Q2. Let $u = x\sin y + y\sin x$, then show that,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 y}{\partial y \partial x}$$

from L.H.S →

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial (x\sin y + y\sin x)}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\cancel{\frac{\partial}{\partial y}}\left(x\cos y + \sin x\right)\right)$$

$$= \cos y + \cos x$$

and, from R.H.S

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(x\sin y + y\sin x)\right)$$

$$= \frac{\partial}{\partial y}(\sin y + y\cos x)$$

$$= \cos y + \cos x$$

$$= L.H.L$$ hence proved

Q.3→ Let $x^x . y^y . z^z = C$, where c ~~is~~ a positive constant, then show that,

$$\frac{\partial^2 z}{\partial x \partial y} = -(x\log ex)^{-1} \text{ at } x=y=z$$

$$\frac{-1}{x\log ex}$$

Soln – given that, $x^x y^y z^z = C$

Taking ~~log~~ log both sides,

we get,

Solⁿ → Given. $x^x y^y z^z = c$

let $z = f(x, y)$ such that,
z depends on x and y (dependent variable)
y and x are independent variable.

So, taking $\log_e$ both side,

$$x \cdot \log_e x + y \log_e y + z \log_e z = \log_e c$$

$$\Rightarrow \quad z \ln z = \ln c - x \ln x - y \ln y \quad \text{------ (1)}$$

partial differentiating both side w.r.t y,
we get,

$$\Rightarrow \left(\frac{z}{z} + \ln z\right)\frac{\partial z}{\partial y} = 0 - 0 - (1 + \ln y)$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = \frac{-(1 + \ln y)}{(1 + \ln z)} \quad \text{----- (2)}$$

Now, again taking partial derivative with respect to x,
we get,

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{+(1 + \ln y)}{(1+\ln z)^2}\left(0 + \frac{1}{z} \cdot \frac{\partial z}{\partial x}\right) \quad \text{----- (3)}$$

Now, if we take partial derivative of eqⁿ (1), with respect to x,
we get,

Similarly,
$$\frac{\partial z}{\partial x} = \frac{-(1 + \ln x)}{(1 + \ln z)} \quad \text{----- (4)}$$

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So, from equation (3) and (4),
we get,

$$\frac{\partial^2 z}{\partial x \, \partial y} = \frac{(1+\ln y)}{(1+\ln z)^2}\left(\frac{1}{z} \cdot \frac{-(1+\ln x)}{(1+\ln z)}\right)$$

Now, at $x = y = z$,

we get,

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{\cancel{(1+\ln x)}}{(1+\ln x)^2} \cdot \frac{1}{x} \cdot \frac{(-1)\cancel{(1+\ln x)}}{\cancel{(1+\ln x)}}$$

$$= -x^{-1}(1+\ln x)^{-1}$$

$$= -x^{-1}(\log_e e + \ln x)^{-1}$$

$$= -x^{-1}(\ln ex)^{-1}$$

$$= -(x \ln ex)^{-1}$$

$= R.H.S$ hence proved

Q. 2, if $z(x+y) = (x^2+y^2)$
then, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

Solution - Given:- $z(x+y) = (x^2+y^2)$
Taking partial derivative with respect to x,
we get,

$$z(1+0) + (x+y)\left(\frac{\partial z}{\partial x}\right) = 2x$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{(2x - z)}{x+y} \quad \text{------} (1)$$

also, taking partial derivative with respect to y

Q. 4) if $\frac{x^2}{(a+u)} + \frac{y^2}{(b+u)} + \frac{z^2}{(c+u)} = 1$

then show that,

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$

i.e. $u_x^2 + u_y^2 + u_z^2 = 2(x u_x + y u_y + z u_z)$

Q. 5) if $u = \log(x^3 + y^3 + z^3 - 3xyz)$
then find that,

i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ (Hint:-

iii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u$

$\frac{\partial^2 u}{\partial x^2} +$ $\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)$

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Q. 6) If $z = f(x+ay) + \phi(x-ay)$

then show that

$$\frac{\partial^2 z}{\partial y^2} = \frac{a^2 \partial^2 z}{\partial x^2}$$

Q. 7. if $V = (x^2+y^2+z^2)^{-1/2}$

then find,

(i) $\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z}$

(ii) $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

Q. 8) if $v = \sqrt{x^2+y^2+z^2}$

then find :-

1) $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}$

2) $x V_x + y V_y + z V_z$

3) $V_{xx} + V_{yy} + V_{zz}$

$(x^2+y^2+z^2)^{1/2}$

$\sqrt{x^2+y^2+z^2} \cdot 1 - x \cdot \frac{1}{2(x^2+y^2+z^2)^{3/2}}$

$\frac{(x^2+y^2+z^2) + x^2}{(x^2+y^2+z^2)^{3/2}}$

Sollutions :

6 - $\frac{\partial z}{\partial y} = af'(x+ay) - a\phi'(x-ay)$

$$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 \phi''(x-ay) \quad \text{--- (1)}$$

and, taking p'd w.r.t x,

$$\frac{\partial z}{\partial x} = f'(x+ay) + \phi'(x-ay)$$

again, taking p'd w.r.t x,

- **Total derivative** → (or total differentiability) →

let $u = f(x, y)$ be a function of two variables x and y and $x = \phi(t)$ and $y = \psi(t)$, then

$$u = f\left(\phi(t), \psi(t)\right)$$

Therefore, $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt},$$

and, $\frac{du}{dt}$ is called total differentiation of u with respect to t.

Example →

Q → $$u = xy^2$$

let, $$x = t^2$$
$$y = t^3$$

~~So~~, $$\frac{\partial u}{\partial x} = y^2 \quad \text{and} \quad \frac{\partial u}{\partial y} = 2xy$$

and, $$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = 3t^2$$

and ~~thus~~, $$\frac{du}{dt} = \frac{d\left(t^2(t^3)^2\right)}{dt} = \frac{d(t^8)}{dt}$$

$$\frac{du}{dt} = 8t^7$$

and, $$\frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} = y^2 \times 2t + 2xy \times 3t^2$$
$$= 2y^2 t + 6xyt^2$$