System of linear equation

Example:

(1) $ax = b$

Case-1: if $a \neq 0$, then

$x = \frac{b}{a}$ is unique solution of the equation $ax = b$.

Case-2 If $a = 0$, and $b \neq 0$ then Equation has no solution.

Case-3 If $a = 0$ and $b = 0$ then the equation $ax = b$ has infinitely many solution.

(2) $x + 2y = 1$ $x + 3y = 1$

$(x, y)^t = (1, 0)$ is unique solution.

(3) $x + 2y = 1$ and $2x + 4y = 2$

$(x, y)^t = (1 - 2y, y)^t$

infinitely many solution.

Example - 3

$$x + 2y = 1 \quad \text{and} \quad 2x + 4y = 3$$

has no solution.

Def: (1) A system of m- linear equations in n- variables (or n. unknowns) is of the following form:

$$a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n = b_1$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

where $x_1, x_2, \ldots x_n$ are unknowns and a_{ij} and b_k are constants.

(2) A sequence of numbers $(s_1, s_2, \ldots s_n)^*$ is a solution of the system if $x_1 = s_1, \ldots x_n = s_n$ satisfy each equation in the system.

(3) If $b_1 = b_2 = \cdots = b_m = 0$, then the system of linear equation is called homogeneous system of linear equation.

So, question is how to find ~~system of linear~~ solution of a given system of linear equation?

Note 1st: ~~A~~ a homogeneous system of linear equation has always a solution, namely

$x_1 = 0 = \cdots = x_n = 0$, called trivial solution.

So, given a homogeneous system of linear equation, we always look for a non-trivial (non-zero) solution.

Def: (1) A system of linear equation is said to be consistent if it has atleast one solution.

(2) A system of linear equation is said to inconsistent if it has no solution.

The idea for solving a system of linear equation is to transfer the system of linear equation in a simpler system of linear equation without changing the solution set.

Examble:

Consider the system of 2- linear equations in 2- unknowns.

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \quad \text{— (A)} \\ a_2x + b_2y &= c_2 \quad \text{— B} \end{aligned}\right\} \text{— (1)}$$

Eliminate x from (B)

(A) $\times -\frac{a_2}{a_1} + B$

$$\left(b_2 - \frac{a_2}{a_1}\right) y = c_2 - \frac{a_2}{a_1} c_1$$

Now the system of linear equation

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ \left(b_2 - \frac{a_2}{a_1}b_1\right) y &= \left(c_2 - \frac{a_2}{a_1}c_1\right) \end{aligned}\right\} \text{— (2)}$$

has the same solution as (1).

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

Now consider the system of linear equation

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$a_{i1}x_1 \cdots + a_{in}x_n = b_i \rightarrow i\text{th row}$$

$$a_{j1}x_1 + \cdots + a_{jn}x_n = b_j \rightarrow j\text{th row}$$

$$\vdots$$

$$a_{m1}x_1 \cdots + a_{mn}x_n = b_m$$

and let $x_1 = s_1, \cdots x_n = s_n$ is a solution of this system.

Note-1 Interchang rows does not alter the solution set:

$$a_{11}x_1 + - + a_{1n}x_n = b_1$$

$$\vdots$$

$$a_{j1}x_1 + - + a_{jn}x_n = b_j.$$

$$\vdots$$

$$a_{i1}x_1 + a_{in}x_n = b_i$$

$$\vdots$$

$$a_{m1}x_1 - + - a_{mn}x_n = b_m$$

Still $x_1 = s_1 \cdots x_n = s_n$ is a solution of this system of linear equation.

Note - 2

Multiply an equation by a constt. 'c' don not change the solution.

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$ca_{i1}x_1 \cdots + ca_{in}x_n = cb_i$$

$$\vdots$$

$$a_{m1}x_1 \quad + a_{mn}x_n = b_m$$

$x_1 = s_1, \cdots x_n = s_n$ is also a solution.

Note-3: Adding a constt. multiply of row to another row don not alter the solution.

$$a_{11}x_1 \cdots + a_{1n}x_n = b_1$$

$$\vdots$$

$$(a_{i1} + ca_{j1})x_1 \cdots + (a_{in} + ca_{jn})x_n = b_i + cb_j$$

$$a_{j1}x_1 \cdots + a_{jn}x_n = b_j$$

$$\vdots$$

$$a_{m1}x_1 \cdots + a_{mn}x_n = b_m$$

$x_1 = s_1 \cdots x_n = s_n$ is also a solution.

Now matrix come into picture.

Consider the system of linear equation

$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$

$\vdots$

$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$

then this system can be written in a matrix form

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{i1} & & a_{in} \\ \vdots & & \\ a_{m1} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The matrix :

$\begin{bmatrix} a_{ij} \end{bmatrix}$ is called the co-efficient matrix of the system.

The matrix:

$$\left[\begin{array}{ccc:c} a_{11} & - & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & - & a_{mn} & b_m \end{array}\right]$$ is called augmented

matrix of the system.

Ex: Put this system of linear equation in matrix form, indicate its co.eff. and augmented mat

$$x_1 + 3x_2 - 2x_3 = 3$$

$$2x_1 + 6x_2 - 2x_3 + 4x_4 = 18$$

$$x_2 + x_3 + 3x_4 = 10.$$

Method of finding solution of a given system of linear equation:

Given a system of linear equation:

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$\vdots$$

$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

We find its augment matrix,

$$[A \mid b] = \begin{bmatrix} a_{11} & \cdots & a_{1n} & \vdots & b_1 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & & a_{mn} & \vdots & b_m \end{bmatrix}$$

Then apply the following elementary row operations to $[A \mid b]$:

(1) Multiply a non-zero constt. throughout a row,

$$R_i \to cR_i$$

(2) Interchange any two rows.

$$R_i \leftrightarrow R_j$$

(3) Add a constt. multiple of a row to another row,

$$R_i \to R_i + cR_j.$$

(4) Make first non-zero entry in each row non-zero to reduce $[A|b]$ into form

$$\left[\begin{array}{ccc|c} c_{11} & \cdots & c_{1n} & d_1 \\ & \ddots & \vdots & d_2 \\ & & \ddots & \vdots \\ & & c_{mi} \cdots c_{mn} & d_m \end{array}\right]$$

This process is called ~~Gauss-elimi~~ ~~meth~~ Gauss-elimination method.

Ex:

$$2y + 4z = 2$$
$$x + 2y + 2z = 3$$
$$3x + 4y + 6z = -1$$

$$[A|b] = \left[\begin{array}{ccc|c} 0 & 2 & 4 & 2 \\ 1 & 2 & 2 & 3 \\ 3 & 4 & 6 & -1 \end{array}\right]$$

Apply $R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 3 & 4 & 6 & -1 \end{array}\right] \longrightarrow R_3 \to R_3 - 3R_1$$

$$\left[\begin{array}{cccc} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & -2 & 0 & -10 \end{array}\right] \longrightarrow R_3 \to R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & -8 \end{array}\right]$$

$z = -2$

$$\Rightarrow \begin{array}{ll} x + 2y + 2z = 3 & \Rightarrow x = -1 \\ 2y + 4z = 2 & \Rightarrow y = +5 \\ 4z = -8 & \Rightarrow z = -2 \end{array}$$

Backward substitution method #

Def:
Two augmented matrices (or system of linear equations) are said to be row-equivalent if one can be transformed to other by a finite sequence of elementary row operations.

→ Note that elementary row operations can be applied to any matrix.

Def:
A row-echelon form of an augmented matrix is of the following form:

(1) The zero rows, if they exist, come last in the order of rows.

(2) The first non-zero entries in the non-zero rows are 1, called leading 1's.

(3) Below each leading 1, is a column of zeros. Thus in any consecutive non-zero rows, the leading in the lower row appears farther to the right than the leading 1 in the upper row.

The reduced row-echelon form of an augmented matrix is of the form:

(4) Above each leading 1 is a column of zeros, in addition to the row-echelon form.

Two matrices are said to be row-equivalent, if one can be obtained from other by apply finitely many elementary row op—

Exercise: Identify which of the following is in row-echelon form or reduced row echelon form;

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 6 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 2 & 6 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Methods of finding solutions of a system of linear equations

* suppose $Ax = b$ ~~be~~ is a system of linear equations given in a matrix form.

let $[A|b]$ be the augmented matrix.

→ Then, we find row-Echelon form of $[A|b]$ and ~~we~~ we use backward substitution method to find the solution.

This process is called Gauss-Elimination method.

→ If we find reduced row-Echelon form of $[A|b]$ and use backward substitution method to find the solution, then this process is called Gauss-Jordan Elimination method.

Ex:

$$y + z = 2$$

$$2x + 3z = 5$$

$$x + y + z = 3$$

$$[A \mid b] = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & 0 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to 2R_3 - R_1} \begin{bmatrix} 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3 & -3 \end{bmatrix}$$

$$\xrightarrow{R_1 \to \frac{1}{2}R_1,\ R_3 \to -\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad z = 1 \quad y = 1, \quad x = 1$$

unique solutn.

Ex: solve the system of linear equation equation by Gaussian elimination method.

$$x + y + z = 3$$

$$x + 2y + 2z = 5$$

$$3x + 4y + 4z = 12$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 3 & 4 & 4 & 12 \end{bmatrix} \xrightarrow{R_2 \to -R_1 + R_2} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 3 & 4 & 4 & 12 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -3R_1 + R_2} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$0 \cdot x + 0 \cdot y + 0 \cdot z = 1$$

Hence this system of linear equation has no solution.

* If given a system of non-homogen linear equations, if row-echelon form of ~~an~~ corresp augmented matrix has a row of the form $[0, 0, 0 \dots 0, b]$ $b \neq 0$, then the system has no solution.

i.e. system is inconsistent.

Example:

$$2y - z = 1$$
$$4x - 10y + 3z = 5$$
$$3x - 3y + 0 = 6$$

$$\begin{bmatrix} 0 & 2 & -1 & 1 \\ 4 & -10 & 3 & 5 \\ 3 & -3 & 0 & 6 \end{bmatrix} \xrightarrow{R_{12}} \begin{bmatrix} 4 & -10 & 3 & 5 \\ 0 & 2 & -1 & 1 \\ 3 & -3 & 0 & 6 \end{bmatrix}$$

$$\xrightarrow{R_3 \to -\frac{3}{4} R_1 + R_3} \begin{bmatrix} 4 & -10 & 3 & 5 \\ 0 & 2 & -1 & 1 \\ 0 & \frac{18}{4} & -\frac{9}{4} & \frac{9}{4} \end{bmatrix} \xrightarrow{R_3 \to -\frac{9}{4} R_2 + R_3} \begin{bmatrix} 4 & -10 & 3 & 5 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & \cancel{0} \end{bmatrix}$$

$6 + \frac{-15}{4}$

$\frac{30}{4} - 3$

has no solution.

Example: Solve the system of linear eqn by Gauss elimination method:

$$x + y + z = 3$$

$$x + 2y + 2z = 5$$

$$3x + 4y + 4z = 11$$

Augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 3 & 4 & 4 & 11 \end{pmatrix}$$

Find the Echelon form of this matrix

$R_2 \to R_2 - R_1$ $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 3 & 4 & 4 & 11 \end{bmatrix}$

$R_3 \to -3R_1 + R_3$
$\longrightarrow$ $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow$

infinitely many solutions

$$x + y + z = 3$$

$$y + z = 2$$

$$(1, 2 - z, z)$$

Ex

$$x+y+z=7$$

$$x+2y+2z=5$$

$$3x+4y+4z=12$$

Def: Among the variables in a system, the ones corresponding to the columns containing leading 1's are called basic variables, and the ones corresponding to the columns without leading in are called the free variables if they are any.

Q. Solve the system of linear equation by Gaussian Elimination method.

Ans.
$$-x + y + 2z = 0$$
$$3x + 4y + z = 0$$
$$2x + 5y + 3z = 0$$

$$[A|b] = \begin{bmatrix} -1 & 1 & 2 & 0 \\ 3 & 4 & 1 & 0 \\ 2 & 5 & 3 & 0 \end{bmatrix} \xrightarrow{R_2 \to 3R_1 + R_2} \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 7 & 7 & 0 \\ 2 & 5 & 3 & 0 \end{bmatrix} \xrightarrow{R_3 \to 2R_1 + R_3}$$

$$\begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 7 & 7 & 0 \end{bmatrix} \xrightarrow{R_3 \to -R_2 + R_3} \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & \boxed{-2} & 0 \\ 0 & \boxed{1} & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x - y + 2z = 0$$
$$y + z = 0$$

$$\Rightarrow y = -z$$

$$x = -z,$$

$(-z, -z, z)$ are the solution.

Ex: Find the reduced row Echelon form of the following matrix.

$$A = \begin{bmatrix} 0 & 1 & 2 & 4 & -1 \\ 1 & 3 & -1 & 0 & 5 \\ 2 & 0 & 4 & 1 & 3 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & 3 & -1 & 0 & 5 \\ 0 & 1 & 2 & 4 & -1 \\ 2 & 0 & 4 & 1 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1}$$

$$\begin{bmatrix} 1 & 3 & -1 & 0 & 5 \\ 0 & 1 & 2 & 4 & -1 \\ 0 & -6 & 6 & 1 & -7 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 6R_2} \begin{bmatrix} 1 & 3 & -1 & 0 & 5 \\ 0 & 1 & 2 & 4 & -1 \\ 0 & 0 & 18 & 25 & -13 \end{bmatrix}$$

$R_3 \to \frac{1}{18} R_3$

$$\begin{bmatrix} 1 & 3 & -1 & 0 & 5 \\ 0 & 1 & 2 & 4 & -1 \\ 0 & 0 & 1 & \frac{25}{18} & \frac{-13}{18} \end{bmatrix} \xrightarrow{R_2 \to R_2 \to 2R_3} \begin{bmatrix} 1 & 3 & -1 & 0 & 5 \\ 0 & 1 & 0 & \frac{11}{9} & \frac{4}{9} \\ 0 & 0 & 1 & \frac{25}{18} & \frac{-13}{18} \end{bmatrix}$$

$$R_1 \to R_1 + R_3 \quad \begin{bmatrix} 1 & 3 & 0 & \frac{25}{18} & 5 - \frac{13}{18} \\ 0 & 1 & 0 & \frac{11}{9} & \frac{4}{9} \\ 0 & 0 & 1 & \frac{25}{18} & \frac{-13}{8} \end{bmatrix} \xrightarrow{R_1 \to R_1 \to 3R_2}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\text{[illegible]} & \text{[illegible]} \\ 0 & 1 & 0 & \frac{11}{9} & \frac{4}{9} \\ 0 & 0 & 1 & \frac{25}{18} & \frac{-13}{18} \end{bmatrix}$$

Example: Find row Echelon form and reduced row Echelon form of the following matrices.

(1)
$$\begin{bmatrix} 1 & -3 & 2 & 1 & 2 \\ 3 & -9 & 10 & 2 & 9 \\ 2 & -6 & 4 & 2 & 4 \\ 2 & -6 & 8 & 1 & 7 \end{bmatrix}$$

(2)
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}$$