Interpolation

Content

- Lagrange's Interpolation
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- Newton Interpolation Formula

Divided Differences

Let $f(x_0)$, $f(x_1)$, …… $f(x_n)$ be the values of the function $y=f(x)$ corresponding to the values $x_0, x_1, \ldots\ldots\ldots x_n$ of the argument x. where, $x_1 - x_0$, $x_2 - x_1$ …… $x_n - x_{n-1}$ are not necessarily equal i.e. when there is a case of unequal intervals.

The first divided difference of f(x) for the argument x_0, x_1 is denoted by $f[x_0, x_1]$ or $\underset{x_1}{\Delta} f(x_0)$

so $$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Divided Differences (cont…)

Similarly the other divided difference of f(x) for the argument x_1, x_2, x_3…..x_{n-1} are

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\vdots$$

$$f[x_{n-1}, x_n] = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

This is called first divided difference,

Again the second and higher divided differences are defined in terms of lower divided differences.

For example the second divided difference of f(x) for the arguments (x_0, x_1, x_2) is given by

Divided Differences (cont…)

Again the second and higher divided differences are defined in terms lower divided differences,

example the second divided difference of f(x) for the arguments (x_0, x_1, x_2) is given by

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \underset{x_1, x_2}{\Delta}^{2} f(x_0)$$

Divided Differences (cont…)

Similarly, the n^th divided difference is given by

$$f[x_0, x_1, \ldots\ldots x_n] =$$

$$\frac{f[x_1, x_2 \ldots. x_n] - f[x_0, x_1 \ldots.. x_{n-1}]}{x_n - x_0} = \underset{x_1, x_2 \ldots. x_n}{\overset{n}{\Delta}} f(x_0)$$

Example

Given the values

x	f(x)
0	2
1	3
2	12
5	147

Prepare Newton's divided difference table.

Solution

$x0=0$, $x1=1$, $x2=2$, $x3=5$

$f(x0)=2$, $f(x1)=3$, $f(x2)=12$, $f(x3)=147$

Divided difference table:

x	f(x)	Δ	Δ^2	Δ^3
0	2	(3-2)/1-0)=1	(9-1)/(2-0) =4	(9-4)/(5-0)=1
1	3	(12-3)/(2-1)=9	(45-9)/(5-1) =9	
2	12	(147-12)/(5-2) =45		
5	147			

Newton's formula for unequal intervals
or
Newton's Divided Differences Interpolation Formula

Let $f(x_0)$, $f(x_1)$, …… $f(x_n)$ be the values of the function $y=f(x)$ corresponding to the values x_0, x_1, ……….x_n of the argument x not necessarily equally.

Let the polynomial of degree n be

$$P_n(x) = a_0 + (x - x_0)a_1 +$$
$$(x - x_0)(x - x_1)a_2 + (x - x_0)(x - x_1)(x - x_2)a_3 + \ldots\ldots\ldots$$
$$\ldots\ldots\ldots + (x - x_0)(x - x_1)\ldots\ldots\ldots\ldots(x - x_{n-1})a_n \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now it should fit the data $P_n(x_i) = f(x_i)$

Newton's formula for unequal intervals
or
Newton's Divided Differences Interpolation Formula

For $x=x_0$

$P_n(x_0)= f(x_0)$

Now putting this in equation (1)

$a_0= f(x_0)$

Now at $x=x_1$

$$P_n(x_1) = f(x_1) = a_0 + (x_1 - x_0)a_1$$

$$\Rightarrow a_1 = \frac{f(x_1) - a_0}{(x_1 - x_0)}$$

$$As, a_0 = f(x_0)$$

$$a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = f[x_0, x_1]$$

Newton's formula for unequal intervals
or
Newton's Divided Differences Interpolation Formula (Cont..)

Now at $x = x_2$

$$P_n(x_2) = f(x_2) = a_0 + (x_2 - x_0)a_1 + (x_2 - x_0)(x_2 - x_1)a_2$$

$$\Rightarrow a_2 = \frac{f(x_2) - a_0 - (x_2 - x_0)a_1}{(x_2 - x_0)(x_2 - x_1)}$$

Newton's formula for unequal intervals
or
Newton's Divided Differences Interpolation Formula (Cont..)

$$As, a_0 = f(x_0)\ldots.and\ldots.a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$$

$$\Rightarrow a_2 = \frac{f(x_2) - f(x_0) - (x_2 - x_0)\dfrac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_1)}{(x_0 - x_1)(x_0 - x_2)}$$

$$= f[x_0, x_1, x_2]$$

$Now, By\ldots Induction$

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$$a_n = f[x_0, x_1, x_2\ldots.x_n]$$

Newton’s formula for unequal intervals
or
Newton’s Divided Differences Interpolation Formula (Cont..)

Now from equation (1)

$$P_n(x) = f[x_0] + (x - x_0) f[x_0, x_1] +$$
$$(x - x_0)(x - x_1) f[x_0, x_1, x_2] + \ldots. + (x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1}) f[x_0, x_1, \ldots. x_n]$$
$$\ldots\ldots\ldots(2)$$

So form the divided difference table and place the values in equation (2) to get the required polynomial.

Example

Given the values

x	f(x)
0	1
1	3
3	55

Find the polynomial of the lowest possible degree using Newton's divided difference interpolation.

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Solution

$x_0=0$, $x_1=1$, $x_2=3$

$f(x_0)=1$, $f(x_1)=3$, $f(x_2)=55$

Divided difference table:

x	f(x)	Δ	$Δ^2$
0	1	(3–1)/1–0)=2	(26–2)/(3–0) =8
1	3	(55–3)/(3–1)=26	
3	55		

Solution (cont..)

x	f(x)	Δ	Δ^2
0	1	(3-1)/1-0)=2	(26-2)/(3-0) =8
1	3	(55-3)/(3-1)=26	
3	55		

The Newton divided difference interpolating polynomial becomes

$$P_2(x) = f(0) + (x-0)f[0,1] + (x-0)(x-1)f[0,1,3]$$
$$= 1 + 2x + x(x-1)8$$
$$= 8x^2 - 6x + 1$$

Practice Problem

Given the values

x	**f(x)**
0	2
1	3
2	12
5	147

Find the polynomial of the lowest possible degree using Newton's divided difference interpolation.

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Suggested books

1. Numerical Methods by **S.R.K Lyenger & R.K. Jain.**

2. Numerical Analysis by **Richard L. Burden.**

3. Introductory methods of Numerical analysis by **S.S. Sastry**.

Thank you