$ht = 2a(m_1 - m_2)\ \{ \ \ 3$

V. Imp

**8·5.** *If the central acceleration is equal to $\frac{\mu}{r^2}$, to find the orbit.*

Here $$\frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^2}.$$

Integrating we have $$\frac{h^2}{p^2} = \frac{2\mu}{r} + C.$$

Comparing with $\frac{b^2}{p^2} = \frac{2a}{r} \mp 1$, which is an ellipse or hyperbola referred to the focus, we get

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{C}{\mp 1}$$

$$\therefore \quad h = \sqrt{\frac{\mu b^2}{a}} = \sqrt{\mu l}$$ where $l$ is the semi-latus rectum

and

$$C = \mp \frac{\mu}{a}.$$

Hence the orbit is an ellipse, parabola $(p^2 = ar)$ or hyperbola according as $C$ is negative, zero or positive.

Also $$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + C = \mu\left(\frac{2}{r} \mp \frac{1}{a}\right).$$

Hence for elliptic orbit, $$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

,, hyperbolic ,, ,, $$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$$

for parabolic orbit $$v^2 = \frac{2\mu}{r}.$$

$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$

Also since $\frac{1}{2}h$ is the areal velocity

$\frac{1}{2}h.\ T$ = area of the ellipse $= \pi ab$

*i.e.*, periodic time $T=\dfrac{2\pi ab}{h}=\dfrac{2\pi ab}{\sqrt{\mu\dfrac{b^2}{a}}}=\dfrac{2\pi}{\sqrt{\mu}}\,a^{3/2}$

When the central acceleration is from the centre, we have

$$\frac{h^2}{p^3}\frac{dp}{dr}=-\frac{\mu}{r^2}$$

$$\therefore \qquad \frac{h^2}{p^2}=-\frac{2\mu}{r}+C.$$

Comparing with $\dfrac{b^2}{p^2}=1-\dfrac{2a}{r}$ which is the farther branch of the hyperbola,

we get $$\frac{h^2}{b^2}=\frac{\mu}{a}=C$$

so that $$h=\sqrt{\mu l} \text{ and } v^2=\frac{h^2}{p^2}=\mu\left(\frac{1}{a}-\frac{2}{r}\right).$$

Thus when the central force is repulsive varying inversely as the square of the distance, the farther branch of the hyperbola is described.

## § 6. Kepler's Laws of Planetary Motion.

1. *Each planet describes an ellipse having the sun in one of its foci.*

2. *The radius vector, drawn from the sun to the planet, describes equal areas in equal times.*

3. *The squares of the periodic times of the various planets are proportional to the cubes of the major axes of their orbits,*

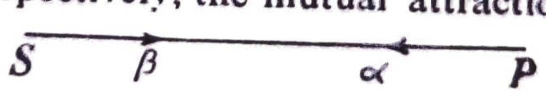
*Deductions.* The second law says that the areal velocity is constant or $r^2\dot\theta$ = constant ; hence the force on planet is entirely directed to the sun.

From the first law it follows that the force on the planet is inversely proportional to the square of its distance from the sun.

From the third law it follows, since $T=\dfrac{2\pi}{\sqrt{\mu}}\,a^{\frac{3}{2}}$ that $\mu$ is constant *i.e.*, acceleration at unit distance from the sun is the same for all planets.

**Meaning of $\mu$.**

From *Newton's law of gravitation*, we know that if $S$, $P$ be the masses of the sun and of the planet respectively, the mutual attraction between them is




$$F=\gamma\,\frac{S\,.\,P}{r^2}$$

where $r$ is the distance between, $\gamma$ the constant of gravitation.

Thus the acceleration of the planet is

$$\alpha=\frac{F}{P}=\gamma\frac{S}{r^2}$$

and that of the sun is $\beta=\dfrac{F}{S}=\gamma\dfrac{P}{r^2}$

Hence the acceleration of planet relative to the sun is

$$\alpha+\beta=\gamma\frac{(S+P)}{r^2}=\frac{\mu}{r^2}$$

$$\therefore \quad \mu=\gamma(S+P).$$

Thus the periodic time of the planet is given by

$$T=\frac{2\pi a^{3/2}}{\sqrt{\gamma(S+P)}}$$

For two planets $P$, $P_1$ having semi-major axis $a$, $a_1$ in their orbits, the periodic times $T$, $T_1$ are related as

$$\frac{T}{T_1}=\sqrt{\frac{S+P_1}{S+P}}\left(\frac{a}{a_1}\right)^{3/2}$$

or
$$\frac{S+P}{S+P_1}\cdot\frac{T^2}{T_1^2}=\frac{a^3}{a_1^3}.$$

If the planet $P$ has a satellite $p$, $d$ being its mean distance from $P$, its periodic time is

$$t=\frac{2\pi}{\sqrt{\gamma(P+p)}}d^{3/2}.$$

$$\therefore \quad \frac{S+P}{P+p}\cdot\frac{T^2}{t^2}=\frac{a^3}{d^3}.$$

Ex. 13. *A particle describes an ellipse under a force* $\dfrac{\mu}{r^2}$ *and has a velocity* $v$ *at a distance* $r$ *from the centre of force, show that its periodic time is*

$$\frac{2\pi}{\sqrt{\mu}}\left(\frac{2}{r}-\frac{v^2}{\mu}\right)^{-3/2}$$

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right)$$

$$\therefore \quad \frac{v^2}{\mu}=\frac{2}{r}-\frac{1}{a}$$

or
$$\frac{1}{a}=\frac{2}{r}-\frac{v^2}{\mu}$$

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \text{etc.}$$

**Ex. 14.** *If $V_1$ and $V_2$ are the velocities of a planet when it is spectively nearest and farthest from the sun, prove that*

$$(1-e)V_1 = (1+e)V_2.$$

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$\therefore \quad V_1^2 = \mu\left\{\frac{2}{a(1-e)} - \frac{1}{a}\right\} = \frac{\mu}{a}\,\frac{1+e}{1-e}$$

$$V_2^2 = \mu\left\{\frac{2}{a(1+e)} - \frac{1}{a}\right\} = \frac{\mu}{a}\,\frac{1-e}{1+e}$$

Hence $\dfrac{V_1^2}{V_2^2} = \text{etc.}$

**Ex. 15.** *If $\omega$ be the angular velocity of a planet at the nearer d of the major axis, prove that its period is*

$$\frac{2\pi}{\omega}\sqrt{\frac{1+e}{(1-e)^3}}$$

$$r^2\dot{\theta} = h$$

$$\therefore \quad a^2(1-e)^2\,\omega = h, \qquad h^2 = \mu l = \mu a(1-e^2)$$

$$\therefore \quad a^4(1-e)^4\,\omega^2 = h^2 = \mu a(1-e^2)$$

$$\therefore \quad \mu = \frac{a^3(1-e)^3\,\omega^2}{1+e}$$

Hence $T = \dfrac{2\pi}{\sqrt{\mu}} a^{3/2} = \text{etc.}$

**Ex. 16.** *If the velocity in a given elliptic orbit (major axis 2a) is same at a certain point P whether the orbit is being described in odic time T about one focus S or in periodic time T′ about the r focus S′, prove that*

$$SP = \frac{2aT'}{T+T'}, \quad S'P = \frac{2aT}{T+T'}.$$

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}, \quad T' = \frac{2\pi}{\sqrt{\mu'}} a^{3/2},$$

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right),$$

$$V'^2 = \mu'\left(\frac{2}{r'} - \frac{1}{a}\right)$$

$$\therefore \quad \mu\left(\frac{2}{r}-\frac{1}{a}\right)=\mu'\left(\frac{2}{r'}-\frac{1}{a}\right)$$

or $$\mu\frac{2a-r}{ar}=\mu'\frac{2a-r'}{ar'}$$

or $$\mu\frac{r'}{r}=\mu'\frac{r}{r'}$$

or $$\left(\frac{r'}{r}\right)^2=\frac{\mu'}{\mu}$$

Also $$\frac{T}{T'}=\sqrt{\frac{\mu'}{\mu}}=\frac{r'}{r}$$

$$\therefore \quad \frac{T+T'}{T'}=\frac{r+r'}{r}=\frac{2a}{r}$$

and $$\frac{T'}{T}=\frac{r}{r'}$$

$$\therefore \quad \frac{T+T'}{T}=\frac{r+r'}{r'}=\frac{2a}{r'}.$$

Ex. 17. *A particle describes an ellipse under a force to the focus S. When the particle is at one extremity of the minor axis, its kinetic energy is doubled without any change in the direction of motion. Prove that the particle proceeds to describe a parabola.*

At the end of minor axis, $r=a$

$$V^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right)=\mu\left(\frac{2}{a}-\frac{1}{a}\right)=\frac{\mu}{a}$$

If $V_1$ be the velocity when the kinetic energy is doubled,

$$\tfrac{1}{2}MV_1^2=2\,.\,\tfrac{1}{2}MV^2$$

i.e., $$V_1^2=2V^2=\frac{2\mu}{a}$$

Hence it moves in a parabola as for a parabola, $v^2=\dfrac{2\mu}{r}$.

Ex. 18. *If the velocity of the earth at any point of its orbit, assumed circular, were increased by about one-half, prove that it would describe a parabola about the sun as focus. Show also that if a body were projected from the earth with a velocity exceeding 7 miles per second, it will not return to the earth.*

For a circular orbit,

$$V^2=\mu\left(\frac{2}{a}-\frac{1}{a}\right)=\frac{\mu}{a}$$

which is parallel to $SP$

Thus the velocity is $e\dfrac{\mu}{h}$ perp. to major axis

and $\dfrac{\mu}{h}$ perp. to radius vector.

## EXAMPLES VIII (C)

1. A particle describes an ellipse about one focus. Show that greatest and least angular velocities occur at the ends of the major axis and that if $\alpha$, $\beta$ be these angular velocities the mean angular velocity is

$$\frac{2(\alpha\beta)^{\frac{3}{4}}}{\sqrt{\alpha}+\sqrt{\beta}}.$$

[Hint. $\alpha=\dfrac{h}{r_1^2}$, $\beta=\dfrac{h}{r_2^2}$, $r_1=a(1-e)$, $r_2=a(1+e)$

mean angular velocity$=\dfrac{2\pi}{T}$. ]

2. If a particle describes an ellipse as a central orbit about a focus, pro that the velocity at the end of the minor axis is a geometric mean between velocities at the ends of any diameter.

3. Prove that if when the particle is at a distance $r$ from the focus, i velocity is $v$ in a direction making an angle $\phi$ with the radius vector then

$$e^2\mu^2=(v^2r-\mu)^2\sin^2\phi+\mu^2\cos^2\phi.$$

[Hint. $v^2=\mu\left(\dfrac{2}{r}-\dfrac{1}{a}\right)$, $h^2=\mu a(1-e^2)$, $h=vp=vr\sin\phi$.]

4. A particle under a central acceleration $\mu/r^2$ is projected with velocity at a distance $R$. Show that the path is a rectangular hyperbola if the angle o projection is

$$\sin^{-1}\frac{\mu}{VR\left(V^2-\dfrac{2\mu}{R}\right)^{\frac{1}{2}}}.$$

5. A particle describes an ellipse about a centre of forces at the foc