Let (a, b) be a point in the domain of definition of a function f. Then $f(a, b)$ is an *extreme value* of f, if for every point (x, y), [other than (a, b)] of some neighbourhood of (a, b), the difference

$$f(x, y) - f(a, b) \qquad \text{...(1)}$$

keeps the same sign.

The extreme value $f(a, b)$ is called *a maximum or a minimum value* according as the sign of (1) is negative or positive.

10.1 A Necessary Condition

A necessary condition for $f(x, y)$ *to have an extreme value at* (a, b) *is that* $f_x(a, b) = 0, f_y(a, b) = 0,$ *provided these partial derivatives exist.*

If $f(a, b)$ is an extreme value of the function $f(x, y)$ of two variables, then it must also be an extreme value of both the functions, $f(x, b)$ and $f(a, y)$ of one variable. But a necessary condition that these have extreme value at $x = a$ and $y = b$ respectively, is

$$f_x(a, b) = 0, \; f_y(a, b) = 0.$$

Notes:

1. The function $f(x, y) = |x| + |y|$ has an extreme value at (0, 0) even though the partial derivatives f_x and f_y do not exist at (0, 0).
2. If $f(x, y) = 0$, if $x = 0$ or $y = 0$, and $f(x, y) = 1$ elsewhere, then both the partial derivatives exist (each equal to zero) at the origin, but $f(0, 0)$ is not an extreme value. Thus the conditions obtained above are *only necessary and not sufficient*.
3. Point at which $f_x = 0, f_y = 0$ (or $df = 0$) are called *Stationary points*.

10.2 Sufficient Conditions for $f(x, y)$ to have an Extreme Value at (a, b)

Let $f_x(a, b) = 0 = f_y(a, b)$. Further, let us suppose that $f(x, y)$ possesses continuous second order partial derivatives in a certain neighbourhood of (a, b) and that these derivatives at (a, b) viz. $f_{xx}(a, b)$, $f_{xy}(a, b), f_{yy}(a, b)$ are not all zero.

Let $(a+h, b+k)$ be a point of this neighbourhood.

Let us write

$$A = f_{xx}(a, b),\; B = f_{xy}(a, b),\; C = f_{yy}(a, b)$$

By Taylor's theorem, we have for $0 < \theta < 1$,

$$f(a + h, b + k) = f(a, b) + [hf_x(a, b) + kf_y(a, b)]$$

$$+ \frac{1}{2!}[h^2 f_{xx}(a + \theta h, b + \theta k) + 2hkf_{xy}(a + \theta h, b + \theta k) + k^2 f_{yy}(a + \theta h, + b + \theta k)]$$

But $f_x(a, b) = 0 = f_y(a, b)$, and

Since the second order partial derivatives are continuous at (a, b), we write

$$f_{xx}(a + \theta h, b + \theta k) - f_{xx}(a, b) = \rho_1$$

$$f_{xy}(a + \theta h, b + \theta k) - f_{xy}(a, b) = \rho_2$$

$$f_{yy}(a + \theta h, b + \theta k) - f_{yy}(a, b) = \rho_3$$

where ρ_1, ρ_2, ρ_3 are functions of h and k, and $\to 0$ as $(h, k) \to (0, 0)$.

$$\therefore \quad f(a + h, b + k) - f(a, b) = \frac{1}{2}[Ah^2 + 2Bhk + Ck^2 + \rho]$$

where $\rho = \rho_1 + \rho_2 + \rho_3 \to 0$ as $(h, k) \to (0, 0)$ and is of unknown sign.

Let $G = Ah^2 + 2Bhk + Ck^2$, so that

$$f(a+h, b+k) - f(a, b) = \frac{1}{2}[G + \rho] \qquad ...(1)$$

There are several cases to consider.

(*i*) G never vanishes and keeps a constant sign. Since $\rho \to 0$ when $(h, k) \to (0, 0)$, therefore ρ is a small number and the sign of $G + \rho$ is same as of G, *i.e.*, $G + \rho$ remains negative or positive according as G is negative or positive. Thus the difference,

$$f(a+h, b+k) - f(a, b) \lesseqgtr 0 \text{ according as } G \lesseqgtr 0$$

But we know by definition that $f(x, y)$ has a maxima or a minima at (a, b) according as the difference $f(a+h, b+k) - f(a, b)$ is negative or positive for all (h, k) except $(0, 0)$.

Thus $f(a, b)$ will be a maximum or a minimum value according as G is negative or positive.

(*ii*) If G can change sign, since $f(a+h, b+k) - f(a, b)$ and G have the same sign when ρ is small, $f(a, b)$ will not be an extreme value.

(*iii*) If G, without ever changing sign, vanishes for certain values of (h, k), the sign of $f(a+h, b+k) - f(a, b)$ will depend upon ρ, which is of unknown sign, and so no conclusion can be drawn. This is the *doubtful case* and requires further investigation.

Let us first take $A \neq 0$.

G may be written in the form:

$$G = \frac{(Ah + Bk)^2 + k^2(AC - B^2)}{A}$$

(1) If $AC - B^2 > 0$, the numerator of G is the sum of two positive quantities and it never vanishes except when $k = 0$, $h = 0$, simultaneously, which is not permissible [see(*i*)]. Hence, G never vanishes and has the same sign as A.

Thus, $f(a, b)$ has a maximum value if $A < 0$, and a minimum value if $A > 0$.

(2) If $AC - B^2 < 0$, the sign of the numerator of G may be positive or negative according as $(Ah + Bk)^2 >$ or $< k^2(B^2 - AC)$, *i.e.*, according to the values of (h, k). Hence, G does not keep the same sign for all values of (h, k), and therefore, $f(a, b)$ is not an extreme value.

(3) If $AC - B^2 = 0$, the numerator of G is a perfect square but may vanish for values of (h, k) for which $Ah + Bk = 0$. Thus G, without changing sign, may vanish for certain values of (h, k).

This is the doubtful case in which the sign of $f(a+h, b+k) - f(a, b)$ depends upon ρ and requires further investigation.

If $A = 0$, then

$$G = 2Bhk + Ck^2 = k(2Bh + Ck)$$

(4) If $A = 0$, $B \neq 0$, G changes sign with k and $(2Bh + Ck)$, and there is no extreme value.

(5) If $A = 0$, $B = 0$, G does not change sign but may vanish when $k = 0$ (without $h = 0$). This is therefore the doubtful case and requires further investigation.

Rule. *$f(a, b)$ is an extreme value of $f(x, y)$, if $f_x(a, b) = 0 = f_y(a, b)$, and*

$$f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0,$$

and this extreme value is a maximum or a minimum according as $f_{xx}(a, b)$ [or $f_{yy}(a, b)$] is negative or positive.

Further investigation is necessary, if

$$f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0$$

Remark: Since at (a, b),

$$df = hf_x(a, b) + kf_y(a, b)$$

and

$$d^2f = h^2f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2f_{yy}(a, b) = Ah^2 + 2Bhk + Ck^2$$

so $f(a, b)$ is an extreme value of $f(x, y)$ if at (a, b), $df = 0$, and d^2f keeps the same sign for all values of $(h, k) \neq (0, 0)$.

Note: Discussion of the doubtful case involves the consideration of terms of higher order than the second in the Taylor expansion of $f(a + h, b + k)$ but this is generally not easy and will not be considered here.

However, it is sometimes possible to decide whether f has a maxima or a minima at (a, b) by algebraic or geometric considerations.

Example 32. Find the maxima and minima of the function

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20$$

■ We have

$$f_x(x, y) = 3x^2 - 3 = 0, \text{ when } x = \pm 1$$

$$f_y(x, y) = 3y^2 - 12 = 0, \text{ when } y = \pm 2$$

Thus, the function has four stationary points:

$$(1, 2), (-1, 2), (1, -2), (-1, -2)$$

Now

$$f_{xx}(x, y) = 6x, f_{xy}(x, y) = 0, f_{yy}(x, y) = 6y$$

At $(1, 2)$

$$f_{xx} = 6 > 0, \text{ and } f_{xx}f_{yy} - (f_{xy})^2 = 72 > 0$$

Hence, $(1, 2)$ is a point of minima of the function.

At $(-1, 2)$

$$f_{xx} = -6, \text{ and } f_{xx}f_{yy} - (f_{xy})^2 = -72 < 0$$

Hence, the function has neither a maxima nor a minima at $(-1, 2)$.

At $(1, -2)$,

$$f_{xx} = 6, \text{ and } f_{xx}f_{yy} - (f_{xy})^2 = -72 < 0.$$

Hence, the function has neither maximum nor minimum at $(1, -2)$.

At $(-1, -2)$,

$$f_{xx} = -6, \text{ and } f_{xx}f_{yy} - (f_{xy})^2 = 72 > 0$$

Hence, the function has a maximum value at $(-1, -2)$.

Note: Stationary points like $(-1, 2)$, $(1, -2)$ which are not extreme points are called the *saddle points*.

Example 33. Show that the function

$$f(x, y) = 2x^4 - 3x^2 y + y^2$$

has neither a maximum nor a minimum at $(0, 0)$, where

$$f_{xx}f_{yy} - (f_{xy})^2 = 0.$$

■ Now

$$f_x(x, y) = 8x^3 - 6xy, \; f_y(x, y) = -3x^2 + 2y$$

$\therefore$

$$f_x(0, 0) = 0 = f_y(0, 0)$$

Also

$$f_{xx}(x, y) = 24x^2 - 6y = 0, \text{ at } (0, 0)$$

$$f_{xy}(x, y) = -6x = 0, \text{ at } (0, 0)$$

$$f_{yy}(x, y) = 2, \text{ at } (0, 0)$$

Thus at $(0, 0)$, $f_{xx}(0, 0) \cdot f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0$.

So that it is a *doubtful* case, and thus requires further examination.

Again

$$f(x, y) = (x^2 - y)(2x^2 - y); \; f(0, 0) = 0$$

or

$$f(x, y) - f(0, 0) = (x^2 - y)(2x^2 - y)$$

$$> 0, \text{ for } y < 0 \text{ or } x^2 > y > 0$$

$$< 0, \text{ for } y > x^2 > \frac{y}{2} > 0$$

Thus $f(x, y) - f(0, 0)$ does not keep the same sign near the origin. Hence f has neither a maximum nor a minimum value at the origin.

Example 34. Show that

$f(x, y) = y^2 + x^2 y + x^4$, has a minimum at $(0, 0)$.

■ It can be easily verified that at the origin,

$$f_x = 0, f_y = 0, f_{xx} = 0, f_{xy} = 0, f_{yy} = 2.$$

Thus at the origin $f_{xx}f_{yy} - (f_{xy})^2 = 0$, so that it is a *doubtful* case and requires further investigation.

But we can write

$$f(x, y) = \left(y + \frac{1}{2}x^2\right)^2 + \frac{3}{4}x^4$$

and

$$f(x, y) - f(0, 0) = \left(y + \frac{1}{2}x^2\right)^2 + \frac{3}{4}x^4$$

which is greater than zero for all values of (x, y). Hence f has a minimum value at the origin.

EXERCISE

1. Examine the following for extreme values:
   (*i*) $4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$
   (*ii*) $x^3y^2(12 - 3x - 4y)$
   (*iii*) $y^2 + 4xy + 3x^2 + x^3$
   (*iv*) $(x^2 + y^2 - 4)^2 - x^2$
   (*v*) $(x^2 + y^2)\, e^{6x+2x^2}$
   (*vi*) $(x - y)^2\,(x^2 + y^2 - 2)$
   (*vii*) $x^3 + y^3 - 63(x + y) + 12xy.1$
2. Investigate the maxima and minima of the functions,
   (*i*) $21x - 12x^2 - 2y^2 + x^3 + xy^2$
   (*ii*) $2(x - y)^2 - x^4 - y^4$
   (*iii*) $x^2 + 3xy + y^2 + x^3 + y^3$
   (*iv*) $x^2 + 4xy + 4y^2 + x^3 + 2x^2y + y^4$
   (*v*) $x^2y^2 - 5x^2 - 8xy - 5y^2$
   (*vi*) $x^2 - 2xy + y^2 + x^3 - y^3 + x^5$
3. Show that the function $(y - x)^4 + (x - 2)^4$ has a minimum at (2, 2).
4. Prove that the function $f(x, y) = x^2 - 2xy + y^2 + x^4 + y^4$ has a minima at the origin.
5. Find and classify the extreme values (if any) of the functions:
   (*i*) $x^2 + y^2 + x + y + xy$
   (*ii*) $x^2 + xy + y^2 + ax + by$
   (*iii*) $y^2 - x^3$
   (*iv*) $x^4 + y^4 - 6(x^2 + y^2) + 8xy$
6. A rectangular box, open at the top, is to have a volume of 32 cu ft. What must be the dimensions so that the total surface is a minimum.
7. Show that the function $f(x, y) = x^2 - 3xy^2 + 2y^4$ has neither a maximum nor a minimum value at the origin.

   [*Hint*: $f(x, y) = (x - y^2)(x - 2y^2)$].

3.4 Maxima and Minima

Recall from 1-dimensional calculus, to find the points of maxima and minima of a function. we first find the critical points i.e where the tangent line is horizontal $f'(x) = 0$. Then

(i) If $f''(x) > 0$ the gradient is increasing and we have a local minimum.

(ii) If $f''(x) < 0$ the gradient is decreasing and we have a local maximum.

(iii) If $f''(x) = 0$ it is inconclusive.

Critical Points of a function of 2 variables

We are now interested in finding points of local maxima and minima for a function of two variables.

DEFINITION. *A function $f(x, y)$ has a relative minimum (resp. maximum) at the point (a, b) if $f(x, y) \geq f(a, b)$ (resp. $f(x. y) \leq f(a. b)$) for all points (x, y) in some region around (a, b)*

DEFINITION. *A point (a, b) is a critical point of a function $f(x, y)$ if one of the following is true*

(i) $f_x(a, b) = 0$ and $f_y(a, b) = 0$

(ii) $f_x(a, b)$ and/or $f_y(a, b)$ does not exist.

Classification of Critical Points

We will need two quantities to classify the critical points of $f(x,y)$:

1. f_{xx}: the second partial derivative of f with respect to x.

2. $H = f_{xx}f_{yy} - f_{xy}^2$ the Hessian

If the Hessian is zero, then the critical point is degenerate. If the Hessian is non-zero, then the critical point is non-degenerate and we can classify the points in the following manner:

case(i) If $H > 0$ and $f_{xx} < 0$ then the critical point is a relative maximum.

case(ii) If $H > 0$ and $f_{xx} > 0$ then the critical point is a relative minimum.

case(iii) If $H < 0$ then the critical point is a saddle point.

EXAMPLE. Find and classify the critical points of

$$12x^3 + y^3 + 12x^2y - 75y.$$

SOLUTION: We first find the critical points of the function.

$$\begin{aligned} f_x &= 36x^2 + 24xy = 12x(3x+2y) \\ f_y &= 3y^2 + 12x^2 - 75 = 3(4x^2 + y^2 - 25). \end{aligned}$$

The critical points are the points where $f_x = 0$ and $f_y = 0$

$$\begin{aligned} f_x &= 0 \\ 12x(3x+2y) &= 0 \end{aligned}$$

Therefore either $x = 0$ or $3x + 2y = 0$. We handle the two cases separately;

case(i) $x = 0$.
Then substituting this in f_y we get $f_y = 3(y^2 - 25) = 0$ implies $y = \pm 5$.

case(ii) $3x + 2y = 0$.

Then $y = -3x/2$ and substituting this in f_y we get,

$$\begin{aligned} f_y &= 3(4x^2 + \frac{9x^2}{4} - 25) \\ &= \frac{3}{4}(16x^2 + 9x^2 - 100) \\ &= \frac{3}{4}(25x^2 - 100) \\ &= \frac{75}{4}(x^2 - 4) \\ f_y &= 0 \\ \frac{75}{4}(x^2 - 4) &= 0 \\ x^2 - 4 &= 0 \\ x &= \pm 2 \end{aligned}$$

Thus we have found four critical points: $(0,5)$, $(0,-5)$, $(2,-3)$, $(-2,3)$.

We must now classify these points.

$$\begin{aligned} f_{xx} &= 72x + 24y = 24(3x + y) \\ f_{xy} &= 24x \\ f_{yy} &= 6y \\ H &= f_{xx}f_{yy} - f_{xy}^2 \\ &= (24)(3x + y)(6y) - (24x)^2 \end{aligned}$$

Points	f_{xx}	H	Type
$(0,5)$	120	3600	Minimum
$(0,-5)$	-120	3600	Maximum
$(2,-3)$	72	-3600	Saddle
$(-2,3)$	-72	-3600	Saddle

1 Single Constraint Lagrange Multipliers

The idea of Lagrange multipliers is simple: we want to optimise a function, $f(x, y)$, under the constraint of another function $g(x, y) = 0$. Lagrange multipliers is a powerful method that can be used to solve this type of "constrained optimisation" problems.

We will firstly look at the the theorem of Lagrange multipliers and then justify it.

Theorem 1 (Method of Lagrange Multipliers) *To optimise for $f(x, y)$ subject to the constraint $g(x, y) = 0$ (assuming that such extrema exist and $\nabla g \neq \mathbf{0}$):*

(a) ***Simultaneously*** *solve for all values of x, y and λ in the following equations:*

$$g(x, y) = 0 \tag{1.1}$$
$$\nabla f(x, y) = \lambda \nabla g(x, y) \tag{1.2}$$

where λ is a non-zero constant; it is called the "Lagrange multiplier".

(b) Evaluate all points in in part (a) to see if these are minimum or maximum points (possibly using Hessian matrix).

We can think of Lagrange multipliers in the following way: if a "constrained critical point" does exist and $\nabla g(x, y) \neq 0$, then the constrained critical point must simultaneously satisfy two criteria: (1.1) and (1.2).

Being a constraint, $g(x, y) = 0$ must always be satisfied, hence, the first criterion is justified. We will justify (1.2) by thinking about the contour plots of $f(x, y)$ and $g(x, y) = 0$. The contour plot of $f(x, y)$ and $g(x, y) = 0$ can look something like Figure 1.

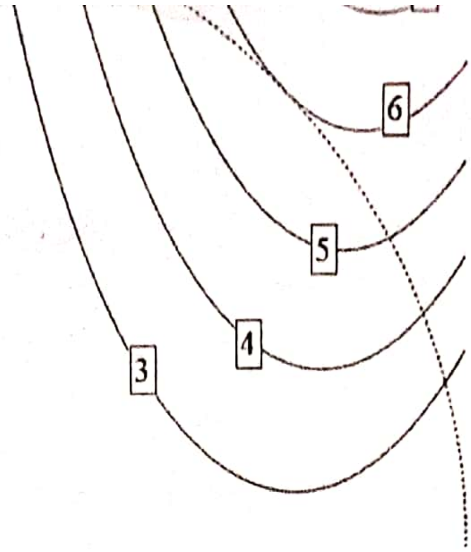

Figure 1: The red line is the contour line of $g(x, y) = 0$ and the blue lines are the contour plot of $f(x, y) = c$, where $c = 3, 4, 5, 6, 7$

We want to optimise $f(x, y)$ under the constraint of $g(x, y) = 0$. Geometrically on the contour plot, this is equivalent to finding the largest (or smallest) value of c such that $f(x, y) = c$ still intersects with $g(x, y) = 0$. And such intersection is our desired "constrained critical point".

From our diagram, it appears such constrained critical point occur when the two contour lines intersect **tangentially**, i.e. when $c = 6$. This can be justified in the following way: imagine moving along the contour line of $g(x, y) = 0$ (this movement is necessary because we must satisfy the constraint at all times); if the two contour lines are tangential to each other, then the value of $f(x, y)$ does not change locally as we move. Hence, the point where the two contour lines intersect tangentially correspond to a constrained critical point of $f(x, y)$ with reference to $g(x, y) = 0$.

If the contour lines of $f(x, y)$ and $g(x, y)$ intersect tangentially at a constrained critical point, this is equivalent to saying that these two contour lines has parallel tangent vectors at this point. Which means their normal vectors must be parallel, or **linearly dependent**. Since the normal vectors for $f(x, y)$ and $g(x, y) = 0$ are $\nabla f(x, y)$ and $\nabla g(x, y)$ respectively, this linear dependence can be expressed as $\nabla f(x, y) = \lambda \nabla g(x, y)$, where λ is a non-zero constant. And hence, we justified the second criterion of Lagrange multiplier method, which is equation (1.2).

Remarks:

Some underlying assumptions are:

1. Such extrema do exist.
2. $f(x, y)$ and $g(x, y)$ are smooth functions. If this were not true, then our geometric justification of for (1.2) might become flawed, due to possible discontinuity.
3. $\nabla g(x, y) \neq \mathbf{0}$, otherwise we will be dealing with an abnormal case. A abnormal case is when Lagrange multipliers method gives inconclusive results. For example:

14.7 Maxima and Minima

Suppose a surface given by $f(x, y)$ has a local maximum at (x_0, y_0, z_0); geometrically, this point on the surface looks like the top of a hill. If we look at the cross-section in the plane $y = y_0$, we will see a local maximum on the curve at (x_0, z_0), and we know from single-variable calculus that $\frac{\partial z}{\partial x} = 0$ at this point. Likewise, in the plane $x = x_0$, $\frac{\partial z}{\partial y} = 0$. So if there is a local maximum at (x_0, y_0, z_0), both partial derivatives at the point must be zero, and likewise for a local minimum. Thus, to find local maximum and minimum points, we need only consider those points at which both partial derivatives are 0. As in the single-variable case, it is possible for the derivatives to be 0 at a point that is neither a maximum or a minimum, so we need to test these points further.

You will recall that in the single variable case, we examined three methods to identify maximum and minimum points; the most useful is the second derivative test, though it does not always work. For functions of two variables there is also a second derivative test; again it is by far the most useful test, though it doesn't always work.

THEOREM 14.7.1 Suppose that the second partial derivatives of $f(x, y)$ are continuous near (x_0, y_0), and $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. We denote by D the **discriminant** $D(x_0, y_0) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$ there is a local maximum at (x_0, y_0); if $D > 0$ and $f_{xx}(x_0, y_0) > 0$ there is a local minimum at (x_0, y_0); if $D < 0$ there is neither a maximum nor a minimum at (x_0, y_0); if $D = 0$, the test fails. ■

EXAMPLE 14.7.2 Verify that $f(x, y) = x^2 + y^2$ has a minimum at $(0, 0)$.

First, we compute all the needed derivatives:

$$f_x = 2x \qquad f_y = 2y \qquad f_{xx} = 2 \qquad f_{yy} = 2 \qquad f_{xy} = 0.$$

The derivatives f_x and f_y are zero only at $(0, 0)$. Applying the second derivative test there:

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 2 \cdot 2 - 0 = 4 > 0$$

and

$$f_{xx}(0,0) = 2 > 0,$$

so there is a local minimum at $(0, 0)$, and there are no other possibilities. □

EXAMPLE 14.7.3 Find all local maxima and minima for $f(x, y) = x^2 - y^2$.

The derivatives:

$$f_x = 2x \qquad f_y = -2y \qquad f_{xx} = 2 \qquad f_{yy} = -2 \qquad f_{xy} = 0.$$

Again there is a single critical point, at $(0,0)$, and

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 2 \cdot -2 - 0 = -4 < 0,$$

so there is neither a maximum nor minimum there, and so there are no local maxima or minima. The surface is shown in figure 14.7.1. □

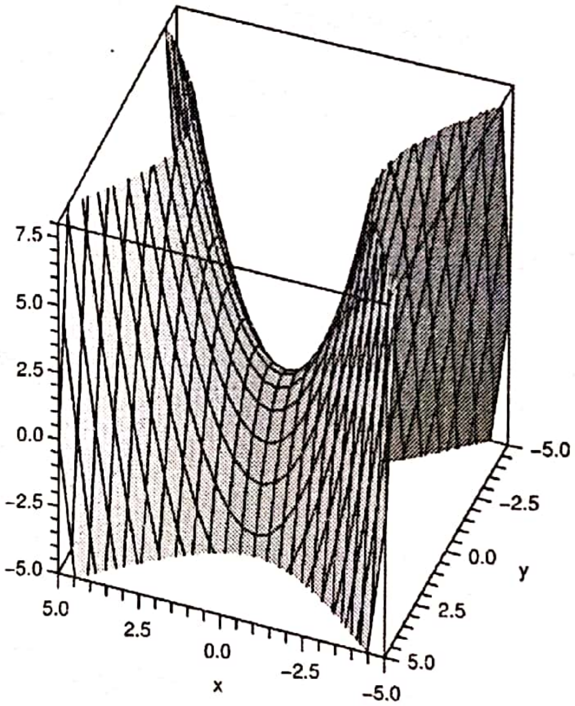

Figure 14.7.1 A saddle point, neither a maximum nor a minimum. (AP)

EXAMPLE 14.7.4 Find all local maxima and minima for $f(x,y) = x^4 + y^4$.

The derivatives:

$$f_x = 4x^3 \qquad f_y = 4y^3 \qquad f_{xx} = 12x^2 \qquad f_{yy} = 12y^2 \qquad f_{xy} = 0.$$

Again there is a single critical point, at $(0,0)$, and

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0 \cdot 0 - 0 = 0,$$

so we get no information. However, in this case it is easy to see that there is a minimum at $(0,0)$, because $f(0,0) = 0$ and at all other points $f(x,y) > 0$. □

EXAMPLE 14.7.5 Find all local maxima and minima for $f(x, y) = x^3 + y^3$.

The derivatives:

$$f_x = 3x^2 \qquad f_y = 3y^2 \qquad f_{xx} = 6x^2 \qquad f_{yy} = 6y^2 \qquad f_{xy} = 0.$$

Again there is a single critical point, at $(0, 0)$, and

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0 \cdot 0 - 0 = 0,$$

so we get no information. In this case, a little thought shows there is neither a maximum nor a minimum at $(0, 0)$: when x and y are both positive, $f(x, y) > 0$, and when x and y are both negative, $f(x, y) < 0$, and there are points of both kinds arbitrarily close to $(0, 0)$. Alternately, if we look at the cross-section when $y = 0$, we get $f(x, 0) = x^3$, which does not have either a maximum or minimum at $x = 0$. □

EXAMPLE 14.7.6 Suppose a box with no top is to hold a certain volume V. Find the dimensions for the box that result in the minimum surface area.

The area of the box is $A = 2hw + 2hl + lw$, and the volume is $V = lwh$, so we can write the area as a function of two variables,

$$A(l, w) = \frac{2V}{l} + \frac{2V}{w} + lw.$$

Then

$$A_l = -\frac{2V}{l^2} + w \quad \text{and} \quad A_w = -\frac{2V}{w^2} + l.$$

If we set these equal to zero and solve, we find $w = (2V)^{1/3}$ and $l = (2V)^{1/3}$, and the corresponding height is $h = V/(2V)^{2/3}$.

The second derivatives are

$$A_{ll} = \frac{4V}{l^3} \qquad A_{ww} = \frac{4V}{w^3} \qquad A_{lw} = 1,$$

so the discriminant is

$$D = \frac{4V}{l^3}\frac{4V}{w^3} - 1 = 4 - 1 = 3 > 0.$$

Since A_{ll} is 2, there is a local minimum at the critical point. Is this a global minimum? It is, but it is difficult to see this analytically; physically and graphically it is clear that there is a minimum, in which case it must be at the single critical point. This applet shows an example of such a graph. Note that we must choose a value for V in order to graph it. □

Recall that when we did single variable global maximum and minimum problems, the easiest cases were those for which the variable could be limited to a finite closed interval, for then we simply had to check all critical values and the endpoints. The previous example is difficult because there is no finite boundary to the domain of the problem—both w and l can be in $(0, \infty)$. As in the single variable case, the problem is often simpler when there is a finite boundary.

THEOREM 14.7.7 If $f(x, y)$ is continuous on a closed and bounded subset of $\mathbb{R}^2$, then it has both a maximum and minimum value. ■

As in the case of single variable functions, this means that the maximum and minimum values must occur at a critical point or on the boundary; in the two variable case, however, the boundary is a curve, not merely two endpoints.

EXAMPLE 14.7.8 The length of the diagonal of a box is to be 1 meter; find the maximum possible volume.

If the box is placed with one corner at the origin, and sides along the axes, the length of the diagonal is $\sqrt{x^2 + y^2 + z^2}$, and the volume is

$$V = xyz = xy\sqrt{1 - x^2 - y^2}.$$

Clearly, $x^2 + y^2 \le 1$, so the domain we are interested in is the quarter of the unit disk in the first quadrant. Computing derivatives:

$$V_x = \frac{y - 2yx^2 - y^3}{\sqrt{1 - x^2 - y^2}}$$

$$V_y = \frac{x - 2xy^2 - x^3}{\sqrt{1 - x^2 - y^2}}$$

If these are both 0, then $x = 0$ or $y = 0$, or $x = y = 1/\sqrt{3}$. The boundary of the domain is composed of three curves: $x = 0$ for $y \in [0, 1]$; $y = 0$ for $x \in [0, 1]$; and $x^2 + y^2 = 1$, where $x \ge 0$ and $y \ge 0$. In all three cases, the volume $xy\sqrt{1 - x^2 - y^2}$ is 0, so the maximum occurs at the only critical point $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. See figure 14.7.2. □

Exercises 14.7.

1. Find all local maximum and minimum points of $f = x^2 + 4y^2 - 2x + 8y - 1$. ⇒
2. Find all local maximum and minimum points of $f = x^2 - y^2 + 6x - 10y + 2$. ⇒
3. Find all local maximum and minimum points of $f = xy$. ⇒
4. Find all local maximum and minimum points of $f = 9 + 4x - y - 2x^2 - 3y^2$. ⇒

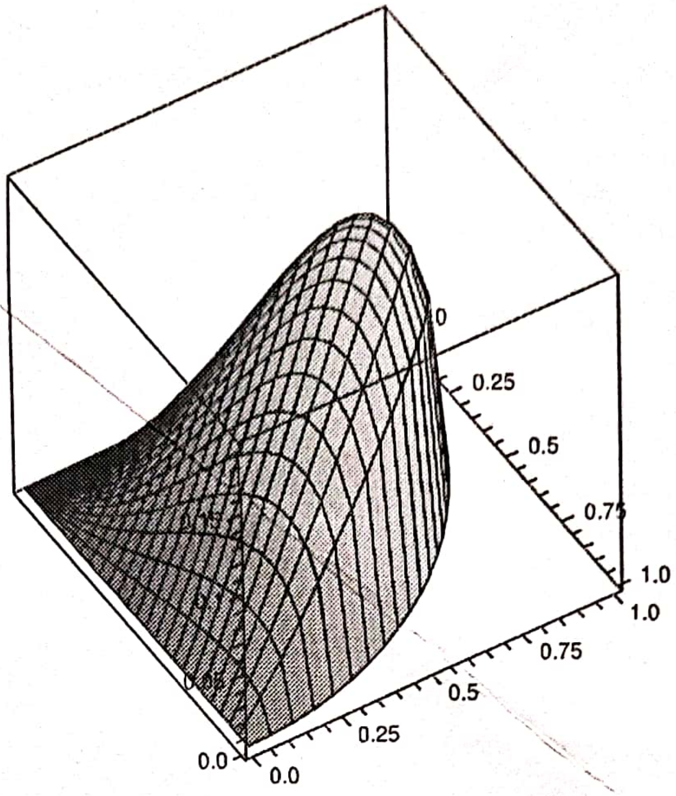

Figure 14.7.2 The volume of a box with fixed length diagonal.

5. Find all local maximum and minimum points of $f = x^2 + 4xy + y^2 - 6y + 1$. ⇒

6. Find all local maximum and minimum points of $f = x^2 - xy + 2y^2 - 5x + 6y - 9$. ⇒

7. Find the absolute maximum and minimum points of $f = x^2 + 3y - 3xy$ over the region bounded by $y = x$, $y = 0$, and $x = 2$. ⇒

8. A six-sided rectangular box is to hold 1/2 cubic meter; what shape should the box be to minimize surface area? ⇒

9. The post office will accept packages whose combined length and girth is at most 130 inches. (Girth is the maximum distance around the package perpendicular to the length; for a rectangular box, the length is the largest of the three dimensions.) What is the largest volume that can be sent in a rectangular box? ⇒

10. The bottom of a rectangular box costs twice as much per unit area as the sides and top. Find the shape for a given volume that will minimize cost. ⇒

11. Using the methods of this section, find the shortest distance from the origin to the plane $x + y + z = 10$. ⇒

12. Using the methods of this section, find the shortest distance from the point (x_0, y_0, z_0) to the plane $ax + by + cz = d$. You may assume that $c \neq 0$; use of Sage or similar software is recommended. ⇒

13. A trough is to be formed by bending up two sides of a long metal rectangle so that the cross-section of the trough is an isosceles trapezoid, as in figure 6.2.6. If the width of the metal sheet is 2 meters, how should it be bent to maximize the volume of the trough? ⇒