B.Sc. Mathematics – 2nd Semester

MTB 202 – Statics and Dynamics

by

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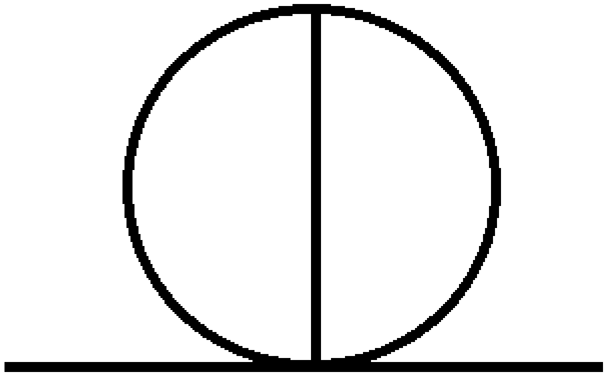
<u>Part – VI</u>

Motion on Smooth Vertical Curves

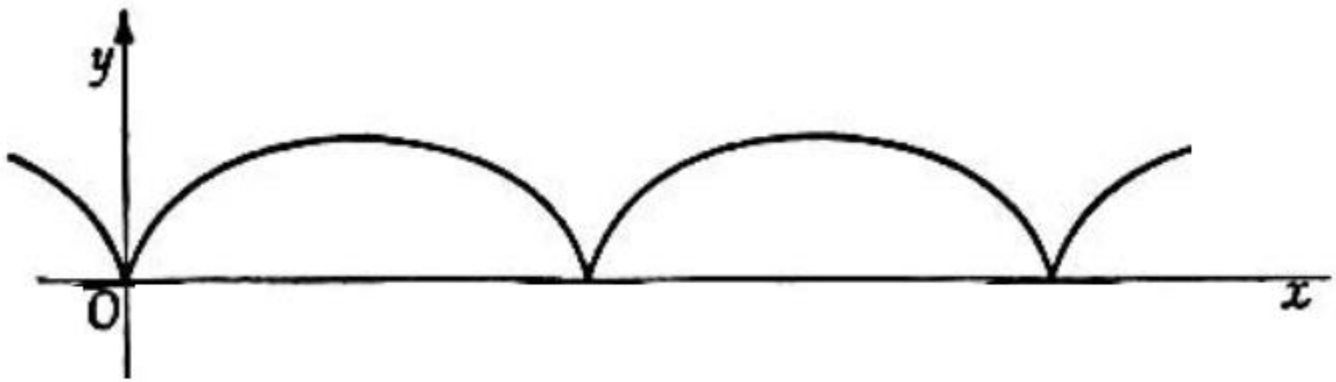
Circle



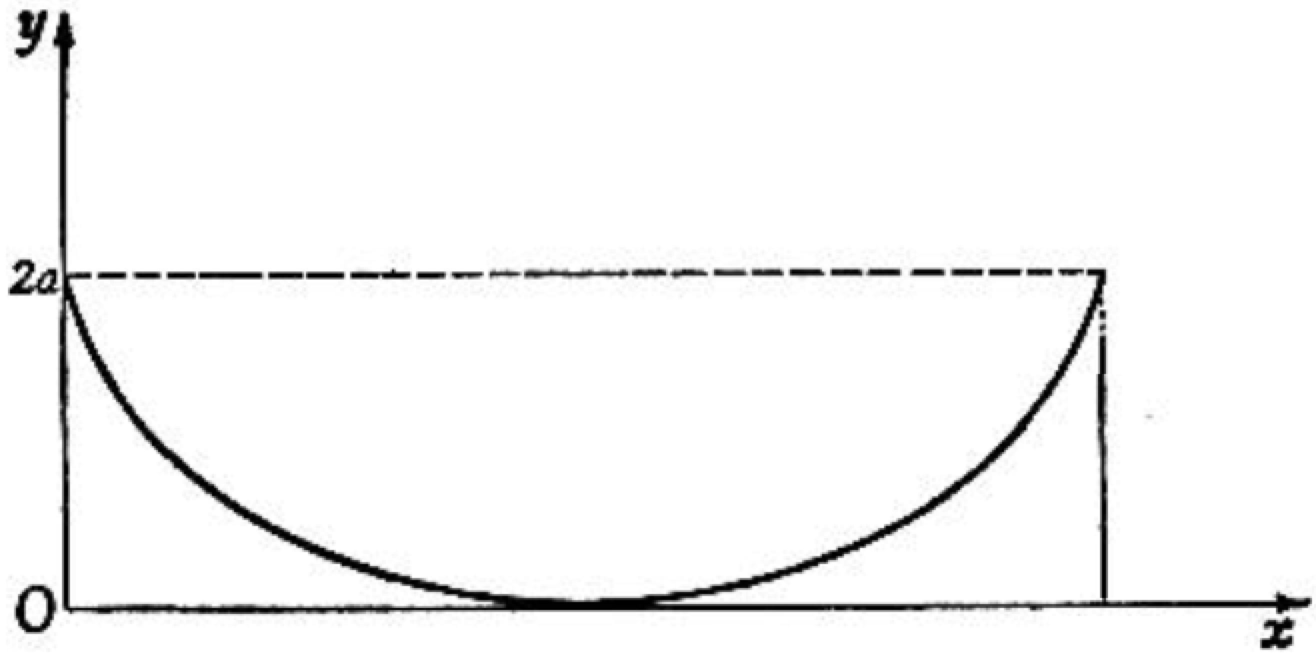
Parametric equation: $x = a\cos\theta, y = a\sin\theta;$

Intrinsic equation: $s = a\psi$

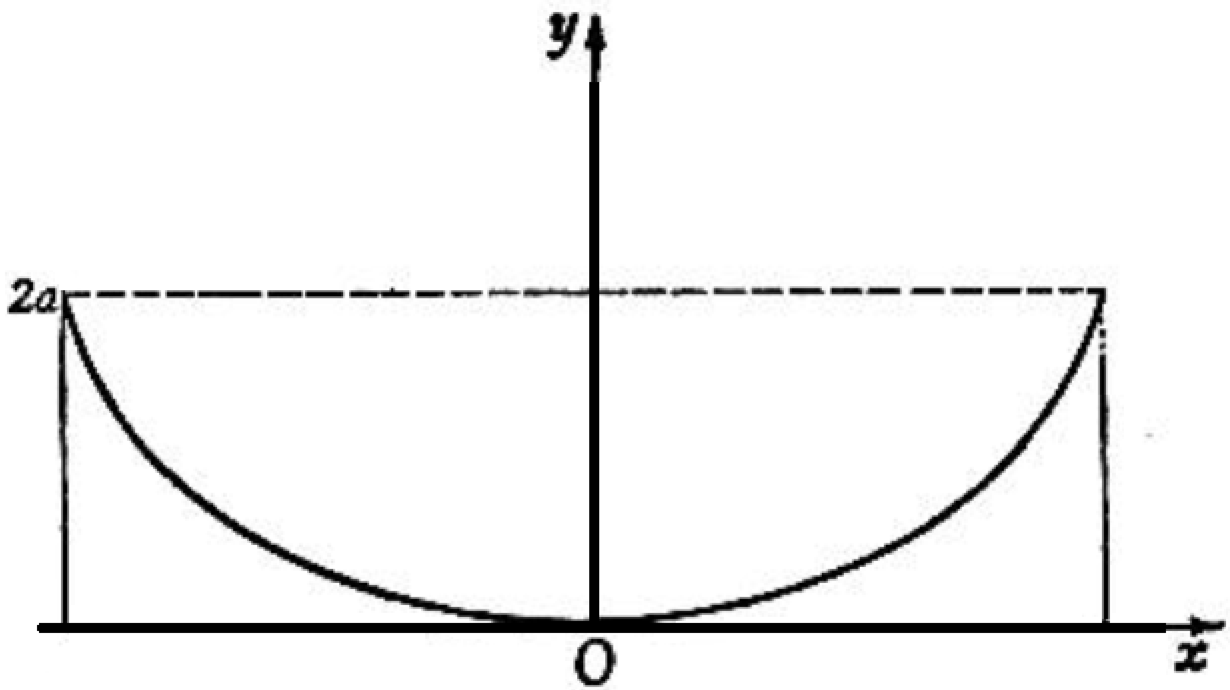
Cycloid



Parametric equation: $x = a(t - \sin t)$ and $y = a(1 - \cos t)$



Parametric equation: $x = a(t - \sin t)$ and $y = a(1 + \cos t)$



Parametric equation: $x = a(t + \sin t)$ and $y = a(1 - \cos t)$

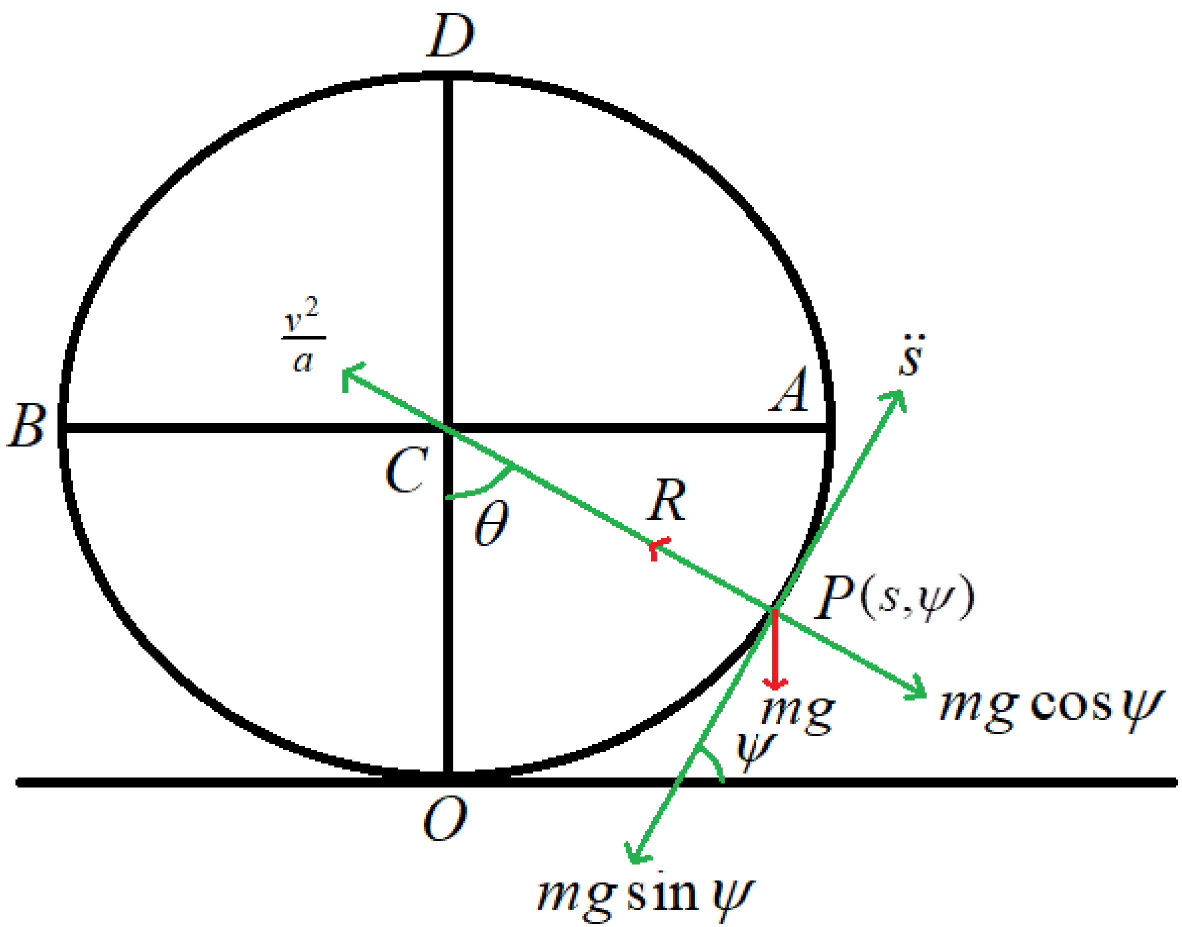
Intrinsic equation: $s = 4a\sin\psi$; with $s^2 = 8ay$

I) Vertical Motion on a Smooth Circle

A. ***Downward motion along the inner surface:***

Let the motion occur towards the bottom point from one end of a horizontal diameter of a circle of radius a and under gravity only. The other force on the particle will be the reaction of the surface in the direction perpendicular to the tangent, i.e., along inward normal.

Let the particle of mass m be moving in the inner surface of the circle $OADB$ with the centre at C and let at time t $P(s,\psi)$ be the position of the particle. Then $OP=s$ and $s = a\psi$.


D
$\frac{v^2}{a}$
$\ddot{s}$
A
B
C
θ
R
$P(s,\psi)$
mg
$mg\cos\psi$
ψ
O
$mg\sin\psi$

Then the equation of motion is

$$m\ddot{s} = -mg\sin\psi \text{ and } m\frac{v^2}{a} = R - mg\cos\psi,$$

where R is the reaction.

So, we have $\ddot{s} = -g\sin\psi$ and $\dfrac{v^2}{a} = \dfrac{R}{m} - g\cos\psi$

If the radius CP makes an angle θ with the vertical diameter OD, then we have $\psi = \theta$ and $s = a\theta$. Then the equation becomes

$$\ddot{\theta} = -\frac{g}{a}\sin\theta \text{ .... (1)} \quad \text{and} \quad \frac{v^2}{a} = \frac{R}{m} - g\cos\theta \text{ .... (2)}$$

Integrating (1) we get

$$\dot{\theta}^2 = 2\frac{g}{a}\cos\theta + c_1 \Rightarrow \left(a\dot{\theta}\right)^2 = 2ag\cos\theta + c_2 \Rightarrow v^2 = \dot{s}^2 = 2ag\cos\theta + c_2$$

Now, at $t = 0, \theta = \pi/2, v = 0 \Rightarrow c_2 = 0$

So, $v^2 = 2ag\cos\theta$ $\quad or, \left(a\dot{\theta}\right)^2 = 2ag\cos\theta$.

From (2),

$$2g\cos\theta = \frac{R}{m} - g\cos\theta \Rightarrow R = 3mg\cos\theta.$$

So, at the starting point the reaction is zero.

Now, if the time taken from $\theta = \dfrac{\pi}{2}$ to $\theta = 0$ is t_1, then

$$\int_0^{t_1} dt = -\sqrt{\frac{a}{2g}} \int_{\theta=\frac{\pi}{2}}^{\theta=0} \sqrt{\sec\theta}\, d\theta$$

[$v = a\dot{\theta} = \pm\sqrt{2ag\cos\theta}$. For this case $a\dot{\theta} = -\sqrt{2ag\cos\theta}$, as t increases θ decreases, i.e., the motion along θ decreasing direction.]

Now, if the particle starts from a position very close to the vertex O, i.e., θ is very small, then we can write $\sin\theta \approx \theta$ then the equation (1) becomes

$$\ddot{\theta} = -\frac{g}{a}\theta$$

This represents a S.H.M about the bottom point with time period

$$\frac{2\pi}{\sqrt{g/a}}, \text{ i.e., } 2\pi\sqrt{\frac{a}{g}}.$$

B. *Upward motion along the inner surface*:

Let the particle be projected from the vertex $O\left(s=0,\theta=0\right)$ with velocity u. The equations of motion are same as previous,

i.e., $m\ddot{s}=-mg\sin\psi$, $m\dfrac{v^2}{a}=R-mg\cos\psi$

i.e., $\ddot{\theta}=-\dfrac{g}{a}\sin\theta$ and $m\dfrac{v^2}{a}=R-mg\cos\theta$

Hence, we have $v^2=2ag\cos\theta+c$.

Now, at $\theta = 0, v = u \Rightarrow c = u^2 - 2ag$.

So, $v^2 = u^2 - 2ag\left(1 - \cos\theta\right)$. Therefore $R = m\left[\dfrac{u^2}{a} - 2g + 3g\cos\theta\right]$

Let at $\theta = \theta_1, u = 0$ and $\theta = \theta_2, R = 0$.

Now $v = 0 \Rightarrow \cos\theta_1 = 1 - \dfrac{u^2}{2ag}$ and $R = 0 \Rightarrow \cos\theta_2 = \dfrac{2}{3}\left(1 - \dfrac{u^2}{2ag}\right) = \dfrac{2}{3}\cos\theta_1$

Thus for $u^2 < 2ag, \cos\theta_1 > \cos\theta_2$ or $\theta_1 < \theta_2$.

These all are true up to $\theta = \pi/2$

So, $v = 0$ before $R = 0$ (up to $\theta = \pi/2$)

It means when particle comes at rest it remains in contact with the surface. Hence, it will start coming down. It will reach the lowest point will maximum velocity and will move on the other side up to same angle and again return back. Thus the particle starts oscillating about lowest point $s = 0$ or $\theta = 0$

Thus if the particle is projected velocity is less than square root of $2ag$, then it will come at rest at some position before A and the particle remains in contact with the surface.

In case $u^2 = 2ag, \cos\theta_1 = \cos\theta_2 = 0, \theta_1 = \theta_2 = \pi/2$

The particles come at rest and its contact with the surface at end A of horizontal diameter also vanishes. From this position it comes down and starts oscillating.

Now, when $\underline{u^2 > 2ag}$. Also, let at the top the particle be at rest.

Then $0 = u^2 - 2ag(1 - \cos\pi) \Rightarrow u^2 = 4ag$

Thus if the particles projected velocity is equal to $\sqrt{4ag}$ then it comes at rest at top position.

Then $R = m\left[2g + 3g\cos\theta\right]$ and it will be zero at

$$\theta = \cos^{-1}\left(-2/3\right) = +131.8^\circ.$$

It means for $u^2 = 4ag$ the contact of the particle with the surface vanishes before the top position.

Thus, the reaction become zero earlier to the linear velocity between $A\left(\theta = \pi/2\right)$ and $D(\theta = \pi)$ if $2ag < u^2 \le 4ag$

If the reaction at the top is zero, then

$$0 = m\left[\frac{u^2}{a} - 2g + 3g\cos\pi\right] \Rightarrow u^2 = 5ag$$

It means if the particle reaches the top then it's projected velocity is equal to $\sqrt{5ag}$.

From it one may conclude that if $u^2 > 5ag$ then particle will move on the other side from the top position and will then complete one round.

Then the reaction of the bottom position is given by

$$R > m\left[\frac{5ag}{a} - 2g + 3g\right] \text{ or } R > 6amg$$

Further if $u^2 > 5ag$, let $u^2 = 6ag$ then

$$v^2 = 4ag + 2ag\cos\theta$$

$$R = \frac{m}{a}\left(4ag + 3ag\cos\theta\right)$$

So, at $\theta = \frac{\pi}{3}$, $v^2 = 5ag$ and $R = 5.5mg$

$\theta = \frac{\pi}{2}$, $v^2 = 4ag$ and $R = 4mg$

$\theta = \frac{2\pi}{3}$, $v^2 = 3ag$ and $R = 2.5mg$

$\theta = \pi$, $v^2 = 2ag$ and $R = mg$.

Thus one may observed that the velocity and the reaction on the particle, both decrease up to $\theta = \pi$. Beyond that both again increase and at the lowest are

$$v^2 = 4ag + 2ag\cos 2\pi = 6ag = u^2 \quad R = \frac{m}{a}\left[4ag + 3ag\cos 2\pi\right] = 7mg.$$

Time of coming at position of no contact when projected with $u^2 = 4ag$

If θ_1 be angle at the point of no contact, then

$$m\left[\frac{4ag}{a} - 2g + 3g\cos\theta_1\right] = 0 \Rightarrow \cos\theta_1 = -\frac{2}{3}. \text{ So, } \theta_1 = \cos^{-1}\left(-\frac{2}{3}\right).$$

We have $v^2 = 4ag - 2ag\left(1 - \cos\theta\right)$

$$\Rightarrow \left(a\dot{\theta}\right)^2 = 2ag\left(1 + \cos\theta\right) = 4ag\cos^2 \theta/2$$

$$\Rightarrow a\dot{\theta} = +2\sqrt{ag}\cos \theta/2 \text{ [as } t \text{ increases } \theta \text{ decrease]}$$

$$\Rightarrow \frac{\dot{\theta}}{\cos\frac{\theta}{2}} = \sqrt{\frac{4g}{a}}$$

If T be the required time, then

$$T\sqrt{\frac{4g}{a}} = \int_0^{\theta_1} \frac{d\theta}{\cos\left(\theta/2\right)} = 2\left[log\left(\sec\theta/2 + \tan\theta/2\right)\right]_0^{\theta_1}$$

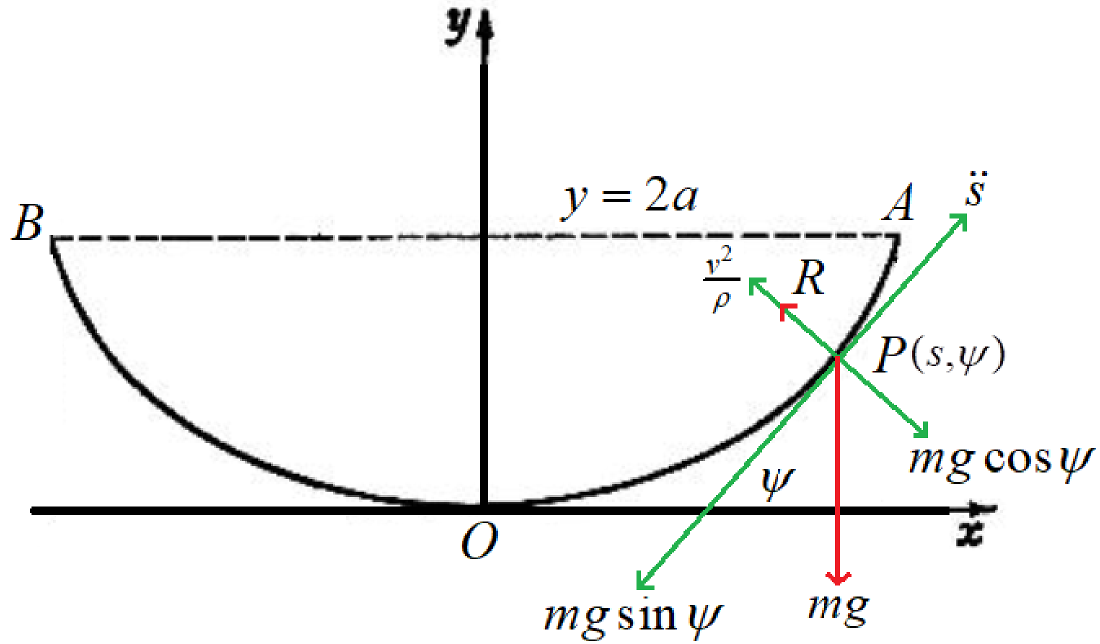
$$\therefore T = \sqrt{\frac{a}{g}}\log\left(\sec\theta_1/2 + \tan\theta_1/2\right)$$

$$\text{i.e., } T = \sqrt{\frac{a}{g}}\log\left(\sqrt{6}+\sqrt{5}\right).$$

II) <u>Motion on a vertical smooth cycloid</u>

A. *Downward motion along the inner surface*:

Let a particle be moved along the inner surface in a direction from cusp $A(s=4a)$ towards the vertex $O(s=0)$ of the cycloid $s=4a\sin\psi$.



Then the equations of motion are

$$m\ddot{s} = -mg\sin\psi \quad \text{and} \quad m\left(\frac{v^2}{\rho}\right) = R - mg\cos\psi$$

$$\Rightarrow \ddot{s} = -g\sin\psi \quad \ldots\ldots..(1) \quad \text{and} \quad \frac{v^2}{\rho} = \frac{R}{m} - g\cos\psi \quad \ldots\ldots..(2)$$

We have from (1) and the equation of cycloid

$$\ddot{s} = -\frac{g}{4a}s,$$

which shows that the motion of the particle is SHM about the vertex O of the time period $T = 2\pi\sqrt{\frac{4a}{g}} = 4\pi\sqrt{\frac{a}{g}}$.

Its general solution may be written as $s = c_1 \cos\sqrt{\frac{g}{4a}}t + c_2 \sin\sqrt{\frac{g}{4a}}t$.

Also we have $v^2 = \dot{s} = -\frac{g}{4a}s^2 + c_3$.

Using $t = 0, s = 4a$ and $t = 0, \dot{s} = 0$, we get $c_2 = 0, c_1 = 4a, c_3 = +4ag$

Thus $s = 4a\cos\sqrt{\frac{g}{4a}}t$ and $\dot{s} = v = -2\sqrt{ag}\sin\sqrt{\frac{g}{4a}}t$.

Also, $\dot{s}^2 = v^2 = \frac{g}{4a}\left(16a^2 - s^2\right) = \frac{g}{4a}\left(16a^2 - 16a^2\sin^2\psi\right)$

i.e., $v^2 = 4ag\cos^2\psi \Rightarrow v = -2\sqrt{ag}\cos\psi$

$\left[v_{\max} = 2\sqrt{ag} \text{ at vertex } s = 0, \psi = 0\right]$

and $R = m\left[\dfrac{v^2}{\rho} + g\cos\psi\right]$

$$= m\left[\frac{4ag\cos^2\psi}{4a\cos\psi} + g\cos\psi\right] \quad \left[\text{as } \rho = \frac{ds}{d\psi} = 4a\cos\psi\right]$$

$$= m\left[g\cos\psi + g\cos\psi\right]$$

i.e., $R = 2mg\cos\psi$

This the reaction at $P(s,\psi)$.

Time period for describing different arc lengths

At first the time in reacting the vertex.

Let it be T_1. Then

$$0 = 4a\cos\sqrt{\frac{g}{4a}}T_1 \Rightarrow T_1 = \frac{\pi}{2}\sqrt{\frac{4a}{g}} = \pi\sqrt{\frac{a}{g}} = \frac{T}{4}.$$

Let the time required in coming down to a position $s = 2a, \psi = \frac{\pi}{3}$, i.e., half of the arc length OA be T_2. Then we have

$$2a = 4a\cos\sqrt{\frac{g}{4a}}T_2 \Rightarrow T_2 = \frac{\pi}{3}\sqrt{\frac{4a}{g}} = \frac{2\pi}{3}\sqrt{\frac{a}{g}}.$$

Again, let the time required in coming to the position whose vertical height is a be T_3. Then we have $s^2 = 8ay$. So, $s^2 = 8a^2 \Rightarrow s = 2\sqrt{2}a$

and $2\sqrt{2}a = 4a\sin\psi \Rightarrow \psi = \dfrac{\pi}{4}$

Hence, $2\sqrt{2}a = 4a\cos\sqrt{\dfrac{g}{4a}}T_3 \Rightarrow T_3 = \dfrac{\pi}{4}\sqrt{\dfrac{4a}{g}} = \dfrac{\pi}{2}\sqrt{\dfrac{a}{g}}$.

If T_4 be time taken by the particle in moving from $s = 2\sqrt{2}a$ to s=0, then

$T_4 = T_1 - T_3 = \dfrac{\pi}{2}\sqrt{\dfrac{a}{g}}$.

So, $T_3 = T_4$

Angular Velocity at a point, $\dot{\psi}$

We have $\dot{s} = \frac{ds}{dt} = \frac{ds}{d\psi}\frac{d\psi}{dt} = \rho\dot{\psi}$

So, $\dot{\psi} = \frac{\dot{s}}{\rho} = \frac{v}{\rho} = \frac{\sqrt{4ag}\cos\psi}{4a\cos\psi} = \sqrt{\frac{g}{4a}}$.

Change initial velocity

Let the particle start from $s = 4a$ with velocity u, then the initial condition will be $t = 0, s = 4a$ and $t = 0, \dot{s} = -u$, where negative sign is assigned as the motion occurs in s decreasing.

Then we have $c_1 = 4a$ and $c_2 = -u\sqrt{\frac{4a}{g}}$

Thus, $s = 4a\cos\sqrt{\frac{g}{4a}}t - u\sqrt{\frac{4a}{g}}\sin\sqrt{\frac{g}{4a}}t$

If the time of reaching the vertex is T_5, then

$$0 = 4a\cos\left(\sqrt{\frac{g}{4a}}T_5\right) - u\sqrt{\frac{4a}{g}}\sin\left(\sqrt{\frac{g}{4a}}T_5\right)$$

$$\Rightarrow \tan\left(\sqrt{\frac{g}{4a}}T_5\right) = \frac{4a}{u}\sqrt{\frac{g}{4a}} = \frac{\sqrt{4ag}}{u} \Rightarrow T_5 = \sqrt{\frac{4a}{g}}\tan^{-1}\left(\frac{\sqrt{4ag}}{u}\right).$$

Another initial condition

Let the particle be started from rest position from $s = b < 4a$

Then we have $s = b\cos\sqrt{\dfrac{g}{4a}}t$ and $\dot{s} = -b\sqrt{\dfrac{g}{4a}}\sin\sqrt{\dfrac{g}{4a}}t$.

Simultaneous motion of two particles

Let two particles start motion from the cusp from rest position at the interval of time t^*. [Then to find the time when both will meet]

If s_1 be the arc travelled by the first particle after time t then

$$s_1 = 4a\cos\sqrt{\frac{g}{4a}}t$$

and s_2 be that by the second particle then

$$s_2 = 4a\cos\sqrt{\frac{g}{4a}}\left(t-t^*\right).$$

At a position where both the particle meet, we have $s_1 = s_2$

i.e., $4a\cos\sqrt{\frac{g}{4a}}t = 4a\cos\sqrt{\frac{g}{4a}}\left(t-t^*\right) \Rightarrow \sqrt{\frac{g}{4a}}t = 2\pi - \sqrt{\frac{g}{4a}}\left(t-t^*\right)$

$$\Rightarrow \sqrt{\frac{g}{a}}t = 2\pi + \sqrt{\frac{g}{4a}}t^* \Rightarrow t = 2\pi\sqrt{\frac{a}{g}} + \frac{1}{2}t^*$$

$$\Rightarrow t = \frac{T}{2} + \frac{t^*}{2} = \frac{1}{2}\left(T + t^*\right)$$

B. *Upward motion along the inner surface:*

Let the particle be projected from the vertex $(s=0)$ with velocity u along the inner surface towards the cusp $A(s=4a)$

Let at time t $P(s,\psi)$ be position of the particle. Then

$$m\ddot{s} = -mg\sin\psi \text{ and } m\frac{v^2}{\rho} = R - mg\cos\psi$$

For the cycloid we have

$$\ddot{s} = -\frac{g}{4a}s \quad [\text{as } s = 4a\sin\psi]$$

$$\dot{s}^2 = v^2 = -\frac{g}{4a}s^2 + c_4 \Rightarrow v^2 = u^2 - \frac{g}{4a}s^2 [\text{ using } \dot{s} = u \text{ at } s = 0]$$

So, the motion of the particle will be a SHM with period $4\pi\sqrt{\frac{a}{g}}$ and amplitude S. The arc length S will depend on u.

If the particle reaches up to cusp and comes at rest there, then $S = 4a,\ v = 0$. Then $u^2 = 4ag$.

And if the particle reaches up to the position whose vertical height is a, then $s = 2\sqrt{2}a,\ v = 0$ (and $S = 2\sqrt{2}a$), which implies $u^2 = \frac{g}{4a}8a^2 = 2ag$.

The expression of R at a point is

$$R = mg\cos\psi + m\frac{1}{\rho}\left(u^2 - \frac{g}{4a}s^2\right) = mg\cos\psi + \frac{m}{4a\cos\psi}\left(u^2 - \frac{g}{4a}s^2\right).$$

So, for $u^2 = 4ag$, the reaction at cusp, $\psi = \frac{\pi}{2}$ is

$$R = mg\cos\frac{\pi}{2} + \frac{m}{4a}(4ag - 4ag) = 0$$

and at $\psi = \frac{\pi}{4}$ $\left(s = 2\sqrt{2}a \text{ or } y = a\right)$

$$R = mg\cos\frac{\pi}{4} + \frac{m}{4a\cos\frac{\pi}{4}}\left(4ag - \frac{g}{4a}8a^2\right)$$

$$\Rightarrow R = \frac{mg}{\sqrt{2}} + \frac{m}{4a\frac{1}{\sqrt{2}}}(4ag - 2ag) = \frac{mg}{\sqrt{2}} + \frac{m}{\sqrt{2}} = \sqrt{2}mg.$$

i.e., the particle is in contact with the surface in all the positions below the cusp.

Time taken in describing certain arc length

We have the general solution as $s = c_1 \cos\sqrt{\frac{g}{4a}}t + c_2 \sin\sqrt{\frac{g}{4a}}t$

Using $t = 0, s = 0, \dot{s} = u$, we have $c_1 = 0$ and $c_2 = u\sqrt{\frac{4a}{g}}$

Thus $s = u\sqrt{\frac{4a}{g}}\sin\sqrt{\frac{g}{4a}}t$. Now, if T_1 be the time required to reach the cusp

when the particle is projected with $u = \sqrt{4ag}$, then we have

$$4a = \sqrt{4ag}\sqrt{\frac{4a}{g}}\sin\sqrt{\frac{g}{4a}}T_1 \Rightarrow 1 = \sin\sqrt{\frac{g}{4a}}T_1 \Rightarrow T_1 = \frac{\pi}{2}\sqrt{\frac{4a}{g}} = \pi\sqrt{\frac{a}{g}} = \frac{T}{4}.$$

***To get the intrinsic equation of cycloid and to prove** $s^2 = 8ay$*

We have $x = a(t + \sin t)$ and $y = a(1 - \cos t)$.

So, $\dfrac{dx}{dt} = a(1 + \cos t), \dfrac{dy}{dt} = a\sin t$

and $\dfrac{dy}{dx} = \dfrac{\sin t}{1 + \cos t} \Rightarrow \tan\psi = \dfrac{2\sin {}^t\!/_2 \cos {}^t\!/_2}{2\cos {}^t\!/_2}$ $\left[\dfrac{dy}{dx} = \tan\psi\right]$

$\Rightarrow \tan\psi = \tan {}^t\!/_2 \Rightarrow \psi = {}^t\!/_2$

$ds = \sqrt{dx^2 + dy^2} \Rightarrow ds = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt$

$$\Rightarrow ds = a\sqrt{(1+\cos t)^2 + \sin^2 t}\,dt = a\sqrt{2+2\cos t}\,dt$$

$$\Rightarrow ds = 2a\cos\frac{t}{2}dt \Rightarrow ds = (2a\cos\psi)(2d\psi) \quad [\text{as } \psi = \frac{t}{2}, d\psi = \frac{dt}{2}]$$

$$\Rightarrow ds = 4a\cos\psi\, d\psi$$

$\Rightarrow s = 4a\sin\psi$, the intrinsic equation of the cycloid.

Now, $y = a(1-\cos t) = a2\sin^2\frac{t}{2} = 2a\sin^2\psi$

$$\Rightarrow y = 2a\left(\frac{s}{4a}\right)^2 \Rightarrow y = \frac{s^2}{8a} \Rightarrow s^2 = 8ay. \quad \text{(Proved)}$$