Interpolation-4

Content

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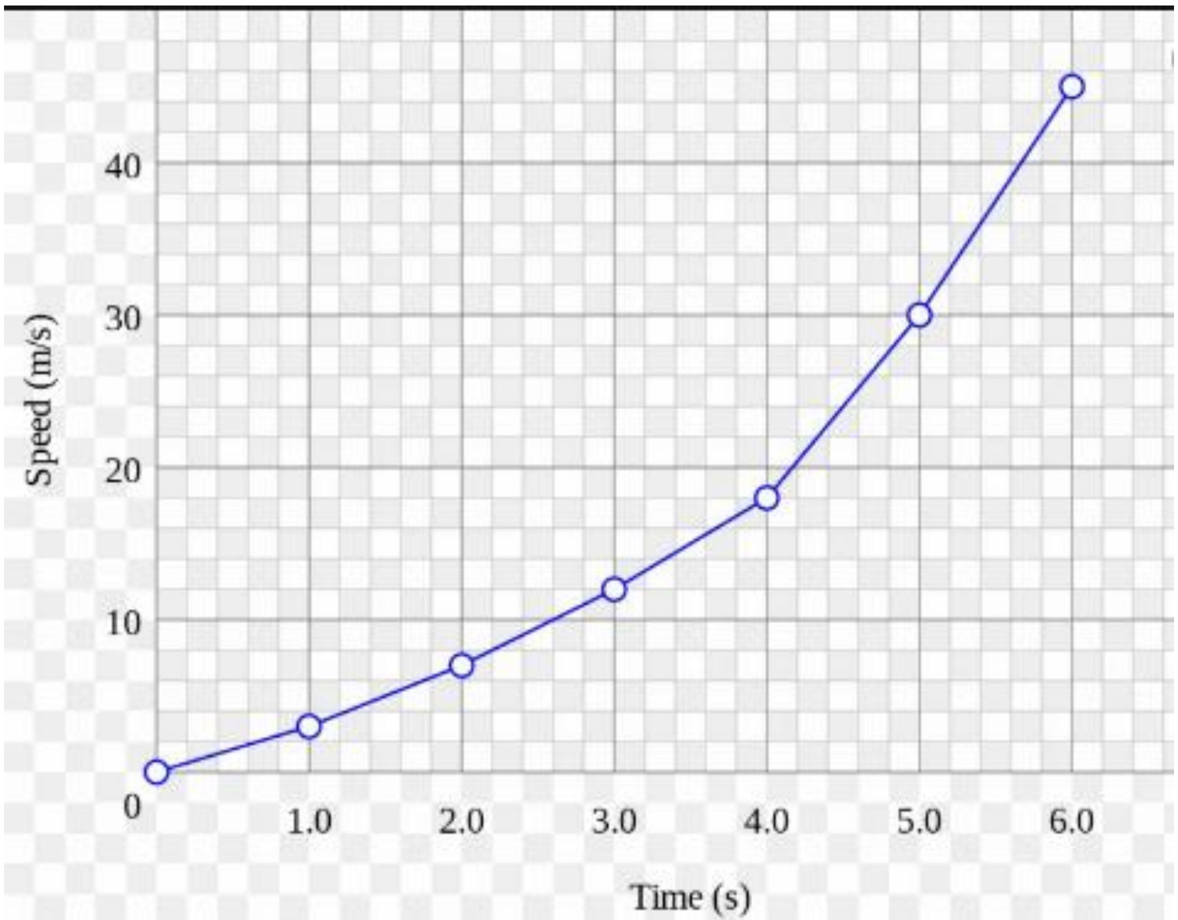

Speed (m/s)
40
30
20
10
0
1.0
2.0
3.0
4.0
5.0
6.0
Time (s)

Finite Differences

Now for equal interval, we consider the more simpler formula. Let for equal spaced (h) data, We have x_0, x_0 +h, x_0 +2h,……… x_0 +nh, Or $x_n = x_0 + nh$, n=0,1,2…….

Some finite difference operators are as follows:

1. Shift Operator
2. Forward difference operator
3. Backward difference operator
4. Central difference operator

Shift Operator (E)

It is defined as

$$Ef(x) = f(x+h)$$

$$EEf(x) = E^2 f(x) = f(x+2h)$$

$$.$$

$$.$$

$$EE..Ef(x) = E^n f(x) = f(x+nh)$$

For example

$$E^{1/2} f(x) = f\left(x+\frac{h}{2}\right)$$

$$E^{-1/2} f(x) = f\left(x-\frac{h}{2}\right)$$

Forward difference Operator(Δ)

It is defined as $\Delta f(x) = f(x+h) - f(x)$

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$.$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

Where, Δ is called the forward difference operator &
Δy_0, Δy_1…. Δy_{n-1} are called first forward differences

Forward difference Operator(Δ) (Cont…)

The difference of first forward difference is called second forward difference and are denoted by $\Delta^2 y_0$, $\Delta^2 y_1$……

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0)$$
$$= \Delta(y_1) - \Delta(y_0)$$
$$= (y_2 - y_1) - (y_1 - y_0)$$
$$= y_2 - 2y_1 + y_0$$

Forward difference Operator(Δ) (Cont…)

The difference of first forward difference is called second forward difference

$$\Delta^2 f(x) = \Delta.\Delta f(x) = \Delta(f(x+h) - f(x))$$
$$= \Delta f(x+h) - \Delta f(x)$$
$$= [f(x+2h) - f(x+h)] - [f(x+h) - f(x)]$$
$$= f(x+2h) - 2f(x+h) + f(x)$$

Similarly, third, fourth and other forward difference can be obtained

Backward difference Operator(∇)

It is defined as

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla^2 f(x) = \nabla.\nabla f(x) = \nabla(f(x) - f(x-h))$$

$$= \nabla f(x) - \nabla f(x-h)$$

$$= [f(x) - f(x-h)] - [f(x-h) - f(x-2h)]$$

$$= f(x) - 2f(x-h) + f(x-2h)$$

$$= f(x-2h) - 2f(x-h) + f(x)$$

Central Difference Operator (δ)

- It is defined as:

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$\delta^2 f(x) = \delta f\left(x+\frac{h}{2}\right) - \delta f\left(x-\frac{h}{2}\right)$$

$$= f\left(x+\frac{h}{2}+\frac{h}{2}\right) - f\left(x+\frac{h}{2}-\frac{h}{2}\right) - \left[f\left(x-\frac{h}{2}+\frac{h}{2}\right) - f\left(x-\frac{h}{2}-\frac{h}{2}\right)\right]$$

$$= f(x+h) - f(x) - f(x) + f(x-h)$$

$$= f(x+h) - 2f(x) - f(x-h)$$

Forward difference Table

x	f(x)	Δ^1	Δ^2	Δ^3
x_0	y_0	$\Delta y_0 = y_1 - y_0$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_1	y_1	$\Delta y_1 = y_2 - y_1$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
x_2	y_2	$\Delta y_2 = y_3 - y_2$		
x_3	y_3			

Backward difference Table

x	f(x)	∇^1	∇^2	∇^3
x_0	y_0	$\nabla y_1 = y_1 - y_0$	$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
x_1	y_1	$\nabla y_2 = y_2 - y_1$	$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
x_2	y_2	$\nabla y_3 = y_3 - y_2$		
x_3	y_3			

Note: $\Delta y_0 = \nabla y_1$ and $\Delta^2 y_0 = \nabla^2 y_2$

Relationship between E, Δ and ∇

As

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta f(x) = Ef(x) - f(x)$$

$$\Delta f(x) = (E-1)f(x)$$

$$E = 1 + \Delta$$

Again,

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla f(x) = f(x) - E^{-1} f(x)$$

$$\nabla f(x) = (1 - E^{-1}) f(x)$$

$$\nabla = (1 - E^{-1})$$

$$E^{-1} = 1 - \nabla$$

Relationship between E and δ

As

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$\delta f(x) = E^{1/2} f(x) - E^{-1/2} f(x)$$

$$\delta f(x) = (E^{1/2} - E^{-1/2}) f(x)$$

$$\delta = (E^{1/2} - E^{-1/2})$$

$$\delta = E^{-1/2}(E-1)$$

Show that $\nabla = E^{-1}\Delta$

R.H.S.

$E^{-1}\Delta$

$E^{-1}(E-1)$ as $\Delta = (E-1)$

$1-E^{-1}$

∇ L.H.S

Show that

1. $\Delta - \nabla = -\Delta\ \nabla$
2. $\Delta + \nabla = \Delta / \nabla - \nabla / \Delta$
3. $\delta = \nabla(1 - \nabla)^{-1/2}$

Gregory Newton Formula or Newton's formula for interpolation

1. Newton's Forward Difference Formula: For equal interval, we consider the more simpler formula. Let for equal spaced (h) data, We have x_0, $x_0 + h$, $x_0 + 2h$, ……… $x_0 + nh$, Or $x_n = x_0 + nh$, n=0,1,2…….

$$f(x) = f\left[x_0 + \left(\frac{x - x_0}{h}\right)h\right]$$

$$Let\ldots: u = \left(\frac{x - x_0}{h}\right)$$

$$f(x) = f(x_0 + uh)$$

$$f(x) = E^u f(x_0)$$

$$f(x) = (1 + \Delta)^u f(x_0)$$

$$f(x) = f(x_0) + u\frac{\Delta f(x_0)}{1!} + u(u-1)\frac{\Delta^2 f(x_0)}{2!} + \ldots u(u-1)\ldots(u-(n-1))\frac{\Delta^n f(x_0)}{n!}$$

2. Newton's Backward Difference Formula: For equal interval, we consider the more simpler formula. Let for equal spaced (h) data, We have x_0, x_0+h, x_0+2h,……… x_0+nh, Or $x_n = x_0+nh$, n=0,1,2…….

$$f(x) = f\left[x_n + \left(\frac{x-x_n}{h}\right)h\right]$$

$$let....u = \left(\frac{x-x_n}{h}\right)$$

$$f(x) = f(x_n + uh)$$

$$f(x) = E^u f(x_n)$$

$$f(x) = (1-\nabla)^{-u} f(x_n)$$

$$f(x) = \left[1 + u\frac{\nabla}{1!} - u(-u-1)\frac{\nabla^2}{2!} + ..u(u+1)...(u+n-1)\frac{\nabla^n}{n!}\right] f(x_n)$$

$$f(x) = f(x_n) + u\frac{\nabla f(x_n)}{1!} + u(u+1)\frac{\nabla^2 f(x_n)}{2!} + ..u(u+1)...(u+n-1)\frac{\nabla^n f(x_n)}{n!}$$

Example-1

Given the values

x	-2	-1	0	1	2	3
f(x)	-4	1	0	-1	4	21

Construct Newton forward difference table for the given data.

.

Solution

We have

x	f(x)	Δ	Δ^2	Δ^3	Δ^4
−2	−4	(1+4)=5	−6	6	0
−1	1	0−1=−1	0	6	0
0	0	−1−0=−1	6	6	
1	−1	4+1=5	12		
2	4	21−4=17			
3	21				

Example-2

Given the values

x	-2	-1	0	1	2	3
f(x)	-4	1	0	-1	4	21

Construct an interpolating polynomial for the data given in the table using Newton forward interpolation formula and compute the value of f(-1.5).

.

Solution (cont..)

x	f(x)	Δ	Δ^2	Δ^3	Δ^4
-2	-4	(1+4)=5	-6	6	0
-1	1	0-1=-1	0	6	0
0	0	-1-0=-1	6	6	
1	-1	4+1=5	12		
2	4	21-4=17			
3	21				

From Newton's forward interpolation formula

$$f(x) = f(x_0) + u\frac{\Delta f(x_0)}{1!} + u(u-1)\frac{\Delta^2 f(x_0)}{2!} + \ldots u(u-1)\ldots(u-(n-1))\frac{\Delta^n f(x_0)}{n!}$$

Solution (cont..)

$$f(x) = f(x_0) + u\frac{\Delta f(x_0)}{1!} + u(u-1)\frac{\Delta^2 f(x_0)}{2!} + ..u(u-1)(u-2)\frac{\Delta^3 f(x_0)}{3!}$$

$$as..u = \frac{(x-x_0)}{h}$$

$$f(x) = f(x_0) + (x-x_0)\frac{\Delta f(x_0)}{1!} + (x-x_0)(x-x_1)\frac{\Delta^2 f(x_0)}{2!} + (x-x_0)(x-x_1)(x-x_2)\frac{\Delta^3 f(x_0)}{3!}$$

$$f(x) = -4 + (x+2)\frac{5}{1!} + (x+2)(x+1)\frac{-6}{2!} + (x+2)(x+1)(x)\frac{6}{3!}$$

$$f(x) = -4 + 5x + 10 + (x^2+3x+2)(-3) + (x^2+3x+2)x$$

$$f(x) = x^3 + 3x^2 + 2x - 3x^2 - 9x - 6 + 5x + 6$$

$$f(x) = x^3 - 2x$$

Solution (cont..)

At x=-1.5

$$f(x) = x^3 - 2x$$

$$f(x) = (-1.5)^3 - 2(-1.5)$$

$$f(x) = -0.375$$

Example-3

Given the values

x	-2	-1	0	1	2	3
f(x)	-4	1	0	-1	4	21

Compute the value of f(2.5) using Newton Backward interpolation formula and

.

Backward difference Table

x	f(x)	∇^1	∇^2	∇^3
x_0	y_0	$\nabla y_1 = y_{1-}y_0$	$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
x_1	y_1	$\nabla y_2 = y_{2-}y_1$	$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
x_2	y_2	$\nabla y_3 = y_{3-}y_2$		
x_3	y_3			

Solution (cont..)

x	f(x)	∇^1	∇^2	∇^3	∇^4
-2	-4	(1+4)=5	-6	6	0
-1	1	0-1=-1	0	6	0
0	0	-1-0=-1	6	6	
1	-1	4+1=5	12		
2	4	21-4=17			
3	21				

From Newton's backward interpolation formula

$$f(x) = f(x_n) + u\frac{\nabla f(x_n)}{1!} + u(u+1)\frac{\nabla^2 f(x_n)}{2!} + ..u(u+1)...(u+n-1)\frac{\nabla^n f(x_n)}{n!}$$

Where, $u = \left(\dfrac{x - x_n}{h}\right)$

Solution (cont..)

From Newton's backward interpolation formula

$$f(x) = f(x_n) + u\frac{\nabla f(x_n)}{1!} + u(u+1)\frac{\nabla^2 f(x_n)}{2!} + ..u(u+1)...(u+n-1)\frac{\nabla^n f(x_n)}{n!}$$

Where, $u = \left(\dfrac{x - x_n}{h}\right)$ u=(2.5-3)/1=-0.5

$$f(2.5) = 21 + (-0.5)\frac{17}{1!} + (-0.5)(-0.5+1)\frac{12}{2!} + (-0.5)(-0.5+1)(-0.5+2)\frac{6}{3!}$$

$$f(2.5) = 21 - 8.5 - 1.5 - 0.375$$

$$f(2.5) = 10.625$$

Practice Problem

Given the values

x	0	1	2	3	4
f(x)	1	7	23	55	109

find **f(0.5)** and **f(1.5)** using Newton's forward difference formula.

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Suggested books

1. Numerical Methods by **S.R.K Lyenger & R.K. Jain.**

2. Numerical Analysis by **Richard L. Burden.**

3. Introductory methods of Numerical analysis by **S.S. Sastry**.

Thank you