Second Exam

Math 8 — Fall 2014
Solution

$\boxed{20}$ 1. **Multiple Choice.**

(a) Evaluate the integral $\displaystyle\int_0^{\pi/3} \cos^2(t)\,dt$

A. $\dfrac{\pi}{6} - \dfrac{\sqrt{3}}{8}$

B. $\dfrac{\pi}{6} + \dfrac{\sqrt{3}}{8}$

C. $\dfrac{\pi}{3}$

D. $\dfrac{\pi}{6} + \dfrac{\sqrt{3}}{4}$

E. $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}$

Solution: A duplication formula implies that

$$\begin{aligned}
\int_0^{\pi/3} \cos^2(t)\,dt &= \int_0^{\pi/3} \frac{1+\cos(2t)}{2}\,dt \\
&= \left.\frac{t}{2} + \frac{\sin(2t)}{4}\right]_0^{\frac{\pi}{3}} \\
&= \frac{\frac{\pi}{3}}{2} + \frac{\sin\left(\frac{2\pi}{3}\right)}{4} \\
&= \frac{\pi}{6} + \frac{\sqrt{3}}{8}.
\end{aligned}$$

(b) By using an appropriate trigonometric substitution, the indefinite integral

$$\int \frac{x^2}{(9x^2-2)^{3/2}}\,dx$$

can be transformed into which one of the following?

A. $\displaystyle \frac{1}{27}\int \frac{\sec^3\theta}{\tan^2\theta}d\theta$

B. $\displaystyle \frac{2}{9}\int \sec^3\theta d\theta$

C. $\displaystyle \frac{1}{9\sqrt{2}}\int \frac{\sec^2\theta}{\tan^3\theta}d\theta$

D. $\displaystyle \frac{1}{27}\int \tan^2\theta d\theta$

E. $\displaystyle \frac{1}{9\sqrt{2}}\int \tan^2\theta\sec\theta d\theta$

Solution: Setting $x = \dfrac{\sqrt{2}}{3}\sec\theta$, so that $dx = \dfrac{\sqrt{2}}{3}\sec\theta\tan\theta\,d\theta$ and assuming $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, the integral becomes

$$\begin{aligned}
\int \frac{x^2}{(9x^2-2)^{3/2}}\,dx &= \int \frac{\frac{2}{9}\sec^2\theta}{\left(9\cdot\frac{2}{9}\sec^2\theta-2\right)^{3/2}}\cdot\frac{\sqrt{2}}{3}\sec\theta\tan\theta\,d\theta \\
&= \frac{2\sqrt{2}}{27\sqrt{2}^3}\int \frac{\sec^3\theta\tan\theta}{(\tan^2\theta)^{3/2}}\,d\theta \\
&= \frac{1}{27}\int \frac{\sec^3\theta}{\tan^2\theta}\,d\theta.
\end{aligned}$$

(c) Suppose $x = \sin\theta$ for $-\pi/2 < \theta < \pi/2$. Which of the following is equal to

$$\csc\theta + \tan\theta?$$

A. $\dfrac{1}{x} + \dfrac{1}{\sqrt{1-x^2}}$

B. $\dfrac{x+1}{\sqrt{1-x^2}}$

C. $\dfrac{1}{x} + \dfrac{x}{\sqrt{1-x^2}}$

D. $x + \sqrt{1+x^2}$

E. $\dfrac{1}{x} + \dfrac{x}{\sqrt{1+x^2}}$

Solution: The relation $x = \sin\theta$ for $-\pi/2 < \theta < \pi/2$ is equivalent to having $\theta = \arcsin x$, so

$$\csc\theta + \tan\theta = \frac{1}{\sin(\arcsin x)} + \frac{\sin(\arcsin x)}{\cos(\arcsin x)} = \frac{1}{x} + \frac{x}{\cos(\arcsin x)}.$$

Since $-\pi/2 < \theta = \arcsin x < \pi/2$, its cosine is positive so

$$\cos(\arcsin x) = \sqrt{1 - \sin^2(\arcsin x)} = \sqrt{1-x^2}$$

and

$$\csc\theta + \tan\theta = \frac{1}{x} + \frac{x}{\sqrt{1-x^2}}.$$

(d) Which of the following integrals corresponds to $\int \frac{1}{e^{2x}-1} dx$ after the substitution $u = e^x$ and some algebraic manipulations?

A. $\int \left(\frac{1}{2(u-1)} + \frac{1}{2(u+1)} - \frac{1}{u} \right) du$

B. $\int \left(\frac{1}{2(u-1)} + \frac{1}{2(u+1)} \right) du$

C. $\int \left(\frac{1}{2(u-1)} - \frac{1}{2(u+1)} \right) du$

D. $\int \left(\frac{1}{2(u+1)} - \frac{1}{2(u-1)} \right) du$

E. $\int \left(\frac{1}{2(u-1)} - \frac{1}{2(u+1)} - \frac{1}{u} \right) du$

Solution: Substituting $u = e^x$ so that $du = e^x dx$ and $dx = \frac{du}{u}$, we get

$$\int \frac{1}{e^{2x}-1}\, dx = \int \frac{du}{u(u^2-1)} = \int \frac{du}{u(u-1)(u+1)}.$$

The integrand is a rational function that admits a partial fractions decomposition of the form $\frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1}$. Let us determine the coefficients A, B and C:

$$\begin{aligned} \frac{1}{u(u-1)(u+1)} &= \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1} \\ &= \frac{A(u-1)(u+1) + Bu(u+1) + Cu(u-1)}{u(u-1)(u+1)} \\ &= \frac{(A+B+C)u^2 + (B-C)u - A}{u(u-1)(u+1)} \end{aligned}$$

Since the equality must hold for infinitely many values of u, we can identify the coefficients of the numerators, which leads to the linear system:

$$\begin{cases} A+B+C &= 0 \\ B-C &= 0 \\ -A &= 1 \end{cases}.$$

We find that $A = -1$ and $B = C = \frac{1}{2}$, so that

$$\frac{1}{u(u-1)(u+1)} = \frac{1}{2(u-1)} + \frac{1}{2(u+1)} - \frac{1}{u}.$$

$\boxed{25}$ 2. **True/False.**

(a) Any two planes in 3-space that do not intersect are parallel. $\boxed{\text{True}}$

Solution: Two non-parallel planes are either identical or admit a line of intersection, directed by the (non-zero) crossed product of respective normal vectors of the planes.

(b) Any two skew lines in 3-space intersect at a point. $\boxed{\text{False}}$

Solution: By definition, skew lines do not intersect.

(c) If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then $\mathbf{u} \cdot \mathbf{v} = 0$. $\boxed{\text{False}}$

Solution: Consider $\mathbf{u} = \mathbf{v} \neq \mathbf{0}$. Then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|^2 \neq 0$.

(d) If $\mathbf{u}$ is the vector projection of $\mathbf{w}$ onto $\mathbf{v}$, then $(\mathbf{u} - \mathbf{w}) \cdot \mathbf{v} = 0$. $\boxed{\text{True}}$

Solution: By definition of the vector projection, $\mathbf{u} = \dfrac{\mathbf{w} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$. Therefore,

$$\mathbf{u} \cdot \mathbf{v} = \frac{\mathbf{w} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} \cdot \mathbf{v} = \frac{\mathbf{w} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\|\mathbf{v}\|^2 = \mathbf{w} \cdot \mathbf{v}$$

and the result follows by distributivity of the dot product.

(e) Any two parallel lines in 3-space are contained in a common plane. $\boxed{\text{True}}$

Solution: Let Δ_1 and Δ_2 be parallel lines in $\mathbb{R}^3$ with a common direction vector $\mathbf{d}$ and consider a vector $\mathbf{u}$ represented by $\overrightarrow{P_1P_2}$ where $P1$ and P_2 are points on Δ_1 and Δ_2 respectively.

The plane with normal vector $\mathbf{n} = \mathbf{d} \times \mathbf{u}$ containing P_1 also contains P_2 because $\mathbf{n}$ is orthogonal to $\mathbf{u}$ hence to $\overrightarrow{P_1P_2}$. Since $\mathbf{n}$ is also perpendicular to $\mathbf{d}$, the plane must contain both lines.

$\boxed{25}$ 3. Let $\mathbf{a} = \langle 1, 2, -1\rangle$, $\quad \mathbf{b} = 3\mathbf{i} - \mathbf{k}$, $\quad \mathbf{c} = \mathbf{a} \times \mathbf{b}$. Compute the following.

(a) $|\mathbf{a}|$

Solution: $|\mathbf{a}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$

(b) $\mathbf{a} \cdot \mathbf{b}$

Solution: $\mathbf{a} \cdot \mathbf{b} = \langle 1, 2, -1\rangle \cdot \langle 3, 0, -1\rangle = 1 \cdot 3 + 2 \cdot 0 + (-1) \cdot (-1) = 4$

(c) $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{c})$

Solution:

$$\begin{aligned}
(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{c}) &= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{a} + \overbrace{\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})}^{=\mathbf{0}} + \overbrace{\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b})}^{=\mathbf{0}} \\
&= |\mathbf{a}|^2 + \mathbf{a} \cdot \mathbf{b} \\
&= 6 + 4 = 10
\end{aligned}$$

(d) $\mathbf{c} - 2\mathbf{a}$

Solution:

$$\begin{aligned}
\mathbf{c} - 2\mathbf{a} &= (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times (3\mathbf{i} - \mathbf{k}) - 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \\
&= 3\overbrace{\mathbf{i} \times \mathbf{i}}^{=\mathbf{0}} - \overbrace{\mathbf{i} \times \mathbf{k}}^{-\mathbf{j}} + 6\overbrace{\mathbf{j} \times \mathbf{i}}^{-\mathbf{k}} - 2\overbrace{\mathbf{j} \times \mathbf{k}}^{\mathbf{i}} - 3\overbrace{\mathbf{k} \times \mathbf{i}}^{j} + \overbrace{\mathbf{k} \times \mathbf{k}}^{=\mathbf{0}} - 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \\
&= \mathbf{j} - 6\mathbf{k} - 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \\
&= -4\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}
\end{aligned}$$

(e) The equation of the plane through the origin containing $\mathbf{c}$ and $\mathbf{j}$.

Solution: Any non-zero vector perpendicular to $\mathbf{c}$ and $\mathbf{j}$ is normal to that plane. Since $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \cdot \mathbf{j} = 0$, we can take $\mathbf{b}$ as a normal vector and the plane has an equation of the form

$$3x + 0y - z + d = 0.$$

The condition that it must contain the origin implies $d = 0$. Finally, an equation of that plane is $3x - z = 0$.

$\boxed{40}$ 4. **Integrate.**

(a) $\displaystyle\int e^{-x}\cos(2x)dx$

Solution: Integrating by parts with $\left| \begin{array}{ll} u = \cos(2x) & dv = e^{-x}dx \\ du = -2\sin(2x)dx & v = -e^{-x} \end{array} \right.$ leads to

$$I = \int e^{-x}\cos(2x)\,dx = -e^{-x}\cos(2x) - 2\int e^{-x}\sin(2x)\,dx.$$

Integrating by parts again, with $\left| \begin{array}{ll} u = \sin(2x) & dv = e^{-x}dx \\ du = 2\cos(2x)dx & v = -e^{-x} \end{array} \right.$, we get

$$\begin{aligned} I &= -e^{-x}\cos(2x) - 2\left(-e^{-x}\sin(2x) + 2\int e^{-x}\cos(2x)\,dx\right) \\ &= -e^{-x}\cos(2x) + 2e^{-x}\sin(2x) - 4I \end{aligned}$$

It follows that $5I = -e^{-x}\cos(2x) + 2e^{-x}\sin(2x)$ and we conclude that

$$\int e^{-x}\cos(2x)\,dx = (2\sin(2x) - \cos(2x))\,\frac{e^{-x}}{5} + C$$

(b) $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$

Solution:

$$\begin{aligned} \int \sin^2(x)\cos^3(x)\,dx &= \int \sin^2(x)\cos^2(x)\cos(x)\,dx \\ &= \int \sin^2(x)\left(1 - \sin^2(x)\right)\cos(x)\,dx \\ &\overset{u=\sin(x)}{=} \int u^2(1-u^2)\,du \\ &= \int u^2 - u^4\,du \\ &= \frac{u^3}{3} - \frac{u^5}{5} + C \\ &= \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C. \end{aligned}$$

(c) $\int e^{\sin(t)} \sin(t)\cos(t)dt$

Solution: The change of variables $x = \sin(t)$ gives

$$\int e^{\sin(t)} \sin(t)\cos(t)dt = \int xe^x\,dx.$$

Then, integrating by parts with $\left| \begin{array}{ll} u = x & dv = e^x dx \\ du = dx & v = e^x \end{array} \right.$, we get

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

and, substituting back $x = \sin(t)$,

$$\int e^{\sin(t)} \sin(t)\cos(t), dt = (\sin(t) - 1)e^{\sin(t)} + C.$$

(d) $\int \cos(t)\cos(2t)dt$

Solution: Rewriting the integrand as:

$$\cos(t)\cos(2t) = \frac{1}{2}\left(\cos(t) + \cos(3t)\right),$$

we get

$$\begin{aligned} \int \cos(t)\cos(2t)\,dt &= \frac{1}{2}\int \left(\cos(t) + \cos(3t)\right)\,dt \\ &= \frac{1}{2}\sin(t) + \frac{1}{6}\sin(3t) + C. \end{aligned}$$

$\boxed{20}$ 5. (a) Determine a suitable partial fraction decomposition for

$$\frac{x^2 - 7x - 10}{x^4 - 4x^3 + 5x^2}$$

Solution: A factorized form of the denominator is $x^2(x^2 - 4x + 5)$ in which the second factor is irreducible because $(-4)^2 - 4 \cdot 1 \cdot 5 < 0$
Therefore, the partial fraction decomposition must be of the form

$$\frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 - 4x + 5},$$

so that

$$\begin{aligned} \frac{x^2 - 7x - 10}{x^4 - 4x^3 + 5x^2} &= \frac{Ax(x^2 - 4x + 5) + B(x^2 - 4x + 5) + (Cx + D)x^2}{x^2(x^2 - 4x + 5)} \\ &= \frac{(A + C)x^3 + (-4A + B + D)x^2 + (5A - 4B)x + 5B}{x^4 - 4x^3 + 5x^2} \end{aligned}$$

which leads, by identification, to

$$\left\{ \begin{array}{rcr} A + C &=& 0 \\ -4A + B + D &=& 1 \\ 5A - 4B &=& -7 \\ 5B &=& -10 \end{array} \right.$$

We then find

$$\left\{ \begin{array}{lcl} B &=& -2 \\ A &=& \frac{1}{5}(-7 + 4(-2)) = -3 \\ D &=& 1 - 4(-3) - (-2) = -9 \\ C &=& 3 \end{array} \right. ,$$

hence the expected decomposition:

$$\frac{x^2 - 7x - 10}{x^4 - 3x^3 + 5x^2} = -\frac{3}{x} - \frac{2}{x^2} + \frac{3x - 9}{x^2 - 3x + 5}.$$

(b) Suppose $f(x) = \dfrac{1}{2x+1} + \dfrac{3x+1}{x^2} + \dfrac{x}{x^2-4x+5}$. Calculate $\displaystyle\int f(x)\,dx$.

Solution: We integrate each term separately:

$$\int \frac{dx}{2x+1} = \frac{1}{2}\ln|2x+1| + \text{const.}$$

$$\int \frac{3x+1}{x^2}\,dx = 3\int \frac{dx}{x} + \int \frac{dx}{x^2} = 3\ln|x| - \frac{1}{x} + \text{const.}$$

$$\begin{aligned}
\int \frac{x}{x^2-4x+5}\,dx &= \int \frac{x-2}{x^2-4x+5}\,dx + \int \frac{2}{x^2-4x+5}\,dx \\
&= \int \frac{x-2}{x^2-4x+5}\,dx + \int \frac{2}{(x-2)^2+1}\,dx \\
&= \frac{1}{2}\ln|(x-2)^2+1| + 2\arctan(x-2) + \text{const.}
\end{aligned}$$

Finally,

$$\begin{aligned}
\int f(x)\,dx &= 2\arctan(x-2) + \frac{\ln|2x+1|}{2} + 3\ln|x| - \frac{1}{x} + \frac{\ln((x-2)^2+1)}{2} + C \\
&= 2\arctan(x-2) - \frac{1}{x} + \ln\left(|x|^3\sqrt{|2x+1|\,((x-2)^2+1)}\right) + C
\end{aligned}$$

$\boxed{20}$ 6. (a) Find the length of the curve given by the parametric equations

$$\begin{cases} x(t) = e^t \\ y(t) = e^t \cos t \\ z(t) = e^t \sin t \end{cases}$$

between $(1, 1, 0)$ and $(e^{2\pi}, e^{2\pi}, 0)$.

Solution: Let $\mathbf{r}(t) = \langle x(t)\,,\, y(t)\,,\, z(t)\rangle$. Then

$$\begin{aligned} \mathbf{r}'(t) &= \langle x'(t)\,,\, y'(t)\,,\, z'(t)\rangle \\ &= \langle e^t\,,\, e^t(\cos(t) - \sin(t))\,,\, e^t(\sin(t) + \cos(t))\rangle. \end{aligned}$$

So that

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{e^{2t} + e^{2t}(\cos(t) - \sin(t))^2 + e^{2t}(\sin(t) + \cos(t))^2} \\ &= \sqrt{3e^{2t}} = \sqrt{3}e^t. \end{aligned}$$

Since $\langle 1\,,\, 1\,,\, 0\rangle = \mathbf{r}(0)$ and $\langle e^{2\pi}\,,\, e^{2\pi}\,,\, 0\rangle = \mathbf{r}(2\pi)$, the length of the curve between these two points is given by

$$\int_0^{2\pi} \|\mathbf{r}'(t)\|\; dt = \sqrt{3}\int_0^{2\pi} e^t\, dt = \sqrt{3}(e^{2\pi} - 1)$$

(b) Find the unit tangent to the curve $\mathbf{r}(t) = \langle 1, t^2, t^3\rangle$ at $(1, \frac{1}{4}, \frac{1}{8})$.

Solution: The general expression for the tangent vector is $\mathbf{r}'(t) = \langle 0, 2t, 3t^2\rangle$. Since $\langle 1, \frac{1}{4}, \frac{1}{8}\rangle = \mathbf{r}\left(\frac{1}{2}\right)$, the unit tangent vector at the considered point is

$$\mathbf{T}\left(\frac{1}{2}\right) = \frac{\mathbf{r}'\left(\frac{1}{2}\right)}{\left\|\mathbf{r}'\left(\frac{1}{2}\right)\right\|} = \frac{\langle 0, 1, \frac{3}{4}\rangle}{\sqrt{0^2 + 1^2 + \left(\frac{3}{4}\right)^2}} = \frac{\langle 0, 1, \frac{3}{4}\rangle}{\frac{5}{4}} = \left\langle 0, \frac{4}{5}, \frac{3}{5}\right\rangle.$$

$\boxed{20}$ 7. (a) Parametrize the curve of intersection between the elliptic cylinder $x^2 + \dfrac{y^2}{4} = 1$ and the plane $x + y + z = 2$.

Solution: The trace of the cylinder in any horizontal plane is the ellipse with equation $x^2 + \dfrac{y^2}{4} = 1$, which can be parametrized by

$$x(t) = \cos(t) \quad , \quad y(t) = 2\sin(t).$$

The condition corresponding to the plane equation implies that $z(t)$ must satisfy $x(t) + y(t) + z(t) = 2$, so the curve of intersection is parametrized by

$$\left\{ \begin{array}{rcl} x(t) & = & \cos(t) \\ y(t) & = & 2\sin(t) \\ z(t) & = & 2 - \cos(t) - 2\sin(t) \end{array} \right. .$$

(b) Find the curvature of the curve given by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^t\mathbf{k}$ at the point $(1, 1, e)$.

Solution:

$$\begin{aligned} \mathbf{r}(t) &= \langle t, t^2, e^t \rangle \\ \mathbf{r}'(t) &= \langle 1, 2t, e^t \rangle \\ \mathbf{r}''(t) &= \langle 0, 2, e^t \rangle \end{aligned}$$

so

$$\begin{aligned} \mathbf{r}(1) &= \langle 1, 1, e \rangle \\ \mathbf{r}'(1) &= \langle 1, 2, e \rangle \\ \mathbf{r}''(1) &= \langle 0, 2, e \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{r}'(1) \times \mathbf{r}''(1) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & e \\ 0 & 2 & e \end{vmatrix} \\ &= \begin{vmatrix} 2 & e \\ 2 & e \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & e \\ 0 & e \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \mathbf{k} \\ &= 0\mathbf{i} - e\mathbf{j} + 2\mathbf{k} \end{aligned}$$

$$\kappa(1) = \frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \frac{(0^2 + (-e)^2 + 2^2)^{1/2}}{(1^2 + 2^2 + e^2)^{3/2}} = \sqrt{\frac{e^2 + 4}{(e^2 + 5)^3}}.$$

$\boxed{30}$ 8. Aliens from the 4-th dimension are invading our 3-space!!!

From a space defense base located at coordinates $(1,-1,0)$, you notice an alien spacecraft traveling in along the path $\mathbf{r}(t) = \langle t^2-1, 2t-t^2, t\rangle$. At time $t=0$, you launch a missile traveling at constant speed along a straight line. The missile hits the spacecraft at time $t=2$ saving 3-space from the alien invaders!

(a) Is the space defense base on the path of the alien spacecraft?

Solution: Let's look for the time at which the spacecraft would land on the base, that is, a parameter t_0 such that $\mathbf{r}(t_0) = \langle 1,-1,0\rangle$. Such a t_0 needs to simultaneously satisfy

$$\left\{\begin{array}{rcl} t_0^2-1 &=& 1 \\ 2t_0-t_0^2 &=& -1 \\ t_0 &=& 0 \end{array}\right. .$$

The last equation is obviously incompatible with the previous ones, so there can be no such t_0.

(b) Find the equation of the path $\mathbf{p}(t)$ of the missile.

Solution: The target is hit at $t=2$ when its coordinates are $(3,0,2)$. The trajectory of the missile is straight, so it is carried by the line containing the space base $(1,-1,0)$ and directed by $\langle 3-1, 0-(-1), 2-0\rangle = \langle 2,1,2\rangle$.

Since the missile is travelling at constant speed, say v, its trajectory is of the form

$$\mathbf{p}(t) = \langle 1,-1,0\rangle + v\cdot t\langle 2,1,2\rangle.$$

The condition $\mathbf{p}(2) = \langle 3,0,2\rangle$ implies that $v = \dfrac{1}{2}$ and we can conclude that

$$\mathbf{p}(t) = \langle 1,-1,0\rangle + t\left\langle 1, \frac{1}{2}, 1\right\rangle$$

or, equivalently

$$\left\{\begin{array}{rcl} x(t) &=& 1+t \\ y(t) &=& -1+\dfrac{t}{2} \\ z(t) &=& t. \end{array}\right.$$

(c) Does the missile hit the spacecraft from the front or from the rear?

Solution: We will consider that the missile hits the alien spacecraft from the rear if its velocity (tangent) vector makes an acute angle with that of the spacecraft, from the front if the angle is obtuse and from the side if the vectors are perpendicular at the time of impact.

Therefore, we only need to determine the sign of the cosine of the angle of incidence $\alpha = (\widehat{\mathbf{p}'(2), \mathbf{r}'(2)})$, which is given by the formula

$$\mathbf{p}'(2) \cdot \mathbf{r}'(2) = \|\mathbf{p}'(2)\| \cdot \|\mathbf{r}'(2)\| \cos \alpha.$$

More precisely, the sign of $\cos \alpha$ is the same as that of $\mathbf{p}'(2) \cdot \mathbf{r}'(2)$.

For any $t \geq 0$, we have $\mathbf{r}'(t) = \langle 2t, 2 - 2t, 1 \rangle$ and $\mathbf{p}'(t) = \langle 1, \frac{1}{2}, 1 \rangle$ so

$$\mathbf{p}'(2) \cdot \mathbf{r}'(2) = \left\langle 1, \frac{1}{2}, 1 \right\rangle \cdot \langle 4, -2, 1 \rangle = 4 > 0.$$

This implies that $\cos \alpha > 0$ and therefore that the angle of incidence is acute: the alien spacecraft was hit from the rear.