Gauss Law for Dielectric Materials

Electrostatic field in the dielectric material is modified due to polarization and is not the same as in vacuum. Hence the Gauss law

$$\nabla E = \frac{\rho}{\varepsilon_0}$$

which is applicable in vacuum is reconsidered for dielectric media. It can be expressed in two forms- A) Integral 2) Differential - as follows

A) Integral Form of Gauss Law

(I) Consider two parallel-plate conductors having plane area S, separation d and vacuum between plates. Let charge +q and –q be the charges on the plates. Due to the charges, $\boldsymbol{E_0}$ is the uniform electric field directed from positive to negative plate (Fig. a).

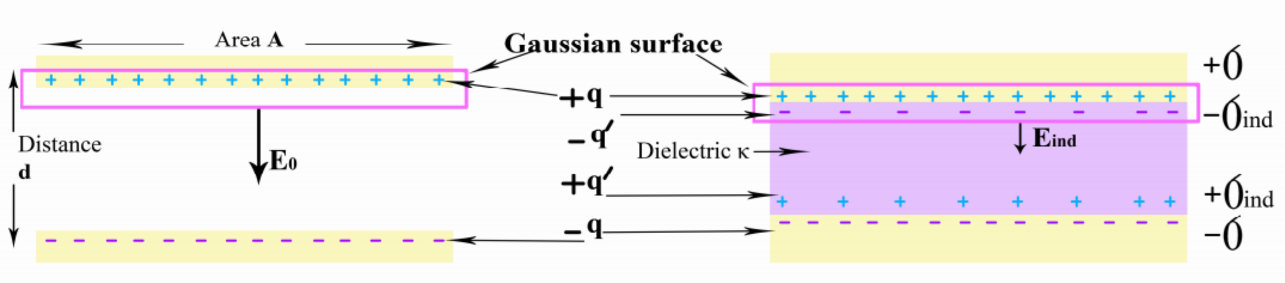

Fig. parallel-plate conductors (a) without dielectric (b) with dielectric

Consider the Gaussian surface around the upper conducting plate of positive charges. Applying Gauss's law the electric flux passing through the closed surface is given by

$\oint E\, ds = \frac{q}{\varepsilon_0}$ or $E_0 A = \frac{q}{\varepsilon_0}$

The field $E_0 = \frac{q}{A\varepsilon_0}$.

It is normal to the plate surfaces.

(II) Consider that a dielectric material of permittivity ε is filled completely between the plates (Fig. b). Charges –q' and +q' are induced on the surfaces of the dielectric that are in the proximity of the plates having charges q and -q respectively. The induced charges set up an electric field E in the dielectric. The dielectric is polarized. It remains as a whole electrically neutral as the positive induced surface charge must be equal to the negative induced surface charge.

If the dielectric is present, the surface encloses two types of charge:

Free charge on the upper conducting plate is q and

Induced charge on the top face of dielectric due to polarization is -q'

The net charge enclosed by the Gaussian surface around the (same upper conducting) plate (of positive charges +q) is q-q'.

According to Gauss's law

$\oint E\, ds = \frac{1}{\varepsilon_0}(q - q')$

$E \cdot A = \frac{1}{\varepsilon_0}(q - q')$

Or $E = \frac{q}{A \cdot \varepsilon_0} - \frac{q'}{A \cdot \varepsilon_0}$

Field **E** in the dielectric is in the opposite direction to that of the applied electric field E_0. The effect of the dielectric is to weaken the original field by the factor $k = \varepsilon/\varepsilon_0$.

$$\frac{E}{E_0} = \frac{1}{k}$$

Or

$$E = \frac{E_0}{k}$$

$$E = \frac{q}{k\varepsilon_0 A}$$

$$q' = q(1 - \frac{1}{k})$$

The magnitude of the net induced charge q` is always less than magnitude of the free charge q applied to the plates and is equal to zero if dielectric is absent.

$$\oint E\, ds = \frac{q}{\varepsilon_0 k}$$

Or $\oint E\, ds = \frac{q}{\varepsilon}$

i.e. $\oint D\, ds = q$ Where $D = \varepsilon E = \varepsilon_0 k E$.

D is called as the displacement vector. The induced surface charge is purposely ignored on the right side of this equation, since it is taken into account fully by introducing the dielectric constant k on the left side.

The equation states that "the surface integral of displacement vector 'D' over a closed surface is equal to the free charge enclosed within the surface" or "The outward flux of D over any closed surface S equals the algebraic sum of the free charges enclosed by S"

This important equation, although derived for parallel plate conductors, is true in general.It is the most general form of Gauss' law. The charge q enclosed by the Gaussian surface is the free charge only, which can be controlled and measured. Hence this form of Gauss law is very useful.

B) Differential Form of Gauss Law

Consider a dielectric material kept in an electric field is polarized. It has bound or polarization charge density ρ_b due to accumulation of bound charges $\rho_b = -\nabla P$.

The applied electric field itself is created by transferring electric charges.They are called as free charges (charges brought from outside e.g. conduction electrons in metals).They give rise to ρf the charge density due to free charges i.e. which is not due to polarization. The total charge density ρ consists of two parts as follows $\rho = \rho_f + \rho_b$.

According to Gauss' law in differential form

$$\nabla E = \frac{\rho}{\varepsilon_0} = \frac{\rho_f + \rho_b}{\varepsilon_0}$$

E is the total electric field due to both types (bound and free) charges.

and Rearranging the expression and substituting

$$D = \varepsilon E + P \text{ .}$$

This is the differential form of Gauss Law in dielectrics. **D** is termed as the electric displacement. **D** has the same dimensions as (dipole moment per unit volume). Above law can be written in terms of **E** using relation $\nabla \varepsilon E = \rho_f$.

The other equation in electrostatics $\nabla \times E = 0$ remains unchanged in dielectrics.

According to the divergence theorem the differential form changes to integral form as

$$\oint_S D \, ds = \int_v \rho_f \, dV$$

The flux of **D** out of a closed surface **S** is equal to the total free charge enclosed within that surface. Thus the statement of Gauss's Law in integral form can be obtained from differential form. It can be derived from first principles also. (Refer to Integral form)