9. TAYLOR'S THEOREM

If $f(x, y)$ is a function which possesses continuous partial derivatives of order n in any domain of point (a, b), and the domain is large enough to contain a point $(a + h, b + k)$ with it, then there exists positive number $0 < \theta < 1$, such that

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$+ \ldots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a, b) + R_n,$$

where $R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k), 0 < \theta < 1.$

Let $x = a + th, y = b + tk$, where $0 \leq t \leq 1$ is a parameter, and

$$f(x, y) = f(a + th, b + tk) = \phi(t)$$

Since the partial derivatives of $f(x, y)$ of order n are continuous in the domain under consideration, $\phi^n(t)$ is continuous in [0, 1], and also

$$\phi'(t) = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f$$

$$\phi''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$$

$$\vdots$$

$$\phi^{(n)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f$$

therefore by Maclaurin's theorem

$$\phi(t) = \phi(0) + t\phi'(0) + \frac{t^2}{2!}\phi''(0) + \ldots + \frac{t^{n-1}}{(n-1)!}\phi^{(n-1)}(0) + \frac{t^n}{n!}\phi^{(n)}(\theta t),$$

where $0 < \theta < 1$.

Now on putting $t = 1$, we get

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2!}\phi''(0) + \ldots + \frac{1}{(n-1)!}\phi^{(n-1)}(0) + \frac{1}{n!}\phi^{(n)}(0)$$

But $\phi(1) = f(a+h, b+k)$, and $\phi(0) = f(a, b)$

$$\phi'(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b)$$

$$\phi''(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$\vdots$$

$$\phi^{(n)}(\theta) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k)$$

$$\therefore \quad f(a+h, b+k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b) + \ldots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a, b) + R_n,$$

where $R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k),\ 0 < \theta < 1.$

R_n is called the *remainder after n terms*, and the theorem, *Taylor's theorem with remainder* or *Taylor's expansion* about the point (a, b).

If we put $a = b = 0$; $h = x$, $k = y$, we get

$$f(x, y) = f(0, 0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0, 0)$$

$$+ \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0, 0) + \ldots + \frac{1}{(n-1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n-1} f(0, 0) + R_n$$

where $R_n = \dfrac{1}{n!}\left(x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\right)^n f(\theta x, \theta y)$, $0 < \theta < 1$, is called the *Maclaurin's theorem* or *Maclaurin's expansion.*

It is easy to see that Taylor's theorem can also be put in the form:

$$f(a+h, b+k) = f(a, b) + df(a, b) + \frac{1}{2!}d^2 f(a, b) + \ldots$$

$$+ \frac{1}{(n-1)!}d^{n-1} f(a, b) + \frac{1}{n!}d^n f(a + \theta h, b + \theta k)$$

The reasoning in the general case of several variables is precisely the same and so the theorem can be easily extended to any number of variables.

9.1 The Theorem can be Stated in Still another Form

$$f(x, y) = f(a, b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a, b)$$

$$+ \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a, b) + \ldots$$

$$+ \frac{1}{(n-1)!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{n-1} f(a, b) + R_n,$$

where $R_n = \dfrac{1}{n!}\left[(x-a)\dfrac{\partial}{\partial x} + (y-b)\dfrac{\partial}{\partial y}\right]^n f(a + (x-a)\theta, b + (y-b)\theta)$, $0 < \theta < 1$, called the Taylor's expansion of $f(x, y)$ about the point (a, b) in powers of $x - a$ and $y - b$.

Example 30. Expand $x^2y + 3y - 2$ in powers of $x - 1$ and $y + 2$.

■ Let us use Taylor's expansion with $a = 1$, $b = -2$. Then

$f(x, y) = x^2y + 3y - 2,$ $\quad f(1, -2) = -10$

$f_x(x, y) = 2xy,$ $\quad f_x(1, -2) = -4$

$f_y(x, y) = x^2 + 3,$ $\quad f_y(1, -2) = 4$

$f_{xx}(x, y) = 2y,$ $\quad f_{xx}(1, -2) = -4$

$f_{xy}(x, y) = 2x,$ $\quad f_{xy}(1, -2) = 2$

$f_{yy}(x, y) = 0,$ $\quad f_{yy}(1, -2) = 0$

$f_{xxx}(x, y) = 0 = f_{yyy}(x, y),$ $\quad f_{yxx}(1, -2) = 2 = f_{xxy}(1, -2)$

All higher derivatives are zero.

$$\therefore \quad x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) + \frac{1}{2}[-4(x-1)^2 + 4(x-1)(y+2)]$$

$$+ \frac{1}{3!}3(x-1)^2(y+2)(2) + 0$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$$

Example 31. If $f(x, y) = \sqrt{|xy|}$, prove that Taylor's expansion about the point (x, x) is not valid in any domain which includes the origin.

■ As was shown earlier in Example II § 4.1,

$$f_x(0, 0) = 0 = f_y(0, 0)$$

$$f_x(x, y) = \begin{cases} \frac{1}{2}\sqrt{|y/x|}, & x > 0 \\ -\frac{1}{2}\sqrt{|y/x|}, & x < 0 \end{cases}$$

$$f_y(x, y) = \begin{cases} \frac{1}{2}\sqrt{|x/y|}, & y > 0 \\ -\frac{1}{2}\sqrt{|x/y|}, & y < 0 \end{cases}$$

$$\therefore \quad f_x(x, x) = f_y(x, x) = \begin{cases} \frac{1}{2}, & x > 0 \\ -\frac{1}{2}, & x < 0 \end{cases}$$

Now Taylor's expansion about (x, x) for $n = 1$, is

$$f(x+h, x+h) = f(x, x) + h[f_x(x+\theta h, x+\theta h) + f_y(x+\theta h, x+\theta h)]$$

or

$$|x+h| = \begin{cases} |x| + h, & \text{if } x + \theta h > 0 \\ |x| - h, & \text{if } x + \theta h < 0 \\ |x|, & \text{if } x + \theta h = 0 \end{cases} \qquad \text{...(1)}$$

If the domain $(x, x; x+h, x+h)$ includes the origin, then x and $x+h$ must be of opposite signs, that is either

$$|x+h| = x+h, \quad |x| = -x$$

or

$$|x+h| = -(x+h), \quad |x| = x$$

But under these conditions none of the inequalities (1) holds. Hence the expansion is not valid.

Ex. 1. Expand $x^4 + x^2y^2 - y^4$ about the point $(1, 1)$ up to terms of the second degree. Find the form of R_2.

Ex. 2. Find the expansion of $\sin x \sin y$ about $(0, 0)$ up to and including the terms of the fourth degree in (x, y). Compare the result with that you get by multiplying the series for $\sin x$ and $\sin y$.

Ex. 3. Expand $e^x \tan^{-1} y$ about $(1, 1)$ up to the second degree in $(x-1)$ and $(y-1)$.

Ex. 4. Show that the expansion of $\sin(xy)$ in powers of $(x-1)$ and $(y-\pi/2)$ up to and including second degree terms is

$$1-\frac{1}{8}\pi^2(x-1)^2-\frac{1}{2}\pi(x-1)\left(y-\frac{\pi}{2}\right)-\frac{1}{2}\left(y-\frac{\pi}{2}\right)^2$$

Ex. 5. Show that, for $0<\theta<1$,

$$\sin x \sin y = xy-\frac{1}{6}[(x^3+3xy^2)\cos\theta x \sin\theta y+(y^3+3x^2y)\sin\theta x\cos\theta y]$$

Ex. 6. Prove that the first four terms of the Maclaurin expansion of $e^{ax}\cos by$ are

$$1+ax+\frac{a^2x^2-b^2y^2}{2!}+\frac{a^3x^3-3ab^2xy^2}{3!}.$$

Ex. 7. Prove that for $0<\theta<1$,

$$e^{ax}\sin by = by+abxy+\frac{1}{6}[(a^3x^3-3ab^2xy^2)\sin(b\theta y)$$
$$+(3a^2bx^2y-b^3y^3)\cos(b\theta y)]\,e^{a\theta x}$$

Ex. 8. Show that if f, f_x, f_y are all continuous in a domain D of (a, b), and D is large enough to contain the point $(a+h, b+k)$, within it, then for $0<\theta<1$,

$$f(a+h, b+k)=f(a,b)+hf_x(a+\theta h, b+\theta k)+kf_y(a+\theta h, b+\theta k).$$

If $f(x,y)=x\sqrt{x^2+y^2}$, $a=b=-1$, $h=k=3$, verify that the above conditions are satisfied and find the value of θ.