_{31}u_{13} + \ell_{32}u_{23} + \ell_{33} \end{bmatrix}$$

$$\ell_{11} = a_{11}, \ell_{11}u_{12} = a_{12}, \ell_{11}u_{13} = a_{13}$$

$$\Rightarrow \ell_{11} = a_{11},$$

$$u_{12} = \frac{a_{12}}{\ell_{11}} = \frac{a_{12}}{a_{11}}$$

$$u_{13} = \frac{a_{13}}{\ell_{11}} = \frac{a_{13}}{a_{11}}$$

Similarly the values of other unknowns can be determined.

Solve the following linear system of equations using LU Decomposition method

$2x_1+3x_2+x_3=9$

$x_1+2x_2+3x_3=6$

$3x_1+x_2+2x_3=8$

We have,

$$A=\begin{bmatrix}2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2\end{bmatrix}$$

$$[A]=[L][U] or [L][U]=[A]$$

$$\begin{bmatrix}1 & 0 & 0\\ \ell_{21} & 1 & 0\\ \ell_{31} & \ell_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}=\begin{bmatrix}2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2\end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

	$u_{11}=2$,	$u_{12}=3$,	$u_{13}=1$
	$l_{21}u_{11}=1$,	$l_{21}=1/u_{11}$,	$l_{21}=1/2$
	$l_{31}u_{11}=3$,	$l_{31}=3/u_{11}$,	$l_{31}=3/2$
Now,	$l_{21}u_{12}+u_{22}=2$,	$u_{22}=2-l_{21}u_{12}$,	$u_{22}=2-(1/2)(3)=1/2$
And	$l_{21}u_{13}+u_{23}=3$,	$u_{23}=3-l_{21}u_{13}$,	$u_{23}=3-(1/2)(1)=5/2$
	$l_{31}u_{12}+l_{32}u_{22}=1$,	$l_{32}u_{22}=1-l_{31}u_{12}$,	$l_{32} =-7$
	$l_{31}u_{13}+l_{32}u_{23}+u_{33} =2$,		$u_{33} =18$

$u_{11}=2,\quad u_{12}=3\ ,\quad u_{13}=1$

$l_{21}u_{11}=1,\quad l_{21}=1/u_{11}\ ,\quad l_{21}=1/2$

$l_{31}u_{11}=3,\quad l_{31}=3/u_{11}\ ,\quad l_{31}=3/2$

Now, $l_{21}u_{12}+u_{22}=2,\quad u_{22}=2-l_{21}u_{12}\ ,\quad u_{22}=2-(1/2)(3)=1/2$

And $l_{21}u_{13}+u_{23}=3,\quad u_{23}=3-l_{21}u_{13}\ ,\quad u_{23}=3-(1/2)(1)=5/2$

$l_{31}u_{12}+l_{32}u_{22}=1,\ l_{32}u_{22}=1-l_{31}u_{12}\ ,\quad l_{32}=-7$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=2,\quad u_{33}=18$

$$A = LU$$

$$A=\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

$$Now.......A = LU$$

$$and...Ax = b$$

$$\Rightarrow LUx = b$$

$$Let Ux = y$$

$$\Rightarrow Ly = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$$\Rightarrow y_1 = 9, y_{2=} 3/2, y_3 = 5$$

$$Now....Ux = y$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

Using backward substitution

$$x_3 = \frac{5}{18}, x_2 = \frac{29}{18}, x_1 = \frac{35}{18}$$

Which is required solution.

Practice Problems

1. Solve the following linear system of equations using LU decomposition method

$$y+z=2$$
$$2x+3z=5$$
$$x+y+z=3$$

2. Solve the following linear system of equations using LU decomposition method

$$7x-2y+z=12$$
$$14x-7y-3z=17$$
$$-7x+11y+18z=5$$

Suggested books

1. Numerical Methods by **S.R.K Lyenger & R.K. Jain.**

2. Numerical Analysis by **Richard L. Burden.**

3. Introductory methods of Numerical analysis by **S.S. Sastry**.

Thank you