Numerical Integration

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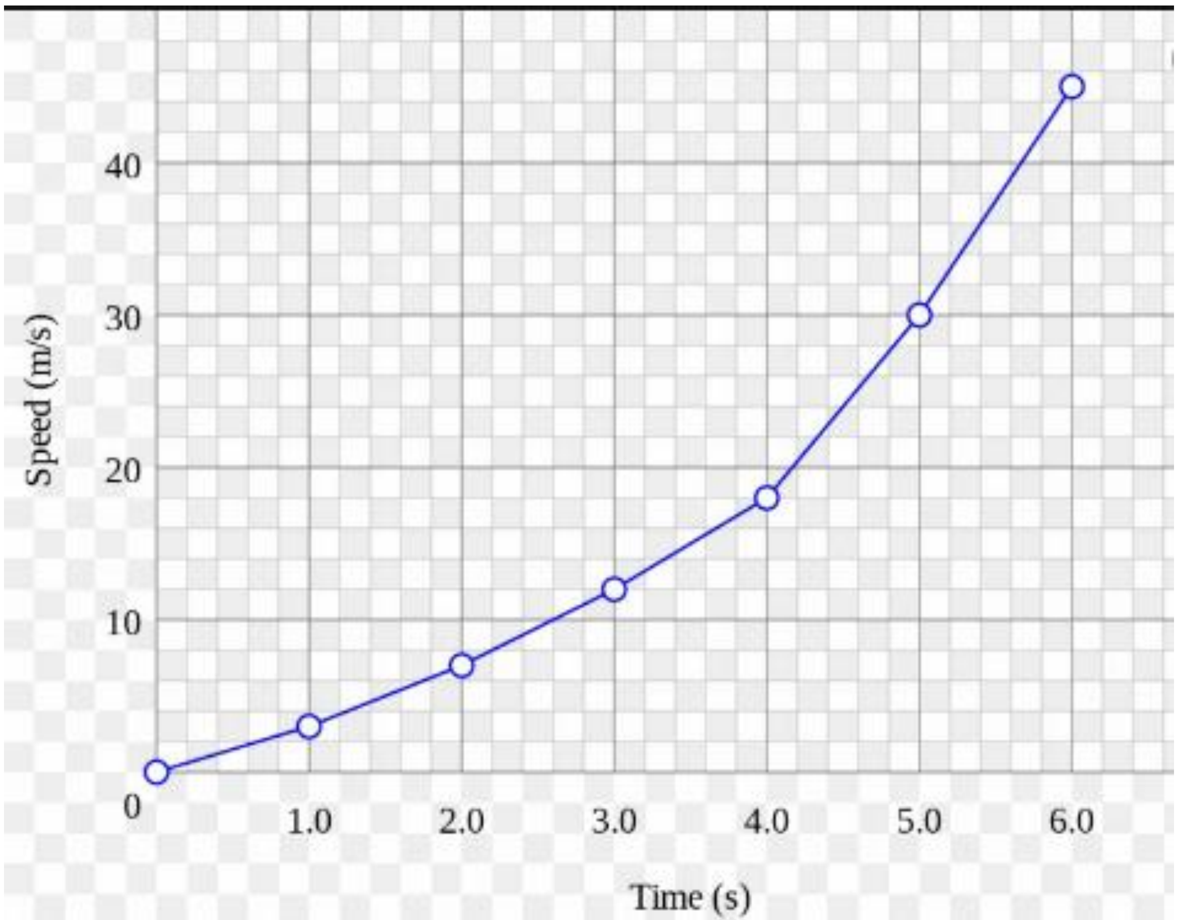


Speed (m/s)
40
30
20
10
0
1.0
2.0
3.0
4.0
5.0
6.0
Time (s)

Gregory Newton Formula or Newton's formula for interpolation

1. Newton's Forward Difference Formula: For equal interval, we consider the more simpler formula. Let for equal spaced (h) data, We have x_0, $x_0 + h$, $x_0 + 2h, \ldots\ldots\ldots$ $x_0 + nh$, Or $x_n = x_0 + nh$, n=0,1,2…….

$$f(x) = f\left[x_0 + \left(\frac{x - x_0}{h}\right)h\right]$$

$$Let \ldots : u = \left(\frac{x - x_0}{h}\right)$$

$$f(x) = f(x_0 + uh)$$

$$f(x) = E^u f(x_0)$$

$$f(x) = (1 + \Delta)^u f(x_0)$$

$$f(x) = f(x_0) + u\frac{\Delta f(x_0)}{1!} + u(u-1)\frac{\Delta^2 f(x_0)}{2!} + \ldots u(u-1)\ldots(u-(n-1))\frac{\Delta^n f(x_0)}{n!}$$

2. Newton's Backward Difference Formula:

For equal interval, we consider the more simpler formula. Let for equal spaced (h) data, We have x_0, $x_0 + h$, $x_0 + 2h$,……… $x_0 + nh$, Or $x_n = x_0 + nh$, n=0,1,2…….

$$f(x) = f\left[x_n + \left(\frac{x - x_n}{h}\right)h\right]$$

$$let....u = \left(\frac{x - x_n}{h}\right)$$

$$f(x) = f(x_n + uh)$$

$$f(x) = E^u f(x_n)$$

$$f(x) = (1 - \nabla)^{-u} f(x_n)$$

$$f(x) = \left[1 + u\frac{\nabla}{1!} - u(-u-1)\frac{\nabla^2}{2!} + ..u(u+1)...(u+n-1)\frac{\nabla^n}{n!}\right] f(x_n)$$

$$f(x) = f(x_n) + u\frac{\nabla f(x_n)}{1!} + u(u+1)\frac{\nabla^2 f(x_n)}{2!} + ..u(u+1)...(u+n-1)\frac{\nabla^n f(x_n)}{n!}$$

Numerical Integration

Numerical Integration

Let f(x) be given for certain data points (x_0,y_0), (x_1,y_1)……(x_n,y_n), where f(x) is not known explicitly.
It is required to compute the value of integral

$$I = \int_a^b y\,dx$$

Let the interval [a,b] be divided into n equal subintervals
$a=x_0<x_1<\ldots\ldots\ldots\ldots x_n=b$,
Clearly, $x_n=x_0+nh$
Hence the integral becomes,

$$I = \int_{x_0}^{x_n} y\,dx$$

Numerical Integration

Approximating y by Newton's forward difference formula

$$I = \int_{x_0}^{x_n} y dx$$

$$I = \int_{x_0}^{x_n} \left[y_0 + u \frac{\Delta y_0}{1!} + u(u-1) \frac{\Delta^2 y_0}{2!} + u(u-1)(u-2) \frac{\Delta^3 y_0}{3!} + \ldots\ldots \right] du$$

$$x = x_0 + uh, dx = hdu$$

$$at x = x_0, u = 0$$

$$at x = x_n, u = \frac{x_n - x_0}{h}$$

$$Let \ldots say \ldots u = \frac{x_n - x_0}{h} = n (no. of sub \operatorname{int} ervals)$$

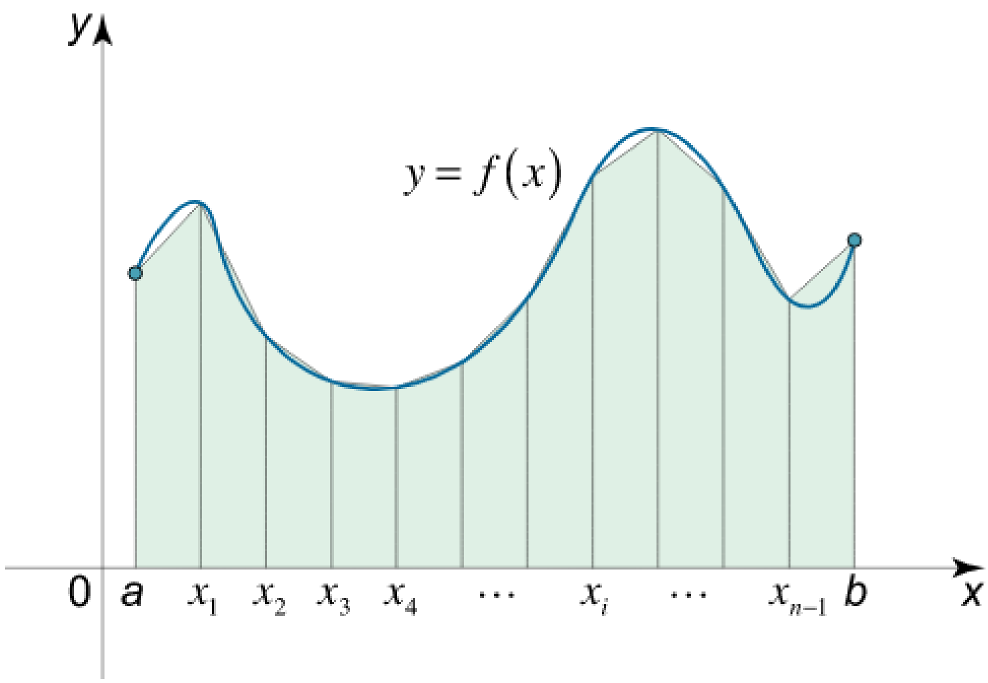
$$Hence,$$

$$I = h \left[y_0 u + \frac{u^2}{2} \frac{\Delta y_0}{1!} + \left(\frac{u^3}{3} - \frac{u^2}{2} \right) \frac{\Delta^2 y_0}{2!} + \ldots\ldots\ldots \right]_0^n$$

$$\int_{x_0}^{x_0 + nh} y dx = nh \left[y_0 + \frac{n}{2} \frac{\Delta y_0}{2} + \left(\frac{n(2n-3)}{12} \right) \frac{\Delta^2 y_0}{2} + \ldots\ldots\ldots \right] \ldots\ldots\ldots(1)$$

Trapezoidal rule

Put n=1 in eq (1) and neglect second and higher differences



Numerical Integration

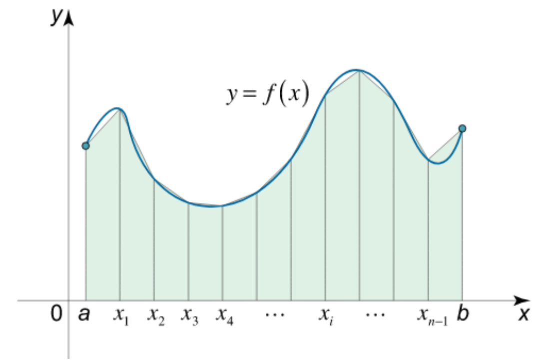
$$I = h\left[y_0 u + \frac{u^2}{2}\frac{\Delta y_0}{1!} + \left(\frac{u^3}{3} - \frac{u^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \ldots\ldots. \right]_0^n$$

$$\int_{x_0}^{x_0+nh} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \left(\frac{n(2n-3)}{12}\right)\Delta^2 y_0 + \ldots\ldots. \right] \qquad \ldots\ldots.(1)$$

Trapezoidal rule

Put n=1 in eq (1) and neglect second and higher differences

$$\int_{x_0}^{x_0+h} y\,dx = h\left[y_0 + \frac{1}{2}\frac{\Delta y_0}{}\right] = h\left[\frac{y_0 + y_1}{2}\right]$$



$similarly$

$$\int_{x_0+h}^{x_0+2h} y\,dx = h\left[\frac{y_1 + y_2}{2}\right]$$

.

.

$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = h\left[\frac{y_{n-1} + y_n}{2}\right]$$

$on, combining$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \ldots y_{n-1}) + y_n\right]$$

Numerical Integration

$$I = h\left[y_0 u + \frac{u^2}{2}\frac{\Delta y_0}{1!} + \left(\frac{u^3}{3} - \frac{u^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \ldots\ldots\ldots \right]_0^n$$

$$\int_{x_0}^{x_0+nh} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \left(\frac{n(2n-3)}{12}\right)\Delta^2 y_0 + \ldots\ldots\ldots \right] \qquad \ldots\ldots\ldots(1)$$

Simpson's 1/3 rule

Put n=2 in eq (1) and neglect third and higher differences
In Simpson's 1/3 rule approximate the given curve by a polynomial of degree 2.
Due to which it gives better result.

$$\int_{x_0}^{x_0+2h} ydx = 2h\left[y_0 + \Delta\, y_0 + \frac{1}{6}\frac{\Delta^2 y_0}{}\right] = 2h\left[y_0 + (y_1 - y_0) + \frac{(y_2 - y_1 + y_0)}{6}\right]$$

$$= \frac{2h}{6}[6y_1 + y_2 - 2y_1 + y_0] = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

similarly

$$\int_{x_0+2h}^{x_0+4h} ydx = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

.

.

$$\int_{x_0+(n-2)h}^{x_0+nh} ydx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

on, combining

$$\int_{x_0}^{x_n} ydx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \ldots y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots y_{n-2}) + y_n]$$

Simpson's 3/8 rule

Put n=3 in eq (1) and neglect higher differences

$$\int_{x_0}^{x_n} y dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_2 + \ldots y_{n-1}) + 2(y_3 + y_6 + \ldots\ldots y_{n-3})\right]$$

*Do it by yourself

Example

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using trapezoidal and simpson's 1/3 rule

x	0	1	2	3	4	5	6
f(x)	1	0.5	0.2	0.1	0.0588	0.0385	0.622

Divide the interval into 6 parts each of width 1.

Solution

x	0	1	2	3	4	5	6
f(x)	1	0.5	0.2	0.1	0.0588	0.0385	0.622

By trapezoidal rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$\int_0^6 \frac{dx}{1+x^2} = \frac{1}{2}\left[1 + 0.622 + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)\right]$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.4108$$

Solution

x	0	1	2	3	4	5	6
f(x)	1	0.5	0.2	0.1	0.0588	0.0385	0.622

By Simpson's 1/3 rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$\int_0^6 \frac{dx}{1+x^2} = \frac{1}{2}\left[1 + 0.622 + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)\right]$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.3662$$

Practice Problems

Evaluate $\int_0^4 e^x dx$ using trapezoidal and simpson's 1/3 rule and compare with the actual value.

Suggested books

1. Numerical Methods by **S.R.K Lyenger & R.K. Jain.**

2. Numerical Analysis by **Richard L. Burden.**

3. Introductory methods of Numerical analysis by **S.S. Sastry**.

Thank you