Stationary values under condi

Ex. 1. Let $F(x,y,z)$ is a function of three variables $x, y \& z$. subject to the constraint condition $G(x,y,z) = 0$. Show that at a stationary point $F_x G_y - F_y G_x = 0$.

Soln: We may consider z as a function of two independent variables x and y.

as a stationary point $dF = 0$

$$dF = F_x\,dx + F_y\,dy + F_z\,dz = 0 \quad \text{———} (1)$$

Now, differentiating $G(x,y,z) = 0$, we have

$$G_x\,dx + G_y\,dy + G_z\,dz = 0 \quad \text{———} (2)$$

Eliminating dz from (1) & (2),

$$F_x G_z\,dx + F_y G_z\,dy + \cancel{F_z G_z\,dz} = 0$$

$$-\;\; F_z G_x\,dx + F_z G_y\,dy + \cancel{F_z} G_z\,dz = 0$$

$$(F_x G_z - F_z G_x)\,dx + (F_y G_z - F_z G_y)\,dy = 0$$

$\because$ dx and dy are arbitrary,

$\therefore$ $F_x G_z - F_z G_x = 0$

$F_y G_z - F_z G_y = 0$

Eliminating G_z,

$$\cancel{F_x F_y G_z} - F_z F_y G_x = 0$$

$$-\;\; \cancel{F_x F_y G_z} \underset{+}{-} F_x F_z G_y = 0$$

$$\Rightarrow \boxed{F_y G_x - F_x G_y = 0} \text{ proved.}$$

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Ex 2. Find the stationary points of the function

a) xy^2z^2 subject to the conditions $x+y+z=6$, $x>0, y>0, z>0$

b) ~~$3x^2+4xy+6y^2=$~~ x^2+y^2 under the condition $3x^2+4xy+6y^2=140$

c) $x^2y^2z^2$ subject to the conditions $x^2+y^2+z^2=a^2$

d) xyz subject to the condition $\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{36}=1.$

Soln. a) Here, $G(x,y,z) = x+y+z-6=0$

and $F(x,y,z) = x^2y^2z^2$

$\therefore$ stationary point is

$$F_x G_y - F_y G_x = 0$$

$$\Rightarrow 2xy^2z^2 \cdot 1 - 2yx^2z^2 \cdot 1 = 0$$

$$\cancel{\Rightarrow 2xy}(yz^2 - xz^2) = 0$$

$$\Rightarrow xy^2 - yx^2 = 0$$

$$xy(y-x) = 0$$

$$\Rightarrow \boxed{y = x \checkmark \quad \text{or} \quad xy = 0.}$$

b) $G(x,y,z) = 3x^2+4xy+6y^2-140=0$

$F(x,y,z) = x^2+y^2$

$\therefore$ stationary points are

$$F_x G_y - F_y G_x = 0$$

$$\Rightarrow 2x\cdot(4x+12y) - 2y(6x+4y) = 0$$

$$\Rightarrow 8x^2+24xy - 12xy - 8y^2 = 0$$

$$\Rightarrow 8x^2+12xy-8y^2=0$$

$$\boxed{2x^2+3xy-2y^2=0}$$

(c) & (d) Do Yourself.

Lagrange's Undetermined Multipliers:

__Rule:__ To find stationary points of the ~~Define~~ ~~a~~ function

$$f(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m) \qquad (1)$$

of $n+m$ variables which are connected ~~with~~ by eq-ns

$$\phi_r(x_1, x_2, \ldots x_n, u_1, \ldots u_m) = 0, \quad r = 1, 2, \ldots m. \qquad (2)$$

1. Define a function

$$F = f + \lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_m\phi_m$$

(λ_i are multipliers)

and consider all the variables $x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m$ as independent.

At a stationary point of F, $dF = 0$.

$$\Rightarrow dF = \frac{\partial F}{\partial x_1}dx_1 + \frac{\partial F}{\partial x_2}dx_2 + \ldots + \frac{\partial F}{\partial u_m}du_m = 0$$

$$\therefore \frac{\partial F}{\partial x_1} = 0, \ldots \frac{\partial F}{\partial x_n} = 0, \quad \frac{\partial F}{\partial u_1} = 0, \ldots \frac{\partial F}{\partial u_m} = 0$$

Now, Stationary points of f may be found by determining stationary points of F, where $F = f + \lambda_1\phi_1 + \ldots + \lambda_m\phi_m$

A stationary point will be an extreme point of f if d^2F keeps the same sign and will be max/min as d^2F is negative or positive.

__Ex. 3.__ Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225$, $z = 0$.

__Soln:__ consider the function $F = x^2 + y^2 + \lambda(x^2 + 8xy + 7y^2 - 225)$

$$dF = (2x + 2x\lambda + 8y\lambda)dx + (2y + 8x\lambda + 14y\lambda)dy$$

$$\Rightarrow \left.\begin{array}{l}(1+\lambda)x + 4\lambda y = 0 \\ \text{and } 4\lambda x + (1+7\lambda)y = 0\end{array}\right\} \therefore \lambda = 1, -\frac{1}{9}$$

For $\lambda = 1$, $\quad 2x + 4y = 0 \Rightarrow x = -2y$

Substitute in $x^2 + 8xy + 7y^2 = 225 \Rightarrow y^2 = -45$

No Real soln

For $\lambda = -\frac{1}{9}$, $\quad (1 - \frac{1}{9})x + 4 \cdot (-\frac{1}{9}y) = 0$

$$\Rightarrow \frac{8}{9}x = \frac{4}{9}y$$

$$\therefore y = 2x$$

Substitute in $x^2 + 8xy + 7y^2 = 225$, $\Rightarrow x^2 = 5$, $y^2 = 20$

$$\therefore x^2 + y^2 = 25$$

$$\therefore d^2F = 2(1+\lambda)dx^2 + 16\lambda\, dx\,dy + 2(1+7\lambda)dy^2$$

$$= \frac{16}{9}dx^2 - \frac{16}{9}dx\,dy + \frac{4}{9}dy^2 \quad \text{at } \lambda = -\frac{1}{9}$$

$$= \frac{4}{9}\underbrace{(2dx - dy)^2}_{>0}$$

$\therefore$ function $z = 0$ or $x^2 + y^2$ has a minimum value $\underline{\underline{25}}$.

Example 10. Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the conditions $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$, and $z=x+y$.

Let us consider a function F of independent variables x, y, z where

$$F=x^2+y^2+z^2+\lambda_1\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right)+\lambda_2(x+y-z)$$

$$\therefore \quad dF=\left(2x+\frac{x}{2}\lambda_1+\lambda_2\right)dx+\left(2y+\frac{2y}{5}\lambda_1+\lambda_2\right)dy+\left(2z+\frac{2z}{25}\lambda_1-\lambda_2\right)dz$$

As x, y, z are independent variables, we get

$$2x+\frac{x}{2}\lambda_1+\lambda_2=0$$

$$2y+\frac{2y}{5}\lambda_1+\lambda_2=0$$

$$2z+\frac{2z}{25}\lambda_1-\lambda_2=0$$

$$\therefore \quad x=\frac{-2\lambda_2}{\lambda_1+4},\quad y=\frac{-5\lambda_2}{2\lambda_1+10},\quad z=\frac{25\lambda_2}{2\lambda_2+50}$$

Substituting in $x+y=z$, we get

$$\frac{2}{\lambda_1+4}+\frac{5}{2\lambda_1+10}+\frac{25}{2\lambda_1+50}=0,\ \lambda_2\neq 0 \quad \text{...(1)}$$

for if, $\lambda_2=0$, $x=y=z=0$, but (0, 0, 0) does not satisfy the other condition of constraint.

Hence from (1), $17\lambda_1^2+245\lambda_1+750=0$, so that $\lambda_1=-10,-75/17$.

For $\lambda_1=-10$,

$$x=\frac{1}{3}\lambda_2,\ y=\frac{1}{2}\lambda_2,\ z=\frac{5}{6}\lambda_2$$

Substituting in $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$, we get

$$\lambda_2^2 = 180/19 \text{ or } \lambda_2 = \pm 6\sqrt{5/19}$$

The corresponding stationary points are

$$(2\sqrt{5/19}, 3\sqrt{5/19}, 5\sqrt{5/19}), (-2\sqrt{5/19}, -3\sqrt{5/19}, -5\sqrt{5/19})$$

The value of $x^2 + y^2 + z^2$ corresponding to these point is 10.

For $\lambda_1 = -75/17$,

$$x = \frac{34}{7}\lambda_2, \; y = -\frac{17}{4}\lambda_2, \; z = \frac{17}{28}\lambda_2,$$

which on substitution in $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ give

$$\lambda_2 = \pm 140/(17\sqrt{646})$$

The corresponding stationary points are

$$(40/\sqrt{646}, -35/\sqrt{646}, 5/\sqrt{646}), (-40/\sqrt{646}, 35/\sqrt{646}, -5/\sqrt{646})$$

The value of $x^2 + y^2 + z^2$ corresponding to these points is 75/17.

Thus, the maximum value is 10 and the minimum 75/17.

Notes:

1. We have not theoretically established the existence of maximum or minimum value. We have simply shown that of all the possible values, 10 is the maximum and 75/17 the minimum.
2. Using constraint conditions, $dz = dx + dy; \frac{x}{4}dx + \frac{y}{5}dy + \frac{z}{25}dz = 0$, so that dz, dy and consequently d^2F may be expressed in terms of dx (or dx^2) alone. It can, then, be easily verified that 10 is a maximum value and 75/17 the minimum.

Example 11. Prove that the volume of the greatest rectangular parallelopiped, that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, is $\frac{8abc}{3\sqrt{3}}$.

■ We have to find the greatest value of $8xyz$ subject to the conditions

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \; x > 0, \; y > 0, \; z > 0 \quad \ldots(1)$$

Let us consider a function F of three independent variables x, y, z, where

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$\therefore \quad dF = \left(8yz + \frac{2x\lambda}{a^2}\right)dx + \left(8zx + \frac{2y\lambda}{b^2}\right)dy + \left(8xy + \frac{2z\lambda}{c^2}\right)dz$$

At stationary points,

$$8yz + \frac{2x\lambda}{a^2} = 0,\ 8zx + \frac{2y\lambda}{b^2} = 0,\ 8xy + \frac{2z\lambda}{c^2} = 0 \qquad \ldots(2)$$

Multiplying by x, y, z respectively and adding,

$$24xyz + 2\lambda = 0 \text{ or } \lambda = -12xyz \qquad \text{[using (1)]}$$

Hence from (2), $x = a/\sqrt{3}$, $y = b/\sqrt{3}$, $z = c/\sqrt{3}$, and so

$$\lambda = -4abc/\sqrt{3}$$

Again

$$d^2F = 2\lambda\left(\frac{dx^2}{a^2} + \frac{dy^2}{b^2} + \frac{dz^2}{c^2}\right) + 16z\,dx\,dy + 16x\,dy\,dz + 16y\,dz\,dx$$

$$= -\frac{8abc}{\sqrt{3}}\,\Sigma\frac{1}{a^2}dx^2 + \frac{16}{\sqrt{3}}\Sigma c\,dx\,dy \qquad \ldots(3)$$

Now from equations. (1), we have

$$x\frac{dx}{a^2} + y\frac{dy}{b^2} + z\frac{dz}{c^2} = 0 \text{ or } \frac{dx}{a} + \frac{dy}{b} + \frac{dz}{c} = 0 \qquad \ldots(4)$$

Hence squaring,

$$\Sigma\frac{dx^2}{a^2} + 2\Sigma\frac{dx\,dy}{ab} = 0$$

or

$$abc\,\Sigma\frac{dx^2}{a^2} = -2\Sigma c\,dx\,dy$$

$$\therefore \qquad d^2F = -\frac{16}{\sqrt{3}}\,abc\,\Sigma\frac{dx^2}{a^2}$$

which is always negative.

Hence $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ is a point of maxima and the maximum value of $8xyz$ is $\frac{8abc}{3\sqrt{3}}$.

Note: The sign of d^2F can also be decided by expressing it in terms of dx and dy alone, by putting into (3) the value of dz from (4).

Example 12. Show that the length of the axes of the section of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by the plane $lx + my + nz = 0$ are the roots of the quadratic in r^2,

$$\frac{l^2a^2}{r^2 - a^2} + \frac{m^2b^2}{r^2 - b^2} + \frac{n^2c^2}{r^2 - c^2} = 0.$$

■ We have to find the stationary values of the function r^2 where $r^2 = x^2 + y^2 + z^2$, subject to the two equations of condition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad ...(1)$$

$$lx + my + nz = 0 \quad ...(2)$$

Let us consider a function F of independent variables x, y, z,

$$F = x^2 + y^2 + z^2 + \lambda_1\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) + 2\lambda_2(lx + my + nz)$$

$$\therefore \quad dF = 2\left(x + \frac{x\lambda_1}{a^2} + \lambda_2 l\right)dx + 2\left(y + \frac{y\lambda_1}{b^2} + \lambda_2 m\right)dy + 2\left(z + \frac{z\lambda_1}{c^2} + \lambda_2 n\right)dz$$

At stationary points,

$$x + \frac{x}{a^2}\lambda_1 + l\lambda_2 = 0,\ y + \frac{y}{b^2}\lambda_1 + m\lambda_2 = 0,\ z + \frac{z}{c^2}\lambda_1 + n\lambda_2 = 0 \quad ...(3)$$

Multiplying by x, y, z, respectively and adding, we get

$$\lambda_1 = -(x^2 + y^2 + z^2) = -r^2$$

$$\therefore \quad x = \frac{a^2 l\lambda_2}{r^2 - a^2},\ y = \frac{b^2 m\lambda_2}{r^2 - b^2},\ z = \frac{c^2 n\lambda_2}{r^2 - c^2}$$

But $$0 = lx + my + nz = \lambda_2\left\{\frac{a^2 l^2}{r^2 - a^2} + \frac{b^2 m^2}{r^2 - b^2} + \frac{c^2 n^2}{r^2 - c^2}\right\}$$

and since $\lambda_2 \neq 0$, we get the quadratic in r^2 giving the stationary values:

$$\frac{a^2 l^2}{r^2 - a^2} + \frac{b^2 m^2}{r^2 - b^2} + \frac{c^2 n^2}{r^2 - c^2} = 0$$

Example 13. If the variables x, y, z satisfy the equation

$$\phi(x)\phi(y)\phi(z) = k^3 \quad ...(1)$$

and $\phi(a) = k \neq 0,\ \phi'(a) \neq 0$, show that the function

$$f(x) + f(y) + f(z) \quad ...(2)$$

has a maximum, when $x = y = z = a$, provided that

$$f'(a)\left\{\frac{\phi''(a)}{\phi'(a)} - \frac{\phi'(a)}{\phi(a)}\right\} > f''(a)$$

■ Let us consider a function

$$F = f(x) + f(y) + f(z) + \lambda\{\phi(x)\phi(y)\phi(z) - k^3\}$$

$$\therefore \quad dF = \Sigma\{f'(x) + \lambda\phi'(x)\phi(y)\phi(z)\}dx$$

1. Show that

   (i) if $2x + 3y + 4z = a$, the maximum value of $x^2y^3z^4$ is $\left(\frac{a}{9}\right)^9$.

   (ii) if $a^2x^2 + 2by^3 + z^4 = c^4$, the maximum value of x^4yz^2 is given by

$$17a^2x^2 = 12c^4,\ 17by^3 = c^4,\ 17z^4 = 3c^4$$

2. If $xyz = abc$, the minimum value of $bcx + cay + abz$ is $3abc$.
3. If $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, the maximum value of xyz is $abc/3\sqrt{3}$.
4. If $xyz = a^2(x + y + z)$, the minimum value of $yz + zx + xy$ is $9a^2$.
5. If $x^2 + y^2 = 1$, the minimum value of $(ax^2 + by^2)/(a^2x^2 + b^2y^2)^{1/2}$ is $2(ab)^{1/2}/(a + b)$.
6. If $xyz = k^3$, the product $(x + a)(y + b)(z + c)$ is a minimum, when $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{k}{(abc)^{1/3}}$; a, b, c are positive.
7. Show that the points on the ellipse $5x^2 - 6xy + 5y^2 = 4$ for which the tangent is at the greatest distance from the origin are $(1, 1)$ and $(-1, -1)$.
8. Show that the point on the sphere $x^2 + y^2 + z^2 = 1$ which is farther from $(2, 1, 3)$ is $(-2/\sqrt{14}, -1/\sqrt{14}, -3/\sqrt{14})$.
9. Show that the shortest distance from the origin to the curve of intersection of the surfaces $xyz = a$ and $y = bx$, where $a > 0, b > 0$, is $\sqrt[3]{a(b^2 + 1)/2b}$.
10. If $ax^2 + by^2 = ab$, show that the maximum and minimum values of $x^2 + xy + y^2$ will be the values of λ, given by the equation

$$4(\lambda - a)(\lambda - b) - ab = 0$$