# Eigen Value Problem

# Content

- Jacobi's method for diagonalization of real symmetric matrix.

## Diagonalization of real symmetric matrix by Jacobi's method or Jacobi's method-

Let A be a given real symmetric matrix. Its eigen values are real and there exists a real orthogonal matrix B such that $B^{-1}AB$ is a diagonal matrix (D).

Jacobi method consists of diagonalizating 'A' by applying series of orthogonal transformations $B_1$, $B_2$,….$B_n$, such that their product satisfies $B^{-1}AB=D$.

## Diagonalization of real symmetric matrix by Jacobi's method or Jacobi's method (Cont..)

For this purpose, we choose the numerically largest non diagonal element $a_{ij}$ and form a $2 \times 2$ submatrix

$$A_1 = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$$

$$where, a_{ij} = a_{ji}$$

Which can easily be diagonalized.

Consider an orthogonal matrix

$$B_1 = \begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix}$$

$$then,\ B_1^{-1}AB_1 = \begin{bmatrix} Cos\theta & Sin\theta \\ -Sin\theta & Cos\theta \end{bmatrix}\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}\begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix}$$

$$B_1^{-1}AB_1 = \begin{bmatrix} a_{ii}Cos^2\theta + a_{jj}Sin^2\theta + a_{jj}Sin2\theta & a_{ij}Cos2\theta + \frac{1}{2}(a_{jj} - a_{ii})Sin2\theta \\ a_{ij}Cos2\theta + \frac{1}{2}(a_{jj} - a_{ii})Sin2\theta & a_{ii}Sin^2\theta + a_{jj}Cos^2\theta - a_{ij}Sin2\theta \end{bmatrix}$$

Now the matrix will be reduce to the diagonal form if

$$a_{ij}Cos2\theta + \frac{1}{2}(a_{jj} - a_{ii})Sin2\theta = 0$$

$$\tan 2\theta = \frac{2a_{ij}}{(a_{ii} - a_{jj})}$$

The value of θ can be calculated, which results diagonal matrix.
At next step the largest non diagonal element in the rotated matrix can be choosen and the above procedure is repeated using orthogonal matrix $B_2$

Advantage-

It gives all Eigen value simultaneously.

Disadvantage-

It is applicable on symmetric matrix only.

Example: Find all eigen values of the matrix

$$A=\begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

Using the Jacobi method.

Solution: Here the largest non diagonal element is $a_{13}=a_{31}=2$

$$\tan 2\theta = \frac{2a_{13}}{a_{11}-a_{33}} = \frac{2\times 2}{1-1} = \infty$$

$$\tan 2\theta = \infty$$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

$$then B_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$then B_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$D_1 = B_1^{-1} A B_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$D_1 = B_1^{-1} A B_1 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Now the largest non diagonal element is $a_{12}=a_{21}=2$

$$\tan 2\theta = \frac{2a_{12}}{a_{11}-a_{22}} = \frac{2\times 2}{3-3} = \infty$$

$$\tan 2\theta = \infty$$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

$$then B_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The second transformation gives

$$D_2 = B_2^{-1} A B_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D_2 = B_2^{-1} A B_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Hence the eigen values of the given matrix are 5, 1.-1 and the corresponding eigen vectors are column of matrix $B_1B_2$

Hence the eigen values of the given matrix are 5, 1.-1 and the corresponding eigen vectors are column of matrix $B_1B_2$

$$B_1B_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_1B_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Practice Problems

Find all the eigen values of the matrix using Jacobi method. Iterate till off diagonal elements in magnitude are less than 0.0005

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

# Suggested books

1. Numerical Methods by **S.R.K Lyenger & R.K. Jain.**

2. Numerical Analysis by **Richard L. Burden.**

3. Introductory methods of Numerical analysis by **S.S. Sastry**.

Thank you