Partial Derivatives

The ordinary derivative of a function ~~$f(x,y)$~~ $f(x_1, x_2, \ldots x_n)$ of ~~several~~ $n-$variables w.r.t. one of the independent variables ~~x or y~~ x_1 or x_2 or $\ldots x_n$, keeping all other independent variable constant is called the partial derivative of the function w.r.t. the variable

Partial derivative of $f(x,y)$ w.r.t. x is denoted by $\frac{\partial f}{\partial x}$ and w.r.t y is $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(a+\delta x, y) - f(x,y)}{\delta x}$$

(when limits exist).

$$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

Similarly, p.d. at a particular point (a,b) is denoted by

$$\left[\frac{\partial f}{\partial x}\right]_{(a,b)}, \quad \frac{\partial f(a,b)}{\partial x} \text{ or } f_x(a,b)$$

and

$$\left[\frac{\partial f}{\partial y}\right]_{(a,b)}, \quad \frac{\partial f(a,b)}{\partial y} \text{ or } f_y(a,b)$$

and it is given by

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

$$f_k(a,b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a,b)}{k}$$

Ex. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the following

a) $f(x,y) = 3x^3 - 2xy + x^2 + y^2$

$$\frac{\partial f}{\partial x} = 9x^2 - 2y + 2x + 0$$

$$\frac{\partial f}{\partial y} = 0 - 2x + 0 + 2y$$

b) $f(x,y) = 2x^2 - xy + 2y^2$ at $(1,2)$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = 4x - y = 2$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,2)} = -x + 4y = 7$$

II

Ex 2. If $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Show that both the partial derivatives exist at $(0,0)$ but the function is not continuous ~~the~~ at origin.

Soln: taking path $y = mx$

$$\lim_{x \to 0} f(x,y) = \frac{m}{1+m^2}$$

Which is path dependent. Hence, simultaneous limit does not exist. Therefore, the function is not continuous at $(0,0)$.

Now,
$$f_x(0,0) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$
$$= \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{0}{h} = 0$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0, 0+k) - f(0,0)}{k}$$
$$= \lim_{k \to 0} \frac{0}{k} = 0.$$

* Partial derivatives may exist at a point but function is not even continuous.

Ex. 3. Find f_x and f_y, if $f(x,y) = x^3y + y + e^{xy^2}$

4. If ~~f(x,y)~~ $f(x,y) = \begin{cases} xy\frac{(x^2-y^2)}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & (x,y) = (0,0) \end{cases}$

Show that $f_x(x,0) = 0 = f_y(0,y)$

and $f_x(0,y) = -y$, $f_y(x,0) = x$.

5. Calculate $f_x, f_y, f_x(0,0), f_y(0,0)$ for

a) $$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x = y = 0 \end{cases}$$

b) $$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & x \neq 0, y \neq 0 \\ 0, & x = y = 0 \end{cases}$$

c) $$f(x,y) = \sqrt{|xy|}$$

6: If $$f(x,y) = \begin{cases} xy \tan\left(\dfrac{y}{x}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Show that $x f_x + y f_y = 2f$.

7. Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}, & x^2+y^2 \neq 0 \\ 0, & x = 0 = y \end{cases}$$

possesses first partial derivatives everywhere, including the origin, but the function is discontinuous at the origin.

8. If $$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & x \neq y \\ 0 & x = y \end{cases}$$

Show that the function is discontinuous at the origin but possesses partial derivatives f_x and f_y at every point, including origin.