## Ex 1(E)

Q3. A small bead slides with constant speed $v$ on a smooth cone in the shape of the cardioid $r = a(1+\cos\theta)$. Show that the value of $\dot\theta$ is $\left(\frac{v\sec\theta/2}{2a}\right)$ and that the radial component of the acceleration is const.

Proof: $r = a(1+\cos\theta) = a\{\cancel{1} + 2\cos^2\theta/2 - \cancel{1}\} = 2a\cos^2\theta/2$

Radial velocity $= \dot r = 2a\,\cancel{2}\cos\theta/2\cdot(-\sin\theta/2)\cdot\frac{\dot\theta}{\cancel{2}} = -2a\sin\theta/2\cos\theta/2\,\dot\theta$

transverse velocity $= r\dot\theta = 2a\cos^2\theta/2\,\dot\theta$

Velocity $v = \sqrt{\dot r^2 + (r\dot\theta)^2} = \sqrt{4a^2\sin^2\theta/2\cos^2\theta/2\,\dot\theta^2 + 4a^2\cos^4\theta/2\,\dot\theta^2}$

$= \dot\theta\, 2a\cos\theta/2\sqrt{\cos^2\theta/2 + \sin^2\theta/2} = 2a\cos\theta/2\,.\,\dot\theta$

$\Rightarrow v = 2a\cos\theta/2\,\dot\theta \Rightarrow \dot\theta = \frac{v}{2a\cos\theta/2} = \frac{v\sec\theta/2}{2a}$ — n

Radial component of ~~velocity~~ acceleration $= \ddot r - r\dot\theta^2$

$\dot r = -2a\sin\theta/2\cos\theta/2\,\dot\theta = -\cancel{2a}\sin\theta/2\,\cancel{\cos\theta/2}\left(\frac{v\,\cancel{\sec\theta/2}}{\cancel{2a}}\right) = -v\sin\theta/2$

$\ddot r = -\frac{d}{dt}(v\sin\theta/2) = -v\cos\theta/2\,\frac{1}{2}\dot\theta = -\frac{v}{2}\cos\theta/2.\,\frac{v\sec\theta/2}{2a} = \frac{-v^2}{4a}$

transverse component of acceleration $= \frac{1}{r}\frac{d}{dt}(r^2\dot\theta) = \frac{1}{r}\{2r\dot r\dot\theta + r^2\ddot\theta\}$

$= 2\dot r\dot\theta + r\ddot\theta$

$= \cancel{2}(-v\sin\theta/2)\frac{v\sec\theta/2}{\cancel{2}a} + \cancel{2}\cos^2\theta/2\cdot\cancel{a}\left(\frac{v}{\cancel{2a}}\sec\theta/2\tan\theta/2\,\frac{1}{2}\dot\theta\right)$

$= -\frac{v^2}{a}\tan\theta/2 + v\cos\theta/2\cdot\tan\theta/2\,\frac{1}{2}\left\{\frac{v\sec\theta/2}{2a}\right\}$

$= -\frac{v^2}{a}\tan\theta/2 + \frac{1}{4a}v^2\tan\theta/2 = -\frac{3v^2}{4a}\tan\theta/2 = -\frac{3v^2}{4a}\tan\theta/2$

Only radial component of acceleration $= \ddot r - r\dot\theta^2 = -\frac{v^2}{4a} - 2a\cos^2\theta/2\left\{\frac{v^2\sec^2\theta/2}{4a^2}\right.$

$= \frac{-v^2}{4a} - \frac{v^2}{2a} = -\frac{3v^2}{4a}$ is constant

Ans.

Q4 If the curve is the equi-angular spiral $r = ae^{\theta \cot\alpha}$ and if the radius vector to the particle has constant angular velocity, show that the resultant acceleration of the particle makes an angle $2\alpha$ with the radius of vector and is of magnitude $v^2/r$ where $v$ is the speed of the particle.

Prof

$$r = ae^{\theta\cot\alpha}$$

Angular velocity $= \frac{d\theta}{dt}$ = Constant, let $\frac{d\theta}{dt} = \omega$ (Constant)

radial velocity $= \frac{dr}{dt} = \underline{ae^{\theta\cot\alpha}}\,\dot{\theta}\cot\alpha = \omega\cot\alpha\, r = \omega r\cot\alpha$ — ①
$\qquad\qquad\qquad\qquad r$

transverse velocity $= r\frac{d\theta}{dt} = r\omega$ — ②

$$V = \sqrt{\dot{r}^2 + (r\dot{\theta})^2} = \sqrt{\omega^2 r^2\cot^2\alpha + r^2\omega^2} = r\omega\operatorname{cosec}\alpha \quad — ③$$

Radial acceleration $= \ddot{r} - r\dot{\theta}^2$
$= \omega^2\cot^2\alpha\, r - r\omega^2$
$= r\omega^2(\cot^2\alpha - 1)$ — ④

$\{\dot{r} = \omega r\cot\alpha$ {by eq ①}
$\ddot{r} = \omega\cot\alpha\,\dot{r}$
$\ddot{r} = \omega\cot\alpha\{\omega r\cot\alpha\}$
$\boxed{\ddot{r} = \omega^2\cot^2\alpha\, r}$ ④

transverse acceleration $= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$
$= \frac{1}{r}\{r^2\ddot{\theta} + \dot{\theta}\,2r\dot{r}\} = r\ddot{\theta} + 2\dot{r}\dot{\theta}$
$= 2\dot{r}\dot{\theta} = 2\{\omega r\cot\alpha\}\omega = 2\omega^2 r\cot\alpha$ — ⑤

$\dot{\theta} = \omega$ (Const)
$\ddot{\theta} = 0$

Resultant acceleration $= \sqrt{(\text{radial Acceler})^2 + (\text{transver Accele})^2}$

$$= \sqrt{r^2\omega^4(\cot^2\alpha - 1)^2 + 4\omega^4 r^2\cot^2\alpha}$$

$$= \omega^2 r\sqrt{\cot^4\alpha + 1 - 2\cot^2\alpha + 4\cot^2\alpha} = \omega^2 r\sqrt{\cot^4\alpha + 2\cot^2\alpha + 1} = \omega^2 r(1+\cot^2\alpha)$$

$$= \omega^2 r\operatorname{cosec}^2\alpha = \frac{\omega^2 r^2\operatorname{cosec}^2\alpha}{r} = \frac{\{\omega r\operatorname{cosec}\alpha\}^2}{r} = \frac{V^2}{r}$$ A

let resultant acceleration makes $\beta$ angle with radius vector

$$\beta = \tan^{-1}\left\{\frac{\text{transver Acceler}}{\text{radial Acceler}}\right\} = \tan^{-1}\left\{\frac{2\omega^2 r\cot\alpha}{\omega^2 r(\cot^2\alpha - 1)}\right\} = \tan^{-1}\left\{\frac{2\tan\alpha}{1-\tan^2\alpha}\right\}$$

$$\beta = \tan^{-1}(\tan 2\alpha) = 2\alpha$$

$$\boxed{\beta = 2\alpha}$$ Ans

Q7. A point starts from the origin in the direction of the initial line with velocity $f/\omega$, and moves with constant angular velocity $\omega$ about the origin, and with constant negative radial acceleration $-f$. Show that the rate of growth of the radial velocity is never positive but tends to the limit zero, and prove that the equation of the path is

$$\omega^2 r = f(1-e^{-\theta}) \quad \text{—①}$$

Prof: Angular velocity $= \frac{d\theta}{dt} = \omega =$ Constant; radial Acceleration $= \ddot{r} - r\dot{\theta}^2 = -f$ —②

$$\ddot{r} - r\omega^2 = -f \Rightarrow \ddot{r} = \omega^2 r - f \quad \text{—③}$$

Multiplying both sides eq ③ by $2\dot{r}$ and integrating w.r.t. 't'.

$$\int 2\dot{r}\ddot{r}\,dt = \int 2\omega^2 r\dot{r}\,dt - \int f\,2\dot{r}\,dt$$

$$\dot{r}^2 = \omega^2 r^2 - 2fr + A \quad \{\text{where } A \text{ is integration const}\}$$

When $r=0$, $\dot{r} = f/\omega \Rightarrow f^2/\omega^2 = A$

$$\dot{r}^2 = \omega^2 r^2 - 2fr + f^2/\omega^2 \Rightarrow \dot{r}^2 = (f/\omega - \omega r)^2 \Rightarrow \boxed{\dot{r} = f/\omega - \omega r}$$

$$\frac{dr}{d\theta}\frac{d\theta}{dt} = f/\omega - \omega r \Rightarrow \frac{dr}{d\theta} = \left(f/\omega^2 - r\right)$$

$$\Rightarrow \int \frac{dr}{(f/\omega^2 - r)} = \int \frac{d\theta}{1} \Rightarrow -\log(f/\omega^2 - r) = \theta + B$$

But initially $\theta = 0$, $r = 0 \Rightarrow B = -\log f/\omega^2$

$$-\log(f/\omega^2 - r) = \theta - \log f/\omega^2 \Rightarrow \log(f/\omega^2 - r) - \log f/\omega^2 = -\theta$$

$$\log\left\{\frac{f/\omega^2 - r}{f/\omega^2}\right\} = -\theta \quad 1 - \frac{r}{f/\omega^2} = e^{-\theta} \Rightarrow f - \frac{\omega^2 r}{1} = fe^{-\theta}$$

$$\omega^2 r = f - fe^{-\theta} = f(1-e^{-\theta}) \Rightarrow \boxed{\omega^2 r = f(1-e^{-\theta})}$$

rate of growth of radial velocity $= \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d^2r}{dt^2} = \omega^2 r - f = \cancel{f} - fe^{-\theta} - \cancel{f}$

$$\frac{d^2r}{dt^2} = -fe^{-\theta} = -\frac{f}{e^{\theta}}$$ Since $f$ and $e^{\theta}$ is always positive $\forall\,\theta$.

which never positive.

$$\lim_{\theta\to\infty}\frac{d^2r}{dt^2} = \lim_{\theta\to\infty}\left(-\frac{f}{e^{\theta}}\right) = 0$$