B.Sc. Mathematics – 2nd Semester

MTB 202 – Statics and Dynamics

by

Dr. Krishnendu Bhattacharyya

Department of Mathematics,

Institute of Science, Banaras Hindu University

Part – VII

Motion in a Moving Frame of Reference

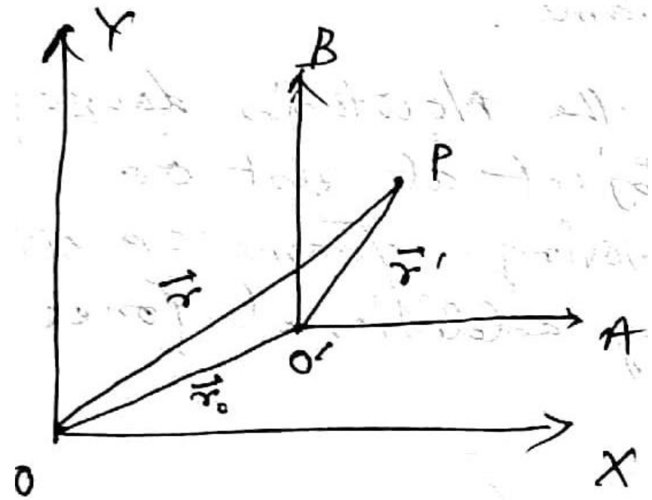
A moving cycle or a running car or a running train and their rotating wheels, a flying bird, a rotating disc, these all are examples of moving frames for an observer in a fixed frame. But for a passenger sitting or moving in a compartment of a running train, the compartment is not a moving frame, instead it is a fixed frame.

We have to see whether the Newton's Laws of motion are applicable on the object at rest or changing its position in a moving frame or not and whether there is any additional force or not on the object.

Motion in a Linearly Moving Frame

A) ***Linear motion with uniform velocity*:**

Let us consider an object of mass m moving in a plane $O'AB$. This frame itself is moving with uniform linear velocity with reference to a fixed plane of reference OXY.



Let any time t, the coordinates of the object at position P, with reference to fixed plane and moving plane be $\vec{r}(x,y)$ and $\vec{r}'(x',y')$ respectively, where as those of O' with reference to O be $\vec{r}_o\left(x_o, y_0\right)$. Then

$$\vec{r} = \vec{r}' + \vec{r}_o, \text{ or } x = x' + x_o, y = y' + y_o.$$

Differentiating the above expressions w.r.t time variable w.r.t the fixed frame of reference we have

$$\vec{v} = \vec{v}' + \vec{v}_o, \text{ or } \dot{x} = \dot{x}' + \dot{x}_o, \dot{y} = \dot{y}' + \dot{y}_o, \qquad \text{(i)}$$

where the terms from the left to right represent velocities of the particle with reference to fixed plane OXY moving plane $O'AB$ and uniform velocity of the moving plane with reference to the fixed plane, respectively.

Further differentiating w.r.t time variable (in the fixed frame of reference) we have the two accelerations equal to each other, i.e., $\vec{a} = \vec{a}'$ as $\dot{\vec{v}}_o = 0$.

Thus according to Newton's law, the equation of motion of the object w.r.t the fixed plane $m\vec{a} = \vec{F}$ may also be written as $m\vec{a}' = \vec{F}$, where $\vec{F}$ is external force acting on the particle.

It means all the frames moving with uniform linear velocity relative to a fixed frame are also a Newtonian frame of references (in which Newton's Laws of motion hold good). So, there is no effect of the uniform linear motion of the frame on the motion of the object in it.

B) <u>Linear Motion with an acceleration</u>:

Let the plane $O'AB$ be moving with an acceleration $\vec{a}$ with reference to the fixed plane. Then differentiating (i) w.r.t time (w.r.t fixed plane), we have

$$\vec{a} = \vec{a}' + \overline{a}_o, \text{ or } \ddot{x} = \ddot{x}' + \ddot{x}_o, \ddot{y} = \ddot{y}' + \ddot{y}_0,$$

where first two terms from left to right represent the accelerations of the particle with reference to fixed plane and moving plane, respectively. So, the equation of motion of the object w.r.t the fixed plane is

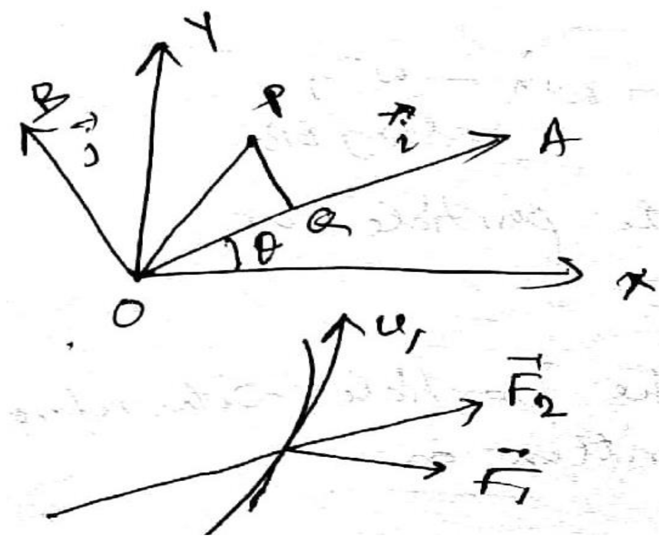
$$m\vec{a} = \vec{F}, \text{ i.e., } m\vec{a}' = \vec{F} - m\vec{a}_o.$$

Since the term on the left hand side represents the product of mass and acceleration (effective force) of the object with reference to moving plane, therefore the equation represent the equation of motion of the object with reference to $O'AB$. Comparing with the previous case, we observe that an additional force $\left(-m\vec{a}_o\right)$ comes into effect. Obviously, it is due to the linear motion of $O'AB$ with an acceleration.

So we may conclude that a frame moving linearly with acceleration is not a Newtonian Frame of Reference. The force is a fictitious force and it vanishes when the acceleration vanishes.

Motion in a Uniformly Rotating Plane

Let a two dimensional frame of reference $O'AB$ be rotating with uniform angular velocity $\vec{\omega}$ in its plane. Let it axis of rotation pass through its origin. Let initially it be coincident with the fixed frame OXY, origin of the both frames being coincident. So, we can name the rotating frame as OAB.



Let $P(\vec{r})$ be the position vector of an object moving in the rotating frame at any time t. Further, let its components in rotating plane

OAB be x=OQ and y=PQ.

Let any time the axis OA make angle θ with the fixed axis OA.

Let $\vec{i}$ and $\vec{j}$ be unit vectors along two axes OA and OB respectively.

Then we have $\vec{r} = x\vec{i} + y\vec{j}$.

The rate of change of $\dot{\vec{r}}$ with time reference to the fixed frame OXY will be

$$\frac{d\vec{r}}{dt} = \dot{\vec{r}} = \dot{x}\vec{i} + \dot{y}\vec{j} + x\frac{d\vec{i}}{dt} + y\frac{d\vec{j}}{dt}.$$

We know that $\dfrac{d\vec{i}}{dt} = \dfrac{d\vec{i}}{d\theta}\dfrac{d\theta}{dt} = \omega\vec{j}$ and $\dfrac{d\vec{j}}{dt} = \dfrac{d\vec{j}}{d\theta}\dfrac{d\theta}{dt} = -\omega\vec{i}$, where ω is the magnitude of the angular velocity vector along the axis of rotation and is the rate of change with respect to time variable w.r.t the fixed frame of the angular displacement of the rotating frame.

$$\text{So, } \dot{\vec{r}} = (\dot{x} - \omega y)\vec{i} + (\dot{y} + \omega x)\vec{j},$$

i.e., the velocity components of the object at P are $(\dot{x} - \omega y)$ and $(\dot{y} + \omega x)$ along OA and OB respectively, when observed from the fixed plane.

Again differentiating w.r.t. t, the acceleration of the object with reference to the fixed frame will have the expression

$$\ddot{\vec{r}} = (\ddot{x} - \omega\dot{y})\vec{i} + (\ddot{y} + \omega\dot{x})\vec{j} + (\dot{x} - \omega y)\omega\vec{j} + (\dot{y} + \omega x)(-\omega\vec{i}),$$

$$\text{i.e., } \ddot{\vec{r}} = \left(\ddot{x} - 2\omega\dot{y} - \omega^2 x\right)\vec{i} + \left(\ddot{y} + 2\omega\dot{x} - \omega^2 y\right)\vec{j}.$$

It means the acceleration components along the rotating axes are

$$\ddot{x} - 2\omega\dot{y} - \omega^2 x \text{ along } OA \text{ and } \ddot{y} + 2\omega\dot{x} - \omega^2 y \text{ along } OB.$$

Let the external force acting on the particle be $F = X\vec{i} + Y\vec{j}$

Then the equation of motion of the particle with reference to fixed frame *OXY* may be written as

$$m\ddot{\vec{r}} = \vec{F}$$

Or, $m\left(\ddot{x} - 2\omega\dot{y} - \omega^2 x\right) = X$ and $m\left(\ddot{y} + 2\omega\dot{x} - \omega^2 y\right) = Y$

Or, $m\ddot{x} = X + 2m\omega\dot{y} + m\omega^2 x$ and $m\ddot{y} = Y - 2m\omega\dot{x} + m\omega^2 y$

If OAB plane is at rest and coincides with OXY the above equation of motion reduces to those in accordance with Newton's laws of motion, i.e., reduce to

$$m\ddot{x} = X, m\ddot{y} = Y.$$

But, the plane OAB is rotating w.r.t fixed plane OXY, hence the equations of motion of the object in the rotating plane w.r.t fixed frame may be written as

$$m\ddot{x} = X + X_1 + X_2 \text{ and } m\ddot{y} = Y + Y_1 + Y_2,$$

where $F_1(X_1, Y_1)$ and $F_2(X_2, Y_2)$ may be interpreted as two additional fictitious forces introduced due to the rotating of the plane.

Thus the plane rotating about its origin, even with uniform angular velocity in its own plane is not a Newtonian frame of reference.

The force $\vec{F}_1$ is of magnitude $2m\omega v_1$, where $\vec{v}_1 = \dot{x}\vec{i} + \dot{y}\vec{j}$ is the linear velocity of the particle along the tangent to the path described by the particle in the rotating plane. It is known as Coriolis force and is perpendicular to direction of $\vec{v}_1$ in a sense opposite to that of angular velocity. Also the second force $\vec{F}_2$ has magnitude $m\omega^2 r$ and acts along the radial direction. It is referred to as Centrifugal force.

If the object is at rest in the rotating frame it will experience only centrifugal force.