... $g(x, y) \neq 0$, for $(x, y) \in N$.

Theorem 1. *If f, g be two functions defined on some neighbourhood of a point (a, b) such that* $\lim f(x, y) = l$, $\lim g(x, y) = m$, *when* $(x, y) \to (a, b)$, *then*

(i) $\lim(f + g) = \lim f + \lim g = l + m$

(ii) $\lim(f - g) = \lim f - \lim g = l - m$

(iii) $\lim(f \cdot g) = \lim f \cdot \lim g = l \cdot m$

(iv) $\lim \dfrac{f}{g} = \dfrac{\lim f}{\lim g} = \dfrac{l}{m}$, *provided* $m \neq 0$, *when* $(x, y) \to (a, b)$

The proofs are exactly similar to those of the corresponding theorems for a single variable.

Example 3 (a). Prove that

$$\lim_{(x, y) \to (1, 2)} (x^2 + 2y) = 5$$

■ *Method* 1. (Using definition of limit). We have to show that for any $\varepsilon > 0$, we can find $\delta > 0$, such that

$$|x^2 + 2y - 5| < \varepsilon, \text{ when } |x - 1| < \delta, |y - 2| < \delta$$

If $|x - 1| < \delta$, and $|y - 2| < \delta$, then

$$1 - \delta < x < 1 + \delta \text{ and } 2 - \delta < y < 2 + \delta, \text{ excluding } x = 1, y = 2$$

Thus

$$1 - 2\delta + \delta^2 < x^2 < 1 + 2\delta + \delta^2$$

and

$$4 - 2\delta < 2y < 4 + 2\delta$$

Adding

$$5 - 4\delta + \delta^2 < x^2 + 2y < 5 + 4\delta + \delta^2$$

or

$$-4\delta + \delta^2 < x^2 + 2y - 5 < 4\delta + \delta^2$$

Now if $\delta \leq 1$, it follows that

$$-5\delta < x^2 + 2y - 5 < 5\delta$$

i.e.,

$$|x^2 + 2y - 5| < 5\delta = \varepsilon$$

so that $\delta = \varepsilon/5$ (or $\delta = 1$ whichever is smaller).

$$\therefore \quad |x^2+2y-5|<\varepsilon \text{ when } |x-1|<\delta, |y-2|<\delta$$

$$\therefore \quad \lim_{(x,y)\to(1,2)} (x^2+2y)=5$$

Method 2. Using above theorem on algebra of limits,

$$\lim_{(x,y)\to(1,2)} (x^2+2y) = \lim_{(x,y)\to(1,2)} x^2 + \lim_{(x,y)\to(1,2)} 2y = 1+4=5.$$

Example 3 (*b*). Show that

(*i*) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2}=0$, (*ii*) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}=\frac{1}{3}$.

■ (*i*) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x \cdot \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 0.1 = 0$

(*ii*) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)} = \lim_{t\to 0}\frac{\sin^{-1}t}{\tan^{-1}3t}$, where $\displaystyle t = xy-2 = \lim_{t\to 0}\frac{1/\sqrt{1-t^2}}{3/(1+9t^2)} = \frac{1}{3}$

Ex. 1. Show that $\displaystyle\lim_{(x,y)\to(0,1)} \tan^{-1}(y/x)$, does not exist.

$\left[\textit{Hint}: \text{ Limit from the left is } -\dfrac{\pi}{2} \text{ and that from the right } \dfrac{\pi}{2}\right]$.

Ex. 2. Show, by using the definition that

$$\lim_{(x,y)\to(1,2)} 3xy = 6$$

Ex. 3. Prove that

(*i*) $\displaystyle\lim_{(x,y)\to(4,\pi)} x^2\sin\frac{y}{8} = 8\sqrt{2}$, (*ii*) $\displaystyle\lim_{(x,y)\to(0,1)} e^{-1/x^2(y-1)^2}=0$,

(*iii*) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x+y-1}{\sqrt{x}-\sqrt{1-y}} = 0,\ x>0,\ y<1.$

1.6 Repeated Limits

If a function f is defined in some neighbourhood of (a, b), then the limit

$$\lim_{y\to b} f(x, y),$$

if it exists, is a function of x, say $\phi(x)$. If then the limit $\lim_{x\to a}\phi(x)$ exists and is equal to λ, we write

$$\lim_{x \to a} \lim_{y \to b} f(x, y) = \lambda$$

and say that λ is a *repeated limit* of f as $y \to b, x \to a$.

If we change the order of taking the limits, we get the other repeated limit

$$\lim_{y \to b} \lim_{x \to a} f(x, y) = \lambda' \text{ (say)}$$

when first $x \to a$, and then $y \to b$.

These two limits may or may not be equal.

Note: In case the simultaneous limit exists, these two repeated limits if they exist are necessarily equal but the converse is not true. However *if the repeated limits are not equal, the simultaneous limit cannot exist.*

Example 4. (*i*) Let

$$f(x, y) = \frac{xy}{x^2 + y^2}, \text{ then}$$

$$\lim_{y \to 0} \lim_{x \to 0} f(x, y) = \lim_{y \to 0} (0) = 0,$$

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) = 0.$$

Thus, the repeated limits exist and are equal. But the simultaneous limit does not exist which may be seen by putting $y = mx$.

(*ii*) Let

$$f(x, y) = \frac{y - x}{y + x} \cdot \frac{1 + x}{1 + y}, \text{ then}$$

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) = \lim_{x \to 0} \left(-\frac{1 + x}{1} \right) = -1,$$

$$\lim_{y \to 0} \lim_{x \to 0} f(x, y) = \lim_{y \to 0} \left(\frac{1}{1 + y} \right) = 1.$$

Thus, the two repeated limits exist but are unequal, consequently the simultaneous limit cannot exist, which may be verified by putting $y = mx$.

Example 5. Show that the limit exists at the origin but the repeated limits do not, where

$$f(x, y) = \begin{cases} x \sin\left(\dfrac{1}{y}\right) + y \sin\left(\dfrac{1}{x}\right), & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$

Here $\lim_{y \to 0} f(x, y)$, $\lim_{x \to 0} f(x, y)$ do not exist and therefore $\lim_{x \to 0} \lim_{y \to 0} f(x, y)$; $\lim_{y \to 0} \lim_{x \to 0} f(x, y)$ do not exist.

Again

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| < |x| + |y| \le 2(x^2 + y^2)^{1/2} < \varepsilon,$$

if

$$x^2 < \frac{\varepsilon^2}{4}, \; y^2 < \frac{\varepsilon^2}{4}$$

or

$$|x| < \frac{\varepsilon}{2} = \delta, \; |y| < \frac{\varepsilon}{2} = \delta$$

Thus for $\varepsilon > 0, \exists\, \delta > 0$ such that

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| < \varepsilon, \text{ when } |x| < \delta, \; |y| < \delta$$

$$\Rightarrow \qquad \lim_{(x,\, y) \to (0,\, 0)} \left(x \sin \frac{1}{y} + y \sin \frac{1}{x} \right) = 0.$$

Example 6. Show that the repeated limits exist at the origin and are equal but the simultaneous limit does not exist, where

$$f(x, y) = \begin{cases} 1, & \text{if } xy \ne 0 \\ 0, & \text{if } xy = 0 \end{cases}$$

■ Here

$$\lim_{y \to 0} f(x, y) = \begin{cases} 1, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$\therefore \qquad \lim_{x \to 0} \lim_{y \to 0} f(x, y) = 1$$

Similarly,

$$\lim_{y \to 0} \lim_{x \to 0} f(x, y) = 1$$

Hence, the repeated limits exist and are equal.

Again, since there are points arbitrarily near (0, 0) at which f is equal to 0 and points arbitrarily near (0, 0) at which f is equal to 1, therefore, there is an $\varepsilon > 0$, such that

$$|f(x, y) - f(0, 0)| = |f(x, y)| \nless \varepsilon,$$

for all points in any neighbourhood of (0, 0).

Hence, $\lim_{(x,\, y) \to (0,\, 0)} f(x, y)$ does not exist.

Ex. 1. Show that $\lim\limits_{(x,\, y)\to(0,\, 0)} f(x, y)$ and $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x, y)$ exist, but $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x, y)$ does not, where

$$f(x, y) = \begin{cases} y + x \sin\left(\dfrac{1}{y}\right), & \text{if } y \neq 0 \\ 0 & , \text{ if } y = 0. \end{cases}$$

Ex. 2. Show that $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x, y)$ exists, but the other repeated limit and the double limit do not exist at the origin, when

$$f(x, y) = \begin{cases} y \sin(1/x) + xy/(x^2 + y^2), & x \neq 0 \\ 0 & , x = 0. \end{cases}$$

Ex. 3. Show that the repeated limits exist but the double limit does not when $(x, y) \to (0, 0)$:

(*i*) $f(x, y) = \dfrac{x - y}{x + y}$, (*ii*) $f(x, y) = \dfrac{x^2 y^2}{x^4 + y^4 - x^2 y^2}$

(*iii*) $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$ (*iv*) $f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & x \neq y \\ 0, & x = y \end{cases}$

Ex. 4. Show that the limit and the repeated limits exist when $(x, y) \to (0, 0)$:

$$f(x, y) = \begin{cases} xy\,\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$