# Inverse of a Matrix

**Def:** (1) An $m \times n$ matrix $A$ is said to have matrix $A^L$ of order $n \times m$ as a left inverse if

$$\cancel{AA^L} =$$

$$A^L A = I_{n \times n}.$$

(2) $A$ is said to have $A^R$ as a right inverse if $AA^R = I_{m \times m}$.

**Note:** Left inverse or Right inverse of a matrix may not be unique.

**Example:**

$$A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}_{3 \times 2}$$

$$A^L = \frac{1}{9}\begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & 11 \end{bmatrix}, \quad A^L = \begin{bmatrix} 0 & -1/2 & 3 \\ 0 & 1/2 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$B^R = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B^R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B^R = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

## Inverse of a square matrix

Let $A_{n\times n}$ be a square matrix, The matrix $B_{n\times n}$ (if exist) is called inverse of $A$

if $AB = BA = I_{n\times n}$.

Usually $B$ is denoted by $A^{-1}$.

Example: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\delta = ad - bc \neq 0$

$$A^{-1} = \frac{1}{\delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

**Note:** Inverse of a square matrix (if exists) is unique.

Let $X_1$ and $X_2$ be inverses of square matrix $A$. Then

$$X_1 = X_1 I = X_1(AX_2) = (X_1A)X_2$$
$$= IX_2 = X_2.$$

**Prob:** Let $A$ be an $n \times n$ matrix. Suppose that there exists $n \times n$ matrices $B$ and $C$ such that

$$AB = I_n \quad \text{and} \quad CA = I_n$$

then $B = C$

**Proof:** $C = CI_n = C(AB) = (CA)B = I_nB$
$= B.$

**Result:** Let $A$ and $B$ be matrices with inverses $A^{-1}$ and $B^{-1}$ respectively.

Then

(1) $(A^{-1})^{-1} = A$

(2) $(AB)^{-1} = B^{-1}A^{-1}$

(3) $(A^t)^{-1} = (A^{-1})^t$

**Pf:** (3) $AA^{-1} = I$

$\Rightarrow (AA^{-1})^t = I^t = I$

$\Rightarrow (A^{-1})^t A^t = I$ — (A)

**Similarly:** $A^{-1}A = I$

$\Rightarrow (A^{-1}A)^t = I$

$\Rightarrow A^t (A^{-1})^t = I$ — (AA)

$\Rightarrow (A^t)^{-1} = (A^{-1})^t$ from (A) & (AA).

Recall :

## Determinant

Let $A$ be an $n \times n$ square matrix. We associate a number (real or complex) corresponding to $A$, called determinant, which is defined recursively as follows;

$$\det(A) = \begin{cases} a & \text{if } A = [a] \ (n=1) \\ \sum_{j=1}^{n} (-1)^{1+j} a_{ij} \det(A(i|j)) \end{cases}$$

(Expansion along 1st row)

Here, $A(i|j)$ is the matrix obtained from $A$ by deleting $i$th row and $j$th column.

Generally, determinant can be calculated by expanding along any row:

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det(A(k|j))$$

(Exercise) → Try atleast

Let $A$ be an $n\times n$ matrix.

<u>Def</u>: (1) $\det(A(i|j)) = A_{ij}$ is called $ij$th minor of $A$ or minor of $a_{ij}$.

(2) $C_{ij} = (-1)^{i+j}\det(A(i|j))$ is called $ij$th cofactor of $A$.

<u>Example</u>:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_{11} = \det(A(1|1)) = \det\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}\right) = 6.$$

$A_{12} = 4$, $A_{13} = 2$

$A_{21} = -4$, $A_{22} = -2$, $A_{23} = 0$

$A_{31} = -14$, $A_{32} = -10$, $A_{33} = -2$

$C_{11} = 6$ $\quad C_{12} = -4$ $\quad C_{13} = 2$

$C_{21} = 4$ $\quad C_{22} = -2$ $\quad C_{23} = 0$

$C_{31} = -14$, $\quad C_{32} = 10$ $\quad C_{33} = -2$

**Def:** (1) The matrix of minors of A is defined as $M = [A_{ij}]$.

(2) The cofactor matrix of A is defined as

$$C = [C_{ij}].$$

**Example**

$$M = \begin{bmatrix} 6 & 4 & 2 \\ -4 & -2 & 0 \\ -14 & -10 & -2 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & -4 & 2 \\ 4 & -2 & 0 \\ -14 & 10 & -2 \end{bmatrix}$$

**Def:** The Adjoint (or Adjugate) of A is defined as the transpose of cofactor matrix of A.

$$Adj(A) = C^t$$

Exercise: Find the Adjoint of $A$

where, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$.

Note:

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} A_{kj}$$

$$= \sum_{j=1}^{n} a_{kj} \underline{(-1)^{k+j} A_{kj}}$$

$$= \sum_{j=1}^{n} a_{kj} C_{kj}$$

$$= a_{k1} C_{k1} + a_{k2} C_{k2} + \cdots + a_{kn} C_{kn}$$

(Cofactor expansion of determinant).

<u>Thm:</u>

$$\sum_{j=1}^{n} a_{ij} C_{kj} = \begin{cases} \det(A) & \text{if } i=k \\ 0 & \text{if } i \neq k. \end{cases}$$

<u>Proof:</u> Let

$$A = \begin{array}{r} \\ \\ \\ k\text{th row} \rightarrow \\ \\ i\text{th row} \rightarrow \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

Let $B$ be a matrix, whose $k$-th row is same as $i$th row of $A$, and all other entries are same as $A$, i.e.

$$B = \begin{array}{r} \\ \\ k\text{th row} \rightarrow \\ \\ i\text{th row} \rightarrow \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{n1} & & \cdots & a_{nn} \end{bmatrix}$$

<u>Note that</u>: (1) if $k = i$ then $A = B$.

In this case;

$$\sum_{j=1}^{n} a_{ij} c_{ij} = \det(A) = \det(B).$$

# (2) If $k \neq i$, then $\det(B) = 0$.

$$\Rightarrow \det(B) = 0 = \sum_{j=1}^{n} b_{kj} (-1)^{k+j} \det(B(k|j)).$$

$$= \sum_{j=1}^{n} a_{ij} (-1)^{k+j} \det(B(k|j)).$$

Now, the submatrix;

$$B(k|j) = A(k|j)$$

Hence, $\det(B(k|j)) = \det(A(k|j))$

So, $$0 = \sum_{j=1}^{n} a_{ij} \underline{(-1)^{k+j} \det(A(k|j))}$$

$$\Rightarrow \boxed{\sum_{j=1}^{n} a_{ij} c_{kj} = 0}$$

#.

Thm: $A(\mathrm{Adj}(A)) = (\mathrm{Adj}(A))A = \det(A) I_n$.

Proof:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad \mathrm{Adj}(A) = \begin{array}{l} \\ \\ i^{th} \rightarrow \\ \\ \end{array} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ & & & \\ c_{1i} & c_{2i} & \cdots & c_{ni} \\ \vdots & & & \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}$$

$$\left(A(\mathrm{Adj}(A))\right)_{ij} = \sum_{k=1}^{n} a_{ik} (\mathrm{Adj}(A))_{kj}$$

$$= \sum_{k=1}^{n} a_{ik} c_{jk}$$

But $\displaystyle\sum_{k=1}^{n} a_{ik} c_{jk} = \begin{cases} \det(A) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$

$$\Rightarrow A(\mathrm{Adj}(A)) = \begin{bmatrix} \det(A) & 0 & 0 & \cdots & 0 \\ 0 & \det(A) & 0 & \cdots & 0 \\ \vdots & & & & \\ & & & & \det(A) \\ 0 & 0 & & & \end{bmatrix}$$

$$\Rightarrow \boxed{A(\mathrm{Adj}(A)) = \det(A) I_n}$$

Similarly,

$$(\mathrm{Adj}(A))A = \det(A) I_n.$$

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## Finding Inverse

(1) Let $\det(A) \neq 0$. ~~then~~

$$A(\mathrm{Adj}(A)) = (\mathrm{Adj}(A))A = \det(A) I_n$$

$$\Rightarrow \boxed{A^{-1} = \frac{1}{\det(A)} \mathrm{Adj}(A)} \quad (*)$$

Assume $\det(A) \neq 0$.

(2) Replace $A$ by $A^{-1}$ in $(*)$

$$A = \det(A)(\mathrm{Adj}(A^{-1})).$$

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)} \text{ if } \det(A) \neq 0.}$$

**Exercise:** Compute $Adj(A)$ and $A^{-1}$

for $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$.

**Exercise:** $A$ is invertible iff $\det(A) \neq 0$

$$A A^{-1} = I$$

$$\det(A A^{-1}) = 1$$

$$\det(A) \det(A^{-1}) = 1$$

$$\Rightarrow \det(A) \neq 0$$

If $\det(A) \neq 0$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} Adj(A)$$

**Exercise:** Show that $A$ is invertible iff $Adj(A)$ is invertible and that if $A$ is invertible then

$$\left(Adj(A)\right)^{-1} = \frac{A}{\det(A)} = Adj(A^{-1}).$$

$$\frac{A(Adj(A))}{\det(A)} = I$$

$$\Rightarrow (Adj(A))^{-1} = \frac{A}{\det(A)}$$

<u>Recall</u>: $A = \det(A)\, Adj(A^{-1})$

$$\Rightarrow \boxed{(Adj(A))^{-1} = Adj(A^{-1})}$$

<u>Ex</u>: Let $A$ be an $n \times n$ invertible matrix.

Prove that (1) $\det(Adj(A)) = \det(A)^{n-1}$

(2) $Adj(Adj(A)) = \det(A)^{n-2} A$

(1) $A(Adj(A)) = \det(A) I_n$

$\Rightarrow \det(A) \cdot \det(Adj(A)) = (\det(A))^n$

$\Rightarrow \boxed{\det(Adj(A)) = \det(A)^{n-1}}$

(2) $\det(Adj(A)) = \det(A)^{n-1}$ from (1)

$A(Adj(A)) = \det(A) I_n$

Replace, $A$ by $Adj(A)$.

$\Rightarrow Adj(A)(Adj(Adj(A))) = \det(Adj(A)) I_n$

$\det(A) \Rightarrow A\, Adj(A)(Adj(Adj(A))) = \cancel{A \det(} A \det A^{n-1} I_n$ #.